QDA 2023 03 13

LINEAR MODELS

$$Y_t = \mu_t + \varepsilon_t$$

$\mu_t \neq \mu$ cost

$\varepsilon_t \overset{iid}{\sim} (0, \sigma_\varepsilon^2)$

N

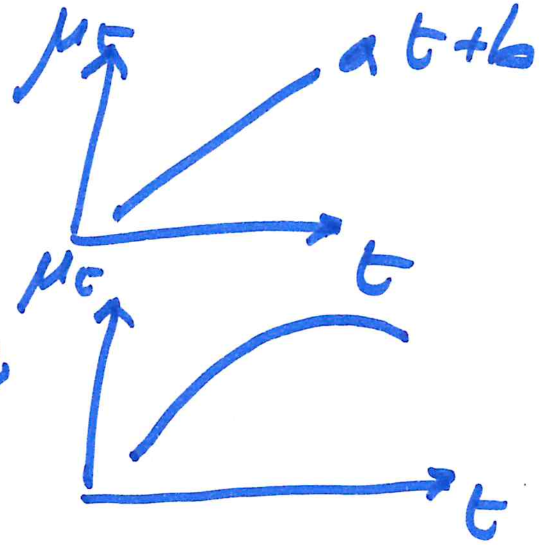
1. Assume a "structure" for μ_t

$$\mu_t = at + b$$

$$\mu_t = at^2 + bt + c$$

$$\mu_t = a \ln t$$

....

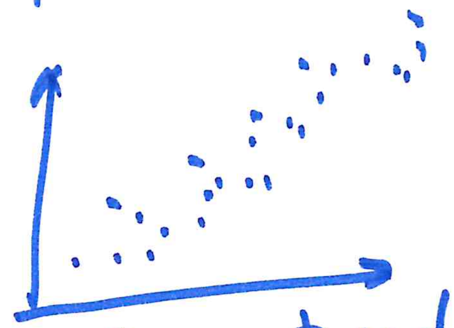


ex

To assume a "structure" for μ_t ②

→ knowledge of the process behind y_t

→ "data snooping"



- linear trend
- quadratic pattern ?

"LINEAR" MODEL

structure for μ_t is LINEAR with respect to the PARAMETERS/COEFFICIENTS that need to be estimated

$\mu_t = a\,t + b$ REGRESSOR

parameters/coeff to be estimated

$\mu_t = a\,t^2 + b\,t + e$ REGRESSORS

x_{1t} x_{2t}

$x_{\ell t}$ ℓ^{th} REGRESSOR

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$$\mu_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \dots + \beta_p x_{pt}$$

REGRESSORS

COEFFICIENTS THAT NEED TO BE ESTIMATED

$p = \#$ regressors

$K = p + 1 = \#$ parameters to be estimated

intercept β_0

LINEAR MODEL

$$Y_t = \mu_t + \varepsilon_t$$

deterministic $\varepsilon_t \sim (0, \sigma_\varepsilon^2)$

but with all the β's unknown → to be estimated

ex $\mu_t = a \underbrace{\ln t}_{x_{1t} = \ln t} = a\, x_{1t}$

$\mu_t = b\,t^{c}$ is linear

$$\underbrace{\ln \mu_t}_{} = \underbrace{c}_{\beta_1}\,\underbrace{\ln t}_{x_{1t}} + \underbrace{\ln b}_{\beta_0}$$

$\mu_t = a\,t + b\,t^{c}$ ~ NON LINEAR

How we can compute the UNKNOWN β's ?

By MINIMIZING the SUM of SQUARED ERRORS

$$SS_E = \sum_{t=1}^{m} (y_t - \mu_t)^2 = \sum_{t=1}^{m} \varepsilon_t^2$$

(5)

