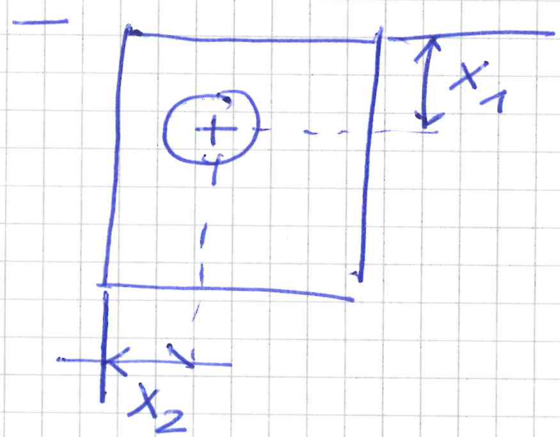
QDA 2023 · 05 · 18

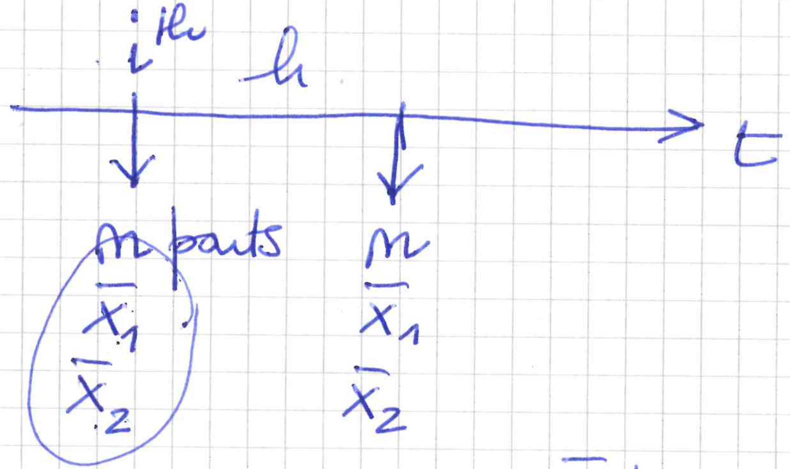
Multivariate control chart (mean - stability)

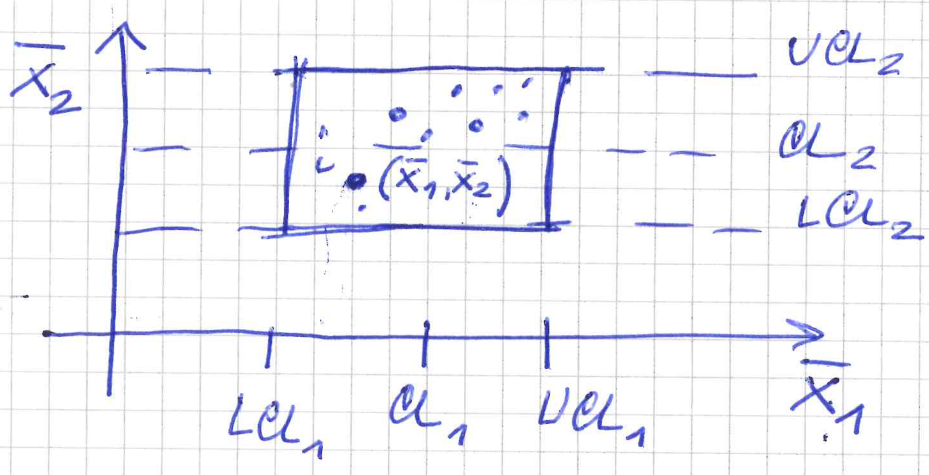
$X_1 \sim N(\mu_1, \sigma_1^2)$

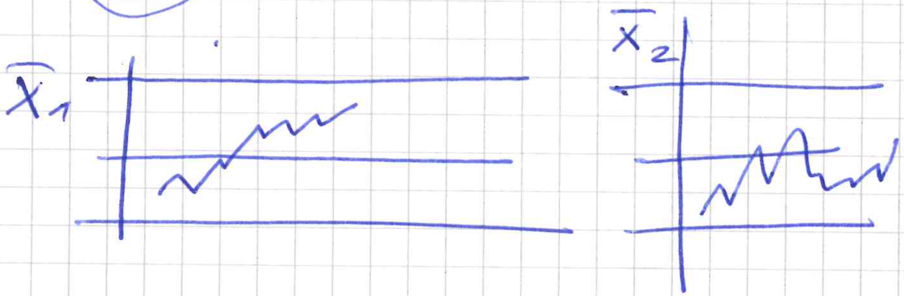
$X_2 \sim N(\mu_2, \sigma_2^2)$



$p = \#$ quality features $= 2$







$$\underline{X}' = [X_1 \dots X_p] \quad \text{VECTOR}$$

$$\underline{X} \sim N_p(\underline{\mu}, \underline{\Sigma}) \quad \text{MULTIVARIATE NORMAL}$$

$$V(\underline{a}'\underline{x}) = \underline{a}' \underline{\Sigma}\, \underline{a}$$

