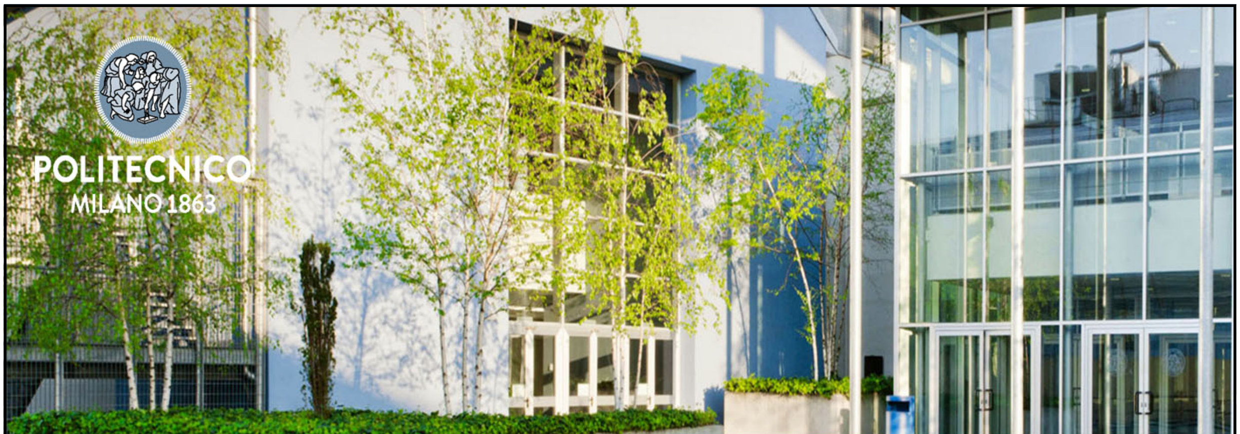


Quality Data Analysis

Multivariate control chart

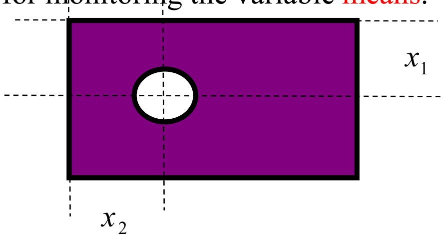
Bianca Maria Colosimo – biancamaria.colosimo@polimi.it

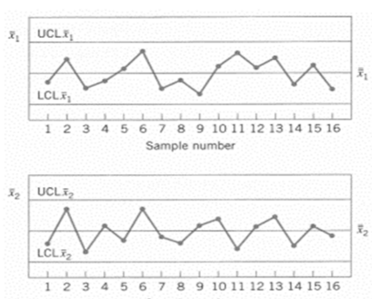
Reference:
Montgomery – Chapter 11

1

Multivariate SPC: control chart for the mean

Assume to be interested in monitoring more than a single variable. We want to simultaneously control all the quality characteristics. Let's apply the known methods for monitoring the variable means.





Quali **Figure 10-1** Control charts for inner ($\bar{x}_1$) and outer ($\bar{x}_2$) bearing diameters.

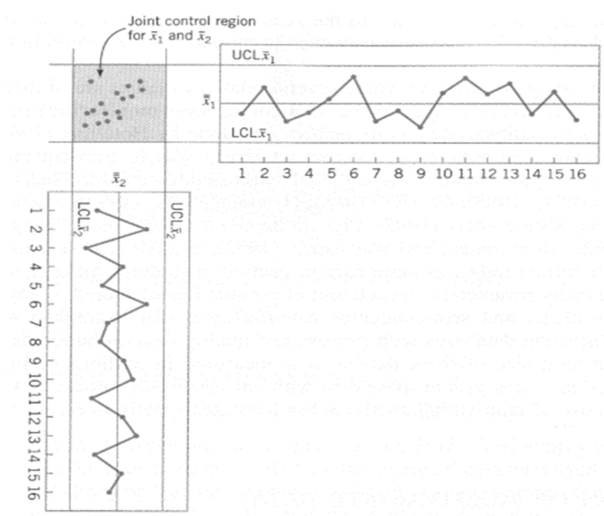


Figure 10-2 Control region using independent control limits for $\bar{x}_1$ and $\bar{x}_2$.

