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A Geometric Framework for Perturbation Theory in Field Theories

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Abstract

In this thesis we will introduce calculus of variations on composite bundles and its applications in the study of perturbation theory in field theories.
Composite bundles can be thought as ladder of regular bundles. Hence, the category of composite bundles extends the category of bundles. As a consequence, the ordinary calculus of variations on bundles can be extended to this new category.
We will use composite calculus of variations to get from a variational point of view the linearised equations associates to a given Lagrangian. This result can be achieved by considering an auxiliary Lagrangian on a specific composite bundle, whose composite variation will give the linearised equations together with field equations. At the end of this work we will also provide a physical example of these computations, in fact we are going to compute linearised Einstein equations starting from the Hilbert Lagrangian.
Moreover, we will be able to check our results by fixing specific background metrics in order to get both the gravitational waves equations and the linearised Friedman equations.

This work will be divided into four chapters, each chapter will be propaedeutic for the next one.
In the first chapter we will recall some basic mathematical physics definitions, as fibrations, tensor and differential forms, and ordinary calculus of variations on bundles.
Second chapter will be about the category of composite bundles and how we can extend ordinary calculus of variations to this category. In particular in section one we will introduce composite bundle, composite vector fields and composite Lagrangians.
In section two we will give some examples of composite bundles, in particular considering the bundle of vertical vectors $V(B)$ over a bundle $B \to M$ as a composite bundle. Finally in section 3 we will compute field equations for a given composite Lagrangian.
In chapter three we will introduce one of the most important objects of this work, that is the *Jacobi Lagrangian*. In particular, in section one, we will see how to define this auxiliary Lagrangian from a given first order Lagrangian, and why the composite variation of Jacobi Lagrangian will give linearised equations. Sections two will extend all previous results in the case of a second order Lagrangian.
Finally, in chapter four we will give a concrete example of linearised equations, starting from the Hilbert Lagrangian. In section one we will get linearised Einstein equations from the Jacobi Lagrangian in the case of a generic background metric on the space-time. At the end of this section we will specialise our result in the case of Minkowsky background, to get the gravitational waves equation. In section two, we will fix as background metric the Friedman Robertson Walker metric, in order to get the linearised Friedman equations.

Sommario

In questa tesi introdurremo il calcolo delle variazioni su fibrati compositi e vedremo come questo formalismo può essere utilizzato per studiare le teorie delle perturbazioni in teorie di campo.
I fibrati compositi possono essere visti come delle ”scale” di normali fibrati, di conseguenza la categoria dei fibrati compositi estende la cateria dei fibrati: è quindi possibile estendere il calcolo delle variazioni ordinario su fibrati a questa nuova categoria.
Utilizzeremo il calcolo delle variazioni su fibrati compositi per ottenere da un punto di vista variazionale le equazioni di campo linearizzate per una certa Lagrangiana. Questo risultato può essere ottenuto considerando una Lagrangiana ausiliaria, su uno specifico fibrato composito, la cui variazione composita restituisce le equazioni di campo e le equazioni linearizzate.
Alla fine di questo lavoro forniremo anche un esempio concreto di questi concetti: calcoleremo le equazioni di Einstein linearizzate a partire dalla Lagrangiana di Hilbert per un background generico.
Infine, saremo anche in grado di controllare i risultati ottenuti fissando delle specifiche metriche di background per ottenere prima l'equazione delle onde gravitazionali, e poi le equazioni di Friedman linearizzate.

Questa tesi è divisa in quattro capitoli, ognuno dei quali è propedeutico per il successivo.
Nel primo capitolo richiameremo alcune nozioni basilari di fisica matematica, come fibrazioni, tensori, forme differenziali ed il calcolo delle variazioni su fibrati.
Nel secondo capitolo definiremo la categoria dei fibrati compositi e vedremo come è possibile estendere a questa categoria il calcolo delle variazioni.
In particolare, nella sezione uno definiremo fibrati compositi, campi vettoriali composite e Lagrangiane composite.
Nella sezione due vedremo alcuni esempi di fibrati compositi, come ad esempio il fibrato dei vettori verticali $V(B) \to B \to M$. Infine, nella sezione tre ricaveremo le equazioni di campo per una Lagrangiana composita.
Nel terzo capitolo introdurremo uno degli oggetti più importanti di questo lavoro, la *Lagrangiana di Jacobi.*
In particolare nella prima sezione vedremo come definire questa Lagrangiana ausiliaria per una data Lagrangiana del prim'ordine e per quale motivo la variazione composita della Lagrangiana di Jacobi fornisce le equazioni di campo linearizzate. Nella sezione due estenderemo i risultati della sezione precedente per una Lagrangiana del second'ordine.

Infine, nel quarto capitolo daremo un esempio concreto di equazioni linearizzate, costruite a partire dalla Lagrangiana di Hilbert.
Nella sezione uno otterremo le equazioni linearizzate a partire dalla Lagrangiana di Jacobi: queste equazioni saranno scritte per una generica metrica di background sullo spazio-tempo che sia anche soluzione delle equazioni di Einstein.
Alla fine di questa sezione fisseremo come background la metrica di Minkowski e faremo vedere come le nostre equazioni si riducano all'equazione delle onde gravitazionali.
Nella sezione seguente fisseremo come background la metrica di Friedman Robertson Walker (nel caso omogeneo, isotropo e spazialmente piatto) e mostreremo che in questo caso le nostre equazioni coincidono con le equazioni di Friedman linearizzate.

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Contents

Chapter 1

General Introduction

In this sections we will recall some standard definitions with the aim to introduce calculus of variations on smooth fiber bundles.
In order to do that, we will introduce smooth bundles, tensors, and hence differential forms on smooth manifold. Finally we will introduce jet bundles.
With these ingredients we will be able to state calculus of variations on bundles. In fact, given a Lagrangian (a differential form on the jet prolongation of the total space of our bundle), we will define its Euler-Lagrange equations as the Kernel of an operator that defines a submanifold on a certain jet prolongation of the total space of the bundle.
In this way we are able to characterise geometric properties of our variational problem, without having to consider the functional space of solutions.
This approach is not better than the standard ”functional” one, they both have advantages and disadvantages. For example with the functional approach it is relatively easy to study regularity and existence of solutions, but it is more difficult to characterise properties that does not depend on the functional space we consider to solve the problem.
On the other side, with ”jet-bundles” approach it is relatively easy to study geometric properties of our problem, but is more difficult to study regularity and existence of solutions for generic initial conditions.
The choice of the approach, clearly depends on what one is interested to study.

1.1 Smooth fiber bundles

A smooth fiber bundle is a 4-tuple (B, M, π, F) where B, M, and F are smooth manifolds called total space, base, and standard fiber, respectively. $\pi : E \to B$ is a surjective maximal rank map that will be called projection.
In order for B to be a fiber bundle, there must exist a collection $\{(U_\alpha, t_\alpha)\}_{\alpha\in I}$ of diffeomorphisms, called trivialization, such that

$$t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \tag{1.1}$$

where $\{U_\alpha\}$ is a covering of M.
We have set $\pi := p_1 \circ t_\alpha$, where $p_1 : U_\alpha \times F \to U_\alpha$, $p_1(x, f) = x$.

Definition 1. *For all $p \in M$, the preimage $\pi^{-1}(\{p\}) \subset B$ will be called fiber over p.*
One can restrict a local trivialization to the fiber over x, to obtain $t_x : \pi^{-1}(x) \to F$, that must be a diffeomorphism, since t is.

Remark 1. *Given two local trivializations $(U_1, t_1), (U_2, t_2)$, one can define transition functions for $x \in U_{12} := U_1 \cap U_2$ by*

$$(t_{12})_x := (t_1)_x \circ (t_2)_x^{-1} : F \to F \tag{1.2}$$

In particular, $(t_{12})_x \in Diff(F)$

We can now define maps between fiber bundles. In particular, the key feature is to require that those morphisms preserve the fiber structures, this leads to the definition:

Definition 2. *Given two smooth fiber bundles, $\pi : B \to M$ and $\pi' : C \to N$, a fibered morphism is a pair (Φ, ϕ) such that the following diagram commutes:*

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & C \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{\phi} & N \end{array} \tag{1.3}$$

ie. $\pi' \circ \Phi = \phi \circ \pi$

Definition 3. *A section of a fiber bundle (B, M, π, F), is a map $f : M \to B$ such that $\pi \circ f = Id_M$.*
We will call the set of all sections of B as $\Gamma(\pi)$; obviously a bundle can also not allow global sections at all, so $\Gamma(\pi)$ can be empty.

Example 1. ***Existence of non-vanishing global sections.*** *Consider for example the tangent bundle $\pi : TM \to M$ with fiber $\mathbb{R}^n$. A non-vanishing global section would be a non singular vector field. It is clear that not every manifold admits a globally defined everywhere non-zero vector field, in fact we have that a non-zero vector field exists if the Euler characteristic $\chi(M)$ of the manifold is zero.*
If we take for example $M = \mathbb{S}^2$, we have $\chi(\mathbb{S}^2) = 2$, so no global sections of the bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ exists other than the trivial one.

Remark 2. *With this convention, the world sections, refers to global sections, while local sections are explicitly said to be local. Local sections are defined on $U \subset M$ as maps $\sigma : U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = Id_U$.*

As a manifold, the total space of a bundle can be covered with an atlas of fibered coordinates (subjected to a local trivialization). Fibered coordinates are defined considering a fibered chart on the total space B.

Definition 4. *We say that a chart (V, ψ) on B is a fibered chart, if there exists a chart (U, ϕ) on M such that $U = \pi(V)$ and*

$$u^1 = x^1 \circ \pi, \quad u^2 = x^2 \circ \pi, ..., u^n = x^n \circ \pi \tag{1.4}$$

where $\psi = (u^1, ..., u^n, y^1, ..., y^k), y_0 \in V$ and $\phi = (x^1, ..., x^n), \pi(y_0) = x_0 \in U$.

Remark 3. *Given a smooth fiber bundle (B, M, π, F) with a trivialization t, when we define fibered coordinates, we are essentially choosing a chart (x^μ) around $x_0 = \pi(y_0)$ and a chart (y^i) around $t_{x_0}(y_0) \in F$. Then (x^μ, y^i) are good coordinates around $y_0 \in B$.*
On bundles we will always work with fibered coordinates.

From now on we will write definitions in two equivalent ways: one using an intrinsic formulation and another in fibered coordinates.
This is done because in this work we will always use fibered coordinates to work with geometric objects. This approach is motivated, among other things, by the fact that we want to compare our results with Physical ones, that in general are written in local coordinates. Although local coordinates plus transition function description may seem less general than intrinsic one, it is easy to prove that they are equivalent.
The two formulations of each definition are aimed to stress this equivalence.
Given a bundle $B \to M$, the most general transition functions between fibered coordinates are in the form:

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \end{cases} \tag{1.5}$$

In fibered coordinates the projection reads as $\pi : B \to M : (x^\mu, y^i) \to x^\mu$, which is clearly surjective and with maximal rank.
A section will be locally expressed as $\sigma : M \to B : (x^\mu) \mapsto (x^\mu, \sigma^i(x))$. In the intersection of two local trivialization, two local sections glue together iff

$$\sigma'^i(x'(x)) = Y^i(x, \sigma(x)) \tag{1.6}$$

These are called transformation rules of sections.

Another class of objects that will have a very important role are *vector fields* (i.e sections of the bundle $\pi : TB \to B$), in particular we will call the set of all vector fields on B as $\mathfrak{X}(B)$, we have the following:

Definition 5. *A vector field Ξ on B is said to be projectable, if $\forall x \in M$, $T_y\pi(\Xi_y)$ is independent of the choice of $y = \pi^{-1}(x)$.*
The set of projectable vector fields on B will be denoted as $\mathfrak{X}_P(\pi)$

In fibered coordinates, this condition reads as a constraint on the form of the vector field: a vector field is projectable iff it is in the form

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x, y)\partial_i \tag{1.7}$$

where $(\partial_\mu := \frac{\partial}{\partial x^\mu}, \partial_i := \frac{\partial}{\partial y^i})$ is the natural basis of TB.

Remark 4. *The projection π is not injective, so one cannot push-forward all vector fields. However, it can happen, for specific vector fields that the tangent map $T\pi$ is constant on points which are projected on the same point $x \in M$, those fields are $\pi-$projectable, and one have $T\pi(\Xi) = \xi^\mu(x)\partial_\mu$ which are constant along the fiber.*

Definition 6. *Let $\pi : B \to M$ a fiber bundle, the vertical bundle $VB \to B$is the sub-bundle of TB defined as*

$$VB := \{\xi \in TB : \pi_*\xi = 0\} \tag{1.8}$$

Its fibers, $V_pB := (VB)_p \subset T_pB$ will be called vertical subspaces.
It is not difficult to show that $V_pE = T_p(E_{\pi(p)})$, so the vertical sub-bundle is the set of all vectors in TB that are tangent to any fiber.
Moreover it is possible to prove that if $\pi : B \to M$ is a smooth fiber bundle, then $VB \to B$ is a smooth real vector bundle.

Remark 5. *As we will see later, we can define vertical vector fields as section of the bundle $VB \to B$, but we can also state another definition, based on tangent maps.*

Definition 7. *A vector field Ξ on B is said to be vertical if is in the kernel of the tangent map of the projection, i.e. $T\pi(\Xi) = 0$.*
Accordingly in fibered coordinated one must have

$$\Xi = \xi^i(x, y)\partial_i \tag{1.9}$$

We will denote the set of vertical vector fields on B as $\mathfrak{X}_V(\pi)$

Theorem 1. *Given a bundle $B \to M$ we have the following:*

1. *The set of vertical fields on B is a subalgebra $\mathfrak{X}_V(\pi) \subset \mathfrak{X}(\pi)$*

2. *The local expression of projectable and vertical vector fields is preserved by transition functions (1.5).*

Proof. Both points are trivial, in fact, the first point follows from the fact that any vertical vector field projects to the zero vector field of M.
The second point can be easily proven considering the bundle $V(B) \to B$ of vertical vector fields over B (sections of this bundle are vertical vector fields on B). Transition functions on $V(B)$ will be

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \\ \xi'^i = J^i_j(x, y)\xi^j \end{cases} \tag{1.10}$$

where we set $J^i_j := \frac{\partial Y^i(x,y)}{\partial y^j}$. Now it is clear that the form of vertical vector fields is preserved from transition function. □

Remark 6. *It is possible to prove that $T(\cdot)$ is a covariant functor from the category of manifolds to the category of vector bundles.*
In fact it associates a vector bundle TM to any manifold, and a fiber morphism $T\Phi$ to any smooth map Φ, that is defines as follows.
Given a smooth map $\Phi : M \to N$ and a tangent vector $(x, v) \in TM$, since $v = [\gamma]$ is an equivalence class of curves based at x we can define a family of curves $\Phi \circ \gamma$ based at $\Phi(x) = y \in N$. Such a map on curves is compatible with the equivalence relation and induces a map in the quotient

$$T\Phi : TM \to TN : (x, v = [\gamma]) \mapsto (\Phi(x), [\Phi \circ \gamma]) \tag{1.11}$$

that will be called tangent map.

Remark 7. *Being $T(\cdot)$ a covariant functor, means that we can iterate is.*
In this way we can consider the bundle $TTM \to M$, although it is possible to define this bundle in an intrinsic way, this is not the aim of this work.
We will only remark that coordinates (x^μ) on M induce a coordinate system on TTM as $(x^\mu, v^\nu, \dot{x}^\alpha, \dot{v}^\beta)$.
Moreover, we can identify a sub-bundle of TTM, that we will call T^2M, with the constraint $v^\alpha = \dot{x}^\alpha$ (it is possible to describe this identification in an intrinsic way using swap maps, but again, this is not the aim of this work).
The bundle $T^2M \to TM \to M$ will be relevant in the next chapter, in fact we will use it as an example of composite bundle.

Remark 8. *Another important bundle, that will be useful in chapter four, is the bundle of Lorentzian metrics over a manifold M.*
This bundle will be called $Lor(M) \to M$. Fibered coordinates on $Lor(M)$ will be $(x^\alpha, g_{\mu\nu})$, and sections are in one-to-one correspondence with global Lorentzian metrics on the space-time M.
Lorentzian metrics are a special case of Pseudo-Riemannian metrics where the signature of the metric is $(1, m-1)$, where $m = dim(M)$.

1.2 Tensors and differential forms

We now want to introduce *differential forms*, that in some sense can be considered as the dual analogous of vector fields.
We introduce them as a specific case of a more general class of objects: tensors.
We first define tensors on vector spaces, and then on a manifold using the fact that the tangent space on each point of the manifold is a vector space.

Definition 8. *Given a vector space V, a covariant $k-$tensor on V is a real valued multilinear function of k elements of V*

$$T : \underbrace{V \times ... \times V}_{k\text{-}times} \to \mathbb{R} \tag{1.12}$$

The set of k covariant tensor on V will be denoted as $T^k(V)$, and the integer k will be called rank of the tensor.

We can now define the main operation between tensor: *tensor product.*

Definition 9. *Given a rank k tensor T and a rank l tensor S, we can define their tensor product as*

$$T \otimes S : \underbrace{V \times ... \times V}_{k+l\ times} \to \mathbb{R} \tag{1.13}$$

by $T \otimes S(X_1, .., X_{k+l}) = T(X_1, ..., X_k)S(X_{k+1}, ..., X_{k+l})$, being T and S multilinear, $T \otimes S$ will be a $(k+l)-$covariant tensor.

Having defined tensor product allows us to define a basis for the space of all $k-$tensors.

Theorem 2. *Let V be a real vector space of dimension n, let (E_i) be any basis for V, and let (ϵ^i) be the dual basis on V^*.*
The set of all $k-$tensors in the form

$$\epsilon^{i_1} \otimes \epsilon^{i_2} \otimes ... \otimes \epsilon^{i_k} \tag{1.14}$$

for $1 \leq i_1, ..., i_k \leq n$ is a basis for $T^k(V)$, which therefore has dimension n^k.

Proof. Let $\mathcal{B} := \{\epsilon^{i_1} \otimes \epsilon^{i_2} \otimes ... \otimes \epsilon^{i_k}, 1 \leq i_1, ..., i_k \leq n\}$. We need to show that $\mathcal{B}$ is independent and spawns $T^k(V)$.
Let us take an arbitrary $T \in T^k(V)$, for any $k-$tuple $(i_1, ...i_k)$ such that $1 \leq i_j \leq n$, we define the numbers $T_{i_1,...,i_k}$ as

$$T_{i_1,...,i_k} = T(E_{i_1}, ..., E_{i_k}) \tag{1.15}$$

We compute

$$\begin{aligned} T_{i_1 \ldots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \left(E_{j_1}, \ldots, E_{j_k}\right) &= T_{i_1 \ldots i_k} \varepsilon^{i_1} \left(E_{j_1}\right) \cdots \varepsilon^{i_k} \left(E_{j_k}\right) \\ &= T_{i_1 \ldots i_k} \delta_{j_1}^{i_1} \cdots \delta_{j_k}^{i_k} \\ &= T_{j_1 \ldots j_k} \\ &= T\left(E_{j_1}, \ldots, E_{j_k}\right) \end{aligned} \tag{1.16}$$

Then, by multilinearity, a tensor is determined by its action on sequences of basis vectors, so this proves that $\mathcal{B}$ spans $T^k(V)$.
To show that $\mathcal{B}$ is independent, suppose some linear combination equals zero,

$$T_{i_1,...,i_k} \epsilon^{i_1} \otimes ... \otimes \epsilon^{i_k} = 0 \tag{1.17}$$

Apply this to any sequence $(E_{j_1}, ..., E_{j_k})$ of basis vector.
By the same computation as above, this implies that each coefficient $T_{i_1,...,i_k}$ is zero. Thus the only linear combination of elements of $\mathcal{B}$ that sums to zero is the trivial one. $\square$

Remark 9. *Because every $k-$covariant tensor can be written as a linear combination of tensor product of covectors ($1-$covariant tensor), it is suggestive to write*

$$T^k(V) = V^* \otimes ... \otimes V^* \tag{1.18}$$

This notation can be fully understood if we define the tensor product in a more general setting: the abstract tensor product of vector space. However, the formal definition of this generalisation is not the aim of this work, so we will review only the main idea.
Given two vector spaces V, W, we can construct a vector space $V \otimes W$ that consists of linear combinations of objects of the form $v \otimes w$ for every $v \in V$ and $w \in W$.

Remark 10. *Using this construction, it is easy to generalise the notion of covariant tensor on a vector space as follows.*
We define the space of contravariant tensor of rank k to be

$$T_k(V) = \underbrace{V \otimes ... \otimes V}_{k\text{-times}} \tag{1.19}$$

Because of the canonical identification $V \simeq V^{**}$, an element of $T_k(V)$ can be canonically identified as a multilinear function from $V^* \times ... \times V^*$ to $\mathbb{R}$.
Of course we can also consider tensors that are in part covariant and in part contravariant, those will be called mixed tensors. The set of all mixed tensors of type (k, l) will be denoted as $T^k_l(V)$, and those tensors can be thought as multilinear applications from

$$\underbrace{V \times ... \times}_{l-times} \times \underbrace{V^* ... \times V^*}_{k-times} \to \mathbb{R} \tag{1.20}$$

Tensors on Manifolds

Having defined tensor fields on a vector space, it is easy to extend the previous definitions to manifolds.

Definition 10. *Let M be a smooth manifold, we define the bundle of $k-$covariant tensors on M as*

$$T^k(M) = \amalg_{p \in M} T^k(T_pM) \tag{1.21}$$

In the same way, we can define the bundle of $k-$contravariant tensors on M as

$$T_k(M) = \amalg_{p \in M} T_k(T_pM) \tag{1.22}$$

and the bundle of mixed tensors of type (k, l) as

$$T^k_l(M) = \amalg_{p \in M} T^k_l(T_pM) \tag{1.23}$$

One can easily show that these are smooth vector bundles over M.
A section of these bundles will be called *(covariant, contravariant or mixed) tensor field.* In any local coordinates (x^i) of M, these fields can be written as

$$\sigma = \sigma^{j_1,...,j_l}_{i_1,...,i_k} dx^{i_1} \otimes ... \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes ... \otimes \frac{\partial}{\partial x^{j_l}} \tag{1.24}$$

This result is stated for mixed tensors and can be easily specified for covariant and contravariant tensors.

Symmetric and alternating tensors

Definition 11. *Given a vector space V, a covariant $k-$tensor is said to be symmetric if its value is unchanged by interchanging any pair of arguments*

$$T(X_1, ..., X_i, ..., X_j, ..., X_k) = T(X_1, ..., X_j, ..., X_i, ...X_k) \tag{1.25}$$

We denote the set of $k-$covariant symmetric tensors as $\Sigma^k(V)$

Clearly $\Sigma^k(V)$ is a vector subspace of $T^k(V)$, so we can define a projection:

$$Sym : T^k(V) \to \Sigma^k(V) \tag{1.26}$$

The projection is defined in this way, first let us consider the symmetric group of k elements, S_k (i.e. the group of all permutations of k elements), then, given a $k-$tensor, we can define a new tensor T^σ as

$$T^\sigma(X_1, ..., X_k) = T(X_{\sigma(1)}, ..., X_{\sigma(k)}) \quad \forall \sigma \in S_k \tag{1.27}$$

Then, the projection will be defined as

$$SymT = \frac{1}{k!} \sum_{\sigma \in S_k} T^\sigma \tag{1.28}$$

In general, the product of two symmetric tensors is not a symmetric tensor, so we can define a new product that is always symmetric, that is:

Definition 12. *Given two symmetric tensors, $T \in \Sigma^k(V)$, $S\ in \Sigma^l(V)$ we can define the symmetric product as*

$$T \odot S := Sym(T \otimes S) \tag{1.29}$$

That is a $k + l$ rank tensor.

More explicitly, the symmetric product is defined as:

$$\begin{aligned} & S \odot T\left(X_1, \ldots, X_{k+l}\right)= \\ & =\frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} S\left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\right) T\left(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}\right) \end{aligned} \tag{1.30}$$

What can we say about tensors that change sign when we permute their arguments?

Definition 13. *A covariant $k-$ tensor is said to be alternating if it has the property*

$$T(X_1, ..., X_i, ..., X_j, ..., X_k) = -T(X_1, ..., X_j, ..., X_i, ...X_k) \tag{1.31}$$

We have a useful lemma about that can help us in finding if a tensor is alternating or not.

Lemma 1. *Suppose Ω is a covariant $k-$tensor on a vector space V with the property that $\Omega(X_1, ..., X_k) = 0$ whenever $(X_1, ..., X_k)$ are linearly independent. Then Ω is alternating.*

Proof. The hypothesis implies, in particular, that Ω gives the value zero whenever two of its arguments are the same. This in turn implies:

$$\begin{aligned} 0 & =\Omega\left(X_1, \ldots, X_i+X_j, \ldots, X_i+X_j, \ldots, X_n\right) \\ & =\Omega\left(X_1, \ldots, X_i, \ldots, X_i, \ldots, X_n\right)+\Omega\left(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_n\right) \\ & +\Omega\left(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_n\right)+\Omega\left(X_1, \ldots, X_j, \ldots, X_j, \ldots, X_n\right) \\ & =\Omega\left(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_n\right)+\Omega\left(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_n\right) \end{aligned} \tag{1.32}$$

Then Ω is alternating. □

We will indicate the vector subspace of all covariant $k-$alternating tensors as $\Lambda^k(V)$. Also in this case we have a natural projection from $T^k(V)$ to the subspace of alternating tensors, defined as

Definition 14. *The alternating projection is a map* $Alt : T^k(V) \to \Lambda^k(V)$*, defined as*

$$AltT = \frac{1}{k!} \sum_{\sigma \in S_k} (sgn(\sigma))T(X_{\sigma(1)}, ..., X_{\sigma(k)}) \tag{1.33}$$

The natural question that could arise is: can we define a product operation for alternating tensors? The answer is yes, and can be defined in a similar way of the symmetric product.

Definition 15. *Given two alternating tensors* $T \in \Lambda^k(V), S \in \Lambda^l(V)$*, we define the wedge product of* T *and* S *as*

$$T \wedge S := \frac{(k+l)!}{k!l!} Alt(T \otimes S) \tag{1.34}$$

With this definition, the wedge product is bilinear, associative and anticommutative.

Differential forms on Manifolds

We are now able to define differential forms on a manifold.
Firstly we introduce the notion of multi-index, that will be very useful.

Definition 16. *A multi-index is a m-tuple* I *of natural numbers. The components of* I *will be denoted as* $I(j)$*;* $1 \leq j \leq m$*. Sum and difference of multi-indexes are defined by components.*
Moreover, we have $|I| := \sum_{i=1}^m I(i)$ *and* $I! := \prod_{i=1}^m (I(i))!$

Definition 17. *The symbol* $\frac{\partial^{|I|}}{\partial x^I}$ *is defined by:*

$$\frac{\partial^{|I|}}{\partial x^I} := \prod_{i=1}^m \left(\frac{\partial}{\partial x^i} \right)^{I(i)} \tag{1.35}$$

If $|I| = 0$ *then* $\frac{\partial^{|I|}}{\partial x^I}$ *is the identity operator.*

Definition 18. *Given a smooth manifold* M*, the subset of* $T^k(M)$ *consisting of alternating tensors of rank k, will be denoted as* $\Lambda^k M$*. We have*

$$\Lambda^k M = \amalg_{p \in M} \Lambda^k(T_p M) \tag{1.36}$$

A smooth section of $\Lambda^k(M)$ is called *differential* $k-$ *form*, or just a $k-$form.
This is just a smooth tensor field whose value at each point is an alternating tensor. In any coordinate chart, a $k-$form can be written locally as

$$\omega = \sum_I \omega_I dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{1.37}$$

where the coeffcients ω_I are smooth functions defined on the coordinate neighbourhood We also have

$$dx^{i_1} \wedge ... \wedge dx^{i_k} (\frac{\partial}{\partial x^{j_1}}, ..., \frac{\partial}{\partial x^{j_k}}) = \delta^I_J \tag{1.38}$$

Moreover, the wedge product of a $k-$form with an $l-$form is a $(k+l)-$form.
A $0-$form is just a real valued function, and we interpret the wedge product $f \wedge \eta$ of a $0-$form f with a $k-$form η, to mean the product $f\eta$.

