23/03/2020

$\underline{E1}$ Dato

$$\begin{cases} \dot{x}_1 = -2x_1 + 6x_2 + u \\ \dot{x}_2 = -2x_1 + 5x_2 \end{cases}$$

1) AS/S/I?

2) $\left.\begin{aligned} x(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ u(t) &= \mathrm{sca}(t) \end{aligned}\right\} \longrightarrow x(t)\ \ t \geq 0$?

---

1) $A = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix}$ $\quad tr(A) > 0 \Rightarrow$ sistema /

2) Calcolo $x_L(t)$ tramite e^{At} e poi $x_F(t)$ con la TDL

- Autovalori di A

$\det(sI - A) = 0$

$s_1 + s_2 = tr(A)$

$\det \begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix} = 0$

$(s+2)(s-5) + 12 = 0$

$s^2 - 3s + 2 = 0 \Rightarrow s = \dfrac{3 \mp \sqrt{9-8}}{2} = \dfrac{3 \mp 1}{2} \begin{cases} s_1 = 1 \\ s_2 = 2 \end{cases}$

2 variazioni $\Rightarrow$ 2 aut. con $Re > 0$

- Autovettori

$s_1 = 1$: $A z = s_1 z$

$$\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 1 \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{cases} -2z_1 + 6z_2 = z_1 \\ -2z_1 + 5z_2 = z_2 \end{cases} \Rightarrow 6z_2 = 3z_1 \Rightarrow \begin{cases} z_1 = 2z_2 \\ \forall z_2 \end{cases}$$

sost. nella 2ª eq.

$-4z_2 + 5z_2 = z_2$

identità ✓

scelgo $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$s_2 = 2:$ $$\begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\begin{cases} -2z_1 + 6z_2 = 2z_1 \\ -2z_1 + 5z_2 = 2z_2 \end{cases} \Rightarrow \quad 6z_2 = 4z_1 \Rightarrow \begin{cases} z_1 = \frac{3}{2} z_2 \\ \forall z_2 \end{cases}$$

Scelgo $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

- Matrice diagonalizzante

$$T = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \Rightarrow T^{-1} A T = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ \begin{matrix} s_1 & \quad s_2 \end{matrix} \end{array}$$

autovettori di s_1 e s_2 $\Rightarrow$

$$T^{-1} = \frac{1}{4-3} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\bullet\quad e^{At} = e^{T\begin{bmatrix}1&0\\0&2\end{bmatrix}T^{-1}t} = T\, e^{\begin{bmatrix}1&0\\0&2\end{bmatrix}t}\, T^{-1}$$

$$= \begin{bmatrix}2&3\\1&2\end{bmatrix}\begin{bmatrix}e^{t}&0\\0&e^{2t}\end{bmatrix}\begin{bmatrix}2&-3\\-1&2\end{bmatrix}$$

$$= \begin{bmatrix}2e^{t}&3e^{2t}\\e^{t}&2e^{2t}\end{bmatrix}\begin{bmatrix}2&-3\\-1&2\end{bmatrix}$$

$$= \begin{bmatrix}4e^{t}-3e^{2t} & -6e^{t}+6e^{2t}\\ 2e^{t}-2e^{2t} & -3e^{t}+4e^{2t}\end{bmatrix}$$

- ML di x

$$x_L(t) = e^{At}\,x(0) = \begin{bmatrix} \bullet & / \\ \bullet & / \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{t} - 3e^{2t} \\ 2e^{t} - 2e^{2t} \end{bmatrix} \quad t \geq 0 \qquad \begin{matrix} \leftarrow x_{L1}(t) \\ \leftarrow x_{L2}(t) \end{matrix}$$

