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3 Modeling Limits to Growth

There will always be limits to growth.
They can be self-imposed.
If they aren't, they will be system-imposed.
Donella H. Meadows, Thinking in Systems: A Primer (2008), p.103

This chapter presents system dynamics models of *limits to growth*. First, a *one-stock* model of limits to growth is presented, where the growth rate varies, and is determined by the system's carrying capacity. Second, an interesting model of economic growth is described, which captures the law of diminishing returns, a feature of many economic systems. Third, a *two-stock* model of limits to growth is specified, where a growing stock consumes its carrying capacity, and this dynamic leads to growth followed by rapid decline. Before introducing the stock and flow models, an explanation of an important formulation method in system dynamics is presented. This allows modelers to construct robust equations to model the effect one variable has on another, as this is a requirement of many system dynamics models, particularly where one system stock influences another system stock, through its flows.

Modeling Causal Relationships Using Effects

An important building block for system dynamics is to model how variables influence one another over time. While some of these may be simple linear relationships, the reality is that real-world effects between variables can also be non-linear, and may also involve multiple variables. System dynamics offers a convenient structure for modeling the effect of one variable on another (Sterman 2000), and these general approach is described in equations (3-1) and (3-2).

$$Y = Y^* \times Effect\ (X_1\ on\ Y)\ \times ... \times\ Effect(X_n\ on\ Y) \tag{3-1}$$

$$Effect\ (X_i\ on\ Y) = f\left({X_i}\big/{X_i^*}\right) \tag{3-2}$$

These two equations are based on the following assumption.

- Variable Y is the dependent variable of a causal relationship, and this is a function of n independent variables $(X_1, X_2, ..., X_n)$
- Variable Y has a reference value Y^*, which is the normal value the variable Y takes on. This reference value is multiplied by a sequence of *effect functions* that are calculated based on the normalized ratio of input term (X_i/X_i^*), where X_i^* is the reference value, and X_i is the actual value.

- The effect function has the normalized ratio (X/X^*) on its x-axis, and always contains the point (1,1). This point (1,1) is important for the following reason: if X equals its reference value X^*, then the effect function will be 1, and therefore Y will then equal its reference value Y^* (from equation 3-1).

Consider the following example, which will shortly form part of this chapter's limits to growth model.
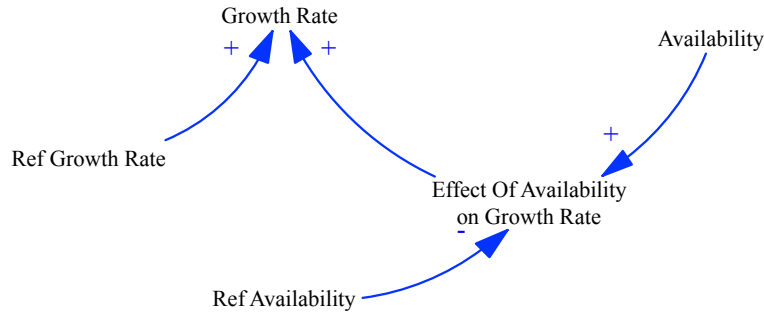
Figure 3.1: Formulating the effect of availability on growth rate

In this example, based on the generic formulation (3-1) with just a single X value, the growth rate of a system (3-3) is the product of the reference growth rate (3-5) times the *effect of resource availability* on the growth rate (3-4). Therefore, if the availability equals the reference availability, the effect will be 1, and the growth rate will equal the reference growth rate.

$$Growth\ Rate = \ Ref\ Growth\ Rate \qquad\qquad (3\text{-}3)$$
$$\times\ Effect\ of\ Availability\ on\ Growth\ Rate$$

$$Effect\ of\ Availability\ on\ Growth\ Rate = \ f(\tfrac{Availability}{Ref\ Availability}) \qquad (3\text{-}4)$$
$$Ref\ Growth\ Rate = \ 0.10 \qquad\qquad (3\text{-}5)$$
$$Ref\ Availability = \ 1.0 \qquad\qquad (3\text{-}6)$$

The effect equation (3-4) is now explored in further detail. The concept is straightforward, and the two extreme cases can be considered in order to explore the relationship between availability and growth rate.

- If availability is 1 (its maximum possible value), then the effect is 1, as the growth rate will take on its maximum value.
- If availability is zero, for example, there are no resources to support further growth, then the effect is zero, and the growth rate is therefore zero.

To keep the model simple, the assumption is that the relationship between availability and growth rate is linear, and this is illustrated in figure 3.2. On the x-axis is

the dimensionless ratio of availability to reference availability, and the y-axis contains the related effect value.
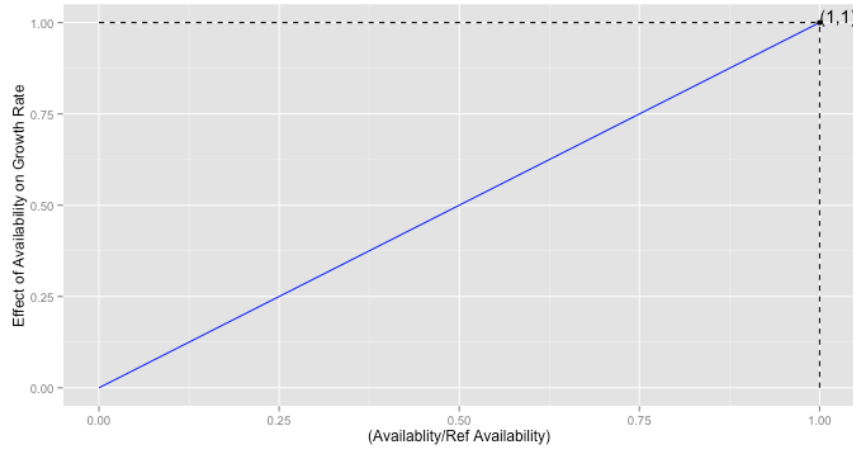
Figure 3.2: The relationship between availability and the effect on growth rate