Remark 11. *Given a $m-$dimensional manifold and an integer $k \leq m$ we have seen that $dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}$ is a basis for $k-$forms on M.*
When $k = m$ the space of $m-$forms is generated by a single element, that we will call

$$d\sigma := dx^1 \wedge ... \wedge dx^m \tag{1.39}$$

this means that any $m-$form can be written as $\omega = \omega(x)d\sigma$.
If we look to transformation laws of $d\sigma$ we get

$$d\sigma' = J^1_{\nu_1} \dots J^1_{\nu_m} dx^{\nu_1} \wedge \dots dx^{\nu_m} = J^1_{\nu_1} \dots J^1_{\nu_m} \epsilon^{\nu_1 \dots \nu_m} d\sigma = \det(J) d\sigma \tag{1.40}$$

Accordingly, the coefficient transforms as a scalar density of weight 1, i.e.

$$\omega' = det(\bar{J})\omega \tag{1.41}$$

where we have defined $J^\mu_\nu := \frac{\partial x'^\mu}{\partial x^\nu}(x)$

Exterior derivative

In this section, we define a natural differential operator on forms, called the *exterior derivative*. It is a generalization of the differential of a function. To give some idea where the motivation for the exterior derivative comes from, let us recall that not all smooth covector fields are differentials of functions: Given a $k-$form ω, a necessary condition for the existence of a function f such that $\omega = df$ is that ω be closed, which means that it satisfies

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0 \tag{1.42}$$

in every coordinate system.
Since this is a coordinate-independent property, one might hope to find a more invariant way to express it. The key is that the expression in (1.42) is antisymmetric in the indices i, j, so they can be interpreted as the component of a $2-$form, we will define this two form as

$$d\omega = \sum_{i \leq j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j \tag{1.43}$$

So it follows that ω is closed if and only if $d\omega = 0$.
This formula has a significant generalization to differential forms of all degrees. For any manifold, we will show that there is a differential operator $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$, satisfying $d(d\omega) = 0$ for all ω.
From this it will follow that a form ω can be written as $d\eta$ for some η if and only if $d\omega = 0$.
The definition of the operator d in coordinates is straightforward:

$$d \left(\sum_I \omega_I dx^I \right) = \sum_I d\omega_I \wedge dx^I \tag{1.44}$$

where we defined $dx^I := dx^{i_1} \wedge ... \wedge dx^{i_k}$, and $d\omega_I$ is just the differential of the function ω_I.
In a more specific way, we have

$$d \left({\sum_I}' \omega_I dx^{i_1} \wedge \dots \wedge dx^{i_k} \right) = \sum_I \sum_i \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \tag{1.45}$$

Remark 12. *When ω is a 1 -form, the previous condition becomes*

$$d\left(\omega_j dx^j\right) = \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j \tag{1.46}$$

The formal definition of the operator d is given in the following theorem, that is a standard theorem of differential geometry

Theorem 3. *On any smooth manifold M, there is a unique linear map $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ for each $k \geq 0$, satisfying the following properties:*

- *If f is a smooth function (a $0-$form), then df is the differential of f, defined as usual*

$$df(X) = X(f) \tag{1.47}$$

  for every vector field X on M

- *If $\omega \in \Lambda^k(M)$, and $\eta \in \Lambda^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \tag{1.48}$$

  for every $k-$form ω, $d(d\omega) = 0$

1.3 Jet Bundles

In order to study Euler-Lagrange equations for a given Lagrangian, we need to define Jet Bundles, to set up variational calculus.
In the following we always assume to work on a bundle (B, π, M) with $dim(M) = m$

Lemma 2. *Let (B, π, M) be a bundle, and let $p \in M$. Suppose that $\phi, \psi \in \Gamma_p(\pi)$ satisfy $\phi(p) = \psi(p)$. Let (x^α, u^i) and (y^α, v^i) be two adapted coordinates systems around $\phi(p)$, suppose also that*

$$\left.\frac{\partial^{|I|}(u^i \circ \phi)}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}(u^i \circ \psi)}{\partial x^I}\right|_p \tag{1.49}$$

for $1 \leq i \leq n$ and for every multi-index I with $1 \leq |I| \leq k$. Then

$$\left.\frac{\partial^{|J|}(v^b \circ \phi)}{\partial y^I}\right|_p = \left.\frac{\partial^{|J|}(v^b \circ \psi)}{\partial y^I}\right|_p \tag{1.50}$$

for $1 \leq b \leq n$ and for every multi-index J with $1 \leq |J| \leq k$

Proof. The first part of the proof uses the induction on the length of the multi- index J. Suppose we have shown that, in some neighbourhood of p

$$\frac{\partial^{|J|}\left(v^\beta \circ \phi\right)}{\partial y^J} = F_J^\beta \circ \left(x^k; \frac{\partial^{|K|}\left(u^\alpha \circ \phi\right)}{\partial x^K}\right) \quad 0 \leq |K| \leq |J| \tag{1.51}$$

where the smooth function F_J^β is independent of the choice of the section ϕ. Then, by the chain rule

$$\begin{aligned}\frac{\partial^{|J|+1}\left(v^\beta\circ\phi\right)}{\partial y^{J+1},} =& \frac{\partial x^i}{\partial y^j}\left(\sum_{|L|=0}^{|J|}\frac{\partial^{|L|+1}\left(u^\gamma\circ\phi\right)}{\partial x^{L+1}}F_{J\gamma}^{\beta L}\circ\left(x^k;\frac{\partial^{|K|}\left(u^\alpha\circ\phi\right)}{\partial x^K}\right)\right.\\ &\left.+\frac{\partial F_J^\beta}{\partial x^i}\circ\left(x^k;\frac{\partial^{|K|}\left(u^\alpha\circ\phi\right)}{\partial x^K}\right)\right)\quad 0\leq|K|\leq|J|\end{aligned}\tag{1.52}$$

where $F_{J\gamma}^{\beta L}$ denotes the partial derivative of F_J^β corresponding to the $\frac{\partial^{|L|(u^\gamma\circ\phi)}}{\partial x^L}$ coordinate. This equation is valid in the same neighbourhood of p.
Now we can write

$$\frac{\partial^{|J|+1}\left(v^\beta\circ\phi\right)}{\partial y^{j+1},}=F_{J+1,}^\beta\circ\left(x^k;\frac{\partial^{|K|}\left(u^\alpha\circ\phi\right)}{\partial x^K}\right)\quad 0\leq|K|\leq|J|+1\tag{1.53}$$

where F_{J+1}^β is again independent of the choice of the section ϕ.
Now, every multi-index of length $|J|+1$ can be written as the sum of a multi-index of length J and one of the form 1_j, so the induction step is valid. Furthermore

$$\begin{aligned}\frac{\partial\left(v^\beta\circ\phi\right)}{\partial y^j} &= \frac{\partial x^i}{\partial y^j}\left(\left(\frac{\partial v^\beta}{\partial x^i}\circ\phi\right)+\left(\frac{\partial v^\beta}{\partial u^\alpha}\circ\phi\right)\frac{\partial\left(u^\alpha\circ\phi\right)}{\partial x^i}\right)\\ &= F_{1,0}^\beta\circ\left(x^k;\frac{\partial\left(u^\alpha\circ\phi\right)}{\partial x^k}\right)\end{aligned}\tag{1.54}$$

We therefore have in general that, for any multi-index J of arbitrary length,

$$\frac{\partial^{|J|}\left(v_\beta\circ\phi\right)}{\partial y^J}=F_J^\beta\circ\left(x^k;\frac{\partial^{|K|}\left(u^\alpha\circ\phi\right)}{\partial x^K}\right)\quad 0\leq|K|\leq|J|\tag{1.55}$$

In some suitably small neighbourhood of p, and thus that

$$\left.\frac{\partial^{|J|}\left(v^\beta\circ\phi\right)}{\partial y^j}\right|_p=F_J^\beta\left(x^k(p);\left.\frac{\partial^{|K|}\left(u^\alpha\circ\phi\right)}{\partial x^K}\right|_p\right)\quad 0\leq|K|\leq|J|\tag{1.56}$$

The result now follows for $1\leq|J|\leq k$ by applying the conditions of the lemma. $\square$

With this lemma we are able to define the notion of $k-$equivalence of sections.

Definition 19. *Let (B,π,M) be a bundle and let $p\in M$. We say that two local sections ϕ,ψ are k-equivalent at p if, $\phi(p)=\phi(p)$ and if, in some adapted coordinates system (x^α,u^i) around $\phi(p)$,*

$$\left.\frac{\partial^{|I|}\phi^i}{\partial x^I}\right|_p=\left.\frac{\partial^{|I|}\psi^i}{\partial x^I}\right|_p\tag{1.57}$$

for $1\leq|I|\leq k$ and $1\leq i\leq n$.
Obviously, the relation "to be k-equivalent" is an equivalence relation.
The equivalence class containing ϕ is called $k-$jet of ϕ at p, and will be denoted as $j_p^k\phi$.

Remark 13. *The equivalence class $j^k_p\phi$ always contains a local section which in coordinates (x^α, u^i) is a polynomial of degree not greater than k. This is of course the $k-th$ order Taylor polynomial of ϕ around p.*

We can now finally define jet manifold.

Definition 20. *The $k-th$ jet manifold of a bundle $\pi : B \to M$ is the set*

$$J^k\pi := \{j^k_p\phi;\ p \in M, \phi \in \Gamma_p(\pi)\} \tag{1.58}$$

If the notation is not ambiguous, given a bundle (B, π, M) we will call the jet manifold of π as J^kB.

We have the following tower of bundles:

$$\begin{array}{ccccccccc} J^k\pi & \xrightarrow{\pi^k_{k-1}} & J^{k-1}\pi & \xrightarrow{\pi^k_{k-1}} & J^{k-2}\pi & & \ldots & J^1\pi & \xrightarrow{\pi^1_0} & E \\ \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{k-1}} & & \downarrow{\scriptstyle \pi_{k-2}} & & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{Id_M} & M & \xrightarrow{Id_M} & M & & \ldots & M & \xrightarrow{Id_M} & M \end{array} \tag{1.59}$$

Remark 14. *The functions π^k_l where $1 \leq l \leq k$ and $\pi^k : J^k\pi \to M$ are surjective smooth submersions.*

Given an adapted coordinates system (U, u) on B, we can induce a coordinate system on $J^k\pi$ as (U^k, u^k) defined by:

$$\begin{aligned} U^k &= \{j^k_p\phi : \phi(p) \in U\} \\ u^k &= (x^\alpha, u^j, u^j_I) \end{aligned} \tag{1.60}$$

where, $x^i(j^k_p\phi) = x^i(p)$, $u^\alpha(j^k_p\phi) = u^\alpha(\phi(p))$, and $u^i : U^k \to \mathbb{R}$ are specified by

$$u^j_I(j^k_p\phi) = \left.\frac{\partial^{|I|}\phi^j}{\partial x^I}\right|_p \tag{1.61}$$

In the following we will denote induced fibered coordinates on J^kB as $(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu}, ..., y^i_{\mu_1...\mu_k})$. We can compute transition functions for J^kB knowing the transition functions of our bundle $\pi : B \to M$ as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \end{cases} \tag{1.62}$$

These induce first partial derivatives

$$y'^i_\mu = \partial'_\mu(Y^i(x, \sigma(x))) = \bar{J}^\rho_\mu(J^i_\rho + J^i_j y^j_\rho) \tag{1.63}$$

And second derivatives as

$$\begin{aligned} y'^i_{\mu\nu} &= \bar{J}^\rho_{\mu\nu}\left(J^i_\rho + J^i_j y^j_\rho\right) + \bar{J}^\rho_\mu \bar{J}^\sigma_\nu \left(J^i_{\rho\sigma} + 2J^i_{j\mu} y^j_\sigma + J^i_{jk} y^j_\rho y^k_\sigma + J^i_j y^j_{\rho\sigma}\right) = \\ &= \left(\bar{J}^\rho_{\mu\nu} J^i_\rho + \bar{J}^\rho_\mu \bar{J}^\sigma_\nu J^i_{\rho\sigma}\right) + \left(\bar{J}^\sigma_{\mu\nu} J^i_j + 2\bar{J}^\rho_\mu \bar{J}^\sigma_\nu J^i_{j\mu}\right) y^j_\sigma + \left(\bar{J}^\rho_\mu \bar{J}^\sigma_\nu J^i_{jk}\right) y^j_\rho y^k_\sigma + \left(\bar{J}^\rho_\mu \bar{J}^\sigma_\nu J'_j\right) y^j_{\rho_\sigma} \end{aligned} \tag{1.64}$$

This procedure can be iterated at any order.
The first thing we notice is that transition functions are smooth, this means that J^kB is a smooth manifold.
Moreover always looking to transition functions, we notice that the bundle $\pi^k_{k-1} : J^kB \to J^{k-1}B$ is an affine bundle for every k.
This result can also be expressed intrinsically (as we said before, coordinates plus transition functions approach is equivalent to global formulation) but in a more complicated way. We only give the main idea of the construction.
To prove that $\pi^k_{k-1} : J^kB \to J^{k-1}B$ is an affine bundle, we first need to construct the vector space on where the affine space is constructed.
It is not easy to prove that the vector space is $(\pi^{k-1})^*S_k(T^*M) \otimes (\pi^{k-1}_0)^*(V(\pi))$ now, considering the set of germs of functions on a point x of our manifold (in particular those where exists a representative, f such that $f(x) = 0$) and using local sections and their deformations, it is possible to prove that for every point $p \in J^{k-1}\pi_{x_0}$, the set $(\pi^k_{k-1})^{-1}(p)$ is an affine space modelled on $S_k(T^*_{x_0}M) \otimes (V_{\pi^{k-1}_0}(\pi))$. In particular, from this result we have the thesis.

1.3.1 Contact Structures

Definition 21. *Given a section $\sigma : M \to B$ we can define a new section $j^k\sigma : M \to J^kB$ as $x \mapsto j^k_x\sigma$, this is called $k-$th jet prolongation of the section σ.*

Remark 15. *In local coordinates a section is the map $\sigma : M \to B,\ x \mapsto (x, y^i = \sigma(x))$. The local expression of the $k-$th jet prolongation will be:*

$$\begin{aligned} &j^k\sigma : M \to J^kB : \\ &x \mapsto (x, y^i = \sigma(x), y^i_\mu = \partial_\mu\sigma^i(x), ..., y^i_{\mu_1,...,\mu_k} = \partial_{\mu_1,...,\mu_k}\sigma^i(x)) \end{aligned} \tag{1.65}$$

Of course, the set of sections of $\Gamma(\pi^k)$ that are prolongations of sections of $\sigma \in \Gamma(\pi)$ are a strict subset $\Gamma(\pi^k)$.
Sections of $\Gamma(\pi^k)$ that are prolongations of a section of $\Gamma(\pi)$ will be called *holonomic sections*.
These sections can be characterised, on a dual context, using *contact forms*, which we will also use later in calculus of variations.

Definition 22. *A $k-$form ω is called a contact form is for any holonomic section $j^k\sigma$, its pull-back along $j^k\sigma$ is zero, that is:*

$$(j^k\sigma)^*\omega = 0 \tag{1.66}$$

The set of all holonomic sections is an ideal of the exterior algebra, and it is denoted as $\Omega_K(J^kB) \subset \Omega(J^kB)$

Example 2. *Let us consider a $1-$form $\omega = \omega_\mu dx^\mu + \omega_i dy^i + \omega^\mu_i dy^i_\mu \in \Omega_1(J^1B)$. If we want ω to be a contact form we have to ask that for any section $\sigma : x \to (x, y^i(x))$,*

$$\begin{aligned} &(j^1\sigma)^*\omega = (\omega_\mu + \omega_i y^i_\mu + \omega^\nu_i y^i_{\mu\nu})dx^\mu = 0 \\ &\iff \omega_\mu + \omega_i y^i_\mu + \omega^\nu_i y^i_{\mu\nu} = 0 \\ &\textit{Being the coefficients functions of } (x^\mu, y^i, y^i_\mu) \textit{ leads to} \\ &\omega_\mu = -\omega_i y^i_\mu; \quad \omega^\nu_i = 0 \end{aligned} \tag{1.67}$$

So the most general contact $1-$*form on* $J^1(B)$ *is* $\omega = \omega_i(dy^i - y^i_\mu dx^\mu)$.
Accordingly any other contact $1-$*form is a linear combination of the basic contact form* $\omega^i = dy^i - y^i_\mu dx^\mu$

In general, we can ask ourself if a generic form can be splitted into a contact part plus something else. We have the following theorem.

Theorem 4. *Let* ω *be a* $p-$*form on* J^kB*, then we have the decomposition*

$$(\pi^{k+1}_k)^*\omega = \omega_K + \omega_H \tag{1.68}$$

where ω_K *is a contact form, and* ω_H *is an horizontal form.*

1.3.2 Total Derivatives

In this section we introduce the notion of total derivative: this is a very useful notion in mathematical physics and geometry, because among other things, it allows to split canonically (without choosing a connection) the bundle $(\pi^1_0)^*(TM)$.

Let us recall that given a bundle

$$\begin{tikzcd}
J^1Y \arrow[d, "\pi^1_0"'] \\
Y \arrow[u, bend right, "\gamma"'] \arrow[d, "\pi"] \\
X \arrow[u, bend right, "\sigma"'] \arrow[uu, bend left=60, "\Gamma"]
\end{tikzcd} \tag{1.69}$$

A section Γ is called holonomic if $\Gamma = j^1\sigma$.
Moreover holonomic sections characterise contact forms for the section π^1_0.

Definition 23. *If the following diagram commute*

$$\begin{tikzcd}
TJ^1Y \arrow[r, "\tau_{J^1Y}"'] \arrow[d, "T\pi^1_0"] \arrow[dd, bend right=60, "T\pi^1"'] & J^1Y \arrow[l, bend right, "\Xi"'] \arrow[d, "\pi^1_0"'] \arrow[dd, bend left=60, "\pi^1"] \\
TY \arrow[r, "\tau_Y"'] \arrow[d, "T\pi"] & Y \arrow[l, bend right, "\bar{\Xi}"'] \arrow[d, "\pi"'] \\
TX \arrow[r, "\tau_X"'] & X \arrow[l, bend right, "\xi"']
\end{tikzcd} \tag{1.70}$$

i.e $T\pi \circ \bar{\Xi} = \xi \circ \pi$*, we will say that* $\bar{\Xi}$ *and* ξ *are* $\pi-$*related.*
Analogous definitions are valid for other projections.

If we write our fields in coordinates we have

$$\begin{aligned}
&\xi = \xi^\alpha(x)\partial_\alpha \\
&z = \bar{\Xi} = z^\alpha(x^\beta, y^i)\partial_\alpha + z^i(x^\beta, y^i)\partial_i
\end{aligned} \tag{1.71}$$

Moreover, if we ask $\bar{\Xi}$ to be projectable we must have $z = z^{\alpha}(x^{\beta})\partial_{\alpha} + z^{i}(x^{\beta}, y^{i})\partial_{i}$, and $z^{\alpha} = \xi^{\alpha}$.
Analogous conditions can be applied to Ξ. For example, if we want Ξ to be projectable on both $z$and Ξ we must have

$$\Xi = \xi^{\alpha}(x) + \bar{\Xi}(x, y) + \Xi^{i}_{\alpha}(j^{1}y)\partial^{i}_{\alpha} \tag{1.72}$$

In general let us consider a vector field $\bar{\Xi} : Y \to TY$, we can define $j^{1}\bar{\Xi} = \frac{d}{dt} j^{1} f_{t}|_{t=0}$, where f_{t} is the flow of automorphisms $Y \to Y$ generated by $\bar{\Xi}$.
Our aim is to decompose this field in horizontal and vertical part.

Definition 24. *The vector field $T\pi^{1}_{0} \circ j^{1}\bar{\Xi} = \Xi \circ \bar{\Xi} : J^{1}Y \to TY$ is called generalized vector field along the map π^{1}_{0}.*
In coordinates, we have

$$j^{1}\bar{\Xi} = \xi^{\alpha}(x)\partial_{\alpha} + \bar{\Xi}^{i}(x, y)\partial_{i} + \hat{\Xi}^{i}_{\beta}(j^{1}y) \tag{1.73}$$

So the generalized vector field will be

$$T\pi^{1}_{0} \circ j^{1}\bar{\Xi} = \xi^{\alpha}(x)\partial_{\alpha} + \bar{\Xi}^{i}(x, y)\partial_{i} \tag{1.74}$$

where, the functions ξ^{α}, Ξ^{i} are considered as functions on $J^{1}Y$.

Using the affine structure of the projection π^{1}_{0} we can now split this composition.
Let us consider $j^{1}_{x}\sigma \in J^{1}Y$ and $j^{1}\bar{\Xi}$ defined as above.
We have the map

$$T_{j^{1}_{x}\sigma}\pi^{1} : T_{j^{1}_{x}\sigma}J^{1}Y \to T_{x}X \tag{1.75}$$

That takes a vector field on $T_{j^{1}_{x}\sigma}J^{1}Y$ and projects it on $\xi^{\alpha}(x)\partial_{\alpha}$ where x is fixed. This map is essentially the total differential.
In fact, given a section $\sigma : X \to Y$ we can consider $T_{x}\sigma$, and in particular

$$T_{x}\sigma \circ T_{j^{1}_{x}\sigma}\pi^{1} \circ j^{1}\bar{\Xi} = \left(j^{1}\bar{\Xi}\right)_{H} = \left(\bar{\Xi}\right)_{H} \tag{1.76}$$

will be called *horizontal part* of $\bar{\Xi}$.
If we write in coordinate the horizontal part we get

$$\left(\bar{\Xi}\right)_{H} = \xi^{\alpha}\frac{d}{dx^{\alpha}} \quad ; \qquad \frac{d}{dx^{\alpha}} = \partial_{\alpha} + y^{i}_{\alpha}\partial_{i} \tag{1.77}$$

Note that here we were able to split a generalised vector field (in particular compute its horizontal part) without using a connection, at the price of considering generalised vector fields. Moreover, the space of horizontal vector field defined above is transverse to vertical vectors.
All there results can be collected in the following theorem.

Theorem 5. *There is a canonical splitting of $J^{1}Y \times_{Y} TY$ as*

$$J^{1}Y \times_{Y} TY = J^{1}Y \times_{M} VY \oplus J^{1}Y \times_{Y} T\sigma(TX) \tag{1.78}$$

Where $T\sigma(TX)$ is the lifting of the image of TX (essentially it is the holonomic lift of $\frac{\partial}{\partial x^{\alpha}}$)

Remark 16. *From what we said before, we can write in coordinates the total derivative. We know that a section is* $\sigma : (x^\alpha, \sigma^i(x^\alpha))$*, so we have*

$$T_x\sigma = \frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} + \frac{\partial \sigma^i}{\partial x^\alpha}\frac{\partial}{\partial y^i} \tag{1.79}$$

So now we have

$$T_x\sigma\left(\frac{\partial}{\partial x^\alpha}\right) = \frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} + \frac{\partial \sigma^i}{\partial x^\alpha}\frac{\partial}{\partial y^i} = \left(\delta^\beta_\alpha\frac{\partial}{\partial x^\beta} + \frac{\partial \sigma^i}{\partial x^\alpha}\frac{\partial}{\partial y^i}\right) \tag{1.80}$$

Now, for the definition of local coordinate on J^1Y*,* $\frac{\partial \sigma^i}{\partial x^\alpha} = y^i_\alpha$

Definition 25. *The generalised vector field* $\frac{d}{dx^\alpha} := T_x\sigma\left(\frac{\partial}{\partial x^\alpha}\right)$ *is called total derivative.*
We have $\frac{d}{dx^\alpha} : J^1Y \to TY$*.*
Moreover, in general, given an open set $V \subset J^kB$*, the total derivative* d_α *will be defined on* $(\pi^{k+1}_k)^{-1}(V)$ *with expression*

$$d_\alpha f = \frac{\partial f}{\partial x^\alpha} + \sum_{t=1}^{k}\sum_{j_1\leq\cdots\leq j_t}\frac{\partial f}{\partial y^\sigma_{j_1\cdots j_t}}y^\sigma_{j_1\ldots jti} \tag{1.81}$$

in general we have

$$\begin{aligned}
&d_\mu f\left(x^\lambda\right) := \partial_\mu f\left(x^\lambda\right)\\
&d_\mu f\left(x^\lambda, y^k\right) := \partial_\mu f\left(x^\lambda, y^k\right) + \partial_i f\left(x^\lambda, y^k\right)y^i_\mu\\
&d_\mu f\left(x^\lambda, y^k, y^k_\lambda\right) := \partial_\mu f\left(x^\lambda, y^k, y^k_\lambda\right) + \partial_i f\left(x^\lambda, y^k, y^k_\lambda\right)y^i_\mu + \partial^\nu_i f\left(x^\lambda, y^k, y^k_\lambda\right)y^i_{\nu\mu}\\
&\cdots\\
&d_\mu f\left(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_k}\right) := \partial_\mu f\left(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_k}\right) + \partial_i f\left(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_k}\right)y^i_\mu +\\
&+ \partial^\nu_i f\left(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_k}\right)y^i_{\nu\mu} + + \ldots + \partial^{\nu_1\ldots\nu_k}_i f\left(x^\lambda, y^k, y^k_\lambda, \ldots, y^k_{\lambda_1\ldots\lambda_k}\right)y^i_{\nu_1\ldots\nu_k\mu}
\end{aligned} \tag{1.82}$$

Remark 17. *Total derivatives have this interesting property, that is very useful in calculus of variations:* $\forall f \in \mathcal{F}(J^kE)$*,*

$$(d_\mu f)\circ j^{k+1}\sigma = \partial_\mu(f\circ j^k\sigma) \tag{1.83}$$

The proof of this property is straightforward and it is based on the derivative rule for composite functions.

1.3.3 Local description of Jet bundles

Previous definitions of jet bundles were given with intrinsic definitions, but in the following it will be very useful to give an alternative (and equivalent) description of jet bundles, using local coordinates and transformation laws.

We will describe in this way first order jet bundles: it is easy to generalize the properties for higher order jet bundles.
Of course, given a bundle $\pi : B \to M$, J^1B is fiber over B and M and has coordinates (x^μ, y^i, y^i_μ). Moreover, transformations of B induce transformations on J^1B as:

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \\ y'^i_\mu = \bar{J}^\nu_\mu(J^i_\nu + J^i_k y^k_\nu) \end{cases} \Rightarrow \begin{cases} dx'^\mu = J^\mu_\nu dx^\nu \\ dy'^i = J^i_\mu dx^\mu + J^i_j dy^j \\ dy'^i_\mu = (\bar{J}^\lambda_{\epsilon\mu} J^\epsilon_\nu d_\lambda Y^i + \bar{J}^\lambda_\mu d_\lambda J^i_\nu)dx^\nu + \bar{J}^\nu_\mu (d_\nu J^i_j)dy^j + \bar{J}^\nu_\mu J^i_k dy^k_\nu \end{cases} \tag{1.84}$$

For the component of vectors, we have

$$\begin{cases} v'^\mu = J^\mu_\nu v^\nu \\ v'^i = J^i_\mu v^\mu + J^i_j v^j \\ v'^i_\mu = (\bar{J}^\lambda_{\epsilon\mu} J^\epsilon_\nu d_\lambda Y^i + \bar{J}^\lambda_\mu d_\lambda J^i_\nu)v^\nu + \bar{J}^\nu_\mu (d_\nu J^i_j)v^j + \bar{J}^\nu_\mu J^i_k v^k_\nu \end{cases} \tag{1.85}$$

while components of vertical vector fields transform a

$$\begin{cases} v'^i = J^i_j v^j \\ v'^i_\mu = \bar{J}^\nu_\mu (d_\nu J^i_j)v^j + \bar{J}^\nu_\mu J^i_k v^k_\nu \end{cases} \tag{1.86}$$

The components of covector, as well as the natural basis for vectors, transform as

$$\begin{cases} \alpha_\nu = J^\mu_\nu \alpha'_\mu + J^i_\nu \alpha'_i + (\bar{J}^\lambda_{\epsilon\mu} J^\epsilon_\nu d_\lambda Y^i + \bar{J}^\lambda_\mu d_\lambda J^i_\nu)\alpha'^\mu_i \\ \alpha_j = J^i_j \alpha'_i + \bar{J}^\nu_\mu (d_\nu J^i_j)\alpha'^\mu_i \\ \alpha^\mu_j = \bar{J}^\mu_\nu J^i_j \alpha'^\nu_i \end{cases} \Rightarrow \begin{cases} \partial_\nu = J^\mu_\nu \partial'_\mu + J^i_\nu \partial'_i + (\bar{J}^\lambda_{\epsilon\mu} J^\epsilon_\nu d_\lambda Y^i + \bar{J}^\lambda_\mu d_\lambda J^i_\nu)\partial'^\mu_i \\ \partial_j = J^i_j \partial'_i + \bar{J}^\nu_\mu (d_\nu J^i_j)\partial'^\mu_i \\ \partial^\mu_j = \bar{J}^\mu_\nu J^i_j \partial'^\nu_i \end{cases} \tag{1.87}$$

1.4 Calculus of Variations

In a field theory, configurations of the system are global sections of a bundle (C, M, π, F), which is called configuration bundle.
In fibered coordinates, local expression of sections will be $\sigma : M \to C$, $x^\mu \mapsto (x^\mu, y^i(x))$.
In general, a field theory will be described by a *Lagrangian*.

Definition 26. *A $k-$order Lagrangian is a horizontal $m-$form on J^kC. Locally a Lagrangian can be represented as*

$$L = L(x^\mu, y^i, ..., y^i_{\mu_1,...,\mu_k})d\sigma \tag{1.88}$$

The key objects that we will use to get field equations are *Actions and variations*

Definition 27. *Given a $m-$region (i.e. a compact sub-manifold of dimension m with a boundary which is a compact sub-manifold of dimension $(m-1)$) $D \subset M$, we call action along a section σ the functional*

$$A_D[\sigma] := \int_D (j^k\sigma)^* L \tag{1.89}$$

This integral makes sense because $(j^k\sigma)^*L$ is a $m-$form on M, that can be integrated on D.

Definition 28. *We define a deformation (or a variation) to be a vertical vector field X on the configuration bundle C. Its prolongation to the $k-$order jet bundle J^kCwill be denoted by j^kX.*
The support of the deformation X will be defined as the closure of the projection in M of set where X is non-zero. It will be denoted by supp(X).

Locally a deformation will be given by

$$X = X^i(x, y)\partial_i \tag{1.90}$$

That can be lifted to

$$j^kX = X^i(x, y)\partial_i + d_\mu X^i(x, y)\partial_i^\mu + ... + d_{\mu_1,...,\mu_k} X^i(x, y)\partial_i^{\mu_1,...,\mu_k} \tag{1.91}$$

In the following, we will use the standard notation

$$\delta y^i := X^i \qquad \delta y^i_\mu := d_\mu X^i =: X^i_\mu \qquad \text{and so on} \tag{1.92}$$

Having the vector field X we can consider its flow, Φ_s, that can be lifted to $j^k\Phi_s$, and this is the flow of j^kX.
Then we can drag any configuration σ to define a $1-$parameter family

$$\sigma_s = \Phi_s \circ \sigma \tag{1.93}$$

Since everything is global, the sections σ_s are global at least in a neighbourhood of $s = 0 \in \mathbb{R}$

Example 3. ***Completeness it is not guaranteed*** *Even if X is global, it is not guaranteed that its flow extends to all $s \in \mathbb{R}$.*
Take for example $M = \mathbb{R}$ and $X = f(x)\frac{\partial}{\partial x}$. A curve $\gamma : \mathbb{R} \to M$, $s \mapsto x(s)$ is an integral curve for X if

$$\frac{dx}{ds} = f(x) \Rightarrow s - s_0 = \int_{x_0}^{x} \frac{1}{f(x)} dx \tag{1.94}$$

In the special case $f(x) = x^2$ we have $s - s_0 = -\frac{1}{x}$, that leads to integral curves

$$x(s) = -\frac{1}{s - \frac{1}{x_0}} = \frac{x_0}{1 - sx_0} \tag{1.95}$$

where we fixed the integration constant s_0 so that $x(0) = x_0$.
The integral curve is defined in $s \in (-\infty, \frac{1}{x_0})$ and when $s \mapsto \frac{1}{x_0}$ $x(s) \mapsto \infty$.
Moreover, the integral curve goes from x_0 to infinity in a finite time s.

We can define

$$\delta_X A_D[\sigma] := \frac{d}{ds} A_D[\sigma_s]\Big|_{s=0} = \int_D \frac{d}{ds}((j^k\sigma_s)^* L)\Big|_{s=0} \tag{1.96}$$

Now we can finally state the *Hamilton principle.*
Hamilton principle. A configuration σ is said to be critical iff for any $m-$region D, and for any deformation X compactly supported in D one has

$$\delta_X A_D[\sigma] = 0 \tag{1.97}$$

We have to stress that this condition is completely expressed intrinsically, i.e. without any reference to coordinates or local objects.

1.4.1 Local Euler-Lagrange Equations

The condition to be a critical section can be expanded in coordinates. Let us consider a second order Lagrangian i.e. $L = L(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu})d\sigma$.
In this case, the Hamilton Principle expands as

$$\delta_X A_D[\sigma] = \int_D \left(\frac{\partial L}{\partial y^i}\delta y^i + \frac{\partial L}{\partial y^i_\mu} d_\mu \delta y^i + \frac{\partial L}{\partial y^i_{\mu\nu}} d_{\mu\nu}\delta y^i \right) d\sigma = 0 \tag{1.98}$$

Integrating by parts and using Stokes theorem we get

$$\begin{aligned} &\delta_X A_D[\sigma] = \\ &\int_D \left(\frac{\partial L}{\partial y^i}\delta y^i - d_\mu \frac{\partial L}{\partial y^i_\mu}\delta y^i - d_\nu \frac{\partial L}{\partial y^i_{\mu\nu}} d_\mu \delta y^i \right) d\sigma + \int_D d_\mu \left(\frac{\partial L}{\partial y^i_\mu}\delta y^i + \frac{\partial L}{\partial y^i_{\mu\nu}} d_\nu \delta y^i \right) d\sigma = \\ &\int_D \left(\frac{\partial L}{\partial y^i} - d_\mu \frac{\partial L}{\partial y^i_\mu} + d_{\mu\nu} \frac{\partial L}{\partial y^i_{\mu\nu}} d_\mu \right) \delta y^i d\sigma + \int_{\partial D} \left(\left(\frac{\partial L}{\partial y^i_\mu} - d_\nu \frac{\partial L}{\partial y^i_{\mu\nu}} \right) \delta y^i + \frac{\partial L}{\partial y^i_{\mu\nu}} d_\nu \delta y^i \right) d\sigma_\mu = 0 \end{aligned} \tag{1.99}$$

If we define the momenta

$$\begin{aligned} &p_i := \frac{\partial L}{\partial y^i} \qquad p^i_\mu := \frac{\partial L}{\partial y^i_\mu} \qquad p^i_{\mu\nu} := \frac{\partial L}{\partial y^i_{\mu\nu}} \\ &f^\mu_i := p^\mu_i - d_\nu p^{\mu\nu}_i \qquad f^{\mu\nu}_i := p^{\mu\nu}_i \end{aligned} \tag{1.100}$$

The equations will become

$$\delta_X A_D[\sigma] = \int_D (p_i - d_\mu p^\mu_i + d_{\mu\nu} p^{\mu\nu}_i)\delta y^i d\sigma + \int_{\partial D} (f^\mu_i \delta y^i + f^{\mu\nu}_i d_\nu \delta y^i) d\sigma_\mu = 0 \tag{1.101}$$

Since the deformation X is compactly supported in D, it vanishes on ∂D. Then one has

$$\delta_X A_D[\sigma] = \int_D (p_i - d_\mu p^\mu_i + d_{\mu\nu} p^{\mu\nu}_i)\delta y^i d\sigma = 0 \tag{1.102}$$

Since the $m-$region is arbitrary, the integrand must be zero.

$$(p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu})\delta y^i d\sigma \tag{1.103}$$

Finally, since $d\sigma$ is a basis for $m-$ forms, the coefficient must vanish, and since the deformation is arbitrary we can define *Euler-Lagrange equations*

$$\mathbb{E}_i(L) := p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu} = 0 \tag{1.104}$$

These PDEs in general are of order 4 since $kth-$order Lagrangians leads to at most $2k-$order equations.

