FOUNDATIONS OF OPERATIONS RESEARCH

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CHAPTER 1: INTRODUCTION

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Branch of applied mathematics in which **mathematical models** and **quantitative methods** (e.g., optimization, game theory, simulation) are used to analyze **complex decision-making problems** and find (near-)optimal solutions.

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Branch of applied mathematics in which **mathematical models** and **quantitative methods** (e.g., optimization, game theory, simulation) are used to analyze **complex decision-making problems** and find (near-)optimal solutions.

Overall goal: to help make better decisions

Interdisciplinary field at the interface of applied mathematics, computer science, economics and industrial engineering.

1.1 Decision-making problems

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1) **Assignment problem**:

   Given m jobs and m machines, suppose that each job can be executed by any machine and that t_{ij} is the execution time of job J_i on machine M_j.

	M_1	M_2	M_3
J_1	2	6	3
J_2	8	4	9
J_3	5	7	8

t_{ij} matrix $(m = 3)$

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Each job must be assigned to exactly one machine, and each machine to exactly one job.

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Number of feasible solutions? $m!$ possible assignments (permutations)

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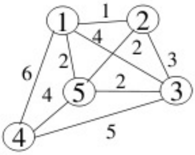

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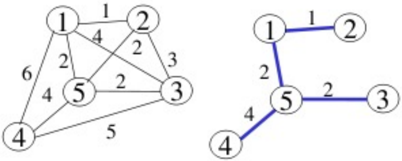

Number of alternative solutions:

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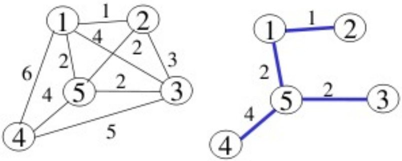
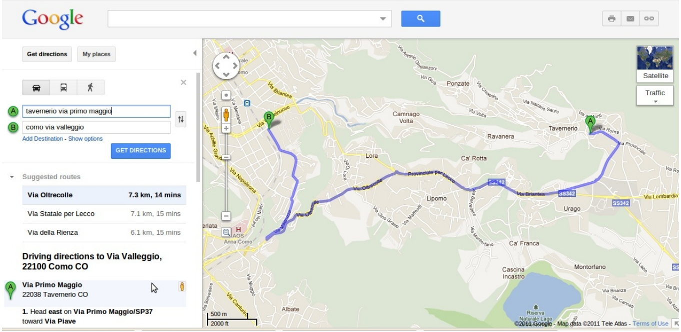


Number of alternative solutions: $\leq 2^m$ where $m = |E|$

3) **Shortest paths**:

Given a graph that represents a road network with distances (traveling times) for each arc, determine the shortest (fastest) path between two points (nodes).



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Determine the week schedule for the hospital personnel so as to minimize the number of people involved (physicians, nurses,...) while meeting the daily requirements.

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Complex decision-making problems that are tackled via a **mathematical modelling approach** (mathematical models, algorithms and computer implementations).

1.2 Scheme of an O.R. study

$$\boxed{\text{Problem}} \rightarrow \boxed{\text{Model}} \rightarrow \boxed{\text{Algorithm}} \rightarrow \boxed{\text{Implementation}}$$

Main steps:

- define the problem
- build the model
- select or develop an appropriate algorithm
- implement or use and efficient computer program
- analyze the results

with feedbacks (the previous steps are reconsidered whenever useful).

A mathematical **model** is a simplified representation of a real-world problem.

To define a model we need to identify the fundamental elements of the problem and the main relationships among them.

1.3 Historical sketch – part I

Origin in World War II: teams of scientists were asked to do **research** on the most efficient way to conduct the **operations**, e.g., to optimize the allocation of the scarse resources.

Examples: radar location, manage convoy operations, logistics,...

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In the decades after the war, the techniques began to be applied more widely to problems in business, industry and society.

During the industrial boom, the substantial increase in the size of the companies and organizations gave rise to more complex decision-making problems.

Favourable circumstances:

- Fast progress in Operations Research and Numerical Analysis methodologies
- Advent and diffusion of computers (available computing power and widespread software).

Operations Research and Management Science are often used as synonyms

1.4 Examples of decision-making problems and models

Simplified versions of three problems:

- production planning
- portfolio selection
- facility location

arising in three important fields of application.

Example 1: Production planning

A company produces three types of electronic devices.

