EXERCISE CLASS 3 (Part 2/3)

EXERCISE 2

A study was performed by ComputerTek Co to determine the time series of order processing durations. Data in the file "dataset_ese3_es2.csv" refer to the period 1995, July – 1997, October. Each datum represents the time (in days) to ship the order.

Design a 95% **prediction interval** for future observations.

> ## Suggestion
>
> Remind that if:
>
> $$X \sim N(\mu, \sigma^2)$$
>
> then:
>
> $$\frac{X-\mu}{s} \sim t_{n-1}$$

Confidence intervals

$L \leq \theta \leq U$ such that $P(L \leq \theta \leq U) = 1 - \alpha$
is called $100(1-\alpha)\%$ confidence interval for the **(unknown) parameter $\boldsymbol{\theta}$**

Interpretation: if, in repeated random samplings, a large number of such intervals is constructed, $100(1-\alpha)\%$ of them will contain the true value of θ.

Prediction intervals

$L \leq Y \leq U$ such that $P(L \leq Y \leq U) = 1 - \alpha$
is called $100(1-\alpha)\%$ prediction interval for the **future process outcome** Y

Interpretation: if, in repeated random samplings, a large number of such intervals is constructed, $100(1-\alpha)\%$ of them will contain the future outcome Y

In [ ]:
```
# Import the necessary libraries
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from scipy import stats

# Import the dataset
data = pd.read_csv('dataset_ese3_es2.csv')
```

```
data.head()
```

Out[ ]:

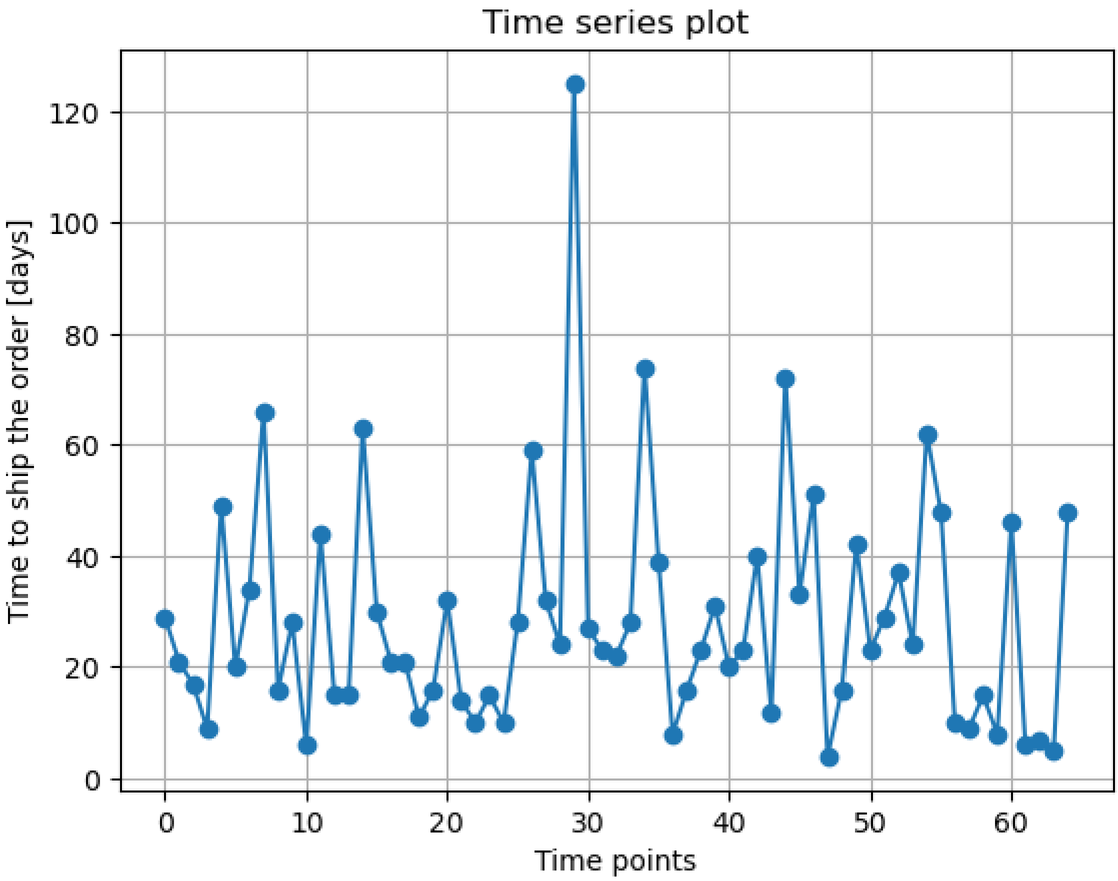
	Ex2
0	29
1	21
2	17
3	9
4	49

> We need a model of process output to make predictions. Which kind of model? **Time series** model or **distributional** model.
>
> We can verify if the process is random by using :
>
> 1. time series plot (qualitative)
> 2. ACF/PCAF (qualitative)
> 3. runs test (quantitative)
> 4. Bartlett's test (quantitative)
> 5. LBQ test (quantitative)

> 1. Time series plot

In [ ]:

```
# Time series plot
plt.plot(data, 'o-')
plt.xlabel('Time points')
plt.ylabel('Time to ship the order [days]')
plt.title('Time series plot')
plt.grid()
plt.show()
```