The algebraic equation of a line with slope *m* and intercept *c* is:

$$y = mx + c$$

In this case, the intercept $c = 0$, and the slope, $m = (y_2 - y_1)/(x_2 - x_1)$, is 1. Elaborating on 3-4, our effect equation is now represented by equation 3-7.

$$Effect\ of\ Availability\ on\ Growth\ Rate = \frac{Availability}{Ref\ Availability} \qquad (3\text{-}7)$$

To highlight possible growth rate values, consider Table 3.1, which shows how the *growth rate* changes depending on the value of *availability* – through the effect variable. It captures the extreme cases of full resource available of 1.0, no resources when availability is 0.0, and the mid-point, when availability is 0.5. For each scenario, the effect equation then determines the actual growth rate. This effect equation now forms part of the first limits to growth model.

| Ref Availability	Availability	Effect of Availability on Growth Rate	Ref Growth Rate	Growth Rate
1.0	1.0	1.0	0.10	0.10
1.0	0.5	0.5	0.10	0.05
1.0	0.0	0.0	0.10	0.00

Table 3.1: Exploring a range of values for the effect function

S-Shaped Growth

Meadows (2008) writes that there will always be limits to system growth, and that these can be self-imposed, or, failing that, imposed by the system. For example, market saturation for a specific product is an example of a limit to growth, as the potential adopters are converted to adopters until (in theory) there are no potential adopters remaining. The spread of a virus is similar, as people change state from being susceptible to infected, and the limit for the virus spread is the total number of susceptible people in the population.

Earlier in chapter one, one-stock customer growth model was presented. A similar one-stock structure is now described, but a new feature is added. This model introduces a limiting factor, which acts as a balancing loop that counteracts growth. From figure 3.3, the model contains the following elements.

- The *stock* (3-8) has a single inflow.
- This inflow (3-9) is the product of the *growth rate* and the stock, where the growth rate is that defined in equation (3-3).
- The growth rate is a product of the reference growth rate and the effect of availability on the growth rate. These variables have already previously defined in equations (3-5, 3-6 and 3-7).
- Availability is a ratio that measures how much capacity remains in the system, and it is specified in equation (3-10), where the capacity is an arbitrary constant value (3-11). When the stock equals the capacity, availability is zero, and there is no further growth in the system.
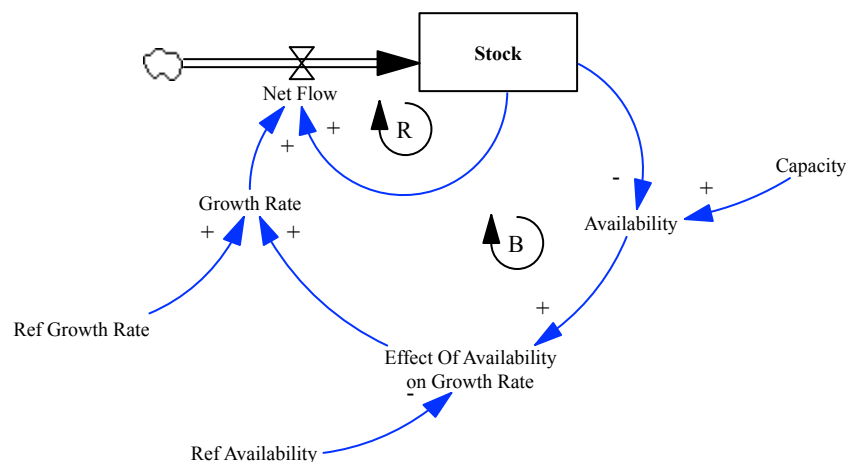
Figure 3.3: A one-stock model of limits to growth

$$Stock = INTEGRAL(Net\ Flow, 100) \qquad \text{(3-8)}$$
$$Net\ Flow = Stock \times Growth\ Rate \qquad \text{(3-9)}$$
$$Availability = 1 - \frac{Stock}{Capacity} \qquad \text{(3-10)}$$
$$Capacity = 10000 \qquad \text{(3-11)}$$

With this model specification complete, the implementation in R is described. The first task is to include the three libraries, **deSolve** for the numerical integration, and **ggplot2** and **gridExtra** for visualizing the output.

```
library(deSolve)
library(ggplot2)
library(gridExtra)
```

Following this, a number of variables are declared that will be used for the simulation run. These include:

- The start time, finish time and simulation step.
- A time vector **simtime**, which contains the sequence of times where the solver must solve for the variables.
- A vector **stocks** for the system stocks, and this must include the initial values for all stocks (3-8).
- A vector **auxs** that contains a list of the auxiliary constants for the model. In this case, we include the capacity (3-11), the reference availability (3-6) and the reference growth rate (3-5).

```
START<-0; FINISH<-100; STEP<-0.25

simtime    <- seq(START, FINISH, by=STEP)
stocks     <- c(sStock=100)
auxs       <- c(aCapacity=10000,
               aRef.Availability=1,
               aRef.GrowthRate=0.10)
```

The model equations are embedded into the R function **model.** This function will be called by the **deSolve** library for each timestep, where each invocation passes in the current time, a vector of stocks with their current simulation values, and a vector of auxiliaries. From these values, the equations are evaluated in the required sequence, starting with availability (3-10), the effect function (3-7), the growth rate (3-3), and, concluding with the net flow (3-9). The integral (3-8) is represented by the variable **dS_dt,** and this is returned in the first element of the list (a vector, as there may be more than one stock in a model). The remaining list elements contain other model variables that will be added to the simulation output.

```
model <- function(time, stocks, auxs){
  with(as.list(c(stocks, auxs)),{
    aAvailability <- 1 - sStock / aCapacity
    aEffect <- aAvailability / aRef.Availability
    aGrowth.Rate <- aRef.GrowthRate * aEffect
    fNet.Flow <- sStock * aGrowth.Rate
    dS_dt  <- fNet.Flow
    return (list(c(dS_dt),NetFlow=fNet.Flow,
                GrowthRate=aGrowth.Rate, Effect=aEffect,
                Availability=aAvailability))
  })
}
```

