II. $$\sum_{spin} |M_s|^2 = Tr[(\not p_3+m)\gamma^\mu(\not p_1+m)\gamma^\nu]\, Tr[(\not p_4-m)\gamma_\mu(\not p_2-m)\gamma_\nu]\,\frac{e^4}{q^4}$$

← ultrarelativistic limit $m \ll E$

$$= Tr[\not p_3\gamma^\mu\not p_1\gamma^\nu]\, Tr[\not p_4\gamma_\mu\not p_2\gamma_\nu]\,\frac{e^4}{q^4}$$

$$Tr[\not p_3\gamma^\mu\not p_1\gamma^\nu] = 4[p_3^\mu p_1^\nu - g^{\mu\nu}(p_1\cdot p_3) + p_3^\nu p_1^\mu]$$

$$Tr[\not p_4\gamma_\mu\not p_2\gamma_\nu] = 4[p_{4\mu} p_{2\nu} - g^{\mu\nu}(p_4\cdot p_2) + p_{4\nu} p_{2\mu}]$$

$$\Rightarrow \sum_{spin}|M_s|^2 = 16\Big[(p_3\cdot p_4)(p_1\cdot p_2) - \cancel{(p_3\cdot p_1)(p_4\cdot p_2)} + (p_1\cdot p_4)(p_3\cdot p_2) +$$
$$- \cancel{(p_1\cdot p_3)(p_4\cdot p_2)} + \cancel{4(p_1\cdot p_3)(p_4\cdot p_2)} - \cancel{(p_1\cdot p_3)(p_4\cdot p_2)} +$$
$$+ (p_3\cdot p_2)(p_4\cdot p_1) - \cancel{(p_3\cdot p_1)(p_4\cdot p_2)} + (p_4\cdot p_3)(p_1\cdot p_2)\Big]\frac{e^4}{q^4}$$

$$= 32[(p_3\cdot p_4)(p_1\cdot p_2) + (p_1\cdot p_4)(p_3\cdot p_2)]\frac{e^4}{q^4} \rightarrow q^2 = t$$

III. $\langle M_a^* M_s\rangle =$

$$= \frac{e^4}{4st}\sum_{rspt}[\bar v^r(p_2)\gamma^\mu u^s(p_1)]^\dagger[\bar u^p(p_3)\gamma_\mu v^t(p_4)]^\dagger[\bar u^p(p_3)\gamma^\nu u^s(p_1)][\bar v^r(p_2)\gamma_\nu v^t(p_4)]$$

$$= \frac{e^4}{4st}\sum_{rspt}\bar u^s(p_1)_a\gamma^\mu_{ab}v^r(p_2)_b\,\bar v^t(p_4)_i\gamma_{\mu ij}u^p(p_3)_j\,\bar u^p(p_3)_k\gamma^\nu_{kl}u^s(p_1)_l\,\bar v^r(p_2)_m\gamma_{\nu mn}v^t(p_4)_n$$

$$= \frac{e^4}{4st}\sum_r v^r_b\bar v^r_m(p_2)\sum_s u^s_l\bar u^s_a(p_1)\sum_p u_j(p_3)\bar u_k(p_3)\sum_t v^t_n\bar v^t_i(p_4)\,\gamma^\mu_{ab}\gamma_{\mu ij}\gamma^\nu_{kl}\gamma_{\nu mn}$$

← completeness relations: $\sum_s u^s\bar u^s(p) = \not p + m$, $\sum_s v^s\bar v^s = \not p - m$

$$= \frac{e^4}{4st}(\not p_2-m)_{bm}(\not p_1+m)_{la}(\not p_3+m)_{jk}(\not p_4-m)_{ni}\,\gamma^\mu_{ab}\gamma_{\mu ij}\gamma^\nu_{kl}\gamma_{\nu mn}$$

$$= \frac{e^4}{4st}(\not p_2-m)_{bm}\gamma_{\nu mn}(\not p_4-m)_{ni}\gamma_{\mu ij}(\not p_3+m)_{jk}\gamma^\nu_{kl}(\not p_1+m)_{la}\gamma^\mu_{ab}$$

← ultrarelativistic limit

$$= \frac{e^4}{4st}Tr[\not p_2\gamma_\nu\not p_4\gamma_\mu\not p_3\gamma^\nu\not p_1\gamma^\mu]$$) Permuting matrices, cyclically

$$= \frac{e^4}{4st}Tr[\not p_1\underbrace{\gamma^\mu\not p_2\gamma_\nu\not p_4\gamma_\mu}\not p_3\gamma^\nu]$$

$\gamma^\mu \not a\not b\not c\gamma_\mu = -2\not c\not b\not a$

$\Rightarrow = -2\not p_4\gamma_\nu\not p_2$ non-cyclical perm.

$$= -\frac{e^4}{2st}Tr[\not p_1\not p_4\gamma_\nu\not p_2\not p_3\gamma^\nu] \overset{\text{non-cyclical perm.}}{=} \frac{e^4}{2st}Tr[\not p_1\not p_4\overbrace{\gamma^\nu\not p_2\not p_3\gamma_\nu}^{(p_2\cdot p_3)}]$$

$$= \frac{e^4}{2st}(p_2\cdot p_3)\,Tr(\not p_1\not p_4) = \frac{e^4}{2st}(p_2\cdot p_3)\underbrace{p_{1\alpha}\, p_{4\beta}\, \underbrace{Tr(\gamma^\alpha\gamma^\beta)}_{4g^{\alpha\beta}}}_{4(p_1\cdot p_4)}$$

$$= \frac{2e^4}{st}(p_2\cdot p_3)(p_1\cdot p_4)$$

Total matrix element:

$$|\mathcal{M}_{TOT}|^2 = |\mathcal{M}_a|^2 + |\mathcal{M}_s|^2 \mp 2\,\mathrm{Re}\{\mathcal{M}_a^* \mathcal{M}_s\}$$

Average $\langle\ \rangle$ is linear:

$$\langle|\mathcal{M}_{TOT}|^2\rangle = \langle|\mathcal{M}_a|^2\rangle + \langle|\mathcal{M}_s|^2\rangle + 2\,\mathrm{Re}\{\langle\mathcal{M}_a^* \mathcal{M}_s\rangle\}$$

$$\overset{!}{=} 4e^4\left[\frac{2(p_2\cdot p_3)(p_1\cdot p_4) + 2(p_1\cdot p_3)(p_2\cdot p_4)}{s^2} + 2\,\frac{(p_3\cdot p_4)(p_1\cdot p_2) + (p_1\cdot p_4)(p_3\cdot p_2)}{t^2} + \mp\frac{(p_2\cdot p_3)(p_1\cdot p_4)}{st}\right]$$

where $s = (p_1+p_2)^2 = (p_3+p_4)^2$
$t = (p_1-p_3)^2 = (p_2-p_4)^2$