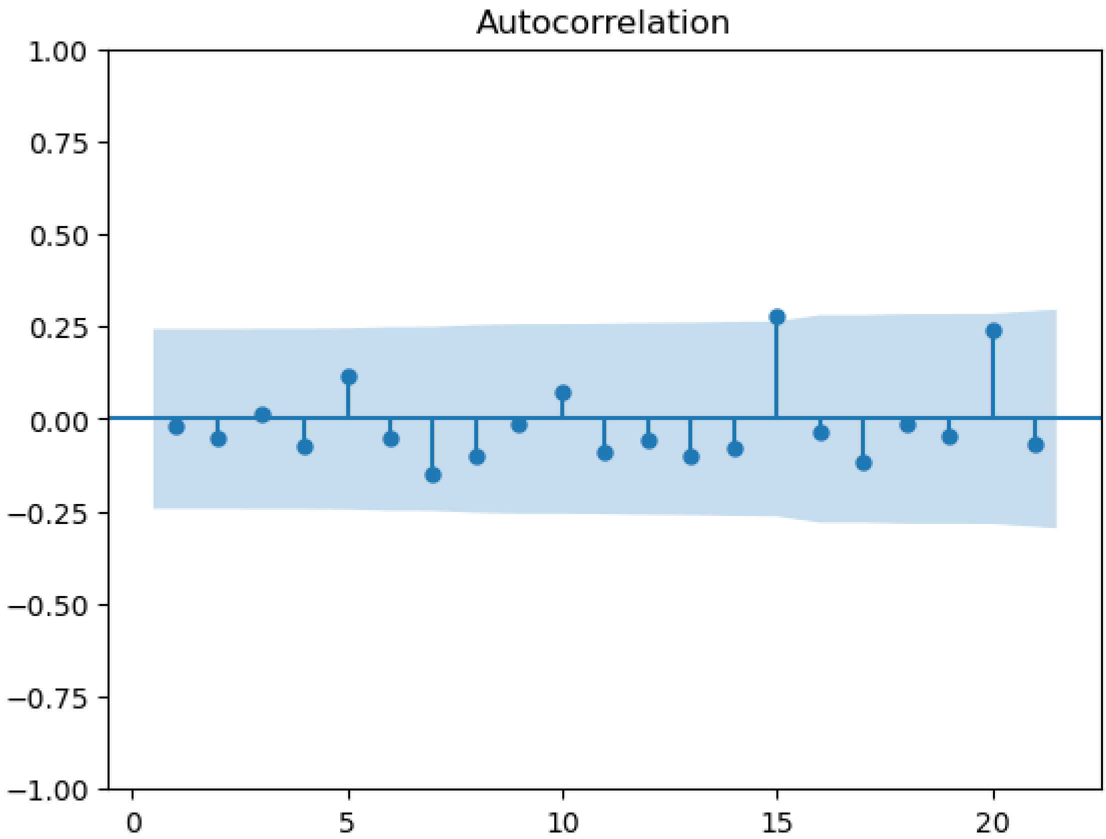
> 1. Runs test

In [ ]:
```
# Import the necessary libraries for the runs test
from statsmodels.sandbox.stats.runs import runstest_1samp

stat_runs, pval_runs = runstest_1samp(data['Ex2'], correction=False)
print('Runs test statistic = {:.3f}'.format(stat_runs))
print('Runs test p-value = {:.3f}'.format(pval_runs))
```

```
Runs test statistic = 0.325
Runs test p-value = 0.745
```

> 1. ACF/PACF

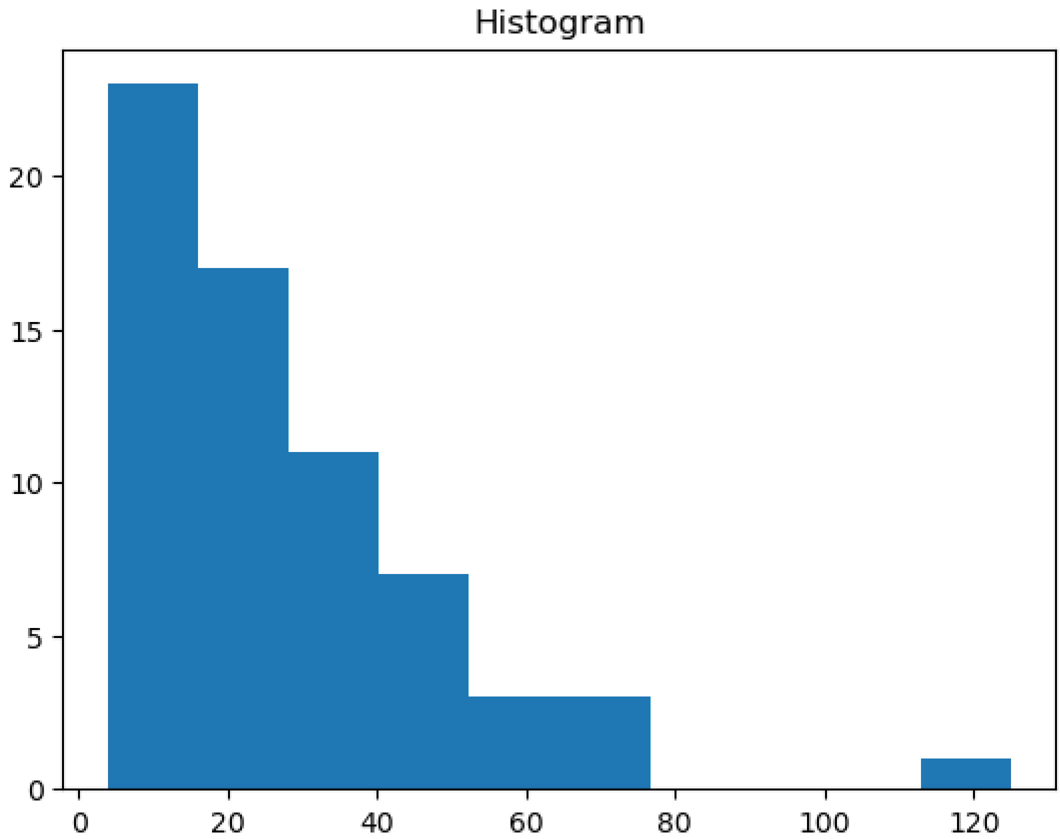
In [ ]:
```
# Plot the acf using the statsmodels library
import statsmodels.graphics.tsaplots as sgt

sgt.plot_acf(data['Ex2'], lags = int(len(data)/3), zero=False)
plt.show()
```

