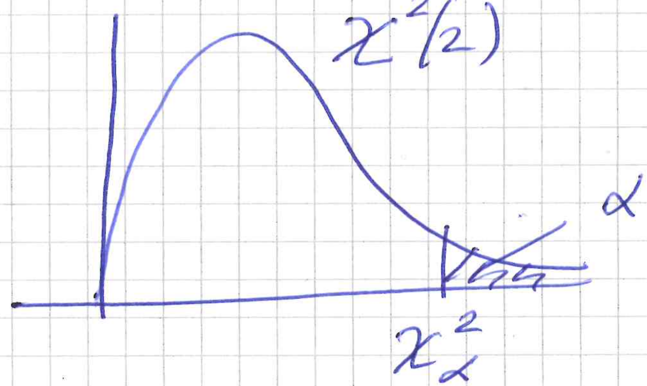
QDA 2023.05.22 ①

$$\chi_0^2 = \begin{bmatrix} \bar{x}_1 - \mu_1 & \bar{x}_2 - \mu_2 \end{bmatrix} \frac{1}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 - \mu_1 \\ \bar{x}_2 - \mu_2 \end{bmatrix}$$

$$\chi_0^2 \leq \chi_{\alpha,2}^2$$

number



$\chi_0^2 = \dots \leq \chi_{\alpha,2}^2$ slide

dividing times $\sigma_1^2 \sigma_2^2$

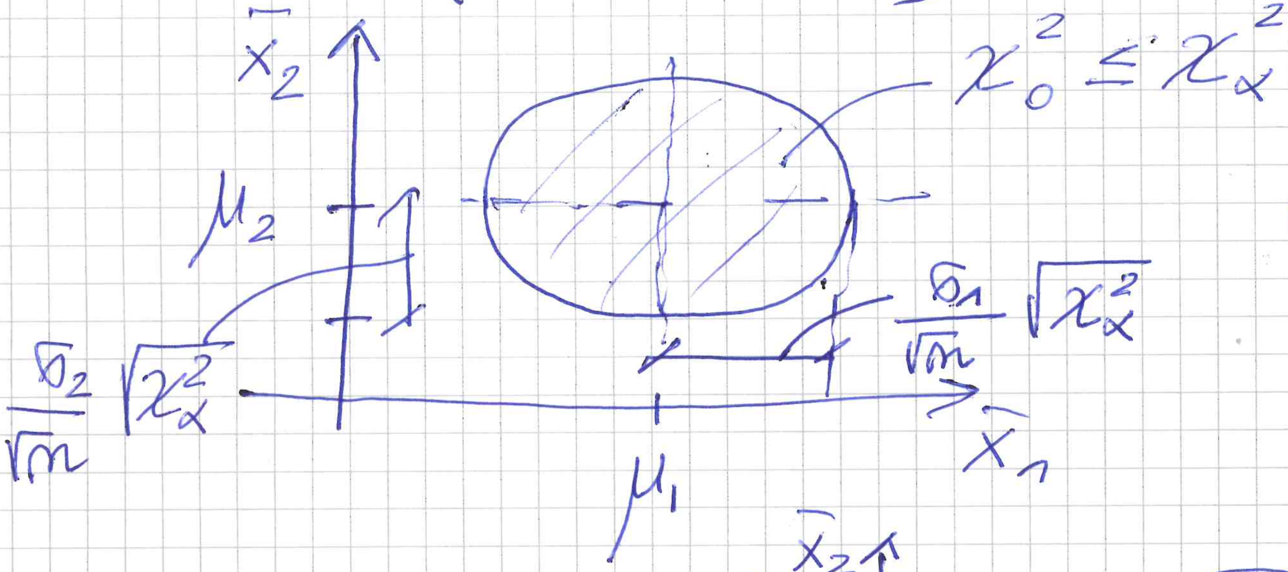
$$\chi_0^2 = \frac{m}{\quad} \dots \leq \chi_{\alpha_2}^2$$

