EXERCISE CLASS 4

Theory Recap

REMINDER (NON RANDOM PROCESSES)

Distributional model
(observations randomly drawn from a population):
$X \sim iid(\mu, \sigma^2)$
To make inference (e.g. hypothesis testing): $X \sim NID(\mu, \sigma^2)$

Time series model
$Y = f(X, \theta) + \varepsilon, \qquad \varepsilon \sim iid(0, \sigma^2)$
To make inference: $\qquad \varepsilon \sim NID(0, \sigma^2)$

Remind:
Independence → absence of autocorrelation
Absence of autocorrelation → independence ONLY IF data are normal

REMINDER (NON RANDOM PROCESSES)

Qualitative analysis of process data
Is it random?

- *Overall process level is constant over time?*
- *Is there a systematic pattern?*
- *The variation around the process level is constant?*

Are there outliers?

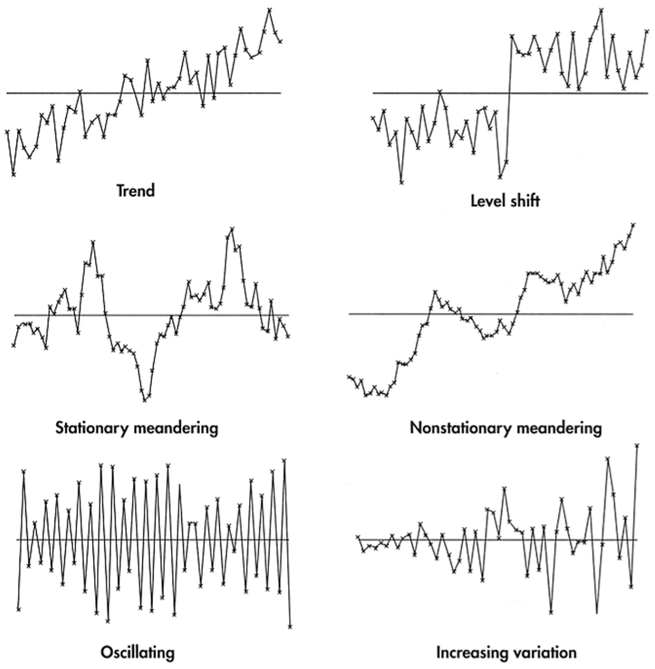


Figure 2.4 Time-series plots of different types of nonrandom process behavior.

Exercise 1

In a chemical process it is necessary to keep constant the pH of a compound. Measurements are made every hour. Data acquired over the first 48 hours are reported in `ESE4_ex1.csv`.

Identify and fit a model for the data.

In a future class:
Design a SCC control chart and a FVC control chart

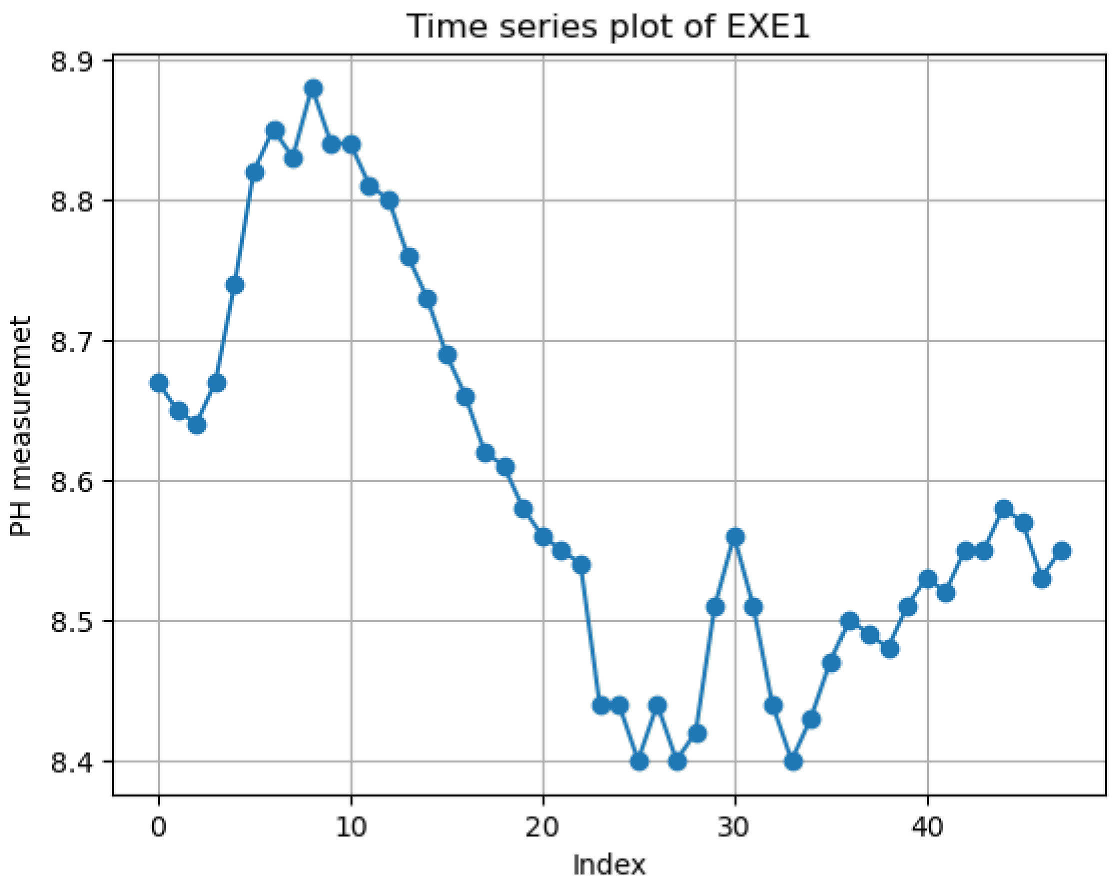
In [ ]:
```python
# Import the necessary libraries
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from scipy import stats
import seaborn as sns

# Import the dataset
data = pd.read_csv('ESE4_ex1.csv')

# Inspect the dataset
data.head()
```

Out[ ]:

	EXE1
0	8.67
1	8.65
2	8.64
3	8.67
4	8.74

> ## Solution
>
> Let's first check if:
>
> - The data are random.
> - (If random) The data are normally distributed.

