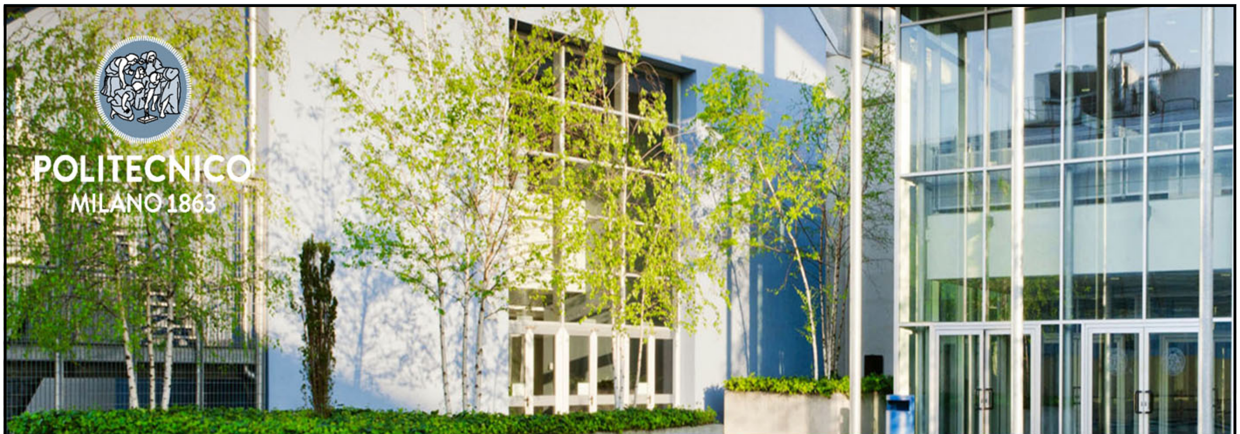

Quality Data Analysis

Control charts for variables - part 1

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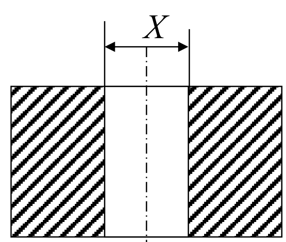
Reference:

Montgomery – Introduction to Statistical Quality Control

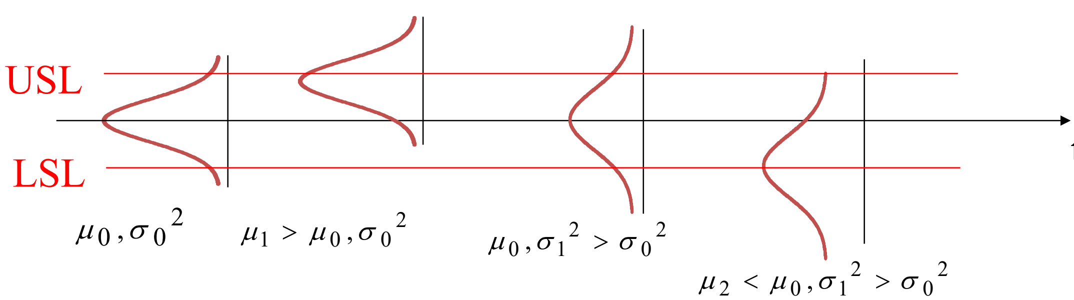
1

Control charts for variables

Quality characteristic: $X \overset{iid}{\sim} N(\mu_0, \sigma_0^2)$



Changes of process mean / dispersion may increase the expected value of non-conformings items



2

Control charts for variables

μ	σ	Chart (n>1)	Chart (n=1)
we use, as variable V, respectively:			
$\bar{X} = \frac{1}{n} \sum_{j=1,\dots,n} X_j$	$R = \max_j X_j - \min_j X_j$	$\boxed{\bar{X} - R}$	$I - MR$ $(X - MR)$
	$S = \sqrt{\sum_{j=1,\dots,n} (X_j - \bar{X})^2 \Big/ (n-1)}$,	$\bar{X} - S$	

Remarks:
- $\bar{X} - R$ chart: easy to compute, with similar performances to $\bar{X} - S$ chart if $n \leq 6$ and n is constant
- For individuals: I-MR chart

Xbar-R Control charts

Shewhart's scheme

$$\text{UCL} = \mu_V + K\sigma_V$$
$$\text{CL} = \mu_V$$
$$\text{LCL} = \mu_V - K\sigma_V$$

1. Control chart for process mean: *$\bar{X}$ Control Chart*

$$\boxed{V = \frac{1}{n} \sum_{j=1,\dots,n} X_j = \bar{X}}$$

$$X_j \overset{\text{iid}}{\sim} N(\mu, \sigma^2) \Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\mu_V = \mu \qquad \sigma_V = \frac{\sigma}{\sqrt{n}}$$

K=3

$$\text{UCL} = \mu + K\frac{\sigma}{\sqrt{n}} = \mu + A(n)\sigma$$
$$\text{CL} = \mu$$
$$\text{LCL} = \mu - K\frac{\sigma}{\sqrt{n}} = \mu - A(n)\sigma$$

Xbar-R Control chart

2. Control chart to detect a change of process dispersion: Carta *R*

$$V = R = \max_j x_j - \min_j x_j$$

$X_j \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2) \Rightarrow$ W relative range

$$W = \frac{R}{\sigma} \qquad E(W) = d_2(n) = \frac{E(R)}{\sigma} \Rightarrow \mu_R = d_2(n)\sigma$$

$$Var(W) = [d_3(n)]^2 = \frac{Var(R)}{\sigma^2} \Rightarrow \sigma_R = d_3(n)\sigma$$

$$\mathrm{UCL} = \mu_R + \mathrm{K}\sigma_R \overset{\mathrm{K}=3}{=} d_2(n)\sigma + 3d_3(n)\sigma = D_2(n)\sigma$$
$$\mathrm{CL} = \mu_R = d_2(n)\sigma$$
$$\mathrm{LCL} = \mu_R - \mathrm{K}\sigma_R = d_2(n)\sigma - 3d_3(n)\sigma = D_1(n)\sigma$$

Xbar-R Control chart

If process mean and variance are unknown- **Phase 1 of control charting:**

1. Pick up *m* (*m*=20-25) samples of size *n* from the process under stable (*a regime*) conditions;
2. Estimate unknown parameters and design the control chart;
3. Plot the control chart (***retrospective*** usage or ***Phase 1*** or ***design phase***) – control limits vs. data used to estimate those limits;
4. If an out-of-control is signalled, look for assignable causes:
   a) If assignable cause is found, remove the observation and go back to step 2
   b) If no assignable cause is found (Alwan): if observation is far beyond the limits it will strongly influence the limit computation itself (and assumption checking) – it may be cautious to remove the observation anyway
5. Assumption checking (why now, after step 4a?)

Xbar-R Control chart

Parameter estimation:

$$\boxed{\hat{\mu} = \overline{\overline{x}} = \frac{\sum_{i=1,\dots,m} \overline{x}_i}{\text{m}}} \qquad \hat{\sigma}\ ?$$

$$X_j \overset{\text{iid}}{\sim} \mathrm{N}(\mu, \sigma^2) \Rightarrow W = \frac{R}{\sigma} \quad \text{relative range}$$

$$E(W) = d_2(n) = \frac{E(R)}{\sigma} \Rightarrow \sigma = \frac{E(R)}{d_2(n)} \Rightarrow \boxed{\hat{\sigma} = \frac{\overline{R}}{d_2(n)}}$$

Xbar Chart

$$\mathrm{UCL} = \hat{\mu} + \underset{\text{K=3}}{\mathrm{K}}\frac{\hat{\sigma}}{\sqrt{n}} = \overline{\overline{x}} + 3\frac{1}{d_2\sqrt{n}}\overline{R} = \overline{\overline{x}} + A_2(n)\overline{R}$$

$$\mathrm{CL} = \hat{\mu} = \overline{\overline{x}}$$

$$\mathrm{LCL} = \hat{\mu} - \mathrm{K}\frac{\hat{\sigma}}{\sqrt{n}} = \overline{\overline{x}} - 3\frac{1}{d_2\sqrt{n}}\overline{R} = \overline{\overline{x}} - A_2(n)\overline{R}$$

Xbar-R Control charts

R Chart

K=3

$$\boxed{\hat{\sigma} = \frac{\overline{R}}{d_2(n)}}$$

$$\mathrm{UCL} = d_2(n)\hat{\sigma} + 3d_3(n)\hat{\sigma} = \overline{R} + 3\frac{d_3(n)}{d_2(n)}\overline{R} = D_4(n)\overline{R}$$

$$\mathrm{CL} = d_2(n)\hat{\sigma} = \overline{R}$$

$$\mathrm{LCL} = d_2(n)\hat{\sigma} - 3d_3(n)\hat{\sigma} = \overline{R} - 3\frac{d_3(n)}{d_2(n)}\overline{R} = D_3(n)\overline{R}$$

Note 1:

Sequential design of the two charts is advocated (*R before and then X*)

Note 2: $D_4(2) = 1 + 3/2\sqrt{2\pi - 4} \cong 3.266$ → Minitab: 3.267; Montgomery: 3.269

$D_3(2) = \max(0, 1 - 3/2\sqrt{2\pi - 4}) = 0$

Appendice A.6 Factors to design control charts for variables

Campione	Carta $\bar{x}$			Carta S						Carta R						
	Fattori per i limiti			Fattori per il centro		Fattori per i limiti				Fattori per il centro		Fattori per i limiti				
n	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.881	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.687	0	3.269
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.357	0	2.574
4	1.5	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.88	0	4.699	0	2.282
5	1.342	0.577	1.427	0.94	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0509	0.03	1.97	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0424	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.205	5.203	0.076	1.924
8	1.061	0.373	1.099	0.965	1.0362	0.185	1.815	0.179	1.751	2.847	0.3512	0.82	0.387	5.307	0.136	1.864
9	1	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.97	0.3367	0.808	0.546	5.394	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0253	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.812	5.534	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.023	0.354	1.646	0.346	1.61	3.258	0.3069	0.778	0.924	5.592	0.284	1.716
13	0.832	0.249	0.85	0.9794	1.021	0.382	1.618	0.374	1.585	3.336	0.2998	0.77	1.026	5.646	0.308	1.692
14	0.802	0.235	0.817	0.981	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.762	1.121	5.693	0.329	1.671
15	0.775	0.223	0.789	0.9823	1.018	0.428	1.572	0.421	1.544	3.472	0.288	0.755	1.207	5.737	0.348	1.652
16	0.75	0.212	0.763	0.9835	1.0168	0.448	1.552	0.44	1.526	3.532	0.2831	0.749	1.285	5.779	0.364	1.636
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.743	1.359	5.817	0.379	1.621
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.64	0.2747	0.738	1.426	5.854	0.392	1.608
19	0.688	0.187	0.698	0.9862	1.014	0.497	1.503	0.49	1.483	3.689	0.2711	0.733	1.49	5.888	0.404	1.596
20	0.671	0.18	0.68	0.9869	1.0132	0.51	1.49	0.504	1.47	3.735	0.2677	0.729	1.548	5.922	0.414	1.586
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.606	5.95	0.425	1.575
22	0.64	0.167	0.647	0.9882	1.012	0.534	1.466	0.528	1.448	3.819	0.2618	0.72	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.71	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.452	1.548
25	0.6	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.42	3.931	0.2544	0.709	1.804	6.058	0.459	1.541