②

$$\left(\frac{\bar{x}_1-\mu_1}{\sigma_1}\right)^2+\left(\frac{\bar{x}_2-\mu_2}{\sigma_2}\right)^2-2\frac{\sigma_{12}}{\sigma_1\sigma_2}\left(\frac{\bar{x}_1-\mu_1}{\sigma_1}\right)\left(\frac{\bar{x}_2-\mu_2}{\sigma_2}\right)\leq$$

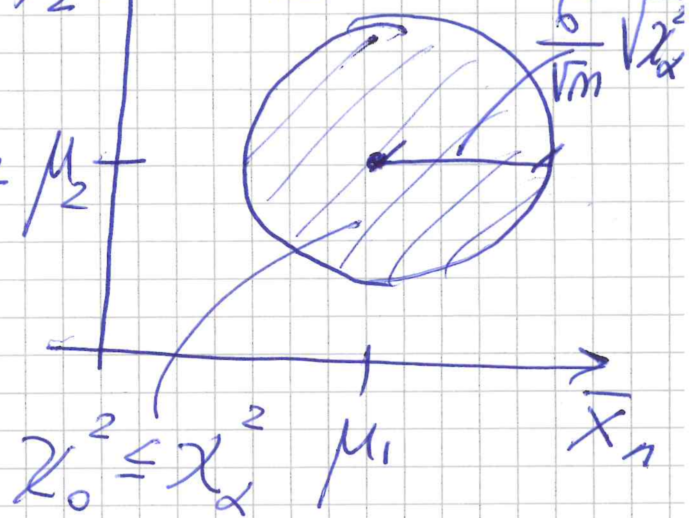
$$\leq \frac{\sigma_1^2\sigma_2^2-\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\quad\frac{\chi^2_\alpha(2)}{m}$$

<u>ELLIPSE</u> $\sigma_{12}=0$ (uncorrelation)

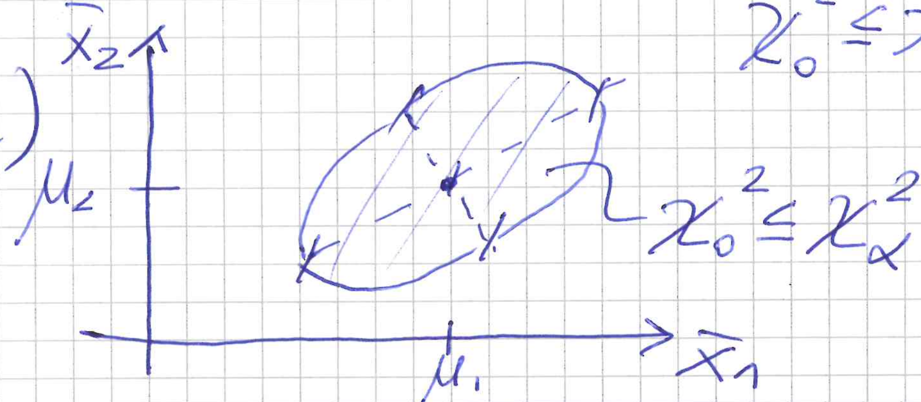


<u>CIRCLE</u>

$\sigma_1=\sigma_2$



$\sigma_{12}\neq 0$ (positive)



③

the simplest case $p=2$ $\sigma_{12}=0$ $\sigma_1=\sigma_2$

↳ in-control region is a CIRCLE (multivariate cc)

↳ in-control region is a square (two univariate cc)

$$\text{radius} = \frac{\sigma}{\sqrt{n}}\sqrt{\chi^2_{\alpha'}}$$

α' OVERALL MULTIV / SINGLE

$$\alpha' \xrightarrow[\sigma_{12}=0]{} (1-\alpha') = (1-\alpha_1)(1-\alpha_2) =$$