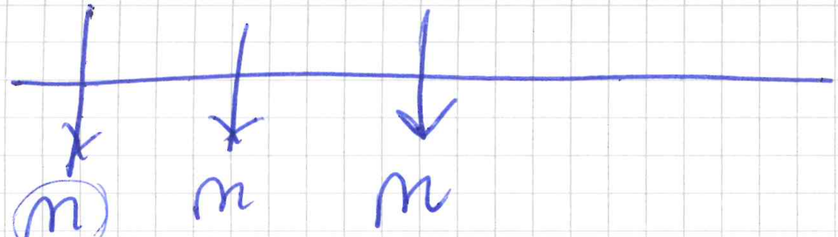
$$\boxed{Q = \underbrace{(\underline{x}-\underline{\mu})'_{1\times p} \; \underline{\Sigma}^{-1}_{p\times p} \; (\underline{x}-\underline{\mu})_{p\times 1}}_{1\times 1} \sim \chi^2(p)}$$

$$= z_1^2 + \dots z_p^2$$

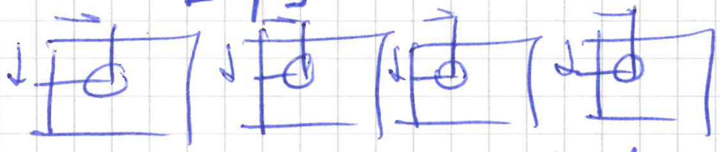
special $p=1$ (univariate case) $X \sim N(\mu, \sigma^2)$

$$(X-\mu)\frac{1}{\sigma^2}(X-\mu) = \frac{(X-\mu)^2}{\sigma^2} = \left(\frac{X-\mu}{\sigma}\right)^2 = z^2 = \chi^2_{(1)}$$

$$z \sim N(0,1)$$



$$\underline{\bar{X}} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \vdots \\ \bar{X}_p \end{bmatrix}$$



m product

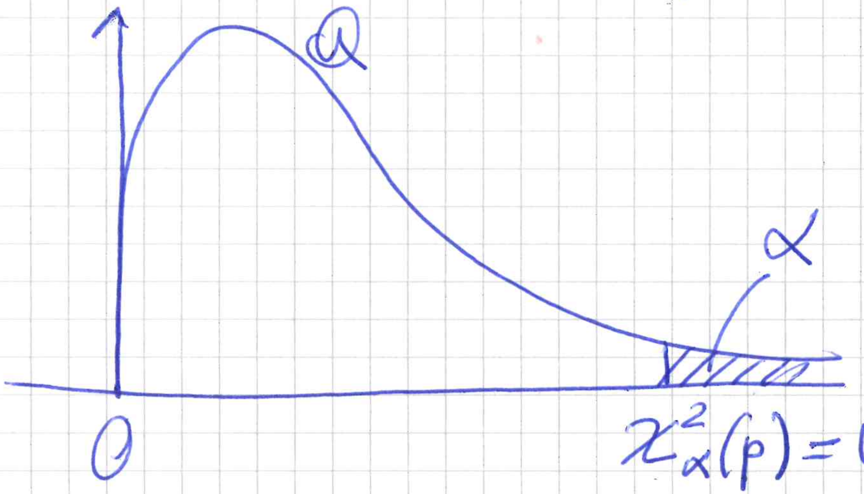
$$\underline{X} \sim N_p(\underline{\mu}, \underline{\Sigma})$$

$$\Downarrow$$

$$\underline{\bar{X}} = \frac{1}{m} \sum_{i=1}^{m} \underline{x}_i \sim N_p\left(\underline{\mu}, \frac{1}{m}\underline{\Sigma}\right)$$

