Fuzzy Logic
Introduction

Andrea Bonarini
andrea.bonarini@polimi.it

Artificial Intelligence and Robotics Lab
Dipartimento di Elettronica, Informazione e Bioingegneria
Politecnico di Milano

Logic

Logic is a tool that has been used since thousands of years to formally represent knowledge.

There are many types of logic.

Propositional logic: truth values for *propositions*

e.g.: (*The grass is green)* is *true*

First order predicate logic: truth values for *predicates*

There are variables and quantifiers

e.g.: ∃ X : (Green X) is *true*

Second order predicate logic: predicates of predicates

e.g.: ∃ X, Y : (Believe Y ((Green X) is *true)) is false*

Meaning of terms

We may notice that the meaning of terms in these logics is not defined together with the formalism, and this is not needed to make the logic work.

E.g.: (*The grass is green)* is *true*

All the terms in the proposition as well as the term *true*, have a meaning only in the mind of the reader. The term *true* has a role in the mechanism of the logic, but the same role could be attributed to *1*, *glog*, or any other term.

E.g.: $\exists$ *X* : (*Green X*) is *true*

The same holds in first order logic: green is just a term as it is X, which takes a role as it is used in the formula. $\exists$ and *true* have a role in the formalism, defined by the axioms of the formalism.

Propositional logic

Propositional logics are concerned with *propositional* (or *sentential*) operators which may be applied to one or more propositions giving new propositions.

The accent is on the **truth value** of propositions, not on their meaning, and on how these truth values are composed.

A logic is *truth functional* if the truth value of a compound sentence depends only on the truth values of the constituent atomic sentences, not on their meaning or structure. For such a logic the only important question about propositions is what truth values they may have.

In a classical, **boolean** or **two-valued** logic every proposition is either *true* or *false* and no other feature of the proposition is relevant.

Operators in propositional logic

For example, the **conjunction** of the two sentences:

(Grass is green)

(Pigs don't fly)

is the sentence:

(Grass is green and pigs don't fly)

The conjunction of two sentences will be true if, and only if, each of the two sentences from which it was formed is true.

Other operators are **disjunction** (OR) and **negation** (NOT)

First order predicate logics

Same as propositional logics augmented with the possibility to define **predicates** on **variables.** Existential and universal quantifiers (Exists, For All) are defined.

A **predicate** is a feature of language that can be used to make a statement about something, e.g. to attribute a property to that thing.

If you say "Peter is tall", then you have *applied* to Peter the predicate "is tall". We also might say that you have *predicated* tallness of Peter or *attributed* tallness to Peter.

Inference

In predicate logics it is possible to **infer** the truth value of a proposition by inferential mechanisms, such as **modus ponens**

E.g.: From

All men are mortal (modus ponens)

and

Socrates is a man (proposition)

we can infer that

Socrates is mortal (proposition)

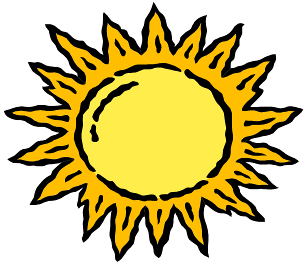
Inference is used to model a mechanism that we have in our minds to store a reduced amount of information and a set of mechanisms that can be applied to derive from information other information, to face everyday situations.
Information and potential **relationships** together compose what we call **knowledge**.

Representational power of classical logic

Aristoteles already had put in evidence problems about the validity of “classical” logic as a **knowledge representation** tool.

For instance, how can we state the truth value of a **proposition in the future**?

E.g.: “Tomorrow it will rain”



Many-valued logics

Let's introduce a third value for undefined situations and define a **three-valued** logic:

true (*1*), *false* (*0*), and *undefined* (*1/2*)

From this to an infinite set of truth values there is just a small step.

Infinite-valued logics: a continuum of truth values T (e.g., $T \in [0..1]$)

E.g., Łukasievicz (1930) logic L1

- $T(\neg a) = 1 - T(a)$
- $T(a \wedge b) = \min(T(a),T(b))$
- $T(a \vee b) = \max(T(a),T(b))$
- $T(a \Rightarrow b) = \min(1, 1 + T(b) - T(a))$
- $T(a \Leftrightarrow b) = 1 - |T(a) - T(b)|$

Notice how much time is passed from Aristoteles to Łukasievicz.