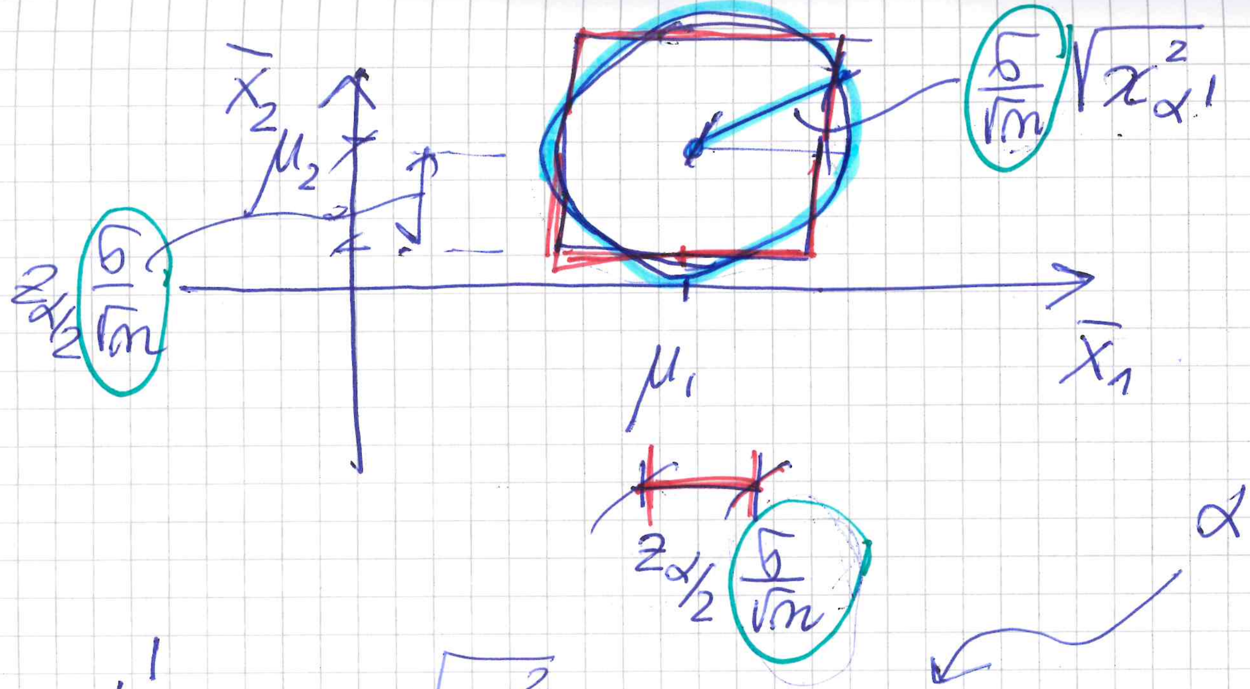
$$= (1-\alpha)^2 \rightarrow \alpha: 1-\alpha = \sqrt{1-\alpha'} \rightarrow \alpha = 1-\sqrt{1-\alpha'}$$

single control chart on $\bar{x}_1$ and $\bar{x}_2$

$$\bar{X}_1 \rightarrow \begin{matrix} LCL_1 \\ UCL_1 \end{matrix} = \mu_1 \mp z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$(\sigma_1=\sigma_2)$

$$\bar{X}_2 = \begin{matrix} LCL_2 \\ UCL_2 \end{matrix} = \mu_2 \mp z_{\alpha/2}\,\sigma/\sqrt{n}$$



$\alpha = 1-\sqrt{1-\alpha'}$

α'	$\sqrt{\chi^2_{\alpha'}}$	α	$z_{\alpha/2}$
$\boxed{0.05}$	$\boxed{2.448}$	0.0253	$\boxed{2.237}$
⋮	⋮		⋮
$\boxed{0.005}$	$\boxed{3.717}$	. . . .	$\boxed{3.481}$

- Phase 1 ? what if μ and Σ are not known → multivariate cc for the mean

- $n = 1$? what about samples of size $n = 1$ (multivariate cc corresponding to the <u>I</u>ndividual I control chart)

