EXERCISE CLASS 6 - SPC for iid data

EXERCISE 1

Data reported in 'ESE06_ex1.csv' represent the diameters of spheres in a recirculating ball mechanism widely used in machine tool industry. The component is very critical and the tolerance is 10 ± 0.5mm.

Verify if the process is in control by using an X-bar and R chart.

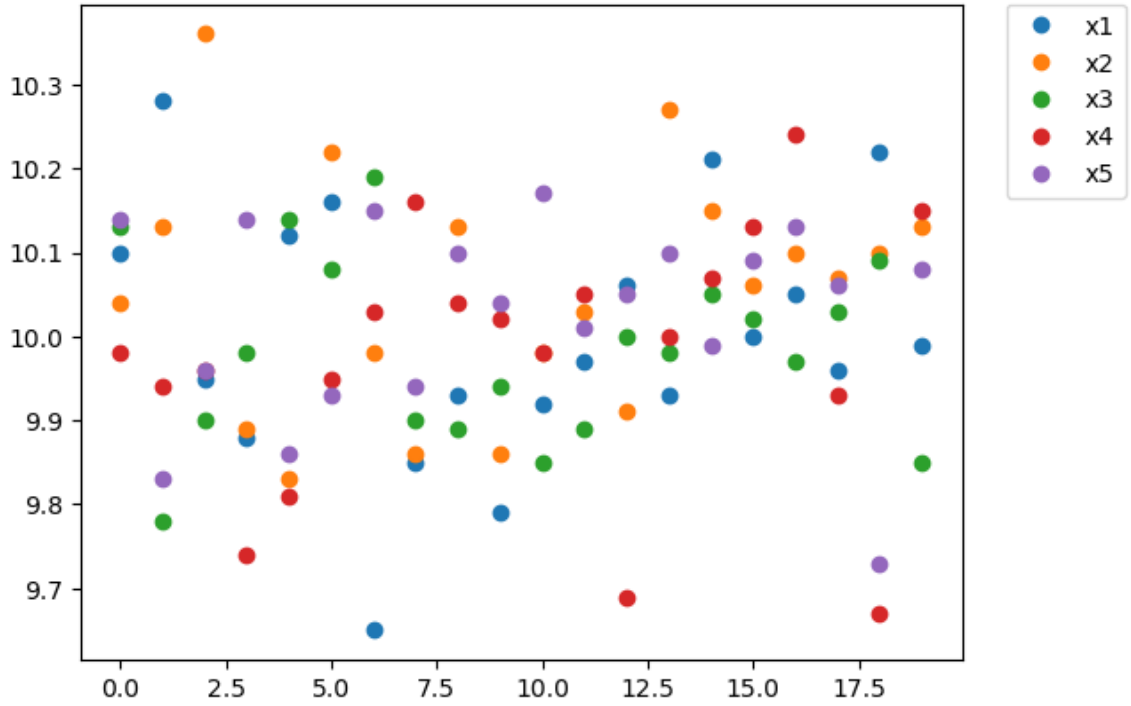
In [ ]:
```
# Import the necessary libraries
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from scipy import stats

# Import the dataset
data = pd.read_csv('ESE06_ex1.csv')

# Inspect the dataset
data.head()
```

Out[ ]:

	x1	x2	x3	x4	x5
0	10.10	10.04	10.13	9.98	10.14
1	10.28	10.13	9.78	9.94	9.83
2	9.95	10.36	9.90	9.96	9.96
3	9.88	9.89	9.98	9.74	10.14
4	10.12	9.83	10.14	9.81	9.86

> ### Solution
>
> Inspect the data by plotting the individual datapoints.

In [ ]:
```
# Make a scatter plot of all the columns against the index
plt.plot(data['x1'], linestyle='none', marker='o', label = 'x1')
plt.plot(data['x2'], linestyle='none', marker='o', label = 'x2')
plt.plot(data['x3'], linestyle='none', marker='o', label = 'x3')
plt.plot(data['x4'], linestyle='none', marker='o', label = 'x4')
plt.plot(data['x5'], linestyle='none', marker='o', label = 'x5')
# place the legend outside the plot
plt.legend(bbox_to_anchor=(1.05, 1), loc=2, borderaxespad=0.)
plt.show()
```

