QDA 2023 04 05

ARIMA

AR(1) $\quad X_t = \xi + \phi_1 X_{t-1} + \varepsilon_t \quad \varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$

Simulate say $m = 1000$ data from an AR(1) process

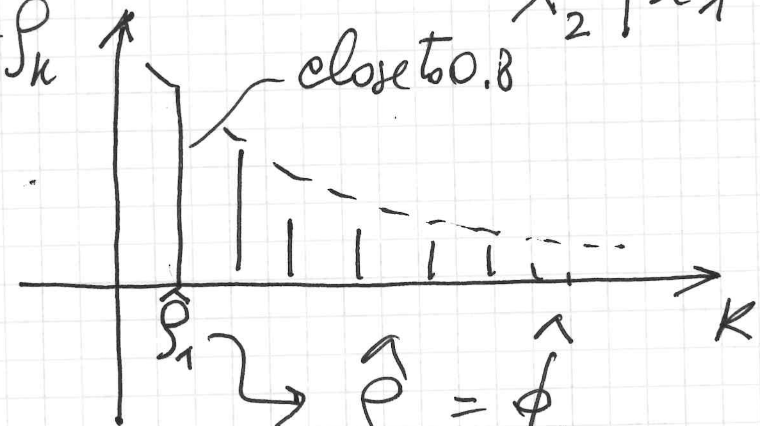
$\boxed{\phi_1 = 0.8} \quad \sigma_\varepsilon^2 = 1 \quad X_0 = 10 \quad \xi = 10$

$X_1 \to x_1 = 10 + 0.8 \cdot x_0 + \varepsilon_1$

$X_2 | x_1 \to x_2 = 10 + 0.8\, x_1 + \varepsilon_2$

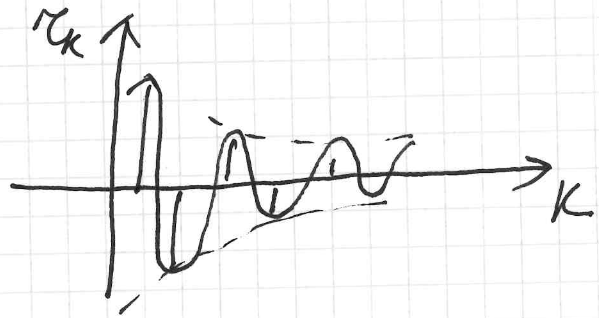
SAMPLE AUTOCORR FUNCTION

$r_k = \hat\rho_k$



$\hat\rho_1 = \hat\phi_1$

$\phi_1 = -0.8$



$AR(p)$ $E(\tilde{X}_t, \tilde{X}_{t-k}) = \gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \dots + \phi_p \gamma_{k-p}$

$k > 1$

$\gamma_0 = \sigma_X^2 = E(\tilde{X}_t \cdot \tilde{X}_t) = \phi_1 \gamma_{-1} + \phi_2 \gamma_{-2} + \dots + \phi_p \gamma_{-p} + \sigma_\varepsilon^2 \overset{*}{=}$

$\phi_1 \tilde{X}_{t-1} + \phi_2 \tilde{X}_{t-2} \dots + \phi_p \tilde{X}_{t-p}$

$\overset{*}{=} \phi_1 \gamma_1 + \phi_2 \gamma_2 + \dots + \phi_p \gamma_p + \sigma_\varepsilon^2$

$\gamma_k = \gamma_{-k}$

$E(X_t, X_{t-k})$ $E(X_{t-k}, X_t)$

let's divide this expression times $\sigma_X^2 = \gamma_0$

$1 = \phi_1 \rho_1 + \phi_2 \rho_2 + \dots + \phi_p \rho_p + \sigma_\varepsilon^2 / \sigma_X^2$ $\sigma_X^2 = \dots$

$$\frac{\sigma_\varepsilon^2}{\sigma_X^2} = 1 - \phi_1 \rho_1 - \phi_2 \rho_2 - \dots - \phi_p \rho_p$$

$$\Rightarrow \quad \sigma_X^2 = \frac{\sigma_\varepsilon^2}{1 - \sum_i \phi_i \rho_i}$$

~~usually~~ as $1 - \underbrace{\sum \phi_i \rho_i} < 1 \implies \sigma_X^2 > \sigma_\varepsilon^2$ (AR(p) with positive autocorr)

$\left[1 - \sum_i \phi_i \rho_i > 1 \implies \sigma_X^2 < \sigma_\varepsilon^2 \right]$ is this possible? Not for an AR(1) but for other AR(p) process? let's think about it

ex: AR(1)

ex: AR(2)

