24/03/2020

E7) Dato il SD NL TI a TC

$$\dot{x}_1 = x_1^2 + x_2 - 2u^3$$
$$\dot{x}_2 = x_1 - x_2$$
$$y = x_1^4 + x_2 + u^2$$

1) $\bar{x}$ e $\bar{y}$ per $u(t) = \bar{u} = 1$?
2) Stabilità equilibri ?
3) Sist. linearizzati ?

1) calcolo stati equilibrio

$$\begin{cases} 0 = \bar{x}_1^2 + \bar{x}_2 - 2\bar{u}^3 \\ 0 = \bar{x}_1 - \bar{x}_2 \end{cases}$$

$$\begin{cases} \bar{x}_2 = \bar{x}_1 \\ \bar{x}_1^2 + \bar{x}_1 - \underset{\uparrow \atop \bar{u}=1}{2} = 0 \end{cases} \rightarrow \begin{cases} \bar{x}_1 = \dfrac{-1 \mp \sqrt{1+8}}{2} = \begin{cases} -2 \\ 1 \end{cases} \\ \bar{x}_2 = \bar{x}_1 \end{cases}$$

Quindi vi sono 2 stati di equilibrio:

$$\bar{x}_a = \begin{bmatrix} -2 \\ -2 \end{bmatrix} \qquad \bar{x}_b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

e in corrispondenza

$$\bar{y}_a = (-2)^4 - 2 + 1^2 = 15$$

$$\bar{y}_b = (1)^4 + 1 + 1^2 = 3$$

2) calcolo matrice A del generico sist. linearizzato

$$f_x = \begin{bmatrix} 2x_1 & 1 \\ 1 & -1 \end{bmatrix}$$

Equilibrio a:

$$\bar{x}_1 = -2 \quad \bar{x}_2 = -2 \quad \bar{u} = 1$$

$$f_x\Big|_{\text{equilibrio a}} = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix} = A_a$$

$$\det(sI - A_a) = 0$$

$$\det \begin{bmatrix} s+4 & -1 \\ -1 & s+1 \end{bmatrix} = 0 \qquad s^2 + 5s + 3 = 0$$

coeff. concordi e 2° grado $\Rightarrow$ 2 radici con Re < 0 $\Rightarrow$ sist. lin. AS $\Rightarrow$ Ep. AS

Equilibrio b:

$$\bar{x}_1 = 1 \,, \quad \bar{x}_2 = 1 \,, \quad \bar{u} = 1$$

$$f_x \Big|_{\text{equilibrio b}} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = A_b$$

traccia > 0 $\Rightarrow$ almeno un autovalore con $\mathrm{Re} > 0$ $\Rightarrow$ Eq. I

$\square$

3) sistemi linearizzati

$$\begin{cases} \delta\dot{x} = f_x\big|_{\bar{x},\bar{u}}\,\delta x + f_u\big|_{\bar{x},\bar{u}}\,\delta u \\ \delta y = g_x\big|_{\bar{x},\bar{u}}\,\delta x + g_u\big|_{\bar{x},\bar{u}}\,\delta u \end{cases}$$

$$\delta u = u - \bar{u}$$
$$\delta x = x - \bar{x}$$
$$\delta y = y - \bar{y}$$

Estraiamo f_u, g_x e g_u

$$f_u = \begin{bmatrix} -6u^2 \\ 0 \end{bmatrix}, \quad g_x = \begin{bmatrix} 4x_1^3 & 1 \end{bmatrix}, \quad g_u = 2u$$

- Sist. lin. attorno a eq. a

$$\begin{cases} \delta \dot{x} = \begin{bmatrix} -4 & 1 \\ 1 & -1 \end{bmatrix} \delta x + \begin{bmatrix} -6 \\ 0 \end{bmatrix} \delta u \\ \delta y = \begin{bmatrix} -32 & 1 \end{bmatrix} \delta x + 2 \delta u \end{cases}$$

$$\delta u = u - 1$$

$$\delta x = x - \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\delta y = y - 15$$

eq. b . . .