Remark 18. *We have stated the Hamilton Principle in a global way, but we derived Euler Lagrange equations from a local computation, in general we can prove that if the Lagrangian is global, then the field equations are global.*
The proof is based on the calculation of the transformation law of $\mathbb{E}_i$, and checking that $\mathbb{E}(L) := \mathbb{E}_i(L)\omega^i \wedge d\sigma$ is a global object.
In the same way we can prove that $\mathbb{F}(L) := f_i^\mu \omega^i \wedge d\sigma_\mu + f_i^{\mu\nu} \omega_\nu^i \wedge d\sigma_\mu$ is a global object as well.

Remark 19. *The forms $\mathbb{E}(L)$ and $\mathbb{F}(L)$ for a $k-$order Lagrangian are related by the following relation, which is called first variation formula*

$$(j^k X) \lrcorner\, dL = X \lrcorner\, \mathbb{E}(L) + d((j^{k-1}X) \lrcorner\, \mathbb{F}(L)) \tag{1.105}$$

Then we can say that a section σ is critical iff

$$\mathbb{E}(L) \circ (j^{2k}\sigma) = 0 \tag{1.106}$$

Chapter 2

Composite bundles

We now want to introduce the category of composite bundles, the framework that we will use to deal with perturbation theory in field theories.
In this introduction we will try to give the general idea behind this new category, while in the following sections we will give all the formal details.
Composite bundles can be viewed as stack of regular bundles, where at each level we have a bundle structure. This is clearly a generalisation of the category of bundles, but the non trivial intuition is that in order to preserve all fibered structures we need to restrict fibered morphisms.
These problems will be faced in the next section.
We can now observe that we already encountered composite bundles, for example, tangent bundles (though as $TB \to B \to M$), jet bundles and vertical bundles. In section 2.2 we will describe these bundles as composite bundles.
Then, we will try to extend the ordinary calculus of variations on this new category. What we will get is a bigger set of Euler Lagrange equations that generalise the standard equations of motion.
At the end, and in the next sections we will explain why composite calculus of variations is important and why we had to introduce this topic in order to study linearised equations.

2.1 The category of composite bundles

Composite bundles can be viewed as ladder of regular bundles, $\pi^k_{k-1} : B_k \to B_{k-1}$, $\forall k \in [1, n]$, that is

$$B_n \to B_{n-1} \to ... \to B_0 \tag{2.1}$$

The number n will be called height of the composite bundle.

This may seem to be a rather trivial generalization of bundles, though let us remark that fibered morphisms need in fact to be restricted to preserve all the fibered structures, and consequently that restricts the notion of flows, jets prolongations need to be adapted, and the prolongation of sections and vector fields to the jet prolongations needs to be adapted as well.

Our aim is to introduce a geometric and systematic framework for variational calculus on composite bundles and to present examples of geometrical and Physical applications.

In particular, composite variational principles provide a framework in which ODE and PDE can coexist in the same setting, as it happens for geodesics and Jacobi fields, or applications to physics, e.g. to n-body problems, when particles and fields should be considered together.

That also provides a unified formalism to discuss linearised equations, hence perturbation theory, as well as stability of solutions from a purely and general variational point of view, again encompassing both curves (ODE) and fields (PDE).

Definition 29. *The objects of the category of composite bundles are a finite ladder of bundles*

$$B_2 \xrightarrow{\pi_1^2} B_1 \xrightarrow{\pi_0^1} B_0 \xrightarrow{\pi} M \tag{2.2}$$

Each level in the ladder is an ordinary bundle, thus fibered coordinates on composite bundles are stratified: we have

$$(x^\mu, y^i, z^A, v^a) \overset{\pi_1^2}{\longmapsto} (x^\mu, y^i, z^A) \overset{\pi_0^1}{\longmapsto} (x^\mu, y^i) \overset{\pi}{\longmapsto} x^\mu \tag{2.3}$$

and their transition functions are stratified as well

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \\ z'^A = Z^A(x, y, z) \\ v'^a = V^a(x, y, z, v) \end{cases} \tag{2.4}$$

A composite bundle is denoted by $[B_2 \to B_1 \to B_0 \to M]$. The request of being a *finite* ladder is here just for the sake of simplicity, to maintain all manifolds ordinary, finite dimensional, smooth manifolds.

Remark 20. *We already know a number of stratified bundles: from any finite sequence of jet bundles, e.g.* $[J^2B \to J^1B \to B \to M]$*, any sequence of higher order tangent bundles, e.g.* $[T^2M \to TM \to M]$*, or the vertical bundles* $[V(B) \to B \to M]$.

Definition 30. *A morphism of composite bundles (of the same height) is a collection of maps which provide a bundle morphism at any step in the ladder.*
This implies that the local expressions of composite maps are stratified as well, for example on a height 4 *bundle we have*

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = \phi_0^i(x, y) \\ z'^A = \phi_1^A(x, y, z) \\ v'^a = \phi_2^a(x, y, z, v) \end{cases}$$

as it happens on jet prolongations.

Often, we refer to a particular bundle within the composite bundle $[B_1 \to B_0 \to M]$ so we need a systematic way to denote projections and fibered coordinates.

Let us denote by $\pi^h_k = \pi^h_{h-1} \circ \cdots \circ \pi^{k+1}_k : B_h \to B_k$ and by $\pi^h = \pi \circ \pi^h_0 : B_k \to M$ the projections of the corresponding bundles and we split the coordinates by semicolon to divide base and fiber coordinates. For example, $(x^\mu, y^i; z^A, v^a)$ will denote fibered coordinates on the bundle $\pi^2_0 : B_2 \to B_0$.

Example 4. ***Not all fibered bundle morphisms are composite***
Let us remark that not all fibered bundle morphisms of the bundle $\pi^2_0 : B_2 \to B_0$ extend to a composite fibered map. In fact, the general bundle morphism of $\pi^2_0 : B_2 \to B_0$ is locally represented by

$$\begin{cases} x'^\mu = x'^\mu(x,y) \\ y'^i = \phi^i_0(x,y) \\ z'^A = \phi^A_1(x,y,z,v) \\ v'^a = \phi^a_2(x,y,z,v) \end{cases}$$

which can extend to a composite map only if $x'^\mu = x'^\mu(x,y) = x'^\mu(x)$ and $z'^A = \phi^A_1(x,y,z,v) = \phi^A_1(x,y,z)$. Hence the category of composite bundles is a genuine new category, since we are really restricting allowed maps. Equivalently, we can say that composite bundles have in fact a richer structure with respect to the bundles which they contain.

As we do in the category of bundles, we consider a class of vector fields which preserve the composite structure, i.e. such that the their flow is made of composite maps. The components of these vector fields are stratified as well as

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^i(x,y)\partial_i + \xi^A(x,y,z)\partial_A + \xi^a(x,y,z,v)\partial_a$$

and they are called *composite vector fields.*

Sections are stratified as well. The local expression of a composite section is a map

$$\sigma : M \to B_2 : x^\mu \mapsto (x, y^i(x), z^A(x,y), v^a(x,y,z))$$

i.e. it is a collection of sections

$$\begin{aligned} &\sigma_0 : M \to B_0 \\ &\sigma_1 : B_0 \to B_1 \\ &\sigma_2 : B_1 \to B_2 \end{aligned} \tag{2.5}$$

Remark 21. *In the same composite bundle we can have various compositions of sections, e.g.*

$$\begin{aligned} \sigma_1 \circ \sigma_0 : M &\to B_1 \\ x &\mapsto (x, y(x), z(x, y(x))) \end{aligned} \tag{2.6}$$

or

$$\begin{aligned} \sigma_2 \circ \sigma_1 : B_0 &\to B_2 \\ (x,y) &\mapsto (x, y; z(x,y), v(x,y,z(x,y))) \end{aligned} \tag{2.7}$$

As a consequence, this is a flexible framework for variational calculus, more general than the usual one, as we will argue.

Of course, we can also consider jet prolongations and higher order tangent bundles.

Example 5. *For example, in mechanics, accelerations play a distinct role. Thus we have a composite bundle*

$$[\mathbb{R} \times T^2M \to \mathbb{R} \times TM \to \mathbb{R} \times M \to \mathbb{R}]$$

which is relevant to (covariant, homogeneous) mechanics.
In this composite bundle we have sections $\sigma_0 : \mathbb{R} \to \mathbb{R} \times M$ which are in one-to-one correspondence with parametrized curves on M, sections $\sigma_1 : \mathbb{R} \times M \to \mathbb{R} \times TM$ which are one-parameter families vector fields on space-time M, as well as $\sigma_2 : \mathbb{R} \times TM \to \mathbb{R} \times T^2M$ which are acceleration (or force) fields. As long as one is interested in mechanics, we are mainly concerned with the composite sections, e.g. $\sigma_2 \circ \sigma_1 \circ \sigma_0 : \mathbb{R} \to \mathbb{R} \times T^2M$ which accounts for position velocities and acceleration along a world-line However, it is sometime useful to regard force fields as fields on space (or space-time) instead of their remnant along the world-lines

Another important example of composite bundle is provided as a construction over the bundle of vertical vector fields. Sections of this bundle will be very important, because can be regarded as the deformations of the ordinary calculus of variations.

2.2 Vertical Vector Bundles

We know how to define the bundle $[V(B) \to B]$ of vertical vectors. The bundle of vertical vectors is by its nature a composite bundle $[V(B) \to B \to M]$. A composite section $(\sigma_0 : M \to B, \sigma_1 : B \to V(B))$ accounts for a section of $[B \to M]$ as well as a vertical vector field on B which, of course, can be restricted to the section σ_0 as one does to define deformations in variational calculus.

Remark 22. *The functor $V(\cdot)$ is a covariant functor, from the category of smooth manifolds to the category of vector bundles.*
Clearly, it associate to every manifold M a vector bundle VM. We will now define how the functor $V(\cdot)$ acts on morphisms.
Given the following diagram

$$\begin{tikzcd} VB \arrow[r, dashed, "V\Phi"] \arrow[d, "\pi_V"] & VB' \arrow[d, "\pi'_V"] \\ B \arrow[r, "\Phi"] \arrow[d, "\pi"] & B' \arrow[d, "\pi'"] \\ M \arrow[r, "\phi"] & M' \end{tikzcd} \tag{2.8}$$

we define $V\Phi : [\gamma] \mapsto [\gamma']$ where $\gamma : \mathbb{R} \to B$ is a curve on B and $\Phi \circ \gamma := \gamma' : \mathbb{R} \to B'$ is a curve on B'.
Finally, it is not difficult to prove that the definition of γ' is well posed because of the commutativity of the first diagram.

Remark 23. *Being $V(\cdot)$ a covariant functor means we can iterate it in order to define the bundle*

$$[VVB \to VB \to B \to M] \tag{2.9}$$

On $[VVB \to VB \to B \to M]$ we have a very rich structure which can be described in a more or less intrinsic fashion. One can discuss it in terms of global swap maps. However, we believe that this intrinsic description turns out to be a bit obscure.

An equivalent way to discuss it is in terms of coordinates. For example, on $V(V(B))$ we have composite coordinates $(x^\mu, y^i, v^i; u^i, w^i)$ to describe a vertical vector $\Xi = u^i \frac{\partial}{\partial y^i} + w^i \frac{\partial}{\partial v^i}$ on $V(B)$. Since $V(V(B))$ is obtained functorially from the bundle B, transition functions on it are generated by transition functions on B and they can easily be obtained in the form

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \end{cases} \qquad \Longrightarrow \begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \\ v'^i = J^i_j(x,y)v^j \\ u'^i = J^i_j(x,y)u^j \\ w'^i = J^i_j(x,y)w^j + J^i_{kj}(x,y)v^j u^k \end{cases}$$

where we set $J^i_j := \frac{\partial}{\partial y^j} Y^i(x,y)$ and $J^i_{kj} := \frac{\partial}{\partial y^j \partial y^k} Y^i(x,y)$. So, first, we notice that transition functions of $V(V(B))$ are stratified as they should in a composite bundle. (Actually, a bit more than stratified since, for example, transition functions on the coordinate u^i do not depend on v.)

Second, we notice that transition functions of the coordinates v^i and u^i are similar in form. Consequently, the local swap maps $S : (x^\mu, y^i, v^i, u^i, \zeta^i) \mapsto (x^\mu, y^i, u^i, v^i, \zeta^i)$ glue together to define a global and intrinsic swap map. Also, one can define a sub-bundle $V^2(B) \subset V(V(B))$ by setting the constraint $v^i = u^i$. The constraint is given in coordinates though it could be intrinsically expressed by the intrinsic swap map.

Then, on the composite bundle $[V^2(B), V(B), B, M]$, we have fibered coordinates (x^μ, y^i, v^i, w^i) and transition functions

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \\ v'^i = J^i_j(x,y)v^j \\ w'^i = J^i_j(x,y)w^j + J^i_{kj}(x,y)v^j v^k \end{cases}$$

Notice how the bundle $\pi^2_1 : V^2(B) \to V(B)$ is an affine bundle, though $\pi^2_0 : V^2(B) \to B$, in general, is not. Of course, a similar procedure can be carried out at higher orders.

2.2.1 Local description of Vertical Bundles

With the same aim of what we did in the case of jet bundles, we provide a coordinate description of vertical bundles; we will do this for first and second order case.
Given a bundle $\pi : B \to M$, the bundle $V(B)$ is fibered over B and M and has coordinates $(x^\mu; y^i, w^i)$. The transformation on B induces a transformation on $V(B)$ as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \\ w'^i = J^i_j(x,y)w^j \end{cases} \Rightarrow \begin{cases} dx'^\mu = J^\mu_\nu dx^\nu \\ dy'^i = J^i_\mu dx^\mu + J^i_j dy^j \\ dw'^i = J^i_{\mu j} w^j dx^\mu + J^i_{kj} w^j dy^k + J^i_j dw^j \end{cases} \tag{2.10}$$

That shows that $V(B)$ is a vector bundle on B.

For components of a vector $\xi = v^\mu \partial_\mu + v^i \partial_i + \hat{v}^i \hat{\partial}_i$, we have

$$\left\{\begin{array}{l} v'^\mu = J^\mu_\nu v^\nu \\ v'^i = J^i_\mu v^\mu + J^i_j v^j \\ \hat{v}'^i = J^i_{\mu j} w^j v^\mu + J^i_{kj} w^j v^k + J^i_j \hat{v}^j \end{array}\right. \tag{2.11}$$

while components of vertical vectors on $V(B)$ transform simply as

$$\left\{\begin{array}{l} v'^i = J^i_j v^j \\ \hat{v}'^i = J^i_{kj} w^j v^k + J^i_j \hat{v}^j \end{array}\right. \tag{2.12}$$

From these transformation rules we see that, also vertical vectors on the bundle $\pi_V : V(B) \to B$ are well defined and their components transforms as

$$\hat{v}'^i = J^i_j \hat{v}^j \tag{2.13}$$

The components of covector, as well as the natural basis for vectors, transform as

$$\left\{\begin{array}{l} \alpha_\nu = J^\mu_\nu \alpha'_\mu + J^i_\nu \alpha'_i + J^i_{\nu j} w^j \hat{\alpha}'_i \\ \alpha_j = J^i_j \alpha'_i + J^i_{kj} w^k \hat{\alpha}'_i \\ \hat{\alpha}_j = J^i_j \hat{\alpha}'_i \end{array}\right. \Rightarrow \left\{\begin{array}{l} \partial_\nu = J^\mu_\nu \partial'_\mu + J^i_\nu \partial'_i + J^i_{\nu j} w^j \hat{\partial}'_i \\ \partial_j = J^i_j \partial'_i + J^i_{kj} w^k \hat{\partial}'_i \\ \hat{\partial}_j = J^i_j \hat{\partial}'_i \end{array}\right. \tag{2.14}$$

In view of transformation rules (2.11), we can set $v^\mu = 0$ independently of the basis to define vertical vectors of $\pi \circ \pi_V : V(B) \to M$, which then transform as

$$\left\{\begin{array}{l} \partial_j = J^i_j \partial'_i + J^i_{kj} w^k \hat{\partial}'_i \\ \hat{\partial}_j = J^i_j \hat{\partial}'_i \end{array}\right. \qquad \left\{\begin{array}{l} v'^i = J^i_j v^j \\ \hat{v}'^i = J^i_{kj} w^j v^k + J^i_j \hat{v}^j \end{array}\right. \tag{2.15}$$

Accordingly, the dual basis $(\bar{d}y^i, \bar{d}w^i)$ transforms as

$$\left\{\begin{array}{l} \bar{d}y'^i = J^i_j \bar{d}y^j \\ \bar{d}w'^i = J^i_{kj} w^j \bar{d}y^k + J^i_j \bar{d}w^j \end{array}\right. \tag{2.16}$$

Also in view of transformation rules (2.11), we can set $v^\mu = 0$ and $v^i = 0$ independently of the basis to define vertical vectors of $\pi_V : V(B) \to B$, which then transform as

$$\hat{\partial}_j = J^i_j \hat{\partial}'_i \qquad \hat{v}'^i = J^i_j \hat{v}^j \tag{2.17}$$

Accordingly, the dual basis $\bar{d}w^i$ transforms as

$$\bar{d}w'^i = J^i_k \bar{d}w^k \tag{2.18}$$

showing that the dual basis $\bar{d}w^i$ on $\pi_V : V(B) \to B$ is different from the element $\bar{d}w^i$ of the dual basis on $\pi \circ \pi_V : V(B) \to M$, although we use the same symbol for them. That is annoying and it needs extra care, though it is standard notation.

2.2.2 Local description of second order vertical bundle

The bundle $V^2(B)$ is fibered over $V(B)$, B, and M and has coordinates $(x^\mu; y^i, w^i, \zeta^i)$. It is defined as a sub-bundle of $V(V(B))$, which has fibered coordinates $(x^\mu; y^i, w^i, \xi^i, \zeta^i)$, by identifying $w^i = \xi^i$ which in fact transform in the same way.

The transformation on B induces a transformation on $V^2(B)$ as

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x,y) \\ w'^i = J^i_j(x,y)w^j \\ \zeta'^i = J^i_{kj}(x,y)w^k w^j + J^i_j(x,y)\zeta^j \end{cases} \Rightarrow \begin{cases} dx'^\mu = J^\mu_\nu dx^\nu \\ dy'^i = J^i_\mu dx^\mu + J^i_j dy^j \\ dw'^i = J^i_{\mu j} w^j dx^\mu + J^i_{kj} w^j dy^k + J^i_j dw^j \\ d\zeta'^i = (J^i_{kj\nu} w^k w^j + J^i_{j\nu}\zeta^j)dx^\nu + (J^i_{kjh} w^k w^j + J^i_{jh}\zeta^j)dy^h + \\ \qquad +2J^i_{kj} w^k dw^j + J^i_j d\zeta^j \end{cases} \tag{2.19}$$

That shows that $V^2(B)$ is an affine bundle on $V(B)$, though not on B, less than ever on M.

For components of a vector $\xi = v^\mu \partial_\mu + v^i \partial_i + \hat{v}^i \hat{\partial}_i + \dot{v}^i \dot{\partial}_i$ we have

$$\begin{cases} v'^\mu = J^\mu_\nu v^\nu \\ v'^i = J^i_\mu v^\mu + J^i_j v^j \\ \hat{v}'^i = J^i_{\mu j} w^j v^\mu + J^i_{kj} w^j v^k + J^i_j \hat{v}^j \\ \dot{v}'^i = (J^i_{kj\nu} w^k w^j + J^i_{j\nu}\zeta^j)v^\nu + (J^i_{kjh} w^k w^j + J^i_{jh}\zeta^j)v^h + 2J^i_{kj} w^k \hat{v}^j + J^i_j \dot{v}^j \end{cases} \tag{2.20}$$

while components of vertical vectors on $V^2(B)$ (over M) transform simply as

$$\begin{cases} v'^i = J^i_j v^j \\ \hat{v}'^i = J^i_{kj} w^j v^k + J^i_j \hat{v}^j \\ \dot{v}'^i = (J^i_{kjh} w^k w^j + J^i_{jh}\zeta^j)v^h + 2J^i_{kj} w^k \hat{v}^j + J^i_j \dot{v}^j \end{cases} \tag{2.21}$$

From these transformation rules we see that, also vertical vectors on the bundle $\pi_V \circ \pi_{V^2} : V^2(B) \to B$ are well defined and their components transforms as

$$\begin{cases} \hat{v}'^i = J^i_j \hat{v}^j \\ \dot{v}'^i = 2J^i_{kj} w^k \hat{v}^j + J^i_j \dot{v}^j \end{cases} \tag{2.22}$$

Also vertical vectors on the bundle $\pi_{V^2} : V^2(B) \to V(B)$ are well defined and their components transforms as

$$\dot{v}'^i = J^i_j \dot{v}^j \tag{2.23}$$

The components of covector, as well as the natural basis for vectors, transform as

$$\begin{cases} \alpha_\nu = J^\mu_\nu \alpha'_\mu + J^i_\nu \alpha'_i + J^i_{\nu j} w^j \hat{\alpha}'_i + \dot{\alpha}'_i (J^i_{kj\nu} w^k w^j + J^i_{j\nu}\zeta^j) \\ \alpha_j = J^i_j \alpha'_i + J^i_{kj} w^k \hat{\alpha}'_i + \dot{\alpha}'_i (J^i_{khj} w^k w^h + J^i_{hj}\zeta^h) \\ \hat{\alpha}_j = J^i_j \hat{\alpha}'_i + 2\dot{\alpha}'_i J^i_{kj} w^k \\ \dot{\alpha}_j = J^i_j \dot{\alpha}'_i \end{cases} \tag{2.24}$$

and hence we have

$$\left\{\begin{array}{l}\partial_\nu = J^\mu_\nu \partial'_\mu + J^i_\nu \partial'_i + J^i_{\nu j} w^j \hat{\partial}'_i + (J^i_{kj\nu} w^k w^j + J^i_{j\nu}\zeta^j)\dot{\partial}'_i \\ \partial_j = J^i_j \partial'_i + J^i_{kj} w^k \hat{\partial}'_i + (J^i_{khj} w^k w^h + J^i_{hj}\zeta^h)\dot{\partial}'_i \\ \hat{\partial}_j = J^i_j \hat{\partial}'_i + 2J^i_{kj} w^k \dot{\partial}'_i \\ \dot{\partial}_j = J^i_j \dot{\partial}'_i \end{array}\right. \tag{2.25}$$

In view of transformation rules (2.20), we can set $v^\mu = 0$ independently of the basis to define vertical vectors of $\pi \circ \pi_V \circ \pi_{V^2} : V^2(B) \to M$, which then transform as

$$\left\{\begin{array}{l}\partial_j = J^i_j \partial'_i + J^i_{kj} w^k \hat{\partial}'_i + (J^i_{khj} w^k w^h + J^i_{hj}\zeta^h)\dot{\partial}'_i \\ \hat{\partial}_j = J^i_j \hat{\partial}'_i + 2J^i_{kj} w^k \dot{\partial}'_i \\ \dot{\partial}_j = J^i_j \dot{\partial}'_i \end{array}\right. \qquad \left\{\begin{array}{l} v'^i = J^i_j v^j \\ \hat{v}'^i = J^i_{kj} w^j v^k + J^i_j \hat{v}^j \\ \dot{v}'^i = (J^i_{kjh} w^k w^j + J^i_{jh}\zeta^j) v^h + 2J^i_{kj} w^k \hat{v}^j + J^i_j \dot{v}^j \end{array}\right. \tag{2.26}$$

Accordingly, the dual basis $(\bar{d}y^i, \bar{d}w^i, \bar{d}\zeta^i)$ transforms as

$$\left\{\begin{array}{l}\bar{d}y'^i = J^i_j \bar{d}y^j \\ \bar{d}w'^i = J^i_{kj} w^j \bar{d}y^k + J^i_j \bar{d}w^j \\ \bar{d}\zeta'^i = (J^i_{kjh} w^k w^j + J^i_{jh}\zeta^j)\bar{d}y^h + 2J^i_{kj} w^k \bar{d}w^j + J^i_j \bar{d}\zeta^j \end{array}\right. \tag{2.27}$$

Also in view of transformation rules (2.20), we can set $v^\mu = 0$ and $v^i = 0$ independently of the basis to define vertical vectors of $\pi_V \circ \pi_{V^2} : V^2(B) \to B$, which then transform as

$$\left\{\begin{array}{l}\hat{\partial}_j = J^i_j \hat{\partial}'_i + 2J^i_{kj} w^k \dot{\partial}'_i \\ \dot{\partial}_j = J^i_j \dot{\partial}'_i \end{array}\right. \qquad \left\{\begin{array}{l}\hat{v}'^i = J^i_j \hat{v}^j \\ \dot{v}'^i = 2J^i_{kj} w^k \hat{v}^j + J^i_j \dot{v}^j \end{array}\right. \tag{2.28}$$

Accordingly, the dual basis $(\bar{d}w^i, \bar{d}\zeta^i)$ transforms as

$$\left\{\begin{array}{l}\bar{d}w'^i = J^i_j \bar{d}w^j \\ \bar{d}\zeta'^i = 2J^i_{kj} w^k \bar{d}w^j + J^i_j \bar{d}\zeta^j \end{array}\right. \tag{2.29}$$

Also in view of transformation rules (2.20), we can set $v^\mu = 0$, $v^i = 0$, and $\hat{v}^i$ independently of the basis to define vertical vectors of $\pi_{V^2} : V^2(B) \to V(B)$, which then transform as

$$\dot{\partial}_j = J^i_j \dot{\partial}'_i \qquad \dot{v}'^i = J^i_j \dot{v}^j \tag{2.30}$$

Accordingly, the dual basis $\bar{d}\zeta^i$ transforms as

$$\bar{d}\zeta'^i = J^i_j \bar{d}\zeta^j \tag{2.31}$$

This time the symbol $\bar{d}\zeta^j$ has three different meanings, depending on the bundle structure we choose on $V^2(B)$. Moreover, also $\bar{d}w^i$ and $\bar{d}y^i$ get extra meanings. The only way out is to be explicit about which bundle one is working on each time.

2.2.3 Higher order variations

Let us consider a bundle $[B \to M]$ and a vertical vector field $X : B \to V(B)$ and let us denote the corresponding vertical flow $\Phi_s : B \to B$ which, in turn, defines the dragging of a section σ as $\sigma_s = \Phi \circ \sigma$.

The local expression of the flow is $\Phi_s : B \to B : (x, y) \mapsto (x, \phi_s(x, y))$, $y' = \phi_s(x, y)$ for short, which relates to the components of the infinitesimal generator X as $X^i(x, y) := \frac{d\phi^i_s}{ds}|_{s=0}$.

The dragged section is locally expressed as

$$\sigma_s : x^\mu \mapsto (x^\mu, y^i_s(x) = \phi^i_s(x, y(x)))$$

which can be expanded to second order recalling that

$$\begin{aligned} &y^i_s(x) = \phi^i_s(x, y(x)) \Rightarrow y^i_s(x)\Big|_{s=0} = y^i(x) \\ &\frac{d}{ds} y^i_s(x) = \frac{d}{ds}\phi^i_s(x, y(x)) = X^i(x, y_s(x)) \\ &\frac{d^2}{ds^2} y^i_s(x) = \frac{d}{ds} X^i(x, y_s(x)) = \partial_j X^i(x, y_s(x)) X^j(x, y_s(x)) \end{aligned} \tag{2.32}$$

We get

$$y^i_s(x) = y^i(x) + X^i(x, y(x))s + \frac{1}{2}s^2 \partial_j X^i(x, y(x)) X^j(x, y(x)) + O(s^3) \tag{2.33}$$

So we can set $\delta y^i(x, y) := X^i(x, y)$ and $\delta^2 y^i(x, y) := \partial_j X^i(x, y) X^j(x, y)$.

Now, while the quantity δy^i can be geometrically well understood on its own as the components of a vertical vector field X on $[B \to M]$, the geometric nature of $\delta^2 y^i$ is considerably less trivial since, for example, we shall see it transforms in a non-linear way, quadratically in δy^i. Accordingly, $\delta^2 y^i$ is not even a geometric object on its own, and can be understood only together with δy^i.

If we change coordinates on B we have

$$\begin{cases} x'^\mu = x'^\mu(x) \\ y'^i = Y^i(x, y) \end{cases} \qquad y'^i_s(x') = Y^i(x, y_s(x)) \tag{2.34}$$

hence expanding we have

$$y'^i_s(x') = y'^i(x') + J^i_j X^j s + \frac{1}{2}s^2 (J^i_{kj} X^j X^k + J^i_j \partial_k X^j X^k) + O(s^3 \tag{2.35}$$

and we can set $\hat{X}^i := \partial_k X^i X^k$ and have

$$X'^i = J^i_j X^j \qquad \hat{X}'^i = J^i_{kj} X^j X^k + J^i_j \hat{X}^j \tag{2.36}$$

From transformation laws of X^i, we can also compute the transformation laws of $\hat{X}^i$ directly, i.e.

$$\hat{X}'^i = \partial'_k X'^i X'^k = J^k_l \partial'_k (J^i_j X^j) X^l = \partial_k (J^i_j X^j) X^k = J^i_{jk} X^j X^k + J^i_j \hat{X}^j \tag{2.37}$$

and, by comparing with transformation rules on $V^2(B)$ in, we see that

$$\hat{X} = X^i \frac{\partial}{\partial y^i} + \hat{X}^i \frac{\partial}{\partial X^i} \tag{2.38}$$

is a good global vertical vector field on $V^2(B)$.

Accordingly, we have the following theorem:

Theorem 6. *For any vertical vector field $X = X^i \frac{\partial}{\partial y^i}$ on B, we can define a global vector field*

$$\hat{X} = X^i \frac{\partial}{\partial y^i} + X^k \partial_k X^i \frac{\partial}{\partial w^i} \tag{2.39}$$

on $V^2(B)$ which is called the vertical lift of X to $V^2(B)$.

As we will show now, the vertical lift of X to $V^2(B)$ accounts for the geometric nature of the second variation of sections.

More precisely, given a vertical vector field $\xi = \xi^i(x,y)\partial_i$ on $(B \to M)$, (i.e. a section of $(V(B) \to B)$), we can lift it to a vertical vector field

$$\hat{\xi} = \xi^i(x,y)\partial_i + \zeta^i(x,y,w)\hat{\partial}_i \tag{2.40}$$

on $(V(B) \to B)$, (i.e. a section of $(V(V(B)) \to V(B) \to B)$), by setting $\zeta^i(x,y,w) := \partial_j \xi^i(x,y) w^j$.

In fact, $\xi'^i = J^i_k \xi^k$ and, in view of (2.14),

$$J^j_l \partial'_j \xi'^i = (J^j_l \partial'_j + J^j_{kl} w^k \hat{\partial}'_j)\xi'^i = \partial_l \xi'^i = J^i_{lk}\xi^k + J^i_k \partial_l \xi^k \tag{2.41}$$

Hence we get

$$\zeta'^i = \partial'_j \xi'^i(x,y) w'^j = J^j_l \partial'_j \xi'^i(x,y) w^l = J^i_{lk}\xi^k w^l + J^i_k \partial_l \xi^k w^l = J^i_{lk}\xi^k w^l + J^i_k \zeta^k \tag{2.42}$$

which in fact agrees with transformation rules (2.28).