PARTIAL AUTOCORRELATION FUNCTION

AUTOCORRELATION FUNCTION : Assume AR(1)

$X_t \leftrightarrow X_{t-1} \rightarrow \rho_1 \rightarrow$ it's like "fitting"

$X_t \leftrightarrow X_{t-2} \rightarrow \rho_2$

$$X_t = \beta_0 + \beta_1 X_{t-1} + \varepsilon_t$$

$$\xi + \phi_1 X_{t-1} + \varepsilon_t = \beta_0 + \beta_1 X_{t-1} + \varepsilon_t$$

$$X_t = \beta_0 + \beta_1 X_{t-2} + \varepsilon_t$$

? It will be significantly different than 0

$H_0 : \beta_1 = 0$

$H_1 : \beta_1 \neq 0 \leftarrow$ Reject H_0

$$\begin{tikzcd} X_t \arrow[r, bend left, leftrightarrow, "\phi_1"] \arrow[rr, bend right, leftrightarrow, "\phi_1^2"'] & X_{t-1} \arrow[r, bend left, "\phi_1"] & X_{t-2} \end{tikzcd}$$

$$X_t = \phi_1 X_{t-1} + \varepsilon_t + \xi$$

$$X_t = \phi_1 \left[\phi_1 X_{t-2} + \xi + \varepsilon_{t-1} \right] + \varepsilon_t + \xi$$

$$X_t = \phi_1^2 X_{t-2} + \phi_1 \xi + \phi_1 \varepsilon_{t-1} + \xi + \varepsilon_t$$

let's assume again the AR(1) model and let us fit a "complete" autoregressive model even if we are interested in X_t X_{t-2}

$$X_t = \beta_0 + \beta_{11} X_{t-1} + \beta_{12} X_{t-2} + \varepsilon_t$$

ϕ_{12}

$\boxed{AR(1)}$ $X_t = \xi + \phi_1 X_{t-1} + \varepsilon_t$

$H_0 : \beta_{12} = 0$

$H_1 : \beta_{12} \neq 0$

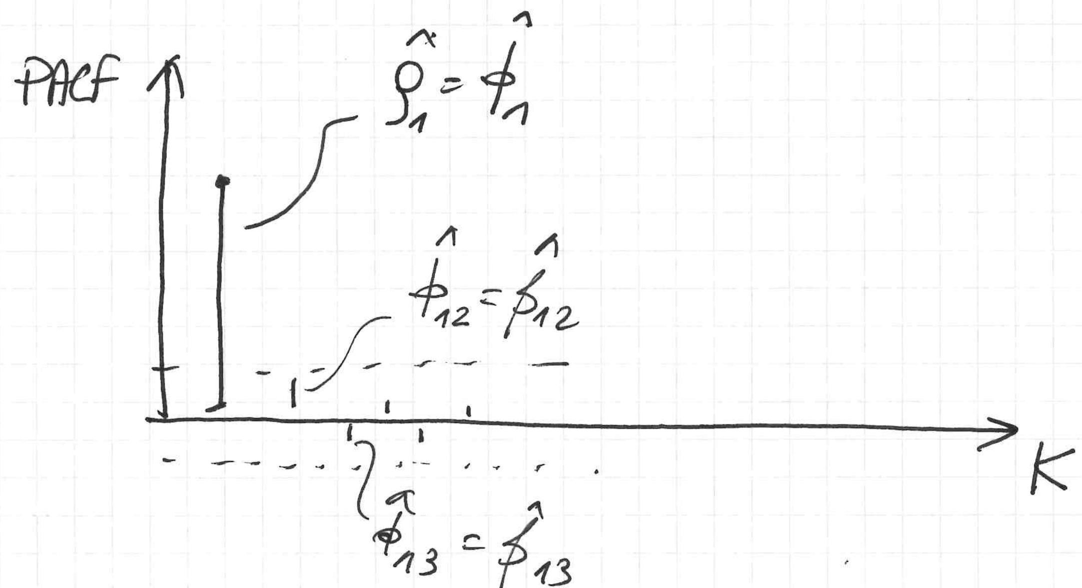
$$X_t = \beta_0 + \beta_{11} X_{t-1} + \beta_{12} X_{t-2} + \beta_{13} X_{t-3} \dots + \varepsilon_t$$