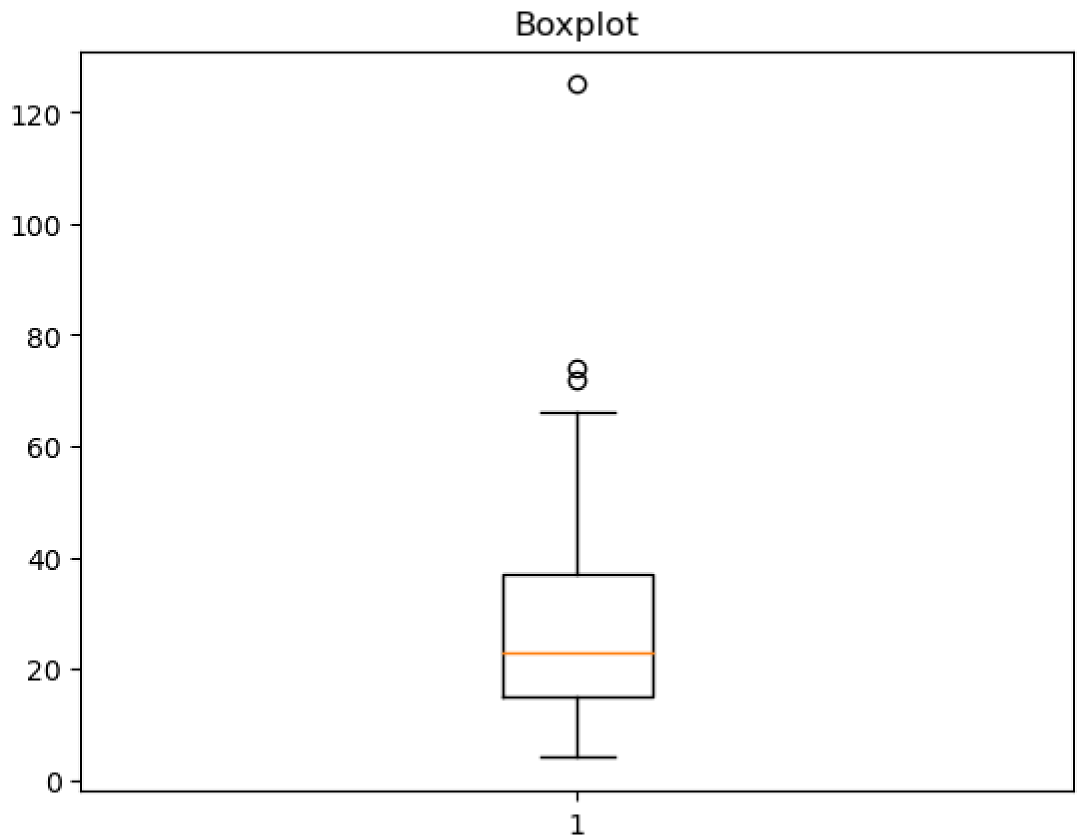


1. Histogram and Boxplot

In [ ]:

```
plt.hist(data)
plt.title('Histogram')
plt.show()

plt.boxplot(data)
plt.title('Boxplot')
plt.show()
```



