Characterization and Optimal Servo-Control of a Peltier Oven

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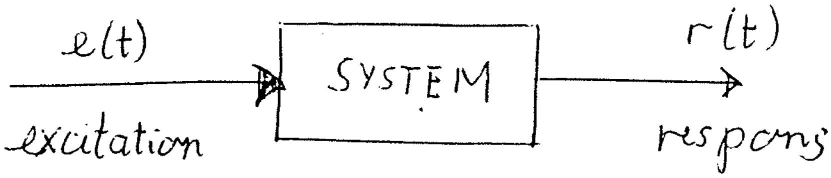
These pages were written as partial background literature for the laboratory course FYS 345. First one finds a general introduction to linear response theory. Then a discussion of closed-loop systems (feedback) is given with special emphasis on analogue PID-control. These topics can be found in many textbooks, but the material is usually spread over many pages. The purpose of this compendium is to collect the material relevant for this exercise, and present it on a minimum number of pages.

Some derivations are left to the students, and these are listed as problems, the solutions of which must be contained in the lab. report.

On the last two pages one finds a pointwise description of how the practical exercise is carried out.

Contents

1 Linear response theory



Above one sees schematically the type of configuration we shal discuss. A system experiences an excitation, $e(t)$, that varies with time. The effect of the excitation is that an observable quantity, $r(t)$, considered as the response, also will vary with time. To be more specific, we can say that $e(t)$ is the power of a heating element, and that $r(t)$ represents the temperature measured a place close to it.

Since real systems always have some inertia it is unrealistic to describe the relation between the response and excitation by

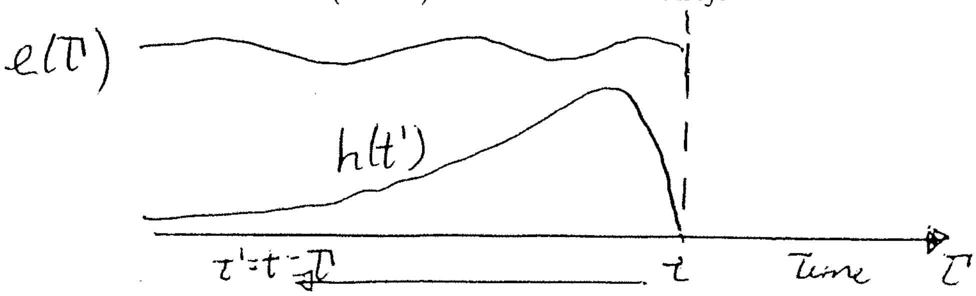
$$r(t) = h\ e(t)\ ,$$

i.e. instantaneous response proportional to the excitation.

It is more realistic to expect a generalized linear response of the form

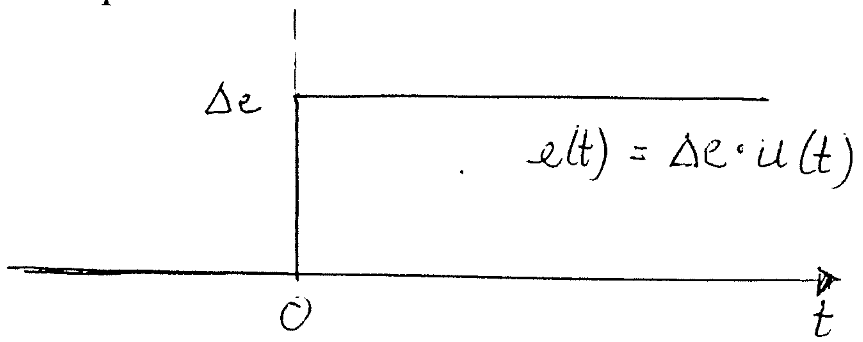
$$r(t) = \int_{-\infty}^{t} h(t-\tau)\ e(\tau)d\tau\ . \tag{1}$$

This expresses that the current state of the system, $r(t)$, depends linearly opun the strength of the excitation at all earlier point in time $\tau \leq t$. The function $h(t')$ is a generalized susceptibility, or a kind of memory function that determines the relative weight of the present as well as the previous excitation values. Note that $h(t' < 0) = 0$ due to causality.



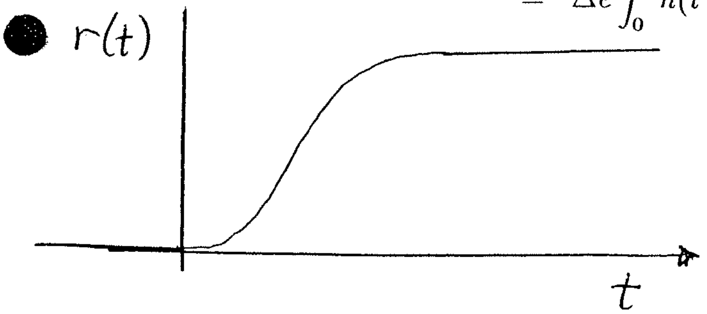
The $h-$function characterizes completely the response properties of a linear system. In very many cases this function can not be calculated from first-principles. (Exceptions are e.g. some electronic networks). How then can one imagine to find the $h-$function experimentally? Here follows two approaches.