Finally, we can restrict the lift $\hat{\xi}$ to a section $\xi \circ \sigma : M \to V(B)$ to obtain

$$\begin{aligned}
&\hat{\xi} \circ \xi \circ \sigma : M \to V^2(B) : \\
&x \mapsto (x, y(x)) \mapsto (x, y(x), w^i = \xi^i(x, y(x))) \mapsto \\
&(x, y(x), \xi^i(x, y(x))) \mapsto (x, y(x), \xi^i(x, y(x)), \zeta^i = \partial_j \xi^i(x, y(x))\xi^j(x, y(x))) \mapsto \\
&(x, y(x), \xi^i(x, y(x))) \mapsto (x, y(x), \delta y^i(x), \delta^2 y^i(x))
\end{aligned} \tag{2.43}$$

Therefore, we have

$$\hat{\xi} = \delta y^i(x)\partial_i + \delta^2 y^i(x)\hat{\partial}_i \tag{2.44}$$

as a vertical vector field on $(V(B) \to M)$ along the section $\xi \circ \sigma : M \to V(B)$.

That would be section of $[V(V(B)) \to M]$, that, in fact, restricts to a section of $(V^2(B) \to M)$.

The lift vector field $\hat{\xi} \circ \xi \circ \sigma : M \to V^2(B)$ is called the *second order deformation* and it accounts for the geometrical meaning of the local quantity $\delta^2 y^i(x)$.

Now we have to check that $\hat{\xi}$ is a global vector field.
This proof is straightforward, since in the intersection of two local coordinates systems we have:

$$\begin{aligned}\hat{\xi} = \delta y^i \frac{\partial}{\partial y^i} + \delta^2 y^i \frac{\partial}{\partial w^i} = \delta y^k (J^i_k \frac{\partial}{\partial y'^i} + J^i_{jk} w^j \frac{\partial}{\partial w'^i}) + \delta^2 y^k J^i_k \frac{\partial}{\partial w'^i} = \\ \delta y^k J^i_k \frac{\partial}{\partial y'^i} + (\delta y^k J^i_{jk} w^j + \delta^2 y^k J^i_k) \frac{\partial}{\partial w'^i} = \delta y'^i \frac{\partial}{\partial y'^i} + \delta^2 y'^i \frac{\partial}{\partial w'^i} = \hat{\xi}'\end{aligned} \tag{2.45}$$

To summarize, the situation is as follows: we start with a configuration bundle $\pi : B \to M$, a configuration, i.e. a section $\sigma : M \to B$, and a deformation, i.e. a vertical vector field ξ on B.
The deformation, (i.e. a section of $\pi_V : V(B) \to B$) induces a second variation $\hat{\xi}$ which is a section of $V^2(B) \subset V(V(B))$, obtained by the constraints $\xi^i = w^i$, as a sub-bundle $V^2(B) \to M$, which has hence coordinates $(x^\mu, y^i, w^i, \zeta^i)$.

We shall see in a while that we can select a dynamics on B by fixing a Lagrangian L on J^1B and, by variation, that induces a Lagrangian $L' = \delta L$ on $J^1V(B)$.

2.3 First variation on composite configuration bundles

We have a good global formulation of geometric variational principles on bundles which can be easily extended to composite bundles.

Let us consider a composite bundle $[C \to B \to M]$ of height 2, called the *configuration bundle*. Let us set $m := \dim(M)$.

Let us denote by (x^μ, y^i, z^A) a set of fibered coordinates on the configuration bundle.
A *configuration* is a pair of (global and smooth) sections $(\sigma, \hat{\rho})$, namely $\sigma : M \to B$ and $\hat{\rho} : B \to C$, and let us denote by Γ the functional space of such configurations. A *deformation* is a composite vertical vector field X on configuration bundle.

The deformation X is locally expressed as $X = X^i(x,y)\frac{\partial}{\partial y^i} + X^A(x,y,z)\frac{\partial}{\partial z^A}$.

Let us denote by $\Phi_\epsilon : C \to C$ its flow.

The flow is stratified and it is locally expressed as

$$x'^\mu = x^\mu \qquad y'^i = \phi^i_\epsilon(x,y) \qquad z'^A = \phi^A_\epsilon(x,y,z) \tag{2.46}$$

If we fix a configuration $(\sigma, \hat{\rho})$, then a deformation X defines a 1-parameter family of configurations $(\sigma_\epsilon, \hat{\rho}_\epsilon) = \Phi_\epsilon(\sigma, \hat{\rho})$.

Locally we have

$$\sigma_\epsilon(x) = (x, \phi^i_\epsilon(x, y(x))) \qquad \hat{\rho}_\epsilon(x, y) = (x, y, \phi^A_\epsilon(x, y, z(x, y))) \tag{2.47}$$

Then we need to decide which derivatives of configurations are relevant for the dynamics that we are going to select.
This is a choice among many possible options, even in the ordinary setting. In fact, if we are discussing a theory with two fields, the configuration bundle is a fibered product $B_1 \times_M B_2$ and we can decide our theory will be of order k_1 with respect to the field in B_1 and of order k_2 with respect to the field in B_2, thus accordingly defining the *Lagrangian phase space* as $\mathbb{P} := J^{k_1}B_1 \times_M J^{k_2}B_2$, hence having a different choice for any pair of natural numbers (k_1, k_2). Of course, one can always consider $k = \max(k_1, k_2)$ and write the dynamics on $J^k(B_1 \times_M B_2)$, though loosing some information about the sharp orders.

Similarly, on a composite bundle we can select the order of derivatives *and* which derivatives will appear in the dynamics. For example, we can treat C as a bundle over B, thus defining the partial derivatives with respect to x and y, or as a bundle over M, thus defining the partial derivatives with respect to x, only. That corresponds to different definitions of the Lagrangian phase space $\mathbb{P}$.

As it happens in the standard case, there is a canonical construction which accounts for all possible partial derivatives of fields. For the sake of simplicity, let us restrict to first order dynamics, thus $k = 1$.

$$\begin{tikzcd}
J^2C \arrow[d, "\pi^2_{1C}"] & J^2C \times_B J^2B \arrow[l] \arrow[d] \arrow[dr] & \\
J^1C \arrow[dr, "\pi^1_C"] & J^1C \times_B J^1B \arrow[l] \arrow[d] & J^2B \arrow[d, "\pi^2_{1B}"] \\
& C \arrow[d, "\pi^1_0"] & J^1B \arrow[dl, "\pi^1_B"] \\
& B \arrow[d, "\pi"'] & \\
& M \arrow[u, bend right, "\sigma"'] \arrow[uu, bend left, "\rho"] \arrow[uuul, bend left, "j^1\rho"] \arrow[uur, bend right, "j^1\sigma"'] &
\end{tikzcd}$$

The construction is by induction on the height of the configuration bundle. For ordinary bundles, we define the Lagrangian phase space as usual, hence to be the ordinary jet prolongation $\mathbb{P}^1_B := J^1B$ as a bundle over B.

Inductively, let us suppose we have defined $\mathbb{P}^1_{B_n}$ as a bundle over B_n; then we can set

$$\mathbb{P}^1_{B_{n+1}} := J^1B_{n+1} \times_{B_n} \mathbb{P}^1_{B_n} \tag{2.48}$$

In our example of a composite bundle $[C \to B \to M]$, we have the canonical, most general, choice, of a first order Lagrangian phase space, i.e.

$$\mathbb{P} := \mathbb{P}^1_C = J^1C \times_B (J^1B) \tag{2.49}$$

which is regarded as a bundle over C, i.e. we consider the following composite bundle

$$[\mathbb{P} \to C \to B \to M] \tag{2.50}$$

On the Lagrangian phase space $\mathbb{P}$, we have natural coordinates $(x^\mu, y^i, z^A; y^i_\mu, z^A_\mu, z^A_i)$ which, in fact, account for partial derivatives of y with respect to x as well as of z with respect to (x, y).

As it happens in the standard case, we can prolong composite maps.

For example, a composite automorphism $\Phi := [\Phi_C, \Phi_B, \phi]$ is such that both (Φ_C, Φ_B) and (Φ_B, ϕ) are bundle morphisms on $[C \to B]$ and $[B \to M]$, respectively.
Accordingly, we have the prolongations $J^1\Phi_C : J^1C \to C$ and $J^1\Phi_B : J^1B \to B$ and, consequently, we have the prolongation $\hat{\Phi} := (J^1\Phi_C \circ \pi^1_0) \times_B J^1\Phi_B : \mathbb{P} \to \mathbb{P}$ to the Lagrangian phase space.

Hence also sections and vector fields, can be prolonged from the configuration bundle to the Lagrangian phase bundle.

In particular, a deformation $X = X^i\partial_i + X^A\partial_A$ is prolonged as.

$$j^1X = X^i\partial_i + X^A\partial_A + X^i_\mu\partial^\mu_i + X^A_\mu\partial^\mu_A + X^A_i\partial^i_A \tag{2.51}$$

If we want to define the prolongations of composite sections we have to pay more attention, in fact we have this remark.

Remark 24. *In the case under consideration, if $(\sigma, \hat{\rho})$ is a configuration, we can define $j^1\hat{\rho} : B \to J^1C$ and $j^1\sigma : M \to J^1B$. But that the pair $(j^1\sigma, j^1\hat{\rho})$ is not a section of $\mathbb{P}$, as one can check by composition with the projections from $\mathbb{P}$ to B or M. However, we can consider the pair $j^1(\sigma, \hat{\rho}) := (j^1\sigma, j^1\hat{\rho} \circ \sigma)$ which in fact defines a section of $[\mathbb{P} \to M]$ which is called the prolongation of the configuration $(\sigma, \hat{\rho})$ to $\mathbb{P}$.*

Let the local expression of $(\sigma, \hat{\rho})$ be $(y^i(x), z^A(x, y))$, then its prolongation to $\mathbb{P}$ is locally expressed as

$$j^1(\sigma, \hat{\rho}) : x \mapsto (x^\mu, y^i(x), z^A(x, y(x)), \partial_\mu y^i(x), \partial_\mu z^A(x, y(x)), \partial_i z^A(x, y(x))) \tag{2.52}$$

We can deform a configuration $(y(x), z(x, y(x)))$ along a deformation as $y^i_\epsilon(x) = \phi^i_\epsilon(x, y(x))$ and $z^A_\epsilon(x) = z^A_\epsilon(x, y_\epsilon(x)) = \phi^A_\epsilon(x, y_\epsilon(x), z(x, y_\epsilon(x)))$ of which the derivatives can be taken as

$$\begin{cases} \frac{d}{d\epsilon} z^A_\epsilon(x, y_\epsilon(x)) = X^A + X^i\partial_i z^A \\ \frac{d}{d\epsilon}\partial_\mu z^A_\epsilon(x, y_\epsilon(x)) = \partial_\mu X^A + X^k\partial_{k\mu} z^A + \partial_\mu z^B\partial_B X^A \\ \frac{d/}{d\epsilon}\partial_i z^A_\epsilon(x, y_\epsilon(x)) = \partial_i X^A + \partial_i z^B\partial_B X^A + X^k\partial_{ki} z^A \end{cases} \tag{2.53}$$

from which we can compute, for later convenience, as

$$\begin{aligned} &\frac{d}{d\epsilon} d_\mu z^A_\epsilon(x, y_\epsilon(x)) = \frac{d}{d\epsilon}(\partial_\mu z^A_\epsilon(x, y_\epsilon(x)) + d_\mu y^k_\epsilon \partial_k z^A_\epsilon(x, y_\epsilon(x))) = \\ &= \partial_\mu X^A + X^k\partial_{\mu k} z^A + \partial_\mu z^B\partial_B X^A + d_\mu X^k\partial_k z^A + y^i_\mu(\partial_i X^A + \partial_i z^B\partial_B X^A + X^k\partial_{ki} z^A) = \\ &= (\partial_\mu X^A + y^i_\mu\partial_i X^A + (\partial_\mu z^B + y^i_\mu\partial_i z^B)\partial_B X^A) + X^i(\partial_{\mu i} z^A + y^k_\mu\partial_{ki} z^A) + d_\mu X^i\partial_i z^A = \\ &= d_\mu X^A + X^i d_\mu\partial_i z^A + d_\mu X^i\partial_i z^A \end{aligned} \tag{2.54}$$

Whatever the definition of Lagrangian phase space $\mathbb{P}$ is, the relevant structure for variational calculus is a subset of sections of it which corresponds to *prolongations* of sections of the configuration bundle, i.e. of configurations. These special sections of $\mathbb{P}$, which arise as prolongations of configurations, are called *holonomic*.

Once one has holonomic sections, one can define the *contact ideal* $\mathcal{K}$ made of forms on $\mathbb{P}$ which vanish on holonomic sections. The contact ideal, in turn, defines *horizontal forms* as what is left when we take out contact part. That is relevant since, as we shall see, contact forms do not contribute to the action functional, which is computed along holonomic sections.

One can easily show that any form on $\mathbb{P}^1$ can be pulled back on $\mathbb{P}^2$ along the projection $\pi^2_1 : \mathbb{P}^2 \to \mathbb{P}^1$ and can be split canonically into a horizontal and a contact part, possibly at the price of increasing the jet prolongation order by 1. In fact, we have contact 1-forms

$$\begin{aligned} \omega^i &:= dy^i - y^i_\nu dx^\nu \\ \omega^A &:= dz^A - z^A_k dy^k - z^A_\nu dx^\nu \end{aligned} \tag{2.55}$$

$$\begin{aligned} \omega^i_\mu &:= dy^i_\mu - y^i_{\mu\nu} dx^\nu \\ \omega^A_\mu &:= dz^A_\mu - z^A_{k\mu} dy^k - z^A_{\mu\nu} dx^\nu \\ \omega^A_i &:= dz^A_i - z^A_{ki} dy^k - z^A_{i\nu} dx^\nu \end{aligned} \tag{2.56}$$

A form which is both horizontal and contact is necessarily zero so the sum is direct and any form ω on $\mathbb{P}$ can be split canonically into its horizontal $h(\omega)$ and contact $k(\omega)$ parts, so that $\omega = h(\omega) \oplus k(\omega)$, only both $h(\omega)$ and $k(\omega)$ are on a 1-higher jet prolongation than ω.

Accordingly, we can focus on horizontal forms only, without loss of generality.

In particular, we define the *Lagrangian* as a horizontal m-form on $\mathbb{P}$.
In this work we shall focus on first order *stratified Lagrangians*, i.e. Lagrangians depending on the first derivatives of z^A only through the combination $d_\mu z^A := z^A_\mu + z^A_k y^k_\mu$, which are locally expressed as

$$L = L(x^\mu, y^i, y^i_\mu, z^A, z^A_\mu + z^A_i y^i_\mu) d\sigma \tag{2.57}$$

and the *action functional* as the integral of the Lagrangian along a configuration, namely

$$A_D[\sigma, \hat\rho] = \int_D (j^1(\sigma, \hat\rho))^* L \tag{2.58}$$

where $D \subset M$ is an *m-region*, i.e. a compact sub-manifold of dimension m with a boundary which is a compact sub-manifold of dimension $(m-1)$.

Notice that the integrand $(j^1(\sigma, \hat\rho))^* L$ of the action functional is, by construction, an m-form on M, thus it can be integrated on D, the result being (intrinsic and) finite since D is assumed to be compact.

We define a *critical configuration* $(\sigma, \hat\rho)$ to be a configuration which makes the action functional stationary, i.e., by *Hamilton principle*, a section such that, for any m-region D and any deformation X supported in D, one has

$$\delta_X A_D[\sigma, \hat\rho] = 0 \tag{2.59}$$

where the variation $\delta_X A_D[\sigma, \hat\rho]$ is defined as the infinitesimal generator

$$\delta_X A_D[\sigma, \hat\rho] = \frac{dA_D}{ds}[\sigma_s, \hat\rho_s]\Big|_{s=0} = \int_D \frac{d}{ds}((j^1(\sigma_s, \hat\rho_s))^* L)\Big|_{s=0} \tag{2.60}$$

One can show that the variation of the Lagrangian δL splits into the sum of two global operators, the *Euler-Lagrange* operator and the divergence of the *Poincaré-Cartan* operator, which captures all features of first order variational principles.

The global Euler-Lagrange operator is defined as $\mathbb{E}(L) : \mathbb{P}^2 \to V^*(C) \otimes A_m(M)$ to be

$$\mathbb{E}(L) := (\mathbb{E}_i \bar{d}y^i + \mathbb{E}_A \bar{d}z^A) \otimes d\sigma$$

where $A_m(M)$ (or $A_k(M)$) is the bundle of which m-forms (or k-forms) over M are sections. The global Poincaré-Cartan operator is defined as $\mathbb{F}(L) : \mathbb{P} \to V^*(C) \otimes A_{m-1}(M)$ to be

$$\mathbb{F}(L) := (\mathbb{F}^\mu_i \bar{d}y^i + \mathbb{F}^\mu_A \bar{d}z^A) \otimes d\sigma_\mu \tag{2.61}$$

Together they define the *first variation formula* which is

$$< \delta L, j^1 X >=< \mathbb{E}(L), X > +d < \mathbb{F}(L), X > \tag{2.62}$$

where we denote by $< \cdot, \cdot >$ the duality on $V(C)$ (as well as the induced duality on $V(\mathbb{P})$).

Locally, we set the *(naive) momenta* to be

$$p_i := \frac{\partial L}{\partial y^i} \qquad p^\mu_i := \frac{\partial L}{\partial y^i_\mu} \qquad p_A := \frac{\partial L}{\partial z^A} \qquad p^\mu_A := \frac{\partial L}{\partial d_\mu z^A} \tag{2.63}$$

and, along a deformation X, we have

$$\begin{aligned}
&\delta L = (p_i X^i + p^\mu_i d_\mu X^i + p_A(X^A + X^i \partial_i z^A) + p^\mu_A(d_\mu X^A + X^i d_\mu \partial_i z^A + d_\mu X^i \partial_i z^A)) = \\
&((p_i + p_A \partial_i z^A + p^\mu_A d_\mu \partial_i z^A) X^i + (p^\mu_i + p^\mu_A \partial_i z^A) d_\mu X^i + p_A X^A + p^\mu_A d_\mu X^A) = \\
&(p_i - d_\mu p^\mu_i + (p_A - d_\mu p^\mu_A)\partial_i z^A) X^i + (p_A - d_\mu p^\mu_A) X^A + \\
&+ d_\mu((p^\mu_i + p^\mu_A \partial_i z^A) X^i + p^\mu_A X^A)
\end{aligned} \tag{2.64}$$

Accordingly, we can define Euler-Lagrange operator by setting

$$\begin{cases} \mathbb{E}_i = p_i - d_\mu p^\mu_i + (p_A - d_\mu p^\mu_A)\partial_i z^A \\ \mathbb{E}_A = p_A - d_\mu p^\mu_A \end{cases} \tag{2.65}$$

as well as Poincaré-Cartan operator by setting

$$\begin{cases} \mathbb{F}^\mu_i = p^\mu_i + p^\mu_A \partial_i z^A \\ \mathbb{F}^\mu_A = p^\mu_A \end{cases} \tag{2.66}$$

Thus we have *Euler-Lagrange equations* to be satisfied by critical sections

$$\begin{cases} p_i - d_\mu p^\mu_i = 0 \\ p_A - d_\mu p^\mu_A = 0 \end{cases} \tag{2.67}$$

These equations are global, every times the Lagrangian is global.

Notice that if we set $M = \mathbb{R}$, $B = \mathbb{R} \times Q$ for same manifold Q of dimension $\dim(Q) = n > 1$ and $C = \mathbb{R} \times X$ for some bundle $[X \to Q]$, then the Euler-Lagrange equations are ODE with respect to σ and PDE with respect to $\hat{\rho}$.
In other words, composite variational principles are a way to have ODE and PDE simultaneously in a single framework, from a single action functional.

As we have just saw, composite bundles is a very flexible framework to deal with variational problems. However there is more: we will use composite bundles to give a global geometric formulation of second variation. In order to do that we will have to compute the composite variation of an auxiliary Lagrangian; that is why we had to develop this theory of composite variations.
This topics will be faced in the next chapter.

Chapter 3

Jacobi Lagrangian and linearised equations

At this point one of the main questions could be: why we had to develop a more generic framework for calculus of variations?
Besides the fact that this theory is relevant just because it extends the usual calculus of variations to a more general setting, our aim is to use this framework to give a global geometric formulation of second variation and then to study linearised equations and perturbation theory for a given Lagrangian.
There are the topics of the next chapters, in particular we will discuss Jacobi equations and Linearised equations for first and second order Lagrangians.

3.1 Linearised equations for first order Lagrangians

Let us consider an ordinary variational principle, defined on a configuration bundle $[B \to M]$ and described by a first order Lagrangian L.
As it is known from the ordinary calculus of variations, we can compute the first variation of L, namely

$$L' = \delta L(j^1 X) = (p_i X^i + p_i^\mu d_\mu X^i) d\sigma \tag{3.1}$$

The crucial observation is that the first variation is also a fibered morphism:

$$\begin{aligned} \delta L : J^1 V(B) &\to A_m(M) \\ j^1 X &\mapsto \delta L(j^1 X) = (p_i X^i + p_i^\mu d_\mu X^i) d\sigma \end{aligned} \tag{3.2}$$

then, we can consider $L' = \langle \delta L, J^1 X \rangle$ as an auxiliary Lagrangian on the composite bundle $[V(B) \to B \to M]$: this Lagrangian is linear in the vertical vector field X, and will be called *Jacobi Lagrangian.*

Then, knowing how to deal with composite calculus of variations, we can consider a deformation $\Xi = \delta y^i \frac{\partial}{\partial y^i} + \delta X^i \frac{\partial}{\partial X^i}$ on $V(B)$ and compute the variation of the Lagrangian L'. We have

$$\begin{aligned}\delta L'(j^1\Xi) &= (p_i\delta X^i + p_i^\mu d_\mu \delta X^i) + (\partial_k p_i \delta y^k + \partial_k^\alpha p_i d_\alpha \delta y^k)X^i + (\partial_k p_i^\mu \delta y^k + \partial_k^\alpha p_i^\mu d_\alpha \delta y^k) d_\mu X^i = \\ &= (p_i\delta X^i + p_i^\mu d_\mu \delta X^i) + (\partial_k p_i X^i + \partial_k p_i^\mu d_\mu X^i)\delta y^k + (\partial_k^\alpha p_i X^i + \partial_k^\alpha p_i^\mu d_\mu X^i) d_\alpha \delta y^k\end{aligned} \tag{3.3}$$

The variation $\delta L'$ can then be integrated by parts as

$$\begin{aligned}\delta L' = &(p_i - d_\mu p_i^\mu)\delta X^i + (\partial_k p_i X^i + \partial_k p_i^\mu d_\mu X^i - d_\alpha(\partial_k^\alpha p_i X^i + \partial_k^\alpha p_i^\mu d_\mu X^i))\delta y^k + \\ &+ d_\mu(p_i^\mu \delta X^i + (\partial_k^\mu p_i X^i + \partial_k^\mu p_i^\nu d_\nu X^i)\delta y^k)\end{aligned} \tag{3.4}$$

We thus obtain two field equations for σ and X as:

$$\begin{cases} p_i - d_\mu p_i^\mu = 0 \\ (\partial_k p_i - d_\alpha \partial_k^\alpha p_i)X^i + (\partial_k p_i^\mu - d_\alpha \partial_k^\alpha p_i^\mu - \partial_k^\mu p_i) d_\mu X^i - \partial_k^\nu p_i^\mu d_{\nu\mu} X^i = 0 \end{cases} \tag{3.5}$$

the first equation says that σ needs to be a critical section of L, the second is equivalent to say that $X = X^i\partial_i$ is tangent to the Euler-Lagrange equation for L (defined as a submanifold in J^2B). In fact, it can be recast as

$$X^i\partial_i(\partial_k p - d_\alpha p_k^\alpha) + d_\mu X^i \partial_i^\mu(\partial_k p - d_\alpha p_k^\alpha) + d_{\mu\nu} X^i \partial_i^{\mu\nu}(\partial_k p - d_\alpha p_k^\alpha) = 0 \tag{3.6}$$

i.e. $j^2X(\mathbb{E}_k(L)) = 0$, which exactly expresses the fact that j^2X is tangent to the submanifold $\mathbb{E}_k(L) = \partial_k p - d_\alpha p_k^\alpha = 0$ in J^2B, which in fact represents the field equations.
Accordingly, the second equation of (3.5) is called the *linearised equation* of L. Its solutions X are vertical vector fields tangent to the equations and are called *Jacobi fields*.
Jacobi field equations (3.6) are linear in X, therefore Jacobi fields form a real vector space, $\mathcal{J}$.
They are second order equations and, if the Lagrangian is non-degenerate (as it happens, for example, in mechanics), they can be put in normal form and Cauchy theorem holds true: there is one Jacobi field for any initial condition $(X, \dot{X})$ on the image of σ.
For the Lagrangian of geodesics, as well as in mechanics, the Cauchy surface is a point and initial conditions are $2k$-numbers. Thus $\mathcal{J}$ is a $2k$-dimensional vector space over $\mathbb{R}$.

Remark 25. *In the previous section we introduced the Composite Euler Lagrange equations.*
Being the Jacobi Lagrangian a composite Lagrangian, we could have derived its field equations as a particular case of (2.65)*. Looking to the form of equations* (3.5) *this is obvious: the first equation of* (2.65) *is the standard Euler Lagrange equation, that corresponds to the first equation of* (3.5)*. The second equation of* (2.65) *is the "new" equation from the framework of composite bundles, that corresponds to the second equation of* (3.5)*, that in this case corresponds to the linearised equations.*

The most important remark is that the second equation of (3.5) represents the second variation of the Lagrangian.
This can be checked by direct computation of the second variation.

Given a section σ and a 1− parameter family of deformations, we have:

$$\begin{aligned}
&\frac{d^2}{ds^2}L = p_i(x, y^i_s, \partial_\mu y^i_s)\frac{d}{ds}X^i(x, y^i_s(s)) + p^\mu_i((x, y^i_s, \partial_\mu y^i_s))\frac{d}{ds}d_\mu X^i(x, y^i_s(x)) + \\
&\left[\partial_k p_i(x, y^i_s, \partial_\mu y^i_s)X^k(x, y^i_s(x)) + \partial^\alpha_k p_i(x, y^i_s, \partial_\mu y^i_s)d_\alpha X^k(x, y^i_s(x))\right] X^i(x, y^i_s(x)) + \\
&\left[\partial_k p^\mu_i(x, y^i_s, \partial_\mu y^i_s)X^k(x, y^i_s(x)) + \partial^\alpha_k p^\mu_i(x, y^i_s, \partial_\mu y^i_s)d_\alpha X^k(x, y^i_s(x))\right] d_\mu X^i(x, y^i_s(x))
\end{aligned} \tag{3.7}$$

where we set $\delta^2 y^i = \partial_j X^i X^j$ and $\delta^3 y^i = \partial_{jk} X^i X^k X^j + \partial_j X^i \partial_k X^j X^k$

If we specialize the equation for $s = 0$ we get the second variation

$$\begin{aligned}
&\delta^2 L = \frac{d^2}{ds^2}L|_{s=0} = \\
&(p_i\delta^2 y^i + p^\mu_i d_\mu \delta^2 y^i) + (\partial_k p_i \delta y^k + \partial^\alpha_k p_i d_\alpha \delta y^k)\delta y^i + (\partial_k p^\mu_i \delta y^k + \partial^\alpha_k p^\mu_i d_\alpha \delta y^k) d_\mu \delta y^i
\end{aligned} \tag{3.8}$$

So we can see how that corresponds to set $X^i = \delta y^i$ (and consequently $\delta X^i = \delta^2 y^i$) in the variation $\delta L'$ of the Jacobi Lagrangian.
The last thing we have to check is that the second variation is a geometrical object, i.e. that is is globally well defined.
We consider a global Lagrangian on B,

$$L = L(x^\mu, y^i, y^i_\mu)d\sigma \tag{3.9}$$

For globality, we have $JL'(x', y'^i, y'^i_\mu) = L(x^\mu, y^i, y^i_\mu)$ and consequently

$$p^\alpha_k = J\bar{J}^\alpha_\mu p'^\mu_i J^i_k \qquad p_k = Jp'_i J^i_k + J\bar{J}^\mu_\alpha p'^\alpha_i (J^i_{\mu k} + J^i_{jk} y^j_\mu) = J(p'_i J^i_k + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_k)) \tag{3.10}$$

at second order, we have

$$\partial^\mu_i p^\alpha_k = J\bar{J}^\alpha_\beta \bar{J}^\mu_\nu \partial'^\nu_j p'^\beta_h J^h_k J^j_i \tag{3.11}$$

$$\partial_i p^\alpha_k = J\bar{J}^\alpha_\mu (\partial'_l p'^\mu_j J^j_k J^l_i + \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_j J^j_k (d_\sigma J^l_i) + p'^\mu_j J^j_{ik}) = \partial^\alpha_k p_i \tag{3.12}$$

Finally, we have

$$\begin{aligned}
&\partial_j p_k = \\
&= J(\partial'_l p'_i J^l_j J^i_k + 2\bar{J}^\sigma_\rho \partial'^\rho_l p'_i J^i_k (d_\sigma J^l_j) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k) + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_{jk}) + p'_i J^i_{jk})
\end{aligned} \tag{3.13}$$

We see that, although p^α_k and $\partial^\mu_i p^\alpha_k$ are well-defined geometrical objects, the other momenta are not. However, there are a number of geometric objects defined out of them which should be pin pointed out, since we expect to provide a good intrinsic formalism for variational calculus.