In [ ]:
```python
# Plot the data first
plt.plot(data['EXE1'], 'o-')
plt.xlabel('Index')
plt.ylabel('PH measuremet')
plt.title('Time series plot of EXE1')
plt.grid()
plt.show()
```

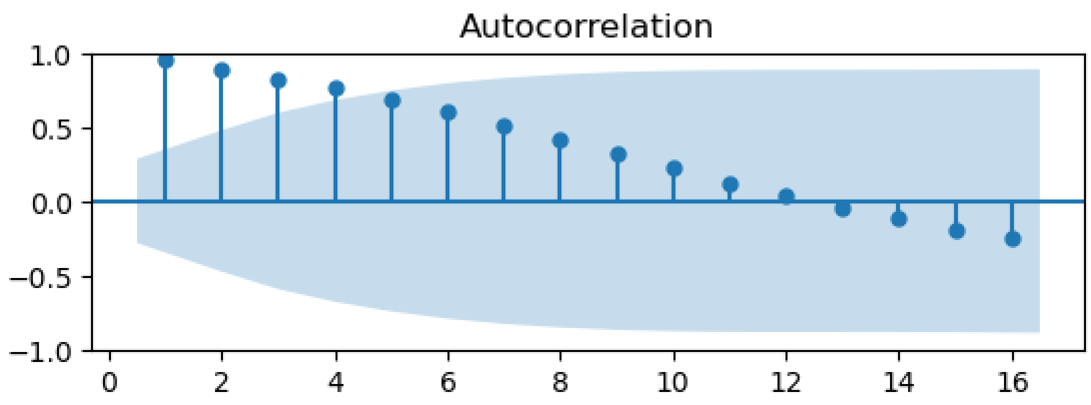


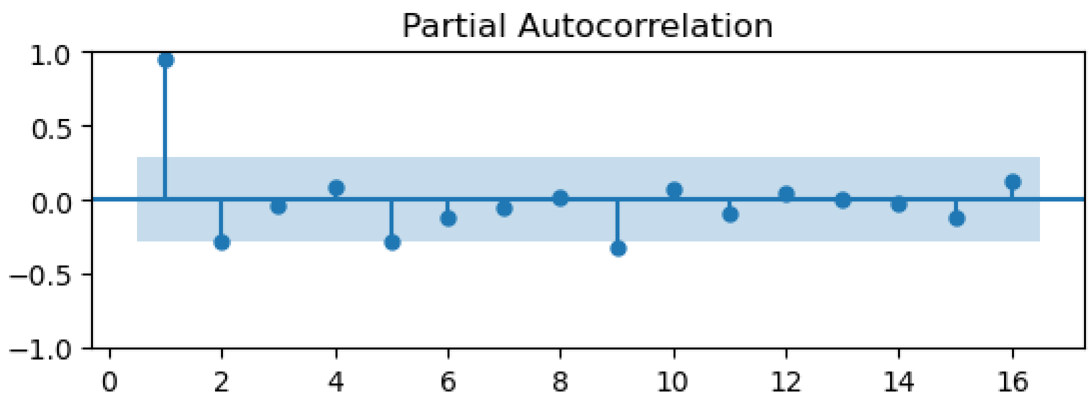
In [ ]:
```python
# Import the necessary libraries for the runs test
from statsmodels.sandbox.stats.runs import runstest_1samp

_, pval_runs = runstest_1samp(data['EXE1'], correction=False)
print('Runs test p-value = {:.3f}'.format(pval_runs))
```

```
Runs test p-value = 0.000
```

> The runs test gives a null p-value, this means that the data are not random .
>
> Plot also the autocorrelation and partial autocorrelation functions of the data. Use the `plot_acf` and `plot_pacf` functions from the `statsmodels` package.

In [ ]:
```python
# Plot the acf and pacf using the statsmodels library
import statsmodels.graphics.tsaplots as sgt

fig, ax = plt.subplots(2, 1)
sgt.plot_acf(data['EXE1'], lags = int(len(data)/3), zero=False, ax=ax[0])
fig.subplots_adjust(hspace=0.5)
sgt.plot_pacf(data['EXE1'], lags = int(len(data)/3), zero=False, ax=ax[1], method
plt.show()
```