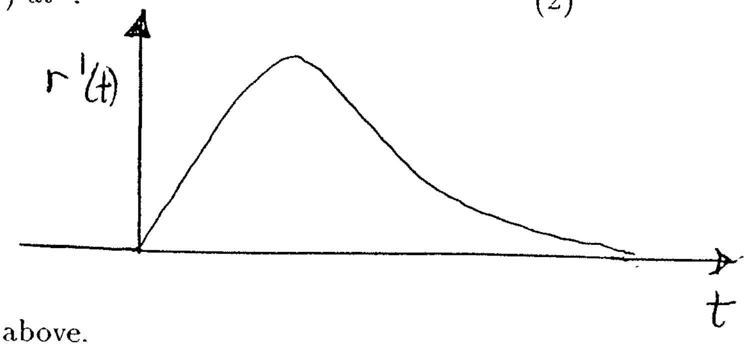
1.1 Step-excitation



We let the system experience a step-excitation as shown above. From eq.(1) the response then becomes

$$\begin{aligned} r(t) &= \int_0^t h(t-\tau)\,\Delta e\,d\tau \quad , \quad t' = t-\tau \\ &= \Delta e \int_0^t h(t')\,dt' \,. \end{aligned} \tag{2}$$





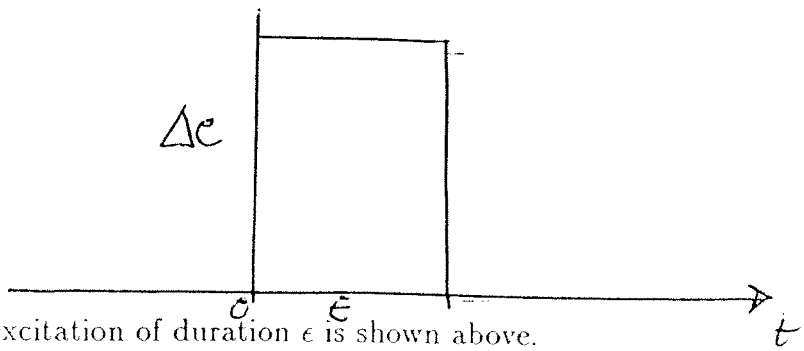
A typical step-response is shown above.

It follows from the expression for $r(t)$ that

$$r'(t) = \Delta e\; h(t) \,,$$

so that $h(t)$ can be found from the derivative of the $r(t)$ curve, as shown to the right.

1.2 Pulse-excitation

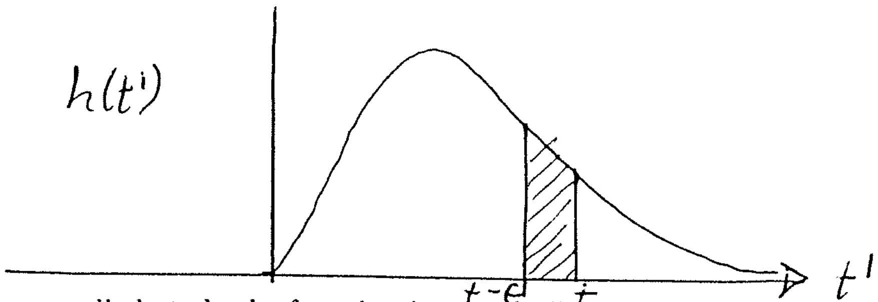


A pulse-excitation of duration ϵ is shown above.
The response for $0 \le t \le \epsilon$ is the same as for the step excitation (the system

does not yet know that the step will end).
For $t > \epsilon$ we get from eq.(1)

$$\begin{aligned} r(t) &= \int_0^\epsilon h(t-\tau)\,\Delta e\,d\tau \quad , \quad t' = t - \tau \\ &= \Delta e \int_{t-\epsilon}^{t} h(t')\,dt' \;. \end{aligned} \tag{3}$$

The response corresponds to the integral of the $h-$function over a small window of width ϵ, as seen in the figure above.



If ϵ is so small that the $h-$function is essentially constant over the window one has

$$r(t) \simeq \Delta e \;\; h(t)\;\epsilon \tag{4}$$

It follows that if ϵ is chosen sufficiently small($\ll$ all relaxation times of the system) vil the short-pulse response will show $h(t)$ directly.

A practical problem arises due to $r(t) \propto \epsilon$, and ideally $\epsilon \rightarrow 0$. One must therefore compensate by increasing the strength of the excitation Δe. (NB; but not so much that the linear approximation is violated.)