2

2

Multivariate SPC

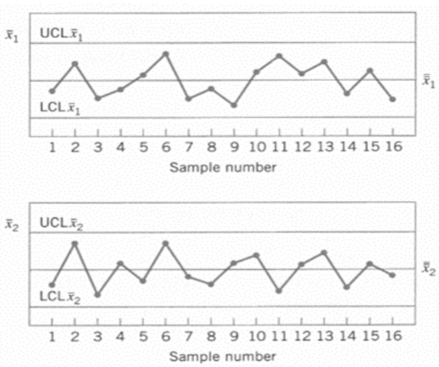


Figure 10-1 Control charts for inner ($\bar{x}_1$) and outer ($\bar{x}_2$) bearing diameters.

Control limit computation:

$$P\{Allarme_{\ 1} | In\ \ Controllo_{\ 1}\} = \alpha_1$$

$$P\{Allarme_{\ 2} | In\ \ Controllo_{\ 2}\} = \alpha_2$$

Assume variables to be independent and:

$$\alpha_1 = \alpha_2 = \alpha$$

$$P\{Allarme | proc.\ in\ controllo\} = 1 - (1-\alpha)(1-\alpha)$$

Generally speaking, for *p* tests on independent variables:

$$\alpha' = 1 - (1-\alpha)^p \qquad \alpha = 1 - (1-\alpha')^{1/p}$$

A few examples:

$$\alpha = 0.05 \begin{cases} p=2 & \alpha' = 0.0975 & ARL = 10 \\ p=3 & \alpha' = 0.143 & ARL = 7 \\ p=4 & \alpha' = 0.185 & ARL = 6 \end{cases}$$

We expected ARL=20

Multivariate SPC

The problem gets more complicated if variables are dependent.

In this case one has to use the Bonferroni's inequality or to determine the joint probability density function.

"familywise" α

$$\boxed{\alpha' \leq \sum_{i=1,..,p} \alpha_i}$$

Bonferroni's inequality

If $\forall$ test *i* we choose $\alpha_i = \alpha/p$

$\Rightarrow \alpha' \leq p\ \alpha/p = \alpha$ being α' the Type I error for the whole set of p tests

Multivariate random variables

Consider a *p*-component vector, i.e. a vector of *p* random variables $\boldsymbol{x}' = [x_1, x_2, ..., x_p]$

Expected value $\boldsymbol{\mu}' = E(\boldsymbol{x}) = [E(x_1), E(x_2), ..., E(x_p)] = [\mu_1, \mu_2, ..., \mu_p]$

Variance-Covariance Matrix

$$\underset{pxp}{V(\boldsymbol{x})} = E[\underset{px1}{(\boldsymbol{x} - E(\boldsymbol{x}))}\underset{1xp}{(\boldsymbol{x} - E(\boldsymbol{x}))'}] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_2^2 & \cdots & \sigma_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_p^2 \end{bmatrix}$$

$$\sigma_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)] = \text{cov}(x_i, x_j)$$

Product between a constant vector and a random vector:

Scalar (1xp)(px1) $V(\mathbf{a}'\mathbf{x}) = E[(\mathbf{a}'\mathbf{x} - E(\mathbf{a}'\mathbf{x}))(\mathbf{a}'\mathbf{x} - E(\mathbf{a}'\mathbf{x}))'] =$

$$= E[(\mathbf{a}'\mathbf{x} - \mathbf{a}'E(\mathbf{x}))(\mathbf{a}'\mathbf{x} - \mathbf{a}'E(\mathbf{x}))'] = \mathbf{a}'E[(\mathbf{x} - E(\mathbf{x}))(\mathbf{a}'(\mathbf{x} - E(\mathbf{x})))'] =$$

$$= \mathbf{a}'E[(\mathbf{x} - E(\mathbf{x}))(\mathbf{x} - E(\mathbf{x}))']\mathbf{a} = \mathbf{a}'\boldsymbol{\Sigma}\mathbf{a}$$

correlation

$$\rho_{ij} = \frac{\text{cov}(x_i x_j)}{\sqrt{V(x_i)V(x_j)}} \qquad \boldsymbol{P} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{12} & 1 & \cdots & \rho_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{1p} & \rho_{2p} & \cdots & 1 \end{bmatrix}$$

Correlation matrix

Bivariate nomal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < +\infty$$

$$\left(\frac{x-\mu}{\sigma}\right)^2 = (x-\mu)\left(\sigma^2\right)^{-1}(x-\mu) \longrightarrow (\boldsymbol{x} - \boldsymbol{\mu})'(\boldsymbol{\Sigma})^{-1}(\boldsymbol{x} - \boldsymbol{\mu})$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'(\boldsymbol{\Sigma})^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \qquad -\infty < x_j < +\infty \quad j=1,2,\ldots,p$$