A change in society: things are no longer stated as true and false, probability (Kolmogorov, probability measure 1929) and stochasticity (Markov, stochastic variables, 1906) became the way to represent the new approach to science and life.

Similarity and differences between L1 and L2

L1 is isomorfic to the **fuzzy set theory** with standard operators as the **classical logic L2** is isomorfic to the **set theory**: there is a one-to-one correspondence of axioms, definitions, operators,....

Tautologies are true by definition, and are used to prove theorems, so to prove the truth of an inferential chain. Some tautologies valid in L2 are no longer valid in L1

- **Third excluded law** ($T(a \vee \neg a) = 1$)
  e.g., if $T(a) = 0.7$, $T(a \vee \neg a) = \max\{0.7, (1 - 0.7)\} = 0.7$

- **Non-contradiction law** ($T(a \wedge \neg a) = 0$)
  e.g., if $T(a) = 0.7$, $T(a \wedge \neg a) = \min\{0.7, (1 - 0.7)\} = 0.3$

The liar paradox

The sentence “I’m a liar” would be a paradox in classical logic, if we give a meaning to the term “liar”, since no formula can have the same truth value of its negation.

This may not be so in many-valued logics.

In Łukasiewicz logic, for instance, it can be that the truth value of a sentence (e.g.: “I’m a liar”) is 0.5, and that of its negation (“I’m not a liar”) is the same, so the proposition is consistent with the axioms, and it is not a paradox.

Fuzzy logic

Fuzzy logic is an infinite-valued logic, with truth values in [0..1]

Propositions are expressed as

A is L

where:

- A is a **linguistic** variable
- L is a label denoting a **fuzzy set**

E.g.: Temperature is HIGH

Linguistic variable

A linguistic variable is defined by a 5-tuple: (X, T(X), U, G, M)

X = name of the variable

T(X) = set of terms for X (linguistic values), each corresponding to a fuzzy variable denoted by T(X) and ranging on U

U = universe of discourse defined on a base variable u

G = syntactic rule to generate the interpretation X of each value u

M = semantic rule to associate to X its meaning

E.g.: X is a linguistic variable labelled “Age”. U = [0..100]. Terms for X are “old”, “middle-aged”, “young”, … The base variable u is the age in years. M is the definition in terms of fuzzy sets of the values of X. G is the fuzzy matching (interpretation) of u.

Simple propositions

p: X is F

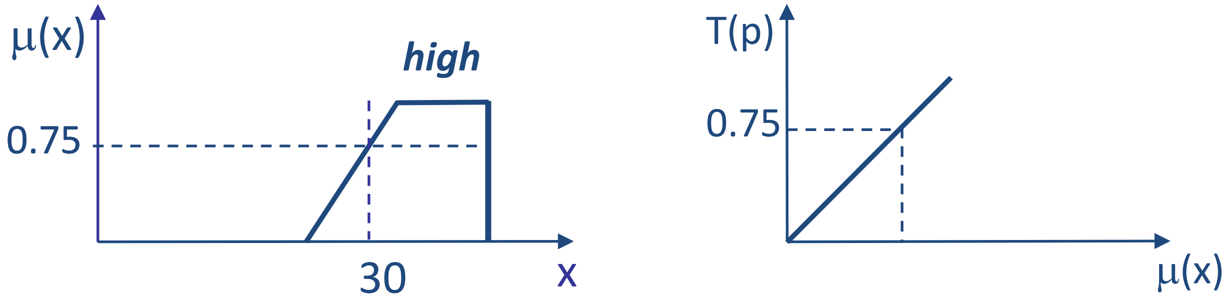
- X is a linguistic variable
- F is the label of a fuzzy set, defined on U, which represents a fuzzy predicate
- $\mu_F(x)$ is the membership function defining F
- $\mu_F(x)$ is interpreted as truth value of the proposition p
- $T(p) = \mu_F(x)$
- Therefore, the truth value of the proposition P is a fuzzy set defined on [0..1]

andrea.bonarini@polimi.it

An example

p: temperature is *high*

X **F**



Degrees of truth and probability

Degrees of truth are often confused with **probabilities**.

They are conceptually distinct: **fuzzy truth represents membership in vaguely defined sets, not likelihood of some event or condition**.