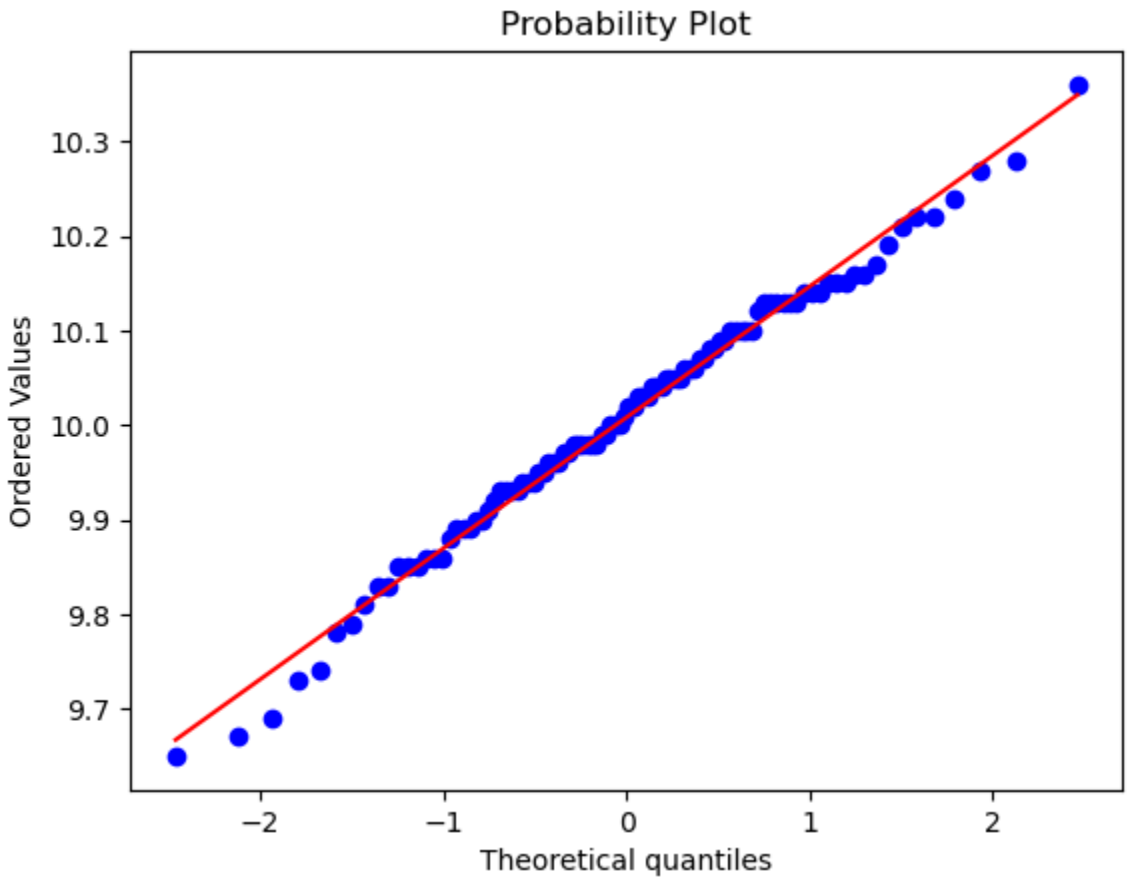


> Doesn't look like strange patterns or outliers are present.
>
> We might also check randomness but we would need to know the within-sample order!

> Verify the assumption of normality, assuming all the data are from the same population.

In [ ]:
```
# Stack the data into a single column
data_stack = data.stack()

# We can use the Shapiro-Wilk test
_, p_value_SW = stats.shapiro(data_stack)
print('p-value of the Shapiro-Wilk test: %.3f' % p_value_SW)

# QQ-plot
stats.probplot(data_stack, dist="norm", plot=plt)
plt.show()
```

```
p-value of the Shapiro-Wilk test: 0.753
```



> Let's compute the mean and the range for each sample.
>
> *Note: we need to apply the mean and range functions to each row of the data frame.*

In [ ]:
```
# Make a copy of the data
data_XR = data.copy()
# Add a column with the mean of the rows
data_XR['sample_mean'] = data.mean(axis=1)
# Add a column with the range of the rows
data_XR['sample_range'] = data.max(axis=1) - data.min(axis=1)

# Inspect the dataset
data_XR.head()
```

Out[ ]:

	x1	x2	x3	x4	x5	sample_mean	sample_range
0	10.10	10.04	10.13	9.98	10.14	10.078	0.16
1	10.28	10.13	9.78	9.94	9.83	9.992	0.50
2	9.95	10.36	9.90	9.96	9.96	10.026	0.46
3	9.88	9.89	9.98	9.74	10.14	9.926	0.40
4	10.12	9.83	10.14	9.81	9.86	9.952	0.33

> Now compute the grand mean and the mean of the ranges.