Bivariate case: p=2

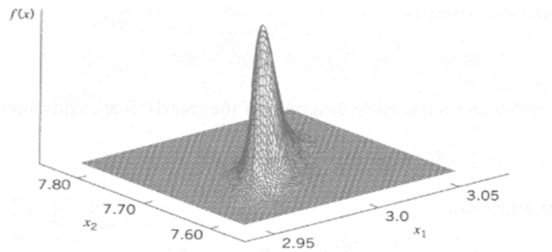
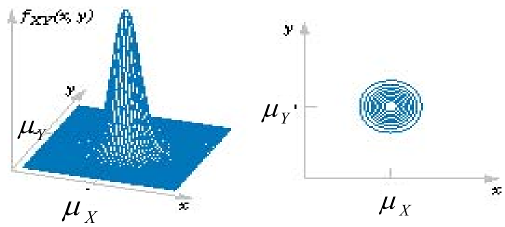
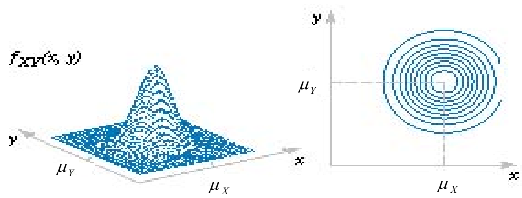
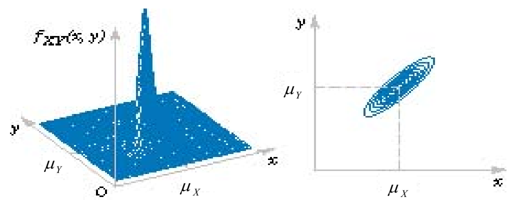


Figure 10-3 A multivariate normal distribution with $p = 2$ variables (bivariate normal).







Some relevant results

Remind:

$$X \sim N(\mu, \sigma^2) \qquad \frac{(X-\mu)^2}{\sigma^2} = Z^2 \sim \chi^2(1)$$

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \qquad n\frac{(\bar{X}-\mu)^2}{\sigma^2} = Z^2 \sim \chi^2(1)$$

Multivariate case:

$$\mathbf{X}_i \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (i = 1, 2, \ldots, n)$$

Has been proven that →

$$(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu}) = Z_1^2 + Z_2^2 + \ldots + Z_p^2 \sim \chi^2(p)$$

↓ Has been proven that

$$\bar{\mathbf{X}} \sim N_p(\boldsymbol{\mu}, (1/n)\boldsymbol{\Sigma}) \qquad n(\bar{\mathbf{X}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{X}}-\boldsymbol{\mu}) = Z_1^2 + Z_2^2 + \ldots + Z_p^2 \sim \chi^2(p)$$

1xp pxp px1

Case p=2

$$\boldsymbol{\bar{x}}' = [\bar{x}_1 \quad \bar{x}_2] \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \rightarrow \boldsymbol{\Sigma}^{-1} = \frac{1}{|\boldsymbol{\Sigma}|} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}$$

$$\boldsymbol{\mu}' = [\mu_1 \quad \mu_2]$$

By solving the matrix multiplications, $n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}) \sim \chi^2(p)$

We got:

$$n[\bar{x}_1 - \mu_1 \quad \bar{x}_2 - \mu_2] \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 - \mu_1 \\ \bar{x}_2 - \mu_2 \end{bmatrix} \sim \chi^2(p)$$

Thus, the statistical quantity:

$$\chi_0^2 = \frac{n}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\left[\sigma_2^2(\bar{x}_1 - \mu_1)^2 - 2\sigma_{12}(\bar{x}_1 - \mu_1)(\bar{x}_2 - \mu_2) + \sigma_1^2(\bar{x}_2 - \mu_2)^2\right]$$

Follows a chi squared distribution with 2 degrees of freedom: $\chi_0^2 \sim \chi^2(2)$

Being known that:

$$\chi_0^2 = \frac{n}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\left[\sigma_2^2(\bar{x}_1 - \mu_1)^2 - 2\sigma_{12}(\bar{x}_1 - \mu_1)(\bar{x}_2 - \mu_2) + \sigma_1^2(\bar{x}_2 - \mu_2)^2\right] \sim \chi^2(2)$$