Per $n \geq 25$: $A = \frac{3}{\sqrt{n}}$, $A_3 = \frac{3}{c_4\sqrt{n}}$, $c_4 = \frac{4(n-1)}{4n-3}$, $B_3 = 1 - \frac{3}{c_4\sqrt{2(n-1)}}$, $B_4 = 1 + \frac{3}{c_4\sqrt{2(n-1)}}$, $B_5 = c_4 - \frac{3}{\sqrt{2(n-1)}}$, $B_6 = c_4 + \frac{3}{\sqrt{2(n-1)}}$.

Design (Phase 1):

If out-of-control with assignable cause -> remove -> re-design ->…

More than 1 or 2 iteractions are uncommon (pay attention to the assumptions)

Use (Phase 2):

How long will control limits be applicable? How often is a revision required?

- Change of sample size
- Change of process conditions
- Periodic revision is often foreseen (e.g., every week, month, 100 samples or new production / product)

Xbar-R control chart if:

- n small (<=10)
- n constant

Recall: estimate of σ

$$\boxed{\hat{\sigma} = \frac{\overline{R}}{d_2(n)}}$$

Example

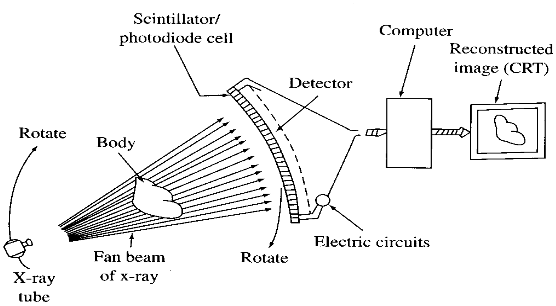
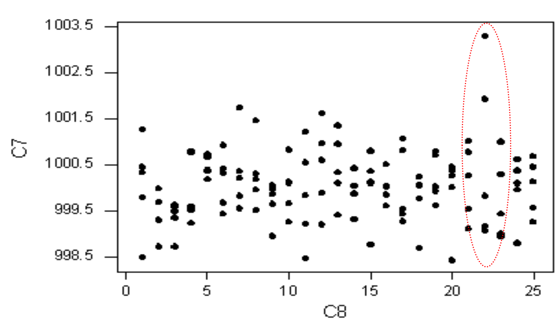


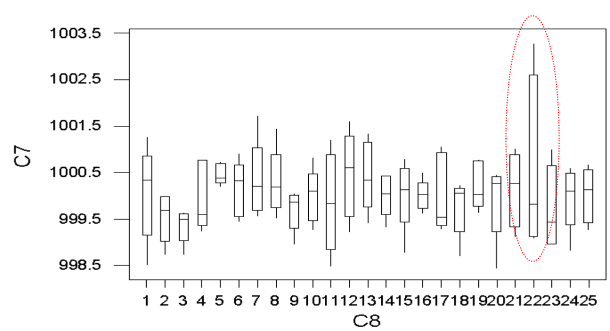
Figure 6.1 Schematic diagram of rotating CT scanner.

(x.ray.dat)

Observations					$\bar{x}_i$	R_i
998.505	1000.327	999.793	1000.446	1001.268	1000.068	2.763
999.986	999.983	999.307	998.721	999.694	999.538	1.265
998.720	999.351	999.490	999.636	999.583	999.356	0.916
1000.755	999.601	999.229	1000.782	999.513	999.976	1.553
1000.381	1000.730	1000.179	1000.363	1000.661	1000.463	0.551
999.433	1000.313	999.676	1000.921	1000.410	1000.151	1.488
999.550	999.818	1000.215	1001.734	1000.353	1000.334	2.184
1001.459	1000.181	1000.310	999.958	999.517	1000.285	1.942
999.859	999.653	998.955	999.975	1000.050	999.698	1.095
1000.831	999.259	1000.107	999.667	1000.131	999.999	1.572
1000.540	1001.213	999.839	998.468	999.215	999.855	2.745
999.212	1000.594	1001.613	1000.957	999.893	1000.454	2.401
1000.337	1000.103	999.402	1001.348	1000.945	1000.427	1.946
999.872	1000.429	1000.409	1000.034	999.309	1000.011	1.120
998.773	1000.128	1000.794	1000.363	1000.103	1000.032	2.021
1000.043	999.615	999.856	1000.018	1000.509	1000.008	0.894
999.538	999.278	1000.810	1001.066	999.439	1000.026	1.788
1000.243	998.696	1000.048	1000.070	999.768	999.765	1.547
999.929	1000.708	1000.024	999.619	1000.787	1000.213	1.168
998.431	1000.260	1000.007	1000.376	1000.448	999.904	2.017
999.112	1000.767	1001.017	1000.262	999.541	1000.140	1.905
1003.290	1001.916	999.170	999.823	999.080	1000.656	4.210
1000.290	1001.004	999.008	999.435	998.937	999.735	2.067
1000.370	998.810	1000.610	999.962	1000.102	999.971	1.800
1000.673	1000.136	999.571	999.263	1000.445	1000.018	1.410

Exploratory analysis



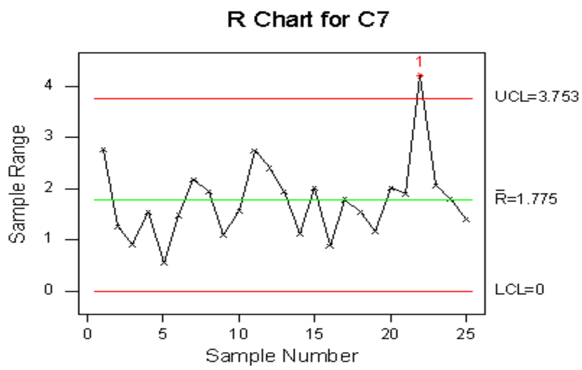


Temporal patterns (for individuals) is not taken into account because the sample is obtained with randomization

R Chart

$$\bar{R} = \frac{\sum_{i=1}^{25} R_i}{25} = 1.775$$

$$UCL = D_4\bar{R} = 2.114(1.775) = 3.752$$
$$LCL = D_3\bar{R} = 0(1.775) = 0$$



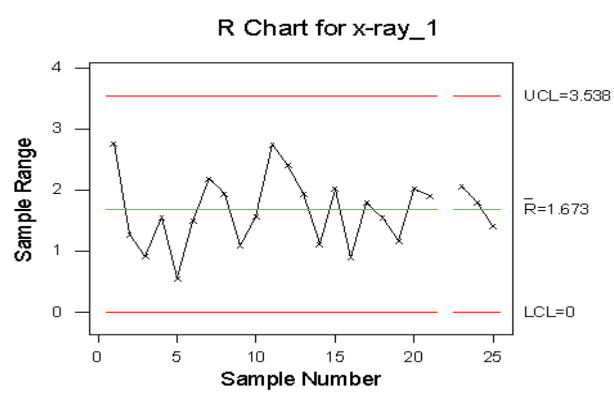
Assignable cause identified:

Remove the 22[nd] sample and re-design the chart

$$\bar{R} = \frac{40.158}{24} = 1.673$$

$\downarrow$

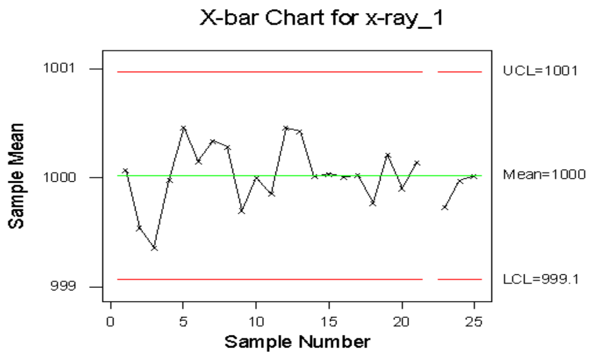
$$UCL = D_4\bar{R} = 2.114(1.673) = 3.537$$
$$LCL = D_3\bar{R} = 0(1.673) = 0$$

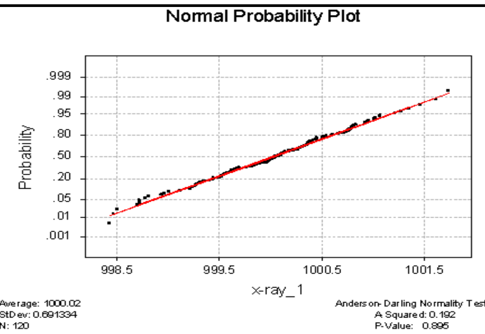


Design of R chart is over: now, design the chart for process mean

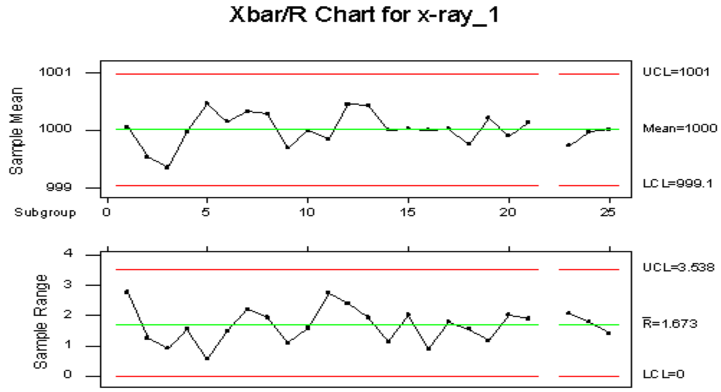
$$\bar{x} = \frac{\sum_{i \neq 22} \bar{x}_i}{24} = \frac{24{,}000.378}{24} = 1000.016$$

$$UCL = \bar{\bar{x}} + A_2\bar{R} = 1000.016 + 0.577(1.673) = 1000.981$$
$$LCL = \bar{\bar{x}} - A_2\bar{R} = 1000.016 - 0.577(1.673) = 999.051$$