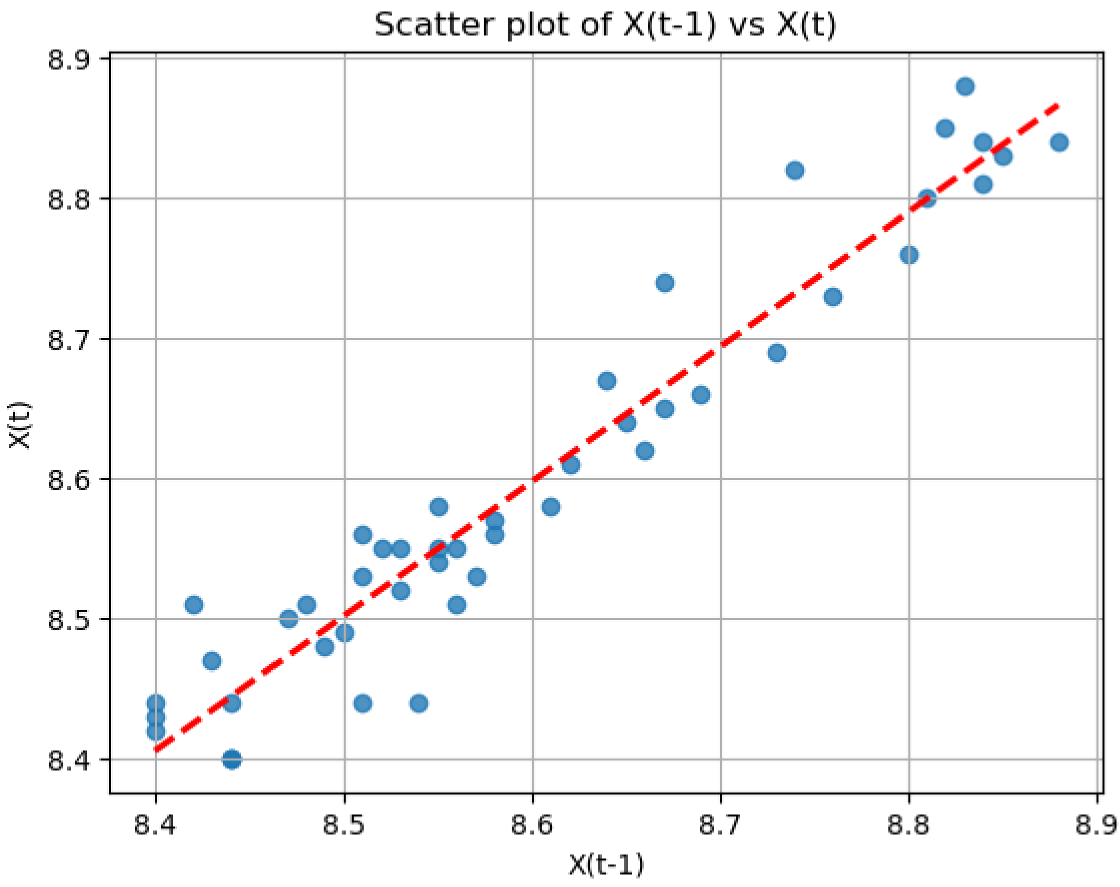




> There is a strong positive correlation. Decay of autocorrelation coefficients is not exponential. Based on ACF analysis, we can state that the process is non-stationary
>
> We can observe with a scatterplot the correlation between $X(t)$ and $X(t-1)$.

In [ ]:

```
#calculate the lag1 from data
data['lag1'] = data['EXE1'].shift(1)

#create scatterplot with regression line using seaborn and set axis labels
sns.regplot(x=data['lag1'], y=data['EXE1'], ci=None, line_kws={'color':'red', 'ls'
plt.title('Scatter plot of X(t-1) vs X(t)')
plt.xlabel('X(t-1)')
plt.ylabel('X(t)')
plt.title('Scatter plot of X(t-1) vs X(t)')
plt.grid()
```


Scatter plot of X(t-1) vs X(t)
X(t)
X(t-1)
8.4
8.5
8.6
8.7
8.8
8.9

EXERCISE 1 (SOLUTION)

REMINDER (previous class)

Autoregressive models: AR(p)

$$X_t = \xi + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \varepsilon_t$$

- ACF "geometrically decays"
- PACF indicates the order p

Moving average models: MA(q)

$$X_t = \mu - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q} + \varepsilon_t$$

$$\tilde{X}_t = X_t - \mu = -\theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q} + \varepsilon_t$$

- PACF "geometrically decays"
- ACF indicates the order q

EXERCISE 1 (SOLUTION)

REMINDER

ARMA(p,q): both AR(p) and MA(q) terms are present

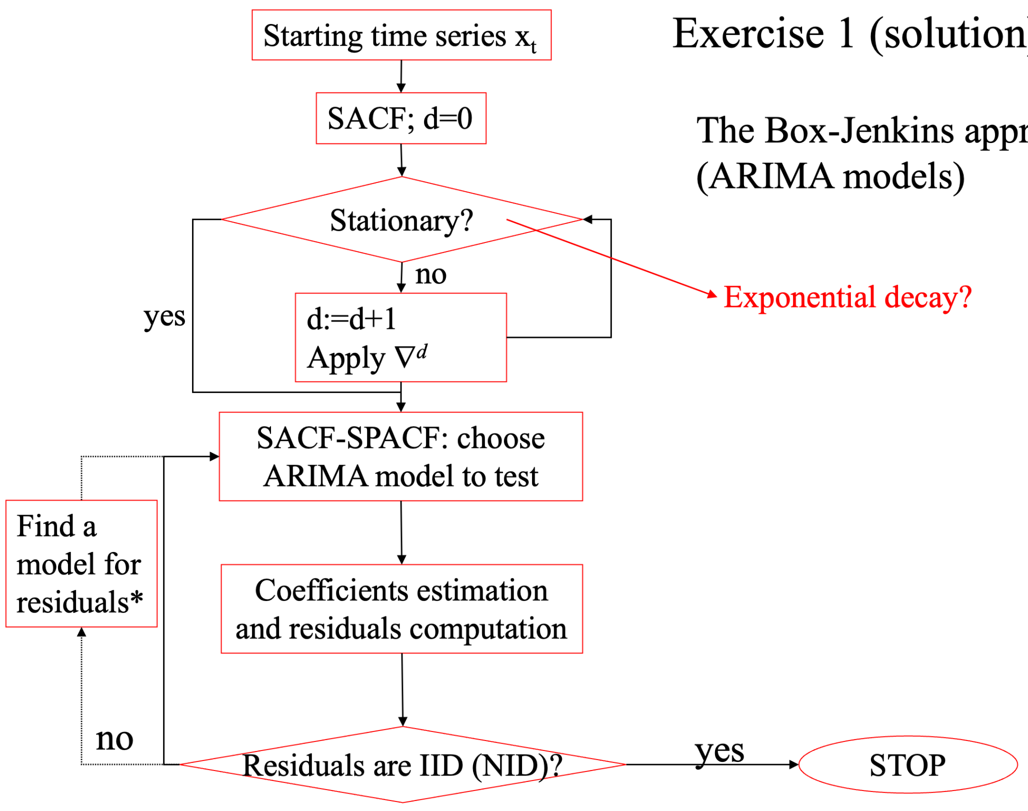
- It resembles an AR(p) after q lags
- Parsimony: low order models are preferred (easier to deal with)
- Model identification is often a *trial and error* problem

