Fuzzy Sets
Introduction

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A bit of history

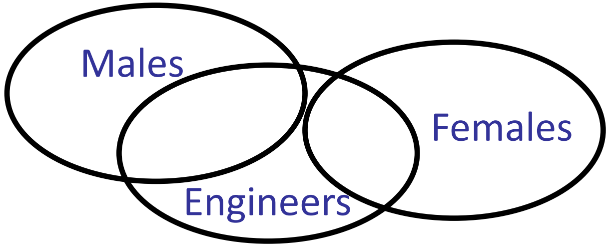
- Fuzzy sets have been defined by Lotfi Zadeh in 1965, as a tool to model approximate concepts
- In 1972 the first “linguistic” fuzzy controller has been implemented
- In the Eighties of last century boom of fuzzy controllers first in Japan, then in USA and in Europe
- In the Nineties applications in many fields: fuzzy data bases, fuzzy decision making, fuzzy clustering, fuzzy learning classifier systems, neuro-fuzzy systems…
  Massive diffusion of fuzzy controllers in end-user goods
- Now, fuzzy systems are the kernel of many “intelligent” devices

What is a fuzzy set?

A «crisp» set is defined by a boolean membership function on some property of the considered elements

μ_{set}: U $\in$ {true, false}

$$\mu_{Engineers}(X) = \begin{cases} true, & if\ degree(X) = Eng \\ false, & otherwise \end{cases}$$

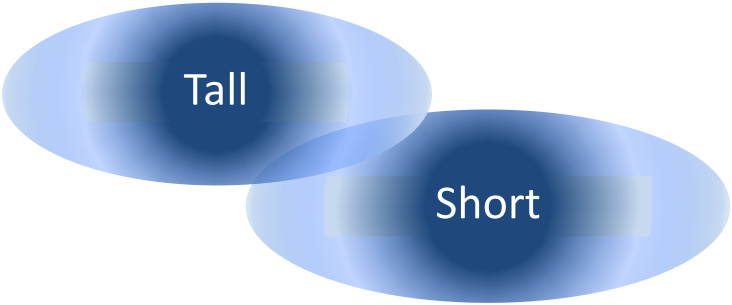


Crisp sets

A «fuzzy set» is a set whose membership function ranges on the interval [0,1].

μ_{set}: U $\in$ [0..1]

$$\mu_{Tall}(X) = f\big(Height(X)\big)$$



Fuzzy sets

Fuzzy membership functions

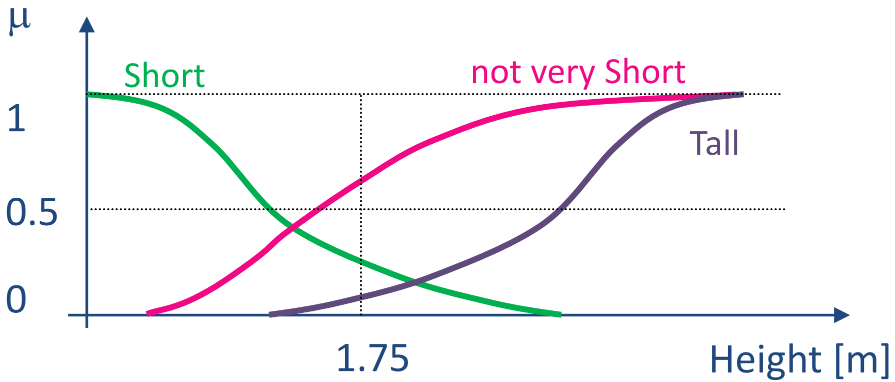
A membership function **defines** a set, by defining the **degree of membership** of an element of the universe of discourse to the set.

A **name** is given to the set to make it possible to refer to it: this is usually called a **label**

So a label **identifies** a set, a membership function (MF) **defines** it.

A person 1.75m high belongs to:

- Short with membership 0.3
- Tall with membership 0.2
- not very Short with membership 0.6



How is it possible to define a membership function?