2 Some Laplace transforms

Since the generalized linear response eq.(1) is written as a convolution integral the mathematical treatment is most effectively done by using Laplace transformed quantities;

$$\mathcal{L}[f(t)] \equiv \int_0^\infty f(t)\;e^{-st}\;dt \equiv F(s)\;.$$

The transformed function, F, depends on a new variable $s = \sigma + i\omega$, a complex frequency. From the convolution theorem it follows that eq.(1) can be written

$$R(s) = H(s)\;E(s)\;,$$

where the functions R, H og E are the Laplace-tranformed of $r(t), h(t)$ and $e(t)$, respectively. The important point here is that the response is expressed simply by a product.

Here follows some basic properties of the $\mathcal{L}$-transform.

$\mathcal{L}$-transform of a derivative

$$\mathcal{L}[f'(t)] = \int_0^\infty \frac{df}{dt}\; e^{-st}\; dt = f\; e^{-st}|_0^\infty + s\int_0^\infty f\; e^{-st}\; dt = sF(s) - f(0)\ , \quad (5)$$

where partial integration was used at the 2. equality.

$\mathcal{L}$-transform of an integral

$$\mathcal{L}[\int_0^t f(\tau)\; d\tau] = \frac{F(s)}{s}\ ,$$

which also is found by partial integration.

Final value theorem

$$\mathcal{L}[f'(t)] = \int_0^\infty df\; e^{-st} \overset{s\to 0}{\to} \int_0^\infty df = f(\infty) - f(0)\ .$$

By comparison with eq.(5) one sees that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)\ .$$

Of specific transformation pairs we will limit ourselves to the following

$$\mathcal{L}[e^{pt}] = \int_0^\infty e^{pt}\; e^{-st}\; dt = \frac{1}{s-p} \quad , \quad \sigma > \Re p,$$

where p is an arbitrary complex number ($1/p$ is measured in seconds).

A special case is $p = 0$;

$$\mathcal{L}[1] = 1/s\ .$$

For a step excitation $e(t) = \Delta e\; u(t)$, where $u(t)$ is the unit step function it follows that the response can be written

$$R(s) = H(s)\; \frac{\Delta e}{s}\ .$$

The final value theorem then gives

$$\lim_{t\to\infty} r(t) = \lim_{s\to 0} sR(s) = \Delta e\; H(0)\ .$$

Thus, we see that $H(0)$ can be found by measuring the asymptotic value of the response for a known step-excitation.

$H(s)$ is called the transfer function (or system function), and gives as $h(t)$ a complete characteristics of the response properties of the system.

3 Models

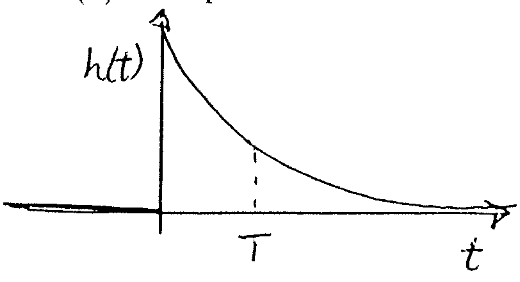
In many cases the systems are so complex that simplified models have to be applied. In that case the $H(s)$-function is given on a phenomenological basis, and parameters are fitted to the actual system. We will now take a look at two models that one often uses.

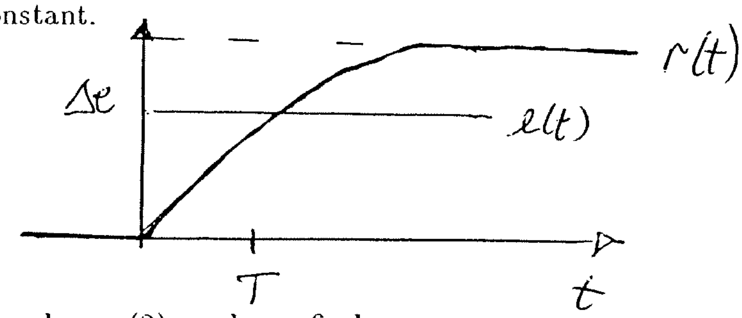
3.1 1. order system

The transfer function is given by

$$H(s) = \frac{H_0}{Ts+1} \qquad \text{or} \qquad h(t) = \frac{H_0}{T}\, e^{-t/T}\, u(t)\ .$$

i.e. two parameters can be fitted to the system. One sees immediately that $H_0 = H(0)$. The parameter T is a timeconstant.