The output from **ode** is converted to a data frame, and plotted (figure 3.4) using the **qplot()** function.
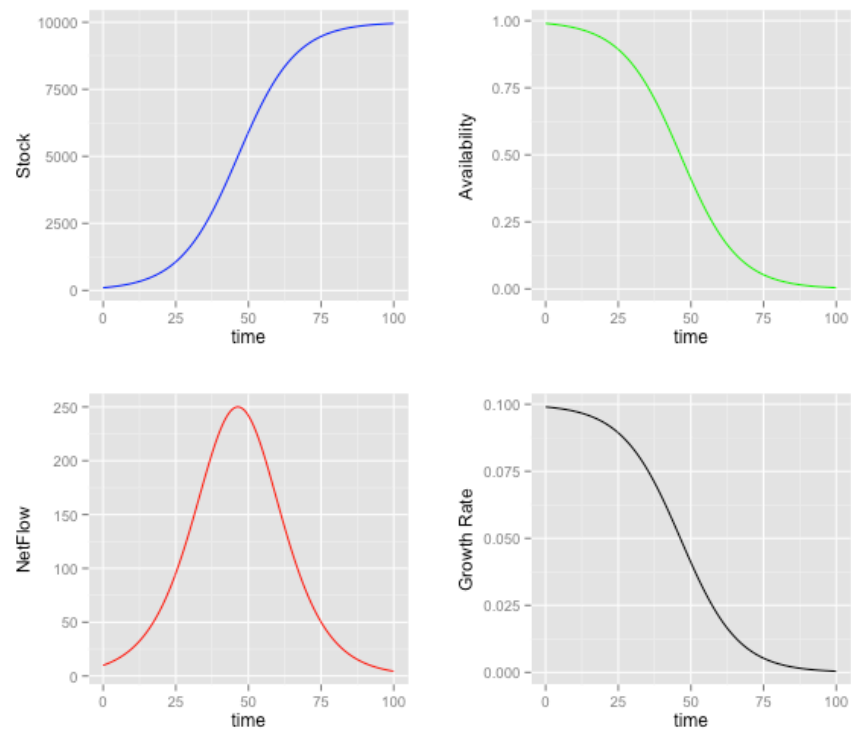

Figure 3.4: Simulation output of limits to growth model

The plots in figure 3.4 can be explored, in a clockwise direction, to gain insight into the workings of this model.

- The first plot, the *stock*, exhibits s-shaped growth behavior, which is the classic mode for limits to growth model. This is characterized by exponential growth in the early phase. However, shortly after time 46, there is a point of inflection, where curve behavior changes to logarithmic growth, and its value then approaches the limit by time 100.
- The second plot, based on equation (3-10) displays the *availability*, and this shows a mirror image of the system stock. Availability is highest when the stock is at its lowest value, and this variable drops as the stock rises, and finishes at zero.
- The third plot, the *growth rate*, which is based on equation (3-3), and is driven by the effect function specified in equation (3-7), starts off very close to its maximum value of 0.10, and then decreases as the availability declines. When the system reaches it fixed capacity, the growth rate drops to zero, and therefore no further growth is possible in the system.
- The fourth plot captures the *net flow*, and this follows a classic bell-shaped growth, where the rate of change increases exponentially, before peaking, and then declining until it reaches zero. This net flow then drives the stock's value.

This system dynamics model has a place in mathematical history, and was proposed by the Belgian mathematician Pierre-François Verhulst (1845, 1847). Verhulst noted that population increase is limited by the size and fertility of the country, with the result that the population gets ever-closer to a steady state. He proposed the following (and somewhat arbitrary) differential equation of the population *P(t)* at time *t:*

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) \tag{3-12}$$

Equation (3-12) is similar to the net flow equation equation (3-8). With a growth rate r and limit K, Verhulst went on to compare his results to empirical data in the populations of France, Belgium, the county of Essex in England, and Russia, and the models reported a good fit to the data (Bacaër 2011).

However, this limits to model also has shortcomings, one of which is that the capacity is assumed to be constant, and is not consumed by the stock growth over time. In many real-world systems this assumption does not hold, and the third growth model will address this scenario. Before that, an insightful small model of economic growth is presented.

Model of Economic Growth

Another fascinating system dynamics model of growth originates from the field of economics, and is based on a simplification of a model formulated by Nobel Prize winning economist Robert Solow (1956). In highlighting the model, Page (2015) proposes a scenario where an economy generates wealth through harvesting coconuts with machines. A fixed percentage of these resources are reinvested to produce more machines, and therefore increase the economic output. Figure 3.5 shows the model structure, which has one stock with an inflow and outflow.
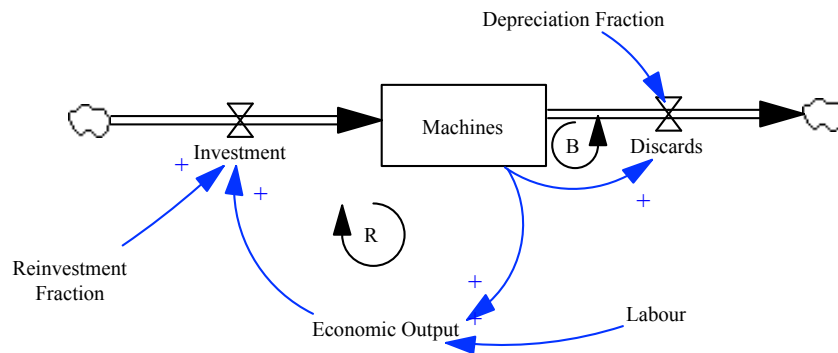
Figure 3.5: A simple economic model of growth

There are with two feedback loops that interact to drive system behavior, one reinforcing and the other balancing. The reinforcing loop, which is similar to the capital model presented in chapter one, is summarized in table 3.2.

↑	Machines	→	Economic Output	↑
↑	Economic Output	→	Investment	↑
↑	Investment	→	Machines	↑

Table 3.2: Positive feedback loop for economic model

The dynamic of this loop mean that the direction of change for the stock is amplified after one loop iteration:

- As the stock (machines) increase, so to does economic output, as there is more capacity to harvest the resource.
- An increase in economic output leads to an increased in investment.
- As the inflow investment increases, the stock of machines grows.

