QDA 2023 03 30

ARIMA models – AR(p)

$$X_t = \xi + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + \varepsilon_t$$

$$\varepsilon_t \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$$

STATIONARITY (weak) .... moments up to 2nd order are constant with time

$\hookrightarrow E(X_t) = \mu_t = \mu$

$E(\xi) = \xi$ $\qquad E(\varepsilon_t) = 0$

$$\mu = E(X_t) = E(\xi + \phi_1 X_{t-1} + \dots + \varepsilon_t) = \xi + \underline{\phi_1 \mu + \dots + \phi_p \mu} + 0$$

$$\mu - \phi_1 \mu - \dots - \phi_p \mu = \xi \rightarrow \mu\left(1 - \sum_{i=1}^{p} \phi_i\right) = \xi$$

$$\Rightarrow \xi = \qquad \mu = \xi / [(1 - \sum \phi_i)]$$

$$\mu = 0 \rightarrow \xi = \underbrace{\mu\left(1 - \sum \phi_i\right)}_{} = 0$$

$$\tilde{X}_t = X_t - \mu \quad \xrightarrow{(*)} \quad X_t = \underbrace{\mu\left(1 - \sum \phi_i\right)}_{\xi} + \phi_1 X_{t-1} + \dots + \phi_p X_{t-p} + \varepsilon_t$$

$$\underbrace{X_t - \mu}_{\tilde{X}_t} = \phi_1 (X_{t-1} - \mu) + \phi_2 (X_{t-2} - \mu) \dots + \phi_p (X_{t-p} - \mu) + \varepsilon_t$$

$$\tilde{X}_t = \phi_1 \tilde{X}_{t-1} + \phi_2 \tilde{X}_{t-2} + \dots + \varepsilon_t$$

AR(p) B = backshift operator $\mathcal{B}$

$$B X_t = X_{t-1} \qquad \underbrace{B(B X_t)}_{B^2 X_t} = B(X_{t-1}) = X_{t-2}$$

$$\dots \quad B^k X_t = X_{t-k}$$

$$X_t = \xi + \phi_1 \underset{\uparrow \atop BX_t}{X_{t-1}} + \dots + \phi_p \underset{\uparrow \atop B^p X_t}{X_{t-p}} + \varepsilon_t$$

$$X_t - \phi_1 B X_t - \phi_2 B^2 X_t - \dots - \phi_p B^p X_t = \xi + \varepsilon_t$$

$$\underbrace{\left(1 - \sum_i \phi_i B^i\right)}_{1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p} X_t = \xi + \varepsilon_t$$

$$\underbrace{1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p}_{A(B)}$$

AR(p)

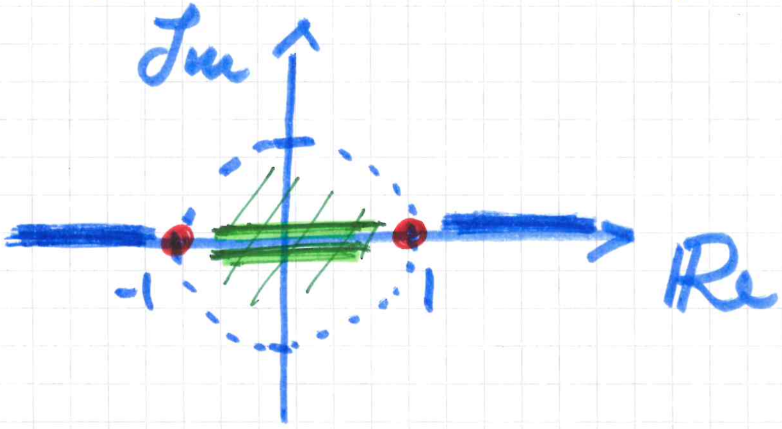
$$\boxed{A(B) X_t = \xi + \varepsilon_t}$$

$$A(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$$

$$A(\underset{\uparrow \atop B}{x}) = 1 - \phi_1 \underset{\uparrow \atop B}{x} - \phi_2 \underset{\uparrow \atop B^2}{x^2} - \dots - \phi_p \underset{\uparrow \atop B^p}{x^p}$$

$$A(B) = 1 - \phi_1 B$$

$$A(B) = 0 \quad 1 - \phi_1 B = 0 \Rightarrow B = 1/\phi_1$$



1) $|B| > 1$ AR(1) STABLE

$\Rightarrow |\phi_1| < 1$

2) $|B| = 1$ AR(1) UNSTABLE

$\Rightarrow |\phi_1| = 1$

(RANDOM WALK)