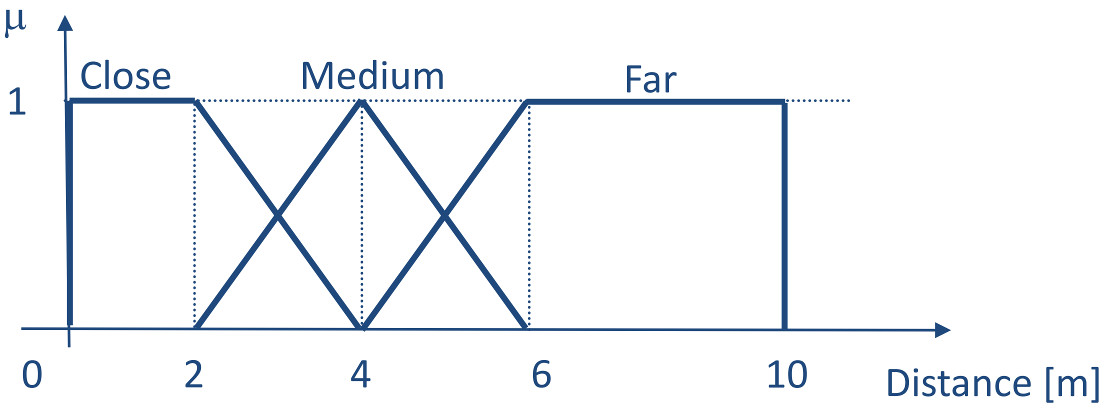
According to the **purpose** of the model, and on the **available data**:

1. **Select a variable** on which the MF to define the fuzzy sets will be defined
2. Define the **range** of the variable
3. Identify the fuzzy sets needed for application and define the **labels**
4. For each fuzzy set identify **characteristic points** for the MF
5. Define the **shapes** of the MF
6. **Check**

Let's try to define some MFs

We would like to model the distance between a soccer robot and the ball

1. First of all, the variable… → Distance
2. Range of the variable → [0..10]
3. Labels that make sense → Close, Medium, Far
4. Characteristic points → 0, max, middle values, where MF=1, …
5. Function shape → Linear



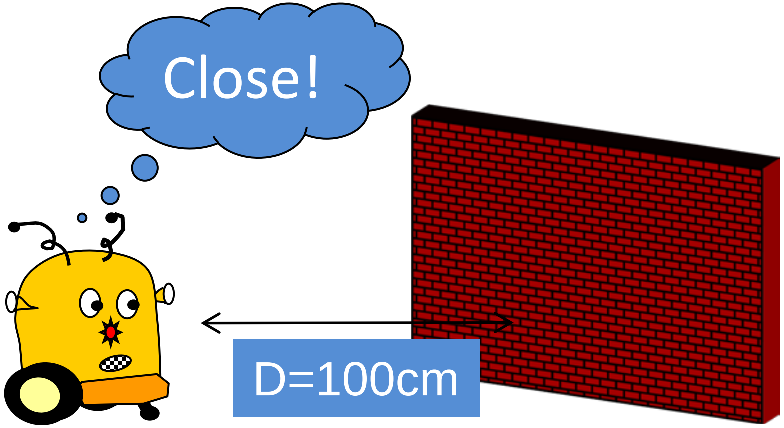
MFs and concepts

MFs **define** fuzzy sets

Labels **denote** fuzzy sets

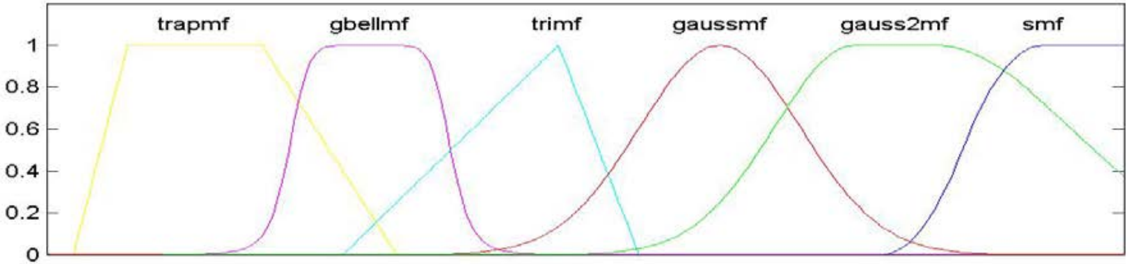
Fuzzy sets can be considered as **representations of concepts**