Eventually:



$$\hat{\sigma} = \frac{\overline{R}}{d_2(n)} = \frac{1.673}{2.326} = 0.7193$$

Quality Data Analysis

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Operating characteristic (OC) curve of the *Xbar* chart

$$H_0 : \overline{X} \sim \mathrm{N}(\mu_0, \sigma^2 / n)$$
$$H_1 : \overline{X} \sim \mathrm{N}(\mu_1, \sigma^2 / n)$$

$$\beta = \Pr(LCL \le \overline{X} \le UCL \mid H_1)$$
$$= \Pr\left(Z \le \frac{UCL - \mu_1}{\sigma / \sqrt{n}}\right) - \Pr\left(Z \le \frac{LCL - \mu_1}{\sigma / \sqrt{n}}\right) =$$
$$= \Pr\left(Z \le 3 - \delta\sqrt{n}\right) - \Pr\left(Z \le -3 - \delta\sqrt{n}\right)$$

where $\delta = \dfrac{\mu_1 - \mu_0}{\sigma}$

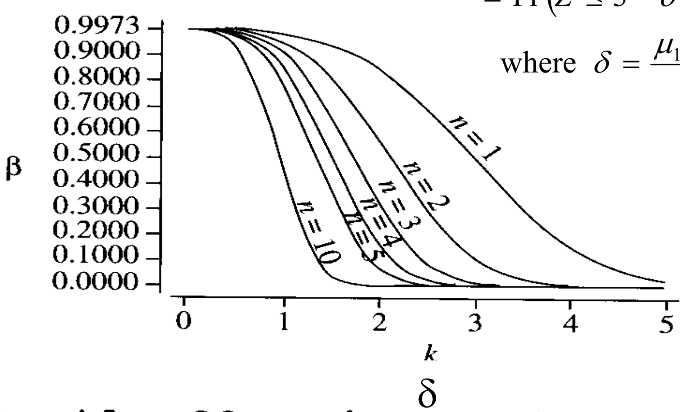

Figure 6.5 OC curves for $\bar{x}$ chart with 3σ limits.

Quality Data Analysis

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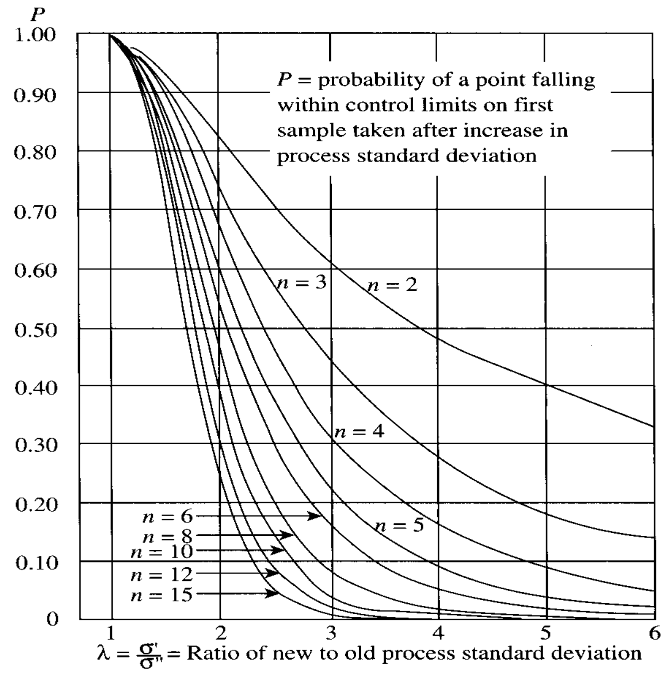
Operating characteristic (OC) curve of the *R* chart

Duncan (1951)
Scheffè (1949)



Average Run Length – Average Time to Signal

$$ARL_0 = \frac{1}{\alpha} \qquad ARL_1 = \frac{1}{1-\beta} \qquad ATS = hARL$$

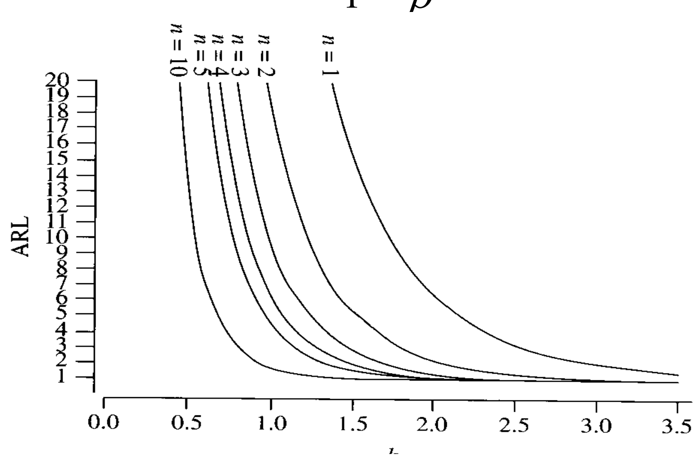


Figure 6.7 ARL curves for $\bar{x}$ chart wih 3σ limits.

Average Run Length – Average Time to Signal

Example:

Data from in-control process with $\mu = 100$ $\sigma = 5$

Quality assurance manager says that a mean shift to 107.5 (1.5σ, δ=1.5) is not accepted. Process monitored via control chart with n=4 (one sample per hour)

$$\beta = P(Z \leq 3 - 1.5\sqrt{4}) - P(Z \leq -3 - 1.5\sqrt{4})$$
$$= P(Z \leq 0) - P(Z \leq -5)$$

0.5 ≅0.000

$$\text{ARL}_1 = \frac{1}{1 - 0.50} = 2$$

$$\text{ATS} = 1(2) = 2 \text{ h}$$

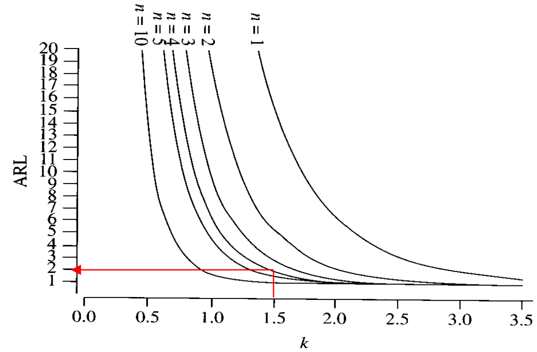


Figure 6.7 ARL curves for $\bar{x}$ chart wih 3σ limits.

Quality Data Analysis

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Average Run Length – Average Time to Signal

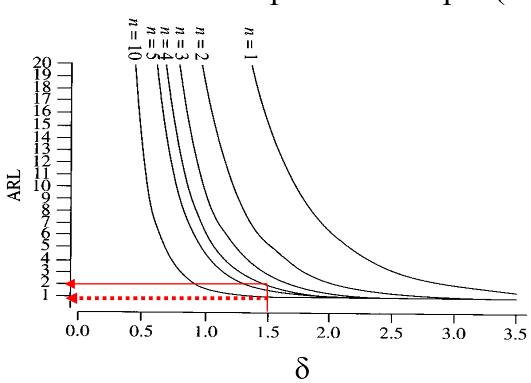
Resulting ATS is deemed too large:

How can we reduce the ATS?

- Alternative control scheme (for small shifts: CUSUM-EWMA)
- More frequent samples

$$\text{ATS} = (\tfrac{1}{2})2 = 1 \text{ h.}$$

Example: one sample (n=4) per ½ hour



$$n = 10 \longrightarrow \beta = 0.0406$$

$$\text{ARL}_1 = \frac{1}{1 - 0.0406} = 1.04$$

ATS=1.04 h

Figure 6.7 ARL curves for $\bar{x}$ chart wih 3σ limits.

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μ	σ	Chart (n>1)	Chart (n=1)
IF, in order to keep under control: we use, as variable V, respectively:			
$\overline{X} = \frac{1}{n} \sum_{j=1,\dots,n} X_j$	$R = \max_j X_j - \min_j X_j$	$\overline{X} - R$	$\boxed{I - MR \; (X - MR)}$
	$S = \sqrt{\sum_{j=1,\dots,n} (X_j - \overline{X})^2 \big/ (n-1)}\,,$	$\overline{X} - S$	

Remarks:

- $\overline{X} - R$ chart: easy to compute, with similar performances to $\overline{X} - S$ chart if $n \leq 6$ and n is constant
- For individuals: I-MR chart

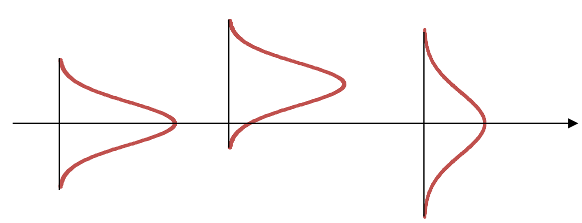
Control charts for individuals (*X* or *I control chart*)

Process with low throughput

Chemical processes

Overall company performance indicators: turnover, amount of provisions, customer satisfaction, …

$$X \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2) \quad (x_1, x_2, \dots, x_n)$$



X Control chart ($V = X$)

With known parameters

$$\mathrm{UCL} = \mu_V + \mathrm{K}\sigma_V$$
$$\mathrm{CL} = \mu_V$$
$$\mathrm{LCL} = \mu_V - \mathrm{K}\sigma_V$$

With unknown parameters

$$\text{UCL} = \mu + \text{K}\sigma$$
$$\text{CL} = \mu$$
$$\text{LCL} = \mu - \text{K}\sigma$$

$$\hat{\mu} = \bar{x} \qquad \hat{\sigma} = \overline{MR} / d_2(2)$$

2. Parameter estimation (collect n observations from the process)

with $MR_i = |x_i - x_{i-1}|$ i $= 2,\ldots, n$ $\qquad \overline{MR} = \frac{1}{n-1}\sum_{i=2,\ldots,n} MR_i$