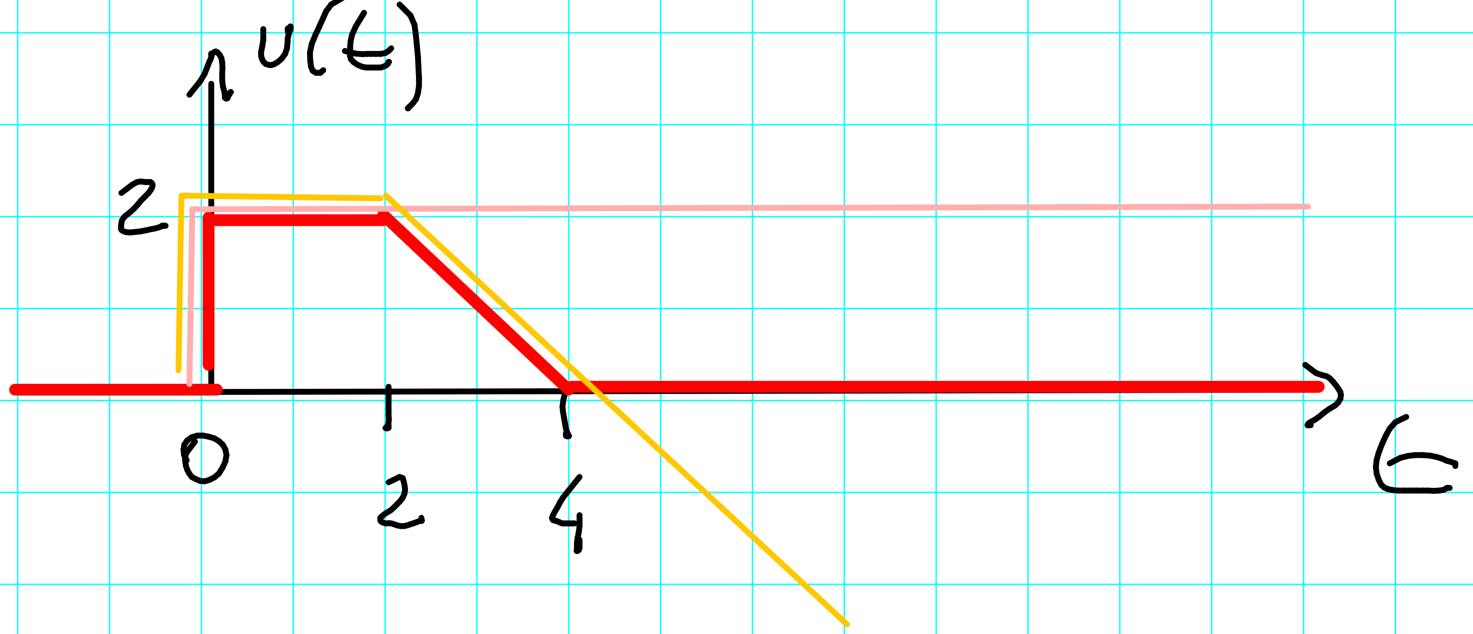
$\square$

E8

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases}$$

$x(0) = 0$

$y(t) \quad t \geq 0 \, ?$



- Esprimo $u(t)$ come somma di segnali canonici

$$u(t) = 2\,\text{sca}(t) - \text{ram}(t-2) + \text{ram}(t-4)$$

Quindi

$$U(s) = \frac{2}{s} - \underbrace{\underbrace{\frac{1}{s^2}}_{\text{ram}(t)} e^{-2s}}_{\text{ram}(t-2)} + \frac{1}{s^2} e^{-4s}$$

- Calcolo la FdT del sistema

$$G(s) = c(sI-A)^{-1}b + d = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{2}{(s+1)(s+2)}$$