3) $|B| < 1$ ~~AR(1)~~ "WILDLY" UNSTABLE

$\Rightarrow \cancel{|\phi_1| > 1}$

AR(1) $\longrightarrow$ $X_t = \xi + \phi_1 X_{t-1} + \varepsilon_t$

$(*)\quad X_{t-1} = \xi + \phi_1 X_{t-2} + \varepsilon_{t-1}$

$(**)\quad X_{t-2} = \xi + \phi_1 X_{t-3} + \varepsilon_{t-2}$

$\vdots$

$$X_t = \xi + \phi_1 \left[\xi + \phi_1 X_{t-2} + \varepsilon_{t-1}\right] + \varepsilon_t \quad (*)$$

$$= \xi + \phi_1 \xi + \phi_1^2 X_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t$$

$$= \xi + \phi_1 \xi + \phi_1^2 \left[\xi + \phi_1 X_{t-3} + \varepsilon_{t-2}\right] + \phi_1 \varepsilon_{t-1} + \varepsilon_t \quad (**)$$

$$= \xi + \phi_1 \xi + \phi_1^2 \xi + \phi_1^3 X_{t-3} + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2}$$

$\vdots$

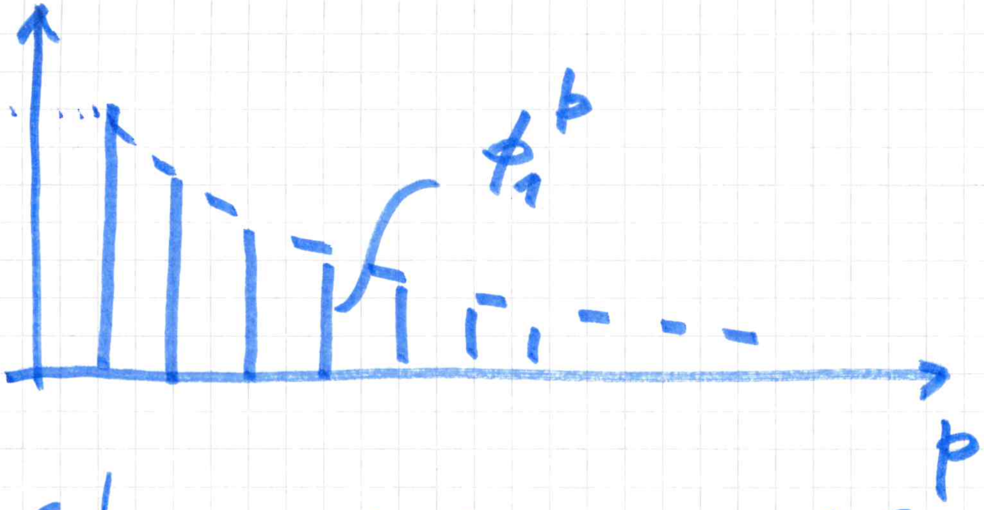
$$X_t = \underbrace{\xi \left(1 + \sum \phi_1^i\right)}_{\xi'} + \underbrace{\varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \dots + \phi_1^p \varepsilon_{t-p}}_{} + \phi_1^p X_t$$