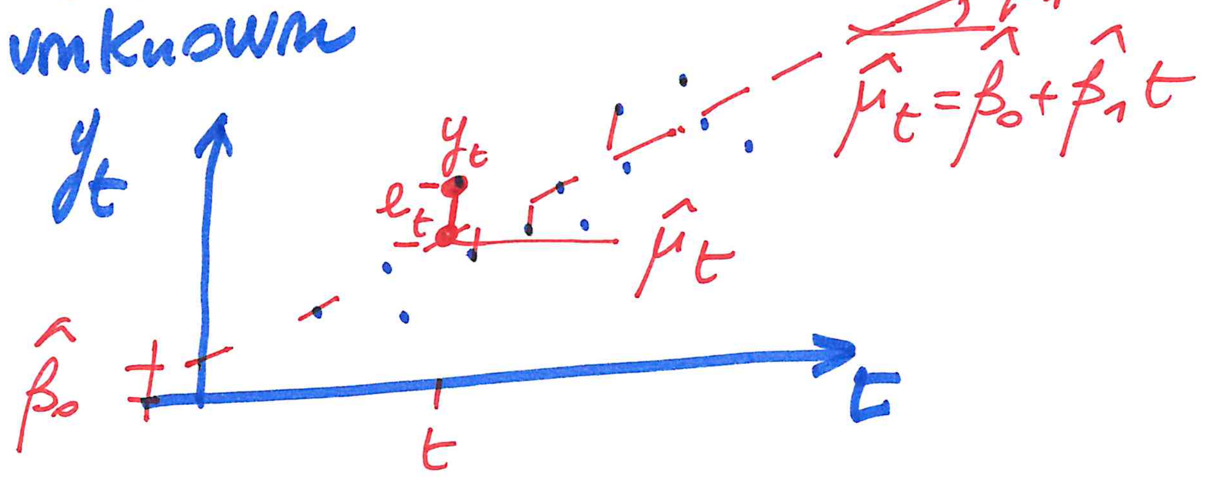
Find $\hat{\beta}_0 \; \hat{\beta}_1 \ldots \hat{\beta}_p \; \ni$

$$\underset{\underline{\beta}}{\text{MIN}} \; SS_E = \min \sum_{t=1}^{m} (y_t - \hat{\mu}_t)^2$$

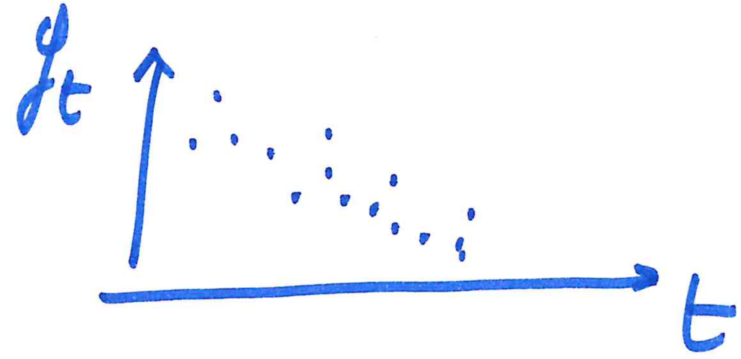
$$\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1 x_{1t} + \ldots \hat{\beta}_p x_{pt}$$

$$Y_t - \mu_t = \varepsilon_t \qquad y_t - \hat{\mu}_t = e_t = \hat{\varepsilon}_t$$

rnd variable ↑ Y_t; real but unknown ↑ μ_t; rnd error ↑ ε_t; data ↑ y_t; estimate ↑ $\hat{\mu}_t$; error t



<u>Ex</u> : deming.dat



<u>ASSUMING</u> : STRUCTURE

$$\mu_t = \beta_0 + \beta_1 \underset{\uparrow}{t}$$

x_{1t} = regressor

$$Y_t = \mu_t + \varepsilon_t$$
$$= \beta_0 + \beta_1 t + \varepsilon_t$$

Find $\hat{\beta}_0 = b_0$ $\hat{\beta}_1 = b_1$ to min $\boxed{SS_E = \sum_t (y_t - \underbrace{\beta_0 - \beta_1 t}_{\mu_t})}$

$$\begin{cases} \left.\dfrac{\partial SS_E}{\partial \beta_0}\right|_{\beta_0 = b_0} = 0 & (1) \\ \left.\dfrac{\partial SS_E}{\partial \beta_1}\right|_{\beta_1 = b_1} = 0 & (2) \end{cases}$$

$$\frac{\partial SS_E}{\partial \beta_0} = \frac{\partial}{\partial \beta_0}\left(\sum_t (y_t - \beta_0 - \beta_1 t)^2\right) =$$

$$= -2 \sum_t (y_t - \hat{\beta}_0 - \hat{\beta}_1 t) = 0$$

$$\underbrace{\sum_t y_t} - \sum_t b_0 - \sum_t b_1 t = 0$$

$\hat{\beta}_0$

$$m\bar{y} - m b_0 - b_1 m \bar{t} = 0$$

$$\bar{y} = \frac{1}{m}\underbrace{\sum y_t}$$

$$b_0 = -b_1 \bar{t} + \bar{y} =$$

$$= \bar{y} - b_1 \bar{t}$$

$\bar{x} = \bar{t}$ if you use $x_t = t$

$$\boxed{b_0 = \bar{y} - m\bar{x}}$$

Similarly $\left.\frac{\partial SS_E}{\partial \beta_1}\right|_{\hat{\beta}_1 = b_1} = 0$ ....