The step-response of the system is given by eq.(2), and one finds

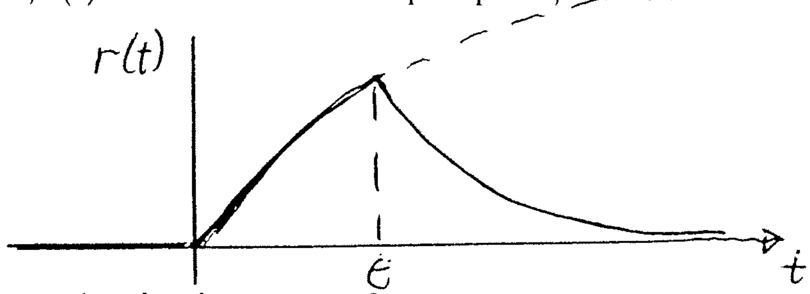
$$r(t) = \Delta e\ H_0\ [1 - e^{-t/T}]\ u(t)\ .$$

This function is shown above.

The pulse-response is given by eq.(3), which applies for $t > \epsilon$

$$r(t) = \Delta e\ H_0\ [e^{\epsilon/T} - 1]\ e^{-t/T}\ u(t)\ .$$

For $t < \epsilon$, $r(t)$ coinsides with the step-response, see below.



One sees that for these types of excitations a 1. order system will respond by linearly increasing $r(t)$ from the moment the system is excited. When this does not correspond to the actual observations, which quite often displays a more gradual initial behaviour, the system is better modelled by a 2. order system.

3.2 2. order system

The transfer function is now given by

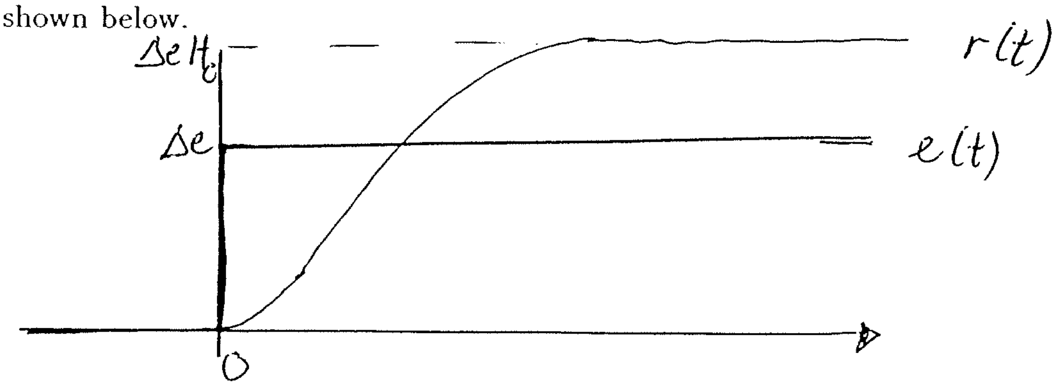
$$H(s) = \frac{H_0}{(T_1 s + 1)(T_2 s + 1)} \qquad \text{or} \qquad h(t) = \frac{H_0}{T_1 - T_2}\left(e^{-t/T_1} - e^{-t/T_2}\right) u(t) \; .$$

i.e., two timeconstants, T_1 and T_2 that can be fitted. As before one has $H_0 = H(0)$.

For the step-response one now finds

$$r(t) = \Delta e \; H_0 \left[1 - \frac{T_1}{T_1 - T_2} e^{-t/T_1} + \frac{T_2}{T_1 - T_2} e^{-t/T_2}\right] u(t) \; ,$$

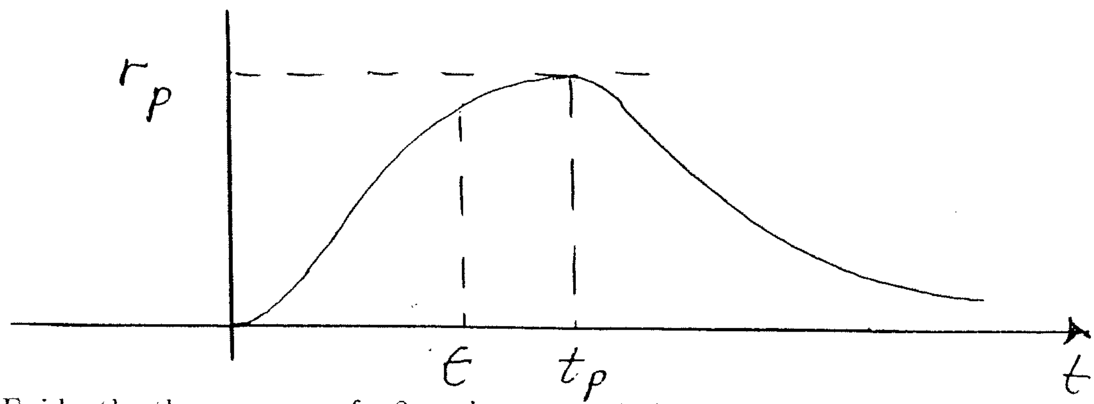
as shown below.