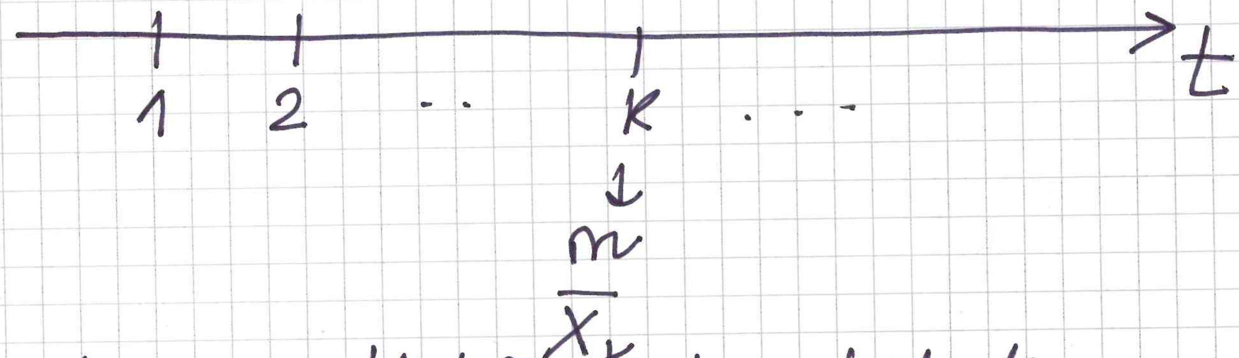
- what about PCA before control charting

Phase 1 control chart (multivariate cc for the means $\bar{\bar{X}}$)

$$T_k^2 \quad n\,(\underline{\bar{x}}_k - \underline{\hat{\mu}})' \, \underline{\underline{\hat{\Sigma}}}^{-1} \, (\underline{\bar{x}}_k - \underline{\hat{\mu}})$$

$\uparrow$ k-th sample

$\uparrow$? $\quad$ $\uparrow$? $\quad$ $\uparrow$?



T^2 statistic $\rightarrow$ Hotelling's statistic

~~Ph1~~ m samples of size n : $\underline{\bar{\bar{x}}} = \underline{\hat{\mu}}$; $\underline{S} = \underline{\underline{\hat{\Sigma}}}$

In Phase 1 (control chart - retrospective way)

$T^2_K \sim c_1 \, F(d_1, d_2)$

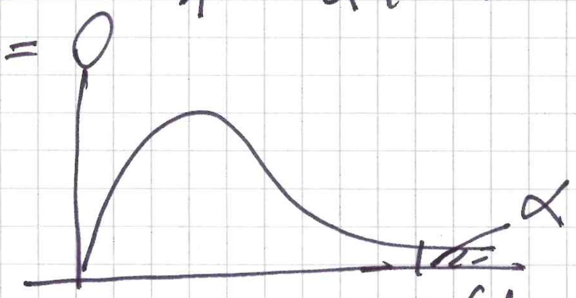
$c_1 = \dfrac{p(m-1)(m-1)}{m(m-1)-(p-1)}$