In [ ]:
```
Xbar_mean = data_XR['sample_mean'].mean()
R_mean = data_XR['sample_range'].mean()

print('Mean of the sample mean: %.3f' % Xbar_mean)
print('Mean of the sample range: %.3f' % R_mean)
```

```
Mean of the sample mean: 10.008
Mean of the sample range: 0.314
```

> Since there is no constraint on the choice of Type I error α, we can set K = 3 (α = 0.0027)
>
> Remember the formulas for the control limits.
>
> **$\bar{\mathbf{X}}$ chart**:
>
> - $UCL = \overline{\overline{X}} + A_2(n)\overline{R}$
> - $CL = \overline{\overline{X}}$
> - $LCL = \overline{\overline{X}} - A_2(n)\overline{R}$
>
> **R chart**:
>
> - $UCL = D_4(n)\overline{R}$
> - $CL = \overline{R}$
> - $LCL = D_3(n)\overline{R}$

Factors for contructing variable control charts

Observations in Sample, n	Chart for Averages: Factors for Control Limits A		A_3	Chart for Standard Deviations: Factors for Center Line c_4	$1/c_4$	Factors for Control Limits B_3	B_4	B_5	B_6	Chart for Ranges: Factors for Center Line d_2	$1/d_2$	d_3	Factors for Control Limits D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.358	0	2.574
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0423	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.806	6.056	0.459	1.541

For $n > 25$.

In [ ]:
```
n = 5
A2 = 0.577
D3 = 0
D4 = 2.114

# Now we can compute the CL, UCL and LCL for Xbar and R
data_XR['Xbar_CL'] = Xbar_mean
```

```
data_XR['Xbar_UCL'] = Xbar_mean + A2 * R_mean
data_XR['Xbar_LCL'] = Xbar_mean - A2 * R_mean

data_XR['R_CL'] = R_mean
data_XR['R_UCL'] = D4 * R_mean
data_XR['R_LCL'] = D3 * R_mean

# Inspect the dataset
data_XR.head()
```

Out[ ]:

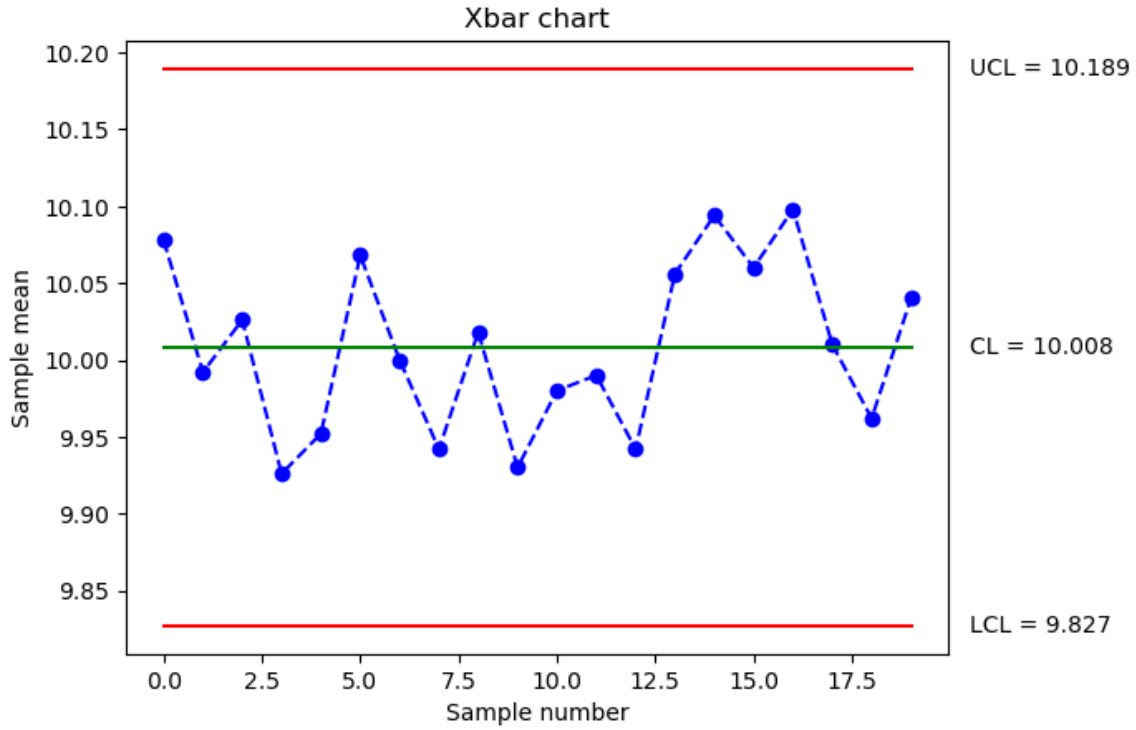
	x1	x2	x3	x4	x5	sample_mean	sample_range	Xbar_CL	Xbar_UCL	Xbar_LCL
0	10.10	10.04	10.13	9.98	10.14	10.078	0.16	10.0082	10.189378	9.827022
1	10.28	10.13	9.78	9.94	9.83	9.992	0.50	10.0082	10.189378	9.827022
2	9.95	10.36	9.90	9.96	9.96	10.026	0.46	10.0082	10.189378	9.827022
3	9.88	9.89	9.98	9.74	10.14	9.926	0.40	10.0082	10.189378	9.827022
4	10.12	9.83	10.14	9.81	9.86	9.952	0.33	10.0082	10.189378	9.827022

> Add two columns to store the violations of the control limits.