$d_2(2) = 1.128$

I Control Chart: $\bar{x} \pm 3\left(\frac{\overline{MR}}{d_2}\right) = \bar{x} \pm 2.66\,\overline{MR}$

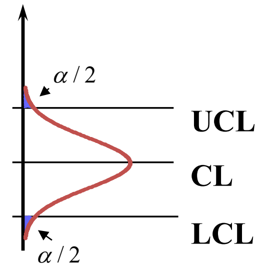
MR Control chart

$$\text{UCL} = D_4(n)\bar{R}$$
$$\text{CL} = \bar{R}$$
$$\text{LCL} = D_3(n)\bar{R}$$

$$UCL = D_4(2)\ \overline{MR} = (3.267)\,\overline{MR}$$
$$LCL = D_3(2)\ \overline{MR} = (0)\,\overline{MR} = 0$$

- Remarks:
  - MR_i are autocorrelated ($\rho_1 = 0.22$): pay attention to run-rules
  - MR_i are not normally distributed (asymmetric and non-negative distribution):
    - LCL=0
    - $\alpha=0.00915$ vs. 0.0027

$$\Pr\left(LCL \le V \le UCL\right) = 1-\alpha \quad \text{by using the distribution of } V$$
$$\Pr\left(V \le LCL\right) = \alpha/2 \qquad \Pr\left(V > UCL\right) = \alpha/2$$



$$X \sim N(\mu, \sigma^2)$$

I Control chart

$$UCL = \hat{\mu} + K\hat{\sigma} = \bar{x} + z_{\alpha/2}\frac{\overline{MR}}{d_2} \qquad LCL = \hat{\mu} - K\hat{\sigma} = \bar{x} - z_{\alpha/2}\frac{\overline{MR}}{d_2}$$

where $d_2 = d_2(2) = 1.128$

R Control Chart

$$UCL = D_{1-\alpha/2}\frac{\overline{MR}}{d_2} \qquad LCL = D_{\alpha/2}\frac{\overline{MR}}{d_2}$$

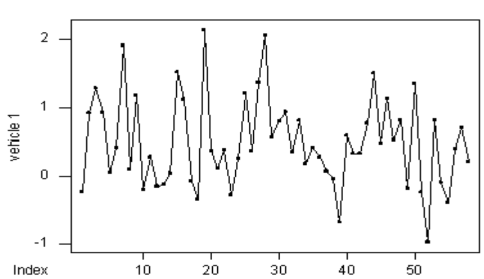
For n = 2

$$D_{1-\alpha/2} = \sqrt{2}\, z_{\alpha/4} \qquad D_{\alpha/2} = \sqrt{2}\, z_{1/2-\alpha/4}$$

Example

Data from a vehicle assembly process: the main vehicle body (not yet painted) passes through an optical control station equipped with 48 laser sensors. 95 dimensional measurements are acquired. In particular, the data in the table refer to the shift (in mm) from the nominal position (along y direction) of a hole for component coupling (very important for vehicle stability).

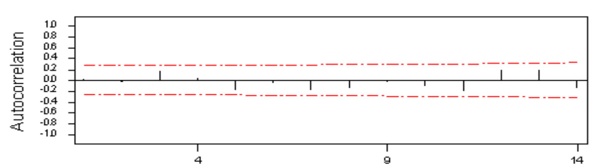
(vehicle1.dat)



Runs Test: vehicle 1

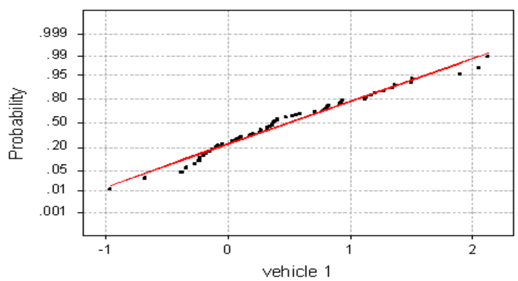
```
K =      0.4886
The observed number of runs =  29
The expected number of runs =
29.1379
24 Observations above K   34 below
The test is significant at  0.9699
Cannot reject at alpha = 0.05
```

Autocorrelation Function for vehicle 1



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.02	0.16	0.03	8	-0.16	-1.10	8.60
2	-0.03	-0.23	0.09	9	-0.04	-0.25	8.69
3	0.16	1.25	1.78	10	-0.10	-0.69	9.45
4	0.04	0.31	1.90	11	-0.20	-1.34	12.39
5	-0.18	-1.34	4.07	12	0.19	1.24	15.11
6	-0.06	-0.44	4.31	13	0.19	1.23	17.99
7	-0.19	-1.38	6.86	14	-0.15	-0.91	19.70

Normal Probability Plot



Average: 0.488643
StDev: 0.661090
N: 58

Anderson-Darling Normality Test
A-Squared: 0.524
P-Value: 0.175

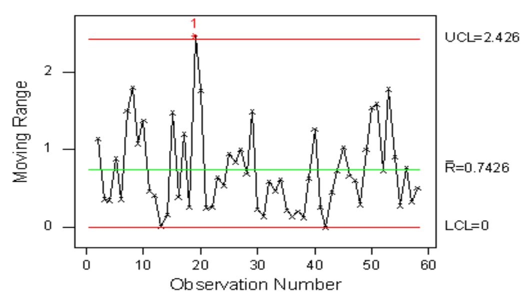
vehicle 1	MR
-0.2302	
0.9196	1.1498
1.2809	0.3613
0.9288	0.3521
0.043	0.8858
0.4074	0.3644
1.9086	1.5012
0.1009	1.8077
1.1807	1.0798
-0.1985	1.3792
0.2714	0.4699
-0.1458	0.4172
-0.1269	0.0189
0.035	0.1619
1.5164	1.4814
1.1236	0.3928
-0.0814	1.205
-0.3408	0.2594
2.1323	2.4731
0.3683	1.764
0.112	0.2563
0.3759	0.2639
-0.2727	0.6486
0.261	0.5337
1.2097	0.9487
0.3638	0.8459
1.3688	1.005
…	…

$$UCL = D_4 \overline{MR} = (3.267)\, 0.7426 = 2.426$$

$$LCL = D_3 \overline{MR} = 0$$

One out-of-control datum:
Assignable cause not found

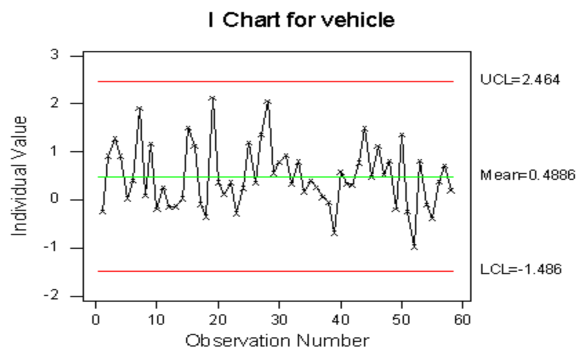
Moving Range Chart for vehicle



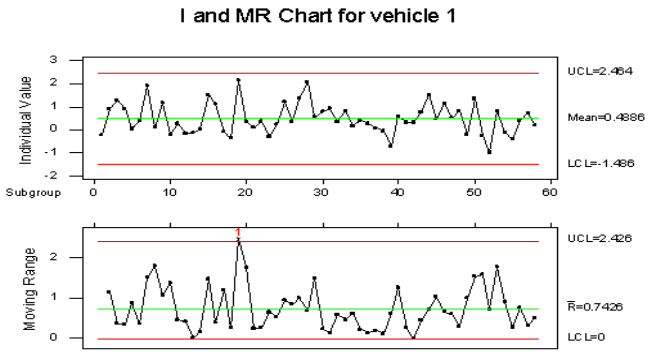
Proceed with *I chart:*

Variable	N	N*	Mean	Median	StDev
vehicle	58	0	0.4886	0.3721	0.4666
MR	57	1	0.7426	0.6258	0.7084

$$\begin{matrix} UCL \\ LCL \end{matrix} = \bar{x} \pm 3\left(\frac{\overline{MR}}{d_2}\right) = 0.4886 \pm 3\left(\frac{0.7426}{1.128}\right) = \begin{matrix} 2.462 \\ -1.486 \end{matrix}$$



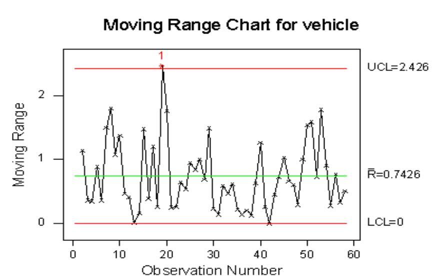
By using the run rules, no out-of-control detected too



The design phase is over:
The control limits will be used for future observations

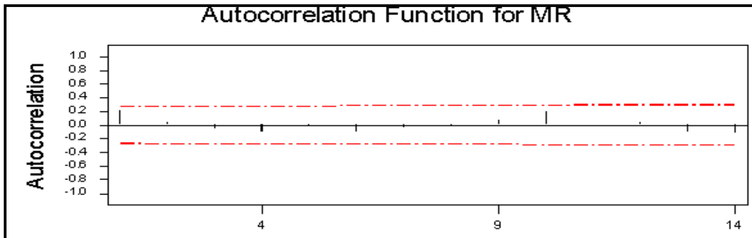
- Remarks:
  - –MR_i are autocorrelated (r_1 =0.22): pay attention to run-rules
  - –MR_i are not normally distributed

    (asymmetric and non-negative distribution):
    - –LCL=0
    - – α=0.00915 vs. 0.0027



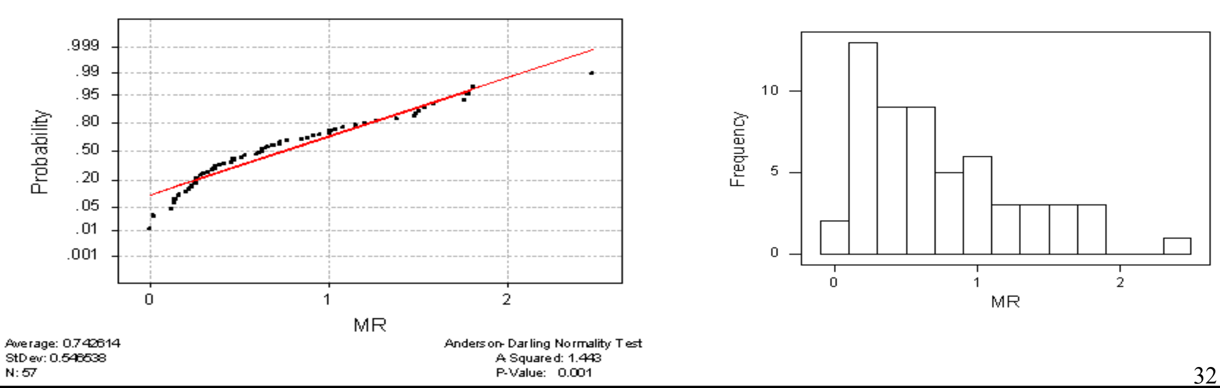
(vehicle 1.dat)

Can the out-of-control obs. be related to problems affecting the use of these limits (as no assignable cause was found)?