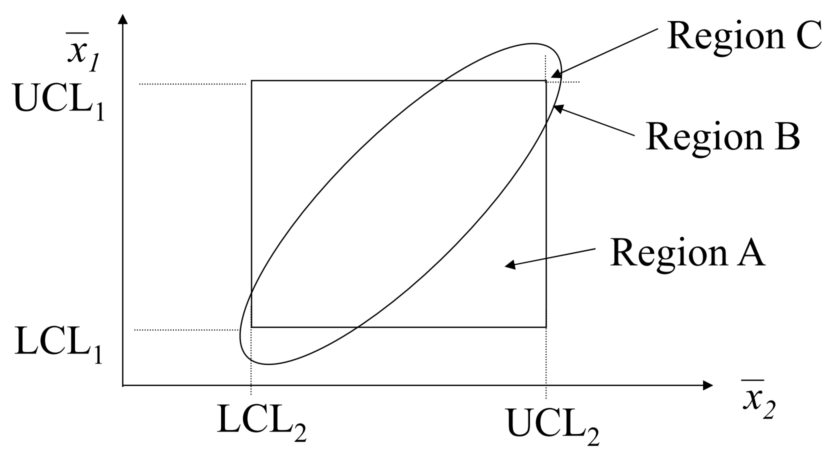
A joint control region is applicable:

$$Alarm\ if: \chi_0^2 > \chi_\alpha^2(2)$$

The above inequality corresponds to an elliptic region in the bivariate space spanned by $\bar{x}_1$ and $\bar{x}_2$
Such ellipse if referred to as ‘control ellipse’

The equation of the control ellipse is:

$$\left(\frac{\bar{x}_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{\bar{x}_2 - \mu_2}{\sigma_2}\right)^2 - 2\frac{\sigma_{12}}{\sigma_1\sigma_2}\left(\frac{\bar{x}_1 - \mu_1}{\sigma_1}\right)\left(\frac{\bar{x}_2 - \mu_2}{\sigma_2}\right) = \frac{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}{\sigma_1^2\sigma_2^2}\frac{\chi_\alpha^2(2)}{n}$$



Region A: out-of-control not signaled by traditional charts
Region B: false out-of-control signaled
Region C: 'twice' false out-of-control (because of $\overline{x}_1$ and $\overline{x}_2$) signaled

χ^2 control chart

A statistical control scheme can be implemented directly into the plane $\overline{x}_1$ - $\overline{x}_2$ by using the control ellipse:

- But the information about the temporal sequence would be lost;
- It would be difficult to depict the control region for 3 variables, and even impossible for larger numbers of variables

A control chart can be designed to monitor the quantity χ_0^2 by using the control limit $\chi_\alpha^2(p)$. In the most general case:

$$\boldsymbol{\mu}' = [\mu_1, \mu_2, \ldots, \mu_p]$$

$$\boldsymbol{\Sigma}$$

$$\overline{\mathbf{x}} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \overline{x}_p \end{bmatrix}$$

$$\chi_0^2 = n(\overline{\mathbf{x}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\overline{\mathbf{x}} - \boldsymbol{\mu})$$ scalare

1xp pxp px1

$$\text{UCL} = \chi^2_{\alpha,p}$$

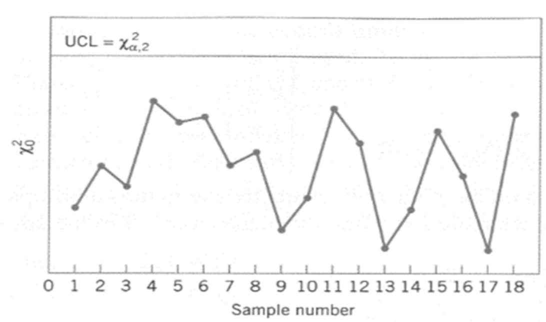


Figure 10-6 A chi-square control chart for $p = 2$ quality characteristics.

Assume we don't know $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and we m samples of size n to be used in design phase

Estimators (considering m samples)

$$\boldsymbol{\mu} \longrightarrow \bar{\bar{\mathbf{x}}}$$
$$\boldsymbol{\Sigma} \longrightarrow \mathbf{S}$$

For each sample k (k=1,…,m):

$$\chi^2_{0k} = n(\bar{\boldsymbol{x}}_k - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{x}}_k - \boldsymbol{\mu}) \longrightarrow T_k^2 = n\underbrace{(\bar{\boldsymbol{x}}_k - \bar{\bar{\boldsymbol{x}}})'}_{1xp} \, \underbrace{\boldsymbol{S}^{-1}}_{pxp}\underbrace{(\bar{\boldsymbol{x}}_k - \bar{\bar{\boldsymbol{x}}})}_{px1}$$