• Esprimo $Y(s)$ (c'è sotto TF)

$$Y(s) = G(s)U(s) = \frac{2}{(s+1)(s+2)}\left(\frac{2}{s} - \frac{1}{s^2}e^{-2s} + \frac{1}{s^2}e^{-4s}\right)$$

$$= \underbrace{\frac{4}{s(s+1)(s+2)}}_{Y_1(s)} + \underbrace{\frac{2}{s^2(s+1)(s+2)}}_{Y_2(s)}\left(e^{-4s} - e^{-2s}\right)$$

$Y_1(s)$, $Y_2(s)$ → ... Heaviside

in generale (N, D polinomi)

$$\sum_i \frac{N_i(s)}{\underbrace{D_i(s)}_{\text{Heaviside} \to g_i(t)}} \sum_j e^{\underbrace{-T_i s}}$$

Heaviside → $g_i(t)$

ritardi con cui appaiono le $g_i(t)$

$$y_1(t) = \left(2 + 2e^{-2t} - 4e^{-t}\right) \mathrm{sca}(t)$$

$$y_2(t) = \left(-\frac{3}{2} + t - \frac{1}{2}e^{-2t} + 2e^{-t}\right) \mathrm{sca}(t)$$

Componendo

$$y(t) = y_1(t) - y_2(t-2) + y_2(t-4) \longrightarrow$$

$$\left[\ \mathcal{L}^{-1}\left[\boxed{\frac{N(s)}{D(s)}}\, e^{-s\tau}\right] = y(t-\tau)\ \right]$$

HEAVISIDE $y(t)$

N, D polinomi

$$y(t) = \underbrace{y_1(t)} - \underbrace{y_2(t-2)} + y_2(t-4)$$

$$= \left(2 + 2e^{-2t} - 4e^{-t}\right)\sigma(t)$$

$$- \left(-\frac{3}{2} + (t-2) - \frac{1}{2}e^{-2(t-2)} + 2e^{-(t-2)}\right)\sigma(t-2)$$

$$+ \left(-\frac{3}{2} + (t-4) - \frac{1}{2}e^{-2(t-4)} + 2e^{-(t-4)}\right)\sigma(t-4)$$

$\square$

E 9

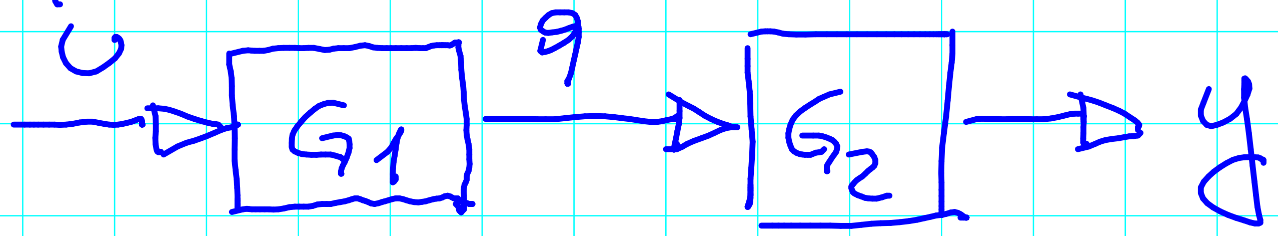
Consideriamo una FdT $G(s)$ già scomposta in somma di fratti semplici

