Electron-positron annihilation

Calculation of the $e^+e^- \rightarrow \mu^+\mu^-$ cross section based on the traces of matrices and the completeness relations for Dirac spinors

- In the calculation of the $e^+e^- \rightarrow \mu^+\mu^-$ cross section
    - Individual matrix elements calculated for each helicity combination using the explicit representations of the spinors and the γ-matrices.
    - resulting squares of the matrix elements were then summed and averaged.
    - approach relatively simple and exposes the underlying physics of the interaction.
    - In the limit where the masses of the particles can be neglected, these calculations are relatively straightforward as they involve only a limited number of helicity combinations.
- However, when the particle masses cannot be neglected,
    - Necessary to consider all possible spin combinations.
    - In this case, calculating the individual helicity amplitudes is not particularly efficient (although it is well suited to computational calculations).
- For more complicated processes,
    - analytic solutions usually most easily obtained using a powerful technique based on the traces of matrices and the completeness relations for Dirac spinors.

Completeness relation

- Completeness relation

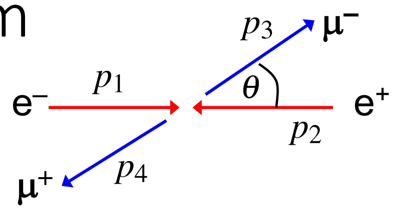
  See 6.5.1 for derivation

  - Calculate sums over spin states of initial – and final-state particle
  - Particle spinors $\sum_{s=1}^{2} u_s \bar{u}_s = (\gamma^\mu p_\mu + mI) = \not{p} + m,$

    $$\not{p} \equiv \gamma^\mu p_\mu = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3$$

  - Antiparticle spinors $\sum_{r=1}^{2} v_r \bar{v}_r = (\gamma^\mu p_\mu - mI) = \not{p} - m,$

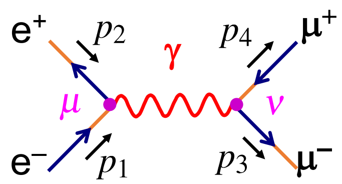
Spin sums and the trace formalism



- QED, QCD, Weak Interactions vertex factors in form

  $$\bar{u}(p)\,\Gamma\,u(p') = \bar{u}(p)_j\,\Gamma_{ji}\,u(p')_i$$ *Simply a complex number!*

  - Γ is a 4x4 matrix (one or more Dirac γ-matrices)
  - For QED $\Gamma=\gamma^\mu$



- Matrix element for the process $e^+e^- \rightarrow \mu^+\mu^-$

$$\mathcal{M}_{fi} = -\frac{e^2}{q^2}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right] g_{\mu\nu} \left[\bar{u}(p_3)\gamma^\nu v(p_4)\right]$$

$$= -\frac{e^2}{q^2}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu v(p_4)\right],$$

$$\mathcal{M}^\dagger_{fi} = \frac{e^2}{q^2}\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]^\dagger \left[\bar{u}(p_3)\gamma_\nu v(p_4)\right]^\dagger$$

$$|\mathcal{M}_{fi}|^2 = \mathcal{M}_{fi}\mathcal{M}^\dagger_{fi}$$

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}\left[\bar{v}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]^\dagger \times \left[\bar{u}(p_3)\gamma_\mu v(p_4)\right]\left[\bar{u}(p_3)\gamma_\nu v(p_4)\right]^\dagger$$

$$\boxed{\begin{aligned}\langle|\mathcal{M}_{fi}|^2\rangle &= \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 \\ &= \frac{e^4}{4q^4}\sum_{s,r}\left[\bar{v}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\bar{v}^r(p_2)\gamma^\nu u^s(p_1)\right]^\dagger \\ &\quad\times\sum_{s',r'}\left[\bar{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\right],\left[\bar{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\right]^\dagger\end{aligned}}$$

$$\langle|\mathcal{M}_{fi}|^2\rangle = \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2$$
$$= \frac{e^4}{4q^4}\sum_{s,r}\left[\overline{v}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\overline{v}^r(p_2)\gamma^\nu u^s(p_1)\right]^\dagger$$
$$\times\sum_{s',r'}\left[\overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\right],\left[\overline{u}^{s'}(p_3)\gamma_\nu v^{r'}(p_4)\right]^\dagger$$

- s, r, s', r' spin (helicity) states
- calculation of the spin-averaged matrix element squared reduced to product of 2 terms of form

$$\sum_{\text{spins}}\left[\overline{\psi}\Gamma_1\phi\right]\left[\overline{\psi}\Gamma_2\phi\right]^\dagger$$

- Γ_1 and Γ_2 are two 4x4 matrices
- for QED: $\Gamma_1 = \gamma^\mu$; $\Gamma_2 = \gamma^\nu$

$$\overline{\Gamma} = \Gamma$$
$$\left[\overline{\psi}\Gamma\phi\right]^\dagger \equiv \overline{\phi}\Gamma\psi.$$

See 6.5.2 for proof

$$\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}\sum_{s,r}\left[\overline{v}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\overline{u}^s(p_1)\gamma^\nu v^r(p_2)\right]$$
$$\times\sum_{s',r'}\left[\overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\right]\left[\overline{v}^{r'}(p_4)\gamma_\nu u^{s'}(p_3)\right]$$

$$\mathcal{L}^{\mu\nu}_{(\mathrm{e})} = \sum_{s,r=1}^{2}\overline{v}^r_j(p_2)\gamma^\mu_{ji}u^s_i(p_1)\,\overline{u}^s_n(p_1)\gamma^\nu_{nm}v^r_m(p_2)$$
$$\mathcal{L}^{\mu\nu}_{(\mathrm{e})} = \left[\sum_{r=1}^{2}v^r_m(p_2)\overline{v}^r_j(p_2)\right]\left[\sum_{s=1}^{2}u^s_i(p_1)\overline{u}^s_n(p_1)\right]\gamma^\mu_{ji}\gamma^\nu_{nm}$$