WILDLY UNSTABLE $|\phi_1| > 1$

UNSTABLE $|\phi_1| = 1$

STABLE $|\phi_1| < 1$

$\phi_1 = 0.8$

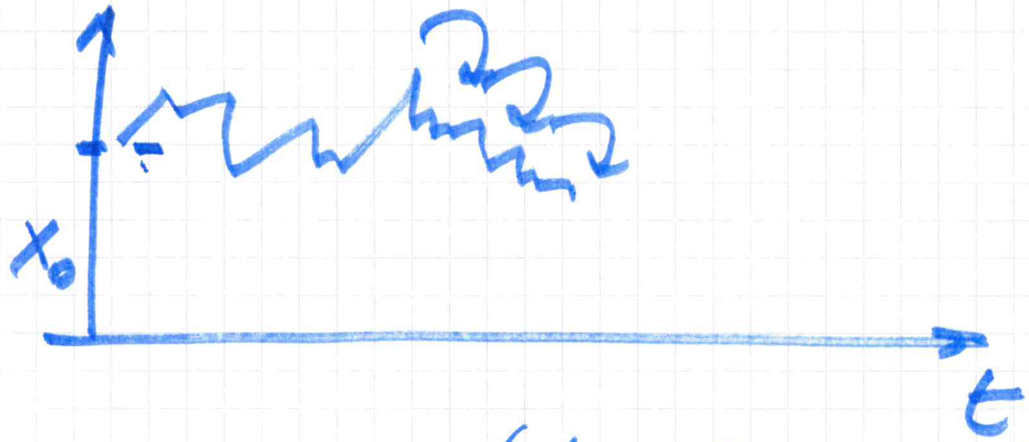


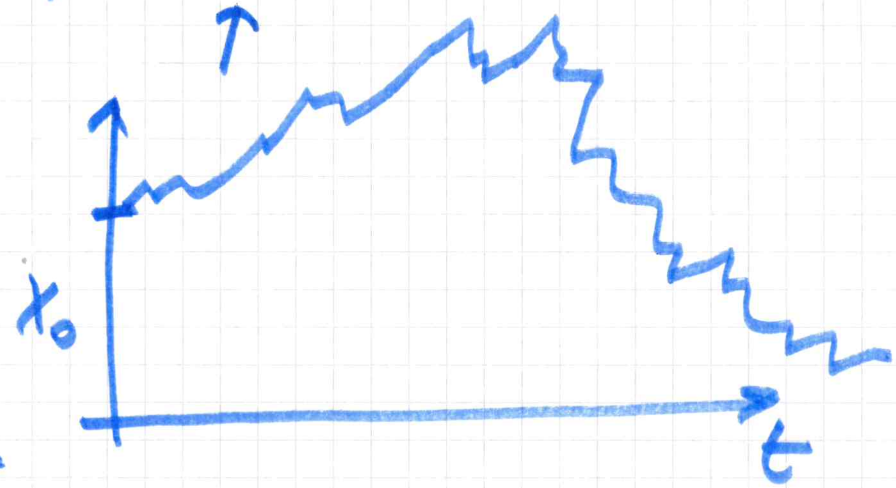
$\phi_1^{\,p}$

$|\phi_1| < 1$

STABLE

$$X_t = \xi' + \varepsilon_t + \phi_1 \varepsilon_{t-1} + \dots + \phi_p \varepsilon_{t-p} + \cup$$





RANDOM WALK (NON STATIONARY)

$$X_t = \xi + X_{t-1} + \varepsilon_t \qquad X_t - X_{t-1} = \xi + \varepsilon_t$$

$$\underbrace{X_t - BX_t}_{} = \xi + \varepsilon_t \qquad X_t \underbrace{(1-B)}_{\nabla X_t} = \xi + \varepsilon_t$$

∇X_t ← $(1-B)$

ARIMA(0,1,0)

p=0 (stable AR(p))

order of unstable part of AR(p)

q=0 (order of MA part)

Random walk $X_t = \xi + X_{t-1} + \varepsilon_t$

AR(1) process unstable

stable ↓

unstable ↓

- Roots $A(B)$ allowed us to check STABILITY
- How can one IDENTIFY that a time series is an $\underline{\underline{AR(p)}}$ process
  - STRUCTURE (AR vs MA vs I)
  - order p

[looking to the AUTOCORRELATION FUNCTION & PARTIAL AUTOCORRELATION FUNCTION]

→ MOMENTS of $AR(p)$
- $E(X_t) = \mu$
- $E[(X_t-\mu)(X_{t-k}-\mu)] = Cov(X_t, X_{t-k})$