The pulse-response becomes for $t < \epsilon$ the same as above, whereas for $t > \epsilon$ eq.(3) gives

$$r(t) = \Delta e \; H_0 \frac{1}{T_1 - T_2} \left[T_1(e^{\epsilon/T_1} - 1) \; e^{-t/T_1} - T_2(e^{\epsilon/T_2} - 1) \; e^{-t/T_2}\right] u(t) \; .$$

This function is sketched below.

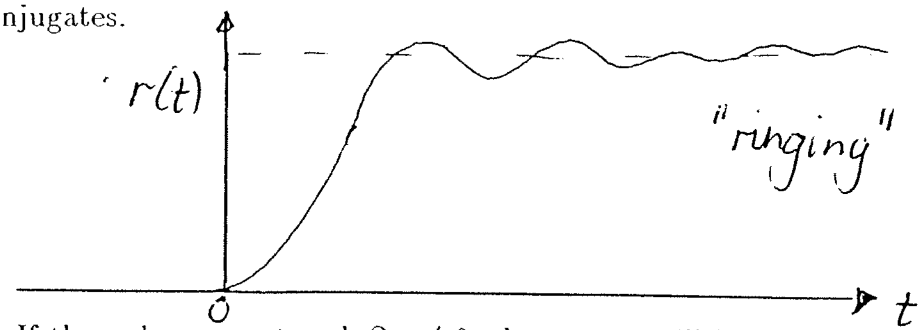


Evidently, the response of a 2. order system is far more "softer" than for a 1. order system, and will in many cases represent an adequate model for real systems.

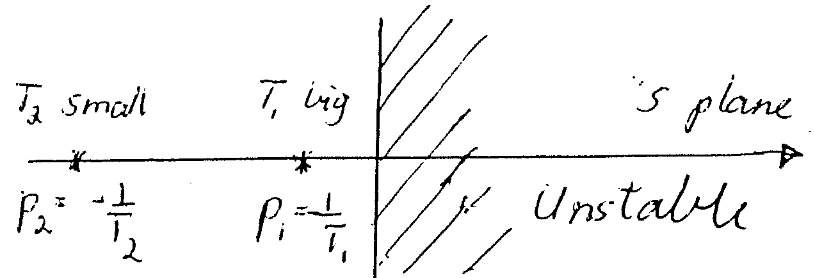
To apply the model the parameters must be fitted according to observations. We will here determine the timeconstants T_1 og T_2 by performing a pulse-excitation, and localize the peak in the response, (t_p, r_p).

4 The significance of the poles

The values of the complex frequency s that makes the denominator of $H(s)$ equal to zero is called the poles of the system. A 2. order system has in general two poles p_1 og p_2. These can either be both real (as assumed in the section above), or they can be comples, and in that case forming a pair of complex conjugates.



If the poles are not real, $\Im p \neq 0$, the system will behave partly "elastic". The step-response will contain an oscillating component (ringing), see above, where the ringing frequency is given directly by the imaginary part of the poles. If the poles are real oscillatory behaviour will not occur, but only exponential transients with characteristic timeconstants. As one sees from $H(s)$ for a 2. order system the poles are given by $p_{1,2} = -1/T_{1,2}$. A diagram over the poles of the system gives complete information of the relaxation times and ringing-frequencies that one will see in the response.

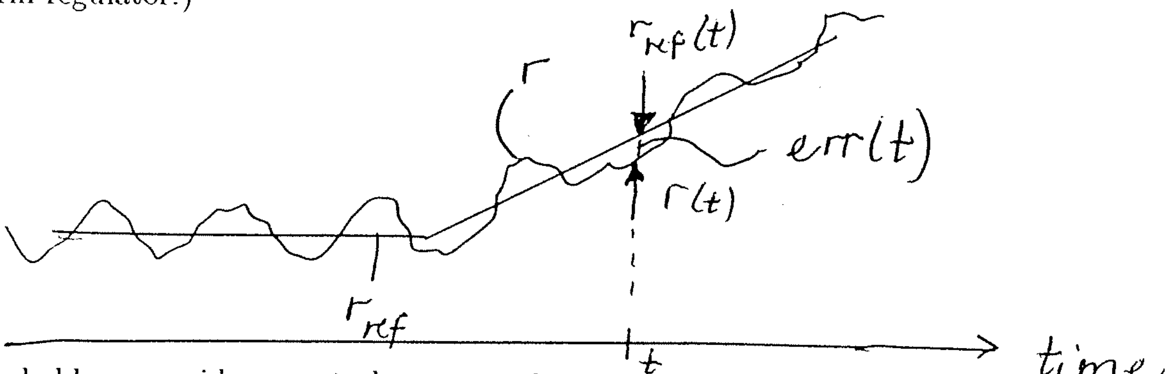


A pole diagram for a 2. order system free from oscillations is shown above. Note that a quick response (small timeconstant) is equivalent to the pole being placed as far to the left as possible. A central point is also that poles placed to the right of the origin (right-half-plane) gives growing exponentials in the time-domain. In other words, one has to do with an unstable system. (Let the time constants T be negative in the 1. and 2. order system calculations , and see that $r(t)$ diverges with time.)