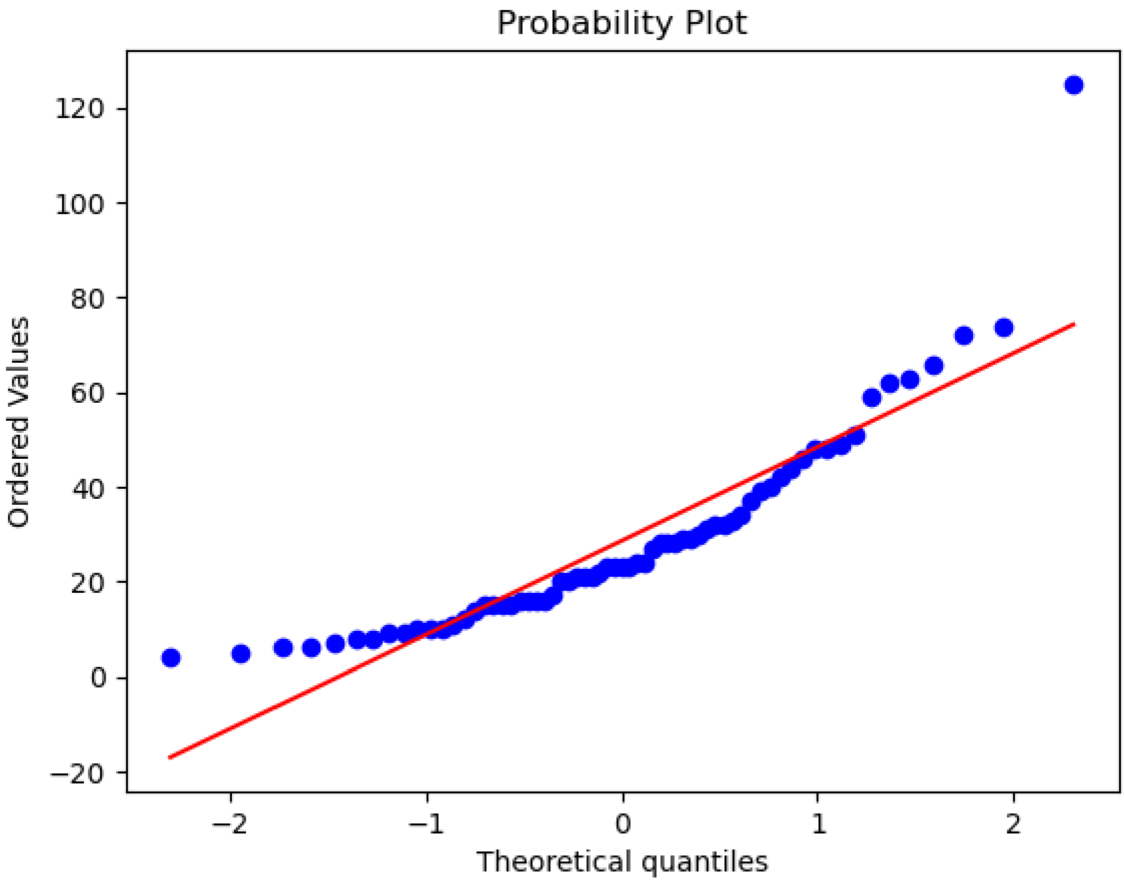


> Let's check normality

In [ ]:
```
#Normality test
#Shapiro-Wilk test
import matplotlib.pyplot as plt

stat_shapiro, p_shapiro = stats.shapiro(data)
```

```
print('Statistic = %.3f, p-val = %.3f' % (stat_shapiro, p_shapiro))
# interpret
alpha = 0.05
if p_shapiro > alpha:
    print('Fail to reject H0')
else:
    print('Reject H0')


# Plot the qqplot
stats.probplot(data['Ex2'], dist="norm", plot=plt)
plt.show()
```

```
Statistic = 0.844, p-val = 0.000
Reject H0
```