HW

HOMEWORK

$$\ldots b_1 = \hat{\beta}_1 = \frac{\sum_t (x_t - \bar{x})(y_t - \bar{y})}{\sum_t (x_t - \bar{x})^2} = \frac{S_{xy}}{S_{xx}}$$

$[\text{if } x_t = t \rightarrow \bar{x} = \bar{t}]$

H_0: $\boxed{Y_t = \mu_t + \varepsilon_t}$ $\quad \varepsilon_t \sim \underline{\underline{NID}}(0, \sigma_\varepsilon^2) \stackrel{*}{\Rightarrow} \boxed{Y_t \sim N(\mu_t, \sigma_\varepsilon^2)}$

$$\Rightarrow b_1 \sim ?(?, ?)$$

$$b_1 = \sum_t \underbrace{\frac{(x_t - \bar{x})}{S_{xx}}}_{c_t} (y_t - \bar{y}) = \sum_t c_t (y_t - \bar{y})$$

(9)

DETERM ↑ ... RANDOM

$$Y_t = \mu_t + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

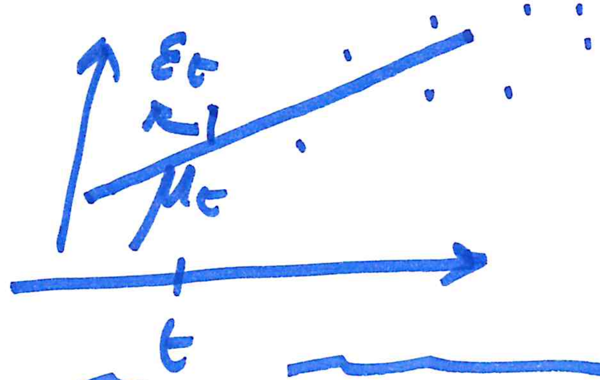
$$\mu_t = \beta_0 + \beta_1 t$$

$$Y_t \sim N(\mu_t, \sigma_\varepsilon^2)$$

$$E(Y_t) = E(\mu_t + \varepsilon_t) = \mu_t$$

$$E(\varepsilon_t) = 0$$

$$V(Y_t) = V(\mu_t + \varepsilon_t) = \sigma_\varepsilon^2$$



$$b_1 = \sum_t c_t (y_t - \bar{y}) \longrightarrow b_1 \sim N(\mu_{b_1}, \sigma_{b_1}^2)$$

$$b_0 = \bar{y} - m\bar{x} \longrightarrow b_0 \sim N(\mu_{b_0}, \sigma_{b_0}^2)$$

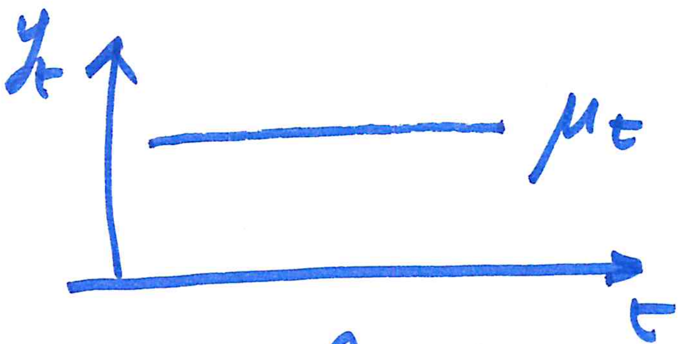
RESULTS IMPORTANT TO TEST $H_0: \beta_i = 0$ $H_1: \beta_i \neq 0$

for given i

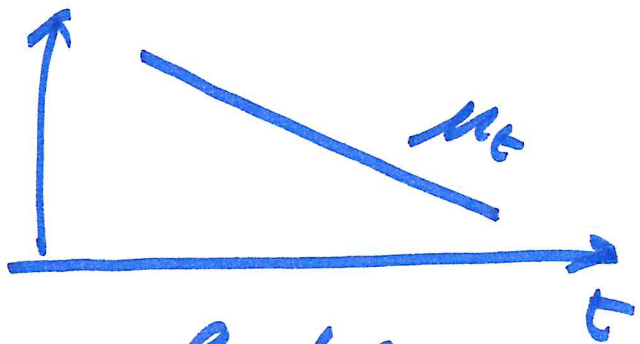
ex ELONGATION TEST

$$b_0 = \bar{y} - m\bar{x} = 0.00136522$$

$$b_1 = \frac{S_{xy}}{S_{xx}} = -0.00001066$$



$\beta_1 = 0$



$\beta_1 \neq 0$

$H_0 : \beta_1 = 0$

$H_1 \quad \beta_1 \neq 0$

$\Longrightarrow$ [ASSUMING: $Y_t \sim N(\mu_t, \sigma_\varepsilon^2)$] (*)