Main phases of the production process: assembly, refinement and quality control

Time in minutes required for each phase and product:

	D_1	D_2	D_3
Assembly	80	70	120
Refinement	70	90	20
Quality control	40	30	20

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Available resources whithin the planning horizion (depend on workforce) in minutes:

Assembly	Refinement	Quality control
30000	25000	18000

Unitary profit expressed in KEuro:

D_1	D_2	D_3
1.6	1	2

Assumption: the company can sell whathever it produces.

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Assumption: the company can sell whathever it produces.

Give a mathematical model for determining a production "plan" which maximizes the total profit.

Production planning model

Decision variables:

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$$\max\ z = 1.6x_1 + 1x_2 + 2x_3$$

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$$80x_1 + 70x_2 + 120x_3 \leq 30000 \qquad \text{(assembly)}$$

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$$\begin{array}{rcll} 80x_1 + 70x_2 + 120x_3 & \leq & 30000 & \text{(assembly)} \\ 70x_1 + 90x_2 + 20x_3 & \leq & 25000 & \text{(refinement)} \end{array}$$

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Non-negative variables:

$$x_1, x_2, x_3 \geq 0 \qquad \text{may take fractional (real) values}$$

Example 2: Portfolio selection problem

An insurance company must decide which investments to select out of a given set of possible assets (stocks, bonds, options, gold certificates, real estate,...).

Investments	area	capital (c_j in KEuro)	expected return (r_j)
A (automotive)	Germany	150	11%
B (automotive)	Italy	150	9%
C (ICT)	U.S.A.	60	13%
D (ICT)	Italy	100	10%
E (real estate)	Italy	125	8%
F (real estate)	France	100	7%
G (short term treasury bonds)	Italy	50	3%
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Available capital = 600 KEuro

At most 5 investments to avoid excessive fragmentation.

Geographic diversification to limit risk: ≤ 3 investments in Italy and ≤ 3 abroad.

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Give a mathematical model for deciding which investments to select so as to maximize the expected return while satisfying the constraints.

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Binary (integer) variables: $x_j \in \{0, 1\} \quad \forall j,\ 1 \leq j \leq 8$

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\frac{x_3 + x_4}{2} & \leq & x_7 + x_8 & \text{(investment in treasury bonds)}
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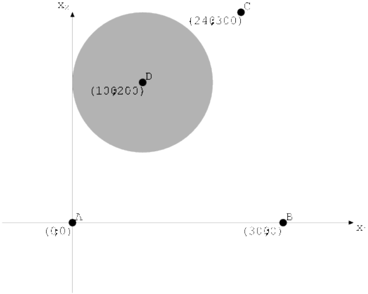
Example 3: Facility location

Consider three oil pits, located in positions A, B and C, from which oil is extracted.



Connect them to a refinery with pipelines whose cost is proportional to the square of their length.

The refinery must be at least 100 km away from point $D = (100, 200)$, but the oil pipelines can cross the corresponding forbidden zone.

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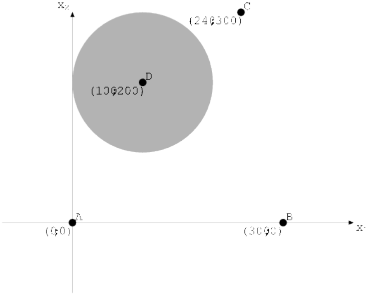

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Give a mathematical model to decide where to locate the refinery so as to minimize the total pipeline cost.

Facility location model

Decision variables:

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x_1, x_2 cartesian coordinates of the refinery

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Objective function:

$$\min\ z = \left[(x_1 - 0)^2 + (x_2 - 0)^2\right] + \left[(x_1 - 300)^2 + (x_2 - 0)^2\right] + \left[(x_1 - 240)^2 + (x_2 - 300)^2\right]$$

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Constraints:

$$\sqrt{(x_1-100)^2+(x_2-200)^2} \geq 100$$

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Constraints:

$$\sqrt{(x_1-100)^2+(x_2-200)^2} \geq 100$$

Variables:

$$x_1, x_2 \in \mathbb{R}$$

1.5 Main features of decision-making problems

- number of decision makers (who decides?)
- number of objectives (based on which criteria?)
- level of uncertainty in the parameters (based on which information?)

You may think of each one of these three features as a dimension of an abstract coordinate system...