Lag	Corr	T	LBQ	Lag	Corr	T	LBQ
1	0.21	1.61	2.74	8	-0.01	-0.11	4.53
2	0.05	0.33	2.86	9	0.06	0.45	4.81
3	-0.05	-0.37	3.02	10	0.20	1.44	7.81

Normal Probability Plot



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μ	σ	Chart (n>1)	Chart (n=1)
IF, in order to keep under control: we use, as variable V, respectively:			
$\bar{X} = \frac{1}{n} \sum_{j=1,\ldots,n} X_j$	$R = \max_j X_j - \min_j X_j$	$\bar{X} - R$	$I - MR$ $(X - MR)$
	$S = \sqrt{\sum_{j=1,\ldots,n} (X_j - \bar{X})^2 / (n-1)}$,	$\boxed{\bar{X} - S}$	

Remarks:

- $\bar{X} - R$ chart: easy to compute, with similar performances to $\bar{X} - S$ chart if $n \leq 6$ and n is constant
- For individuals: I-MR chart

Quality Data Analysis

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Xbar-S Control charts

μ_S ? σ_S ?

With unknown parameters (with m samples of size n)

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad Y = \sqrt{\frac{(n-1)S^2}{\sigma^2}} = \frac{S\sqrt{(n-1)}}{\sigma}$$

$$E(Y) = \sqrt{2}\frac{\Gamma(n/2)}{\Gamma[(n-1)/2]} \Rightarrow E(S) = \sqrt{\frac{2}{n-1}}\frac{\Gamma(n/2)}{\Gamma[(n-1)/2]}\sigma = c_4(n)\sigma$$

$$\boxed{\mu_S = c_4(n)\sigma} \qquad \boxed{\hat{\sigma} = \frac{\bar{S}}{c_4(n)}}$$

Being $E(S^2) = \sigma^2$

$$\Rightarrow \text{Var}(S) = E(S^2) - [E(S)]^2 = \sigma^2 - (c_4(n)\sigma)^2 = [1 - c_4(n)^2]\sigma^2$$

$$\boxed{\sigma_S = \sigma\sqrt{1 - c_4(n)^2}}$$

Xbar-S Control charts

Xbar chart (in *Xbar*-S)

K=3

$$\text{UCL} = \hat{\mu} + \text{K}\frac{\hat{\sigma}}{\sqrt{n}} = \bar{\bar{x}} + 3\frac{1}{c_4\sqrt{n}}\bar{s} = \bar{\bar{x}} + A_3(n)\bar{s}$$

$$\text{CL} = \hat{\mu} = \bar{\bar{x}}$$

$$\text{LCL} = \hat{\mu} - \text{K}\frac{\hat{\sigma}}{\sqrt{n}} = \bar{\bar{x}} - 3\frac{1}{c_4\sqrt{n}}\bar{s} = \bar{\bar{x}} - A_3(n)\bar{s}$$

Analogously: *S chart*

	parameters	
	known	*unknown*
UCL	$B_6(n)\sigma$	$B_4(n)\bar{s}$
CL	$c_4(n)\sigma$	$\bar{s}$
LCL	$B_5(n)\sigma$	$B_3(n)\bar{s}$

1. Exercise: find the expression of B_3, B_4, B_5, B_6 constants
2. With regard to the same chart, find relations between B_3 and B_5, and B_4 and B_6.

X-S Control charts

1. Solution:

S chart Known parameters

$$\text{UCL} = \mu_S + \text{K}\sigma_S = c_4\sigma + 3\sqrt{1-c_4^{\,2}}\,\sigma = B_6\sigma \Rightarrow B_6 = c_4 + 3\sqrt{1-c_4^{\,2}}$$
$$\text{CL} = \mu_S = c_4\sigma$$
$$\text{LCL} = \mu_S - \text{K}\sigma_S = c_4\sigma - 3\sqrt{1-c_4^{\,2}}\,\sigma = B_5\sigma \Rightarrow B_5 = c_4 - 3\sqrt{1-c_4^{\,2}}$$

Unknown parameters

$$\text{UCL} = c_4\hat{\sigma} + 3\sqrt{1-c_4^{\,2}}\,\hat{\sigma} = \bar{s} + 3\frac{\sqrt{1-c_4^{\,2}}}{c_4}\bar{s} = B_4\bar{s} \Rightarrow B_4 = 1 + 3\frac{\sqrt{1-c_4^{\,2}}}{c_4}$$
$$\text{CL} = c_4\hat{\sigma} = \bar{s}$$
$$\text{LCL} = c_4\hat{\sigma} - 3\sqrt{1-c_4^{\,2}}\,\hat{\sigma} = \bar{s} - 3\frac{\sqrt{1-c_4^{\,2}}}{c_4}\bar{s} = B_3\bar{s} \Rightarrow B_3 = 1 - 3\frac{\sqrt{1-c_4^{\,2}}}{c_4}$$

2. Solution:

$$B_6\sigma = B_6\frac{\bar{s}}{c_4} = B_4\bar{s} \Rightarrow B_6 = B_4 c_4$$
$$\text{analogousl y} \quad \Rightarrow B_5 = B_3 c_4$$

Xbar-S control charts with variable sample size

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{m} n_i \bar{x}_i}{\sum_{i=1}^{m} n_i} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}}{\sum_{i=1}^{m} n_i} \qquad s_p = \sqrt{\frac{\sum_{i=1}^{m}(n_i-1)s_i^{\,2}}{\sum_{i=1}^{m}(n_i-1)}}$$

$$d = \sum_{i=1}^{m} n_i - m + 1$$
$$\Rightarrow \hat{\sigma} = \frac{s_p}{c_4(d)}$$

$\overline{X}$ Chart

$$UCL_i = \bar{\bar{x}} + \frac{3}{\sqrt{n_i}}\hat{\sigma}$$
$$CL_i = \bar{\bar{x}}$$
$$LCL_i = \bar{\bar{x}} - \frac{3}{\sqrt{n_i}}\hat{\sigma}$$

S Chart

$$UCL_i = B_6(n_i)\hat{\sigma}$$
$$CL_i = c_4(n_i)\hat{\sigma}$$
$$LCL_i = B_5(n_i)\hat{\sigma}$$

or, analogously

$$UCL_i = \frac{c_4(n_i)}{c_4(d)}B_4(n_i)s_p$$
$$CL_i = \frac{c_4(n_i)}{c_4(d)}s_p$$
$$LCL_i = \frac{c_4(n_i)}{c_4(d)}B_3(n_i)s_p$$

Example

Precise Tech Inc.:

- Produces valves for motorbike engines via hot forging.
- Before chip removal processes (milling and grinding), the operator randomly collects a subgroup of valves produced in 1 hour and measures the diameters (in cm)
- New input batch : sample of 10 valves produced in 1 hour
- If 5 consecutive subgroups exhibit an in-control pattern: sample of 5 valves produced in 1 hour

Table 6.3 Valve measurement data

Subgroup Number	Observations										$\bar{x}_i$	s_i
1	4.92	4.96	5.00	4.82	5.11						4.962	0.106395
2	4.91	5.02	5.05	4.96	4.90						4.968	0.066106
3	5.17	4.82	4.81	5.15	4.92						4.974	0.175300
4	4.87	5.11	5.06	5.04	4.98						5.012	0.092033
5	4.93	4.98	4.94	5.09	4.99						4.986	0.063482
6	4.91	4.86	5.08	5.20	4.96						5.002	0.137550
7	5.04	4.98	4.96	5.12	5.16						5.052	0.086718
8	4.88	4.99	5.19	5.09	4.88						5.006	0.135019
9	4.98	4.88	5.03	4.98	4.94						4.962	0.055857
10	5.00	4.84	4.79	4.98	5.15						4.952	0.142373
11	5.08	4.99	5.06	5.01	4.92						5.012	0.063008
12	4.95	5.08	4.88	4.88	5.00						4.958	0.084971
13	4.90	4.99	4.90	4.92	5.01						4.944	0.052249
14	5.07	4.93	5.21	4.99	4.98						5.036	0.109453
15	4.98	5.02	5.14	4.93	5.17						5.048	0.103296
16	4.96	4.86	4.87	4.93	4.89						4.902	0.042071
17	4.88	5.11	5.03	5.11	4.98						5.022	0.096799
18	4.99	5.08	4.94	5.12	5.05						5.036	0.071624
19	5.20	4.98	4.99	4.87	5.04	5.00	4.89	5.04	5.05	4.80	4.986	0.112171
20	4.95	4.89	5.08	4.79	4.85	5.09	5.03	5.13	5.04	5.04	4.989	0.113279
21	5.09	4.96	4.97	5.03	4.98	5.12	4.96	5.04	5.11	4.98	5.024	0.063456
22	5.14	4.99	4.90	5.03	5.05	4.78	4.95	4.84	4.94	5.01	4.963	0.105204
23	5.04	4.97	5.10	4.92	4.95	5.01	4.83	5.21	4.83	5.06	4.992	0.118491
24	4.83	5.11	4.98	4.89	4.96						4.954	0.105499
25	5.06	5.00	4.89	5.04	5.27						5.052	0.138456
26	4.93	5.01	4.96	5.18	4.92						5.000	0.106536
27	5.02	4.96	4.99	4.82	4.89						4.936	0.080808
28	4.94	5.20	5.12	5.03	4.89						5.036	0.127004
29	5.22	5.04	5.01	5.08	4.92						5.054	0.109909
30	5.05	4.90	5.06	4.99	4.76						4.952	0.124780