The second feedback loop is described in table 3.3. This is a balancing loop, as can be seen when tracing the change in the stock through the loop elements, as the direction of the stock is changed after one loop iteration.

- As machines increase, the number of discards also increase.
- An increase in discards reduces the machine stock.

↑	Machines	→	Discards	↑
↑	Discards	→	Machines	↓

Table 3.3: Negative feedback loop for economic model

Following the feedback loop polarity calculation, the model equations are formulated. The stock is specified in equation (3-13), with an initial value of 100 machines. The inflow (3-14) is the product of the reinvestment fraction and the economic output. The outflow (3-15) is a depreciation measure that reduces the stock of machines by a constant fraction for each time unit.

$$Machines\ (M) = INTEGRAL(Investment - Disards, 100) \qquad (3\text{-}13)$$
$$Investment\ = Economic\ Output\ \times\ Reinvestment\ Fraction \qquad (3\text{-}14)$$
$$Discards = Machines\ \times\ Depreciation\ Fraction \qquad (3\text{-}15)$$
$$Reinvestement\ Fraction\ (R) = 0.20 \qquad (3\text{-}16)$$
$$Depreciation\ Fraction(D) = 0.10 \qquad (3\text{-}17)$$

An important model equation is the economic output (3-18), and this is based on a fundamental of economics. This equation is a convenient model of *diminishing returns*, as the rate of increase in productivity decreases as additional machines are added. This is captured mathematically by using the square root function, which is a widely used concave function (i.e. where the slope is decreasing).

$$Economic\ Output(O) = Labour\ \times\ \sqrt{Machines} \qquad (3\text{-}18)$$
$$Labour\ (L) = 100 \qquad (3\text{-}19)$$

These seven equations are now implemented as a system dynamics model in R. First, the simulation time, stock (with initial value) and constant auxiliaries are defined in the usual vector format, and the model equations are encapsulated in the **model** function, which is called by the **ode** function.

```
START   <-0; FINISH<-100; STEP<-0.25
simtime <- seq(START, FINISH, by=STEP)
stocks  <- c(sMachines=100)
auxs    <- c(aDepFraction=0.1, aLabour=100,
            aReinvestFraction=0.20)
```

```
model <- function(time, stocks, auxs){
  with(as.list(c(stocks, auxs)),{
    aEconomicOutput <- aLabour * sqrt(sMachines)
    fInvestment     <- aEconomicOutput * aReinvestFraction
    fDiscards       <- sMachines * aDepFraction
    dM_dt           <- fInvestment − fDiscards

    return (list(c(dM_dt),
                Investment=fInvestment, Discards=fDiscards,
                EconomicOutput=aEconomicOutput))
  })
}
o<-data.frame(ode(y=stocks, times=simtime, func = model,
                  parms=auxs, method="euler"))
```

The output is captured in displayed in figure 3.6. What is of interest is that over time the stock of machines converges to a constant value, even though more machines are being added. Therefore the marginal benefit, in terms of economic output, of adding new machines decreases until it reaches zero. This is due to the impact of discards, which is the balancing feedback loop in the model.
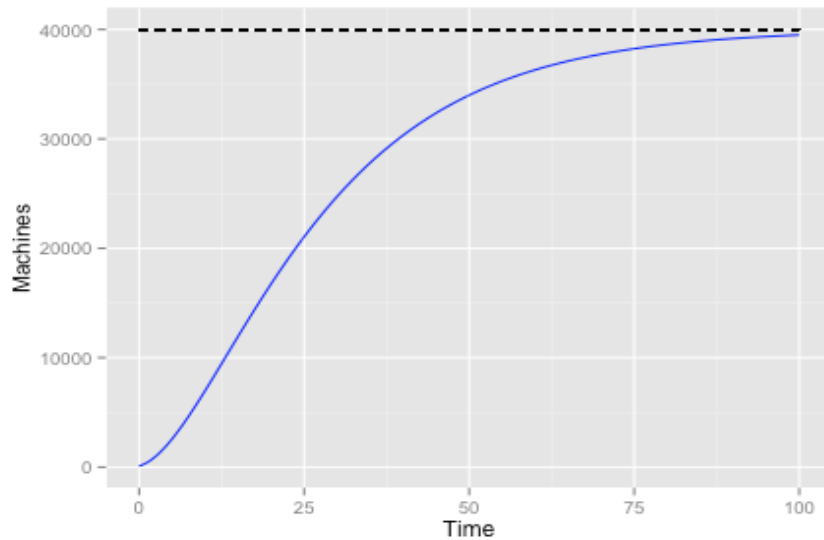
Figure 3.6: Model output showing limit to growth for machines

Interestingly, if the discard rate was set to zero, the balancing loop would be deactivated, and the number of machines would grow exponentially. However, with the balancing loop active, as the number of machines rise, so too does the discard rate, and over time the model reaches an equilibrium point where the discards

equals the investment. When this happens, the machine level is constant (i.e. a dynamic equilibrium), and so economic output also remains constant.

System dynamics also provides the capability to perform equilibrium analysis for this model. A basic principle of system dynamics is that, under equilibrium conditions for any stock, the sum of all inflows will equal the sum of all outflows. This relationship between inflow and outflow is represented in equation (3-20), and re-arranged to show the value for M^* in equilibrium (3-21). Interestingly, with L and D, constant, this equation (3-21) demonstrates that the number of machines increases with the square of the reinvestment rate. This shows that economic output, which is a function of the square root of the machines (3-18), will only increase linearly as the investment fraction increases.

$$R \times L \times \sqrt{M^*} = M^* \times D \qquad (3\text{-}20)$$
$$M^* = \left(\frac{RL}{D}\right)^2 \qquad (3\text{-}21)$$

The actual equilibrium value M can be calculated with values of $R = 0.2$, $L = 100$ and $D = 0.1$, and this is 40,000, which is the same result as the steady state value computed in the simulation model, and shown in figure 3.6. While this basic growth model finally generates a fixed level of output (i.e. growth halts), the model is useful as it models growth that results from exploiting a technology to its limits (Page 2015). As such, this initial model does not cater for future innovations in machine technology (e.g. increased productivity and longer life-span), and Solow's more detailed model accommodates this, and so can be used to model further increases in growth (Solow 1956).