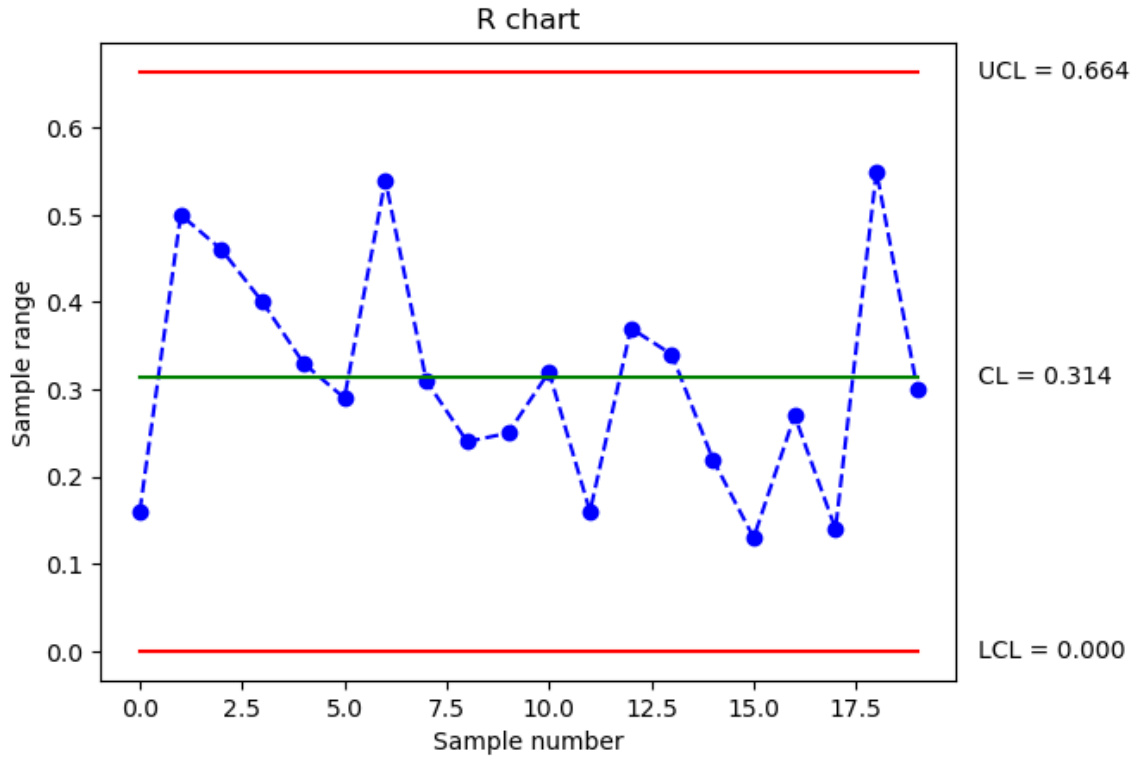
In [ ]:
```
data_XR['Xbar_TEST1'] = np.where((data_XR['sample_mean'] > data_XR['Xbar_UCL']) |
                (data_XR['sample_mean'] < data_XR['Xbar_LCL']), data_XR['sample_
data_XR['R_TEST1'] = np.where((data_XR['sample_range'] > data_XR['R_UCL']) |
                (data_XR['sample_range'] < data_XR['R_LCL']), data_XR['sample_ra
```

> Now plot the limits and the data in the charts.

In [ ]:
```
# Plot the Xbar chart
plt.title('Xbar chart')
plt.plot(data_XR['sample_mean'], color='b', linestyle='--', marker='o')
plt.plot(data_XR['Xbar_UCL'], color='r')
plt.plot(data_XR['Xbar_CL'], color='g')
plt.plot(data_XR['Xbar_LCL'], color='r')
plt.ylabel('Sample mean')
plt.xlabel('Sample number')
# add the values of the control limits on the right side of the plot
plt.text(len(data_XR)+.5, data_XR['Xbar_UCL'].iloc[0], 'UCL = {:.3f}'.format(dat
plt.text(len(data_XR)+.5, data_XR['Xbar_CL'].iloc[0], 'CL = {:.3f}'.format(data_
plt.text(len(data_XR)+.5, data_XR['Xbar_LCL'].iloc[0], 'LCL = {:.3f}'.format(dat
# highlight the points that violate the alarm rules
plt.plot(data_XR['Xbar_TEST1'], linestyle='none', marker='s', color='r', markers
plt.show()
```