Finally we can check transformation rules for second variation.

$$
\begin{aligned}
&\delta^2 L := (p_i\delta^2 y^i + p_i^\mu d_\mu \delta^2 y^i) + (\partial_k p_i \delta y^i \delta y^k + 2\partial_i p_k^\alpha \delta y^i d_\alpha \delta y^k + \partial_k^\alpha p_i^\mu d_\mu \delta y^i d_\alpha \delta y^k) d\sigma = \\
&= J\Big[((p'_i J^i_k + \bar{J}^\lambda_\mu p'^\mu_i d_\lambda J^i_k)\delta^2 y^k + \bar{J}^\alpha_\mu p'^\mu_i J^i_k d_\alpha \delta^2 y^k) + \\
&+ (\partial'_l p'_i J^l_j J^i_k + 2\bar{J}^\sigma_\rho \partial'^\rho_l p'_i J^i_k (d_\sigma J^l_j) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k) + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_{jk}) + p'_i J^i_{jk})\delta y^k \delta y^j + \\
&+ 2\bar{J}^\alpha_\mu (\partial'_l p'^\mu_j J^j_k J^l_i + \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_j J^j_k (d_\sigma J^l_i) + p'^\mu_j J^j_{ik})\delta y^i d_\alpha \delta y^k + \\
&+ \bar{J}^\alpha_\beta \bar{J}^\mu_\nu \partial'^\nu_j p'^\beta_h J^h_k J^j_i d_\alpha \delta y^k d_\mu \delta y^i\Big] d\sigma = \\
&= \Big[(p'_i(\delta^2 y'^i - J^i_{kj}\delta y^j \delta y^k) + p'^\mu_i(\bar{d}'_\mu \delta^2 y'^i - \bar{J}^\alpha_\mu d_\alpha (J^i_{kj}\delta y^j \delta y^k))) + (\partial'_j p'_i \delta y'^i \delta y'^j) + \\
&+ (2\bar{J}^\sigma_\rho \partial'^\rho_l p'_i J^i_k (d_\sigma J^l_j) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k) + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_{jk}) + p'_i J^i_{jk})\delta y^k \delta y^j + \\
&+ 2\bar{J}^\alpha_\mu (\partial'_l p'^\mu_j J^j_k J^l_i + \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_j J^j_k (d_\sigma J^l_i) + p'^\mu_j J^j_{ik})\delta y^i \bar{J}^k_i (J^\mu_\alpha \delta y'^i_\mu - d_\alpha J^i_j \delta y^j) + \\
&+ \partial'^\nu_j p'^\beta_l (\delta y'^l_\beta \delta y'^j_\nu - \bar{J}^\alpha_\beta d_\alpha J^l_k \delta y^k \delta y'^j_\nu - \bar{J}^\mu_\nu d_\mu J^j_n \delta y^n \delta y'^l_\beta + \bar{J}^\alpha_\beta d_\alpha J^l_k \bar{J}^\mu_\nu d_\mu J^j_n \delta y^k \delta y^n)\Big] d\sigma' = \\
&= \Big[(p'_i \delta^2 y'^i + p'^\mu_i \bar{d}'_\mu \delta^2 y'^i) + (\partial'_j p'_i \delta y'^i \delta y'^j + \partial'^\nu_j p'^\beta_l \delta y'^l_\beta \delta y'^j_\nu) + \\
&- \bar{J}^\alpha_\mu p'^\mu_i (d_\alpha J^i_{kj})\delta y^j \delta y^k - 2p'^\mu_j J^j_{ki} \bar{J}^k_n \delta y^j i \delta y'^n_\mu + 2\bar{J}^\alpha_\mu p'^\mu_j J^j_{ki} \bar{J}^k_l d_\alpha J^l_h \delta y^i \delta y^h + \\
&+ (2\bar{J}^\sigma_\rho \partial'^\rho_l p'_i J^i_k (d_\sigma J^l_j) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_i (d_\sigma J^l_j)(d_\lambda J^i_k) + \bar{J}^\lambda_\mu p'^\mu_i (d_\lambda J^i_{jk}))\delta y^k \delta y^j + \\
&+ 2(\partial'_l p'^\mu_n J^l_i + \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_n (d_\sigma J^l_i) + p'^\mu_j J^j_{ik} \bar{J}^k_n)\delta y^i \delta y'^n_\mu + \\
&- 2(\bar{J}^\sigma_\rho \partial'_l p'^\rho_n J^l_i (d_\sigma J^n_h) + \bar{J}^\lambda_\mu \bar{J}^\sigma_\rho \partial'^\rho_l p'^\mu_n (d_\sigma J^l_i)(d_\lambda J^n_h) + \bar{J}^\alpha_\mu p'^\mu_j J^j_{ik} \bar{J}^k_l d_\alpha J^l_h)\delta y^i \delta y^h + \\
&- \bar{J}^\sigma_\rho \partial'^\mu_n p'^\rho_l (d_\sigma J^l_i)\delta y^i \delta y'^n_\mu - \bar{J}^\sigma_\nu \partial'^\nu_j p'^\mu_n (d_\sigma J^j_i)\delta y^i \delta y'^n_\mu + \\
&+ \bar{J}^\sigma_\rho \bar{J}^\lambda_\mu \partial'^\mu_i p'^\rho_l (d_\sigma J^l_j)(d_\lambda J^i_k)\delta y^k \delta y^j\Big] d\sigma' = \Big[(p'_i \delta^2 y'^i + p'^\mu_i \bar{d}'_\mu \delta^2 y'^i) + \\
&+ (\partial'_j p'_i \delta y'^i \delta y'^j + 2\partial'_i p'^\alpha_k \delta y'^i \delta y'^k_\alpha + \partial'^\nu_j p'^\beta_l \delta y'^l_\beta \delta y'^j_\nu)\Big] d\sigma' = \delta^2 L'
\end{aligned}
\tag{3.14}
$$

This proves that the second variation is a good geometric object.

3.2 Linearised equations for second order Lagrangians

In the previous section we defined Jacobi Lagrangian associated to a first order Lagrangian. However our interest is toward second order Lagrangians. This is because we want to apply linearised equations theory to general relativity, which is a second order theory.
We introduced the Jacobi Lagrangian for a first order Lagrangian only to simplify the treatment and point out the relevant points, such as how linearised equations can be obtained and how Jacobi Lagrangian is related to second variation.
In this section we want to extend our definitions to the case of a second order Lagrangian. There is not any conceptual complication, knowing what happens in the first order case.
Let us consider a Lagrangian $L = L(x^\mu, y^i, y^i_\mu, y^i_{\mu\nu})d\sigma$ on a bundle $B \to M$ and a vertical vector field $X = X^i(x, y)\partial_i \in \mathfrak{X}(B)$.

We can compute Euler-Lagrange equations associated to L as

$$\delta_X A_D(\sigma) = \int_D (p_i X^i + p_i^\mu d_\mu X^i + p_i^{\mu\nu} d_{\mu\nu} X^i) d\sigma = 0$$

Integrating by parts we get

$$\delta_X A_D(\sigma) = \int_D (p_i X^i - d_\mu p_i^\mu X^i - d_\mu p_i^{\mu\nu} d_\nu X^i) d\sigma + \int_D d_\mu (p_i^\mu X^i + p_i^{\mu\nu} d_\nu X^i) d\sigma_\mu =$$
$$= \int_D (p_i X^i - d_\mu p_i^\mu X^i + d_{\mu\nu} p_i^{\mu\nu} X^i) d\sigma + \int_D d_\mu (p_i^\mu X^i + p_i^{\mu\nu} d_\nu X^i - d_\nu p_i^{\mu\nu} X^i) d\sigma_\mu = 0 \tag{3.15}$$

that leads to the equations

$$\mathbb{E}_i = p_i X^i - d_\mu p_i^\mu X^i + d_{\mu\nu} p_i^{\mu\nu} X^i = 0 \tag{3.16}$$

Now we can define, as we did in the first order case, the Jacobi Lagrangian.

$$L' =< \delta L, j^2 X >= p_i X^i + p_i^\mu d_\mu X^i + p_i^{\mu\nu} d_{\mu\nu} X^i \tag{3.17}$$

Then we get

$$\begin{aligned}
&\delta L'(j^2\Xi) = p_i \delta X^i + p_i^\mu d_\mu \delta X^i + p_i^{\mu\nu} d_{\mu\nu} \delta X^i + \\
&(\partial_k p_i \delta y^k X^i + \partial_k p_i^\mu \delta y^k d_\mu X^i + \partial_k p_i^{\mu\nu} \delta y^k d_{\mu\nu} X^i) + \\
&+ (\partial_k^\alpha p_i d_\alpha \delta y^k X^i + \partial_k^\alpha p_i^\mu d_\alpha \delta y^k d_\mu X^i + \partial_k^\alpha p_i^{\mu\nu} d_\alpha \delta y^k d_{\mu\nu} X^i) + \\
&+ (\partial_k^{\alpha\beta} p_i d_{\alpha\beta} \delta y^k X^i + \partial_k^{\alpha\beta} p_i^\mu d_{\alpha\beta} \delta y^k d_\mu X^i + \partial_k^{\alpha\beta} p_i^{\mu\nu} d_{\alpha\beta} \delta y^k d_{\mu\nu} X^i) = \\
&= (p_i \delta X^i + p_i^\mu d_\mu \delta X^i + p_i^{\mu\nu} d_{\mu\nu} \delta X^i) + \\
&+ (\partial_k p_i X^i + \partial_k p_i^\mu d_\mu X^i + \partial_k p_i^{\mu\nu} d_{\mu\nu} X^i) \delta y^k + \\
&+ (\partial_k^\alpha p_i X^i + \partial_k^\alpha p_i^\mu d_\mu X^i + \partial_k^\alpha p_i^{\mu\nu} d_{\mu\nu} X^i) d_\alpha \delta y^k + \\
&+ (\partial_k^{\alpha\beta} p_i X^i + \partial_k^{\alpha\beta} p_i^\mu d_\mu X^i + \partial_k^{\alpha\beta} p_i^{\mu\nu} d_{\mu\nu} X^i) d_{\alpha\beta} \delta y^k
\end{aligned} \tag{3.18}$$

The second variation can be integrated by parts as

$$\begin{aligned}
&\delta L' = (p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu}) \delta X^i + \\
&+ \Big[(\partial_k p_i X^i + \partial_k p_i^\mu d_\mu X^i + \partial_k p_i^{\mu\nu} d_{\mu\nu} X^i) - d_\alpha (\partial_k^\alpha p_i X^i + \partial_k^\alpha p_i^\mu d_\mu X^i + \partial_k^\alpha p_i^{\mu\nu} d_{\mu\nu} X^i) + \\
&d_{\alpha\beta} (\partial_k^{\alpha\beta} p_i X^i + \partial_k^{\alpha\beta} p_i^\mu d_\mu X^i + \partial_k^{\alpha\beta} p_i^{\mu\nu} d_{\mu\nu} X^i) \Big] \delta y^k + \\
&+ d_\alpha \Big[(p_i^\alpha \delta X^i + p^{\alpha\nu} d_\nu \delta X^i - d_\mu p^{\mu\alpha} \delta X^i) + (\partial_k^\alpha p_i X^i + \partial_k^\alpha p_i^\mu d_\mu X^i + \partial_k^\alpha p_i^{\mu\nu} d_{\mu\nu} X^i) \delta y^k + \\
&+ (\partial_k^{\alpha\beta} p_i X^i + \partial_k^{\alpha\beta} p_i^\mu d_\mu X^i + \partial_k^{\alpha\beta} p_i^{\mu\nu} d_{\mu\nu} X^i) d_\beta \delta X^k - d_\beta (\partial_k^{\alpha\beta} p_i X^i + \partial_k^{\alpha\beta} p_i^\mu d_\mu X^i + \partial_k^{\alpha\beta} p_i^{\mu\nu} d_{\mu\nu} X^i \delta y^k) \Big]
\end{aligned} \tag{3.19}$$

This leads to the equations:

$$\begin{cases}
p_i - d_\mu p_i^\mu + d_{\mu\nu} p_i^{\mu\nu} = 0 \\
\\
(\partial_k p_i - d_\alpha \partial_k^\alpha p_i + d_{\alpha\beta} \partial_k^{\alpha\beta} p_i) X^i + (\partial_k p_i^\mu - \partial_k^\mu p_i - d_\alpha \partial_k^\alpha p_i^\mu + d_{\alpha\beta} \partial_k^{\alpha\beta} p_i^\mu) d_\mu X^i + \\
+ (\partial_k p_i^{\mu\nu} - \partial_k^\nu p_i^\mu - d_\alpha \partial_k^\alpha p_i^{\mu\nu} + \partial_k^{\mu\nu} p_i + d_{\alpha\beta} \partial_k^{\alpha\beta} p_i^{\mu\nu}) + \\
(\partial_k^{\alpha\nu} p_i^\mu - \partial_k^\alpha p_i^{\mu\nu}) d_{\mu\nu\alpha} + \partial_k^{\alpha\beta} p_i^{\mu\nu} d_{\alpha\beta\mu\nu} X^i = 0
\end{cases} \tag{3.20}$$

In particular we can recast the last equation as:

$$X^i\partial_i(p_k - d_\mu p_k^\mu + d_{\mu\nu}p_k^{\mu\nu}) + d_\mu X^i\partial_i^\mu(p_k - d_\alpha p_k^\alpha + d_{\alpha\nu}p_k^{\alpha\nu}) + d_{\mu\nu}X^i\partial_i^{\mu\nu}(p_k - d_\alpha p_k^\alpha + d_{\alpha\beta}p_k^{\alpha\beta}) +$$
$$+ d_{\mu\nu\eta}X^i\partial_i^{\mu\nu\eta}(p_k - d_\alpha p_k^\alpha + d_{\alpha\beta}p_k^{\alpha\beta}) + d_{\mu\nu\eta\gamma}X^i\partial_i^{\mu\nu\eta\gamma}(p_k - d_\alpha p_k^\alpha + d_{\alpha\beta}p_k^{\alpha\beta}) = 0 \tag{3.21}$$

And that is equivalent to say that the field X is tangent to the sub-manifold defined by Euler Lagrange equations, namely $j^4X(\mathbb{E}_k) = 0$.
Of course, one can also check that the second variation of a second order Lagrangian is a geometric object. This proof is conceptually straightforward recalling how we proved the same statement in the first order case.
To clarify these constructions, we think it is better to provide a direct example of computation of the second variation associated to a specific Lagrangian.
Although a more relevant (and complicated) physical example will be provided in the next chapter, it is useful to start dealing with these computations with an easy example. Moreover, this example directly shows that composite bundles framework allows to study, in an unified framework, Euler-Lagrange equations for critical sections and linearised equations for fields which flows preserve the space of solutions.

Example 6. ***Jacobi Lagrangian for Geodesics Lagrangian*** *Let us consider the Lagrangian for geodesics on* $[B := \mathbb{R} \times Q \to \mathbb{R}]$ *given by*

$$L = \frac{1}{2} g_{\mu\nu}u^\mu u^\nu ds \tag{3.22}$$

Varying this Lagrangian we obtain an auxiliary Lagrangian on $[\mathbb{R} \times TQ \to \mathbb{R} \times Q \to \mathbb{R}]$

$$L' = \delta L = u^\mu g_{\mu\alpha}\left(\frac{dX^\alpha}{ds} + \Gamma^\alpha_{\lambda\nu}X^\lambda u^\nu\right) ds \tag{3.23}$$

which is regarded as a Lagrangian on $V(B) = \mathbb{R} \times TQ$*, which happens to be linear in* $X = X^\mu(x)\partial_\mu$ *which is assumed to be a vector field on* Q*.*

The variation of the auxiliary Lagrangian L' *defines Jacobi equation for* X*, and can be* *computed as*

$$
\begin{aligned}
\delta L' :=\ & \delta u^\mu g_{\alpha\mu}(\dot X^\alpha + \Gamma^\alpha_{\sigma\nu} X^\sigma u^\nu) + u^\mu \partial_\epsilon g_{\alpha\mu}(\dot X^\alpha + \Gamma^\alpha_{\sigma\nu} X^\sigma u^\nu)\delta x^\epsilon + \\
& + u^\mu g_{\alpha\mu}((\frac{d}{ds}\delta X^\alpha + \delta x^\beta \frac{d}{ds}\partial_\beta X^\alpha + \delta u^\beta \partial_\beta X^\alpha) + \Gamma^\alpha_{\sigma\nu}(\delta X^\sigma + \delta x^\beta \partial_\beta X^\sigma)u^\nu) + \\
& + u^\mu g_{\alpha\mu}\partial_\epsilon \Gamma^\alpha_{\sigma\nu} X^\sigma u^\nu \delta x^\epsilon + u^\mu g_{\alpha\mu}\Gamma^\alpha_{\sigma\nu} X^\sigma \delta u^\nu = \\
=& \delta u^\epsilon g_{\epsilon\alpha}(u^\beta(\partial_\beta X^\alpha + \Gamma^\alpha_{\sigma\beta} X^\sigma)) + u^\mu \partial_\epsilon g_{\mu\alpha}(u^\beta(\partial_\beta X^\alpha + \Gamma^\alpha_{\sigma\beta} X^\sigma))\delta x^\epsilon + \\
& + u^\mu g_{\alpha\mu}((\frac{d}{ds}\delta X^\alpha + \delta x^\epsilon(u^\lambda \partial_{\lambda\epsilon} X^\alpha) + \delta u^\epsilon \partial_\epsilon X^\alpha) + \Gamma^\alpha_{\sigma\nu}(\delta X^\sigma + \delta x^\epsilon \partial_\epsilon X^\sigma)u^\nu) + \\
& + u^\mu g_{\mu\alpha}\partial_\epsilon \Gamma^\alpha_{\sigma\nu} X^\sigma u^\nu \delta x^\epsilon + u^\mu g_{\mu\alpha}\Gamma^\alpha_{\sigma\epsilon} X^\sigma \delta u^\epsilon = \\
=& u^\mu u^\nu(\partial_\epsilon g_{\mu\alpha}\nabla_\nu X^\alpha + g_{\alpha\mu}\partial_\epsilon \nabla_\nu X^\alpha)\delta x^\epsilon + u^\mu(g_{\epsilon\alpha}\nabla_\mu X^\alpha + g_{\alpha\mu}\nabla_\epsilon X^\alpha)\delta u^\epsilon + \\
& + \frac{d}{ds}(u^\mu g_{\mu\alpha}\delta X^\alpha) - (\dot u^\lambda g_{\lambda\sigma} + u^\mu u^\nu(g_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} + g_{\mu\lambda}\Gamma^\lambda_{\sigma\nu}) - u^\mu u^\nu g_{\lambda\mu}\Gamma^\lambda_{\sigma\nu})\delta X^\sigma = \\
=& u^\mu u^\nu((g_{\lambda\alpha}\Gamma^\lambda_{\mu\epsilon} + g_{\mu\lambda}\Gamma^\lambda_{\alpha\epsilon})\nabla_\nu X^\alpha + g_{\alpha\mu}\nabla_\epsilon \nabla_\nu X^\alpha - g_{\lambda\mu}\Gamma^\lambda_{\alpha\epsilon}\nabla_\nu X^\alpha + g_{\alpha\mu}\Gamma^\lambda_{\nu\epsilon}\nabla_\lambda X^\alpha)\delta x^\epsilon + \\
& - (\dot u^\lambda(g_{\epsilon\alpha}\nabla_\lambda X^\alpha + g_{\alpha\lambda}\nabla_\epsilon X^\alpha) + u^\mu u^\nu(g_{\epsilon\alpha}\nabla_\nu \nabla_\mu X^\alpha + g_{\alpha\mu}\nabla_\nu \nabla_\epsilon X^\alpha))\delta x^\epsilon + \\
& + u^\mu u^\nu(\Gamma^\lambda_{\epsilon\nu}(+g_{\alpha\mu}\nabla_\lambda X^\alpha) + \Gamma^\lambda_{\mu\nu}(g_{\epsilon\alpha}\nabla_\lambda X^\alpha + g_{\alpha\lambda}\nabla_\epsilon X^\alpha))\delta x^\epsilon + \\
& + \frac{d}{ds}(g_{\epsilon\alpha}\partial X^\alpha \delta x^\epsilon + u^\mu(g_{\epsilon\alpha}\nabla_\mu X^\alpha + g_{\alpha\mu}\nabla_\epsilon X^\alpha)\delta x^\epsilon + u^\mu g_{\mu\alpha}\delta X^\alpha) - g_{\lambda\sigma}(\dot u^\lambda + \Gamma^\lambda_{\mu\nu}u^\mu u^\nu)\delta X^\sigma = \\
=& - u^\mu u^\nu(g_{\epsilon\alpha}\nabla_{\mu\nu} X^\alpha + g_{\alpha\mu}[\nabla_\nu, \nabla_\epsilon] X^\alpha)\delta x^\epsilon - (\dot u^\lambda + \Gamma^\lambda_{\alpha\beta}u^\alpha u^\beta)(g_{\epsilon\alpha}\nabla_\lambda X^\alpha + g_{\alpha\lambda}\nabla_\epsilon X^\alpha)\delta x^\epsilon + \\
& + \frac{d}{ds}(g_{\epsilon\alpha}\partial X^\alpha \delta x^\epsilon + u^\mu(g_{\epsilon\alpha}\nabla_\mu X^\alpha + g_{\alpha\mu}\nabla_\epsilon X^\alpha)\delta x^\epsilon + u^\mu g_{\mu\alpha}\delta X^\alpha) - g_{\lambda\sigma}(\dot u^\lambda + \Gamma^\lambda_{\mu\nu}u^\mu u^\nu)\delta X^\sigma = \\
=& - u^\mu u^\nu g_{\epsilon\alpha}(\nabla_{\mu\nu} X^\alpha - R^\alpha{}_{\nu\mu\lambda} X^\lambda)\delta x^\epsilon - (\dot u^\lambda + \Gamma^\lambda_{\alpha\beta}u^\alpha u^\beta)(g_{\epsilon\alpha}\nabla_\lambda X^\alpha + g_{\alpha\lambda}\nabla_\epsilon X^\alpha)\delta x^\epsilon + \\
& + \frac{d}{ds}(g_{\epsilon\alpha}\partial X^\alpha \delta x^\epsilon + u^\mu(g_{\epsilon\alpha}\nabla_\mu X^\alpha + g_{\alpha\mu}\nabla_\epsilon X^\alpha)\delta x^\epsilon + u^\mu g_{\mu\alpha}\delta X^\alpha) - g_{\lambda\sigma}(\dot u^\lambda + \Gamma^\lambda_{\mu\nu}u^\mu u^\nu)\delta X^\sigma
\end{aligned}
\tag{3.24}
$$

Therefore, Euler-Lagrange equations for L', i.e. linearised equation for L, are

$$
\begin{cases}
\dot u^\lambda + \Gamma^\lambda_{\alpha\beta}u^\alpha u^\beta = 0 \\
\nabla_{\mu\nu} X^\alpha = R^\alpha{}_{(\nu\mu)\lambda} X^\lambda
\end{cases}
\tag{3.25}
$$

and the solution vector field X are called Jacobi fields on (M, g) and their flow sends (affinely parameterised) geodesics into (affinely parameterised) geodesics.
The variation $\delta L'$ is something on $V(V(B))$ which can be restricted on $V^2(B)$ to get the second variation of the original Lagrangian L, evaluated along a critical configuration, and neglecting boundary term that will vanish for boundary conditions, that is

$$
\begin{aligned}
\delta^2 L &= -(u^\mu u^\nu g_{\epsilon\alpha}\nabla_\mu \nabla_\nu X^\alpha - u^\mu u^\nu R_{\epsilon\nu\mu\lambda} X^\lambda) X^\epsilon = \\
&= (g_{\epsilon\alpha}\hat X^\alpha \hat X^\epsilon + u^\mu u^\nu R_{\epsilon\nu\mu\lambda} X^\lambda X^\epsilon) - \frac{d}{ds}(u^\nu g_{\epsilon\alpha}\nabla_\nu X^\alpha X^\epsilon)
\end{aligned}
\tag{3.26}
$$

where we set $\hat X^\alpha := \dot X^\alpha + u^\mu \Gamma^\alpha_{\mu\nu} X^\nu = u^\mu \nabla_\mu X^\alpha$.

Therefore, to summarize, using composite bundles we were able to define an auxiliary Lagrangian, $L' :=< \delta L, j^1 X >$ whose composite variation leads to linearised field equations.
As we will see in the next section, this construction allows to justify, from a variational point of view, the linearised field equations of a field theory.
In the next chapter we will see a specific example of this construction in the context of General Relativity.

Chapter 4

Linearised Einstein Equations

In this chapter we will compute the linearised Einstein equations, and we will check that our results match those that can be found in literature.
Starting from the Hilbert Lagrangian we will apply the methods described in the previous chapter to get the linearised equations. These equations will be written for a generic background metric $g_{\mu\nu}$ that is a solution of Einstein equations.
To check our computations we will then specialize our results choosing two specific backgrounds.
The first background we will consider will be the Minkowski background, where we chose the background metric to be $\eta_{\mu\nu} = diag(-1,1,1,1)$. In this case we expect that linearised equations became the gravitational waves equation.
The second background we will consider is the Friedman Robertson Walker background. Again, specializing our formula taking as background metric the FRW metric we expect to find the linearised Einstein equations in Cosmology.
In order to make this chapter complete we will also directly compute the gravitational waves equation and the linearised FRW equations in the classical way.

4.1 Changing coordinates on $J^2Lor(M)$

As we will see in the next section, it is easier to write the Jacobi Lagrangian if, instead of considering the derivatives of the metric tensor, we take new coordinates on $J^2Lor(M)$.
We will build our new coordinates by counting independent components of Riemann tensor.
Let us consider a $m-$dimensional manifold M with a metric g. Being symmetric, the metric has $\frac{m}{2}(m+1)$ independent components.
Moreover, while a generic connection has m^3 components, a torsionless connection has $\frac{m^2}{2}(m+1)$ components.
Then, since the first derivative of the metric has $\frac{m^2}{2}(m+1)$ components we can invert the definition of Christoffel symbols. We have in fact

$$2g_{\alpha\epsilon}\{g\}^{\epsilon}_{\beta\mu} = -\partial_\alpha g_{\beta\mu} + \partial_\mu g_{\alpha\beta} + \partial_\beta g_{\mu\alpha} \tag{4.1}$$

permuting indices we get

$$2g_{\beta\epsilon}\{g\}^{\epsilon}_{\mu\alpha} = -\partial_\beta g_{\mu\alpha} + \partial_\alpha g_{\beta\mu} + \partial_\mu g_{\alpha\beta} \tag{4.2}$$

Then, summing we get

$$\begin{aligned}
&2g_{\alpha\epsilon}\{g\}^{\epsilon}_{\beta\mu}+2g_{\beta\epsilon}\{g\}^{\epsilon}_{\mu\alpha}=\\
&-\partial_\alpha g_{\beta\mu}+\partial_\mu g_{\alpha\beta}+\partial_\beta g_{\mu\alpha}-\partial_\beta g_{\mu\alpha}+\partial_\alpha g_{\beta\mu}+\partial_\mu g_{\alpha\beta}=2\partial_\mu g_{\alpha\beta}\\
&\Rightarrow \partial_\mu g_{\alpha\beta}=g_{\alpha\epsilon}\{g\}^{\epsilon}_{\beta\mu}+g_{\beta\epsilon}\{g\}^{\epsilon}_{\mu\alpha}
\end{aligned} \tag{4.3}$$

if we drop the hypothesis of torsionless connection, we have to consider the torsion tensor $T^{\alpha}_{\beta\mu}$, that has $\frac{m^2}{2}(m-1)$ components. We have

$$\frac{m^2}{2}(m-1)+\frac{m^2}{2}(m+1)=m^3 \tag{4.4}$$

this shows that a generic connection can be splitted into a torsionless connection plus a torsion.
If we now consider the covariant Riemann tensor associated to a general torsionless connection, it has to obey to the identities

$$R_{\alpha\beta(\mu\nu)}=0 \qquad R_{\alpha[\beta\mu\nu]}=0 \tag{4.5}$$

That are $\frac{m^2}{2}(m+1)+m\binom{m}{3}$ linear identities.
However, in order to count independent components, we have to discuss if they are independent. As the α index is not involved in these identities we will set it aside for the moment.
We know that a 3 tensor $T_{\beta\mu\nu}$ can be split as

$$T_{\beta\mu\nu}=T_{\beta(\mu\nu)}\oplus T_{\beta[\mu\nu]} \tag{4.6}$$

This splits the space of rank 3 tensors (that has m^3 components) in two subspaces of dimension $\frac{m^2}{2}(m+1)$ and $\frac{m^2}{2}(m-1)$, respectively.
However, in the first subspace there is still an invariant subspace with respect to the permutation group Π_3 acting on indices, namely the space of completely symmetric tensors with 3 indices.
This subspace has dimension

$$\binom{m+2}{3}=\frac{m}{6}(m+2)(m+1) \tag{4.7}$$

Hence

$$T_{\beta(\mu\nu)}=T_{(\beta\mu\nu)}\oplus(T_{\beta(\mu\nu)}-T_{(\beta\mu\nu)}) \tag{4.8}$$

in the same way, also the space of antisymmetric tensors can be split into two subspaces,

$$T_{\beta[\mu\nu]}=T_{[\beta\mu\nu]}\oplus(T_{\beta[\mu\nu]}-T_{[\beta\mu\nu]}) \tag{4.9}$$

in other words, we are considering the action of the permutation group Π_3 on the space of 3 tensors. This representation is not irreducible but it can be written as the sum of 4 irreducible representations of Π_3 acting on subspaces.

$$T_{\beta\mu\nu}=T_{(\beta\mu\nu)}\oplus(T_{\beta(\mu\nu)}-T_{(\beta\mu\nu)})\oplus T_{[\beta\mu\nu]}\oplus(T_{\beta[\mu\nu]}-T_{[\beta\mu\nu]}) \tag{4.10}$$

Now, the identities (4.5) tells us that the Riemann tensor belong to the fourth subspace, which, accounting for the index α has dimension

$$\frac{m^2}{3}(m^2-1) \tag{4.11}$$

Remark 26. *If we consider Levi Civita Connection, the Riemann tensor has more constraints, namely*

$$\begin{aligned} &R_{\alpha\beta(\mu\nu)} = 0\\ &R_{(\alpha\beta)\mu\nu} = 0\\ &R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = 0\\ &R_{\alpha[\beta\mu\nu]} = 0 \end{aligned} \tag{4.12}$$

Of course $R_{(\alpha\beta)\mu\nu} = 0$ follows from$R_{\alpha\beta(\mu\nu)} = 0$ and $R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = 0$, so its not independent.
Moreover, since we have $R_{[\alpha\beta\mu\nu]} = \frac{1}{4}\left(R_{\alpha[\beta\mu\nu]} - R_{\nu[\alpha\beta\mu]} + R_{\mu[\nu\alpha\beta]} - R_{\beta[\mu\nu\alpha]}\right)$, also $R_{[\alpha\beta\mu\nu]} = 0$ follows.
On the other side, if one assumes $R_{\alpha\beta(\mu\nu)} = 0$, $R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta} = 0$ and $R_{[\alpha\beta\mu\nu]} = 0$ we have

$$\begin{aligned} R_{\alpha[\beta\mu\nu]} &= \frac{1}{24}(4\alpha\beta\mu\nu + 4\alpha\mu\nu\beta + 4\alpha\nu\beta\mu) =\\ &= \frac{1}{24}(\alpha\beta\mu\nu + \alpha\mu\nu\beta + \alpha\nu\beta\mu - 3\beta\alpha\mu\nu - 3\beta\nu\alpha\mu - 3\beta\mu\nu\alpha) =\\ &= \frac{1}{24}(\alpha[\beta\mu\nu] - \beta[\alpha\mu\nu] + \mu[\nu\alpha\beta] - \nu[\mu\alpha\beta]) = R_{[\alpha\beta\mu\nu]} = 0 \end{aligned} \tag{4.13}$$

So the constraints are equivalent to the conditions $R_{[\alpha\beta\mu\nu]} = 0$ on the space of components $R_{\alpha\beta\mu\nu}$ skew symmetric in each pair and symmetric for the exchange of the pairs, which is of dimension $\frac{m}{8}(m-1)(m^2-m+2)$. Knowing that the conditions are $\frac{m}{24}(m-1)(m-2)(m-3)$ we can conclude that the Riemann tensor for a Levi Civita connection has

$$\frac{m^2}{12}(m^2-1) \tag{4.14}$$

independent components.