$$G(s) = \frac{1}{s+1} + \frac{2}{s} + \frac{3}{(s+2)^2}$$

$$= \frac{1}{s+1} + \frac{2}{s} + \frac{3}{s+2} \cdot \frac{1}{s+2}$$

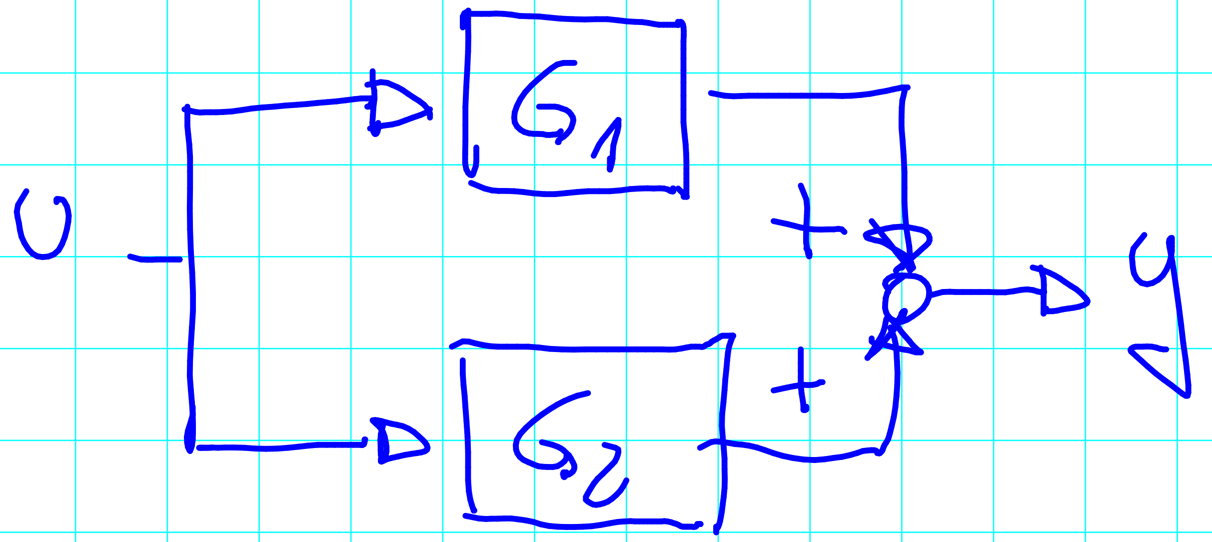
Somma di ~~prodotti~~ potenze di F. semplici con den. di 1° grado