• Calcolo $x_F(t)$ tramite TDL

$$\dot{x}_F = A x_F + b u \Rightarrow s X_F \cancel{- x(0)} = A X_F + b U$$

$$(sI - A) X_F = b U$$

$$X_F = (sI - A)^{-1} b U = \underbrace{\begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix}^{-1}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b} \underbrace{\frac{1}{s}}_{U(s)}$$

$$X_F = \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-5 & 6 \\ -2 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{s}$$

$$= \frac{1}{\,''\,} \begin{bmatrix} s-5 \\ -2 \end{bmatrix} \frac{1}{s}$$

$$= \begin{bmatrix} \dfrac{s-5}{s(s-1)(s-2)} \\[2ex] \dfrac{-2}{s(s-1)(s-2)} \end{bmatrix} \begin{array}{l} \leftarrow X_{F1}(s) \\[4ex] \leftarrow X_{F2}(s) \end{array}$$

Antitrasformo con Heaviside

$$X_{F1}(s) = \frac{s-5}{s(s-1)(s-2)} = \frac{\alpha}{s} + \frac{\beta}{s-1} + \frac{\gamma}{s-2}$$

Eguagliamo: $\alpha(s-1)(s-2) + \beta s(s-2) + \gamma s(s-1) = s-5$

$s=0 \Rightarrow 2\alpha = -5 \Rightarrow \alpha = -5/2$

$s=1 \Rightarrow -\beta = -4 \Rightarrow \beta = 4$

$s=2 \Rightarrow 2\gamma = -3 \Rightarrow \gamma = -3/2$

Quindi

$$X_{F1}(s) = \frac{-5/2}{s} + \frac{4}{s-1} - \frac{3/2}{s-2}$$

$$x_{F1}(t) = \left(-\frac{5}{2} + 4e^{t} - \frac{3}{2}e^{2t}\right) \operatorname{sca}(t)$$

$$X_{F2}(s) = \dots$$

$$x_{F2}(t) = \dots$$

$$x(t) = x_L(t) + x_F(t)$$

$\square$

OPPURE

Calcolo tutto con la TDL

$$\left.\begin{array}{l} \dot{x} = Ax + bu \\ x(0) \\ u(t) \quad t \geq 0 \end{array}\right\} \longrightarrow \quad sX(s) - x(0) = AX(s) + bU(s)$$

$$\Rightarrow \quad X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} b\, U(s)$$

$$X(s) = \underbrace{\begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix}^{-1}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x(0)} + \underbrace{\begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix}^{-1}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b} \underbrace{\frac{1}{s}}_{U(s)}$$

$$= \underbrace{\begin{bmatrix} s+2 & -6 \\ 2 & s-5 \end{bmatrix}^{-1}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1+1/s \\ 0 \end{bmatrix}}_{x(0)+b\,U(s)}$$

$$= \ldots = \begin{bmatrix} \dfrac{(s-5)(s+1)}{s(s-1)(s-2)} \\ \dfrac{-2(s+1)}{s(s-1)(s-2)} \end{bmatrix} \ldots$$

$\square$

$\boxed{E2}$

$\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

AS/S/I ?

Calcolo $x_L(t)$ per $x(0)$ generico

A non è diagonalizzabile $\Rightarrow$ TDL

$$X_L(s) = (sI - A)^{-1} x(0) = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

$$= \frac{1}{s^2} \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} x_{01} + \frac{1}{s^2} x_{02} \\ \frac{1}{s} x_{02} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = sca(t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = ram(t) = t\, sca(t)$$

$$\Rightarrow x_L(t) = \begin{bmatrix} (x_{01} + x_{02} t)\, sca(t) \\ x_{02}\, sca(t) \end{bmatrix}$$

Se $x_{02} \neq 0$ $x_L(t)$ diverge $\Rightarrow$ sistema I

$\square$

E3

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

AS/S/I?

Calcolo x_L:

$$X_L(s) = (sI - A)^{-1} x(0) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \frac{1}{s^2} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} x_{01} \\ \frac{1}{s} x_{02} \end{bmatrix}$$

$$\Rightarrow x_L(t) = \begin{bmatrix} x_{01}\, \mathrm{sca}(t) \\ x_{02}\, \mathrm{sca}(t) \end{bmatrix} \qquad \text{limitato } \forall x(0)$$

$\Rightarrow$ sistema S

$\square$

E4

Dato $\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -3x_2 + u \\ y = 2x_2 \end{cases}$

FdT $G(s)$?