By letting the system be part of a closed-loop, i.e. one lets the reponse determine the current value of the excitation, the transfer function will in general be different. The poles can be changed both in their values but also in their number. Clearly, one has the possibility of modifying the response properties of the original system by choosing feedback strategy. A good choise brings the poles further to the left, whereas bad choises shifts them to the right (really bad if they end up in the right-half-plane). In what follows we shal see that one can make optimal choises if one knows in advance the timeconstants for the system *without* feedback.

5 PID-control

In control contexts one wishes that the response , $r(t)$, traces a preprogrammed reference value , $r_{\text{ref}}(t)$, as well as possible. It is implicite here that the system lives under noisy conditions, so that when left by itself $r(t)$ varies in an unacceptable fashion. When the reference, or set-point, varies with time one uses the term servo for the total feedback system (when $r_{\text{ref}}(t)$ = constant one uses the term regulator.)



We shal here consider a control strategy wherew the feedback is determined by the error,

$$err(t) = r_{\text{ref}}(t) - r(t) \ ,$$

in the following way

$$e(t) = K_P \ err(t) + K_I \int_0^t err(t')dt' + K_D \ err'(t) \ .$$

The first term is a contribution proprtional to the current error, the second is an integral over previous errors, and the last term considers the current rate of change in the error. The three contributions to the excitation is weighted according to the factors K_P, K_I og K_D. Feedback of this type is called PID-control.

Laplace transformation of the above equations gives

$$Err(s) = R_{\text{ref}}(s) - R(s) \ ,$$

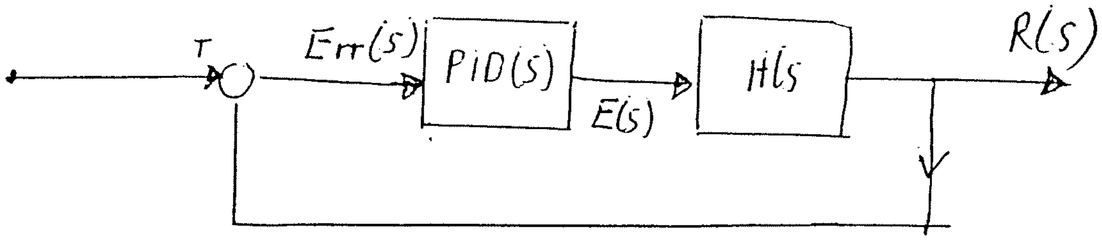
and

$$E(s) = PID(s) \ \ Err(s) \ ,$$

where

$$PID(s) = K_P + K_I/s + K_D s \ .$$

The situation is seen in the block-diagram below.



For optimal control the K-factors must be chosen the bast way. This is done by considering the transfer function, $G(s)$, of the servo, ie. for the entire closed-loop system. We then let the reference signal, $r_{\mathrm{ref}}(t)$ play the role of the excitation of the servo, so that

$$G(s) = \frac{R(s)}{R_{\mathrm{ref}}(s)} \ .$$

It follows from the equations above that

$$R(s) = H(s)\ E(s) = H(s)\ PID(s)\ [R_{\mathrm{ref}}(s) - R(s)] \ ,$$

where division by R_{ref} gives

$$G(s) = \frac{PID(s)}{H(s)^{-1} + PID(s)} \ .$$

We will chose K–factors by considering the behaviour of the system when the reference is changed in a step, $r_{\mathrm{ref}}(t) = \Delta r\ u(t)$, or

$$R_{\mathrm{ref}}(s) = \Delta r / s \ .$$

The response becomes

$$R(s) = \Delta r\ \frac{1}{s}\ \frac{K_P + K_I/s + K_D s}{H(s)^{-1} + K_P + K_I/s + K_D s}$$

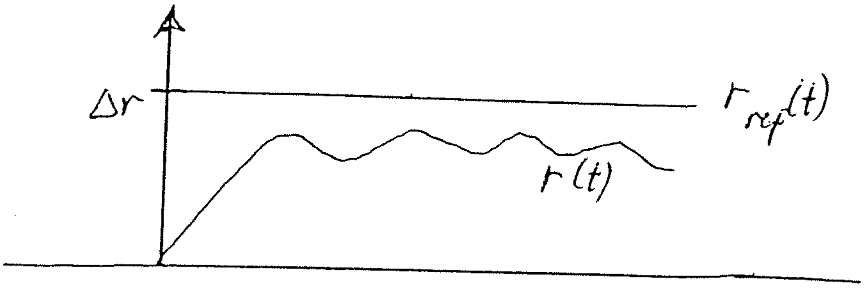
Normally one does not want an asymptotic static deviation between r and r_{ref}, i.e. one would like to have