To illustrate the difference, consider this scenario:

> Bob is in a house with two adjacent rooms: the kitchen and the dining room. In many cases, Bob's status within the set of things "kitchen" is completely plain: he's either "in the kitchen" or "not in the kitchen".
>
> What about when Bob stands in the doorway? He may be considered "partially in the kitchen". Quantifying this partial state yields a fuzzy set membership. With only his little toe in the dining room, we might say Bob is 0.99 "in the kitchen", for instance.
>
> No event (like a coin toss) will resolve Bob to being completely "in the kitchen" or "not in the kitchen", as long as he's standing in that doorway.

Fuzzy modifiers

Fuzzy modifiers modify truth values

x is *Young* actually means (*x* is *Young*) is *true*

It can be modified as:

x is *very Young* is *true*

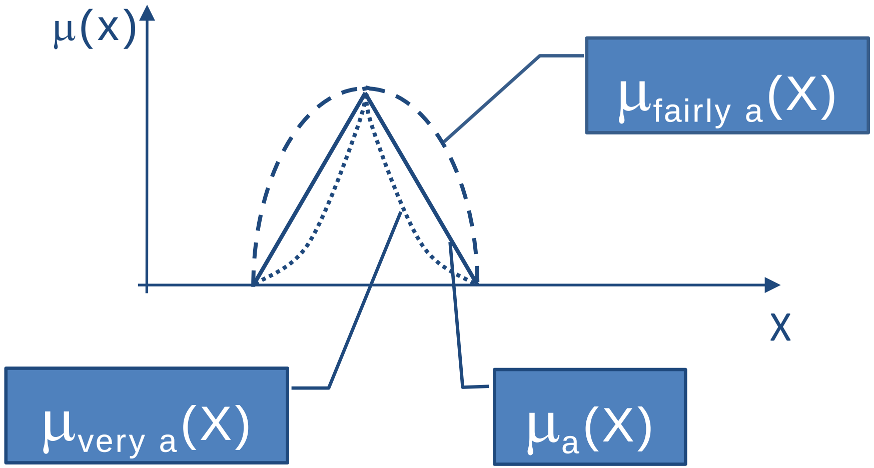
x is *Young* is *very true*

x is *very Young* is *very true*

Examples of fuzzy modifiers

$\mu_{very\ a}(x) = \mu_a(x)^2$

$\mu_{fairly\ a}(x) = \mu_a(x)^{1/2}$



Kind of modifiers

Strong modifiers

- m(a) <= a for each a in [0..1]
- They make the predicate stronger, so they reduce the truth of the proposition

Weak modifiers

- m(a) >= a for each a in [0..1]
- They make the predicate weaker, so they increase the truth of the proposition

Properties of fuzzy modifiers

- m(0)=0 and m(1)=1
- m is a continuous function
- if m is strong m^{-1} is weak, and the other way round
- given another modifier g, the composition of g and m and the other way round are modifiers, too, and, if both are strong (weak), so is their composition.

Qualified, non-conditional propositions

p: (X is F) is S

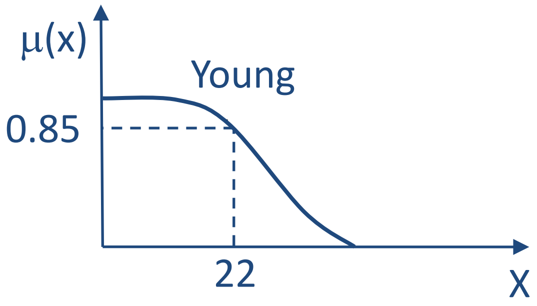
- S is a fuzzy truth qualifier
- F is a fuzzy set

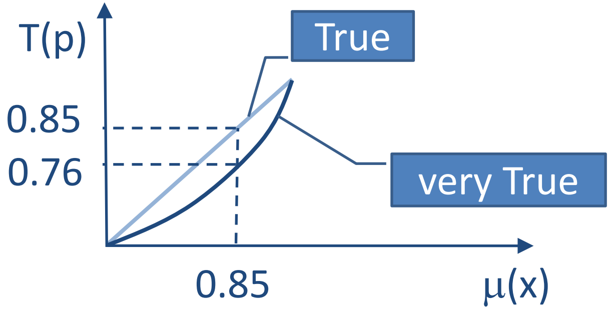
- p is truth qualified

Example

p: the age of Tina is Young is very true

X F S





What to remember from these slides?

- What is a fuzzy logic
- The conceptual path from fuzzy sets to fuzzy logic
- Fuzzy modifiers