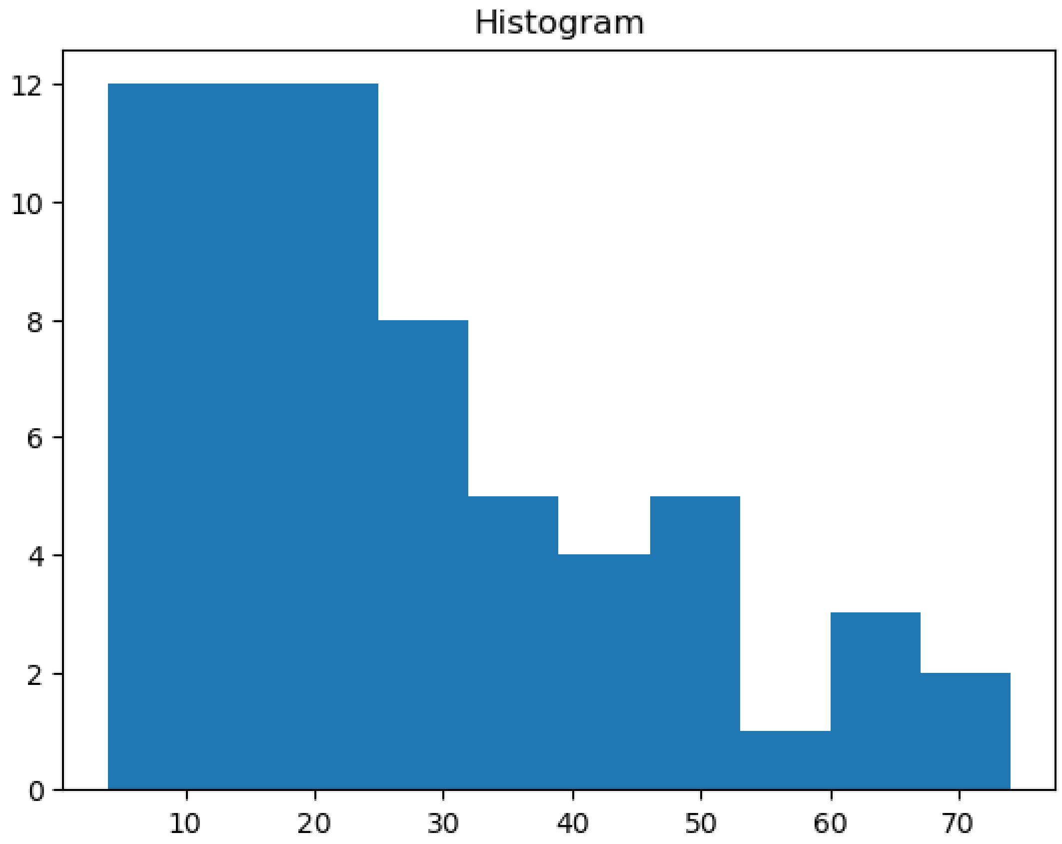


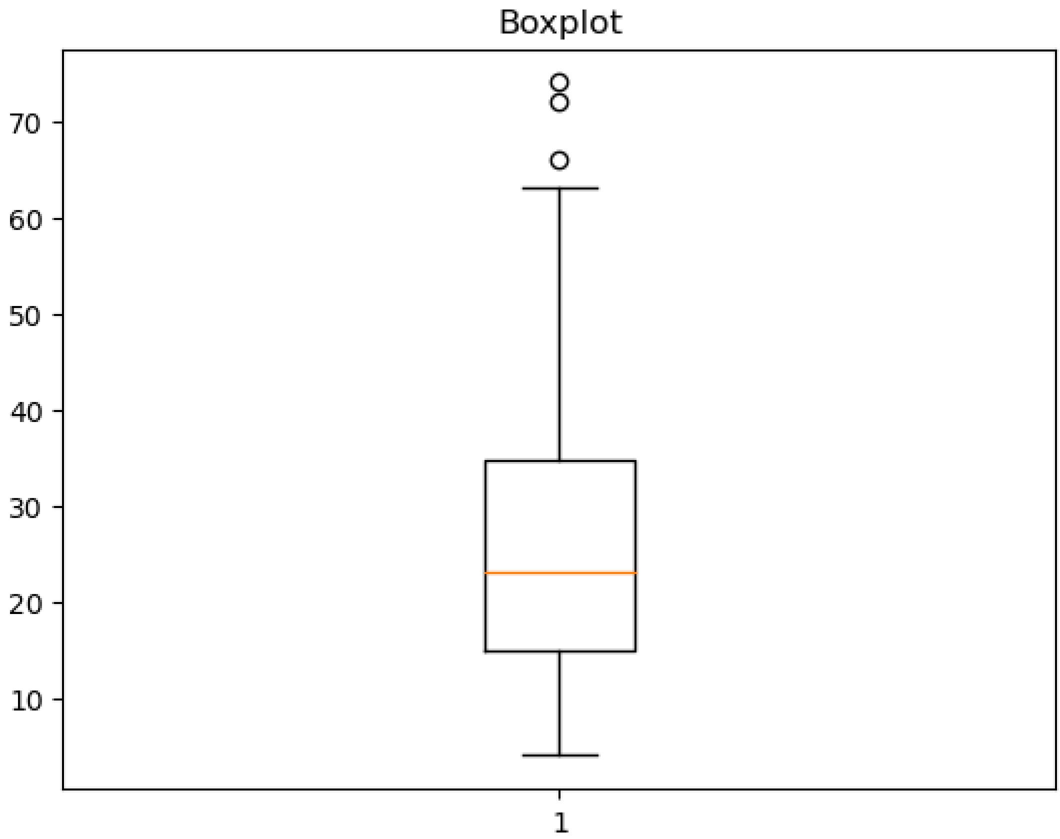
> How much is this result influenced by the outlier? We can try to remove the outlier and check for normality again.

In [ ]:

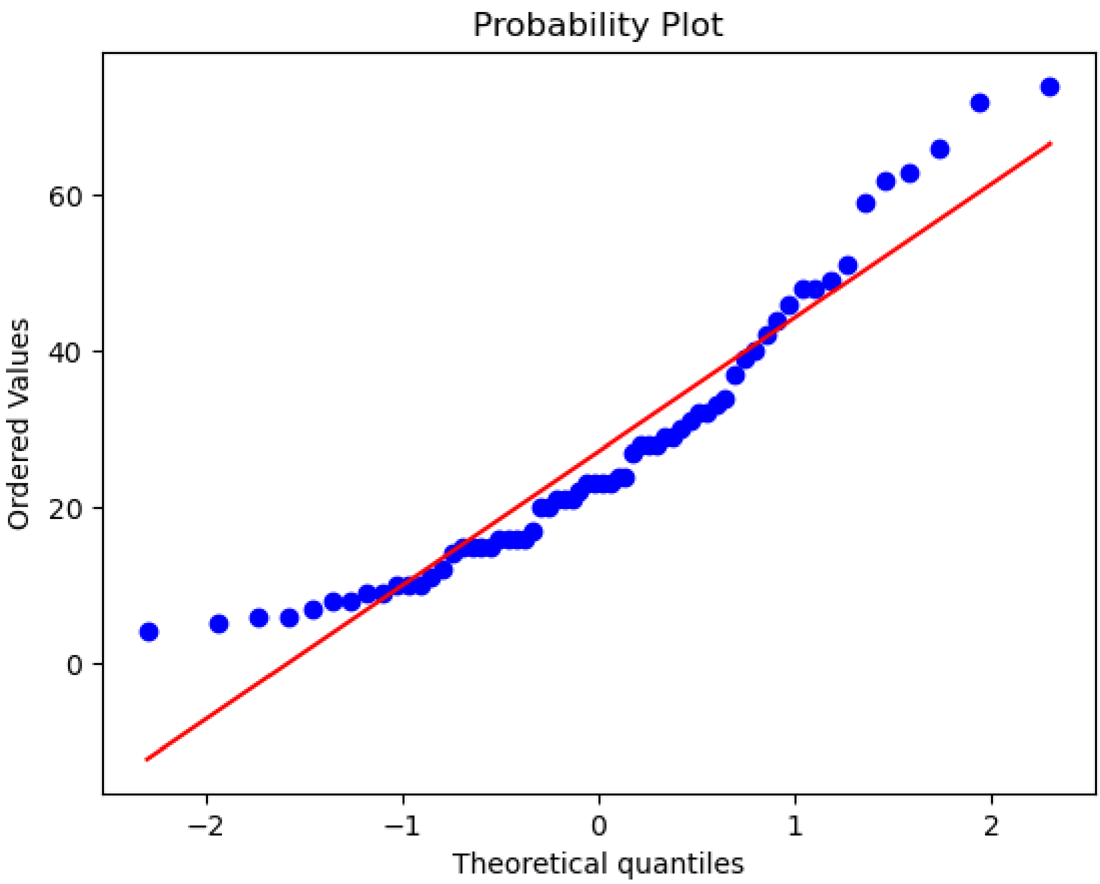
```
# Remove outlier (point 30) and check normality
data_out = data.drop(index=29)

plt.hist(data_out)
plt.title('Histogram')
plt.show()

plt.boxplot(data_out)
plt.title('Boxplot')
plt.show()

#Normality test
#Shapiro-Wilk test
from scipy.stats import shapiro
stat_shapiro_out, p_shapiro_out = shapiro(data_out)
```

```
print('Statistic = %.3f, p-val = %.3f' % (stat_shapiro_out, p_shapiro_out))
# interpret
alpha = 0.05
if p_shapiro > alpha:
    print('Fail to reject H0')
else:
    print('Reject H0')

# Plot the qqplot
stats.probplot(data_out['Ex2'], dist="norm", plot=plt)
plt.show()
```






```
Statistic = 0.914, p-val = 0.000
Reject H0
```