$= 0$

PARTIAL ~~CO~~ AUTOCORRELATION COEFFICIENTS

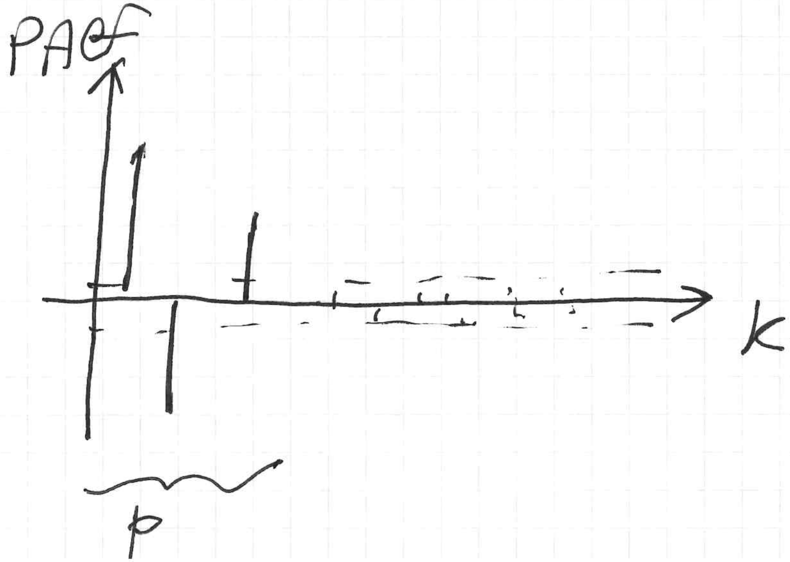
(SAMPLE) PARTIAL AUTOCORRELATION FUNCTION

AR(1)



PACF → provides us the "guess" about the order (p) of an AR(p) model

AR(p)



For an AR(p) process the ACF decays with an exponential pattern while the PACF has only the first p terms that are significantly different from 0

HINT to show that x AR(p) the $\rho_k = \ldots$

$$\gamma_k = Cov(X_t, X_{t-k}) = E(\tilde{X}_t \tilde{X}_{t-k}) =$$

$\tilde{X}_t = X_t - \mu$, $\tilde{X}_{t-k} = X_{t-k} - \mu$

$t-(t-k) = k$

$$= E\left[(\phi_1 \tilde{X}_{t-1} + \phi_2 \tilde{X}_{t-2} + \ldots + \phi_p \tilde{X}_{t-p} + \varepsilon_t)\tilde{X}_{t-k}\right] =$$

$$= E\left[\phi_1 \tilde{X}_{t-1} \cdot \tilde{X}_{t-k} + \phi_2 \tilde{X}_{t-2} \cdot \tilde{X}_{t-k} + \ldots + \phi_p \tilde{X}_{t-p} \tilde{X}_{t-k} + \varepsilon_t \tilde{X}_{t-k}\right.$$

$$= \phi_1 \underbrace{E\left[\tilde{X}_{t-1} \tilde{X}_{t-k}\right]}_{\underbrace{\cancel{t}-1-\cancel{t}+k}_{\underbrace{k-1}_{\gamma_{k-1}}}} + \phi_2 \underbrace{E\left[\tilde{X}_{t-2} \tilde{X}_{t-k}\right]}_{\gamma_{k-2}} + \ldots + \phi_p \underbrace{E\left[\tilde{X}_{t-p} \tilde{X}_{t-k}\right]}_{\gamma_{k-p}} + E\left[\varepsilon_t \tilde{X}\right.$$

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$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \dots + \phi_p \gamma_{k-p} + E(\varepsilon_t \underbrace{\tilde{x}_{t-k}})$$

$$E(\varepsilon_t \tilde{x}_{t-k}) = E[\varepsilon_t (\overbrace{\phi_1 \tilde{x}_{t-k-1} + \dots + \phi_p \tilde{x}_{t-k-p} + \varepsilon_{t-k}}^{\tilde{x}_{t-k}})]$$

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \longrightarrow E(\varepsilon_t \varepsilon_{t-k}) = 0 \qquad k = 1, 2 \dots$$

$$Cov(\varepsilon_t \varepsilon_{t-k}) = 0 \qquad k = 1 \dots$$

$$\boxed{E(\varepsilon_t \varepsilon_{t-k}) = \begin{cases} 0 & k \geq 1 \\ \sigma_\varepsilon^2 & k = 0 \end{cases}}$$

$$E(\varepsilon_t \varepsilon_t) = \sigma_\varepsilon^2 \qquad k = 0$$

$$E[(\varepsilon_t - \underset{\mu_\varepsilon}{0})(\varepsilon_t - \underset{\mu_\varepsilon}{0})] = E[\varepsilon_t^2] = \sigma_\varepsilon^2$$