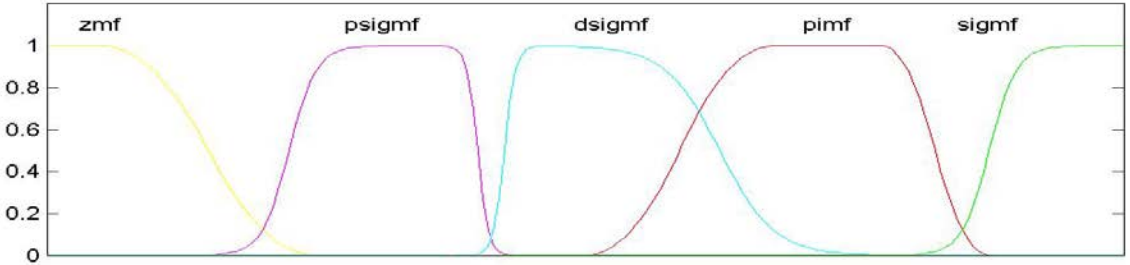
Symbol grounding: reason in terms of concepts and ground them on objective reality



Different MF shapes

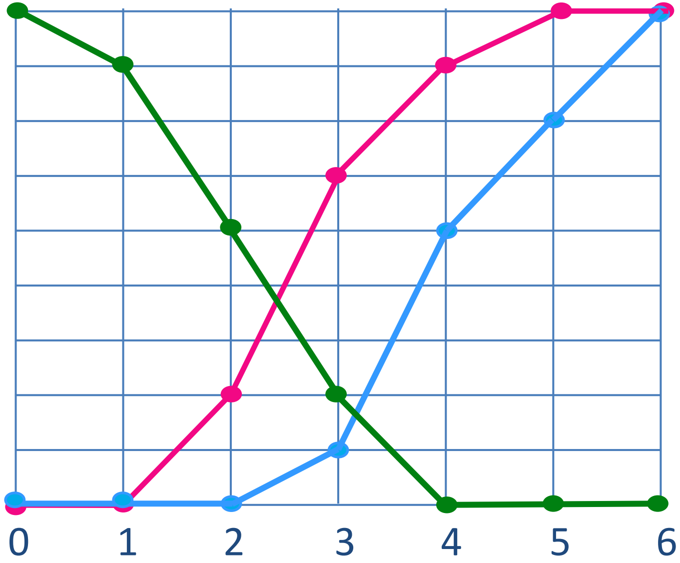
The shapes of the MF are arbitrary. Here are some common shapes (from Matlab)





Fuzzy sets on ordinal variables

It is possible to define fuzzy sets also for variables with **discrete** values



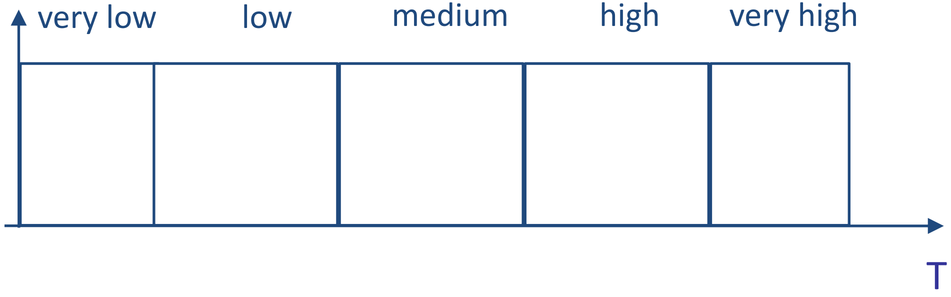
0 - no education
1 - elementary school
2 - high school
3 - two years college
4 - bachelor's degree
5 - master's degree
6 - doctoral degree

poorly educated
highly educated
very highly educated

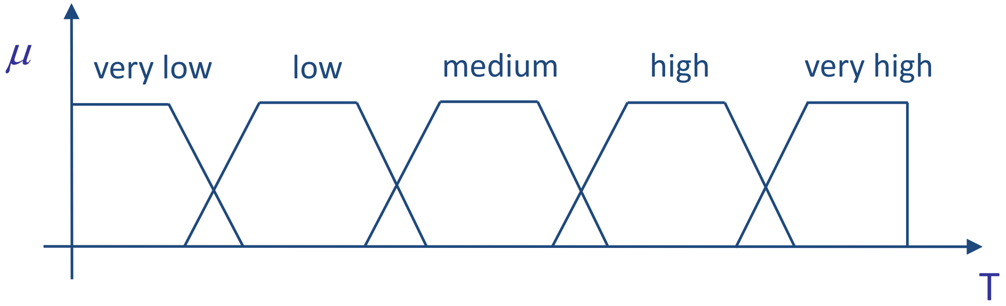
Fuzzy sets and intervals

Why using fuzzy sets instead than intervals?