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One decision maker, one objective $\Rightarrow$ Mathematical programming

One decision maker, several objectives $\Rightarrow$ Multi-objective programming

Uncertainty level $> 0 \Rightarrow$ Stochastic programming

Several decision makers $\Rightarrow$ Game theory

1.6 Mathematical Programming/Optimization (MP)

Decision-making problems with a **single decision maker**, a **single objective** and reliable parameter estimates

$$\mathrm{opt}\ f(\mathbf{x}) \quad \text{with} \quad \mathbf{x} \in X \quad \text{and} \quad \mathrm{opt} = \left\{ \begin{array}{c} \text{min} \\ \text{max} \end{array} \right\}$$

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- **decision variables** $\mathbf{x} \in \mathbb{R}^n$: numerical variables whose values identify a solution of the problem

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$$\text{opt } f(\mathbf{x}) \quad \text{with} \quad \mathbf{x} \in X \quad \text{and} \quad \text{opt} = \left\{ \begin{array}{c} \min \\ \max \end{array} \right\}$$

- **decision variables** $\mathbf{x} \in \mathbb{R}^n$: numerical variables whose values identify a solution of the problem
- **feasible region** $X \subseteq \mathbb{R}^n$ distinguishes between feasible and infeasible solutions (via constraints)

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \left\{ \begin{array}{c} = \\ \leq \\ \geq \end{array} \right\} 0, i = 1, \ldots, m \right\}$$

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Observation: $\max\{f(\mathbf{x}) : \mathbf{x} \in X\} = -\min\{-f(\mathbf{x}) : \mathbf{x} \in X\}$

Global optima

Solving a mathematical programming (MP) problem consists in finding a feasible solution which is **globally optimum**, i.e., a vector $\mathbf{x}^* \in X$ such that

$$
\begin{aligned}
f(\mathbf{x}^*) \leq f(\mathbf{x}) \qquad & \forall \mathbf{x} \in X \qquad \text{if opt} = \min \\
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3. the problem has a single optimal solution,
4. the problem has a large number (even an infinite number) of optimal solutions (with the same optimal value!).

Local optima

When the problem at hand is very hard we must settle for a feasible solution that is a **local optimum**, i.e., a vector $\hat{\mathbf{x}} \in X$ such that

$$f(\hat{\mathbf{x}}) \leq f(\mathbf{x}) \qquad \forall \mathbf{x} \text{ with } \mathbf{x} \in X \text{ and } ||\mathbf{x} - \hat{\mathbf{x}}|| \leq \epsilon \qquad \text{if opt} = \min$$
$$f(\hat{\mathbf{x}}) \geq f(\mathbf{x}) \qquad \forall \mathbf{x} \text{ with } \mathbf{x} \in X \text{ and } ||\mathbf{x} - \hat{\mathbf{x}}|| \leq \epsilon \qquad \text{if opt} = \max$$

for an appropriate value $\epsilon > 0$.

An optimization problem can have many local optima.

Special cases of Mathematical Programming

- **Linear Programming** (LP)

  $f(\mathbf{x})$ linear

  $$X = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \left\{ \begin{array}{c} = \\ \leq \\ \geq \end{array} \right\} 0, i = 1, \ldots, m \right\} \text{ with } g_i(\mathbf{x}) \text{ linear for each } i$$

  Example: Production planning

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  Example: Production planning

- **Integer Linear Programming** (ILP)

  $f(\mathbf{x})$ linear

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  Examples: Portfolio selection

ILP coincides with LP with the additional integrality constraint on the variables

- **Nonlinear Programming** (NLP)

  $f(\mathbf{x})$ convex/regular or non convex/regular

  $$X = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \left\{ \begin{array}{c} = \\ \leq \\ \geq \end{array} \right\} 0, i = 1, \dots, m \right\}$$ with for each i $g_i(\mathbf{x})$

  convex/regular or non convex/regular

  Example: Facility location (with g_i convex)

Historical sketch: Mathematical Programming

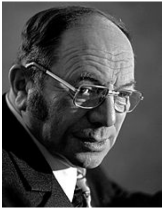
1826/1827: **Joseph Fourier** presents a method to solve systems of linear inequalities (Fourier-Motzkin) and discusses some LPs with 2-3 variables.

1939: **Leonid Kantorovitch** lays the bases of Linear Programming (Nobel prize 1975).