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{m} n_i \bar{x}_i}{\sum_{i=1}^{m} n_i} = \frac{5(4.952) + \ldots + 10(4.986) + \ldots + 10(4.992) + \ldots + 5(4.952)}{5 + \ldots + 10 + \ldots + 10 + \ldots + 5} = \frac{873.63}{175} = 4.992$$

$$s_p = \sqrt{\frac{\sum_{i=1}^{m} (n_i - 1) s_i^2}{\sum_{i=1}^{m} (n_i - 1)}} =$$

$$= \sqrt{\frac{4(0.106395)^2 + \ldots + 9(0.112171)^2 + \ldots + 9(0.118491)^2 + \ldots + 4(0.12780)^2}{4 + \ldots + 9 + \ldots + 9 + \ldots 4}} =$$

$$= \sqrt{\frac{1.5781}{145}} = 0.10432$$

$$d = \sum_{i=1}^{30} n_i - 30 + 1 = 175 - 30 + 1 = 146$$

$$c_4(146) = \frac{4(146) - 4}{4(146) - 3} = 0.9983$$

$$\hat{\sigma} = \frac{0.10432}{0.9983} = 0.104498$$

Appendice A.6 Factors for the design of control charts for variables

Campione	Carta $\bar{x}$			Carta S						Carta R						
	Fattori per i limiti			Fattori per il centro		Fattori per i limiti				Fattori per il centro		Fattori per i limiti				
n	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.881	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.687	0	3.269
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.357	0	2.574
4	1.5	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.88	0	4.699	0	2.282
→ 5	1.342	0.577	1.427	$\boxed{0.94}$	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0509	0.03	1.97	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0424	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.205	5.203	0.076	1.924
8	1.061	0.373	1.099	0.965	1.0362	0.185	1.815	0.179	1.751	2.847	0.3512	0.82	0.387	5.307	0.136	1.864
9	1	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.97	0.3367	0.808	0.546	5.394	0.184	1.816
→ 10	0.949	0.308	0.975	$\boxed{0.9727}$	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0253	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.812	5.534	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.023	0.354	1.646	0.346	1.61	3.258	0.3069	0.778	0.924	5.592	0.284	1.716
13	0.832	0.249	0.85	0.9794	1.021	0.382	1.618	0.374	1.585	3.336	0.2998	0.77	1.026	5.646	0.308	1.692
14	0.802	0.235	0.817	0.981	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.762	1.121	5.693	0.329	1.671
15	0.775	0.223	0.789	0.9823	1.018	0.428	1.572	0.421	1.544	3.472	0.288	0.755	1.207	5.737	0.348	1.652
16	0.75	0.212	0.763	0.9835	1.0168	0.448	1.552	0.44	1.526	3.532	0.2831	0.749	1.285	5.779	0.364	1.636
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.743	1.359	5.817	0.379	1.621
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.64	0.2747	0.738	1.426	5.854	0.392	1.608
19	0.688	0.187	0.698	0.9862	1.014	0.497	1.503	0.49	1.483	3.689	0.2711	0.733	1.49	5.888	0.404	1.596
20	0.671	0.18	0.68	0.9869	1.0132	0.51	1.49	0.504	1.47	3.735	0.2677	0.729	1.548	5.922	0.414	1.586
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.606	5.95	0.425	1.575
22	0.64	0.167	0.647	0.9882	1.012	0.534	1.466	0.528	1.448	3.819	0.2618	0.72	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.71	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.452	1.548
25	0.6	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.42	3.931	0.2544	0.709	1.804	6.058	0.459	1.541

Per $n \geq 25$: $A = \frac{3}{\sqrt{n}}$, $A_3 = \frac{3}{c_4\sqrt{n}}$, $c_4 = \frac{4(n-1)}{4n-3}$, $B_3 = 1 - \frac{3}{c_4\sqrt{2(n-1)}}$, $B_4 = 1 + \frac{3}{c_4\sqrt{2(n-1)}}$, $B_5 = c_4 - \frac{3}{\sqrt{2(n-1)}}$, $B_6 = c_4 + \frac{3}{\sqrt{2(n-1)}}$.

From Montgomery

	n=5	n=10
c4	0.9400	0.9727
B3	0	0.284
B4	2.089	1.716
B5	0	0.276
B6	1.964	1.669

$$\hat{\sigma} = \frac{0.10432}{0.9983} = 0.104498$$

$\overline{X}$ Chart

$$UCL = 4.992 + \frac{3}{\sqrt{5}} 0.104498 = 5.132$$
$$CL = 4.992$$
$$LCL = 4.992 - \frac{3}{\sqrt{5}} 0.104498 = 4.852$$

$$UCL = 4.992 + \frac{3}{\sqrt{10}} 0.104498 = 5.091$$
$$CL = 4.992$$
$$LCL = 4.992 - \frac{3}{\sqrt{10}} 0.104498 = 4.893$$

S Chart

$$UCL = B_6(5)\hat{\sigma} = (1.964)(0.104498) = 0.2052$$
$$CL = c_4(5)\hat{\sigma} = (0.940)(0.104498) = 0.0982$$
$$LCL = B_5(5)\hat{\sigma} = (0)(0.104498) = 0$$

$$UCL = B_6(10)\hat{\sigma} = (1.669)(0.104498) = 0.1744$$
$$CL = c_4(10)\hat{\sigma} = (0.9727)(0.104498) = 0.1016$$
$$LCL = B_5(10)\hat{\sigma} = (0.276)(0.104498) = 0.0288$$

Stat>>Control Charts>>Xbar-S

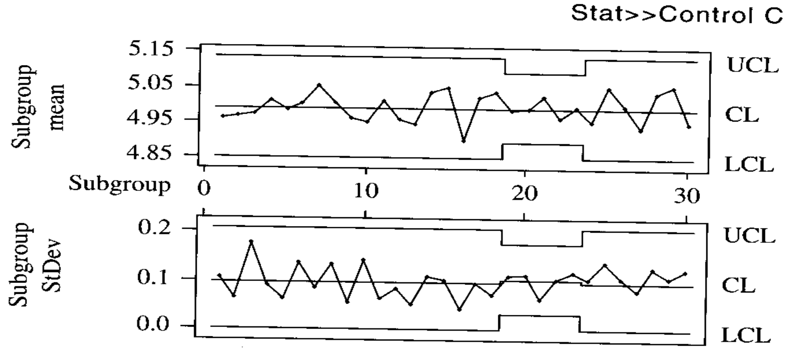


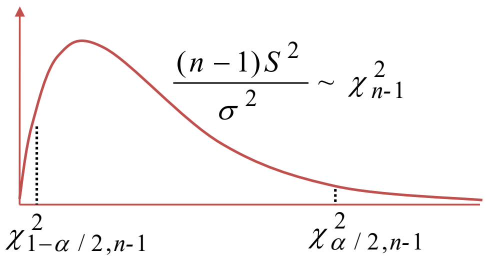
Figure 6.10 $\bar{x}$ and s Control chart for valve data with variable subgroup size.

No out of controls. If out of controls were observed: look for assignable causes. If assignable causes were found: remove the observation/the sample and re-design the chart. The process ends when no more assignable causes are found or no new out of controls are observed.

S^2 Control Chart

To monitor the process variance

For a probabilistic control chart (strong asymmetry of the distribution)



$$\Pr\left(\chi^2_{1-\alpha/2,n-1} \le (n-1)\frac{S^2}{\sigma^2} \le \chi^2_{\alpha/2,n-1}\right) = 1-\alpha$$

$$UCL = \frac{\sigma^2}{n-1}\chi^2_{\alpha/2,n-1}$$

$$CL = \sigma^2$$

$$LCL = \frac{\sigma^2}{n-1}\chi^2_{1-\alpha/2,n-1}$$

where
$$\hat{\sigma}^2 = \begin{cases} \overline{s^2} = \dfrac{1}{m}\sum_{i=1,..,m} s_i^2 & \text{if n is constant} \\ s_p^2 = \dfrac{\sum_{i=1}^{m}(n_i-1)s_i^2}{\sum_{i=1}^{m}(n_i-1)} & \text{if n is not constant} \end{cases}$$

Operating characteristic curve for the S control chart

$H_0: \sigma = \sigma_0$

$H_1: \sigma = \sigma_1 = \lambda\sigma_0$

$$\text{UCL} = \sigma_0\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}} \qquad \text{LCL} = \sigma_0\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}}$$

$$\boxed{\lambda = \frac{\sigma_1}{\sigma_0}}$$

$$\beta(\lambda) = \Pr\left(S \in [LCL, UCL] \,\middle|\, \overbrace{\frac{(n-1)S^2}{\sigma_1^2} \sim \chi^2_{n-1}}^{*}\right) = \Pr\left(S \le UCL \,|*\right) - \Pr\left(S < LCL \,|*\right)$$

$$\Pr\left(S \le UCL \,|*\right) = \Pr\left(S^2 \le UCL^2 |*\right) = \Pr\left(\frac{(n-1)S^2}{\sigma_1^2} \le \frac{(n-1)UCL^2}{\sigma_1^2} \,\middle|\, *\right) =$$

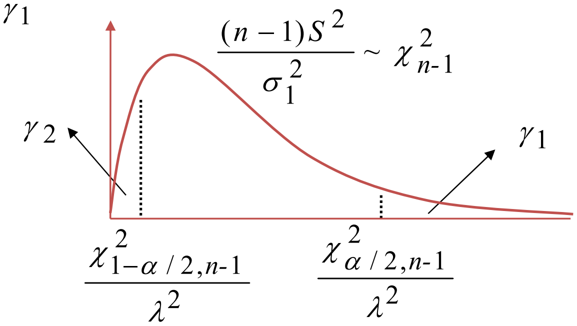
$$= \Pr\left(\chi^2_{n-1} \le \frac{(n-1)\sigma_0^2\chi^2_{\alpha/2,n-1}}{\sigma_1^2(n-1)}\right) = \Pr\left(\chi^2_{n-1} \le \frac{\chi^2_{\alpha/2,n-1}}{\lambda^2}\right)$$

$$\Pr\left(S \le UCL \mid *\right) = \Pr\left(\chi^2_{n-1} \le \frac{\chi^2_{\alpha/2,n-1}}{\lambda^2}\right) = 1-\gamma_1$$

$$\Pr\left(S < LCL \mid *\right) = \Pr\left(S^2 \le LCL^2 \mid *\right) =$$

$$= \Pr\left(\frac{(n-1)S^2}{\sigma_1^2} \le \frac{(n-1)LCL^2}{\sigma_1^2} \middle| *\right) =$$

$$= \Pr\left(\chi^2_{n-1} \le \frac{\chi^2_{1-\alpha/2,n-1}}{\lambda^2}\right) = \gamma_2$$



$$\beta(\lambda) = \Pr\left(\chi^2_{n-1} \le \frac{\chi^2_{\alpha/2,n-1}}{\lambda^2}\right) - \Pr\left(\chi^2_{n-1} \le \frac{\chi^2_{1-\alpha/2,n-1}}{\lambda^2}\right) = 1-\gamma_1-\gamma_2$$