The T^2 statistic is referred to as Hotelling's statistic and it follows the F distribution (not the squared chi distribution), corrected by a constant

In particular:

- Phase 1 (design phase)

$$T_k^2 = n(\bar{\boldsymbol{x}}_k - \bar{\bar{\boldsymbol{x}}})' \, \boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_k - \bar{\bar{\boldsymbol{x}}}) \sim c_1(m,n,p)F(p, m(n-1)-(p-1))$$

$$c_1(m,n,p) = \frac{p(n-1)\boxed{(m-1)}}{m(n-1)-(p-1)} \Rightarrow \begin{array}{l} \text{UCL} = c_1(m,n,p)F_\alpha(p, m(n-1)-(p-1)) \\ \text{LCL} = 0 \end{array}$$

- Phase 2 (future observations):

Under the assumption that m* samples during the design phase

$$T_k^2 = n(\bar{\boldsymbol{x}}_k - \bar{\bar{\boldsymbol{x}}})' \, \boldsymbol{S}^{-1}(\bar{\boldsymbol{x}}_k - \bar{\bar{\boldsymbol{x}}}) \sim c_2(m^*,n,p)F(p, m^*(n-1)-(p-1))$$

$$c_2(m^*,n,p) = \frac{p(n-1)\boxed{(m^*+1)}}{m^*(n-1)-(p-1)} \Rightarrow \begin{array}{l} \text{UCL} = c_2(m^*,n,p)F_\alpha(p, m^*(n-1)-(p-1)) \\ \text{LCL} = 0 \end{array}$$

It is possible to prove that, for large values of m:

$$c_1(m,n,p)F_\alpha(p,m(n-1)-(p-1)) \rightarrow \chi^2_\alpha(p)$$

Remind that:

$$\lim_{\nu_2 \to \infty} \mathrm{F}(\nu_1,\nu_2) = \frac{\chi^2(\nu_1)}{\nu_1}$$

Example: ultimate tensile strenght and diameter of a textile fiber

n=10; m=20; p=2.

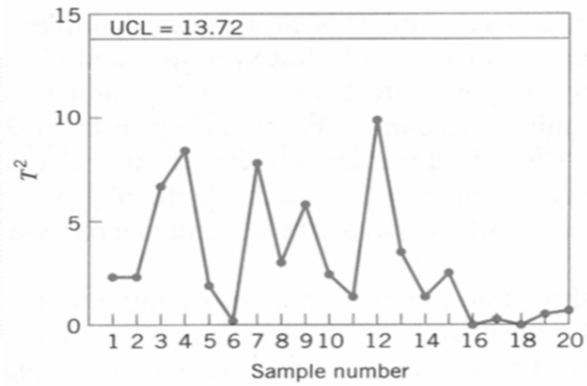
Sample Number k	Sample Means: Tensile Strength ($\bar{x}_{1k}$)	Sample Means: Diameter ($\bar{x}_{2k}$)	Variances and Covariances: S^2_{1k}	Variances and Covariances: S^2_{2k}	Variances and Covariances: S_{12k}	Statistics: T^2_k	Statistics: $\lvert \mathbf{S}_k \rvert$
1	115.25	1.04	1.25	0.87	0.80	2.16	0.45
2	115.91	1.06	1.26	0.85	0.81	2.14	0.41
3	115.05	1.09	1.30	0.90	0.82	6.77	0.50
4	116.21	1.05	1.02	0.85	0.81	8.29	0.21
5	115.90	1.07	1.16	0.73	0.80	1.89	0.21
6	115.55	1.06	1.01	0.80	0.76	0.03	0.23
7	114.98	1.05	1.25	0.78	0.75	7.54	0.41
8	115.25	1.10	1.40	0.83	0.80	3.01	0.52
9	116.15	1.09	1.19	0.87	0.83	5.92	0.35
10	115.92	1.05	1.17	0.86	0.95	2.41	0.10
11	115.75	0.99	1.45	0.79	0.78	1.13	0.54
12	114.90	1.06	1.24	0.82	0.81	9.96	0.36
13	116.01	1.05	1.26	0.55	0.72	3.86	0.17
14	115.83	1.07	1.17	0.76	0.75	1.11	0.33
15	115.29	1.11	1.23	0.89	0.82	2.56	0.42
16	115.63	1.04	1.24	0.91	0.83	0.70	0.19
17	115.47	1.03	1.20	0.95	0.70	0.19	0.65
18	115.58	1.05	1.18	0.83	0.79	0.00	0.36
19	115.72	1.06	1.31	0.89	0.76	0.35	0.59
20	115.40	1.04	1.29	0.85	0.68	0.62	0.63
Averages	$\bar{\bar{x}}_1 = 115.59$	$\bar{\bar{x}}_2 = 1.06$	$\bar{S}^2_1 = 1.23$	$\bar{S}^2_2 = 0.83$	$\bar{S}_{12} = 0.79$		