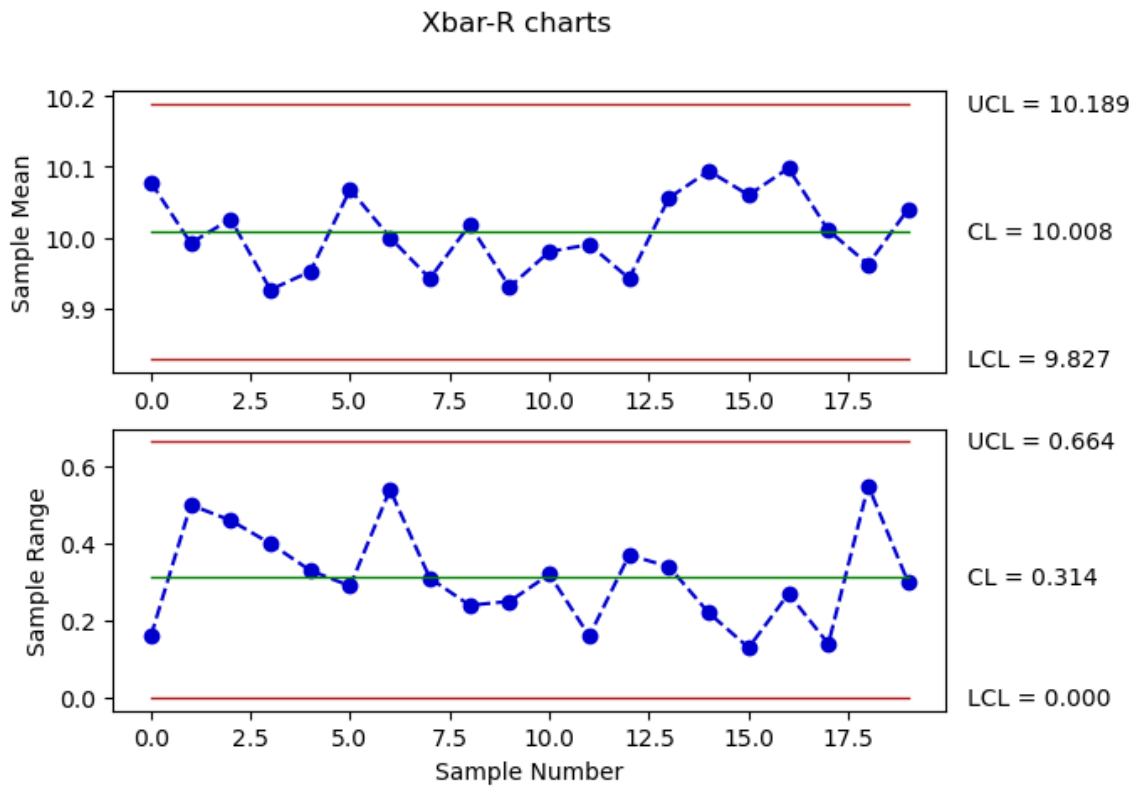
In [ ]:
```
# Plot the R chart
plt.title('R chart')
plt.plot(data_XR['sample_range'], color='b', linestyle='--', marker='o')
plt.plot(data_XR['R_UCL'], color='r')
plt.plot(data_XR['R_CL'], color='g')
plt.plot(data_XR['R_LCL'], color='r')
plt.ylabel('Sample range')
plt.xlabel('Sample number')
# add the values of the control limits on the right side of the plot
plt.text(len(data_XR)+.5, data_XR['R_UCL'].iloc[0], 'UCL = {:.3f}'.format(data_X
plt.text(len(data_XR)+.5, data_XR['R_CL'].iloc[0], 'CL = {:.3f}'.format(data_XR[
plt.text(len(data_XR)+.5, data_XR['R_LCL'].iloc[0], 'LCL = {:.3f}'.format(data_X
# highlight the points that violate the alarm rules
plt.plot(data_XR['R_TEST1'], linestyle='none', marker='s', color='r', markersize
plt.show()
```



> The process is in control.

> Alternatively, we can use the `XbarR` function from the `qda.ControlCharts` package.

In [ ]:
```
import qda

# Create the control chart with the data
data_XR_qda = qda.ControlCharts.XbarR(data)
```



EXERCISE 1 (continued)

Given the previous dataset:

1. Redesign the X-bar and R chart in order to achieve in both the charts a Type I error equal to 0.002 (assuming that the normal approximation applies for both of them).
2. Determine the operating characteristic curve (OC) for the X-bar chart (by using K=3 and expressing the shift of the mean in standard deviation units)
3. Determine the corresponding ARL curve.
4. Estimate the standard deviation through the statistic R.
5. Design the confidence interval on the process mean that corresponds to the control limits computed in point 1.

Point 1

Redesign the X-bar and R chart in order to achieve in both the charts a Type I error equal to 0.002 (assuming that the normal approximation applies for both of them).