Analogously: $$ARL(\lambda) = \frac{1}{1-\beta(\lambda)} = \frac{1}{\gamma_1+\gamma_2}$$

We know that: $$\boxed{\hat{\sigma} = \frac{\overline{S}}{c_4(n)}}$$

Thus, a further possible approach to design control charts for individuals:

$$\hat{\mu} \pm 3\hat{\sigma} = \overline{x} \pm 3\left(\frac{s}{c_4}\right)$$ where $$s = \sqrt{\sum_{j=1,\dots,n} (x_j - \overline{x})^2 \Big/ (n-1)},$$

$c_4(n)$ in table for $2 \le n \le 24$; for $n \ge 25$: $c_4(n) \cong \frac{4n-4}{4n-3}$

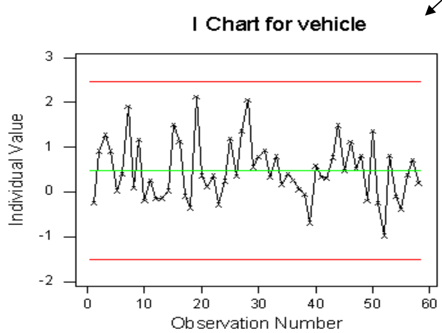
Two options are available:

$$\overline{x} \pm 3\left(\frac{s}{c_4}\right) \qquad \overline{x} \pm 3\left(\frac{\overline{MR}}{d_2}\right) = \overline{x} \pm 2.66\,\overline{MR}$$

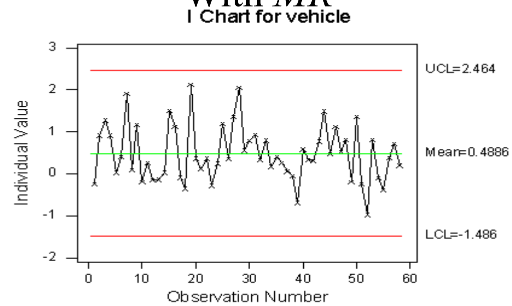
Previous example

$$\begin{matrix}UCL\\LCL\end{matrix} = \bar{x} \pm 3\left(\frac{s}{c_4(58)}\right) = 0.4886 \pm 3\left(\frac{0.6611}{0.9956}\right) = \begin{matrix}2.481\\-1.503\end{matrix}$$

(vehicle1.dat)



With *MR*



NID process: two exchangeable approaches. Otherwise:

–Estimator based on MR is less efficient (larger variance) is process is IID– (Cryer and Ryan, 1990)

–Some authors advocate using the method based on MR because it is more robust (no bias) to Phase I analysis in the presence of out-of-control conditions (Rigdon, Cruthis Champ, 1994)

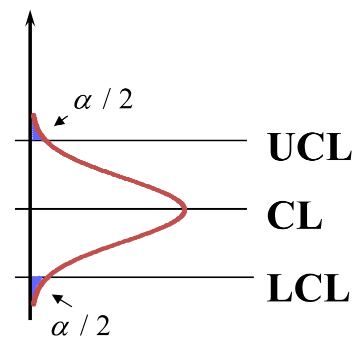
Probabilistic control chart

$$\Pr(LCL \le V \le UCL) = 1 - \alpha \qquad \text{by using the distribution of } V$$
$$\Pr(V \le LCL) = \alpha/2 \qquad \Pr(V > UCL) = \alpha/2$$

Xbar Chart

$$X \sim \mathrm{N}(\mu, \sigma^2) \Rightarrow \overline{X} \sim \mathrm{N}(\mu, \sigma^2/n)$$
approx if X non-normal (central limit theorem)

$$\boxed{\begin{aligned} \mathrm{UCL} &= \hat{\mu} + \mathrm{K}\frac{\hat{\sigma}}{\sqrt{n}} = \overline{\overline{x}} + z_{\alpha/2}\frac{1}{d_2\sqrt{n}}\overline{R} \\ \mathrm{CL} &= \hat{\mu} = \overline{\overline{x}} \\ \mathrm{LCL} &= \hat{\mu} - \mathrm{K}\frac{\hat{\sigma}}{\sqrt{n}} = \overline{\overline{x}} - z_{\alpha/2}\frac{1}{d_2\sqrt{n}}\overline{R} \end{aligned}}$$



$\alpha = 0.002 \Rightarrow z_{0.002/2} = z_{0.001} \Rightarrow z_{0.001} = 3.09.$

In the example:

$$\mathrm{UCL} = \overline{\overline{x}} + z_{\alpha/2}\frac{1}{d_2\sqrt{n}}\overline{R} = 1000.016 + 3.09\left(\frac{1.673}{2.326\sqrt{5}}\right) = 1001.010$$
$$\mathrm{CL} = \overline{\overline{x}} = 1000.016$$
$$\mathrm{LCL} = \overline{\overline{x}} - z_{\alpha/2}\frac{1}{d_2\sqrt{n}}\overline{R} = 1000.016 - 3.09\left(\frac{1.673}{2.326\sqrt{5}}\right) = 999.022$$

Probabilistic limits for R control chart

Control chart with normality approximation:

- LCL=0 for n≤6: we cannot detect variability reductions!
- α_{real} different from α_{design} (2 or 3 times α_{design}=0.0027)

$$UCL = D_{1-\alpha/2}\frac{\overline{R}}{d_2} \qquad LCL = D_{\alpha/2}\frac{\overline{R}}{d_2}$$

Harter (1960)

D_α

n	α=0.001	0.005	0.025	0.975	0.995	0.999
3	0.06	0.13	0.30	3.68	4.42	5.06
4	0.20	0.34	0.59	3.98	4.69	5.31
5	0.37	0.55	0.85	4.20	4.89	5.48
6	0.53	0.75	1.07	4.36	5.03	5.62
7	0.69	0.92	1.25	4.49	5.15	5.73
8	0.83	1.08	1.41	4.60	5.25	5.82
9	0.97	1.21	1.55	4.70	5.34	5.90
10	1.08	1.33	1.67	4.78	5.42	5.97

For n = 2

$$D_{1-\alpha/2} = \sqrt{2}z_{\alpha/4} \qquad D_{\alpha/2} = \sqrt{2}z_{1/2-\alpha/4}$$

Quality Data Analysis

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$\alpha = 0.002$

In the example:

n	α=0.001	0.005	0.025	0.975	0.995	0.999
3	0.06	0.13	0.30	3.68	4.42	5.06
4	$\boxed{0.20}$	0.34	0.59	3.98	4.69	$\boxed{5.31}$
5	0.37	0.55	0.85	4.20	4.89	5.48
6	0.53	0.75	1.07	4.36	5.03	5.62
7	0.69	0.92	1.25	4.49	5.15	5.73
8	0.83	1.08	1.41	4.60	5.25	5.82
9	0.97	1.21	1.55	4.70	5.34	5.90
10	1.08	1.33	1.67	4.78	5.42	5.97

D_α

$$UCL = D_{0.999}\left(\frac{\overline{R}}{d_2}\right) = 5.48\left(\frac{1.775}{2.326}\right) = 4.182$$

$$LCL = D_{0.001}\left(\frac{\overline{R}}{d_2}\right) = 0.37\left(\frac{1.775}{2.326}\right) = 0.282$$

Quality Data Analysis

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Method 1: control chart with probabilistic limits MR

1. Use the true distribution: Half-Normal

(if original data were normal)

$$UCL_{HN} = \sqrt{2}\, z_{\alpha/4}\sigma = \sqrt{2}\, z_{\alpha/4}\overline{MR} / d_2(2)$$

$$LCL_{HN} = \sqrt{2}\, z_{1/2-\alpha/4}\sigma = \sqrt{2}\, z_{1/2-\alpha/4}\overline{MR} / d_2(2)$$

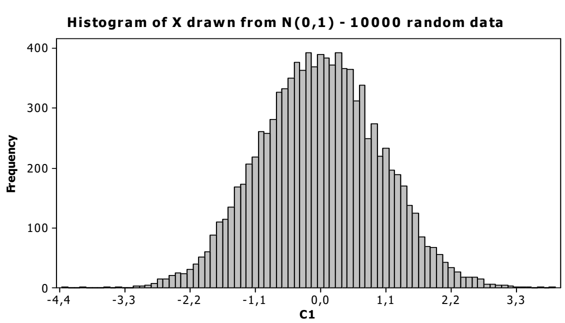
$\alpha = 0.0027$ $\quad z_{\alpha/4} = z_{0.000675} = 3.20515$ $\quad z_{1/2-\alpha/4} = z_{0.499325} = 0.00169$

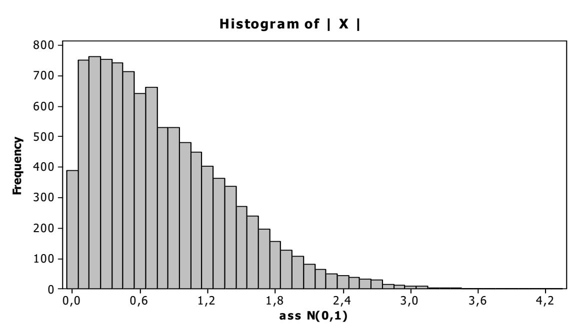
$\alpha = 0.00915$ $\quad z_{\alpha/4} = z_{0.0022875} = 2.8355$ $\quad z_{1/2-\alpha/4} = z_{0.4977125} = 0.00573$