Homogeneous nonstationary ARMA (p,q) = ARIMA(p,d,q)

- ACF with slow (e.g. linear) decay, not a "geometrical decay"

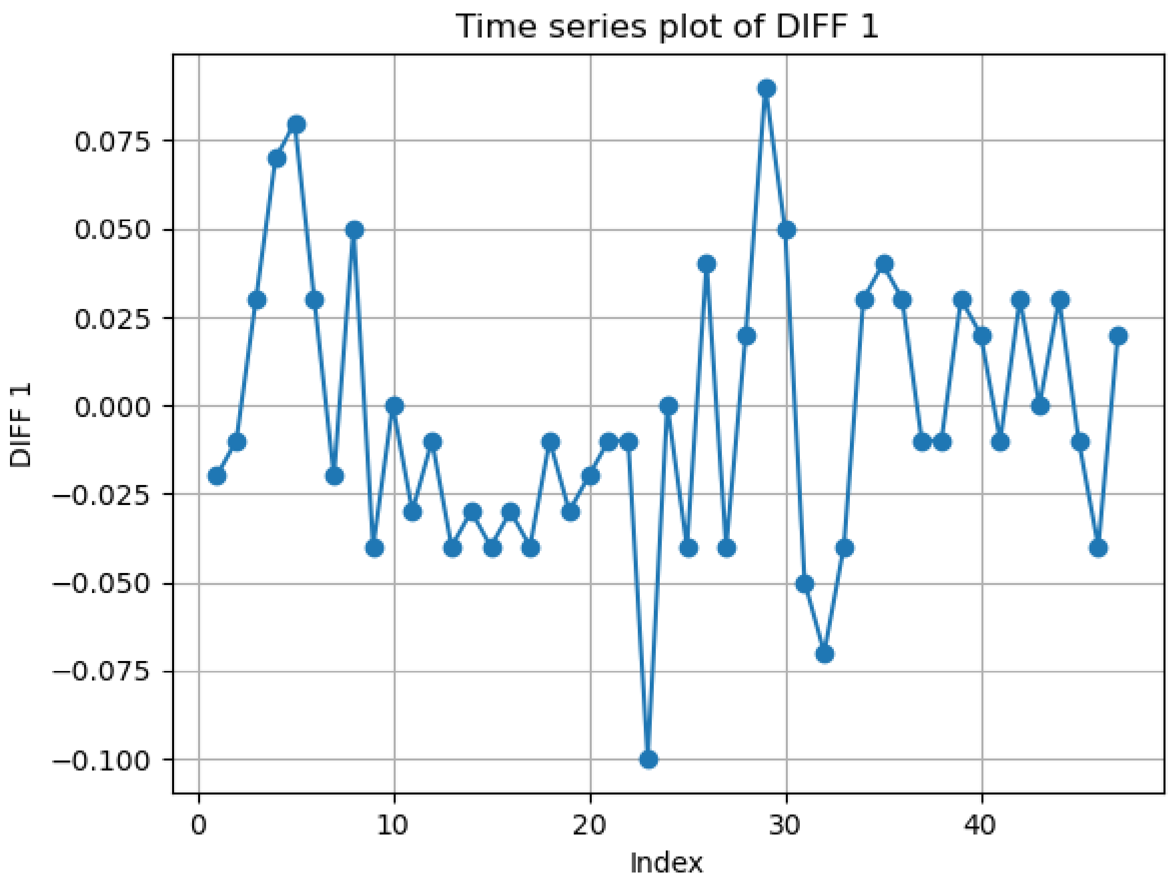
Exercise 1 (solution)

The Box-Jenkins approach (ARIMA models)


Starting time series x_t
SACF; d=0
Stationary?
Exponential decay?
yes
no
d:=d+1
Apply ∇^d
SACF-SPACF: choose ARIMA model to test
Find a model for residuals*
Coefficients estimation and residuals computation
no
Residuals are IID (NID)?
yes
STOP

> Let's apply the time difference operator to our time serie data

```
In [ ]: #calculate the difference between the data and the lag1
        data['diff1'] = data['EXE1'] - data['lag1']

        plt.plot(data['diff1'], 'o-')
        plt.xlabel('Index')
        plt.ylabel('DIFF 1')
        plt.title('Time series plot of DIFF 1')
        plt.grid()
        plt.show()
```