HW Try to show that:

$$\gamma_k = Cov(X_t, X_{t-k}) = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \dots + \phi_p \gamma_{k-p}$$

↑STATIONARY $\mu_t = \mu$ (otherwise $\gamma_{k,t}$)

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \dots + \phi_p \rho_{k-p}$$

ex $AR(1) \Rightarrow p = 1$ $\qquad \rho_k = \phi_1 \rho_{k-1}$

$\boxed{\text{STABLE } |\phi_1| < 1}$

$$\rho_1 = \phi_1 \cdot \rho_0 = \phi_1 \cdot \frac{\gamma_0}{\gamma_0} = \phi_1$$

$\rho_0 = 1$

$$\rho_2 \stackrel{\Delta}{=} \phi_1 \rho_1 = \phi_1^2$$

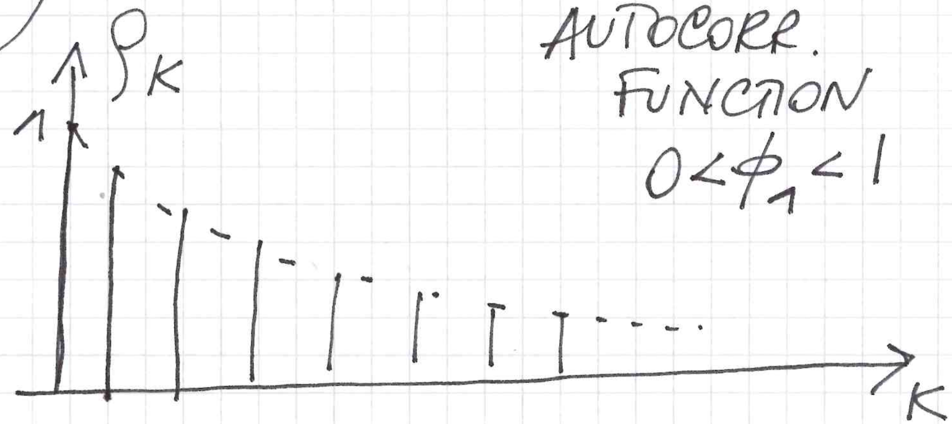
$$\rho_3 \stackrel{\Delta}{=} \phi_1 \rho_2 = \phi_1^3$$

$\vdots$

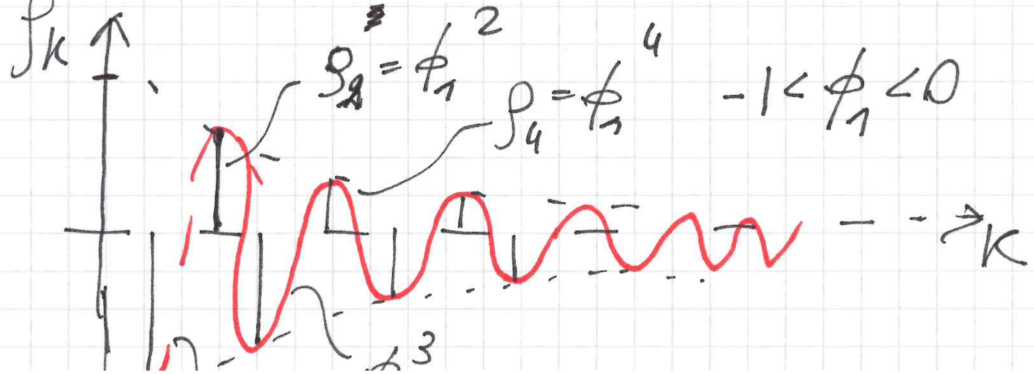
$$\rho_k = \phi_1^k$$

AUTOCORR. FUNCTION

$0 < \phi_1 < 1$



$-1 < \phi_1 < 0$



We can "detect" or "suspect" a (STABLE) AR(p) process when the (sample) autocorrelation function has an exponential decay or

$\hat{\rho}_k = r_k$

sinusoidal path with exponential decay in module

$\Longrightarrow$ order "p" will be clear from the (sample) PARTIAL autocorrel-function.