> ### Solution
>
> Assuming that the normal approximation applies for both of them, we need to find the value of K such that $\alpha = 0.002$:
>
> $$K = z_{\alpha/2}$$

In [ ]:
```
# Compute the new K_alpha value
alpha = 0.002
K_alpha = stats.norm.ppf(1-alpha/2)

print('K = %.3f' % K_alpha)
```

```
K = 3.090
```

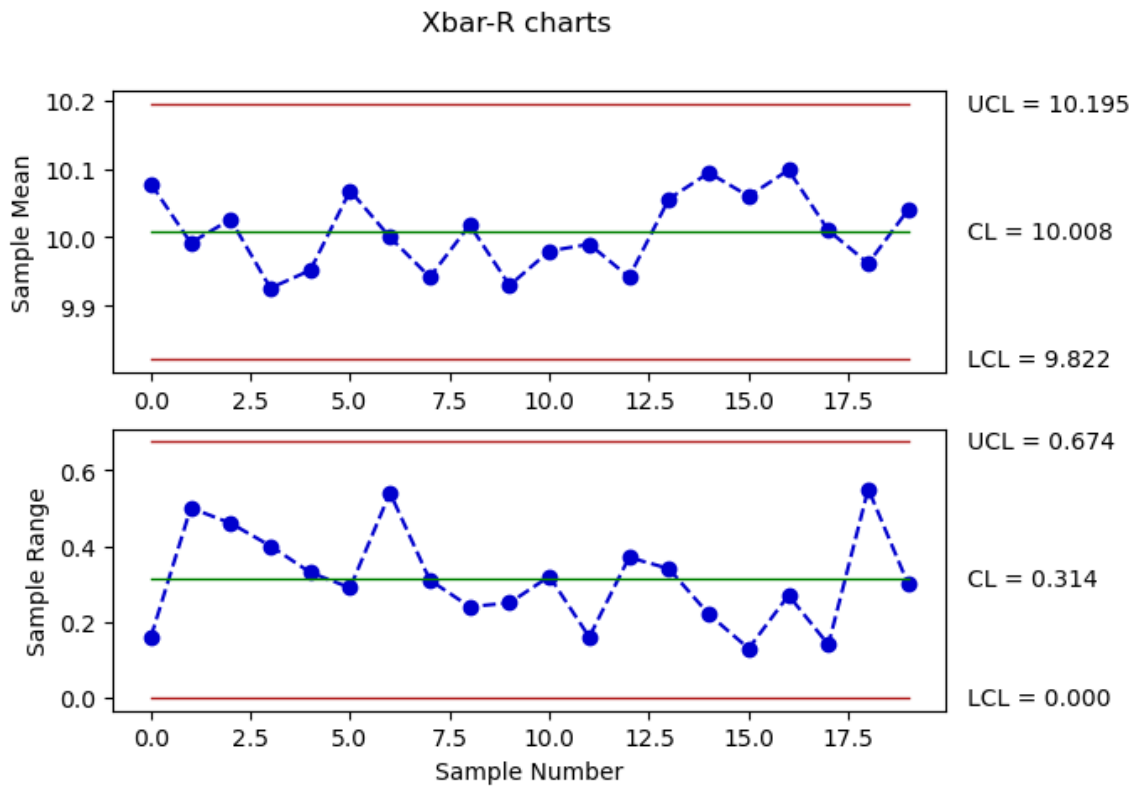
> Now let's design the control charts with the new value of K.
>
> Remember the formulas for the control limits for $K \neq 3$.
>
> **$\bar{\mathbf{X}}$ chart**:
>
> - $UCL = \overline{\overline{X}} + z_{\alpha/2}\frac{1}{d_2\sqrt{n}}\overline{R}$
> - $CL = \overline{\overline{X}}$
> - $LCL = \overline{\overline{X}} - z_{\alpha/2}\frac{1}{d_2\sqrt{n}}\overline{R}$
>
> **R chart**:
>
> - $UCL = \overline{R} + z_{\alpha/2}\frac{d_3}{d_2}\overline{R}$
> - $CL = \overline{R}$
> - $LCL = max(0;\ \overline{R} - z_{\alpha/2}\frac{d_3}{d_2}\overline{R})$

Factors for contructing variable control charts

Observations in Sample, n	Chart for Averages: Factors for Control Limits A	A_2	A_3	Chart for Standard Deviations: Factors for Center Line c_4	$1/c_4$	Factors for Control Limits B_3	B_4	B_5	B_6	Chart for Ranges: Factors for Center Line d_2	$1/d_2$	d_3	Factors for Control Limits D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.358	0	2.574
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0423	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.806	6.056	0.459	1.541

For $n > 25$.

In [ ]:
```
# We can use the same function again. This time we need to specify the new K_alp
data_XR_alpha = qda.ControlCharts.XbarR(data, K = K_alpha)
```



Point 2

Determine the operating characteristic curve (OC) for the X-bar chart (by using K=3 and expressing the shift of the mean in standard deviation units).

Solution

To determine the OC curve, we need to compute the probability of β for each value of the shift μ.

We are testing the null hypothesis H_0 that the sample mean $\bar{X}$ is normally distributed with mean μ_0 and variance σ^2/n.

$$H_0 : \bar{X} \sim N(\mu_0, \sigma^2/n)$$

The alternative hypothesis is that the sample mean is normally distributed with mean μ_1 and variance σ^2/n.

$$H_1 : \bar{X} \sim N(\mu_1, \sigma^2/n)$$

So β is the probability of rejecting H_0 when H_1 is true.