We can see that for $m > 1$ one has $\frac{m^2}{3}(m^2-1) > \frac{m^2}{12}(m^2-1)$.
This tells us that the metric Riemann tensors live in a smaller space than curvature tensors of torsionless connections.
Using this fact we can try to break the derivatives of Christoffel symbols into the Riemann tensor plus something else.
Let us consider a tensor $S^\alpha_{(\beta\mu\nu)}$, it lives in a space with dimension $\frac{m^2}{6}(m+2)(m+1)$, then

$$\frac{m^2}{3}\left(m^2-1\right) \oplus \frac{m^2}{6}(m+2)(m+1) = \frac{m^3}{2}(m+1) \tag{4.15}$$

That suggests that we can break (in a one to one way) $\partial_\mu\Gamma^\alpha_{\beta\mu}$ into its curvature $R^\alpha_{\beta\mu\nu}$ plus $S^\alpha_{\beta\mu\nu} = \partial_{(\mu}\Gamma^\alpha_{\beta\mu)}$.

using these facts we can map $(g_{\mu\nu}, \partial_\nu g_{\alpha\mu}, \partial_{\mu\nu} g_{\alpha\beta})$ into $(g_{\mu\nu}, \{g\}^\alpha_{\beta\nu}, R^\alpha_{\beta\mu\nu}, S^\alpha_{\beta\mu\nu})$ by setting

$$\begin{aligned} R^\alpha_{\beta\mu\nu} &= \partial_\mu \{g\}^\alpha_{\beta\nu} - \partial_\nu \{g\}^\alpha_{\beta\mu} + \{g\}^\alpha_{\gamma\mu}\{g\}^\gamma_{\beta\nu} - \{g\}^\alpha_{\gamma\nu}\{g\}^\gamma_{\beta\mu} \\ S^\alpha_{\beta\mu\nu} &= \partial_\mu \{g\}^\alpha_{\beta\nu} + \partial_\nu \{g\}^\alpha_{\mu\beta} + \partial_\beta \{g\}^\alpha_{\nu\mu} \end{aligned} \tag{4.16}$$

and the transformation is one to one because it can be inverted as

$$\begin{aligned} 3\partial_\nu \{g\}^\alpha_{\mu\beta} &= S^\alpha_{\beta\mu\nu} - 2R^\alpha_{(\beta\mu)\nu} + 2\{g\}^\alpha_{\gamma(\mu}\{g\}^\gamma_{\beta)\nu} - 2\{g\}^\alpha_{\gamma\nu}\{g\}^\gamma_{\beta\mu} \\ &\Rightarrow \partial_{\mu\nu} g_{\alpha\beta} = \partial_\nu g_{\alpha\epsilon}\{g\}^\epsilon_{\beta\mu} + g_{\alpha\epsilon}\partial_\nu\{g\}^\epsilon_{\beta\mu} + \partial_\nu g_{\beta\epsilon}\{g\}^\epsilon_{\mu\alpha} + g_{\beta\epsilon}\partial_\nu\{g\}^\epsilon_{\mu\alpha} = \\ &= g_{\epsilon\lambda}\{g\}^\lambda_{\nu\alpha}\{g\}^\epsilon_{\beta\mu} + g_{\epsilon\lambda}\{g\}^\lambda_{\nu\beta}\{g\}^\epsilon_{\mu\alpha} + \frac{1}{3}S^*_{\alpha\beta\mu\nu} - \\ &- \frac{2}{3}R_{\alpha(\beta\mu)\nu} + \frac{1}{3}S^*_{\beta\alpha\mu\nu} - \frac{2}{3}R_{\beta(\alpha\mu)\nu} + g_{\alpha\epsilon}\{g\}^\epsilon_{\gamma(\mu}\{g\}^\gamma_{\beta\nu)} + g_{\beta\epsilon}\{g\}^\epsilon_{\gamma(\mu}\{g\}^\gamma_{\alpha\nu)} \end{aligned} \tag{4.17}$$

we are now ready to compute the Jacobi Lagrangian of the Hilbert Lagrangian: the first part of the computation will be done taking $(g_{\mu\nu}, \{g\}^\alpha_{\beta\nu}, R^\alpha_{\beta\mu\nu}, S^\alpha_{\beta\mu\nu})$ as coordinates on $J^2 Lor(M)$.

4.2 Linearised Einstein Equations

This section can be considered as one of the cores of this work: we are going to apply to a specific case the construction we did in chapter 3.
Although this premise may seem unnecessary it is very important because it highlights the steps we have to follow.
Given a Lagrangian we first have to evaluate its first variation, then we have to consider the deformation (vertical vector field) as a "true variable" in order to define the Jacobi Lagrangian as an auxiliary Lagrangian on a composite bundle. Then we can compute the composite variation of Jacobi Lagrangian to get linearised equations.

Let us begin by considering the Hilbert Lagrangian.

$$L_H = \sqrt{g} g^{\beta\nu} \delta^\mu_\alpha R^\alpha_{\beta\mu\nu} d\sigma \tag{4.18}$$

If we take as coordinates on $J^2 Lor(M)$, $(x^\mu, g_{\mu\nu}, \{g\}^\alpha_{\beta\nu}, R^\alpha_{\beta\mu\nu}, S^\alpha_{\beta\mu\nu})$ we have

$$\delta L_H = \sqrt{g}\left(-\frac{1}{2} g_{\eta\theta} g^{\beta\nu}\delta^\mu_\alpha R^\alpha_{\beta\mu\nu} X^{\eta\theta} + \delta^\mu_\alpha R^\alpha_{\beta\mu\nu}\delta^\beta_{(\eta}\delta^\nu_{\theta)} X^{\eta\theta} + g^{\mu\nu}\delta^\eta_\alpha\left[\delta^\alpha_\eta\delta^\theta_\mu\delta^\zeta_\eta\delta^\gamma_\nu\right]\delta R^\eta_{\theta\zeta\gamma}\right) \tag{4.19}$$

where we set $£_\xi g^{\mu\nu} = X^{\mu\nu} = \delta g^{\mu\nu}$ and $£_\xi R^\alpha_{\beta\mu\nu} = \delta R^\alpha_{\beta\mu\nu}$.
Now we ask to $R^\alpha_{\beta\mu\nu}$ to be the metric Riemann tensor of $g_{\mu\nu}$ (i.e we ask to ∇ to be the Levi Civita connection of $g_{\mu\nu}$). We recall that in this case we have

$$\delta R^\alpha_{\beta\mu\nu} = \nabla_\mu \delta\Gamma^\alpha_{\beta\nu} - \nabla_\nu \delta\Gamma^\alpha_{\beta\mu} \tag{4.20}$$

$$\delta\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\nabla_\mu \delta g_{\lambda\nu} + \nabla_\nu \delta g_{\lambda\mu} - \nabla_\lambda \delta g_{\mu\nu}) \tag{4.21}$$

Moreover, we can also contract equation (4.20) in order to get the variation of the Ricci tensor

$$\begin{aligned}\delta R_{\beta\nu} &= \delta^{\mu}_{\alpha}\delta R^{\alpha}_{\eta\mu\nu} = \nabla_\mu \delta\Gamma^{\mu}_{\beta\nu} - \nabla_\nu \delta\Gamma^{\sigma}_{\beta\sigma} = \\ &= \nabla_\mu\left(\frac{1}{2}g^{\mu\lambda}(\nabla_\beta\delta g_{\lambda\nu} + \nabla_\nu\delta g_{\lambda\beta} - \nabla_\lambda\delta g_{\beta\nu})\right) - \nabla_\nu\left(\frac{1}{2}g^{\sigma\lambda}(\nabla_\beta\delta g_{\lambda\sigma} + \nabla_\sigma\delta g_{\lambda\beta} - \nabla_\lambda\delta g_{\beta\sigma})\right) \\ &= \frac{1}{2}\left(\nabla_\mu\nabla_\beta X^{\mu}_{\nu} + \nabla_\mu\nabla_\nu X^{\mu}_{\beta} - \Box X_{\beta\nu} - \nabla_\nu\nabla_\beta X - \nabla_\nu\nabla_\sigma X^{\sigma}_{\beta} + \nabla_\nu\nabla_\lambda X^{\sigma}_{\beta}\right) = \\ &= \frac{1}{2}\left(\nabla_\mu\nabla_\beta X^{\mu}_{\nu} + \nabla_\mu\nabla_\nu X^{\mu}_{\beta} - \Box X_{\beta\nu} - \nabla_\nu\nabla_\beta X\right)\end{aligned} \tag{4.22}$$

and the variation of the scalar curvature

$$\begin{aligned}\delta R &= g^{\beta\nu}\delta R_{\beta\nu} = g^{\beta\nu}\frac{1}{2}\left(\nabla_\mu\nabla_\beta X^{\mu}_{\nu} + \nabla_\mu\nabla_\nu X^{\mu}_{\beta} - \Box X_{\beta\nu} - \nabla_\nu\nabla_\beta X\right) = \\ &= \frac{1}{2}\left(\nabla_\alpha\nabla_\beta X^{\alpha\beta} + \nabla_\alpha\nabla_\beta X^{\alpha\beta} - \Box X - \Box X\right) = \\ &= \nabla_\alpha\nabla_\beta X^{\alpha\beta} - \Box X\end{aligned} \tag{4.23}$$

where we have defined $\Box X_{\mu\nu} := g^{\alpha\beta}\nabla_\alpha\nabla_\beta X_{\mu\nu}$, $X := g^{\mu\nu}X_{\mu\nu}$ and $X^{\alpha}_{\beta} := g^{\alpha\lambda}X_{\lambda_\beta}$.
We now have to write the explicit dependence of $\delta L'$ from $X^{\mu\nu}$ and its derivatives. We have:

$$\begin{aligned}L' &=< \delta L_H, j^2X >= -\frac{1}{2}\sqrt{g}g_{\eta\theta}g^{\beta\nu}\delta^{\mu}_{\alpha}R^{\alpha}_{\beta\mu\nu}X^{\eta\theta} + \delta^{\mu}_{\alpha}\sqrt{g}R^{\alpha}_{\beta\mu\nu}\delta^{\beta}_{(\eta}\delta^{\nu}_{\theta)}X^{\eta\theta} + \sqrt{g}g^{\mu\nu}\delta^{\eta}_{\alpha}\left[\delta^{\alpha}_{\eta}\delta^{\theta}_{\mu}\delta^{\zeta}_{\eta}\delta^{\gamma}_{\nu}\right]\delta R^{\eta}_{\theta\zeta\gamma} = \\ &= -\frac{1}{2}\sqrt{g}g_{\eta\theta}g^{\beta\nu}\delta^{\mu}_{\alpha}R^{\alpha}_{\beta\mu\nu}X^{\eta\theta} + \delta^{\mu}_{\alpha}\sqrt{g}R^{\alpha}_{\beta\mu\nu}\delta^{\beta}_{(\eta}\delta^{\nu}_{\theta)}X^{\eta\theta} + \sqrt{g}\delta R = \\ &= -\frac{1}{2}\sqrt{g}g_{\eta\theta}g^{\beta\nu}\delta^{\mu}_{\alpha}R^{\alpha}_{\beta\mu\nu}X^{\eta\theta} + \delta^{\mu}_{\alpha}\sqrt{g}R^{\alpha}_{\beta\mu\nu}\delta^{\beta}_{(\eta}\delta^{\nu}_{\theta)}X^{\eta\theta} + \sqrt{g}\left(\nabla_\alpha\nabla_\beta X^{\alpha\beta} - g^{\mu\nu}\nabla_\mu\nabla_\nu(g_{\alpha\beta}X^{\alpha\beta})\right) = \\ &= \sqrt{g}\left(-\frac{1}{2}g_{\eta\theta}X^{\eta\theta}R + R_{\eta\theta}X^{\eta\theta} + (\delta^{\eta}_{\alpha}\delta^{\theta}_{\beta} - g^{\eta\theta}g_{\alpha\beta})\nabla_\eta\nabla_\theta X^{\alpha\beta}\right)\end{aligned} \tag{4.24}$$

then, the Jacobi Lagrangian associated to the Hilbert Lagrangian is

$$L' = \sqrt{g}\left(-\frac{1}{2}g_{\eta\theta}X^{\eta\theta}R + R_{\eta\theta}X^{\eta\theta} + (\delta^{\eta}_{\alpha}\delta^{\theta}_{\beta} - g^{\eta\theta}g_{\alpha\beta})\nabla_\eta\nabla_\theta X^{\alpha\beta}\right) \tag{4.25}$$

This is a Lagrangian on the composite bundle $[V(Lor(M)) \to Lor(M) \to M]$.
If we want to compute its composite variation, we first have to know what is the variation of the second covariant derivative.
We have:

$$\begin{aligned}\delta\nabla_\alpha\nabla_\beta X^{\mu\nu} &= \delta\nabla_\alpha\left(\partial_\beta X^{\mu\nu} + \Gamma^{\mu}_{\epsilon\beta}X^{\epsilon\nu} + \Gamma^{\nu}_{\epsilon\beta}X^{\epsilon\mu}\right) = \\ &= \delta\{\partial_\alpha(\partial_\beta X^{\mu\nu} + \Gamma^{\mu}_{\epsilon\beta}X^{\epsilon\nu} + \Gamma^{\nu}_{\epsilon\beta}X^{\epsilon\mu}) + \Gamma^{\mu}_{\epsilon\alpha}(\partial_\beta X^{\epsilon\nu} + \Gamma^{\epsilon}_{\sigma\beta}X^{\sigma\nu} + \Gamma^{\nu}_{\sigma\beta}X^{\sigma\epsilon}) + \\ &+ \Gamma^{\nu}_{\epsilon\alpha}(\partial_\beta X^{\epsilon\mu} + \Gamma^{\epsilon}_{\sigma\beta}X^{\sigma\mu} + \Gamma^{\mu}_{\sigma\beta}X^{\sigma\epsilon}) - \Gamma^{\epsilon}_{\beta\alpha}(\partial_\epsilon X^{\mu\nu} + \Gamma^{\mu}_{\sigma\epsilon}X^{\sigma\nu} + \Gamma^{\nu}_{\sigma\epsilon}X^{\sigma\mu})\}\end{aligned} \tag{4.26}$$

that can be written as

$$
\begin{aligned}
&\partial_\alpha(\partial_\beta\delta X^{\mu\nu}+\Gamma^\mu_{\epsilon\beta}\delta X^{\epsilon\nu}+\Gamma^\nu_{\epsilon\beta}\delta X^{\epsilon\mu})+\Gamma^\mu_{\epsilon\alpha}(\partial_\beta\delta X^{\epsilon\nu}+\Gamma^\epsilon_{\sigma\beta}\delta X^{\sigma\nu}+\Gamma^\nu_{\sigma\beta}\delta X^{\sigma\epsilon})+\\
&+\Gamma^\nu_{\epsilon\alpha}(\partial_\beta\delta X^{\epsilon\mu}+\Gamma^\epsilon_{\sigma\beta}\delta X^{\sigma\mu}+\Gamma^\mu_{\sigma\beta}\delta X^{\sigma\epsilon})-\Gamma^\epsilon_{\beta\alpha}(\partial_\epsilon\delta X^{\mu\nu}+\Gamma^\mu_{\sigma\epsilon}\delta X^{\sigma\nu}+\Gamma^\nu_{\sigma\epsilon}\delta X^{\sigma\mu})+\\
&+\partial_\alpha(\delta\Gamma^\mu_{\epsilon\beta}X^{\epsilon\nu}+\delta\Gamma^\nu_{\epsilon\beta}X^{\epsilon\mu})+\delta\Gamma^\mu_{\epsilon\alpha}(\partial_\beta X^{\epsilon\nu}+\Gamma^\epsilon_{\sigma\beta}X^{\sigma\nu}+\Gamma^\nu_{\sigma\beta}X^{\sigma\epsilon})+\\
&+\delta\Gamma^\nu_{\epsilon\alpha}(\partial_\beta X^{\epsilon\mu}+\Gamma^\epsilon_{\sigma\beta}X^{\sigma\mu}+\Gamma^\mu_{\sigma\beta}X^{\sigma\epsilon})-\delta\Gamma^\epsilon_{\beta\alpha}(\partial_\epsilon X^{\mu\nu}+\Gamma^\mu_{\sigma\epsilon}X^{\sigma\nu}+\Gamma^\nu_{\sigma\epsilon}X^{\sigma\mu})+\\
&+\Gamma^\mu_{\epsilon\alpha}(\delta\Gamma^\epsilon_{\sigma\beta}X^{\sigma\nu}+\delta\Gamma^\nu_{\sigma\beta}X^{\sigma\epsilon})+\\
&+\Gamma^\nu_{\epsilon\alpha}(\delta\Gamma^\epsilon_{\sigma\beta}X^{\sigma\mu}+\delta\Gamma^\mu_{\sigma\beta}X^{\sigma\epsilon})-\Gamma^\epsilon_{\beta\alpha}(\delta\Gamma^\mu_{\sigma\epsilon}X^{\sigma\nu}+\delta\Gamma^\nu_{\sigma\epsilon}X^{\sigma\mu})=\\
&=\nabla_\alpha\nabla_\beta\delta X^{\mu\nu}+\delta\Gamma^\mu_{\epsilon\beta}\nabla_\alpha X^{\epsilon\nu}+\delta\Gamma^\nu_{\epsilon\beta}\nabla_\alpha\delta X^{\epsilon\mu}+X^{\epsilon\nu}\nabla_\alpha\delta\Gamma^\mu_{\epsilon\beta}+X^{\epsilon\mu}\nabla_\alpha\delta\Gamma^\nu_{\epsilon\beta}+\delta\Gamma^\mu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\nu}+\\
&+\delta\Gamma^\nu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\mu}-\delta\Gamma^\epsilon_{\beta\alpha}\nabla_\epsilon X^{\mu\nu}
\end{aligned}
\tag{4.27}
$$

integrating by parts (4.27) and neglecting boundary terms we get:

$$
\begin{aligned}
&\delta\Gamma^\mu_{\epsilon\beta}\nabla_\alpha X^{\epsilon\nu}+\delta\Gamma^\nu_{\epsilon\beta}\nabla_\alpha X^{\epsilon\mu}+X^{\epsilon\nu}\nabla_\alpha\delta\Gamma^\mu_{\epsilon\beta}+X^{\epsilon\mu}\nabla_\alpha\delta\Gamma^\nu_{\epsilon\beta}+\delta\Gamma^\mu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\nu}+\\
&+\delta\Gamma^\nu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\mu}-\delta\Gamma^\epsilon_{\beta\alpha}\nabla_\epsilon X^{\mu\nu}=\\
&=\delta\Gamma^\mu_{\epsilon\beta}\nabla_\alpha X^{\epsilon\nu}+\delta\Gamma^\nu_{\epsilon\beta}\nabla_\alpha X^{\epsilon\mu}-\nabla_\alpha X^{\epsilon\nu}\delta\Gamma^\mu_{\epsilon\beta}-\nabla_\alpha X^{\epsilon\mu}\delta\Gamma^\nu_{\epsilon\beta}+\delta\Gamma^\mu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\nu}+\\
&+\delta\Gamma^\nu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\mu}-\delta\Gamma^\epsilon_{\beta\alpha}\nabla_\epsilon X^{\mu\nu}=\\
&=\delta\Gamma^\mu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\nu}+\delta\Gamma^\nu_{\epsilon\alpha}\nabla_\beta X^{\epsilon\mu}-\delta\Gamma^\epsilon_{\beta\alpha}\nabla_\epsilon X^{\mu\nu}
\end{aligned}
\tag{4.28}
$$

the full object we will have to compute is:

$$
\begin{aligned}
&(\delta^i_\alpha\delta^j_\beta-g^{ij}g_{\alpha\beta})\delta\nabla_i\nabla_j X^{\alpha\beta}=\\
&(\delta^i_\alpha\delta^j_\beta-g^{ij}g_{\alpha\beta})\left(\delta\Gamma^\alpha_{\epsilon i}\nabla_j X^{\epsilon\beta}+\delta\Gamma^\beta_{\epsilon i}\nabla_j X^{\epsilon\alpha}-\delta\Gamma^\epsilon_{ij}\nabla_\epsilon X^{\alpha\beta}\right)
\end{aligned}
\tag{4.29}
$$

we do the computation evaluating each term individually.
The first term gives,

$$
\begin{aligned}
&\delta^i_\alpha\delta^j_\beta\left(\delta\Gamma^\alpha_{\epsilon i}\nabla_j X^{\epsilon\beta}+\delta\Gamma^\beta_{\epsilon i}\nabla_j X^{\epsilon\alpha}-\delta\Gamma^\epsilon_{ij}\nabla_\epsilon X^{\alpha\beta}\right)=\\
&=\delta\Gamma^\alpha_{\epsilon\alpha}\nabla_\beta X^{\epsilon\beta}+\delta\Gamma^\beta_{\epsilon\alpha}\nabla_\beta X^{\epsilon\alpha}-\delta\Gamma^\epsilon_{\alpha\beta}\nabla_\epsilon X^{\alpha\beta}
\end{aligned}
\tag{4.30}
$$

that is

$$
\begin{aligned}
&\nabla_\beta X^{\epsilon\beta}\delta\Gamma^\alpha_{\epsilon\alpha}=\frac{1}{2}g^{\alpha\lambda}\left(\nabla_\alpha\delta g_{\epsilon\lambda}+\nabla_\epsilon\delta g_{\alpha\lambda}-\nabla_\lambda\delta g_{\epsilon\alpha}\right)\nabla_\beta X^{\epsilon\beta}=\\
&=-\frac{1}{2}g^{\alpha\lambda}\left(-\nabla_\alpha\nabla_\beta X^{\epsilon\beta}g_{\epsilon\mu}g_{\lambda\nu}-\nabla_\epsilon\nabla_\beta X^{\epsilon\beta}g_{\alpha\mu}g_{\lambda\nu}+\nabla_\lambda\nabla_\beta X^{\epsilon\beta}g_{\epsilon\mu}g_{\alpha\nu}\right)\delta g^{\mu\nu}=\\
&=-\frac{1}{2}\left(-\nabla_\nu\nabla_\beta X^\beta_\mu-g_{\mu\nu}\nabla_\epsilon\nabla_\beta X^{\epsilon\beta}+\nabla_\nu\nabla_\beta X^\beta_\mu\right)=\\
&=\frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta X^{\alpha\beta}\delta g^{\mu\nu}
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
&\delta\Gamma^\beta_{\epsilon\alpha}\nabla_\beta X^{\epsilon\alpha}=\frac{1}{2}g^{\beta\lambda}\left(\nabla_\alpha\delta g_{\epsilon\lambda}+\nabla_\epsilon\delta g_{\alpha\lambda}-\nabla_\lambda\delta g_{\epsilon\alpha}\right)\nabla_\beta X^{\epsilon\alpha}=\\
&=-\frac{1}{2}g^{\beta\lambda}\left(-\nabla_\alpha\nabla_\beta X^{\epsilon\alpha}g_{\epsilon\mu}g_{\lambda\nu}-\nabla_\epsilon\nabla_\beta X^{\epsilon\alpha}g_{\alpha\mu}g_{\lambda\nu}+\nabla_\lambda\nabla_\beta X^{\epsilon\alpha}g_{\epsilon\mu}g_{\alpha\nu}\right)\delta g^{\mu\nu}=\\
&-\frac{1}{2}\left(-\nabla_\alpha\nabla_\nu X^\alpha_\mu-\nabla_\alpha\nabla_\nu X^\alpha_\mu+\Box X_{\mu\nu}\right)\delta g^{\mu\nu}
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
&\delta\Gamma^{\epsilon}_{\alpha\beta}\nabla_{\epsilon}X^{\alpha\beta} = \frac{1}{2}g^{\epsilon\lambda}\left(\nabla_{\alpha}\delta g_{\beta\lambda} + \nabla_{\beta}\delta g_{\alpha\lambda} - \nabla_{\lambda}\delta g_{\alpha\beta}\right)\nabla_{\epsilon}X^{\alpha\beta} = \\
&= -\frac{1}{2}g^{\epsilon\lambda}\left(-\nabla_{\alpha}\nabla_{\epsilon}X^{\alpha\beta}g_{\beta\mu}g_{\lambda\nu} - \nabla_{\beta}\nabla_{\epsilon}X^{\alpha\beta}g_{\alpha\mu}g_{\lambda\nu} + \nabla_{\lambda}\nabla_{\epsilon}X^{\alpha\beta}g_{\alpha\mu}g_{\beta\nu}\right)\delta g^{\mu\nu} = \\
&-\frac{1}{2}\left(-\nabla_{\alpha}\nabla_{\nu}X^{\alpha}_{\mu} - \nabla_{\alpha}\nabla_{\nu}X^{\alpha}_{\mu} + \Box X_{\mu\nu}\right)\delta g^{\mu\nu}
\end{aligned}
\tag{4.33}
$$

By summing these last three terms we get

$$
\begin{aligned}
&\delta^{i}_{\alpha}\delta^{j}_{\beta}\left(\delta\Gamma^{\alpha}_{\epsilon i}\nabla_{j}X^{\epsilon\beta} + \delta\Gamma^{\beta}_{\epsilon i}\nabla_{j}X^{\epsilon\alpha} - \delta\Gamma^{\epsilon}_{ij}\nabla_{\epsilon}X^{\alpha\beta}\right) = \\
&= \delta\Gamma^{\alpha}_{\epsilon\alpha}\nabla_{\beta}X^{\epsilon\beta} + \delta\Gamma^{\beta}_{\epsilon\alpha}\nabla_{\beta}X^{\epsilon\alpha} - \delta\Gamma^{\epsilon}_{\alpha\beta}\nabla_{\epsilon}X^{\alpha\beta} = \\
&= \frac{1}{2}g_{\mu\nu}\nabla_{\alpha}\nabla_{\beta}X^{\alpha\beta}\delta g^{\mu\nu}
\end{aligned}
\tag{4.34}
$$

Remark 27. *To avoid some calculations one can rename indices of*

$$
\delta\Gamma^{\beta}_{\epsilon\alpha}\nabla_{\beta}X^{\epsilon\alpha} - \delta\Gamma^{\epsilon}_{\alpha\beta}\nabla_{\epsilon}X^{\alpha\beta}
\tag{4.35}
$$

to see that the second term is equal to the third, so their difference is zero.

The other term we have to compute is,

$$
g^{ij}g_{\alpha\beta}\left(\delta\Gamma^{\alpha}_{\epsilon i}\nabla_{j}X^{\epsilon\beta} + \delta\Gamma^{\beta}_{\epsilon i}\nabla_{j}X^{\epsilon\alpha} - \delta\Gamma^{\epsilon}_{ij}\nabla_{\epsilon}X^{\alpha\beta}\right)
\tag{4.36}
$$

that gives

$$
\begin{aligned}
&g^{ij}g_{\alpha\beta}\left(\delta\Gamma^{\alpha}_{\epsilon i}\nabla_{j}X^{\epsilon\beta}\right) = \frac{1}{2}g^{\lambda\alpha}\left(\nabla_{\epsilon}\delta g_{\lambda i} + \nabla_{i}\delta g_{\lambda\epsilon} - \nabla_{\lambda}\delta g_{\epsilon i}\right)\nabla_{j}X^{\epsilon\beta}g^{ij}g_{\alpha\beta} = \\
&= -\frac{1}{2}g^{\lambda\alpha}\left(-\nabla_{\epsilon}\nabla_{j}X^{\epsilon\beta}g^{ij}g_{\alpha\beta}g_{\mu\lambda}g_{i\nu} - \nabla_{i}\nabla_{j}X^{\epsilon\beta}g^{ij}g_{\alpha\beta}g_{\lambda\mu}g_{\epsilon\nu} + \nabla_{\lambda}\nabla_{j}X^{\epsilon\beta}g^{ij}g_{\alpha\beta}g_{\mu\epsilon}g_{\nu i}\right) = \\
&= -\frac{1}{2}(-\nabla_{\epsilon}\nabla_{\nu}X^{\epsilon}_{\mu} - \Box X_{\mu\nu} + \nabla_{\beta}\nabla_{\nu}X^{\beta}_{\mu})\delta g^{\mu\nu}
\end{aligned}
\tag{4.37}
$$

$$
\begin{aligned}
&g^{ij}g_{\alpha\beta}\left(\delta\Gamma^{\beta}_{\epsilon i}\nabla_{j}X^{\epsilon\alpha}\right) = \frac{1}{2}g^{\lambda\beta}\left(\nabla_{\epsilon}\delta g_{\lambda i} + \nabla_{i}\delta g_{\lambda\epsilon} - \nabla_{\lambda}\delta g_{\epsilon i}\right)\nabla_{j}X^{\epsilon\alpha}g^{ij}g_{\alpha\beta} = \\
&= -\frac{1}{2}g^{\lambda\beta}\left(-\nabla_{\epsilon}\nabla_{j}X^{\epsilon\alpha}g^{ij}g_{\alpha\beta}g_{\mu\lambda}g_{i\nu} - \nabla_{i}\nabla_{j}X^{\epsilon\alpha}g^{ij}g_{\alpha\beta}g_{\lambda\mu}g_{\epsilon\nu} + \nabla_{\lambda}\nabla_{j}X^{\epsilon\alpha}g^{ij}g_{\alpha\beta}g_{\mu\epsilon}g_{\nu i}\right) = \\
&= -\frac{1}{2}(-\nabla_{\epsilon}\nabla_{\nu}X^{\epsilon}_{\mu} - \Box X_{\mu\nu} + \nabla_{\beta}\nabla_{\nu}X^{\beta}_{\mu})\delta g^{\mu\nu}
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
&g^{ij}g_{\alpha\beta}\left(\delta\Gamma^{\epsilon}_{ij}\nabla_{\epsilon}X^{\alpha\beta}\right) = \frac{1}{2}g^{\lambda\epsilon}\left(\nabla_{i}\delta g_{\lambda j} + \nabla_{j}\delta g_{\lambda i} - \nabla_{\lambda}\delta g_{ij}\right)\nabla_{\epsilon}X^{\alpha\beta}g^{ij}g_{\alpha\beta} = \\
&= -\frac{1}{2}g^{\lambda\epsilon}\left(-\nabla_{i}\nabla_{\epsilon}X^{\alpha\beta}g^{ij}g_{\alpha\beta}g_{\mu\lambda}g_{j\nu} - \nabla_{j}\nabla_{\epsilon}X^{\alpha\beta}g^{ij}g_{\alpha\beta}g_{\lambda\mu}g_{i\nu} + \nabla_{\lambda}\nabla_{\epsilon}X^{\alpha\beta}g^{ij}g_{\alpha\beta}g_{i\mu}g_{j\nu}\right) = \\
&= -\frac{1}{2}(-\nabla_{\mu}\nabla_{\nu}X - \nabla_{\mu}\nabla_{\nu}X + g_{\mu\nu}\Box X)\delta g^{\mu\nu}
\end{aligned}
\tag{4.39}
$$

then, we get

$$g^{ij}g_{\alpha\beta}\left(\delta\Gamma^{\alpha}_{\epsilon i}\nabla_j X^{\epsilon\beta}+\delta\Gamma^{\beta}_{\epsilon i}\nabla_j X^{\epsilon\alpha}-\delta\Gamma^{\epsilon}_{ij}\nabla_\epsilon X^{\alpha\beta}\right)=\Box X_{\mu\nu}-\nabla_\mu\nabla_\nu X+\frac{1}{2}g_{\mu\nu}\Box X \tag{4.40}$$

Remark 28. *Again, to avoid some calculations, these two terms*

$$g^{ij}g_{\alpha\beta}\left(\delta\Gamma^{\alpha}_{\epsilon i}\nabla_j X^{\epsilon\beta}+\delta\Gamma^{\beta}_{\epsilon i}\nabla_j X^{\epsilon\alpha}\right) \tag{4.41}$$

are equal in view of the symmetry of the metric tensor.