Modeling Constraints – A Non-Renewable Stock

Continuing with an economic theme, a two-stock system is presented where the growth of one stock depends on the level of a second stock, which is non-renewable. This scenario depicts that the growth of oil wells is ultimately constrained by the availability of oil. This is characterized by an initial growth phase, as the underlying resource is abundant, followed by a sharp decline as the resource is consumed and depleted. This two-stock example, based on Meadows (2008, p.60), and shown in figure 3.7, focuses on a system that generates revenue through the extraction of a non-renewable resource.
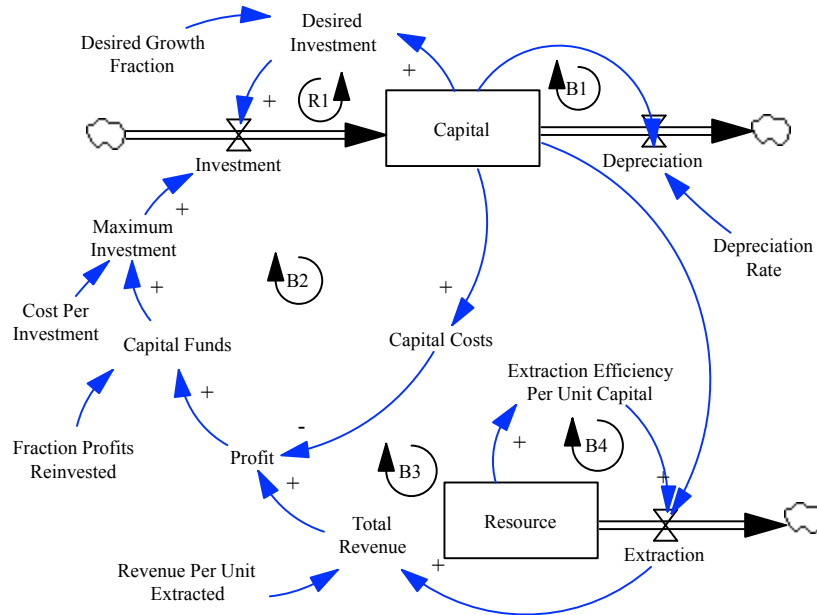
Figure 3.7: Limits to growth for capital, constrained by a non-renewable resource

This example demonstrates that systems with limits to growth have a reinforcing loop driving the growth, and a counteracting balancing loop that constrains growth. The model captures the growth and decline dynamics of a company discovering a new oil field, where the stock of oil could last for up to 200 years. The key features of the model are:

- The capital stock (e.g. oil wells) provides the capability to extract the resource. Investment is needed in capital stock, because equipment degrades over time, and must be replaced. The investment rate is initially determined by the growth goal, but this investment rate is impacted as the resource depletes, which results in limits to further growth.
- The resource stock is non-renewable, which features a single outflow, as it can only be consumed. Resource extraction is based on the amount of available capital. However, extraction rates are impacted by the amount of the available resource. As the resource level drops, the amount of resource extracted per unit capital declines. In the case of oil, this is an important dynamic. As oil resource becomes more dilute, there is less natural pressure to force it to the surface, and therefore more costly and technically sophisticated measures are required for extraction (Meadows 2008).

As with limits to growth systems, there is a reinforcing loop that drives growth, and a set of balancing loops that limit growth. The positive feedback loop (R_1) is summarized in table 3.4, and this show an exponential growth process, whereby higher capital leads to further investment, and in turn, higher capital. If this loop is unchecked, capital would grow exponentially over time. However, will soon be evident from examining the model equations, the power of the reinforcing loop is weakened as the balancing loops strengthen.

↑	Capital	→	Desired Investment	↑
↑	Desired Investment	→	Investment	↑
↑	Investment	→	Capital	↑

Table 3.4: Positive feedback loop (R_1) – capital growth

The first balancing loop (B_1) is captured in table 3.5. This is a familiar depreciation loop already encountered in the previous economic model. Given the wear and tear on equipment, it will have a finite life span, and the negative feedback loop models the depreciation effect on capital.

↑	Capital	→	Depreciation	↑
↑	Depreciation	→	Capital	↓

Table 3.5: Negative feedback loop (B_1) – capital depreciation

As capital increases, so too does the cost of capital, and this in turn will reduce profits, which is shown in loop (B_2). Reduction in profits lead to lower investment levels, and hence lower capital. Table 3.6 shows the causal links that combine to have a balancing effect on the accumulation of capital.

↑	Capital	→	Capital Costs	↑
↑	Capital Costs	→	Profit	↓
↓	Profit	→	Capital Funds	↓
↓	Capital Funds	→	Maximum Investment	↓
↓	Maximum Investment	→	Investment	↓
↓	Investment	→	Capital	↓

Table 3.6: Negative feedback loop (B_2) – increasing capital, increasing costs

Finally, two more balancing loops (B_3 and B_4) combine to impact the growth potential of capital. The logic of these loops are intuitive. More capital leads to more extraction, which depletes the resource. With a lower resource, extraction efficiency declines, which lowers the extraction rate further. This leads to reduced revenue and profits, which negatively impacts capital funds. Reduced capital investment leads to a reduction in capital, therefore the direction of change for the capital stock has reversed after one iteration through the loop structure.