Intervals are equivalent to rectangular MFs

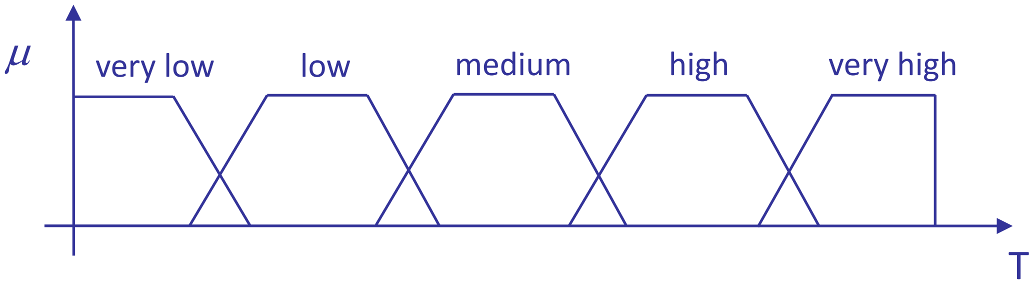


Fuzzy sets enable a smoother transition when labeling a value



Frame of cognition

A set of fuzzy sets fully covering the universe of discourse, i.e., the range of a variable, is called **frame of cognition**



Properties of a frame of cognition

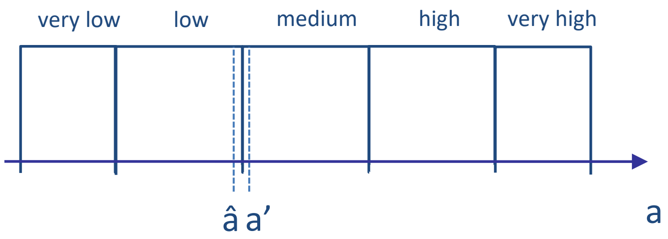
- **Coverage:** each element of the universe of discourse is assigned to at least one granule with membership > 0
- **Unimodality** of fuzzy sets: there is a unique set of values for each granule with maximum membership

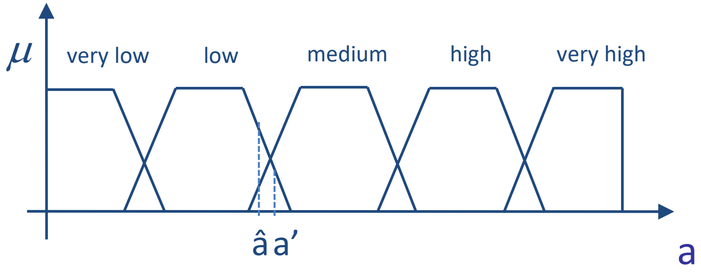
Fuzzy partition

A frame of cognition for which the sum of the membership values of each value of the base variable is equal to 1 is called a **fuzzy partition**

Let's consider a punctual error as the sum of the errors of interpretation by fuzzy sets due to imprecise measurements, noise, … : $e(\hat{a}) = |\mu_1(\hat{a}) - \mu_1(a')| + \ldots + |\mu_n(\hat{a}) - \mu_n(a')|$

and the integral error, as the integral of e(a) over the range of the base variable a: $e_i = \int e(a)\, da$