We are now ready to compute the variation of the Jacobi Lagrangian.
We get,

$$\begin{aligned}
\delta L'=\Bigg\{&-\frac{1}{2}g_{\mu\nu}\left(-\frac{1}{2}g_{ij}X^{ij}R+R_{ij}X^{ij}+(\delta^i_\alpha\delta^j_\beta-g^{ij}g_{\alpha\beta})\nabla_i\nabla_j X^{\alpha\beta}\right)+\\
&-\frac{1}{2}\delta^\alpha_\mu\delta^\beta_\nu R_{\alpha\beta}g_{ij}X^{ij}+\frac{1}{4}(g_{i\nu}g_{j\mu}+g_{i\mu}g_{j\nu})X^{ij}R-\delta^i_\mu\delta^j_\nu g_{\alpha\beta}\nabla_i\nabla_j X^{\alpha\beta}+\\
&\frac{1}{2}g^{ij}(g_{\alpha\nu}g_{\beta\mu}+g_{\alpha\mu}g_{\beta\nu})\nabla_i\nabla_j X^{\alpha\beta}+\frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta X^{\alpha\beta}-\Box X_{\mu\nu}+\nabla_\mu\nabla_\nu X-\frac{1}{2}g_{\mu\nu}\Box X\Bigg\}\sqrt{g}\delta g^{\mu\nu}+\\
&+\left(-\frac{1}{2}Rg_{\mu\nu}+R_{\mu\nu}\right)\delta X^{\mu\nu}-\frac{1}{2}(\nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta})-\Box(\delta g_{\alpha\beta})g^{\alpha\beta})g_{ij}X^{ij}+\\
&+X^{\alpha\beta}\frac{1}{2}\left(\nabla_\lambda\nabla_\alpha\delta g^\lambda_\beta+\nabla_\lambda\nabla_\beta\delta g^\lambda_\alpha-\Box\delta g_{\alpha\beta}-g^{ij}\nabla_\alpha\nabla_\beta\delta g_{ij}\right)
\end{aligned} \tag{4.42}$$

integrating by parts the last two lines containing $\delta g_{\alpha\beta}$, neglecting boundary terms (that vanish for appropriate boundary conditions), and asking $\delta L'=0$ we get

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}(\nabla_\alpha\nabla_\beta X^{\alpha\beta}-\Box X)-\frac{1}{2}R_{\mu\nu}X+\frac{1}{2}RX_{\mu\nu}-\frac{1}{2}\nabla_\mu\nabla_\nu X+\frac{1}{2}g_{\mu\nu}\Box X+\\
&+\frac{1}{2}\nabla_\alpha\nabla_\lambda X^{\alpha\beta}g^{\lambda k}g_{\mu k}g_{\nu\beta}+\frac{1}{2}\nabla_\beta\nabla_\lambda X^{\alpha\beta}g^{\lambda k}g_{\mu\alpha}g_{\nu k}-\frac{1}{2}\Box X_{\mu\nu}-\frac{1}{2}\nabla_\beta\nabla_\alpha X^{\alpha\beta}g_{\mu\nu}+\\
&-\nabla_\mu\nabla_\nu X+\Box X_{\mu\nu}+\frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta X^{\alpha\beta}-\Box X_{\mu\nu}+\nabla_\mu\nabla_\nu X-\frac{1}{2}g_{\mu\nu}\Box X=0
\end{aligned} \tag{4.43}$$

and

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=0 \tag{4.44}$$

As expected, the variation of the Jacobi Lagrangian with respect to $X^{\mu\nu}$ gives field equations (4.44).
On the other hand, equations (4.43) can be recasted as

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta X^{\alpha\beta}+\frac{1}{2}g_{\mu\nu}\Box X+\frac{1}{2}\nabla_\lambda\nabla_\mu X^\lambda_\nu+\frac{1}{2}\nabla_\lambda\nabla_\nu X^\lambda_\mu-\\
&-\frac{1}{2}R_{\mu\nu}X+\frac{1}{2}RX_{\mu\nu}-\frac{1}{2}\nabla_\mu\nabla_\nu X-\frac{1}{2}\Box X_{\mu\nu}=0
\end{aligned} \tag{4.45}$$

These last equations are called *linearised Einstein equations.*

Remark 29. *Equations (4.45) are written for a generic background metric on space-time that is a solution of Einstein equations.*
We can check that this result is consistent by fixing a specific background metric and comparing what we get with results found in literature.
This kind of "congruence proofs" will be provided in the next two sections.

4.3 Gravitational waves equation

We now want to derive the gravitational waves equation, in order to see that it is the same equation we get by fixing as background metric $\eta_{\mu\nu} = diag(-1,1,1,1)$ in (4.45).
We will provide a computation based on the standard tools used in physics to deal with perturbation theory.
Let us consider a weak-field situation, where our metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.46}$$

where $h_{\mu\nu}$ is a small perturbation.
Of course, (4.56) makes sense only locally since the space of metrics has no linear or affine structure. This is however the way the computation is usually carried over. Also "small" perturbation does not mean much.

Remark 30. *One should explain what does it means that $h_{\mu\nu}$ is a small perturbation.*
In general this is done requiring "$|h_{\mu\nu}| << 1$"; although this request may seem a bit obscure, in physics, one is able to compute numerical values of $h_{\mu\nu}$ by solving the gravitational waves equation with retarded potentials.
As an example, for a static field outside a spherical mass one has

$$-h_{00} = h_{11} = h_{22} = h_{33} = \frac{2GM}{Rc^2} \tag{4.47}$$

Where G is the Newton constant ($G = 6{,}67 \times 10^{-11} \frac{Nm^2}{kg^2}$), M is the source's mass and R the distance from the center of the source.
For example, on the surface of the Sun one has $-h_{00} = h_{11} = h_{22} = h_{33} = 4 \cdot 10^{-6}$.

In this situation one can expand field equations in powers of $h_{\mu\nu}$ using a coordinate frame where (4.56) holds, and keep only linear terms.
Following this idea we will expand at first order the Einstein tensor, in order to get linearised equations.
Let us recall that we can raise and lower the indices of $h_{\mu\nu}$ as

$$h^{\mu}_{\nu} = \eta^{\mu\alpha} h_{\alpha\nu} \qquad h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\eta_{\alpha\beta} \tag{4.48}$$

Recalling that Christoffel symbols are defined as

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}\left(-\partial_\nu h_{\alpha\beta} + \partial_\beta h_{\alpha\nu} + \partial_\alpha h_{\beta\nu}\right) = \tag{4.49}$$

and that $\partial_\alpha \eta_{\mu\nu} = 0$, using the linearisation we get the variation of Christoffel symbols as

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\left(-\partial^{\mu}h_{\alpha\beta} + \partial_{\beta}h^{\mu}_{\alpha} + \partial_{\alpha}h^{\mu}_{\beta}\right) \tag{4.50}$$

We can then proceed linearising the Ricci curvature. Knowing that

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} \tag{4.51}$$

and applying the linearisation we get

$$\begin{aligned}\delta R_{\mu\nu} &= \frac{1}{2}\left(-\partial^{\alpha}\partial_{\alpha}h_{\mu\nu} + \partial_{\nu}\partial_{\alpha}h^{\alpha}_{\mu} + \partial_{\mu}\partial_{\alpha}h^{\alpha}_{\nu} + \partial^{\alpha}\partial_{\nu}h_{\mu\alpha} - \partial_{\alpha}\partial_{\nu}h^{\alpha}_{\mu} - \partial_{\mu}\partial_{\nu}h^{\alpha}_{\alpha}\right)\\ &= \frac{1}{2}\left(-\partial^{\alpha}\partial_{\alpha}h_{\mu\nu} + \partial_{\mu}\partial_{\alpha}h^{\alpha}_{\nu} + \partial^{\alpha}\partial_{\nu}h_{\mu\alpha} - \partial_{\mu}\partial_{\nu}h^{\alpha}_{\alpha}\right)\end{aligned} \tag{4.52}$$

now, simplifying the notation as $h = h^{\alpha}_{\alpha}$ and $\Box = \partial^{\alpha}\partial_{\alpha}$ we have

$$\delta R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_{\mu\alpha}h^{\alpha}_{\nu} + \partial_{\nu\alpha}h^{\alpha}_{\mu} - \partial_{\mu\nu}h\right) \tag{4.53}$$

the Scalar curvature is the trace of Ricci tensor, so we get

$$R = R^{\mu}_{\mu} = \frac{1}{2}\left(-2\Box h + 2\partial_{\mu\alpha}h^{\mu\alpha}\right) = -\Box h + \partial_{\mu\alpha}h^{\mu\alpha} \tag{4.54}$$

finally, the linearised Einstein tensor is

$$\begin{aligned}\delta G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R =\\ &= \frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_{\mu\alpha}h^{\alpha}_{\nu} + \partial_{\nu\alpha}h^{\alpha}_{\mu} - \partial_{\mu\nu}h + \Box h\eta_{\mu\nu} - \eta_{\mu\nu}\partial_{\beta\alpha}h^{\beta\alpha}\right)\end{aligned} \tag{4.55}$$

therefore the linearised Einstein equations will be

$$\frac{1}{2}\left(-\Box h_{\mu\nu} + \partial_{\mu\alpha}h^{\alpha}_{\nu} + \partial_{\nu\alpha}h^{\alpha}_{\mu} - \partial_{\mu\nu}h + \Box h\eta_{\mu\nu} - \eta_{\mu\nu}\partial_{\beta\alpha}h^{\beta\alpha}\right) = 0 \tag{4.56}$$

Remark 31. *Let us specialize equation (4.45) in the case of Minkowski background. This is not difficult, since the metric $\eta_{\mu\nu}$ is flat, so its Christoffel symbols vanish and covariant derivatives become ordinary partial derivatives.*
Moreover, $R_{\mu\nu}$ and R, that are the curvatures of the background metric also vanish.
The new equation (4.45) will be

$$-\frac{1}{2}\eta_{\mu\nu}\partial_{\alpha}\partial_{\beta}X^{\alpha\beta} + \frac{1}{2}\eta_{\mu\nu}\Box X + \frac{1}{2}\partial_{\lambda}\partial_{\mu}X^{\lambda}_{\nu} + \frac{1}{2}\partial_{\lambda}\partial_{\nu}X^{\lambda}_{\mu} - \frac{1}{2}\partial_{\mu}\partial_{\nu}X - \frac{1}{2}\Box X_{\mu\nu} = 0 \tag{4.57}$$

That is the same of (4.56).

4.4 Cosmological Perturbation theory

We now want to derive a first order perturbation theory of Einstein equations in spatially flat homogeneous and isotropic cosmology. This is done considering a background metric $\bar{g}_{\mu\nu}$ and considering a perturbation g of this metric.
Then, as we did before, we first evaluate the variation of Christoffel symbols and then the variation of curvatures.
Putting all together we will get the first order perturbation of the Einstein tensor.

Remark 32. *In this chapter we will change notations, in order to be consistent with what can be found in literature.*
We will work in dimension 4, and we will write all the objects we are going to define in components. Spatial partial derivatives will be denoted as d_k, $k = 1,2,3$ and temporal partial derivatives as d_0 or with a dot.
As a convention; Greek indices run from 0 to 3 while Latin indices run from 1 to 3.

Let us begin by considering the background metric as

$$\bar{g}_{\mu\nu} = a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i \otimes dx^j) \tag{4.58}$$

Remark 33. *This may not seem the original Friedman Roberton Walker metric in the spatially flat case, that is*

$$g = -dt^2 + a(t)\delta_{ij}dx^i \otimes dx^j \tag{4.59}$$

in fact we will always work with conformal time, η, such that $a(\eta)d\eta = dt$. Conformal time is well-defined each time we consider spatially homogeneous, isotropic, flat solutions

The most general perturbation to the background metric can be written as

$$g = a^2\{-(1+2A)\,d\eta^2 + B_i\left(d\eta \otimes dx^i + dx^i \otimes d\eta\right) + (\delta_{ij} + h_{ij})\,dx^i \otimes dx^j\} \tag{4.60}$$

where A, B_i, h_{ij} are functions of x^i and η, and h_{ij} is symmetric.

Remark 34. *One can further decompose B_i and h_{ij} using Helmholtz decomposition as*

$$\begin{aligned} B_i &= \partial_i\hat{B} \oplus \hat{B}_i \\ h_{ij} &= 2(C\delta_{ij} \oplus e_{ij}) \\ e_{ij} &= H_{ij} \oplus \partial_{(i}\hat{E}_{j)} \oplus \hat{E}_{ij} \\ H_{ij} &= \partial_{ij}\hat{E} - \frac{1}{3}\delta_{ij}\Delta\hat{E} \end{aligned} \tag{4.61}$$

Where, $\partial_i\left(\sqrt{g}g^{ij}B_j\right) = 0$, H_{ij} is trace free, $\hat{E}_i$ is divergence-free and $\hat{E}_{ij}$ is symmetric, traceless ($\delta^{ij}\hat{E}_{ij} = 0$) and divergence free ($g^{ij}\hat{E}_{ij} = 0$ and $\delta^{kl}\partial_k\hat{E}_{li} = 0$).
This decomposition will not be crucial for the next computations (although it will be used to fix Newtonian gauge later), but is relevant because one can show that $(A, \hat{B}, C, \hat{E}, \hat{B}_i, \hat{E}_i, \hat{E}_{ij})$ are 10 independent fields that give an irreducible representation of $Aut_V(M)$ (vertical automorphisms of M)

It is easy to see that

$$\delta g_{00} = -2a^2A \quad \delta g_{0i} = a^2B_i \quad \delta g_{ij} = a^2h_{ij} \tag{4.62}$$

while, for the perturbation of the inverse metric we have

$$\delta g^{\alpha\beta} = -\bar{g}^{\alpha\rho}\bar{g}^{\beta\sigma}\delta g_{\rho\sigma} \tag{4.63}$$

that leads to

$$
\begin{aligned}
\delta g^{00} &= -\frac{1}{a^4}(-2a^2 A) = \frac{2}{a^2}A \\
\delta g^{0i} &= \frac{a^2}{a^4}\delta^{ij}B_j = \frac{1}{a^2}B_k\delta^{ki} \\
\delta g^{ij} &= -\frac{a^2}{a^4}\delta^{ik}\delta^{ij}h_{kl} = \frac{1}{a^2}h_{kl}\delta^{ik}\delta^{ij}
\end{aligned}
\tag{4.64}
$$

The background Christoffel symbols will be:

$$
\begin{aligned}
\bar{\Gamma}^{\alpha}_{\beta\nu} &= \frac{1}{2}\bar{g}^{\alpha\epsilon}\left(-\partial_\epsilon \bar{g}_{\beta\nu} + \partial_\beta \bar{g}_{\epsilon\nu} + \partial_\nu \bar{g}_{\beta\epsilon}\right) = \\
&= \frac{1}{2a^2}\eta^{\alpha\epsilon}\left(-\partial_\epsilon a^2\eta_{\beta\nu} + \partial_\beta a^2\eta_{\epsilon\nu} + \partial_\nu a^2\eta_{\epsilon\beta}\right) = \\
&= \frac{1}{2a^2}2a^2\mathcal{H}\left(-\eta^{\alpha 0}\eta_{\beta\nu} + \delta^0_\beta\delta^\alpha_\nu + \delta^0_\nu\delta^\alpha_\beta\right)
\end{aligned}
\tag{4.65}
$$

where we set $\mathcal{H} := \frac{\dot{a}}{a}$ for the *conformal Hubble parameter*.
More explicitly, we can compute all components of Christoffel symbols.

$$
\begin{aligned}
\bar{\Gamma}^0_{00} &= \mathcal{H}(-1+1+1) = \mathcal{H} \\
\bar{\Gamma}^0_{0i} &= \mathcal{H}\cdot 0 = 0 \\
\bar{\Gamma}^0_{ij} &= \mathcal{H}\delta_{ij} \\
\bar{\Gamma}^k_{00} &= \mathcal{H}\cdot 0 = 0 \\
\bar{\Gamma}^k_{0i} &= \mathcal{H}\delta^k_i \\
\bar{\Gamma}^k_{ij} &= \mathcal{H}\cdot 0 = 0
\end{aligned}
\tag{4.66}
$$

Also, it will be useful to compute

$$
\bar{\Gamma}_a = \bar{\Gamma}^\epsilon_{\alpha\epsilon} = \bar{\Gamma}^0_{\alpha 0} + \bar{\Gamma}^k_{\alpha k}
\tag{4.67}
$$

that in ADM will be

$$
\begin{aligned}
\bar{\Gamma}_0 &= \mathcal{H} + 3\mathcal{H} = 4\mathcal{H} \\
\bar{\Gamma}_i &= \mathcal{H}\cdot 0 = 0
\end{aligned}
\tag{4.68}
$$

now we have to evaluate the covariant derivatives of the variation of the metric: they will be used to compute the variation of Ricci tensor.
We have

$$
\bar{\nabla}_\alpha \delta g_{\mu\nu} = d_\alpha \delta g_{\mu\nu} - \Gamma^\epsilon_{\mu\alpha}\delta g_{\epsilon\nu} - \Gamma^\epsilon_{\nu\alpha}\delta g_{\epsilon\mu}
\tag{4.69}
$$

that can be written in components as

$$
\begin{aligned}
\bar{\nabla}_0\delta g_{00} &= d_0\delta g_{00} - \bar{\Gamma}^\epsilon_{00}\delta g_{\epsilon 0} - \bar{\Gamma}^\epsilon_{00}\delta g_{\epsilon 0} = -2a^2\dot{A} - 4a^2\mathcal{H}A + 4a^2\mathcal{H} = -2a^2\dot{A} \\
\bar{\nabla}_0\delta g_{0i} &= d_0\delta g_{0i} - \bar{\Gamma}^\epsilon_{00}\delta g_{\epsilon i} - \bar{\Gamma}^\epsilon_{i0}\delta g_{\epsilon 0} = a^2\dot{B}_i + 2a^2\mathcal{H}B_i - 2a^2\mathcal{H}B_i \\
\bar{\nabla}_0\delta g_{ij} &= d_0\delta g_{ij} - \bar{\Gamma}^\epsilon_{i0}\delta g_{\epsilon j} - \bar{\Gamma}^\epsilon_{j0}\delta g_{\epsilon i} = a^2\dot{h}_{ij} + 2a^2\mathcal{H}h_{ij} - 2a^2\mathcal{H}h_{ij} \\
\bar{\nabla}_k\delta g_{00} &= d_k\delta g_{00} - \bar{\Gamma}^\epsilon_{0k}\delta g_{\epsilon 0} - \bar{\Gamma}^\epsilon_{0k}\delta g_{\epsilon 0} = -2a^2\bar{D}_k A - 2\mathcal{H}A^2 B_k = -2a^2\left(\bar{D}_k A + \mathcal{H}B_k\right) \\
\bar{\nabla}_k\delta g_{0i} &= d_k\delta g_{0i} - \bar{\Gamma}^\epsilon_{0k}\delta g_{\epsilon i} - \bar{\Gamma}^\epsilon_{ik}\delta g_{\epsilon 0} = a^2\bar{D}_k B_i - \mathcal{H}a^2 h_{ki} - \mathcal{H}\delta_{ij}(-2a^2 A) = \\
&= a^2\left(\bar{D}_k B_i - \mathcal{H}h_{ki} + 2\mathcal{H}A\delta_{ki}\right)
\end{aligned}
\tag{4.70}
$$

$$\bar{\nabla}_k \delta g_{ij} = d_\alpha \delta g_{\mu\nu} - \Gamma^\epsilon_{\mu\alpha}\delta g_{\epsilon\nu} - \Gamma^\epsilon_{\nu\alpha}\delta g_{\epsilon\mu} = a^2 \left(\bar{D}_k h_{ij} - 2\mathcal{H}\delta_{k(i}B_{j)}\right) \tag{4.71}$$

where $\bar{D}_k$ denotes the spatial covariant derivative, which in these coordinates coincide with the partial derivative d_k.
Knowing the covariant derivatives of $\delta g_{\mu\nu}$ we can compute the variation of Christoffel symbols.

$$\delta\Gamma^\alpha_{\beta\mu} = \frac{1}{2}\bar{g}^{\alpha\epsilon}\left(-\bar{\nabla}_\epsilon \delta g_{\beta\nu} + \bar{\nabla}_\nu \delta g_{\beta\epsilon} + \bar{\nabla}_\beta \delta g_{\epsilon\nu}\right) \tag{4.72}$$

With the same reasoning in (4.70) we can write the components of the variation as

$$\begin{aligned}
\delta\Gamma^0_{00} &= \frac{1}{2}\bar{g}^{00}\left(-\bar{\nabla}_0\delta g_{00} + \bar{\nabla}_0\delta g_{00} + \bar{\nabla}_0\delta g_{00}\right) = -\frac{1}{2a^2}(-2a^2\dot{A}) = \dot{A}\\
\delta\Gamma^0_{0i} &= \frac{1}{2}\bar{g}^{00}\left(-\bar{\nabla}_0\delta g_{0i} + \bar{\nabla}_0\delta g_{i0} + \bar{\nabla}_i\delta g_{00}\right) = \frac{1}{2a^2}(-2a^2(D_iA + \mathcal{H}B_i)) = D_iA + \mathcal{H}B_i\\
\delta\Gamma^0_{ij} &= \frac{1}{2}\bar{g}^{00}\left(-\bar{\nabla}_0\delta g_{ij} + \bar{\nabla}_i\delta g_{j0} + \bar{\nabla}_j\delta g_{0i}\right) = -\frac{1}{2a^2}(-a^2\dot{h}_{ij} + a^2(2D_{(j}B_{i)} - 2\mathcal{H}h_{ij} + 4\mathcal{H}A\delta_{ij})) =\\
&= \frac{1}{2}\dot{h}_{ij} - D_{(j}B_{i)} + \mathcal{H}h_{ij} - 2\mathcal{H}A\delta_{ij}\\
\delta\Gamma^k_{00} &= \frac{1}{2a^2}\eta^{kl}\left(-\bar{\nabla}_l\delta g_{00} + 2\bar{\nabla}_0\delta g_{0l}\right) = \frac{1}{2a^2}\delta^{kl}\left(2a^2(D_lA + B_l) + 2a^2B_l\right) = (D_iA + \mathcal{H}B_i + B_i)\delta^{ik}\\
\delta\Gamma^k_{0i} &= \frac{1}{2a^2}\eta^{kl}\left(-\bar{\nabla}_l\delta g_{0i} + \bar{\nabla}_0\delta g_{il} + \bar{\nabla}_i\delta g_{l0}\right) =\\
&= \frac{a^2}{2a^2}\delta^{kl}\left(-D_lB_i + \mathcal{H}h_{li} - 2\mathcal{H}A\delta_{li} + \dot{h}_{il} + D_iB_l - \mathcal{H}h_{li} + 2\mathcal{H}A\delta_{il}\right) =\\
&= \left(\frac{1}{2}\dot{h}_{il} + D_{[i}B_{l]}\right)\delta_{kl}\\
\delta\Gamma^k_{ij} &= \frac{1}{2a^2}\eta^{kl}\left(-\bar{\nabla}_l\delta g_{ij} + \bar{\nabla}_i\delta g_{jl} + \bar{\nabla}_j\delta g_{li}\right) =\\
&= \frac{a^2}{2a^2}\delta^{kl}\left(-D_lh_{ij} + \mathcal{H}\delta_{ki}B_j + \mathcal{H}\delta_{kj}B_i + D_ih_{jl} - \mathcal{H}\delta_{ij}B_l - \mathcal{H}\delta_{il}B_j + D_jh_{li} - \mathcal{H}\delta_{jl}B_i - \mathcal{H}\delta_{ji}B_l\right) =\\
&= \frac{1}{2}\delta^{kl}\left(-D_lh_{ij} + D_ih_{jl} + D_jh_{li}\right) - \mathcal{H}\delta_{ij}B_l\delta^{kl}
\end{aligned} \tag{4.73}$$

Remark 35. *One should be more precise when writes (4.72). How can we be sure that it is the right formula?*
To prove this, we have to expand the Levi Civita connection at first order as

$$\begin{aligned}
\Gamma^\alpha_{\beta\nu} &= \frac{1}{2}g^{\alpha\epsilon}\left(-d_\epsilon g_{\beta\nu} + d_\nu g_{\epsilon\beta} + d_\beta g_{\nu\epsilon}\right) =\\
&= \bar{\Gamma}^\alpha_{\beta\nu} + \delta g^{\alpha\epsilon}\bar{g}_{\epsilon\sigma}\bar{\Gamma}^\sigma_{\beta\nu} + \frac{1}{2}\bar{g}^{\alpha\epsilon}\left(-d_\epsilon\delta g_{\beta\nu} + d_\nu\delta g_{\epsilon\beta} + d_\beta\delta g_{\nu\epsilon}\right) =\\
&= \bar{\Gamma}^\alpha_{\beta\nu} + \frac{1}{2}\bar{g}^{\alpha\epsilon}\left(-\bar{\nabla}_\epsilon\delta g_{\beta\nu} + \bar{\nabla}_\nu\delta g_{\epsilon\beta} + \bar{\nabla}_\beta\delta g_{\nu\epsilon}\right)
\end{aligned} \tag{4.74}$$

Hence we have

$$\delta\Gamma^\alpha_{\beta\nu} = \Gamma^\alpha_{\beta\nu} - \bar{\Gamma}^\alpha_{\beta\nu} = \frac{1}{2}\bar{g}^{\alpha\epsilon}\left(-\bar{\nabla}_\epsilon\delta g_{\beta\nu} + \bar{\nabla}_\nu\delta g_{\epsilon\beta} + \bar{\nabla}_\beta\delta g_{\nu\epsilon}\right) \tag{4.75}$$

That leads to the Palatini formula we used.

Remark 36. *As we previously did, we evaluate also*

$$\begin{aligned}
&\delta\Gamma_\beta = \delta\Gamma^\alpha_{\alpha\beta} + \delta\Gamma^i_{\beta i} \\
&\textit{whose components are} \\
&\delta\Gamma_0 = \delta\Gamma^0_{00} + \delta\Gamma^k_{0k} = \dot{A} + \frac{1}{2}\dot{h}_{ij}\delta^{ij} \\
&\delta\Gamma_i = \delta\Gamma^0_{i0} + \delta\Gamma^k_{ik} = D_i A + \mathcal{H}B_i + \frac{1}{2}\delta^{kl}D_i h_{kl} - \mathcal{H}B_i
\end{aligned} \tag{4.76}$$

Remark 37. *A check that we can do is to control if $\bar{\nabla}_\nu \delta\Gamma_\beta = \bar{\nabla}_\beta \delta\Gamma_\nu$. But this is true, since it is trivial for the* (00) *component and then we have*

$$\begin{aligned}
\bar{\nabla}_0 \delta\Gamma_i &= \bar{D}_i \dot{A} + \frac{1}{2}\delta^{kl}\bar{D}_i \dot{h}_{kl} \\
\bar{\nabla}_i \delta\Gamma_0 &= \bar{D}_i \dot{A} + \frac{1}{2}\bar{D}_i \dot{h}_{kl}\delta^{kl} \\
\bar{\nabla}_i \delta\Gamma_j &= \bar{D}_{ij} A + \frac{1}{2}\bar{D}_{ij} h_{kl}\delta^{kl} = \bar{\nabla}_j \delta\Gamma_i
\end{aligned} \tag{4.77}$$

The next object we have to compute is the background curvature.

$$\bar{R}^\alpha_{\beta\mu\nu} = d_\mu \bar{\Gamma}^\alpha_{\beta\nu} + \bar{\Gamma}^\alpha_{\epsilon\mu}\bar{\Gamma}^\epsilon_{\beta\nu} - d_\nu \bar{\Gamma}^\alpha_{\beta\mu} + \bar{\Gamma}^\alpha_{\epsilon\nu}\bar{\Gamma}^\epsilon_{\beta\mu} \tag{4.78}$$

$$\bar{R}_{\beta\nu} = \bar{R}^\alpha_{\beta\alpha\nu} = d_\alpha \bar{\Gamma}^\alpha_{\beta\nu} + \bar{\Gamma}^\alpha_{\epsilon\alpha}\bar{\Gamma}^\epsilon_{\beta\nu} - d_\nu \bar{\Gamma}^\alpha_{\beta\alpha} + \bar{\Gamma}^\alpha_{\epsilon\nu}\bar{\Gamma}^\epsilon_{\beta\alpha} \tag{4.79}$$

in components we have

$$\begin{aligned}
\bar{R}_{00} &= d_\alpha \bar{\Gamma}^\alpha_{00} + \bar{\Gamma}_\epsilon \Gamma^\epsilon_{00} - d_0 \bar{\Gamma}_0 - \bar{\Gamma}^0_{\epsilon 0}\bar{\Gamma}^\epsilon_{00} - \bar{\Gamma}^k_{\epsilon 0}\bar{\Gamma}^\epsilon_{k0} \\
&= \dot{\mathcal{H}} + 4\mathcal{H}^2 - 4\dot{\mathcal{H}} - \mathcal{H}^2 - 3\mathcal{H}^2 = -3\dot{\mathcal{H}} \\
\bar{R}_{0i} &= d_\alpha \bar{\Gamma}^\alpha_{0i} + \bar{\Gamma}_\epsilon \Gamma^\epsilon_{0i} - d_i \bar{\Gamma}_0 - \bar{\Gamma}^0_{\epsilon i}\bar{\Gamma}^\epsilon_{00} - \bar{\Gamma}^k_{\epsilon i}\bar{\Gamma}^\epsilon_{k0} = 0 \\
\bar{R}_{ij} &= d_\alpha \bar{\Gamma}^\alpha_{ij} + \bar{\Gamma}_\epsilon \Gamma^\epsilon_{ij} - d_j \bar{\Gamma}_i - \bar{\Gamma}^0_{\epsilon j}\bar{\Gamma}^\epsilon_{0i} - \bar{\Gamma}^k_{\epsilon j}\bar{\Gamma}^\epsilon_{ki} = \\
&= \dot{\mathcal{H}}\delta_{ij} + 4\mathcal{H}^2\delta_{ij} - \mathcal{H}^2\delta_{ij} - \mathcal{H}^2\delta_{ij} = \\
&= (\dot{\mathcal{H}} + 2\mathcal{H}^2)\delta_{ij}
\end{aligned} \tag{4.80}$$

$$\begin{aligned}
\bar{R} &= \frac{1}{a^2}\eta^{00}\bar{R}_{00} + \frac{2}{a^2}\eta^{0i}\bar{R}_{0i} + \frac{1}{a^2}\delta^{ij}\bar{R}_{ij} = \\
&= \frac{1}{a^2}(-3\dot{\mathcal{H}}) + \frac{3}{a^2}(\dot{\mathcal{H}} + 2\mathcal{H}^2)
\end{aligned} \tag{4.81}$$

so the background Einstein tensor in components will be

$$\begin{aligned}
\bar{G}_{00} &= -3\dot{\mathcal{H}} + \frac{1}{2}\frac{6}{a^2}(\dot{\mathcal{H}} + \mathcal{H}^2)a^2 = 3\mathcal{H}^2 \\
\bar{G}_{0i} &= -\frac{1}{2}\bar{R}\bar{g}_{0i} = 0 \\
\bar{G}_{ij} &= (\dot{\mathcal{H}} + 2\mathcal{H}^2)\delta_{ij} - \frac{1}{2}\frac{6}{a^2}(\dot{\mathcal{H}} + \mathcal{H}^2)a^2 = -(2\dot{\mathcal{H}} + \mathcal{H}^2)\delta_{ij}
\end{aligned} \tag{4.82}$$

Remark 38. *We can check our computations, in fact from the background Einstein tensor we expect to get Friedman equations written in conformal coordinates in which $ad\eta = dt$ This true since, the first Friedman equation in vacuum with $k = 0$ reads as*

$$\frac{1}{a^2}\mathcal{H}^2 = \frac{1}{a^4}\frac{da^2}{d\eta^2} = 0 \tag{4.83}$$

That is the same of the first Einstein equation in (4.82).
Moreover, recalling that $\mathcal{H} = \frac{1}{a}\frac{da}{d\eta} = aH$, one can also check that the second Friedman equation is equivalent to the third equation of (4.82).