$$\lim_{t\to\infty} r(t) = \Delta r \ .$$

From the final value theorem it follows that

$$\begin{aligned}
\lim_{t\to\infty} r(t) &= \lim_{s\to 0} sR(s) \\
&= \Delta r\ \lim_{s\to 0} \frac{K_P + K_I/s + K_D s}{H(s)^{-1} + K_P + K_I/s + K_D s} \\
&= \Delta r \begin{cases} \frac{K_P}{H(0)^{-1} + K_P} & \text{for } K_I = 0. \\ 1 & \text{for } K_I \neq 0. \end{cases}
\end{aligned}$$

Thus, one sees that when $H(0) \neq 0$ the integral term is necessary for the response to reach correct final value, see figure below.



6 Optimal PI-control of a 2. order system

Finally we shall see that the theory helps us to make an optimal choise for the feedback parameters, where we for simplicity leave out the derivative term, $K_D = 0$, meaning we only consider PI-control.ontroll. When the system is modelled by a 2. order system, it follows that the transfer function of the servo becomes

$$\begin{aligned} G(s) &= \frac{PI(s)}{H(s)^{-1} + PI(s)} = \frac{K_P + K_I/s}{(T_1 s + 1)(T_2 s + 1)/H(0) + K_P + K_I/s} \\ &= \frac{1}{T_1 T_2} \frac{Ps + I}{s^3 + (1/T_1 + 1/T_2)s^2 + [(P+1)/T_1 T_2]s + I/T_1 T_2} , \end{aligned}$$

where $P = K_P\ H(0)$ and $I = K_I\ H(0)$.
As one sees, the servo is a 3. order system having 3 poles, p_1, p_2 and p_3. One of them has to be real, whereas the remaining two can either be real or a pair of complex conjugate poles.

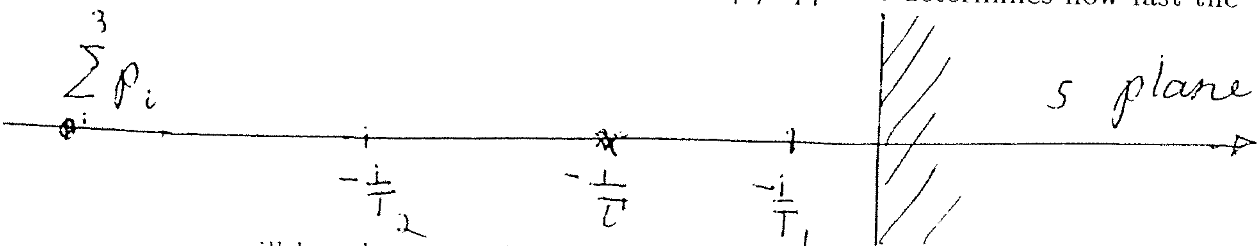
We recall that a quick response requires that the poles are placed as far to the left as possible in the s-plane. Note then that there is a constraint on where we can place them. Since

$$\frac{1}{(s-p_1)(s-p_2)(s-p_3)} = \frac{1}{s^3 - (p_1 + p_2 + p_3)s^2 + (\cdots)s + p_1 p_2 p_3} ,$$

one sees from the expression for $G(s)$ that the sum of the poles must obay

$$p_1 + p_2 + p_3 = -(\frac{1}{T_1} + \frac{1}{T_2}) .$$

Since it is the largest timeconstant $\tau = |1/\Re p|$ that determines how fast the



servo will be, the constraint leads us to choose all poles at the same distance from the imaginary axis. If we in addition will avoid "ringing", we place the poles all on the real axis, see below, meaning that we put all the three poles equal

$$p_1 = p_2 = p_3 = -\frac{1}{3}\left(\frac{1}{T_1} + \frac{1}{T_2}\right) \equiv -1/\tau \ .$$

The polynomial in the denominator now becomes

$$s^3 + 3\tau^{-1}s^2 + 3\tau^{-2}s + \tau^{-3} \ ,$$

and by comparison with the expression for G one then gets

$$\frac{P+1}{T_1T_2} = 3\tau^{-2} \quad \text{og} \quad \frac{I}{T_1T_2} = \tau^{-3} \ ,$$

or

$$P = \frac{1}{3}\frac{(T_1+T_2)^2}{T_1T_2} - 1 \quad \text{og} \quad I = \frac{1}{27}\frac{(T_1+T_2)^3}{(T_1T_2)^2} \ .$$

7 Problems

Problem 1:
Assume an ac-excitation; $e(t) = \Delta e \; cos\omega t = \Delta e \; \Re e^{i\omega t}$.
Show that the response becomes of the form

$$r(t) = \Delta e \cdot M(\omega) \; \cos[\omega t + \phi(\omega)].$$

What is the relation between $M(\omega), \phi(\omega)$ *and the memory function* h*?*

Problem 2:
Give a proof for the convolution theorem.