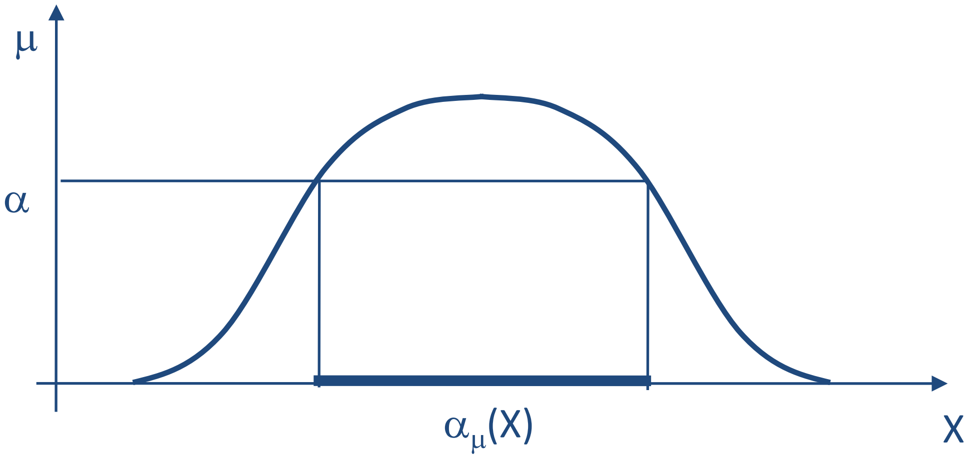


It can be demonstrated that the integral error of a fuzzy partition is smaller than that of a boolean partition, and that it is minimum w.r.t. any other frame of cognition.

α-cut

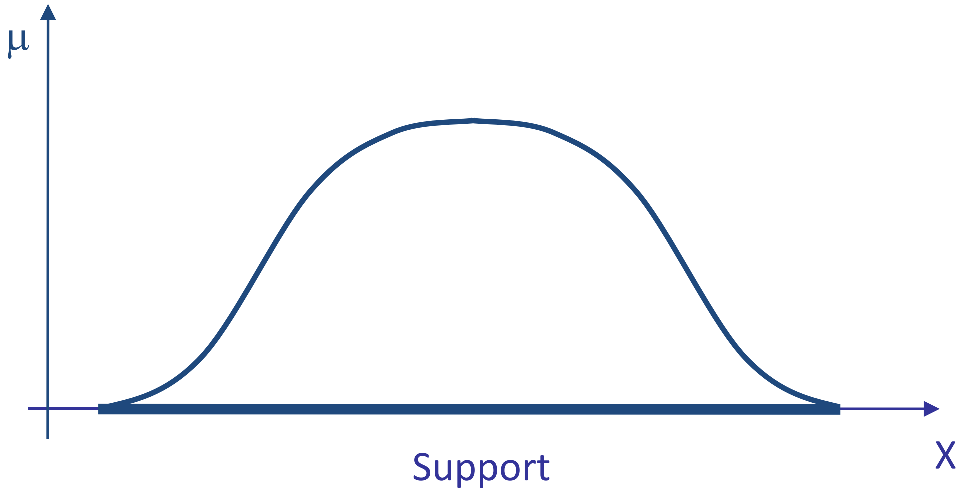
The α-cut of a fuzzy set is the crisp set of the values of x such that $\mu(x) \geq \alpha$

$$\alpha_\mu(X) = \{x \mid \mu(x) \geq \alpha\}$$



Support of a fuzzy set

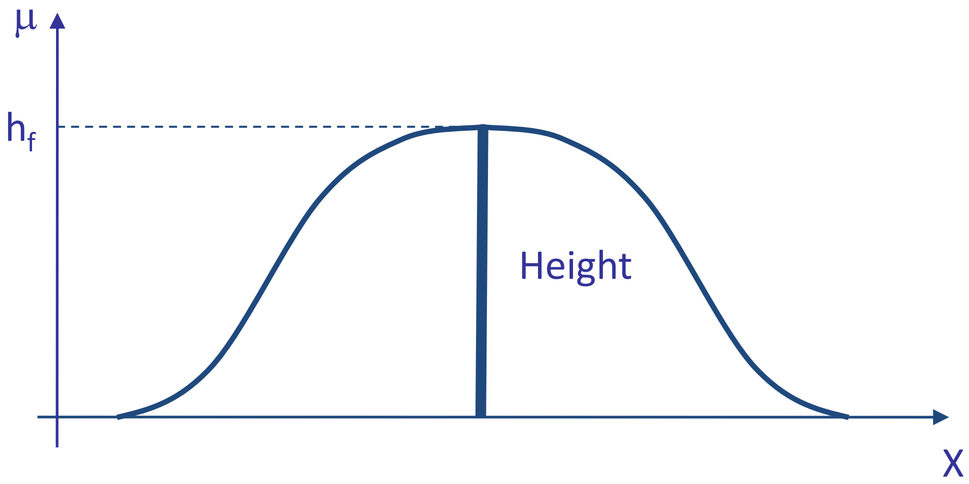
The crisp set of values x of X such that $\mu_f(x) > 0$ is the support of the fuzzy set f on the universe X