↑	Capital	→	Extraction	↑
↑	Extraction	→	Resource	↓
↓	Resource	→	Extraction Efficiency Per Unit Capital	↓
↓	Extraction Efficiency Per Unit Capital	→	Extraction	↓
↓	Extraction	→	Total Revenue	↓
↓	Total Revenue	→	Profits	↓
↓	Profits	→	Capital Funds	↓
↓	Capital Funds	→	Maximum Investment	↓
↓	Maximum Investment	→	Investment	↓
↓	Investment	→	Capital	↓

Table 3.7: Negative feedback loop (B₃ and B₄) – resource depletion

The model equations are now presented, starting with the representation of the capital stock (3-22), which has an initial value of 5. This stock accumulates the net difference of investments and depreciation (3-23). The depreciation rate is constant at 5% (3-24).

$$Capital\ =\ INTEGRAL(Investments - Depreciation, 5) \qquad (3\text{-}22)$$
$$Depreciation = Capital\ \times\ Depreciation\ Rate \qquad (3\text{-}23)$$
$$Depreciation\ Rate = 0.05 \qquad (3\text{-}24)$$

Desired investment represents the target investment rate for capital, in order to stimulate growth. It is modeled as a fixed proportion of the capital stock (3-25), where the initial goal is 7% (3-26), and as this is greater that the depreciation rate of 5%, the capital stock should initially grow at an exponential rate.

$$Desired\ Investment = Desired\ Growth\ Fraction\ \times\ Capital \qquad (3\text{-}25)$$
$$Desired\ Growth\ Fraction = 0.07 \qquad (3\text{-}26)$$

However, non-renewable resource will ultimately limit this growth, and the stock and flow model is designed to capture this interplay. The integral equation for the resource (3-27) has an initial value of 1000, and a single outflow, which is the extraction rate (3-28). This extraction rate depends on the amount of available capital, which is multiplied by the extraction efficiency per unit of capital (3-29).

$$Resource =\ INTEGRAL(-Extraction, 1000) \qquad (3\text{-}27)$$
$$Extraction = Capital\ \times\ Extraction\ Efficiency\ Per\ Unit\ Capital \qquad (3\text{-}28)$$
$$Extraction\ Efficiency\ Per\ Unit\ Capital = GRAPH(Resource) \qquad (3\text{-}29)$$
$$(0,0), (100,0.25), (200,0.45), (300,0.63), (400,0.75), (500,0.85),$$
$$(600,0.92), (700,0.96), (800,0.98), (900,0.99), (1000,1.0)$$
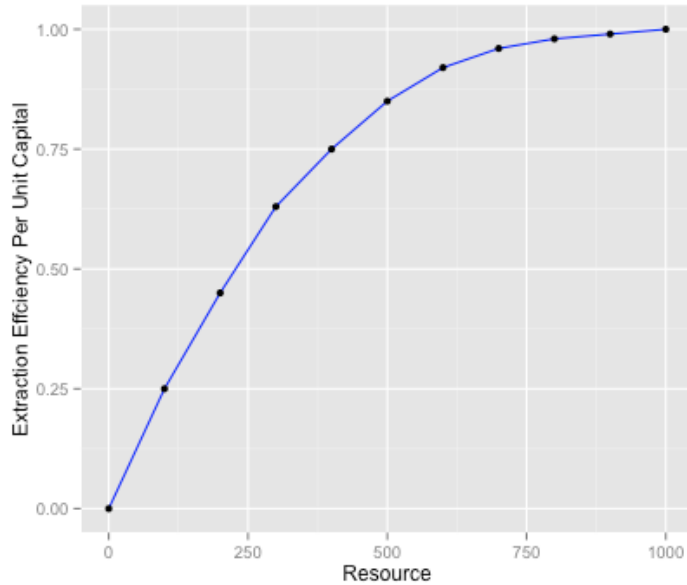
Figure 3.8: Relationship between resource level and extraction efficiency

The extraction efficiency equation (3-29) captures an important relationship between the resource level and the extraction efficiency. From a technical viewpoint, it is a good example of how a stock can be used to influence a flow. This is similar to the effect equation formulation discussed earlier in the chapter, as the efficiency value ranges from 1 to 0, where a value of zero will "switch off" the flow, and no further resources will be extracted, and revenues will drop to zero. This non-linear relationship is plotted in figure 3.8. It shows a maximum efficiency when the resource is at its maximum value of 1000. Once the resource declines, so to does the efficiency. Initially the rate of decline is small and gradual, but once it passes the half-way mark, the efficiency drops sharply, thus impacting the outflow for the extraction process. Again, this conveniently models the scenario whereby the capability of capital extraction reduces as the oil reserve diminishes.

Once the rate for extraction is calculated, the revenue and investment side of the model can be completed. The total revenue (3-30) is the amount extracted times revenue per unit extracted (3-31). The capital costs (3-32), with an arbitrary constant of 10% used, are then deducted from the revenue to generate a value for profits (3-33).

$$Total\ Revenue = Revenue\ Per\ Unit\ Extracted\ \times\ Extraction \qquad (3\text{-}30)$$
$$Revenue\ Per\ Unit\ Extracted = 3 \qquad (3\text{-}31)$$
$$Capital\ Costs = Capital\ \times\ 0.10 \qquad (3\text{-}32)$$
$$Profit = Total\ Revenue - Capital\ Costs \qquad (3\text{-}33)$$

A fixed percentage of profits (3-34) are available as capital funds (3-35). The cost per unit of investment (3-36) then determines the maximum investment in capital possible (3-37).

$$Fraction\ Profits\ Reinvested = 0.12 \qquad (3\text{-}34)$$
$$Capital\ Funds = Profit \times Fraction\ Profits\ Reinvested \qquad (3\text{-}35)$$
$$Cost\ Per\ Investment = 2 \qquad (3\text{-}36)$$
$$Maximum\ Investment = Capital\ Funds\ /\ Cost\ Per\ Investment \qquad (3\text{-}37)$$

The investment equation (3-38) is now formulated. There are two factors determining this. First, there is the desired level of investment (3-25) that is required to maintain the growth target. In a world without limits, this value would always be used in the model, and if it was, the capital stock would rise exponentially (once the growth rate exceeded the depreciation rate).

However, depending on the resources extracted and the available funding, there is the maximum possible investment that can be made (3-27), and this is the "reality check" for the system. Table 3.8 captures the required decision logic for investment. It follows the rule that the company does not invest more than its target, and that it cannot invest more than the maximum value.