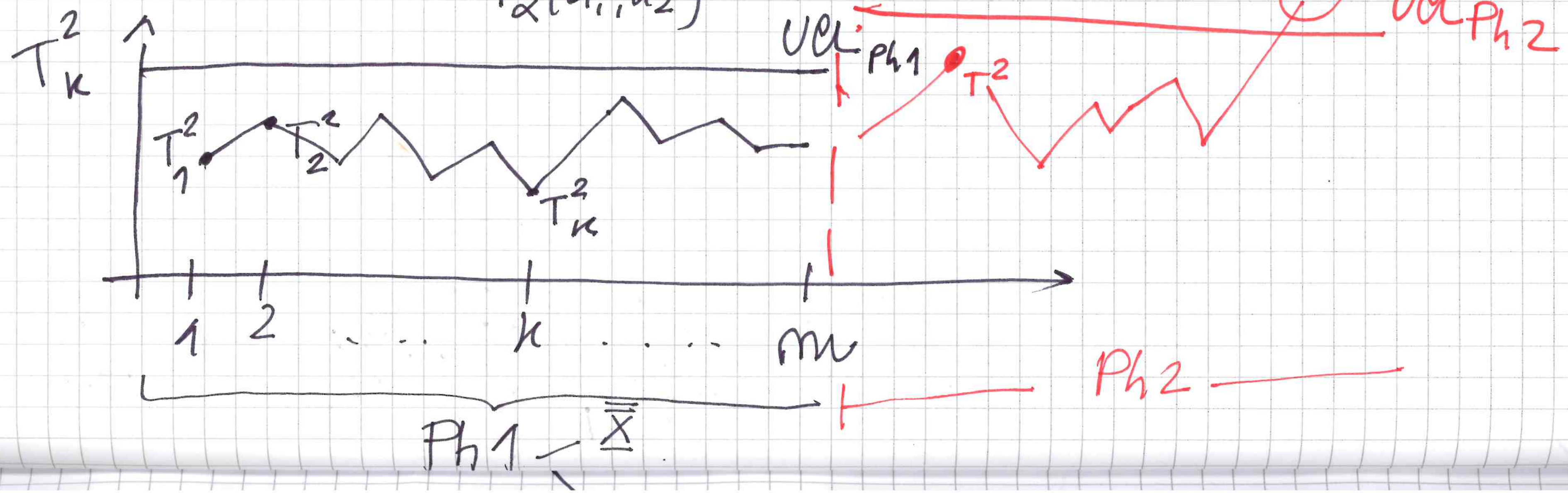
$UCL = c_1 \cdot F_\alpha(d_1, d_2)$

$LCL = 0$

$d_1 = p$

$d_2 = m(m-1) - (p-1)$





Ph 1

$$\hat{\underline{\Sigma}} = \underline{S}$$

If one sample (or more) is out of control in Ph 1
→ look for the assignable cause (i.e. operator, machine, environment, supplier, material ...) if I find the assignable cause I have to remove that sample k^* from Ph 1 samples and recompute everything
→ $\hat{\underline{\mu}}$ $\hat{\underline{\Sigma}}$ control limit etc

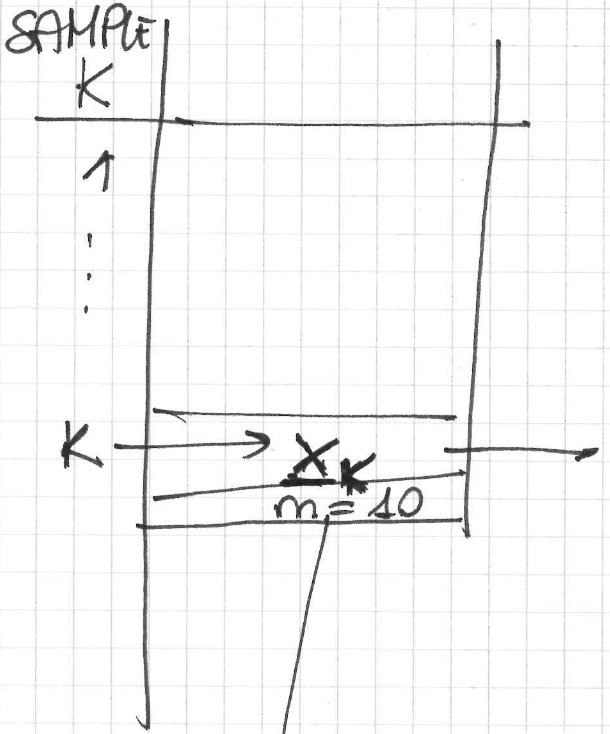
↑ I have m

m^* final # of samples in Ph 1, $\hat{\underline{\mu}}$, $\hat{\underline{\Sigma}}$

Phase 2

$$T_k^2 = m\left(\bar{\underline{x}}_k - \hat{\underline{\mu}}\right)' \hat{\underline{\Sigma}}^{-1}\left(\bar{\underline{x}}_k - \hat{\underline{\mu}}\right) \sim c_2\, F(d_1, d_2)$$

$$c_2 = \frac{p(m-1)(m^*+1)}{\ldots} \qquad d_1 = p \qquad d_2 = m^*(m-1) - (p-1)$$