$$\chi^2_0 = \frac{n}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\left[\sigma_2^2(\overline{x_1}-\mu_1)^2 - 2\sigma_{12}(\overline{x_1}-\mu_1)(\overline{x_2}-\mu_2) + \sigma_1^2(\overline{x_2}-\mu_2)^2\right]$$

$$p = 2 \rightarrow T^2 = \frac{n}{\bar{S}_1^2\bar{S}_2^2 - \bar{S}_{12}^2}\left[\bar{S}_2^2(\overline{x_1}-\overline{\overline{x_1}})^2 - 2\bar{S}_{12}(\overline{x_1}-\overline{\overline{x_1}})(\overline{x_2}-\overline{\overline{x_2}}) + \bar{S}_1^2(\overline{x_2}-\overline{\overline{x_2}})^2\right]$$

$$T^2 = \frac{10}{(1.23)(0.83)-(0.79)^2}[0.83(\bar{x}_1 - 115.59)^2 + 1.23(\bar{x}_2 - 1.06)^2 - 2(0.79)(\bar{x}_1 - 115.59)(\bar{x}_2 - 1.06)]$$

$$\begin{aligned}\text{UCL} &= \frac{p(m-1)(n-1)}{mn-m-p+1}F_{\alpha,p,mn-m-p+1}\\ &= \frac{2(19)(9)}{20(10)-20-2+1}F_{0.001,2,20(10)-20-2+1}\\ &= \frac{342}{179}F_{0.001,2,179}\\ &= (1.91)7.18\\ &= 13.72\end{aligned}$$



No out-of-control data.

By using the Phase II limit advocated by Montgomery we have UCL=15.16.

If we used the χ^2 approximation we would get UCL=13.816, close to the Phase I limit, but different from the Phase II limit.

Problem:
When a control limit violation occurs, how to search for an assignable cause?

We could exploit univariate control charts (one for each variable), with Bonferroni's control limits, i.e., by replacing $Z_{\alpha/2}$ with $Z_{\alpha/(2p)}$

A different approach consists of decomposing the T^2 statistic into components that reflect the contribution of each individual variable.

If T^2 is the current value of the statistic, and $T^2_{(i)}$ is the value of the statistic for all process variables except the ith one, then:

$d_i = T^2 - T^2_{(i)}$ is an indicator of the relative contribution of the ith variable.

Thus, when an out-of-control signal is generated, we can compute the values of d_i for each process variable and to search for assignable causes associated to the variables for which are relatively large.

T² control chart

Example:

$p = 3$ $\quad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.9 & 0.9 \\ 0.9 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{bmatrix}$ $\quad \boldsymbol{\mu}' = [0, 0, 0]$

Standardized vars:

$$y_{ij} = \frac{x_{ij} - \mu_j}{\sqrt{(m-1)\sigma_j^2}}$$

$\chi^2_{0.005,3} = 12.84$ Out-of-control

Observation Vector y'	Control Chart Statistic $T_0^2 (= \chi_0^2)$	d_1	d_2	d_3
(2, 0, 0)	27.14	27.14	6.09	6.09
(1, 1, −1)	26.79	6.79	6.79	25.73
(1, −1, 0)	20.00	14.74	14.74	0

$d_i = T^2 - T^2_{(i)}$

Cut-off for d_i $\quad \chi^2_{0.01,1} = 6.63$

Quality Data Analysis

Individual observations

In some SPC applications (e.g., chemical industry) the size of the sample is *n=1.* In this case, the Hotelling's statistic is defined as follows:

$$n = 1 \quad T^2 = (\mathbf{x} - \overline{\mathbf{x}})'\mathbf{S}^{-1}(\mathbf{x} - \overline{\mathbf{x}})$$

The Phase II control limits are:

$$\begin{cases} \text{UCL} = \dfrac{p(m+1)(m-1)}{m^2 - mp} F_{\alpha,p,m-p} \\ \text{LCL} = 0 \end{cases}$$