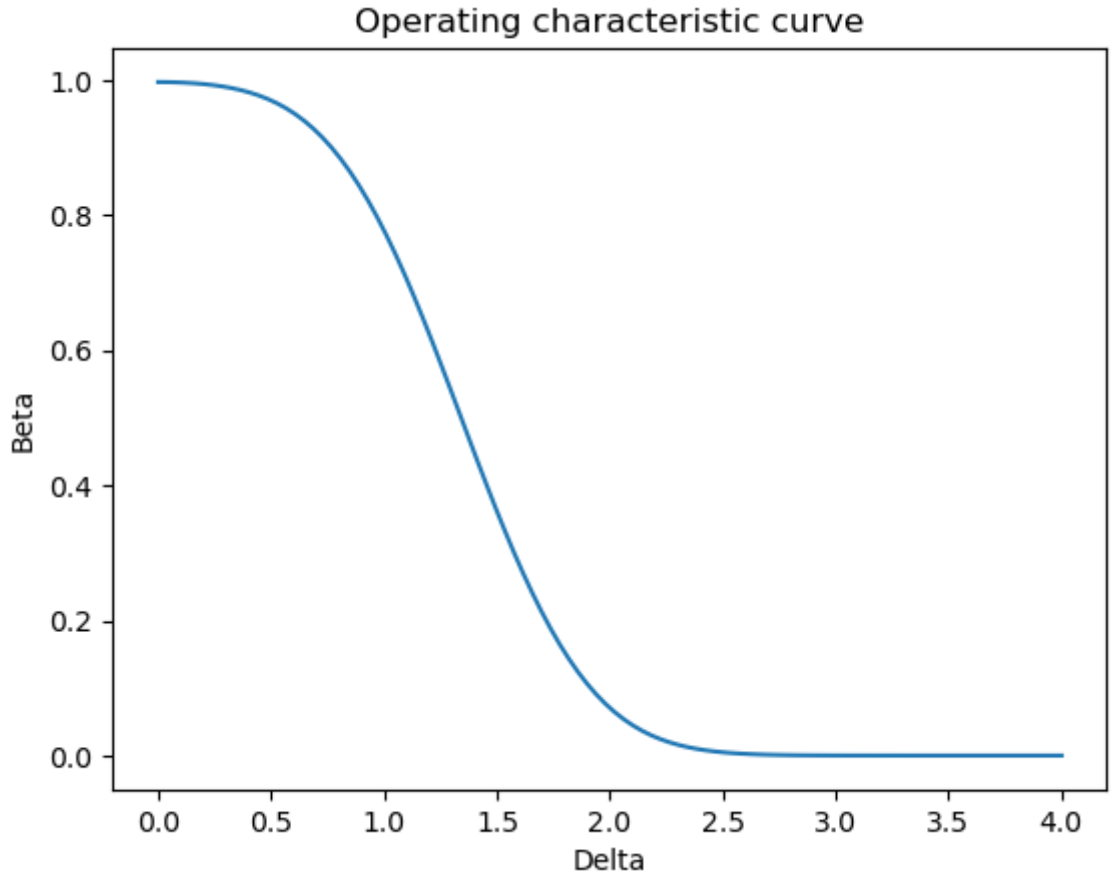
$$\beta = P(LCL \leq \bar{X} \leq UCL | H_1)$$

$$\beta = P(Z \leq \frac{UCL - \mu_1}{\sigma/\sqrt{n}}) - P(Z \leq \frac{LCL - \mu_1}{\sigma/\sqrt{n}})$$

If we define $\delta = (\mu_1 - \mu_0)/\sigma$, we can write:

$$\beta = P(Z \leq 3 - \delta\sqrt{n}) - P(Z \leq -3 - \delta\sqrt{n})$$

In [ ]:
```
# Define a range of values for beta
delta = np.linspace(0, 4, 100)
# Compute the corresponding beta values
beta = stats.norm.cdf(3 - delta*np.sqrt(n)) - stats.norm.cdf(-3 - delta*np.sqrt(

# Plot the beta values
plt.plot(delta, beta)
plt.xlabel('Delta')
plt.ylabel('Beta')
plt.title('Operating characteristic curve')
plt.show()
```