Schema a blocchi?

$$G(s) = c\,(sI-A)^{-1}b + d = [0 \;\; 2] \begin{bmatrix} s+1 & 0 \\ 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0$$

$$= \frac{1}{(s+1)(s+3)} [0 \;\; 2] \begin{bmatrix} s+3 & 0 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{''} [0 \;\; 2(s+1)] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{2\,\cancel{(s+1)}}{\cancel{(s+1)}(s+3)} = \frac{2}{s+3}$$

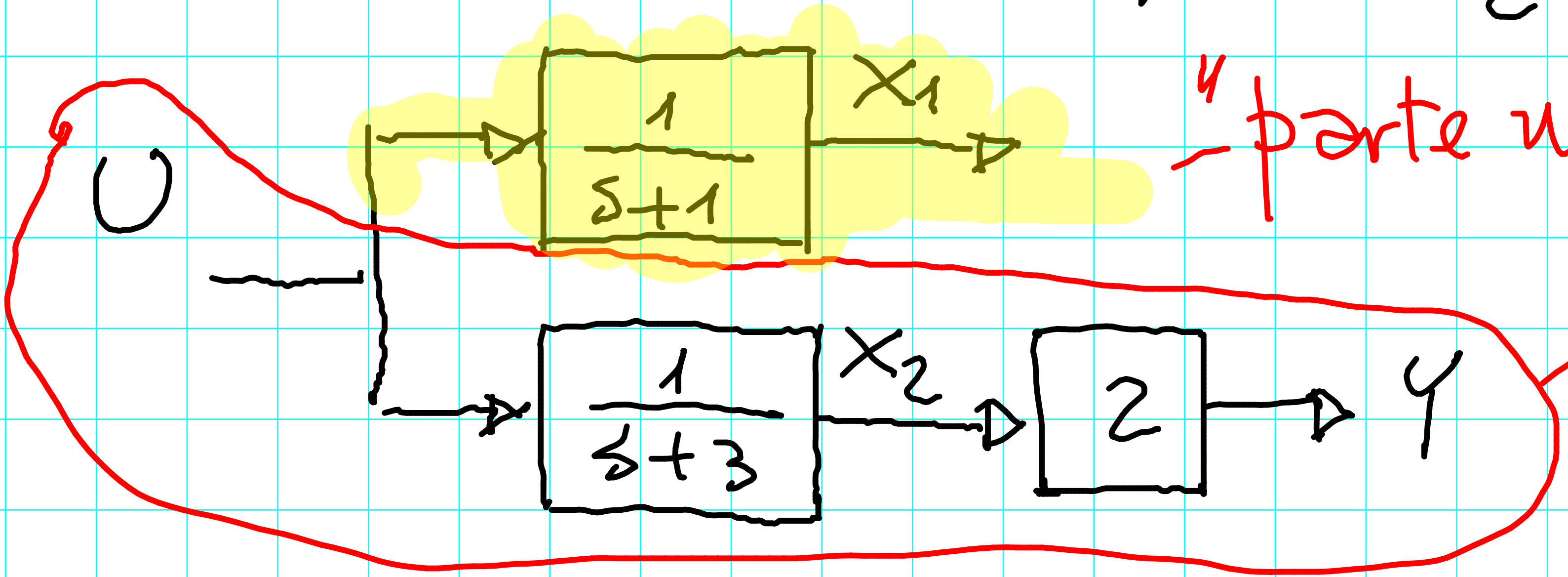
cancellazione polo/zero

Dal sist. scalare

(TF, considerando FdT)

$$\begin{cases} s X_1 = -X_1 + U \\ s X_2 = -3X_2 + U \\ Y = 2X_2 \end{cases}$$

$$\Rightarrow \begin{cases} (s+1) X_1 = U \\ (s+3) X_2 = U \\ Y = 2X_2 \end{cases} \Rightarrow \begin{cases} X_1 = \dfrac{1}{s+1} U \\ X_2 = \dfrac{1}{s+3} U \\ Y = 2X_2 \end{cases}$$