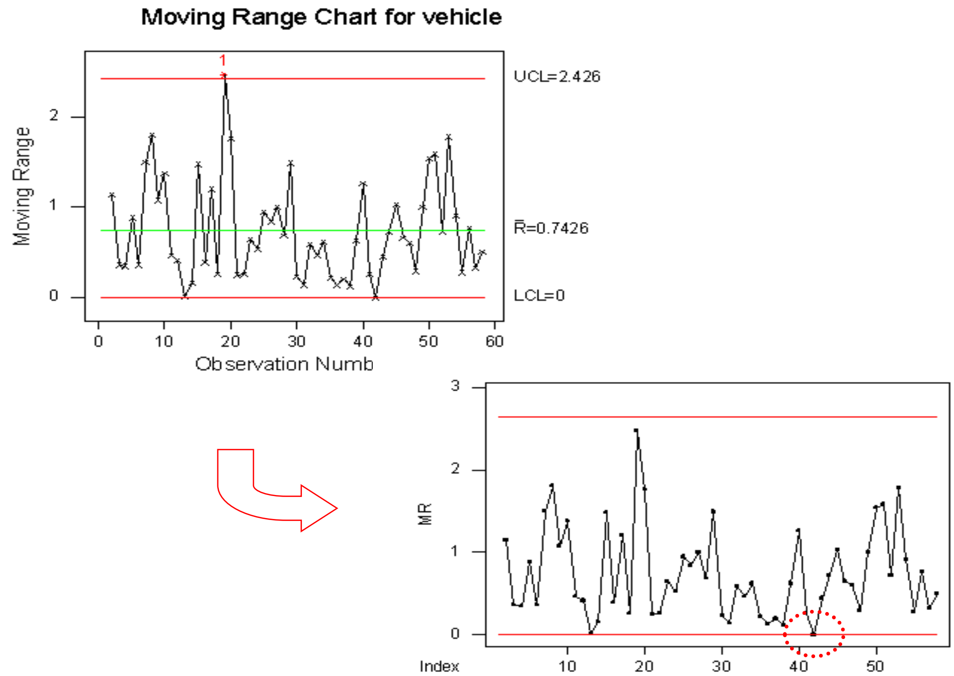
In the example:

$$\mathrm{UCL_{HN}} = \frac{\sqrt{2}(2.835)(0.7426)}{1.128} = 2.639$$

$$\mathrm{LCL_{HN}} = \frac{\sqrt{2}(0.00573)(0.7426)}{1.128} = 0.0053$$







UCL=2.639

LCL=0.0053

42°obs. is out-of-control!

Quality Data Analysis

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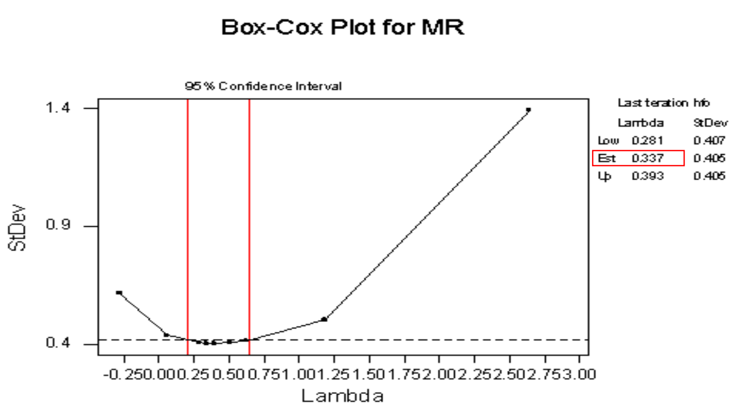
Method 2: directly consider the MR time series: transformation

2. Transform MR_i data to have normality:

We know the real distribution (half-normal) and hence the "true" transformation exponent (Alwan and Radson, 1993, via simulation) is:

$$\boxed{\lambda = 0.4}$$

Apply the power transformation to the MR_i values in the example (vehicle1.dat):



Use: $\lambda = 0.4$

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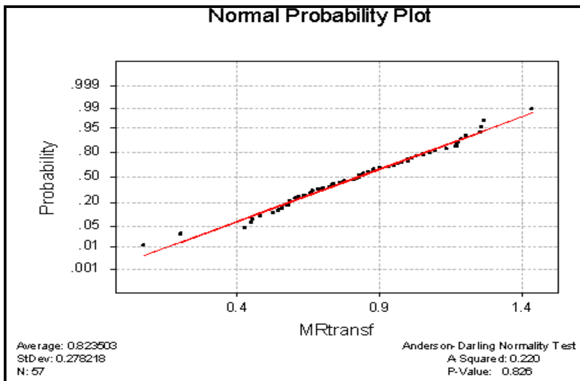
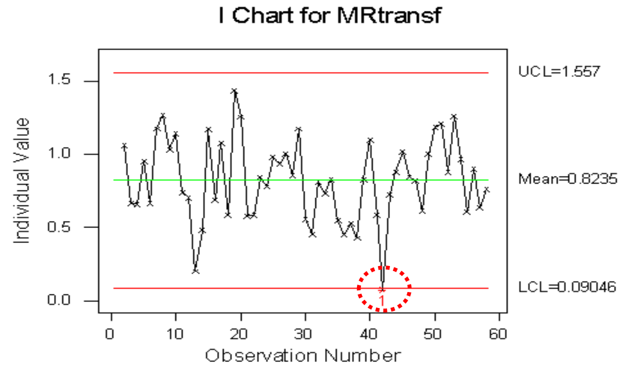
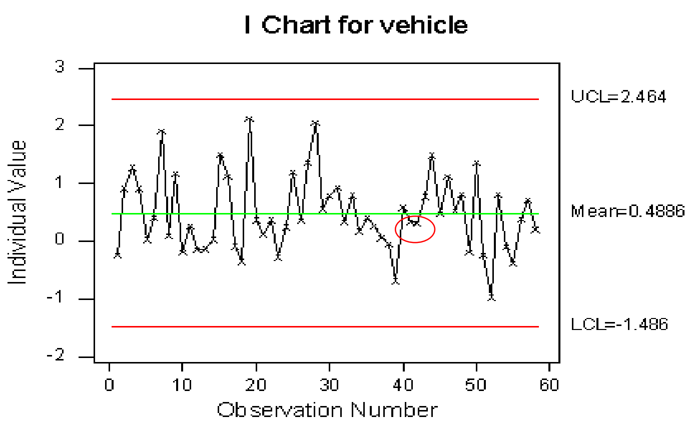


Also in this case, analogously to Method 1, 42° obs is out of control



Look for assignable cause for MR_{42}:



If an assignable cause is found, the observation is removed from the dataset and the control charts (both I and MR) are re-designed, ecc.

Appendice A.6 Factors for the design of control charts for variables

	Carta $\bar{x}$			Carta S						Carta R						
Campione	Fattori per i limiti			Fattori per il centro		Fattori per i limiti				Fattori per il centro		Fattori per i limiti				
n	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.881	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.687	0	3.269
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.357	0	2.574
4	1.5	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.88	0	4.699	0	2.282
5	1.342	0.577	1.427	0.94	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0509	0.03	1.97	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0424	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.205	5.203	0.076	1.924
8	1.061	0.373	1.099	0.965	1.0362	0.185	1.815	0.179	1.751	2.847	0.3512	0.82	0.387	5.307	0.136	1.864
9	1	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.97	0.3367	0.808	0.546	5.394	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0253	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.812	5.534	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.023	0.354	1.646	0.346	1.61	3.258	0.3069	0.778	0.924	5.592	0.284	1.716
13	0.832	0.249	0.85	0.9794	1.021	0.382	1.618	0.374	1.585	3.336	0.2998	0.77	1.026	5.646	0.308	1.692
14	0.802	0.235	0.817	0.981	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.762	1.121	5.693	0.329	1.671
15	0.775	0.223	0.789	0.9823	1.018	0.428	1.572	0.421	1.544	3.472	0.288	0.755	1.207	5.737	0.348	1.652
16	0.75	0.212	0.763	0.9835	1.0168	0.448	1.552	0.44	1.526	3.532	0.2831	0.749	1.285	5.779	0.364	1.636
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.743	1.359	5.817	0.379	1.621
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.64	0.2747	0.738	1.426	5.854	0.392	1.608
19	0.688	0.187	0.698	0.9862	1.014	0.497	1.503	0.49	1.483	3.689	0.2711	0.733	1.49	5.888	0.404	1.596
20	0.671	0.18	0.68	0.9869	1.0132	0.51	1.49	0.504	1.47	3.735	0.2677	0.729	1.548	5.922	0.414	1.586
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.606	5.95	0.425	1.575
22	0.64	0.167	0.647	0.9882	1.012	0.534	1.466	0.528	1.448	3.819	0.2618	0.72	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.71	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.452	1.548
25	0.6	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.42	3.931	0.2544	0.709	1.804	6.058	0.459	1.541

Per $n \geq 25$: $A = \frac{3}{\sqrt{n}}$, $A_3 = \frac{3}{c_4\sqrt{n}}$, $c_4 = \frac{4(n-1)}{4n-3}$, $B_3 = 1 - \frac{3}{c_4\sqrt{2(n-1)}}$, $B_4 = 1 + \frac{3}{c_4\sqrt{2(n-1)}}$, $B_5 = c_4 - \frac{3}{\sqrt{2(n-1)}}$, $B_6 = c_4 + \frac{3}{\sqrt{2(n-1)}}$.

Rational subgroups

Samples of size *n* are collected at regular intervals; sample statistics are computed (either to control the process mean or dispersion) and compared (plot) against control limits

How to manage the sampling operation?

1. Sampling strategy
2. Sample size
3. Time interval between samples

Shewhart: "rational subgroups"

Subgroups must be chosen such that the observations within the sample represent measurements made in the same conditions

- Reduced time between observations *within-the-sample* (AT&T Statistical Quality Handbook: consecutive measurements): attention to be paid to AUTOCORRELATION
- If time between observations *within-the-sample* is large: sampling from a mixture of:
  - Process in the presence of assignable causes and process in stable conditions;
  - Process in different conditions that are 'masked' within the sample.

Sample size

n ranging between 2 and 25 (in most applications, n between 2 and 6)

1. Process condition / measuring system (e.g., process having a low throughput: n=1)
2. Computational effort
3. Size of the shift to be detected by using the control chart: being equal α, n shall increase to detect small shifts *(OC or ARL curve)*
4. Statistical properties (central limit theorem for sample mean)

Sampling frequency

1. Process dynamics (autocorrelation)
2. Frequent samples: frequent check of system conditions

Size vs. sampling frequency:

Sampling costs:

- Variable costs;
- Fixed costs;
- Loss due to delayed detection of an out-of-control event