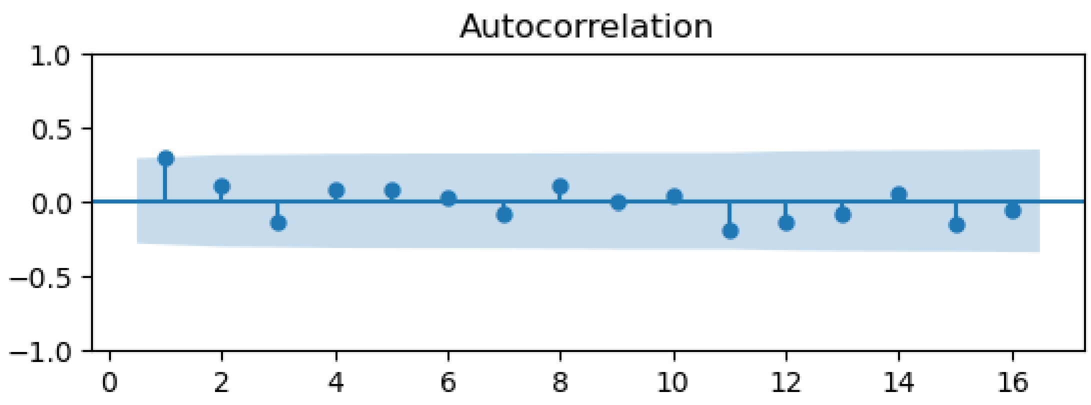


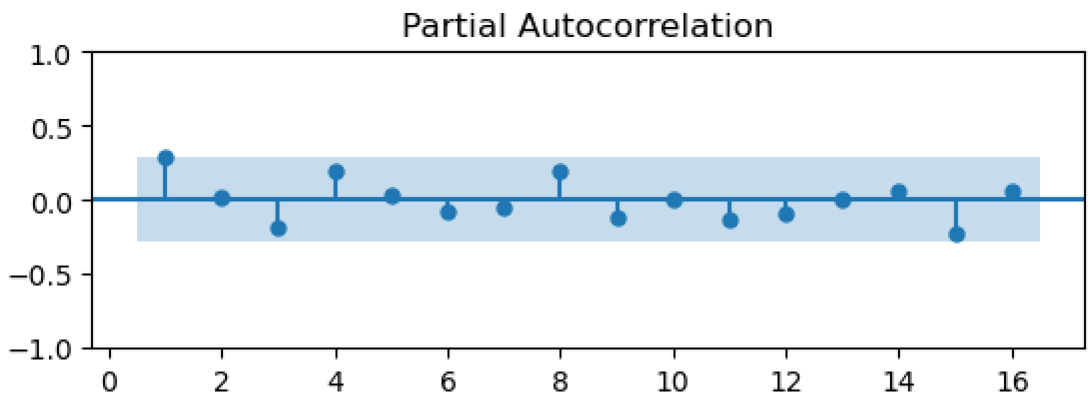
> Let's check if differences at lag 1 are NID

```
In [ ]: #Let's calculate the p-value (exclude the first value because it is null)
        _, pval_runs = runstest_1samp(data['diff1'][1:], correction=False)
        print('Runs test p-value = {:.3f}'.format(pval_runs))
```

```
Runs test p-value = 0.230
```

```
In [ ]: fig, ax = plt.subplots(2, 1)
        sgt.plot_acf(data['diff1'][1:], lags = int(len(data)/3), zero=False, ax=ax[0])
        fig.subplots_adjust(hspace=0.5)
        sgt.plot_pacf(data['diff1'][1:], lags = int(len(data)/3), zero=False, ax=ax[1], me
        plt.show()
```





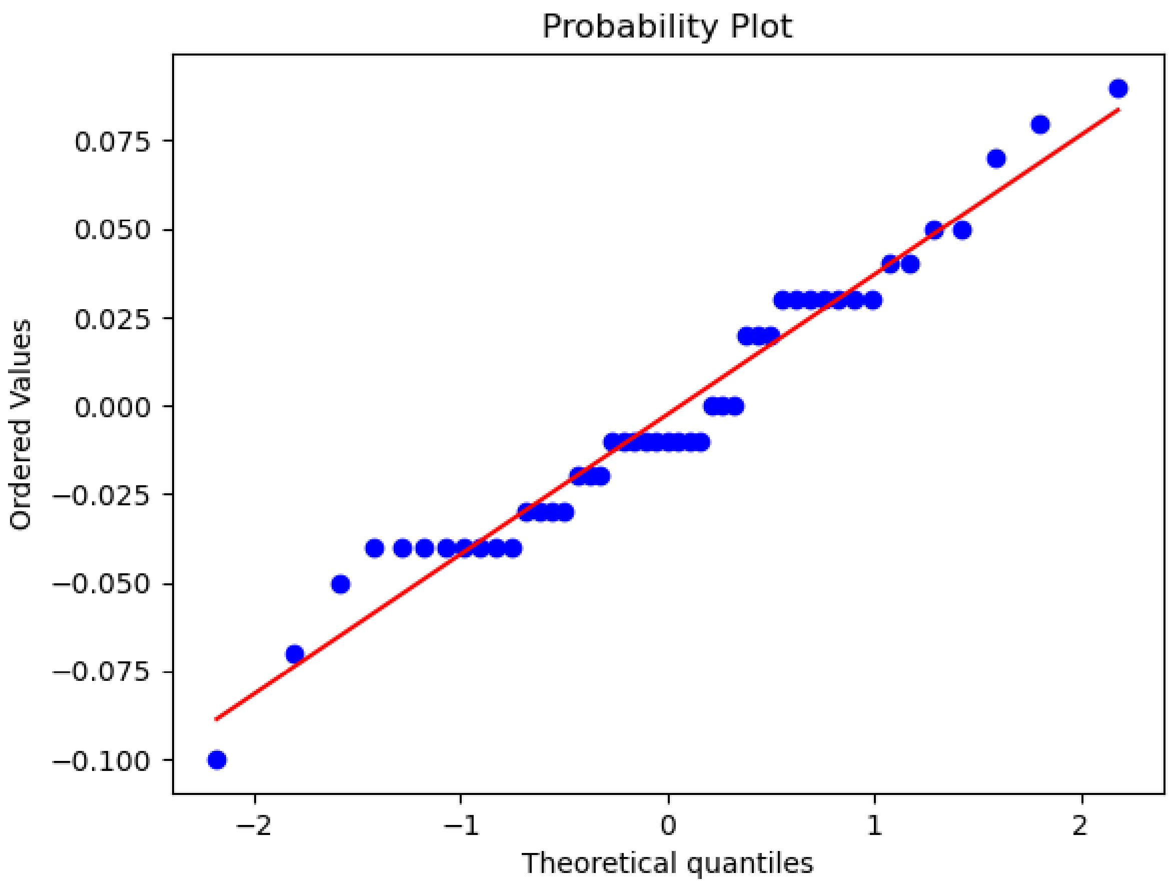
In [ ]:
```
from statsmodels.tsa.stattools import acf

n = len(data['diff1'][1:])

#autocorrelation function
[acf_values, lbq, _] = acf(data['diff1'][1:], nlags = int(np.sqrt(n)), qstat=True,

#Bartlett's test at lag 1
alpha = 0.05
lag_test = 1
rk = acf_values[lag_test]
z_alpha2 = stats.norm.ppf(1-alpha/2)
print('Test statistic rk = %f' % rk)
print('Rejection region starts at %f' % (z_alpha2/np.sqrt(n)))
```

```
Test statistic rk = 0.288387
Rejection region starts at 0.285890
```

This is a typical borderline situation. At 95% confidence, we should reject the randomness assumption. In this case, we may proceed by fitting an AR(1) model on the data trasnformed by applying the difference operator, and check if residuals are NID. If residuals are NID, the resulting model would be an ARIMA(1,1,0) Another option is to accept the transformed data are barely random. If we follow this second path, we shall verify if they are also normal. Let's follow this second route (we'll see examples of ARIMA(1,1,0) models later on).