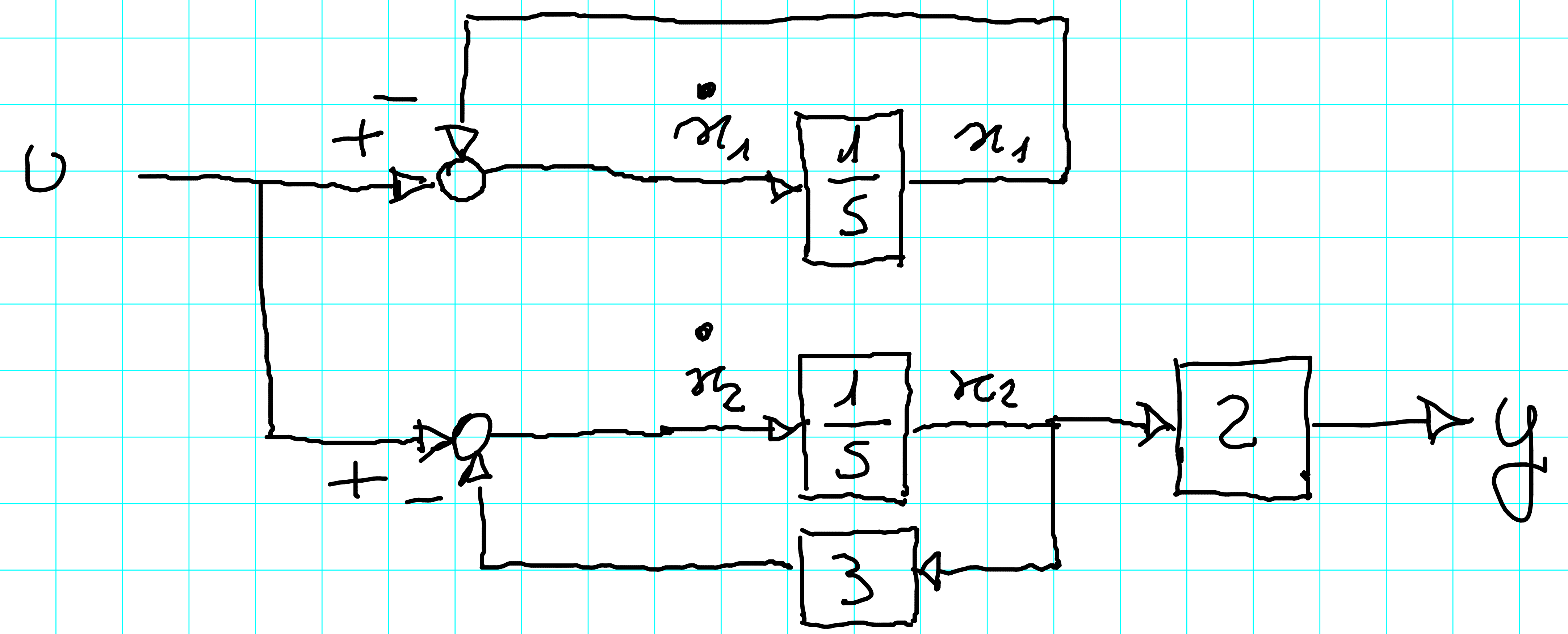
"parte nascosta"