When the number of samples (individuals) is very large (*m>100*) the following Phase II control limits may be used:

$$\begin{cases} \text{UCL} = \dfrac{p(m-1)}{m-p} F_{\alpha,p,m-p} \\ \text{or} \\ \text{UCL} = \chi^2_{\alpha,p} \end{cases}$$

Some authors suggest the use of Beta distribution for Phase I control limit computation.

$$\begin{cases} \text{UCL} = \dfrac{(m-1)^2}{m} \beta_{\alpha,\, p/2,\, (m-p-1)/2} \\ \text{LCL} = 0 \end{cases}$$

Quality Data Analysis

Individual observations

A relevant issue in the presence of individual observations is the variance-covariance matrix Σ estimation

The 'usual' estimator (*long period*) is:

$$\mathbf{S}_1 = \frac{1}{m-1}\sum_{i=1}^{m}(\mathbf{x}_i - \overline{\mathbf{x}})(\mathbf{x}_i - \overline{\mathbf{x}})'$$

This estimator is particularly sensitive to outliers or out-of-control data in the original sample of *m* observations

Alternative:

$$\mathbf{v}_i = \mathbf{x}_{i+1} - \mathbf{x}_i \qquad i = 1, 2, \ldots, m-1$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1' \\ \mathbf{v}_2' \\ \vdots \\ \mathbf{v}_{m-1}' \end{bmatrix}$$

A different estimator (*short period*) is:

$$\mathbf{S}_2 = \frac{1}{2}\frac{\mathbf{V}'\mathbf{V}}{(m-1)}$$

Example (Sullivan e Woodall – 1996)
Composition:

L=percentage classified as large
M=percentage classified as medium
S=percentage classified as small
Control chart just on the first two components (sum=100%)

$$\overline{\mathbf{x}}' = [5.682, 88.22]$$

$$\mathbf{S}_1 = \begin{bmatrix} 3.770 & -5.495 \\ -5.495 & 13.53 \end{bmatrix}$$

$$\mathbf{S}_2 = \begin{bmatrix} 1.562 & -2.093 \\ -2.093 & 6.721 \end{bmatrix}$$

Individual observations

Table 10-2 Example from Sullivan and Woodall (1996) Using the Data from Holmes and Mergen (1993) and the T^2 Statistics Using Estimators $\mathbf{S}_1$ and $\mathbf{S}_2$

i	$L = x_{i,1}$	$M = x_{i,2}$	$S = x_{i,3}$	$T^2_{1,i}$	$T^2_{2,i}$	i	$L = x_{i,1}$	$M = x_{i,2}$	$S = x_{i,3}$	$T^2_{1,i}$	$T^2_{2,i}$
1	5.4	93.6	1.0	4.496	6.439	29	7.4	83.6	9.0	1.594	3.261
2	3.2	92.6	4.2	1.739	4.227	30	6.8	84.8	8.4	0.912	1.743
3	5.2	91.7	3.1	1.460	2.200	31	6.3	87.1	6.6	0.110	0.266
4	3.5	86.9	9.6	4.933	7.643	32	6.1	87.2	6.7	0.077	0.166
5	2.9	90.4	6.7	2.690	5.565	33	6.6	87.3	6.1	0.255	0.564
6	4.6	92.1	3.3	1.272	2.258	34	6.2	84.8	9.0	1.358	2.069
7	4.4	91.5	4.1	0.797	1.676	35	6.5	87.4	6.1	0.203	0.448
8	5.0	90.3	4.7	0.337	0.645	36	6.0	86.8	7.2	0.193	0.317
9	8.4	85.1	6.5	2.088	4.797	37	4.8	88.8	6.4	0.297	0.590
10	4.2	89.7	6.1	0.666	1.471	38	4.9	89.8	5.3	0.197	0.464
11	3.8	92.5	3.7	1.368	3.057	39	5.8	86.9	7.3	0.242	0.353
12	4.3	91.8	3.9	0.951	1.986	40	7.2	83.8	9.0	1.494	2.928
13	3.7	91.7	4.6	1.105	2.688	41	5.6	89.2	5.2	0.136	0.198
14	3.8	90.3	5.9	1.019	2.317	42	6.9	84.5	8.6	1.079	2.062
15	2.6	94.5	2.9	3.099	7.262	43	7.4	84.4	8.2	1.096	2.477
16	2.7	94.5	2.8	3.036	7.025	44	8.9	84.3	6.8	2.854	6.666
17	7.9	88.7	3.4	3.803	6.189	45	10.9	82.2	6.9	7.677	17.666
18	6.6	84.6	8.8	1.167	1.997	46	8.2	89.8	2.0	6.677	10.321
19	4.0	90.7	5.3	0.751	1.824	47	6.7	90.4	2.9	2.708	3.869
20	2.5	90.2	7.3	3.966	7.811	48	5.9	90.1	4.0	0.888	1.235
21	3.8	92.7	3.5	1.486	3.247	49	8.7	83.6	7.7	2.424	5.914
22	2.8	91.5	5.7	2.357	5.403	50	6.4	88.0	5.6	0.261	0.470
23	2.9	91.8	5.3	2.094	4.959	51	8.4	84.7	6.9	1.995	4.731
24	3.3	90.6	6.1	1.721	3.800	52	9.6	80.6	9.8	4.732	11.259
25	7.2	87.3	5.5	0.914	1.791	53	5.1	93.0	1.9	2.891	4.303
26	7.3	79.0	13.7	9.226	14.372	54	5.0	91.4	3.6	0.989	1.609
27	7.0	82.6	10.4	2.940	4.904	55	5.0	86.2	8.8	1.770	2.495
28	6.0	83.5	10.5	3.310	4.771	56	5.9	87.2	6.9	0.102	0.166