Anticipazione



$$Q = G_1 \cdot U$$

$$Y = G_2 Q = G_2 G_1 U$$



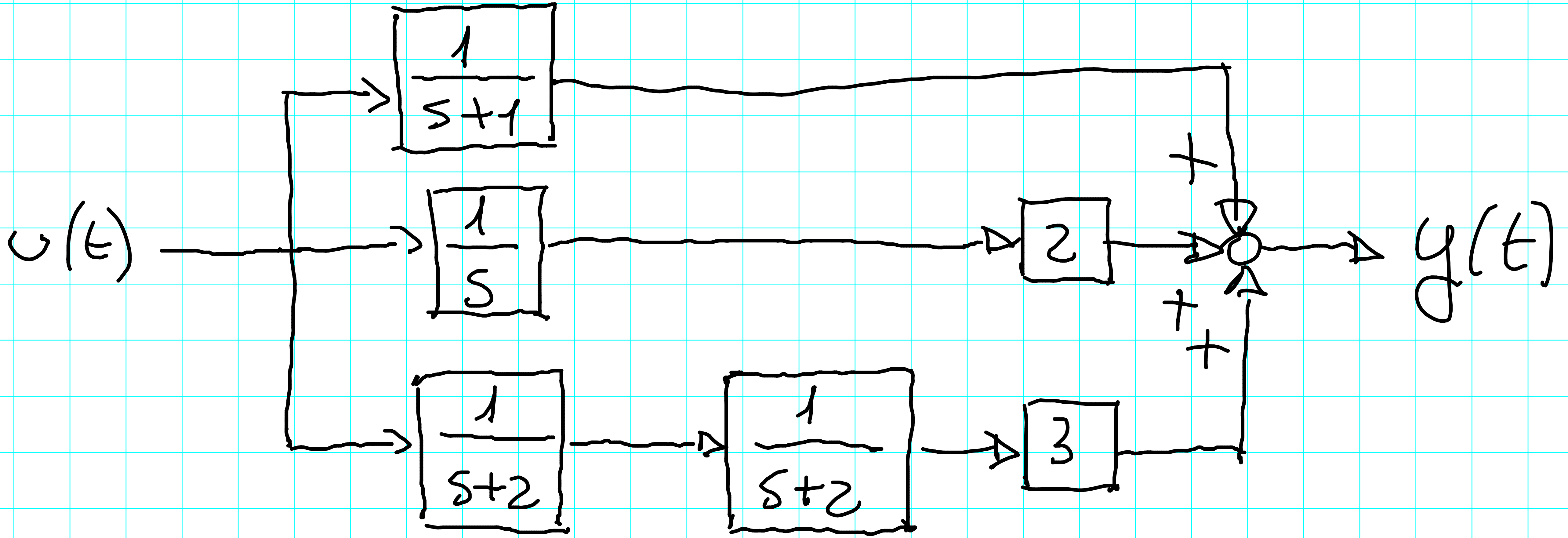
$$Y = G_1 U + G_2 U$$

$$= (G_1 + G_2) U$$

$y(t) = G(s)\, u(t)$ scrittura "operazionale"

Leggasi $\mathcal{L}[y(t)] = G(s)\, \mathcal{L}[u(t)]$

$$G(s) = \frac{1}{s+1} + \frac{2}{s} + \frac{3}{s+2} \cdot \frac{1}{s+2} = \frac{Y(s)}{U(s)}$$



Intermezzo:

$$\begin{cases} \dot{x} = a x + b u \\ y = c x \end{cases}$$

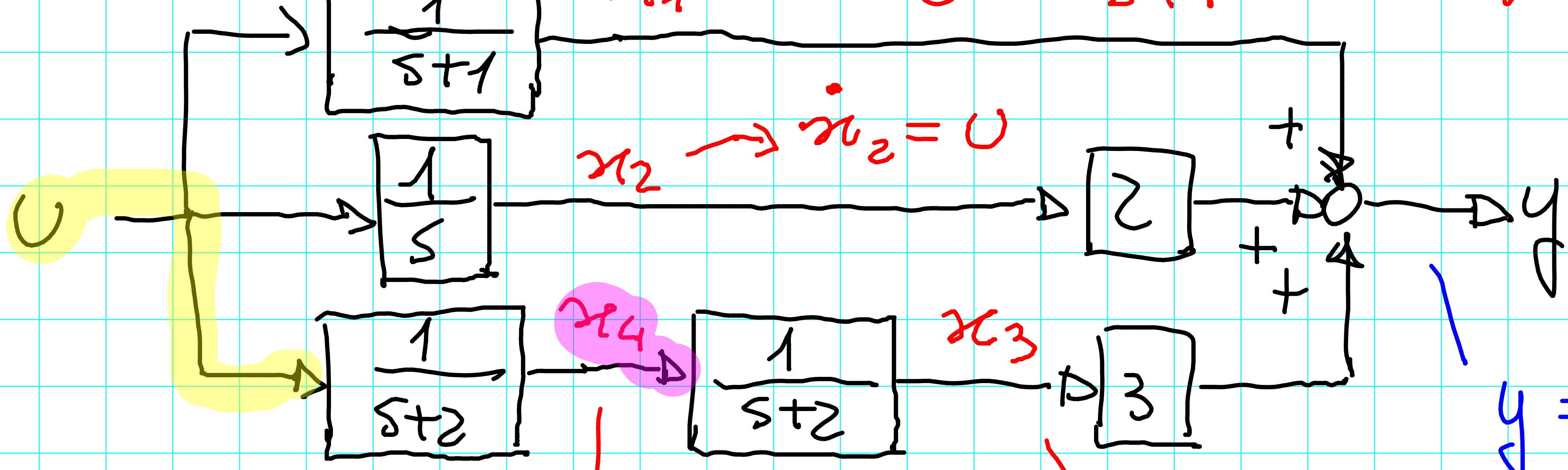
ordine 1, (abc) scalari
s. propria

$$G(s) = c(s-a)^{-1} b = \frac{bc}{s-a}$$

Quindi

$$G(s) = \frac{r}{s-p} \longrightarrow \begin{cases} \dot{x} = p x + r u \\ y = x \end{cases}$$