In [ ]:
```
# Perform the Shapiro-Wilk test
_, pval_SW = stats.shapiro(data['diff1'][1:])
print('Shapiro-Wilk test p-value = %.3f' % pval_SW)

# Plot the qqplot
stats.probplot(data['diff1'][1:], dist="norm", plot=plt)
plt.show()
```

```
Shapiro-Wilk test p-value = 0.188
```



> The process is modeled as a RANDOM WALK
>
> **Random Walk**:
>
> - $Y_t = Y_{t-1} + \epsilon_t$

Point b

It is the best model? Is this the best one? We can also try an AR(1) model directly on the original data (no application of the difference operator)

In [ ]:
```
#calculate a regression model with constant and lag1
import statsmodels.api as sm
import qda

x = data['lag1'][1:]
x = sm.add_constant(data['lag1'][1:]) # this command is used to consider a constan
y = data['EXE1'][1:]
model = sm.OLS(y, x).fit()

qda.summary(model)
```

```
REGRESSION EQUATION
-------------------
EXE1 =  + 0.344 const + 0.960 lag1

COEFFICIENTS
------------
 Term   Coef  SE Coef  T-Value    P-Value
const 0.3439   0.3510   0.9799 3.3238e-01
 lag1 0.9597   0.0408  23.5067 6.5730e-27

MODEL SUMMARY
-------------
     S   R-sq  R-sq(adj)
0.0392 0.9247      0.923

ANALYSIS OF VARIANCE
---------------------
    Source   DF  Adj SS  Adj MS  F-Value    P-Value
Regression  1.0  0.8497  0.8497 552.5667 6.5730e-27
     const  1.0  0.0015  0.0015   0.9602 3.3238e-01
      lag1  1.0  0.8497  0.8497 552.5667 6.5730e-27
     Error 45.0  0.0692  0.0015      NaN        NaN
     Total 46.0  0.9189     NaN      NaN        NaN
```

> Note that the p-value of the constant term is 0.332.
>
> Let's re-calculate the model removing the constant term

```
In [ ]:
x = data['lag1'][1:]
y = data['EXE1'][1:]
model = sm.OLS(y, x).fit()
qda.summary(model)
```

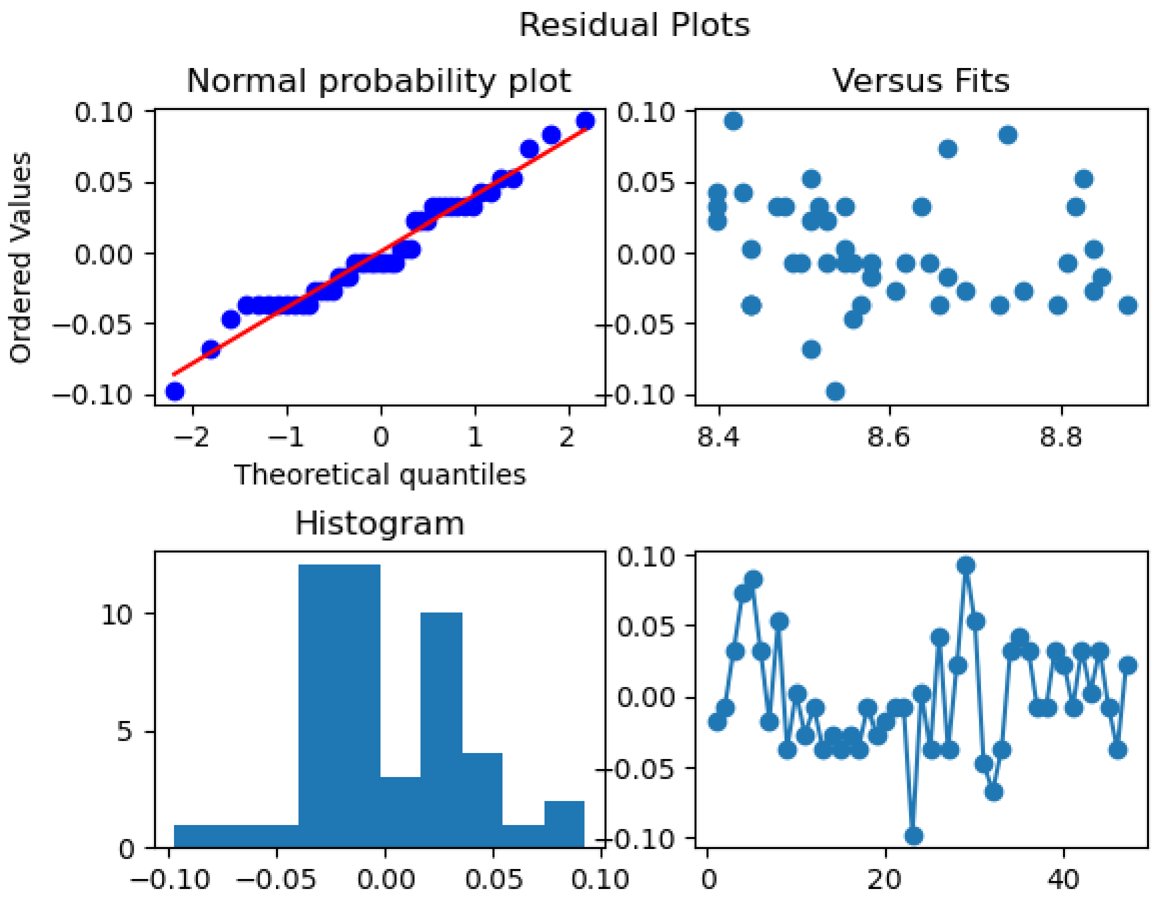
```
REGRESSION EQUATION
-------------------
EXE1 =  + 1.000 lag1

COEFFICIENTS
------------
Term   Coef  SE Coef  T-Value     P-Value
lag1 0.9997   0.0007 1503.228 1.4733e-109

MODEL SUMMARY
-------------
     S  R-sq  R-sq(adj)
0.0392   1.0        1.0

ANALYSIS OF VARIANCE
---------------------
    Source   DF    Adj SS    Adj MS    F-Value     P-Value
Regression  1.0 3471.6382 3471.6382 2.2597e+06 1.4733e-109
      lag1  1.0 3471.6382 3471.6382 2.2597e+06 1.4733e-109
     Error 46.0    0.0707    0.0015        NaN         NaN
     Total 47.0 3471.7089       NaN        NaN         NaN
```