Desired Investment	Maximum Investment	Investment
10	20	10
10	5	5

Table 3.8: Decision logic for finalizing investment

In system dynamics, the conventional way to represent this type of decision between what is desired, and what can be achieved subject to constraints, is to utilize the *MIN* function, and this final equation is listed in (3-38).

$$Investment = MIN(Desired\ Investment, Maximum\ Investment) \qquad (3\text{-}38)$$

The R model is now presented, and initially the time vectors, stocks and auxiliaries (constants) are defined. An interesting feature of this implementation is the way in which non-linear functions can be represented.

```
library(deSolve)

START<-0; FINISH<-200; STEP<-0.25
simtime <- seq(START, FINISH, by=STEP)
stocks  <- c(sCapital=5, sResource=1000)
```

```
auxs      <- c(aDesired.Growth      = 0.06,
               aDepreciation         = 0.05,
               aCost.Per.Investment  = 2.00,
               aFraction.Reinvested  = 0.12,
               aRevenue.Per.Unit     = 3.00)
```

Next, the nonlinear relationship between resource and extraction efficiency (3-29) is defined, and for this, R's **approxfun()** interpolation function is used. This accepts a set of x (input) and y (output) vectors, the interpolation method ("linear"), and parameters indicating the values to be returned when x is less than the minimum (yleft), and when x is greater than the maximum (yright). Two vectors are created, on for the x-axis value (**x.Resource**), and the other for the corresponding y-axis values (**y.Efficiency**). The **approxfun** function takes these vectors and creates a function that can interpolates an individual resource value to its corresponding efficiency value.

```
x.Resource<-seq(0,1000,by=100)
y.Efficiency<-c(0,0.25,0.45,0.63,0.75,0.85,0.92,
                0.96,0.98,0.99,1.0)

func.Efficiency<-approxfun(x=x.Resource,
                           y=y.Efficiency,
                           method="linear",
                           yleft=0, yright=1.0)
```

Therefore, the function **func.Efficiency()** implements equation (3-29). This can also be tested in advance for the range of values, and also for extreme cases, where the input value is outside the expected range. The following console output confirms the new function's behavior.

```
> func.Efficiency(-1)
[1] 0

> func.Efficiency(500)
[1] 0.85

> func.Efficiency(1000)
[1] 1
```

The model is now defined, where all the equations are implemented in the correct order.

```
model<-function(time, stocks, auxs){
  with(as.list(c(stocks,auxs)),
  {
    aExtr.Efficiency  <-func.Efficiency(sResource)

    fExtraction       <- aExtr.Efficiency * sCapital

    aTotal.Revenue      <- aRevenue.Per.Unit * fExtraction
    aCapital.Costs      <- sCapital * 0.10
    aProfit             <- aTotal.Revenue - aCapital.Costs
    aCapital.Funds      <- aFraction.Reinvested * aProfit
    aMaximum.Investment <- aCapital.Funds/
                            aCost.Per.Investment

    aDesired.Investment <-sCapital * aDesired.Growth

    fInvestment       <- min(aMaximum.Investment,
                                aDesired.Investment)
    fDepreciation     <- sCapital * aDepreciation

    dS_dt       <- fInvestment - fDepreciation
    dR_dt       <- -fExtraction

    return (list(c(dS_dt, dR_dt),
                  DesiredInvestment=aDesired.Investment,
                  MaximumInvestment=aMaximum.Investment,
                  Investment=fInvestment,
                  Depreciation=fDepreciation,
                  Extraction=fExtraction))
    })
}
```
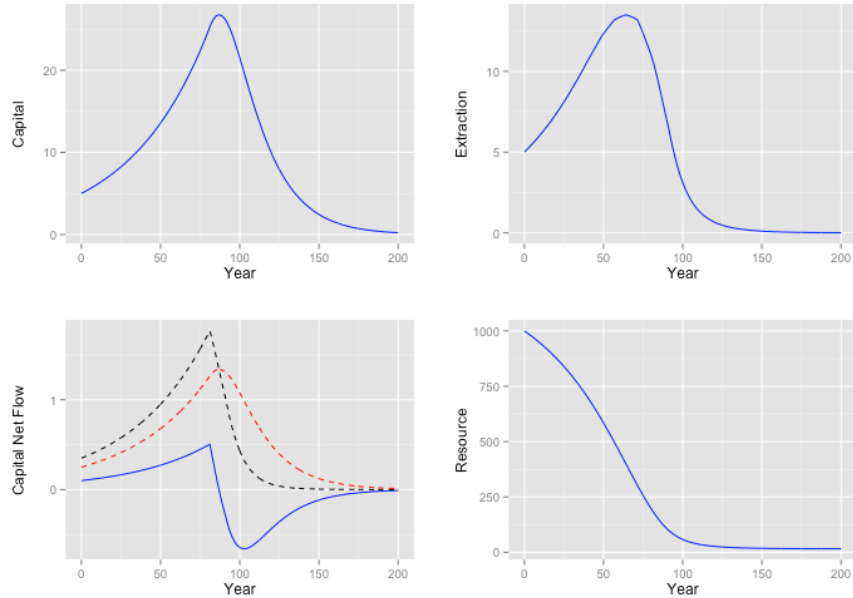
The **ode** function is called, passing in the required arguments and the result returned as a data frame.

```
o<-data.frame(ode(y=stocks, times=simtime, func = model,
                  parms=auxs, method="euler"))
```

Figure 3.9: Simulation output showing stocks and flows

The plots in figure 3.9 are now examined, in order to explore the interplay between the capital and resource stocks.

- The capital initially increases exponentially, as there are sufficient resources available to ensure that growth fraction remains at the desired level, which is 7% per annum.
- The increase in capital drives a corresponding increase in extraction, which in turn reduces the resource stock.
- Declining resources impact the capital net flow. Initially, the investment (inflow) dominates the depreciation (outflow). A critical point can be observed on the capital net flow graph, where the black line (investments) initially exceeds the red line (depreciation), and this continues until a crossover point after about 87 years, and from there on the capital falls, although the level of capital is sufficient to keep extracting the resource.