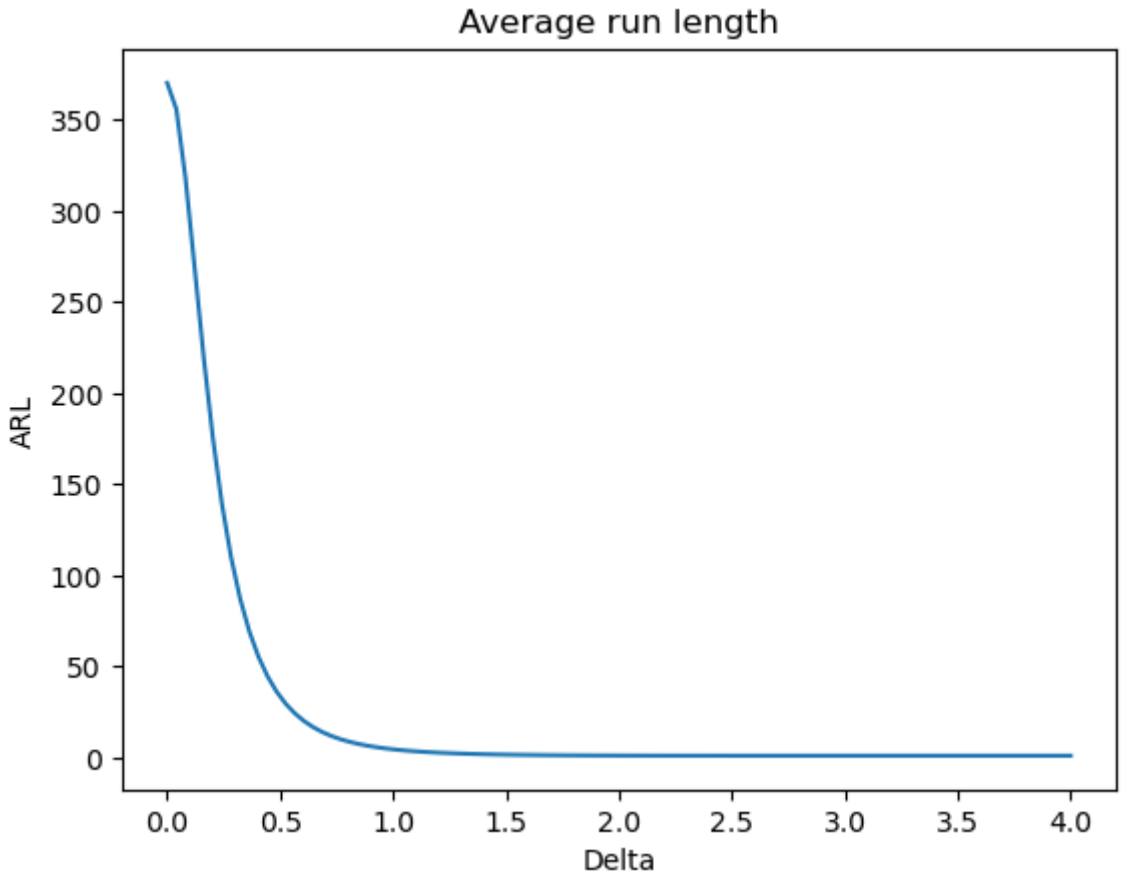
Point 3

Determine the corresponding ARL curve.

> ## Solution
>
> We know that the ARL curve is defined as:
>
> $$ARL = \frac{1}{1-\beta}$$

In [ ]:
```
# Compute ARL using the previous values of beta
ARL = 1/(1-beta)

# Plot the ARL values
plt.plot(delta, ARL)
```

```
plt.xlabel('Delta')
plt.ylabel('ARL')
plt.title('Average run length')
plt.show()
```



Point 4

Estimate the standard deviation through the statistic R.

> ## Solution
>
> The standard deviation is estimated through the statistic R as:
>
> $$\hat{\sigma} = \frac{\bar{R}}{d_2(n)}$$

> You can use the function `getd2` from `qda.constants` to get the value of $d_2(n)$.

In [ ]:
```
d2 = qda.constants.getd2(n)
sigma_hat = R_mean / d2
print('Sigma_hat = %.3f' % sigma_hat)
```

```
Sigma_hat = 0.135
```

Point 5

Design the confidence interval on the process mean that corresponds to the control limits computed in point 1.

> ## Solution
>
> The confidence interval corresponding to the control limits computed in point 1 uses:
>
> - $n = 5$
> - $\alpha = 0.002$
> - $\hat{\sigma} = 0.135$ (computed from the data)
> - $\overline{\overline{X}} = 10.008$ (computed from the data)
>
> Remember the formula of the confidence interval (assume that $\hat{\sigma}$ is the real population variance):
>
> $$\bar{X} - z_{\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}$$

> You can compute the CI using the formula or using the `interval` function from the `stats.norm` package.

In [ ]:
```
CI = stats.norm.interval(1-alpha, loc=Xbar_mean, scale=sigma_hat/np.sqrt(n))
print('CI = (%.3f, %.3f)' % CI)
```

```
CI = (9.822, 10.195)
```

The CI limits correspond to the LCL and UCL of the control chart. Indeed, the X-bar control chart can be interpreted as a recursive application of the confidence interval on the mean along time t, assuming that the estimated sample variance is the real (population) variance.