We can now compute the variation of the Einstein tensor.

$$\begin{aligned} \delta G_{\mu\nu} &= \delta\left(R_{\mu\nu} - \frac{1}{2}g^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu}\right) = \\ &= \delta R_{\mu\nu} + \frac{1}{2}\left(\delta g^{\alpha\beta}\bar{R}_{\alpha\beta} + \bar{g}^{\alpha\beta}\delta\bar{R}_{\alpha\beta}\right) - \frac{1}{2}\bar{R}\delta g_{\mu\nu} \end{aligned} \tag{4.84}$$

where,

$$\begin{aligned} \delta g^{\alpha\beta}\bar{R}_{\alpha\beta} &= \delta g^{00}\bar{R}_{00} + \delta g^{ij}\bar{R}_{ij} = \\ &= \frac{2}{a^2}A(-3\dot{\mathcal{H}}) + \frac{1}{a^2}h_{kl}\delta^{ki}\delta^{lj}\delta_{ij}(-2\dot{\mathcal{H}} - \mathcal{H}^2) = \\ &= -\frac{6}{a^2}A\dot{\mathcal{H}} - \frac{1}{a^2}h_{ij}\delta^{ij}(\dot{\mathcal{H}} + \mathcal{H}^2) \end{aligned} \tag{4.85}$$

To get the variation of the Einstein tensor we have to evaluate $\delta R_{\mu\nu}$. Recalling the variation formula for the Riemann tensor,

$$\delta R^\alpha_{\beta\mu\nu} = \bar{\nabla}_\mu\delta\Gamma^\alpha_{\beta\nu} - \bar{\nabla}_\nu\delta\Gamma^\alpha_{\beta\mu} \tag{4.86}$$

and contracting, we get the variation of Ricci tensor

$$\delta R_{\beta\nu} = \bar{\nabla}_\alpha\delta\Gamma^\alpha_{\beta\nu} - \bar{\nabla}_\nu\delta\Gamma^\alpha_{\beta\alpha} \tag{4.87}$$

therefore, Ricci tensor in ADM will be:

$$\begin{aligned} \delta R_{00} &= \bar{\nabla}_0\delta\Gamma^0_{00} + \bar{\nabla}_k\delta\Gamma^k_{00} - \bar{\nabla}_0\delta\Gamma_0 = \\ &= \ddot{A} + \bar{\Gamma}^0_{\epsilon 0}\delta\Gamma^\epsilon_{00} - \bar{\Gamma}^\alpha_{00}\delta\Gamma^0_{\alpha 0} - \bar{\Gamma}^\alpha_{00}\delta\Gamma^0_{0\alpha} + (D_{ki}A + \mathcal{H}D_kB_i + D_k\dot{B}_i)\delta^{ki} + \\ &+ \bar{\Gamma}^k_{\alpha k}\delta\Gamma^\alpha_{00} - \bar{\Gamma}^\alpha_{0k}\delta\Gamma^k_{\alpha 0} - \bar{\Gamma}^\alpha_{0k}\delta\Gamma^k_{0\alpha} - (\ddot{A} + \frac{1}{2}\ddot{h}_{ij}\delta^{ij}) + \bar{\Gamma}^\alpha_0\delta\Gamma_\alpha = \\ &= -\mathcal{H}\dot{A} + (D_{ki}A + \mathcal{H}D_kB_i + D_k\dot{B}_i)\delta^{ki} - \frac{1}{2}\ddot{h}_{ij}\delta^{ij} + \\ &+ 3\mathcal{H}\dot{A} - \mathcal{H}(\frac{1}{2}\dot{h}_{ij}\delta^{ij}) - \mathcal{H}(\frac{1}{2}\dot{h}_{ij}\delta^{ij}) + \mathcal{H}(\dot{A} + \frac{1}{2}\dot{h}_{ij}\delta^{ij}) = \\ &= -\frac{1}{2}(\ddot{h}_{ij} + \mathcal{H}\dot{h}_{ij})\delta^{ij} + 3\mathcal{H}\dot{A} + D_{ki}A\delta^{ki} + (D_k\dot{B}_i + \mathcal{H}D_kB_i)\delta^{ki} \end{aligned} \tag{4.88}$$

$$
\begin{aligned}
\delta R_{0i} &= \bar{\nabla}_0 \delta\Gamma^0_{0i} + \bar{\nabla}_k \delta\Gamma^k_{0i} - \nabla_i \delta\Gamma_0 = \\
&= D_i \dot{A} + \dot{\mathcal{H}} B_i + \mathcal{H}\dot{B}_i + \bar{\Gamma}^0_{\alpha 0}\delta\Gamma^\alpha_{0i} - \bar{\Gamma}^\alpha_{00}\delta\Gamma^0_{\alpha i} - \bar{\Gamma}^\alpha_{i0}\delta\Gamma^0_{\alpha 0} + \frac{1}{2}(D_k \dot{h}_{il} + D_{ki}B_l - D_{kl}B_i)\delta^{kl} \\
&+ \bar{\Gamma}^k_{\alpha k}\delta\Gamma^\alpha_{0i} - \bar{\Gamma}^\alpha_{0k}\delta\Gamma^k_{\alpha i} - \bar{\Gamma}^\alpha_{ik}\delta\Gamma^k_{\alpha 0} - D_i\dot{A} - \frac{1}{2}D_i \dot{h}_{kl}\delta^{kl} + \bar{\Gamma}^\alpha_{0i}\delta\Gamma_k = \\
&= \mathcal{H}\dot{B} - i + \dot{\mathcal{H}}B_i - \mathcal{H}(D_i A + \mathcal{B}_i) + D_{[k}\dot{h}_{i]l}\delta^{kl} + D_{k[i}B_{l]}\delta^{kl} + 3\mathcal{H}(D_i A + \mathcal{H}B_i) - \\
&- \mathcal{H}\left(\frac{1}{2}\delta^{kl}(-D_l h_{ki} + D_k h_{il} + D_i h_{lk}) - \mathcal{H}B_i\right) - \mathcal{H}\delta_{ik}\delta^{jk}\left(D_j A + \mathcal{H}B_j + \dot{B}_j\right) + \\
&+ \mathcal{H}\left(D_i A + \frac{1}{2}\delta^{kl}D_i h_{kl}\right) = \\
&= (\dot{\mathcal{H}} + 2\mathcal{H}^2)B_i + 2\mathcal{H}D_i A + D_{[k}\dot{h}_{i]l}\delta^{kl} + D_{k[i}B_{l]}\delta^{kl}
\end{aligned}
\tag{4.89}
$$

$$
\begin{aligned}
\delta R_{ij} &= \bar{\nabla}_0 \delta\Gamma^0_{ij} + \bar{\nabla}_k \delta\Gamma^k_{ij} - \nabla_i \delta\Gamma_j = \\
&= \frac{1}{2}\ddot{h}_{ij} - D_{(i}\dot{B}_{j)} + \dot{\mathcal{H}}h_{ij} + \mathcal{H}\dot{h}_{ij} - 2\dot{\mathcal{H}}A\delta_{ij} - 2\mathcal{H}\dot{A}\delta_{ij} + \\
&+ \frac{1}{2}\delta^{kl}(-D_{kl}h_{ij} + D_{ki}h_{kl} + D_{kj}h_{li}) - \mathcal{H}\delta_{ij}D_k B_l \delta^{kl} + \\
&+ \bar{\Gamma}^0_{\alpha 0}\delta\Gamma^0_{ij} - \bar{\Gamma}^\alpha_{i0}\delta\Gamma^0_{\alpha j} - \bar{\Gamma}^\alpha_{j0}\delta\Gamma^0_{i\alpha} + \bar{\Gamma}^k_{\alpha k}\delta\Gamma^\alpha_{ij} - \bar{\Gamma}^\alpha_{ik}\delta\Gamma^k_{\alpha j} - \bar{\Gamma}^\alpha_{jk}\delta\Gamma^k_{i\alpha} - \\
&- D_{ji}A - \frac{1}{2}\delta^{kl}D_{ji}h_{kl} + \bar{\Gamma}^\alpha_{ij}\delta\Gamma_\alpha = \\
&= \frac{1}{2}\ddot{h}_{ij} - D_{(i}\dot{B}_{j)} + \dot{\mathcal{H}}h_{ij} + \mathcal{H}\dot{h}_{ij} - 2\dot{\mathcal{H}}A\delta_{ij} - 2\mathcal{H}\dot{A}\delta_{ij} + \\
&+ \frac{1}{2}\delta^{kl}(-D_{kl}h_{ij} + D_{ki}h_{kl} + D_{kj}h_{li}) - \mathcal{H}\delta_{ij}D_k B_l \delta^{kl} - D_{ij}A + \\
&- \frac{1}{2}\delta^{kl}D_{ji}h_{kl} + 2\mathcal{H}\left(\frac{1}{2}\dot{h}_{ij} - D_{(i}B_{j)} + \mathcal{H}h_{ij} - 2\mathcal{H}A\delta_{ij}\right) + \\
&- \mathcal{H}\delta_{kj}\left(\frac{1}{2}\dot{h}_{il} + D_{[i}B_{l]}\right)\delta^{kl} - \mathcal{H}\left(\frac{1}{2}\dot{h}_{ji} + D_{[j}B_{i]}\right) + \mathcal{H}\delta_{ij}\left(\dot{A} + \frac{1}{2}\dot{h}_{kl}\delta^{kl}\right) = \\
&= \frac{1}{2}\ddot{h}_{ij} + \left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right)h_{ij} + \mathcal{H}\dot{h}_{ij} + \frac{1}{2}\mathcal{H}\dot{h}_{kl}\delta^{kl}\delta_{ij} + \\
&+ \frac{1}{2}\delta^{kl}\left(-D_{kl}h_{ij} - D_{ij}h_{kl} + D_{ki}h_{jl} + D_{kj}h_{li}\right) + \\
&- D_{(i}B_{j)} - 2\mathcal{H}D_{(i}B_{j)} - \mathcal{H}\delta_{ij}D_k B_l \delta^{kl} + \\
&+ \mathcal{H}\delta_{ij}\left(-\dot{A} - 4\mathcal{H}A\right) - 2\dot{\mathcal{H}}A\delta_{ij} - D_{ij}A
\end{aligned}
\tag{4.90}
$$

and then, we can compute the variation of Ricci scalar.

$$
\begin{aligned}
\delta R = & \delta g^{\beta\sigma}\bar{R}_{\sigma\beta} + \bar{g}^{\beta\sigma}\delta R_{\beta\sigma} = \\
& = a^{-2}\left(-6A\dot{\mathcal{H}} - \delta R_{00} - \left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right) h_{kl}\delta^{kl} + \delta^{ij}\delta R_{ij}\right) = \\
& = a^{-2}\left(-6A\dot{\mathcal{H}} - \left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right) h_{kl}\delta^{kl} + \right. \\
& + \frac{1}{2}\left(\ddot{h}_{ij} + \mathcal{H}\dot{h}_{ij}\right)\delta^{ij} - D_i\dot{B}_j\delta^{ij} - \mathcal{H}D_iB_j\delta^{ij} - \delta^{ij}D_{ij}A - 3\mathcal{H}\dot{A} + \\
& + \frac{1}{2}\ddot{h}_{ij}\delta^{ij} + \left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right) h_{ij}\delta^{ij} + \mathcal{H}\dot{h}_{ij}\delta^{ij} + \frac{3}{2}\mathcal{H}\dot{h}_{kl}\delta^{kl} + \\
& + \frac{1}{2}\delta^{kl}\delta^{ij}\left(-\bar{D}_{kl}h_{ij} - \bar{D}_{ij}h_{kl} + \bar{D}_{ki}h_{jl} + \bar{D}_{kj}h_{li}\right) + \\
& - \bar{D}_i\dot{B}_j\delta^{ij} - 2\mathcal{H}\bar{D}_iB_j\delta^{ij} - 3\mathcal{H}\bar{D}_kB_n\delta^{kn} + \\
& \left. -\delta^{ij}\bar{D}_{ij}A - 3\mathcal{H}\dot{A} - 6\left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right)A\right) = \\
& = a^{-2}\left(\ddot{h}_{ij}\delta^{ij} + 3\mathcal{H}\dot{h}_{ij}\delta^{ij} + \delta^{kl}\delta^{ij}\left(\bar{D}_{ki}h_{jl} - \bar{D}_{kl}h_{ij}\right) + \right. \\
& - 2\left(D_i\dot{B}_j + 3\mathcal{H}\bar{D}_iB_j\right)\delta^{ij} + \\
& \left. -2\delta^{ij}D_{ij}A - 6\mathcal{H}\dot{A} - 12\left(\dot{\mathcal{H}} + \mathcal{H}^2\right)A\right)
\end{aligned}
\tag{4.91}
$$

Recalling that the variation of the Einstein tensor is

$$
\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\delta R\bar{g}_{\mu\nu} - \frac{1}{2}\bar{R}g_{\mu\nu} \tag{4.92}
$$

we can compute its components,

$$
\begin{aligned}
\delta G_{00} = & \delta R_{00} + \frac{a^2}{2}\delta R + 6\left(\dot{\mathcal{H}} + \mathcal{H}^2\right) = \\
& - \frac{1}{2}\left(\ddot{h}_{ij} + \mathcal{H}\dot{h}_{ij}\right)\delta^{ij} + \left(D_i\dot{B}_j + \mathcal{H}D_iB_j\right)\delta^{ij} + \delta^{ij}D_{ij}A + 3\mathcal{H}\dot{A} + \\
& + \frac{1}{2}\ddot{h}_{ij}\delta^{ij} + \frac{3}{2}\mathcal{H}\dot{h}_{ij}\delta^{ij} + \frac{1}{2}\delta^{kl}\delta^{ij}(\bar{D}_{ki}h_{jl} - \bar{D}_{kl}h_{ij}) + \\
& - \left(D_i\dot{B} - j + 3\mathcal{H}\bar{D}_iB_j\right)\delta^{ij} \\
& - \delta^{ij}DijA - 3\mathcal{H}\dot{A} - 6\left(\dot{\mathcal{H}} + \mathcal{H}^2\right)A + 6\left(\dot{\mathcal{H}} + \mathcal{H}^2\right)A = \\
& = \mathcal{H}\dot{h}_{ij}\delta^{ij} + \frac{1}{2}\delta^{kl}\delta^{ij}\left(\bar{D}_{ki}h_{jl} - \bar{D}_{kl}h_{ij}\right) - 2\mathcal{H}\bar{D}_iB_j\delta^{ij}
\end{aligned}
\tag{4.93}
$$

$$
\begin{aligned}
\delta G_{0i} = & \delta R_{0i} - 3\left(\dot{\mathcal{H}} + \mathcal{H}^2\right)B_i = \\
& = \bar{D}_{[k}\dot{h}_{i]j}\delta^{kj} + \bar{D}_{k[i}B_{j]}\delta^{kj} - \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)B_i + 2\mathcal{H}D_iA
\end{aligned}
\tag{4.94}
$$

$$
\begin{aligned}
\delta G_{ij} &= \delta R_{ij} - \frac{a^2}{2}\delta R \delta_{ij} - 3\left(\dot{\mathcal{H}} + \mathcal{H}^2\right) h_{ij} = \\
&= \frac{1}{2}\ddot{h}_{ij} + \left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right) h_{ij} + \mathcal{H}\dot{h}_{ij} + \frac{1}{2}\mathcal{H}\dot{h}_{kl}\delta^{kl}\delta_{ij} + \\
&+ \frac{1}{2}\delta^{kl}\left(-\bar{D}_{kl}h_{ij} - \bar{D}_{ij}h_{kl} + \bar{D}_{ki}h_{jl} + \bar{D}_{kj}h_{li}\right) + \\
&- \bar{D}_{(i}\dot{B}_{j)} - 2\mathcal{H}\bar{D}_{(i}B_{j)} - \mathcal{H}\delta_{ij}\bar{D}_k B_n \delta^{kn} + \\
&- \bar{D}_{ij}A - \mathcal{H}\dot{A}\delta_{ij} - 2\left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right) A\delta_{ij} + \\
&+ (-\frac{1}{2}\ddot{h}_{kl}\delta^{kl} - \frac{3}{2}\mathcal{H}\dot{h}_{kl}\delta^{kl} - \frac{1}{2}\delta^{kl}\delta^{mn}\left(\bar{D}_{km}h_{nl} - \bar{D}_{kl}h_{mn}\right) + + \left(D_k\dot{B}_l + 3\mathcal{H}\bar{D}_k B_l\right)\delta^{kl} + \\
&+ \delta^{kl}D_{kl}A + 3\mathcal{H}\dot{A} + 6\left(\dot{\mathcal{H}} + \mathcal{H}^2\right) A)\delta_{ij} + \\
&- 3\left(\dot{\mathcal{H}} + \mathcal{H}^2\right) h_{ij} = \\
&= \frac{1}{2}\ddot{h}_{ij} - \frac{1}{2}\ddot{h}_{kl}\delta^{kl}\delta_{ij} - \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right) h_{ij} + \mathcal{H}\dot{h}_{ij} - \mathcal{H}\dot{h}_{kl}\delta^{kl}\delta_{ij} + \\
&+ \frac{1}{2}\delta^{kl}\left(-\bar{D}_{kl}h_{ij} - \bar{D}_{ij}h_{kl} + \bar{D}_{ki}h_{jl} + \bar{D}_{kj}h_{li} + \delta^{mn}\left(\bar{D}_{kl}h_{mn} - \bar{D}_{km}h_{nl}\right)\right) + \\
&- \bar{D}_{(i}\dot{B}_{j)} + D_k\dot{B}_l\delta^{kl}\delta_{ij} - 2\mathcal{H}\bar{D}_{(i}B_{j)} + 2\mathcal{H}\bar{D}_k B_l\delta^{kl}\delta_{ij} + \\
&- \bar{D}_{ij}A + \delta^{kl}D_{kl}A\delta_{ij} + 2\mathcal{H}\dot{A}\delta_{ij} + 2\left(2\dot{\mathcal{H}} + \mathcal{H}^2\right) A\delta_{ij}
\end{aligned}
\tag{4.95}
$$

So, in vacuum and at first order we ask

$$
\delta G_{\mu\nu} = 0 \tag{4.96}
$$

This leads to equations

$$
\mathcal{H}\dot{h}_{ij}\delta^{ij} + \frac{1}{2}\delta^{kl}\delta^{ij}\left(\bar{D}_{ki}h_{jl} - \bar{D}_{kl}h_{ij}\right) - 2\mathcal{H}\bar{D}_i B_j\delta^{ij} = 0 \tag{4.97}
$$

$$
\bar{D}_{[k}\dot{h}_{i]j}\delta^{kj} + \bar{D}_{k[i}B_{j]}\delta^{kj} - \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right) B_i + 2\mathcal{H}D_i A = 0 \tag{4.98}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\ddot{h}_{ij} - \frac{1}{2}\ddot{h}_{kl}\delta^{kl}\delta_{ij} - \left(2\dot{\mathcal{H}} + \mathcal{H}^2\right) h_{ij} + \mathcal{H}\dot{h}_{ij} - \mathcal{H}\dot{h}_{kl}\delta^{kl}\delta_{ij} + \\
&+ \frac{1}{2}\delta^{kl}\left(-\bar{D}_{kl}h_{ij} - \bar{D}_{ij}h_{kl} + \bar{D}_{ki}h_{jl} + \bar{D}_{kj}h_{li} + \delta^{mn}\left(\bar{D}_{kl}h_{mn} - \bar{D}_{km}h_{nl}\right)\right) + \\
&- \bar{D}_{(i}\dot{B}_{j)} + D_k\dot{B}_l\delta^{kl}\delta_{ij} - 2\mathcal{H}\bar{D}_{(i}B_{j)} + 2\mathcal{H}\bar{D}_k B_l\delta^{kl}\delta_{ij} + \\
&- \bar{D}_{ij}A + \delta^{kl}D_{kl}A\delta_{ij} + 2\mathcal{H}\dot{A}\delta_{ij} + 2\left(2\dot{\mathcal{H}} + \mathcal{H}^2\right) A\delta_{ij} = 0
\end{aligned}
\tag{4.99}
$$

Example 7. *We can simplify these last equations by fixing a specific gauge.*
In order to compare our results with what can be found in literature, we will fix the Newtonian Gauge, that is

$$
B_i = 0 \qquad h_{ij} = 2C\delta_{ij} \tag{4.100}
$$

In this gauge, equations (4.97),(4.98),(4.99) become

$$
\begin{aligned}
&3\mathcal{H}\dot{C} - \delta^{kl}\bar{D}_{kl}C = 0\\
&\bar{D}_i\dot{C} - \mathcal{H}D_iA = 0\\
&-2\ddot{C}\delta_{ij} - 2\left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)C\delta_{ij} - 4\mathcal{H}\dot{C}\delta_{ij} - \bar{D}_{ij}C + \delta^{kl}\bar{D}_{kl}C\delta_{ij} +\\
&-\bar{D}_{ij}A + \delta^{kl}D_{kl}A\delta_{ij} + 2\mathcal{H}\dot{A}\delta_{ij} + 2\left(2\dot{\mathcal{H}} + \mathcal{H}^2\right)A\delta_{ij} = 0
\end{aligned}
\tag{4.101}
$$

4.4.1 Comparison with background-fixed linearised Einstein equations

We now want to fix the Friedman Robertson Walker background in equation (4.45), that is

$$
-\frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta X^{\alpha\beta} + \frac{1}{2}g_{\mu\nu}\Box X + \frac{1}{2}\nabla_\lambda\nabla_\mu X^\lambda_\nu + \frac{1}{2}\nabla_\lambda\nabla_\nu X^\lambda_\mu - \frac{1}{2}R_{\mu\nu}X + \frac{1}{2}RX_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu X - \frac{1}{2}\Box X_{\mu\nu} = 0
\tag{4.102}
$$

In order to see that the result matches what we get by explicit calculation in the previous section.
Let us start by computing covariant derivatives for the variation of the inverse metric.
We have

$$
\begin{aligned}
\bar{\nabla}_0\delta g^{00} &= d_0\delta g_{00} + \Gamma^0_{\epsilon 0}\delta g^{\epsilon 0} + \Gamma^0_{0\epsilon}\delta g_{\epsilon 0} = -4a^2\mathcal{H}A + 2a^{-2}\dot{A} + 4a^2\mathcal{H} = 2a^{-2}\dot{A}\\
\bar{\nabla}_0\delta g^{0i} &= d_0\delta g^{0i} + \Gamma^0_{\epsilon 0}\delta g^{\epsilon i} + \Gamma^i_{\epsilon 0}\delta g^{\epsilon 0} = a^{-2}\dot{B}_k\delta^{ki} - 2a^{-2}\mathcal{H}B_k\delta^{ki} + 2a^{-2}\mathcal{H}B_k\delta^{ki} =\\
&= a^{-2}\dot{B}_k\delta^{ki}\\
\bar{\nabla}_0\delta g^{ij} &= d_0\delta g^{ij} + \Gamma^i_{\epsilon 0}\delta g^{\epsilon j} + \Gamma^j_{\epsilon 0}\delta g^{\epsilon i} = a^{-2}\delta^{im}\delta^{jm}\dot{h}_{mn} + 2a^{-2}\mathcal{H}\delta^{im}\delta^{jm}h_{lm} - 2a^{-2}\mathcal{H}\delta^{im}\delta^{jm}h_{lm} =\\
&= a^{-2}\delta^{im}\delta^{jm}\dot{h}_{mn}\\
\bar{\nabla}_k\delta g^{00} &= d_k\delta g^{00} + \Gamma^0_{\epsilon k}\delta g^{\epsilon 0} + \Gamma^0_{\epsilon k}\delta g^{\epsilon 0} = 2a^{-2}\bar{D}_kA + 2\mathcal{H}a^2B_l\delta^l_k = 2a^2\left(\bar{D}_kA + \mathcal{H}B_l\delta^l_k\right)\\
\bar{\nabla}_k\delta g^{0i} &= d_k\delta g^{0i} + \Gamma^0_{\epsilon k}\delta g^{\epsilon i} + \Gamma^i_{\epsilon k}\delta g^{\epsilon 0} = a^{-2}\bar{D}_kB_l\delta^{li} - \mathcal{H}a^{-2}\delta_{lk}\delta^{lm}\delta^{in}h_{mn} + 2\mathcal{H}\delta^i_k(a^{-2}A) =\\
&= a^{-2}\left(\bar{D}_kB_l\delta^{li} - \mathcal{H}\delta_{lk}\delta^{lm}\delta^{in}h_{mn} + 2\mathcal{H}A\delta^i_k\right)\\
\bar{\nabla}_k\delta g^{ij} &= d_k\delta g^{ij} + \Gamma^i_{\epsilon k}\delta g^{\epsilon j} + \Gamma^j_{\epsilon k}\delta g^{\epsilon i} = -a^{-2}\delta^{im}\delta^{jn}\bar{D}_kh_{mn} + \mathcal{H}\delta^i_ka^{-2}B_l\delta^{lj} + \mathcal{H}\delta^j_kB_l\delta^{lj} =\\
&= a^{-2}\left(\mathcal{H}\delta^i_kB_j\delta^{lj} + \mathcal{H}\delta^j_kB_j\delta^{li} - \delta^{im}\delta^{jn}\bar{D}_kh_{mn}\right)
\end{aligned}
\tag{4.103}
$$

now we can compute the second covariant derivative as

$$\begin{aligned}
\nabla_\alpha\nabla_\beta X^{\alpha\beta} &= \nabla_0\nabla_0 X^{00} + \nabla_0\nabla_i X^{i0} + \nabla_i\nabla_0 X^{0i} + \nabla_i\nabla_j X^{ij} = \\
&= a^{-2}\Big\{ -4\mathcal{H}\dot{A} + 2\ddot{A} + D_i\dot{B}_k\delta^{ki} + 6\mathcal{H}A - 2\mathcal{H}\left(D_iB_l\delta^{li} - \mathcal{H}\delta_{li}\delta^{lm}\delta^{in}h_{mn} + 6A\right) + \\
&+ D_i\dot{B}_l\delta^{li} - \dot{\mathcal{H}}\delta_{li}\delta^{lm}\delta^{in}h_{mn} - \mathcal{H}\delta_{li}\delta^{lm}\delta^{in}\dot{h}_{nm} + 6\dot{\mathcal{H}}A + 6\mathcal{H}(\dot{A}) + 2\mathcal{H}A \\
&+ 3\mathcal{H}\bar{D}_jB_l\delta^{lj} + \mathcal{H}\bar{D}_iB_l\delta^{li} - \delta^{im}\delta^{jn}D_{ij}h_{mn} + 6\mathcal{H}A\Big\} = \\
&= a^{-2}\Big\{2\dot{A}\mathcal{H} + 2\ddot{A} + 2\bar{D}_i\dot{B}_l\delta^{li} + 2\mathcal{H}A + 2\mathcal{H}\bar{D}_jB_l\delta^{jl} + 2\mathcal{H}\delta_{li}\delta^{lm}\delta^{in}h_{mn} - \\
&- \dot{\mathcal{H}}\delta_{li}\delta^{lm}\delta^{in}h_{mn} - \mathcal{H}\delta_{li}\delta^{lm}\delta^{in}\dot{h}_{mn} + 6\dot{\mathcal{H}}A - \delta^{lm}\delta^{in}\bar{D}_{li}h_{mn}\Big\}
\end{aligned} \tag{4.104}$$

The other objects we have to evaluate are

$$\begin{aligned}
X = g_{\alpha\beta}X^{\alpha\beta} &= g_{00}X^{00} + g_{i0}X^{i0} + g_{0i}X^{0i} + g_{ij}X^{ij} = \\
&= -a^2(2a^{-2}A) + a^2\delta_{ij}(-a^{-2}\delta^{im}\delta^{jn}h_{mn}) = \\
&= -2A - \delta_{ij}\delta^{im}\delta^{jn}h_{mn}
\end{aligned} \tag{4.105}$$

and

$$RX_{\mu\nu} - R_{\mu\nu}X \tag{4.106}$$

To begin, let us start by considering the (00) component of the Einstein tensor.
We have

$$RX_{00} - R_{00}X = -\left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right)(-2A - \delta^{im}\delta^{jn}\delta_{ij}h_{mn}) + 6a^{-2}\left(\dot{\mathcal{H}} + \mathcal{H}^2\right)(-2a^2A) \tag{4.107}$$

putting all together, and considering the (00) component of (4.102) we get

$$\begin{aligned}
&-\frac{1}{2}g_{00}\nabla_\alpha\nabla_\beta X^{\alpha\beta} + \frac{1}{2}g_{00}\Box X + \frac{1}{2}\nabla_\lambda\nabla_0 X^\lambda_0 + \frac{1}{2}\nabla_\lambda\nabla_0 X^\lambda_0 - \frac{1}{2}R_{00}X + \frac{1}{2}RX_{00} - \\
&-\frac{1}{2}\nabla_0\nabla_0 X - \frac{1}{2}\Box X_{00} = 0 \Longrightarrow \\
&-\Big\{2\dot{A}\mathcal{H} + 2\ddot{A} + 2\bar{D}_i\dot{B}_l\delta^{li} + 2\mathcal{H}A - 2\mathcal{H}\bar{D}_jB_l\delta^{jl} + 2\mathcal{H}\delta_{li}\delta^{lm}\delta^{in}h_{mn} - \\
&- \dot{\mathcal{H}}\delta_{li}\delta^{lm}\delta^{in}h_{mn} - 2\mathcal{H}\delta_{li}\delta^{lm}\delta^{in}\dot{h}_{mn} + 6\dot{\mathcal{H}}A - \delta^{lm}\delta^{in}\bar{D}_{li}h_{mn}\Big\} + 2\ddot{A} + \\
&+ \delta_{ij}\delta^{im}\delta^{jn}\ddot{h}_{mn} + 2\mathcal{H}\delta_{li}\delta^{lm}\delta^{in}h_{mn} - 2\mathcal{H}\dot{A} + 2\ddot{A} + 2D_i\dot{B}_k\delta^{ki} + 2\mathcal{H}(A) + \\
&+ 6\dot{\mathcal{H}}A - 6A\mathcal{H}^2 + 2A\dot{\mathcal{H}} - \dot{\mathcal{H}}\delta^{im}\delta^{jn}h_{mn}\delta_{ij} + 6A\mathcal{H}^2 - \delta^{lm}\delta^{in}(\bar{D}_{lm}h_{in}) + \\
&- \delta_{ij}\delta^{im}\delta^{jn}\ddot{h}_{mn} + 4\mathcal{H}\dot{A} - 2\ddot{A} = 0
\end{aligned} \tag{4.108}$$

after simplifying and renaming indices we get

$$2\mathcal{H}D_jB_l\delta^{ij} + 2\mathcal{H}\dot{h}_{mn}\delta^{nm} + \delta^{lm}\delta^{in}\bar{D}_{il}h_{mn} - \delta^{lm}\delta^{in}\bar{D}_{lm}h_{in} = 0 \tag{4.109}$$

that is the same equation of (4.97).
In the same way it is possible to check that also the $(0, i)$ and (i, j) components match equations (4.98) and (4.99) respectively.

To summarise, in this chapter we introduced a direct computation of linearised equations taking as an example the Hilbert Lagrangian.
Computing linearised equations showed that with our method it is possible to construct the linearised equations *for every* background metric on the spcetime that is a solution of Einstein equations.
This result is remarkable because it allows to know the vector field X, solution of linearised equations, for every section on the bundle $Lor(M)$ (that is, for every metric that is also a solution of Einstein equations).
This is much more than computing the linearisation with standard techniques, in fact, all the standard techniques require to fix the background metric. This implies that, with standard techniques, one can know the vector field X only on a section of the bundle $Lor(M)$ (that is, the background metric we fixed).
As we will better explain in conclusions, this is the direction we are working at the moment: our aim is to provide an example of a vector field X that is a solution of linearised Einstein equations.

Chapter 5

Conclusions

Composite bundles provide a very flexible framework that allows to define a global geometric setting to study the second variation as well, more generally, higher degree variational principles.
That defines a unified framework to discuss Euler-Lagrange equations for critical sections and linearised equations for fields. The flow of such fields, solution of linearised equations, preserves the space of solutions, i.e. it drags solutions into solutions.

In this work, we developed the general theory of composite variations as well as a way to get linearised field equations from a composite variational point of view.
These results allow to justify, from a variational point of view, computations of linearised field equations based on ”small perturbations”.
In particular, we provide a direct computation of linearised equations (starting from the Hilbert Lagrangian) where we get linearised Einstein equations for a generic background metric.
This result is remarkable, because it can be specialised by fixing a background metric in order to get specific linearised equations. In particular, in order to recover results known for gravitational waves and perturbation theory, we specialised our results in the case of Minkowski and Friedman Robertson Walker background.
Moreover, in this last case, we also computed linearised Friedman equations in the standard way (i.e. considering a generic perturbation of the Friedman Robertson Walker metric) without fixing a gauge.

Although this work may seem complete, there are still some topics that have to be studied in-depth.
In particular, remarking that a vector field that is a solution of linearised Einstein equations drags solutions into solutions in an exact way, we are now trying to prove that these vector fields allow to reconstruct all perturbative orders (and not only the first) of the Einstein tensor.

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