$p = 2$

K | 10 values x_1 | 10 values x_2

UTS fiber

diameter fiber

$$\to \underline{\bar{X}}_K = \begin{bmatrix} \bar{x}_{1,K} \\ \bar{x}_{2,K} \end{bmatrix}$$

$$S_K = \begin{bmatrix} \hat{\sigma}^2_{1,K} & \hat{\sigma}_{12,K} \\ \hat{\sigma}_{12,K} & \hat{\sigma}^2_{2,K} \end{bmatrix}$$

$$S_k = \begin{bmatrix} s_{1,k}^2 & s_{12,k} \\ s_{12,k} & s_{2,k}^2 \end{bmatrix}$$

in the univariate case

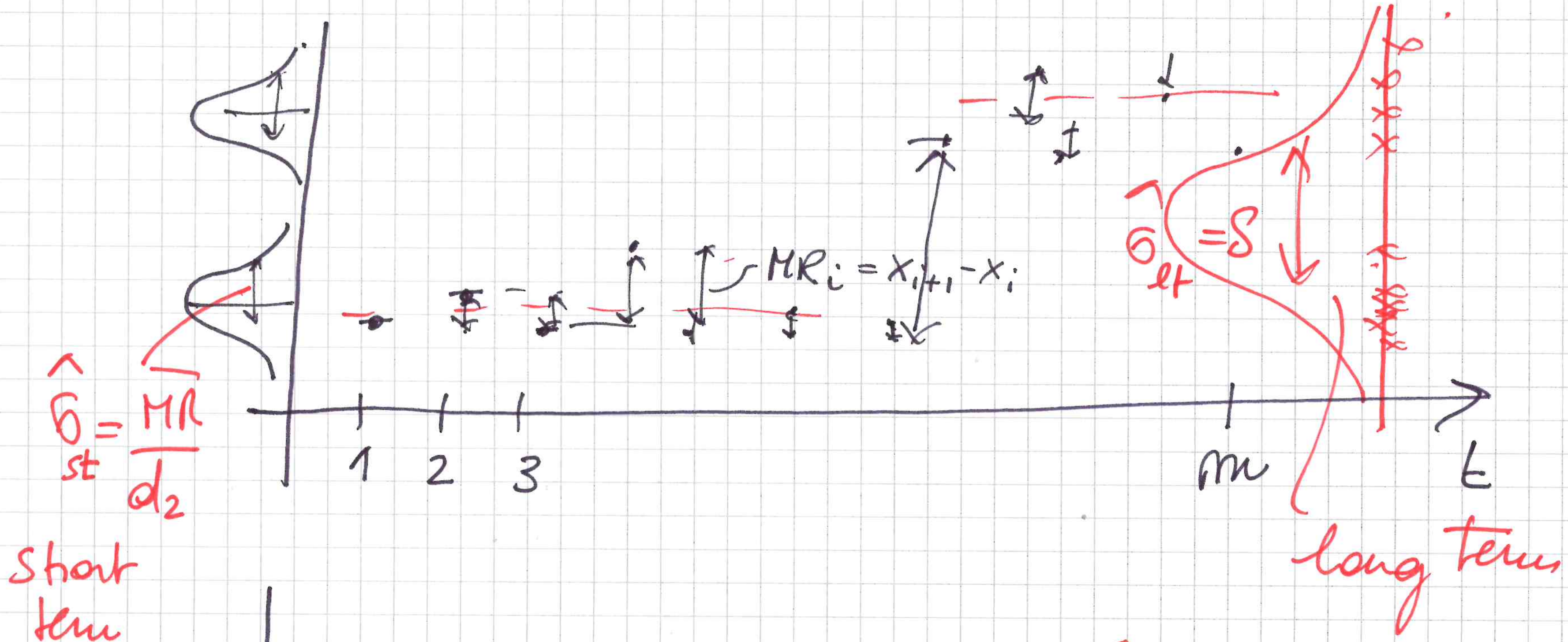
sample $k \longrightarrow \bar{x}_k$, R_k or S_k

with $n = 1$ $\quad \bar{x}_k = x_k$ $\quad R_k$?

S_k ?