→ Reject H_0 → $\beta_1 \neq 0$ SLOPE MATTERS!
REGRESSOR

(11)

$$* \quad b_1 \sim N(\mu_{b_1}, \sigma^2_{b_1}) \qquad b_1 = \frac{S_{xy}}{S_{xx}}$$

$$\mu_{b_1} = \beta_1 \qquad \hat{\sigma}^2_{b_1} = \frac{\hat{\sigma}^2_{\varepsilon}}{S_{xx}} \qquad \hat{\sigma}^2_{\varepsilon} = \frac{SS_E}{n-K}$$

GIVEN H_0 $\beta_1 = 0$

$$SS_E = \sum e_t^2$$

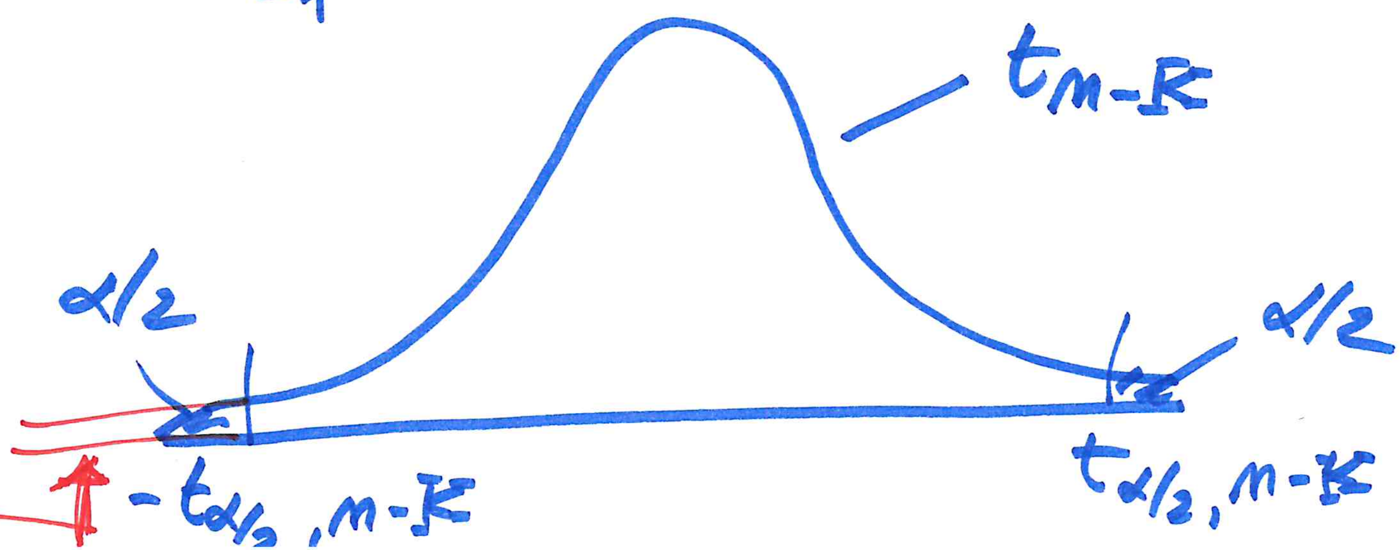
$$K = p + 1$$

regressors

$0 \,/\, H_0 : \beta_1 = 0$

$$t_{b_1} = \frac{b_1 - \mu_{b_1}}{\hat{\sigma}_{b_1}} \sim t_{n-K}$$

$$t_{b_1} = \frac{b_1}{\hat{\sigma}_{b_1}}$$



(12)

$\Rightarrow$ REJECT $H_0: \beta_1 = 0$ $\Longrightarrow$ "ACCEPT" $\beta_1 \neq 0$

$\Longrightarrow$ TIME IS AFFECTING ELONGATION!!!