Height of a fuzzy set

The **height** h_f of a fuzzy set f on the universe X is the highest membership degree of an element of X to the fuzzy set

$h_f(X) = max_{x \in X}\, \mu_f(x)$



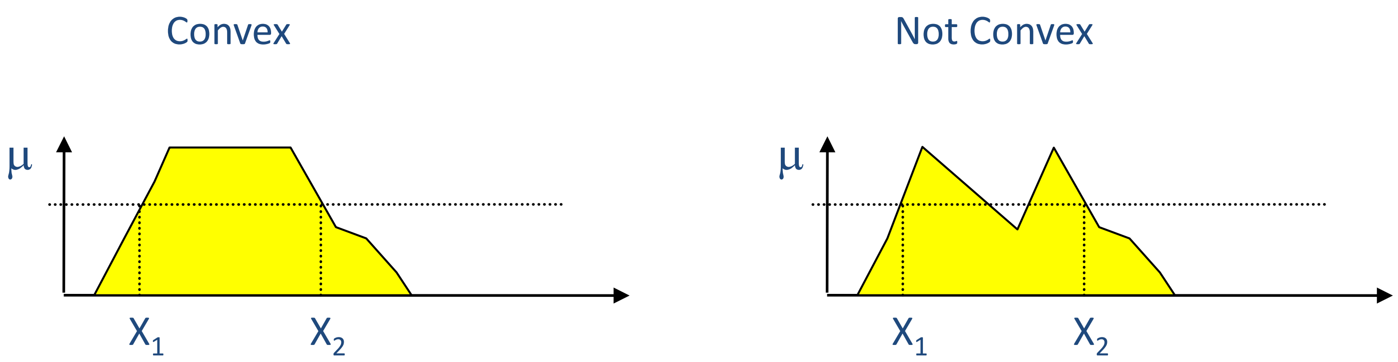
A fuzzy set f is **normal** iff $h_f(X)=1$

Convex fuzzy sets

A fuzzy set is *convex* iff

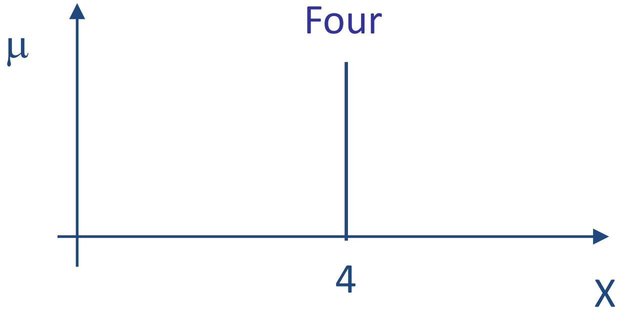
$$\mu(\lambda x_1 + (1-\lambda)\, x_2) \geq \min\,[\mu(x_1),\, \mu(x_2)]$$

for any x_1, x_2 in $\Re$ and any λ belonging to [0,1]