> Even after removing the outlier, normality is still violated.
>
> Try with the Box-Cox transformation.

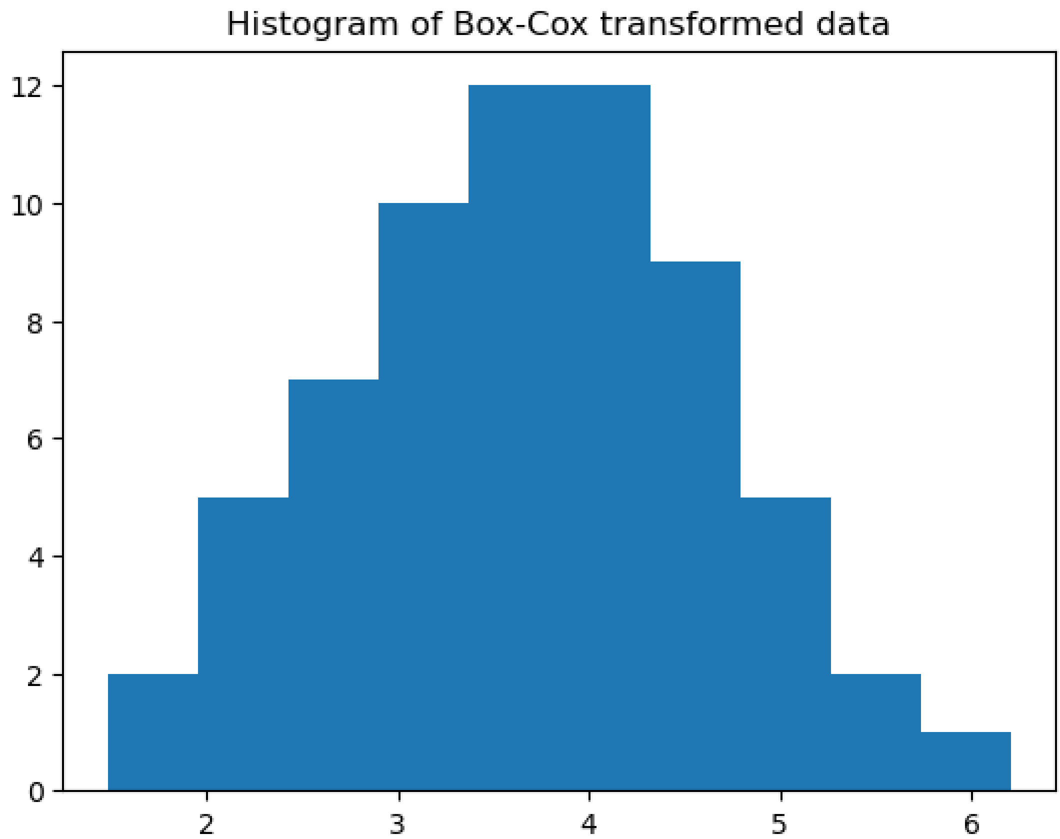
Remind:

$y = (x^\lambda - 1)/\lambda$, for $\lambda \neq 0$

$y = ln(x)$, for $\lambda = 0$

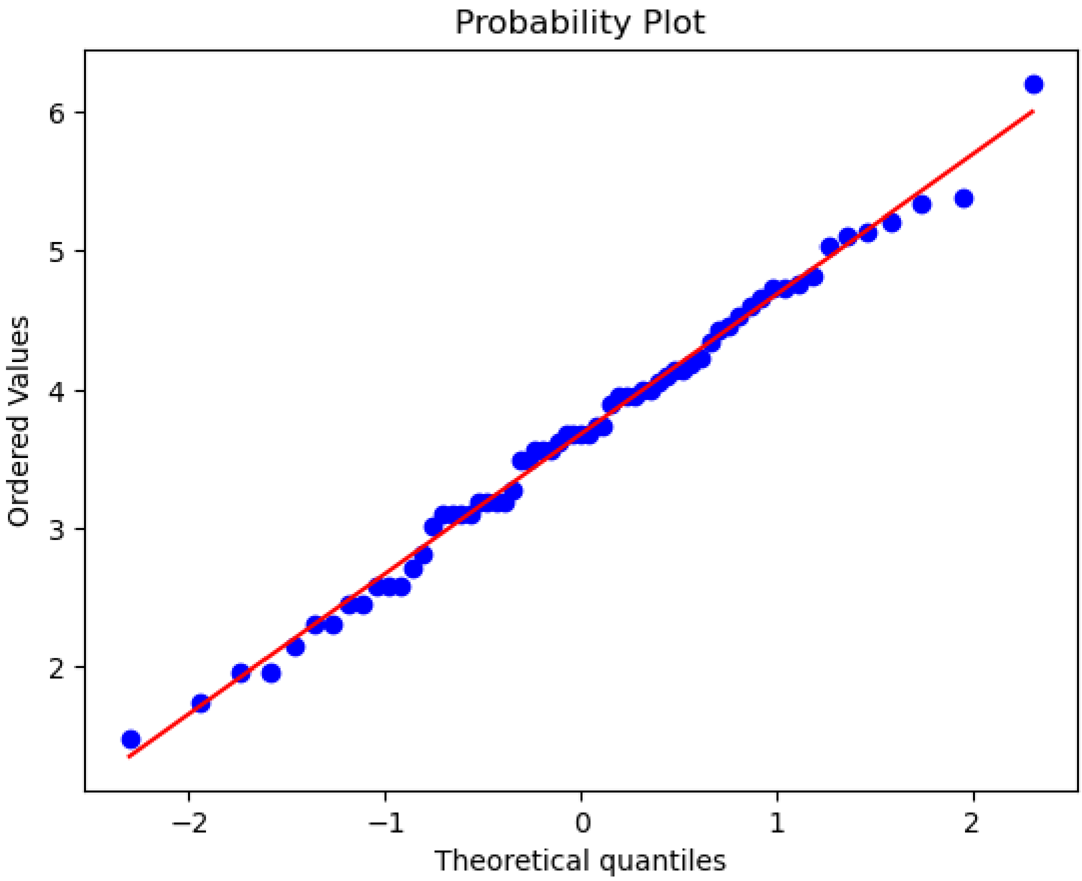
In [ ]:
```
#Box-Cox transformation
[data_norm, lmbda]=stats.boxcox(data['Ex2'])

print('Lambda = %.3f' % lmbda)

plt.hist(data_norm)
plt.title('Histogram of Box-Cox transformed data')
plt.show()
```

```
Lambda = 0.100
```

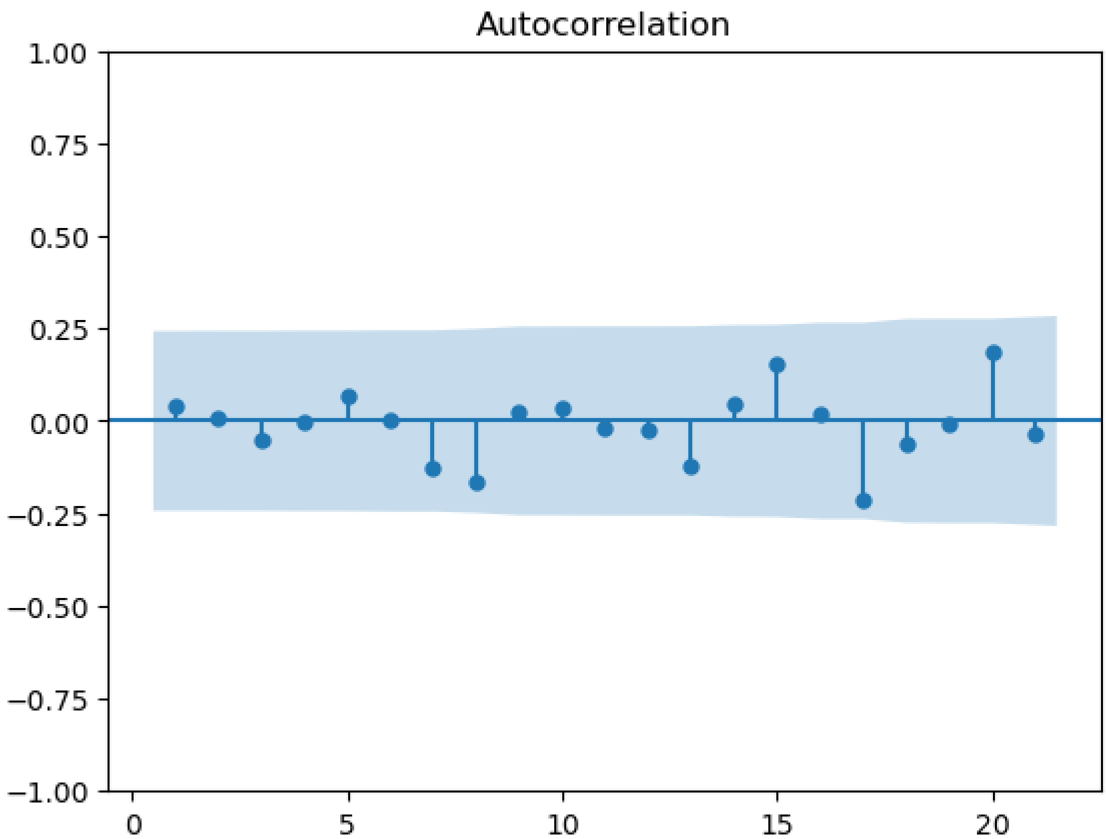


In [ ]:
```
stat, p_shapiro = stats.shapiro(data_norm)
print('Statistics=%.3f, p=%.3f' % (stat, p_shapiro))
# interpret
alpha = 0.05
if p_shapiro > alpha:
    print('Fail to reject H0')
else:
    print('Reject H0')

# Plot the qqplot
stats.probplot(data_norm, dist="norm", plot=plt)
plt.show()
```

```
Statistics=0.992, p=0.959
Fail to reject H0
```