$$\frac{Y(s)}{U(s)} = \frac{2}{s+3} = G(s)$$

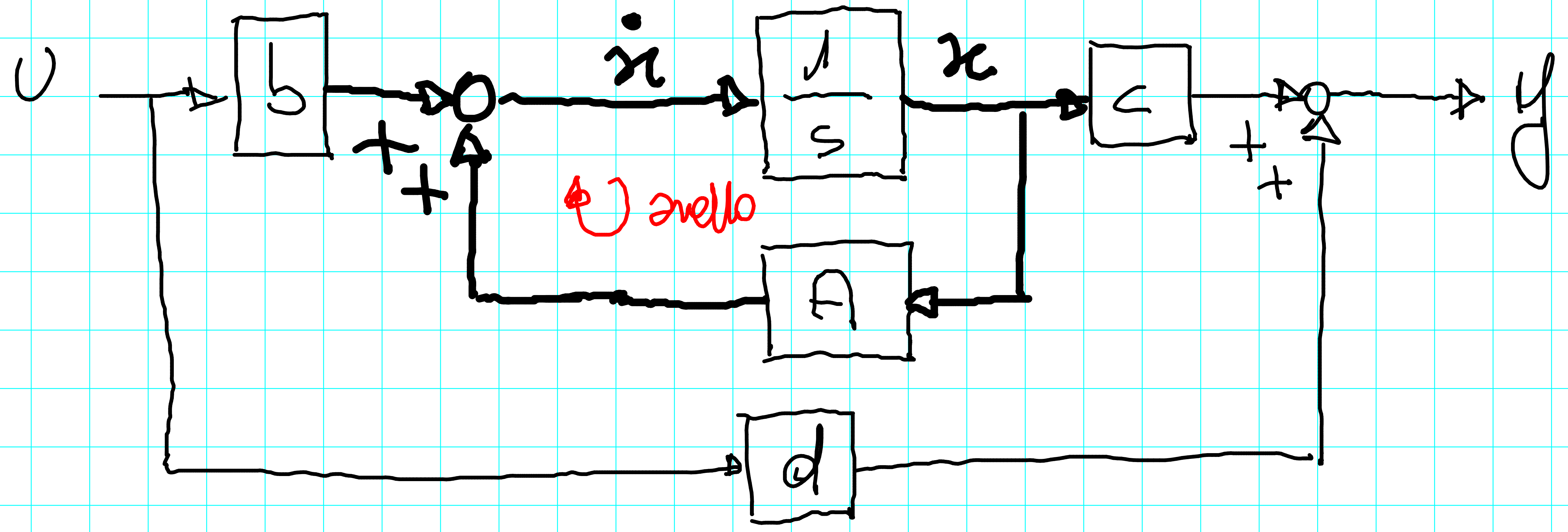
Altro modo di vedere il sistema:

Ricordiamo che se $y(t) = \int_0^t u(\tau)\, d\tau$ allora $Y(s) = \frac{1}{s} U(s)$

Quindi $\frac{1}{s}$ è la FdT del sistema "integratore"