$$E[\varepsilon_t \tilde{x}_{t-k}] = E[\underbrace{\varepsilon_t \cdot \varepsilon_{t-k}}_{=0 \; k \geq 1}] + \phi_1 E[\varepsilon_t \tilde{x}_{t-k-1}] + \dots]$$

$$\underbrace{f(\varepsilon_{t-k-1}, \varepsilon_{t-k+2} \dots)}$$

$$\sigma_X^2 = \underset{\hookrightarrow k=0}{\gamma_k} = \phi_1 \gamma_{k-1} + \dots + \phi_p \gamma_{k-p} + \sigma_\varepsilon^2 \qquad k=0$$

$$\gamma_k = \phi_1 \gamma_{k-1} + \dots + \phi_p \gamma_{k-p} \qquad k \geq 1$$

— MA(q) —

$$X_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \dots - \theta_q \varepsilon_{t-q} + \xi \qquad \varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

$$E(X_t) = 0 - \theta_1 \cdot 0 - \dots + \xi = \xi$$

$$X_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \dots - \theta_q \varepsilon_{t-q} + \mu$$

$$\tilde{X}_t = X_t - \mu = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \dots - \theta_q \varepsilon_{t-q} + \cancel{\xi}$$

Introduce B $\qquad \tilde{X}_t = \varepsilon_t - \theta_1 B \varepsilon_t - \theta_2 B^2 \varepsilon_t - \dots - \theta_q B^q \varepsilon_t$

$$= \underbrace{\left(1 - \sum_{i=1}^{q} \theta_i B^i\right)}_{\Theta(B)} \varepsilon_t$$

AR(p):

$$A(B) = 1 - \sum^{p} \phi_i B^i$$

$AR(p) \qquad A(B)\tilde{X}_t = \varepsilon_t$

$MA(q) \qquad \tilde{X}_t = \Theta(B)\cdot\varepsilon_t$

$MA(q)$: can this be nonstationary ? No

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2) \longrightarrow \tilde{X}_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \dots - \theta_q \varepsilon_{t-q}$$

$$\boxed{X_t = \mu - \theta_1 \varepsilon_{t-1} \dots - \theta_q \varepsilon_{t-q} + \varepsilon_t}$$

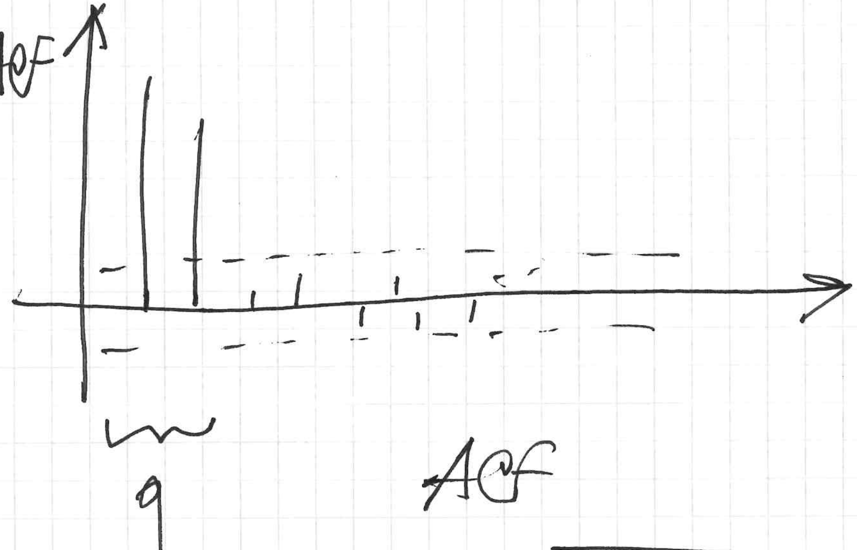
$$E(X_t) = \mu$$

$$Cov(X_t, X_{t-k}) = \gamma_k = \begin{cases} \underline{\neq 0} & k = 1, 2 \dots q \\ 0 & k > q \end{cases}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} \neq 0 & k = 1 \dots q \\ 0 & \underline{k > q} \end{cases}$$

$\Rightarrow$ IDENTIFY q thanks to the $\underline{ACF}$

MA(q)



	ACF	PACF
MA(q)	first q $\rho_k \neq 0$	exponential decay in module
AR(p)	exponential decay in module	first p pacf terms $\neq 0$
unstable AR → I(d)	linear decay	