The simulation data set can be queried for exact information on peak values for capital and extraction. Using R, and the function **which.max()**, the time when capital is at its maximum can be calculated as follows.

```
> o[which.max(o$sCapital),"time"]
[1] 87
```

An important and widely used measure is when the peak of the resource extraction is reached, and this is calculated by finding the index of the maximum value of the extraction flow.

```
> o[which.max(o$Extraction),"time"]
[1] 64.25
```

An informative set of scenarios can be examined, in terms of setting different targets for the growth rate, and observing the resulting impact on the extraction pattern. In R, this can be done by running successive simulations with a different growth value, adding a new column variable to label an individual simulation run, and then joining all the simulation data into one large data frame. The R function **rbind()** is used to append data sets together. The **ggplot** attribute *color* can then be used to visualize the scenarios, which are run for the following growth rates (0.05, 0.06, 0.07, 0.10, and 0.12).

```
auxs["aDesired.Growth"]<-0.05
o1<-data.frame(ode(y=stocks, times=simtime, func = model,
                   parms=auxs, method="euler"))
o1$GR<-"GR=5%"
base<-o1

auxs["aDesired.Growth"]<-0.06
o2<-data.frame(ode(y=stocks, times=simtime, func = model,
                    parms=auxs, method="euler"))
o2$GR<-"GR=6%"
base<-rbind(base,o2)

auxs["aDesired.Growth"]<-0.07
o3<-data.frame(ode(y=stocks, times=simtime, func = model,
                   parms=auxs, method="euler"))
o3$GR<-"GR=7%"
base<-rbind(base,o3)

auxs["aDesired.Growth"]<-0.10
o4<-data.frame(ode(y=stocks, times=simtime, func = model,
                   parms=auxs, method="euler"))
o4$GR<-"GR=10%"
base<-rbind(base,o4)
auxs["aDesired.Growth"]<-0.12
o5<-data.frame(ode(y=stocks, times=simtime, func = model,
                   parms=auxs, method="euler"))
```

```
o5$GR<-"GR=12%"
base<-rbind(base,o5)

ggplot(data=base,aes(x=time,y=Extraction,color=base$GR))+
        geom_line()+ xlab("Year")+ylab("Extraction Rate")+
        theme(legend.position="bottom")+
        guides(color=guide_legend(title=NULL))
```

The scenarios for the extraction rate based in the five growth rates are visualized in figure 3.10. It confirms the view of Meadows (2008) that "a quantity growing exponentially toward a constraint or a limit reaches that limit in a surprising short time", and that "the higher and faster you grow, the farther and faster you fall."
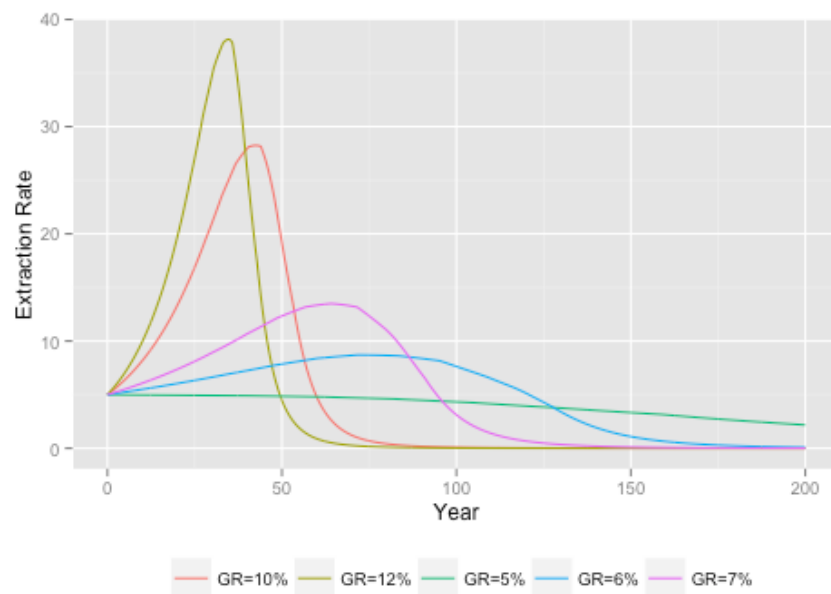
Figure 3.10: Impact of desired growth rates on resource extraction

Data on the individual simulation runs is summarized in table 3.6. This shows the impact of an increasing desired growth rate on the peak value, and peak time, of the extraction rate.

Desired Growth Rate	5%	6%	7%	10%	12%
Peak Value	5.0	8.70	13.49	28.24	38.15
Peak Time	0	71.875	64.125	42.5	34.625

Table 3.6: Comparing simulation output across growth rate scenarios

Summary

This chapter showed how to formulate models of limits to growth, where the availability of a resource impacts a system's growth potential. These models are relevant in many constraint-based disciplines, including business, healthcare and resource extraction industries. This chapter also demonstrated an important system dynamics technique which allows a number of independent variables to influence the value of a dependent variable. The next chapter will build upon these insights, and present further modeling insights that will enable the modeling of higher-order system dynamics models, with a practical application in healthcare systems.

Exercises

1. Build a set of equations to model *Experienced Programmer Productivity,* based on the following scenario. The appropriate effect equations can be sketched to show the overall impact as the variable is (1) at its reference value, (2) less than its reference value and (3) greater than its reference value.
   – Productivity is influenced by three variables: *Overtime*, *Rookie Proportion* and *Average Time to Promotion*. As these variables increase, productivity declines.
   – The reference value for *Experienced Programmer Productivity* is 200 lines of code (LOC)/Day.
   – The reference value for overtime is 5 hours per week
   – The reference rookie proportion is 20%
   – The average time for promotion is 24 months.

2. Find an analytical solution to the following representation of the logistic growth model, where P is the population, r is the growth rate, and K is the carrying capacity.

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

3. Based on the non-renewable stock model, and assuming a capital growth rate of 10%, run two additional scenarios whereby the resource is doubled and quadrupled. What impact does these additional scenarios have on the time of peak extraction?

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