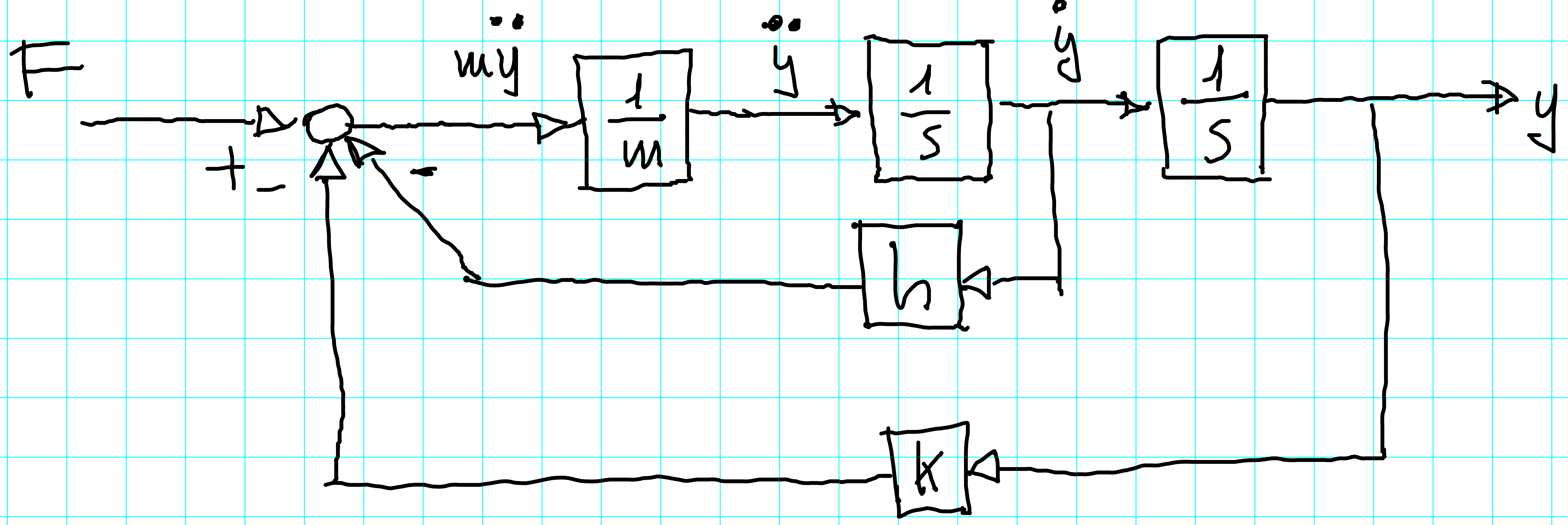
In generale



$$\begin{cases} \dot{x} = Ax + bu \\ y = cx + du \end{cases}$$

E5

Massa - molla $\qquad m\ddot{y} = F - ky - h\dot{y}$



$\square$

E6

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$x_L(t) \quad t \geq 0 \ ?$$

$$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \Rightarrow \det(sI - A) = (s-\alpha)^2 + \beta^2$$

radici $\alpha \mp j\beta$

- Autovalori di A: $\quad 1 \mp j2$

- Autovettori

$$s_1 = 1 - j2$$

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = (1-j2) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\cancel{z_1} - 2z_2 = \cancel{z_1} - 2jz_1 \Rightarrow z_2 = jz_1 \quad \forall z_1 \quad \Rightarrow \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$s_2 = 1 + j2 \qquad \Rightarrow \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

...

- Matrice diagonalizzante

$$T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$$

$$T^{-1} = \frac{1}{-2j} \begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -j/2 \\ 1/2 & j/2 \end{bmatrix}$$

$$\bullet \quad e^{At} = e^{T \begin{bmatrix} 1-2i & 0 \\ 0 & 1+2j \end{bmatrix} T^{-1} t} = T e^{[\cdot 4]t} T^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{(1-2j)t} & 0 \\ 0 & e^{(1+2j)t} \end{bmatrix} \begin{bmatrix} 1/2 & -j/2 \\ 1/2 & j/2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{(1-2j)t} & e^{(1+2j)t} \\ j e^{(1-2j)t} & -j e^{(1+2j)t} \end{bmatrix} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{(1-2j)t} + e^{(1+2j)t} & -j e^{(1-2j)t} + j e^{(1+2j)t} \\ j e^{(1-2j)t} - j e^{(1+2j)t} & e^{(1-2j)t} + e^{(1+2j)t} \end{bmatrix}$$

• ML di x

$$x_L(t) = e^{At}x(0) = \left[\,\cdot\,\right]\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix} e^{(1-2j)t} + e^{(1+2j)t} \\ j\,e^{(1-2j)t} - j\,e^{(1+2j)t} \end{bmatrix}$$

$$x_{L1}(t) = e^{t}\left(\cos(2t) - j\sin(2t)\right) + e^{t}\left(\cos(2t) + j\sin(2t)\right)$$

$$= 2e^{t}\cos(2t)$$

$$x_{L2}(t) = \ldots = 2e^{t}\sin(2t)$$

$\square$