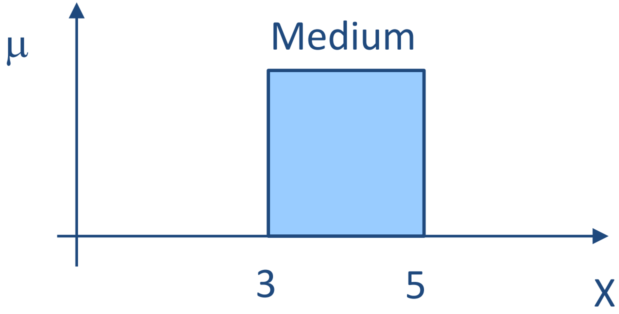


Particular MFs

Singleton: a fuzzy set with one member



Interval: a fuzzy set whose members have all membership = 1



Standard operators on Fuzzy Sets

Complement

$$\mu_{\bar{f}}(x) = 1 - \mu_f(x)$$

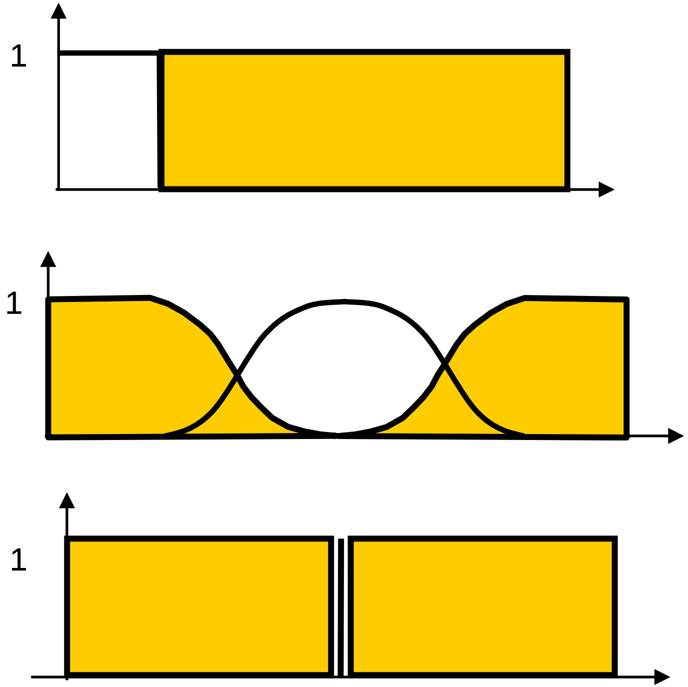
Union

$$\mu_{f_1 \cup f_2}(x) = \max\left[\mu_{f_1}(x), \mu_{f_2}(x)\right]$$

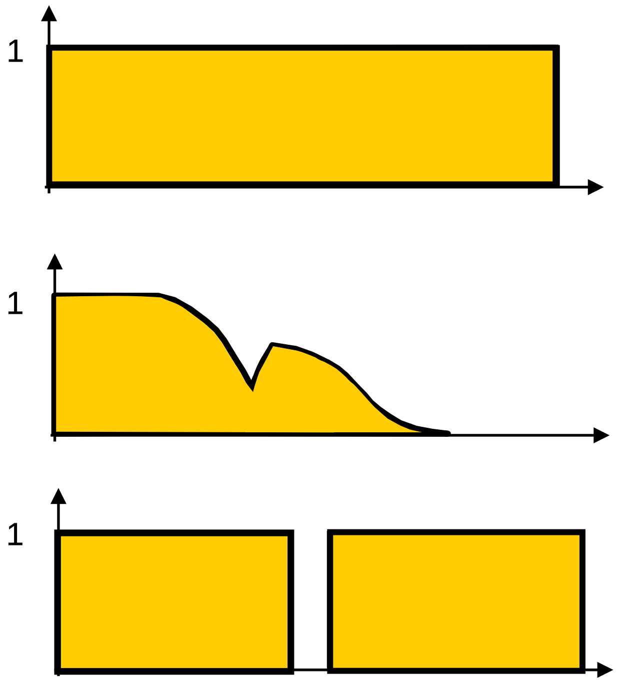
Intersection

$$\mu_{f_1 \cap f_2}(x) = \min\left[\mu_{f_1}(x), \mu_{f_2}(x)\right]$$

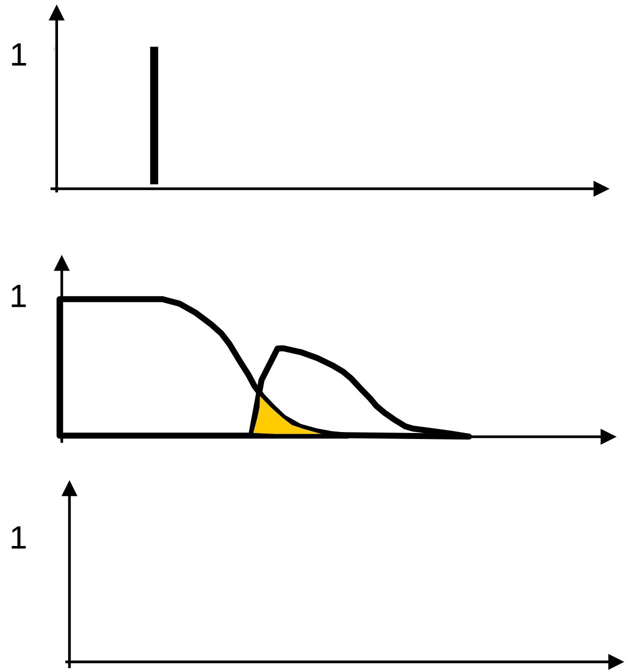
Examples of operator application: complement