Individual observations

Example (Sullivan e Woodall – 1996)

$$\overline{\mathbf{x}}' = [5.682, 88.22]$$

$$\mathbf{S}_1 = \begin{bmatrix} 3.770 & -5.495 \\ -5.495 & 13.53 \end{bmatrix}$$

$$\mathbf{S}_2 = \begin{bmatrix} 1.562 & -2.093 \\ -2.093 & 6.721 \end{bmatrix}$$

$$\overline{\mathbf{x}}'_{1-24} = [4.23, 90.8]$$

$$\overline{\mathbf{x}}'_{25-56} = [6.77, 86.3]$$

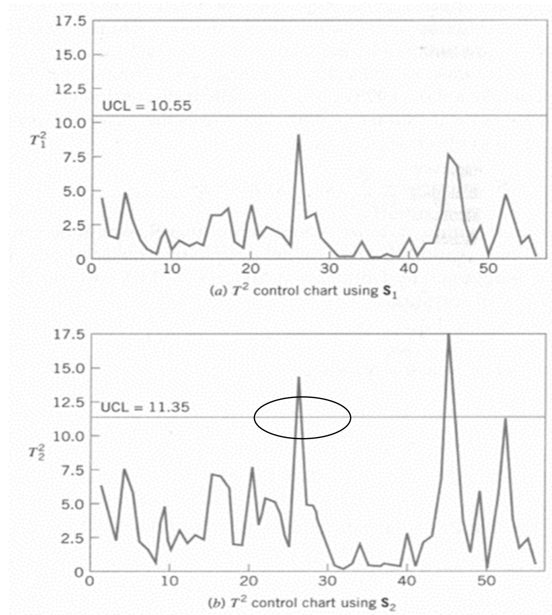


Figure 10-9 T^2 control charts for the data in Table 10-2.

Latent Structure methods (briefly)

Problem: The control-charting schemes previously presented become less and less effective as the number of monitoring variables (p) increases
(satisfactory performances for *p*<=10).

Example: Multivariate control chart
The ARL value when the process is out-of-control increases as *p* increases

Consider PCA and then control chart on the retained components

p	δ	0.05	0.10
		H = 7.35	8.64
2	0.0	199.93	199.98
	0.5	26.61	28.07
	1.0	11.23	10.15
	1.5	7.14	6.11
	2.0	5.28	4.42
	3.0	3.56	2.93
		H = 11.22	12.73
4	0.0	199.84	200.12
	0.5	32.29	35.11
	1.0	13.48	12.17
	1.5	8.54	7.22
	2.0	6.31	5.19
	3.0	4.23	3.41
		H = 14.60	16.27
6	0.0	200.11	200.03
	0.5	36.39	40.38
	1.0	15.08	13.66
	1.5	9.54	8.01
	2.0	7.05	5.74
	3.0	4.72	3.76
		H = 20.72	22.67
10	0.0	199.91	199.95
	0.5	42.49	48.52
	1.0	17.48	15.98
	1.5	11.04	9.23
	2.0	8.15	6.57
	3.0	5.45	4.28
		H = 27.82	30.03
15	0.0	199.95	199.89
	0.5	48.20	56.19
	1.0	19.77	18.28
	1.5	12.46	10.41
	2.0	9.20	7.36
	3.0	6.16	4.78