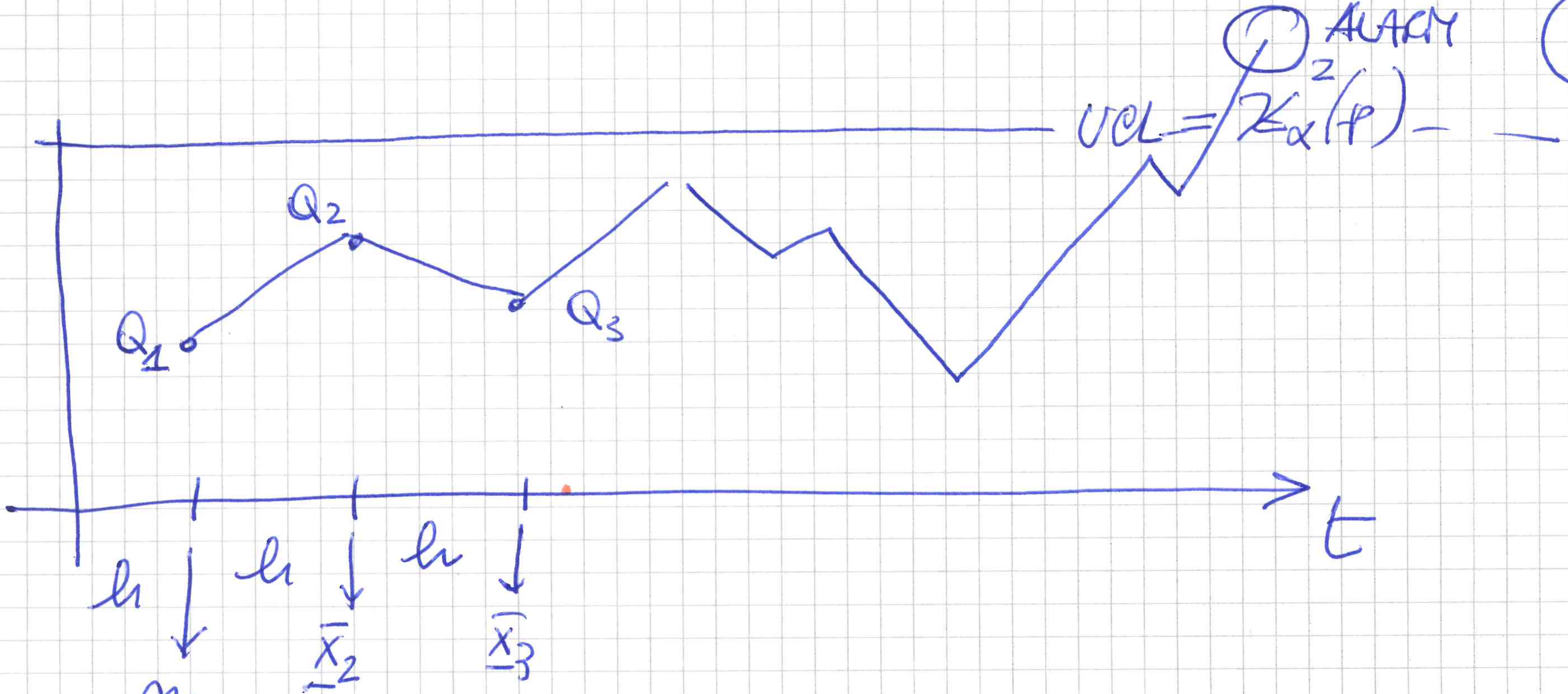
$$Q = \underset{1\times p}{\left(\underline{\bar{X}} - \underline{\mu}\right)'} \underset{p\times p}{\left(\frac{\underline{\Sigma}}{m}\right)^{-1}} \underset{p\times 1}{\left(\underline{\bar{X}} - \underline{\mu}\right)} =$$

$$Q = m \left(\underline{\bar{X}} - \underline{\mu}\right)' \underline{\Sigma}^{-1} \left(\underline{\bar{X}} - \underline{\mu}\right) \sim \chi^2(p)$$



$\underline{\bar{X}} \sim \underline{\mu}$ $\quad Q \to 0$

$\underline{\bar{X}}$ far from $\underline{\mu}$ $\quad Q$ large



n

$$\underline{\bar{x}}_{1=\text{time}} = \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{21} \\ \vdots \\ \bar{x}_{p1} \end{bmatrix} \longrightarrow Q_1 = n\left(\underline{\bar{x}}_1 - \underline{\mu}\right)' \cdot \underline{\underline{\Sigma}}^{-1}\left(\underline{\bar{x}}_1 - \underline{\mu}\right)$$

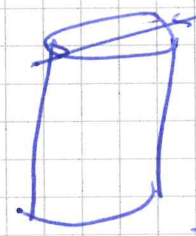
quality variable

time

time 1

5

$p=2$ Let's see how the Q statistic looks like when $p=2$



$$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N_2\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \underline{\underline{\Sigma}}\right)$$

$$Q = m\,(\underline{\bar{X}} - \underline{\mu})' \, \underline{\underline{\Sigma}}^{-1} (\underline{\bar{X}} - \underline{\mu}) =$$

$$Q = m \underbrace{\begin{bmatrix} \bar{x}_1 - \mu_1 & \bar{x}_2 - \mu_2 \end{bmatrix}}_{\substack{1 \times 2 \\ \| \\ p}} \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}_1 - \mu_1 \\ \bar{x}_2 - \mu_2 \end{bmatrix} =$$

$$Q = m \underbrace{\overbrace{\begin{bmatrix} \bar{x}_1 - \mu_1 & \bar{x}_2 - \mu_2 \end{bmatrix}}^{1\times 2} \overbrace{\frac{1}{\underset{\downarrow\ (\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}{\det \Sigma}} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}}^{2\times 2}}_{1\times 2} \underbrace{\begin{bmatrix} \bar{x}_1 - \mu_1 \\ \bar{x}_2 - \mu_2 \end{bmatrix}}_{2\times 1}$$

... (see slides)

$$Q = \ldots\ldots \leq UCL = \chi^2_\alpha(2)$$