Examples of operator application: union (as max)



Examples of operator application: intersection (as min)



Complement

$$c : [0,1] \rightarrow [0,1]$$

$$c(\mu_A(x)) = \mu_{\neg A}(x)$$

Axioms:

1. c(0)=1; c(1)=0 (*boundary conditions*)
2. For all pair of values of x, a and b, in [0,1], if a < b then c(a) > c(b) (*monotonicity*)
3. c is a *continuous function*
4. c is *involutive*, i.e., c(c(a))=a for all a in [0,1]

Intersection and T-norms

$$\mu_{A \cap B}(x) = i[\mu_A(x), \mu_B(x)]$$

Axioms:

1. i[a, 1]=a (*boundary conditions*)

2. $d \geq b$ implies $i(a,d) \geq i(a,b)$ (*monotonicity*)

3. i(b,a) = i(a,b) (*commutativity*)

4. i(i(a,b),d) = i(a,i(b,d)) (*associativity*)

5. i is continuous

6. $a \geq i(a,a)$ (*sub-idempotency*)

7. $a_1 < a_2$ and $b_1 < b_2$ implies that $i(a_1,b_1) < i(a_2,b_2)$ (*strict monotonicity*)

T-norms: examples

Given the axioms, the intersection operator i is defined as a T-norm

A parametric formulation of a class of T-norms:

$$T_\alpha(a,b) = \frac{ab}{max[a,b,\alpha]}$$

for α=1 we have *ab*

for α=0 we have *min(a, b)*

Other T-norms:

$$T_1(a,b) = max(0, a+b-1)$$

$$T_{2.5}(a,b) = \frac{ab}{a+b-ab}$$

Union and S-norms (or T-conorms)

$$\mu_{A \cup B}(x) = u[\mu_A(x), \mu_B(x)]$$

Axioms:

1. u[a, 0]=a (*boundary conditions*)

2. $b \leq d$ implies $u(a,b) \leq u(a,d)$ (*monotonicity*)

3. u(a,b) = u(b,a) (*commutativity*)

4. u(a,u(b,d)) = u(u(a,b),d) (*associativity*)

5. u is continuous

6. $u(a,a) \geq a$ (*super-idempotency*)

7. $a_1 < a_2$ e $b_1 < b_2$ implies that $u(a_1,b_1) < u(a_2,b_2)$ (strict *monotonicity*)

S-norms: examples

The most common ones:

$$S_3(a,b) = max(a,b)$$

$$S_+(a,b) = a + b - ab$$

Other S-norms:

$$S(a,b) = min(1, a^p, b^p)^{1/p} \qquad p \geq 1$$

$$S_1(a,b) = min(1, a + b)$$

Aggregation

Aggregation is the operator that aggregates the values of membership for the same fuzzy set, coming from different knowledge sources. It is used in Fuzzy Rule Systems.

$$\mu_A(x) = h[\mu_{A1}(x), \ldots, \mu_{An}(x)]$$

Axioms

1. h[0,..., 0]=0, h[1,..., 1]=1 (*boundary conditions*)
2. *monotonicity*
3. h is continuous
4. h(a,..,a) = a (*idempotency*)
5. *simmetricity*

Property of aggregation and examples

$$\min(a_1, \ldots, a_n) \le h(a_1, \ldots, a_n) \le \max(a_1, \ldots, a_n)$$

Common aggregation operators:

$h(a_1, \ldots, a_n) = \max(a_1, \ldots, a_n)$

$h(a_1, \ldots, a_n) = \Sigma\, a_i - I(a_1, \ldots, a_n)$ Probabilistic sum $h(a_1, a_2) = a_1 + a_2 - (a_1 * a_2)$

Another aggregation operator: generalized average

$$h(a_1, \ldots, a_n) = (a_1^{\alpha} + \ldots + a_n^{\alpha})^{1/\alpha} / n$$

What to remember from these slides?

- Definition of fuzzy set
- Definition of membership function, support, height, α-cut, convex fuzzy set
- Main operators

https://kahoot.com