$m=1$ — short term estimate of $\hat{\sigma}_{st} = \frac{\overline{MR}}{d_2}$

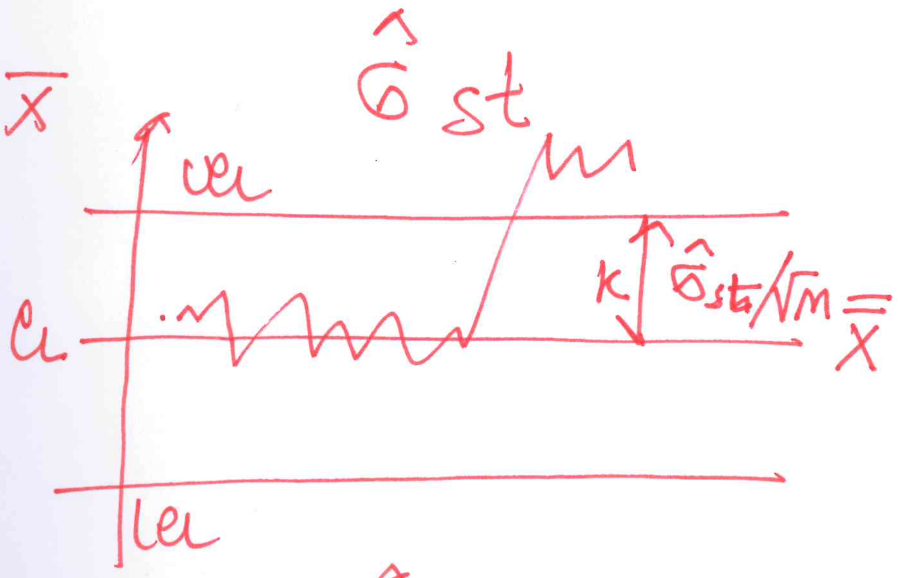
— long term estimate $\hat{\sigma}_{lt} = S = \frac{1}{m-1} \Sigma (x_i - \bar{x})^2$

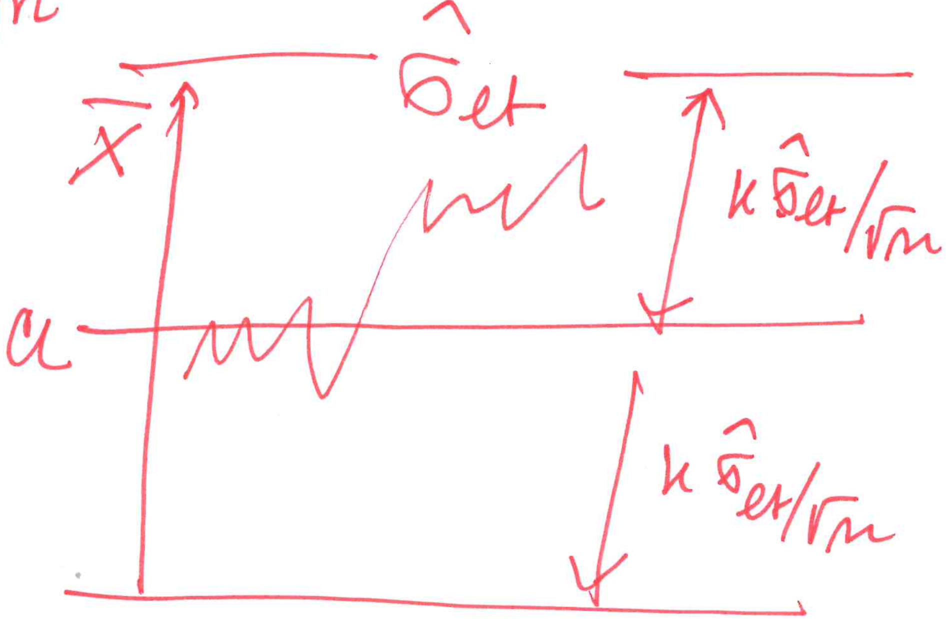


$\hat{\sigma}_{lt} = \hat{\sigma}_{sh}$ mean of the process is stable

$\hat{\sigma}_{lt} > \hat{\sigma}_{st}$ mean of the process is changing

$$\overline{X} - \begin{matrix} ue \\ le \end{matrix} = \bar{\bar{x}} \pm k \frac{\hat{\sigma}}{\sqrt{m}}$$





$m = 1 \quad p = 1 \qquad \frac{\overline{MR}}{d_2}$

$m = 1 \quad p > 1 \qquad \underline{V} = \begin{bmatrix} \underline{v}_1' \\ \underline{v}_2' \\ \vdots \end{bmatrix}$