1947: **George Dantzig** proposes independently LP and invents the Simplex algorithm.

1958: **Ralph Gomory** proposes a cutting plane method for ILP problems.

Joseph Fourier (1786-1830)



L.V. Kantorovitch (1912-1986)

G.B. Dantzig (1914-2005)

R.E. Gomory (1929-)

1.6 Multi-objective programming

Multiple objectives can be taken into account in different ways

Suppose we wish to minimize $f_1(\mathbf{x})$ and maximize $f_2(\mathbf{x})$ (laptop example: f_1 is cost and f_2 is performance)

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Suppose we wish to minimize $f_1(\mathbf{x})$ and maximize $f_2(\mathbf{x})$ (laptop example: f_1 is cost and f_2 is performance)

1. Turn it into a **single objective problem** by expressing the two objectives in terms of the same unit (e.g., monetary unit)

$$\min \ \lambda_1 f_1(\mathbf{x}) - \lambda_2 f_2(\mathbf{x})$$

   for appropriate scalars λ_1 and λ_2.

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2. Optimize the **primary objective** function and turn the other objective into a constraint

$$\max_{\mathbf{x} \in \tilde{X}} \ f_2(\mathbf{x}) \qquad \text{where} \ \ \tilde{X} = \{\mathbf{x} \in X : f_1(\mathbf{x}) \leq c\}$$

   for an appropriate constant c.

1.7 Mathematical programming versus simulation

Both approaches involve constructing a **model** and designing an **algorithm**.

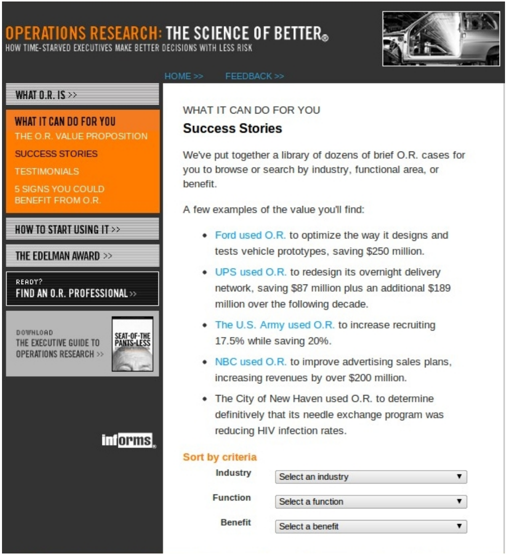
Mathematical programming	**Simulation**
Problem can be "well" formalized	Problem is difficult to formalize
Algorithm yields a(n optimal) solution	Algorithm simulates the evolution of the real system and allows to evaluate the performance of alternative solutions.
The results are "certain"	The results need to be interpreted
Example: assignment	Example: service counters

1.8 The impact of Operations Research

year	company	sector	results
90	Taco Bell (fast food)	personnel scheduling	7.6M\$ annual savings
92	American Arlines	design fare structure, overbooking and flights coordination	+ 500M\$
92	Harris Corp. (semiconductors)	production planning	50% $\Rightarrow$ 95% orders on time
95	GM - car rental	use of car park	+50M\$ per year avoided banckruptcy
96	HP - printers	modify production line	doubled production
97	Bosques Arauco	harvesting logistics and transport	5M\$ annual savings
99	IBM	supply chain re-engineering	750M\$ annual savings

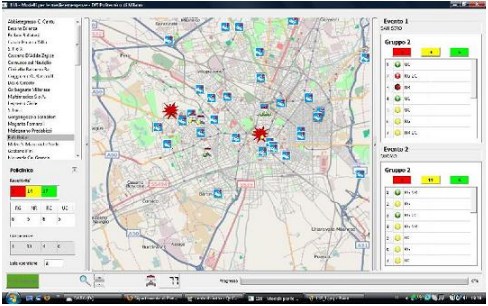
Most of the top managers interviewed by "Fortune 500" declared that they used O.R. methodologies.

Video: http://www.bnet.com/videos/operations-research-critical-applications-for-business/178846



http://www.scienceofbetter.org/

Not just for money. Example: a regional project with Milan Emergency Service 118



Significant impact not only for large companies and organizations.

In rapidly evolving contexts, characterized by strong competitivity, high levels of complexity and uncertainty, it is crucial to identify and implement efficient solutions.

The huge amount of data available with modern information systems (Big data) opens new avenues...