In [ ]:
```
# Plot the acf using the statsmodels library

sgt.plot_acf(data_norm, lags = int(len(data_norm)/3), zero=False)
plt.show()
```



No graphical evidence of auto-correlation from the sample acf plot. We can verify it with quantitative tests, e.g.:

- Bartlett's test
- LBQ test

> On transformed data
>
> **Bartlett's test for a specific lag k**
>
> $H_0 : \rho_k = 0$
>
> $H_1 : \rho_k \neq 0$
>
> ρ_k: true autocorr at lag k
>
> r_k: sample autocorr at lag k
>
> Rejection region $|r_k| > \frac{z_{\alpha/2}}{\sqrt{n}}$

In [ ]:
```python
from statsmodels.tsa.stattools import acf

n = len(data_norm)

#autocorrelation function
[acf_values, lbq, _] = acf(data_norm, nlags = int(np.sqrt(n)), qstat=True, fft = Fa
```

In [ ]:
```python
#Bartlett's test at lag 1
lag_test = 1
rk = acf_values[lag_test]
print('Test statistic rk = %f' % rk)
print('Rejection region starts at %f' % (1.96/np.sqrt(n)))

if rk>1.96/np.sqrt(n):
    print('The null hypothesis is rejected')
else: print('The null hypothesis is accepted')
```

```
Test statistic rk = 0.040639
Rejection region starts at 0.243108
The null hypothesis is accepted
```

> **LBQ test**
>
> $H_0 : \rho_k = 0, k = 1, \ldots, L$
>
> $H_1 : \exists k \in [1, L]$ such that $\rho_k \neq 0$
>
> LBQ test statistic:
>
> $$LBQ = n(n+2)\sum_{k=1}^{L}\frac{r_k^2}{n-k}$$
>
> Under H_0 ($\rho_k = 0, k = 1, \ldots L$), $LBQ \sim \chi^2_L$, and its rejection region is:
>
> $$LBQ > \chi^2_{\alpha,L}$$

In [ ]:
```python
lag_test=6 # this is just an example;
```

```python
# Generally speaking: how many lags?
# Rule of thumb: L<sqrt(n)

Q0_LBQ = lbq[lag_test-1]
print('Q0_LBQ = %f' % Q0_LBQ)

#Rejection region for chi square distribution
dof = lag_test
chi2_alfa= stats.chi2.ppf(1-alpha,dof)
print('Rejection region starts at %f' % chi2_alfa)

if Q0_LBQ>chi2_alfa:
  print('The null hypothesis is rejected')
else:
  print('The null hypothesis is accepted')

# Compute the p-value for the LBQ test
pval = 1 - stats.chi2.cdf(Q0_LBQ, lag_test)
print('p-value = %f' % pval)
```

```
Q0_LBQ = 0.621926
Rejection region starts at 12.591587
The null hypothesis is accepted
p-value = 0.996024
```

> Alternatively, you can use the `acorr_ljungbox` function.

In [ ]:
```python
#LBQ test for autocorrelation
from statsmodels.stats.diagnostic import acorr_ljungbox

lbq_test = acorr_ljungbox(data_norm, lags=[lag_test], return_df=True)
print('LBQ test statistic at lag %d = %f' % (lag_test, lbq_test.loc[lag_test,'lb_s
print('LBQ test p-value at lag %d = %f' % (lag_test, lbq_test.loc[lag_test,'lb_pva
```

```
LBQ test statistic at lag 6 = 0.621926
LBQ test p-value at lag 6 = 0.996024
```

...to finally answer the question (Design a 95% **prediction interval** for future observations):

Process data are **normal and independent** (NID)

$$X \sim \mathrm{N}(\mu,\sigma^2) \rightarrow \frac{X-\mu}{s} \sim \mathrm{t}_{\text{n-1}}$$

Attention: this is not $T = \dfrac{\bar{X}-\mu}{S/\sqrt{n}}$

Prediction interval:

$$\Pr\left(-t_{\alpha/2,n-1} \le \frac{X-\mu}{s} \le t_{\alpha/2,n-1}\right) = 1-\alpha \quad \Rightarrow \quad \mu - t_{\alpha/2,n-1}s \le X \le \mu + t_{\alpha/2,n-1}s$$

Approximated prediction interval (95%):

Approximated because we use the sample mean in place of the true mean

$$\bar{X} \pm t_{0.025,64} \cdot s$$

In [ ]:
```python
alpha = 0.05
df = len(data_norm) - 1
Xbar = data_norm.mean()
s = data_norm.std()
```

```
t_alpha = stats.t.ppf(1 - alpha/2, df)


[pred_lo, pred_up] = [Xbar-t_alpha*s,Xbar+t_alpha*s]
print('Two-sided confidence interval for transformed data: [%.3f %.3f]' % (pred_lo
```

```
Two-sided confidence interval for transformed data: [1.719 5.640]
```

Attention: this is the prediction interval on the transformed data. To estimate the prediction interval on the **original data** we need to back transform.

Remind: $y = (x^\lambda - 1)/\lambda$, for $\lambda \neq 0$

Thus: $x = (y\lambda + 1)^{(1/\lambda)}$

In [ ]:
```
[pred_lo_ORIG, pred_up_ORIG] = [(pred_lo*lmbda+1)**(1/lmbda),(pred_up*lmbda+1)**(1
print('Two-sided confidence interval for original data: [%.3f %.3f]' % (pred_lo_ORI
```

```
Two-sided confidence interval for original data: [4.887 87.689]
```