> We have found again the random walk model!
>
> $$EXE1 = 0.9997 \cdot lag1$$
>
> Let's check assumptions on residuals:

> - Normality
> - Time independence

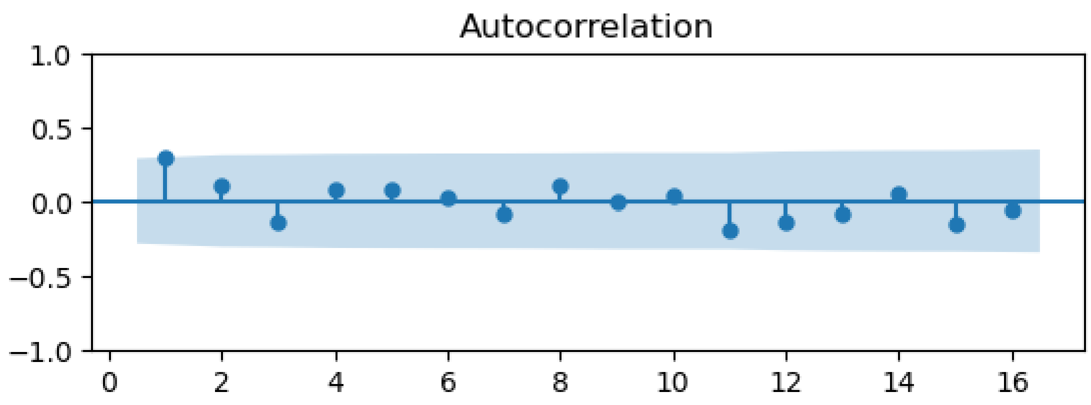
In [ ]:
```
fig, axs = plt.subplots(2, 2)
fig.suptitle('Residual Plots')
stats.probplot(model.resid, dist="norm", plot=axs[0,0])
axs[0,0].set_title('Normal probability plot')
axs[0,1].scatter(model.fittedvalues, model.resid)
axs[0,1].set_title('Versus Fits')
fig.subplots_adjust(hspace=0.5)
axs[1,0].hist(model.resid)
axs[1,0].set_title('Histogram')
axs[1,1].plot(np.arange(1, len(model.resid)+1), model.resid, 'o-')
_, pval_SW_res = stats.shapiro(model.resid)
print('Shapiro-Wilk test p-value on the residuals = %.3f' % pval_SW_res)
```

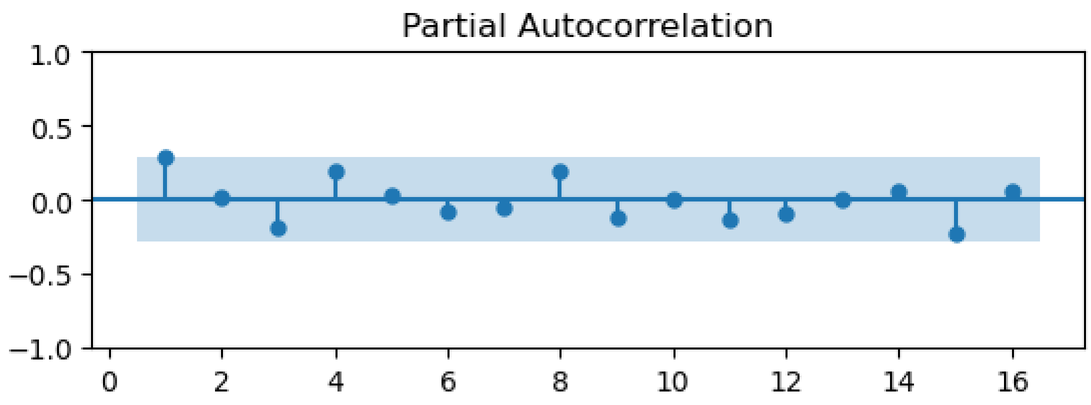
```
Shapiro-Wilk test p-value on the residuals = 0.192
```



In [ ]:
```
_, pval_runs_res = runstest_1samp(model.resid, correction=False)
print('Runs test p-value on the residuals = {:.3f}'.format(pval_runs_res))
fig, ax = plt.subplots(2, 1)
sgt.plot_acf(model.resid, lags = int(len(data)/3), zero=False, ax=ax[0])
fig.subplots_adjust(hspace=0.5)
sgt.plot_pacf(model.resid, lags = int(len(data)/3), zero=False, ax=ax[1],
            method = 'ywm')
plt.show()
```

```
Runs test p-value on the residuals = 0.230
```





Let's check autocorrelation at lag 1 with Bartlett. At 5% significance level:

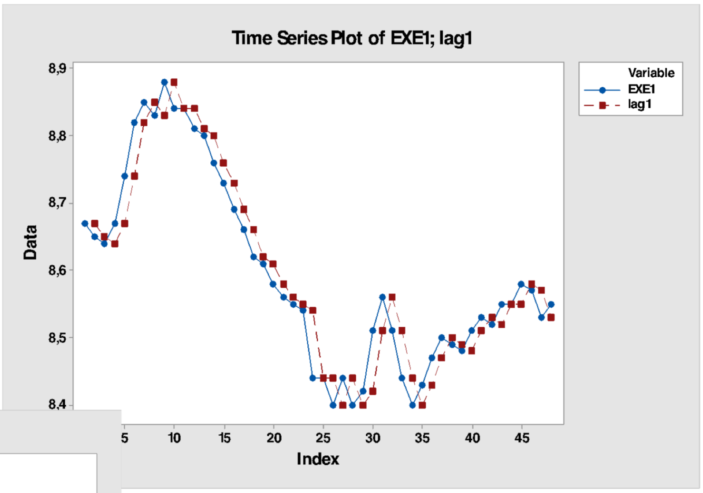
In [ ]:
```
from statsmodels.tsa.stattools import acf

#autocorrelation function
[acf_value, lbq, _] = acf(model.resid, nlags = int(len(model.resid)/3) , qstat=Tru

#Bartlett's test at lag 1
lag_test=1
rk=abs(acf_value[lag_test])
alpha = 0.05 # significance level
z_alpha2 = stats.norm.ppf(1-alpha/2)
print('Test statistic rk = %f' % rk)
print('Rejection region starts at %f' % (z_alpha2/np.sqrt(len(model.resid))))
```

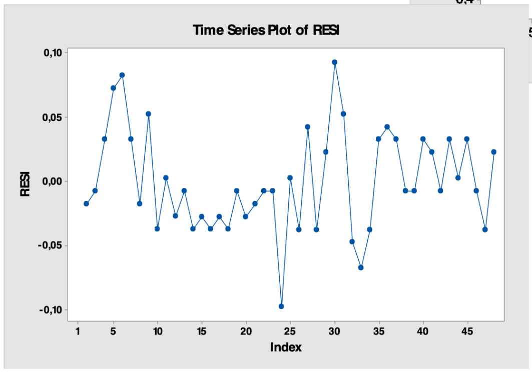
```
Test statistic rk = 0.288386
Rejection region starts at 0.285890
The null hypothesis is rejected
```

Same result as before (indeed the model is the same, i.e., random walk) The Bartlett's null hypothesis is barely rejected at 95% confidence but not rejected at 99% confidence.

Exercise 1 (solution)

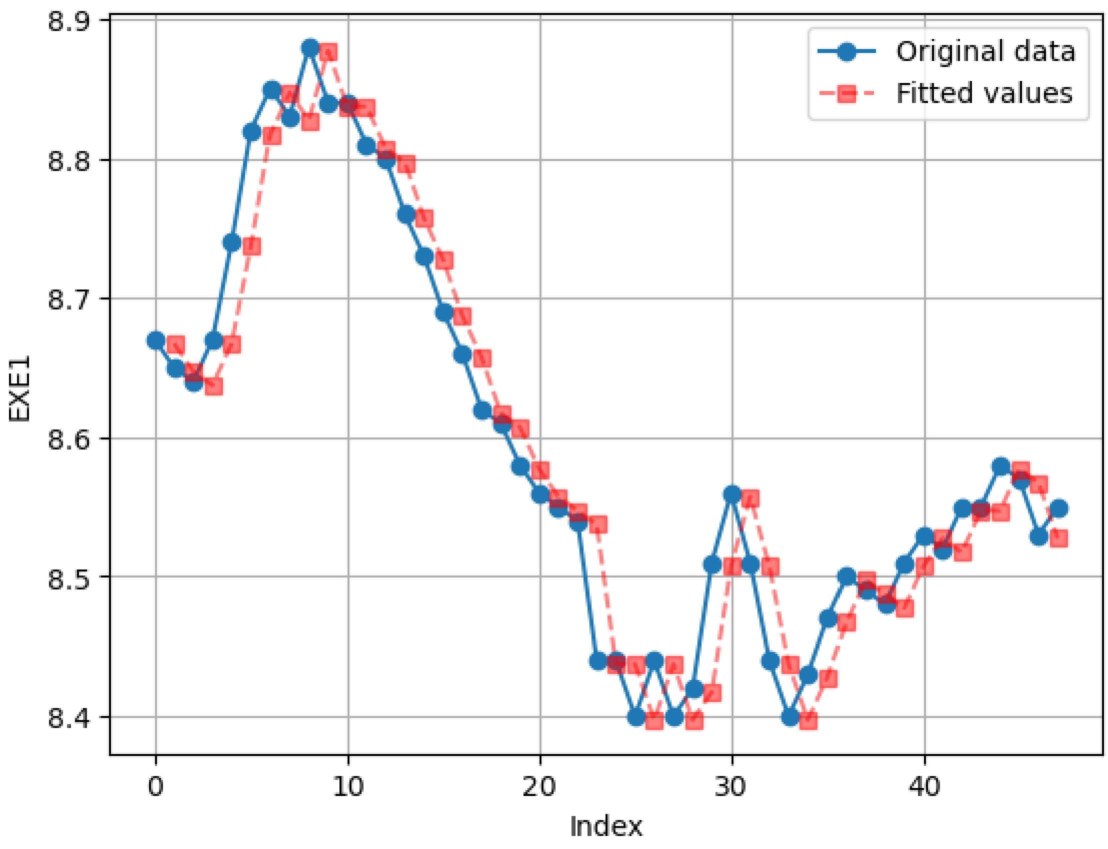
In the random walk model, the lagged variable Y_{t-1} is the estimate of Y_t





This is the time series plot of the residuals: notice that they range from -0.1 and 0.1, which means that their variability is very small.
This is why SSe is very close to 0.

```
In [ ]: plt.plot(data['EXE1'], 'o-', label='Original data')
        plt.xlabel('Index')
        plt.ylabel('EXE1')
        plt.plot(model.fittedvalues, 's--', color='red', label='Fitted values', alpha=0.5)
        plt.legend()
        plt.grid()
        plt.show()
```



Try at home

Try to fit an ARIMA(1,1,0) on this time series and verify if the model is appropriate or not