Problem 3:
Verify eq.(4) for a 1. order system.

Problem 4:
a) Show by calculation all the expressions given in the section for the 2. order system.
b) Show that the following relation applies for the peak-point in the pulse-response

$$r_p = \Delta e \; H_0 \; (e^{\epsilon/T_{1,2}} - 1) \; e^{-t_p/T_{1,2}} \;,$$

where $T_{1,2}$ *means* T_1 *or* T_2.

Problem 5:
Show that the step-response for the servo becomes

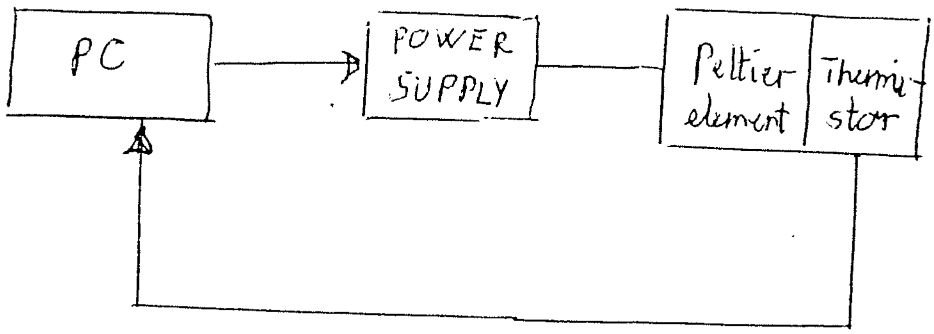
$$\frac{r(t)}{\Delta r} = 1 + \left[(1 - \frac{3}{2} \frac{\tau}{T_1 + T_2}) \; (\frac{t}{\tau})^2 - \frac{t}{\tau} - 1 \right] \; e^{t/\tau} \quad , \quad t \geq 0 \; .$$

Let $T_1 = 20$ *seconds and* $T_2 = 1$ *seconds, and make a graph over the step-response of the system with and without feedback.*

Hint:
Use the $\mathcal{L}$-transform of an integral, and also that

$$\mathcal{L}[t^2 \; e^{-t/\tau}] = \frac{2}{(s + 1/\tau)^3}$$

8 Laboratory-exercise pointwise



1. *Supervisor gives an orientation about each part in the block diagram above.*

2. *Supervisor demonstrates and explains the BASIC program that will be used.*

3. *Perform a series of temperature measurements over a period of about 20 minutes. Give a characteristics of the variations (noise) in the temperature.*

4. *Modify the program so that it can perform a step-excitation in N, (DAC-number), and measure the step-response.*
   Discuss whether the oven should be modelled by a 1. or 2. order system.

5. *Measure the temperature response to a suitable pulse-excitation.*

6. *Consider the system as a 2. order system, i.e.,*

$$\frac{\Theta(s)}{\Delta N(s)} \equiv H(s) = \frac{H(0)}{(T_1 s + 1)(T_2 s + 1)},$$

   where Θ is the temperature relative to the basis-value, ie. the temperature when $\Delta N = 0$.
   Determine from the measurements of pt.4 og 5, $H(0)$, and the timeconstants T_1 og T_2, respectively.

7. *The oven is to be controlled by the PC using PI-servocontrol.*

Here $PI(s) = K_P + K_I/s$.
Find the theoretically optimal values for K_P og K_I, the weight factors for the proprtional and integral terms, respectively.

8. Modify the program so that it performs the PI-control.
NB: The PC measures the temperature and updates the correction at descrete times , nT_s. (n is a time-index $n = 0, 1, 2, ..$). The signal to the DAC is therefore a discrete-time signal given by

$$\Delta N(n) = -\left(K_P \operatorname{err}(n) + K_I \, T_s \, \sum_n \operatorname{err}(n) \right) .$$

where $\operatorname{err}(n)$ is the sampled value for the deviation $\Theta - \Theta_{ref}$

9. Let the PC control the oven. Repeat the measurement of the temperature fluctuations, this time for the closed-loop system. Make a comparison and comment.

10. Measure the response to a step-excitation

$$\Theta_{ref} \rightarrow \Theta_{ref} + \Delta\Theta .$$

Compare with the theoretical result for optimal control.

11. Repeat pt.10 using 10 times as long samplinginterval T_s.
Make a comment.

12. Let $K_I = 0$ in the program, and use the same value for K_P.
Verify that the response to a step-excitation gets an asymptotic deviation between Θ and Θ_{ref}.
Compare with the theory.

13. It is easy to make the system unstable by increasing the corrective feedback.
Experience this!