Applico questo concetto al sistema in questione



$x_1 \rightarrow \frac{X_1}{U} = \frac{1}{s+1} \Rightarrow \dot{x}_1 = -x_1 + u$

$x_2 \rightarrow \dot{x}_2 = u$

$y = x_1 + 2x_2 + 3x_3$

$\dot{x}_4 = -2x_4 + u$

$\dot{x}_3 = -2x_3 + x_4$

Quindi

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 3 & 0 \end{bmatrix} x$$

autovalori

autovalori multipli $\Rightarrow$ blocco sulla diagonale

Forma di Jordan

$$A = \begin{bmatrix} \lambda_1 & & & & & & & & & \\ & \lambda_2 & & & & & & 0 & & \\ & & \ddots & & & & & & & \\ & & & \lambda_p & \bullet & 0 & & & & \\ & & & & \lambda_p & \bullet & & & & \\ & & & 0 & & \lambda_p & & & & \\ & 0 & & & & & \ddots & & & \\ & & & & & & & \lambda_q & \bullet & 0 \\ & & & & & & & & \lambda_q & \bullet \\ & & & & & & & 0 & & \lambda_q \ \bullet \\ & & & & & & & & & \lambda_q \end{bmatrix}$$

BLOCCO DI JORDAN

• 1 oppure 0

MINIBLOCCO DI JORDAN

generico blocco

$$\begin{bmatrix} \lambda_h & 1 & & & & & \\ & \ddots & 1 & & & & \\ & & \ddots & 1 & & & \\ & & & \lambda_h & 0 & & \\ & 0 & & & \ddots & 1 & \\ & & & & & \ddots & 0 \ \ 1 \\ & & & & & & \lambda_h \end{bmatrix}$$

Il + piccolo mini blocco di Jordan (di dim. >1)
possibile è

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

studiamolo

Studiamo cioè $e^{\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}t}$

1 miniblocco di dim. 2 o 2 di dim. 1

$$\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix} = \underbrace{\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}}_{M} + \underbrace{\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}}_{N}$$

dipende solo da 1 o 0 in pos. (1,2) nel BLOCCO di J

$$e^{\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}t} = e^{(M+N)t} = e^{Mt}e^{Nt} = \begin{bmatrix}e^{\lambda t} & 0\\ 0 & e^{\lambda t}\end{bmatrix} e^{Nt}$$

M ed N commutano ($MN = NM$)

Osservo che N è nilpotente : $N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad e^{Nt} = I + Nt \quad \text{E BASTA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Allora

$$e^{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = Q$$

In questo caso
$Re(\lambda) < 0 \iff Q \to \underline{0}$ per $t \to \infty$
$Re(\lambda) > 0 \Rightarrow Q$ diverge " " " " in modo EXP
$Re(\lambda) = 0 \Rightarrow Q$ diverge " " " " " " LIN

miniblocco di dim 2

Quindi detti λ_i gli autovalori di A

- $Re(\lambda_i) < 0 \ \forall i \iff$ sistema AS
- $\exists i \mid Re(\lambda_i) > 0 \implies$ sistema I
- $Re(\lambda_i) \leq 0 \ \forall i$
  $\exists i \mid Re(\lambda_i) = 0$, $\implies$ sistema S
  ma in tal caso il + grande miniblocco di Jordan ha dim. 1
- Altrimenti $\implies$ sistema I

$\square$