- written in index form
- rearrange
- Since all quantities are just numbers, with indices keeping track of matrix multiplication

$$\mathcal{L}^{\mu\nu}_{(\mathrm{e})} = \left[\sum_{r=1}^{2}v^r_m(p_2)\overline{v}^r_j(p_2)\right]\left[\sum_{s=1}^{2}u^s_i(p_1)\overline{u}^s_n(p_1)\right]\gamma^\mu_{ji}\gamma^\nu_{nm}$$

- Use completeness relation $\sum_{s=1}^{2}u_s\overline{u}_s = (\gamma^\mu p_\mu + mI) = \not{p} + m,$

$$\not{p} \equiv \gamma^\mu p_\mu = E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3$$

$$\mathcal{L}^{\mu\nu}_{(\mathrm{e})} = (\not{p}_2 - m)_{mj}(\not{p}_1 + m)_{in}\gamma^\mu_{ji}\gamma^\nu_{nm}$$

- Then put back into normal matrix multiplication →

$$\boxed{\begin{aligned}\mathcal{L}^{\mu\nu}_{(\mathrm{e})} &= (\not{p}_2 - m)_{mj}\gamma^\mu_{ji}(\not{p}_1 + m)_{in}\gamma^\nu_{nm}\\ &= \left[(\not{p}_2 - m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right]_{mm}\\ &= \mathrm{Tr}\left([\not{p}_2 - m]\gamma^\mu[\not{p}_1 + m]\gamma^\nu\right).\end{aligned}}$$

- Sum over spins of initial-state particles replaced by calculation of traces of 4x4 matrices
  - one for each of the sixteen possible combinations of indices μ and ν.
- order in which the two /p terms appear in trace calculation follows order in which spinors appear in original four-vector currents
  - /p term associated with the adjoint spinor appears first
- In constructing traces associated with Feynman diagram, remember:
  - order in which different terms appear from following the arrows in the fermion currents in the backwards direction!

- Sum over spins of final-state particles – muon tensor

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4} \sum_{s,r} \left[\overline{v}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\overline{u}^s(p_1)\gamma^\nu v^r(p_2)\right] \times \sum_{s',r'} \left[\overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\right]\left[\overline{v}^{r'}(p_4)\gamma_\nu u^{s'}(p_3)\right]$$

$$\boxed{\mathcal{L}^{(\mu)}_{\mu\nu} = \sum_{s',r'} \left[\overline{u}^{s'}(p_3)\gamma_\mu v^{r'}(p_4)\right]\left[\overline{v}^{r'}(p_4)\gamma_\nu u^{s'}(p_3)\right]}$$

- In terms of traces

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4} \mathcal{L}^{\mu\nu}_{(e)} \mathcal{L}^{(\mu)}_{\mu\nu} = \frac{e^4}{q^4} \mathrm{Tr}\left([\not{p}_2 - m]\gamma^\mu[\not{p}_1 + m]\gamma^\nu\right) \times \mathrm{Tr}\left([\not{p}_3 + M]\gamma_\mu[\not{p}_4 - M]\gamma_\nu\right)$$

- Now question of Trace calculations ...

Trace theorems

See 6.5.3 (and problems) for derivations

- Set of trace theorems

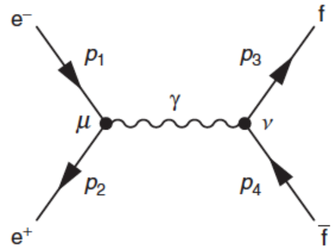
(a) $\mathrm{Tr}(I) = 4$;
(b) the trace of any odd number of γ-matrices is zero;
(c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$;
(d) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$;
(e) the trace of γ^5 multiplied by an odd number of γ-matrices is zero;
(f) $\mathrm{Tr}\left(\gamma^5\right) = 0$;
(g) $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0$; and
(h) $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4i\varepsilon^{\mu\nu\rho\sigma}$, where $\varepsilon^{\mu\nu\rho\sigma}$ is antisymmetric under the interchange of any two indices.

- With these, expressions can be evaluated relatively easily

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4} \mathcal{L}^{\mu\nu}_{(e)} \mathcal{L}^{(\mu)}_{\mu\nu} = \frac{e^4}{q^4} \mathrm{Tr}\left([\not{p}_2 - m]\gamma^\mu[\not{p}_1 + m]\gamma^\nu\right) \times \mathrm{Tr}\left([\not{p}_3 + M]\gamma_\mu[\not{p}_4 - M]\gamma_\nu\right)$$

- worth going through examples of a matrix element calculation using the trace methodology in gory detail
- Later, in FYS5555 calculation and simulation of more complicated processes such $e^+e^- \to \mu^+\mu^-$ also including Z ... and Z' exchange

Electron-positron annihilation revisited



$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4} \mathcal{L}_{(e)}^{\mu\nu} \mathcal{L}^{(\mu)}_{\mu\nu}$$

$$= \frac{e^4}{q^4} \mathrm{Tr}\left([\not{p}_2 - m]\gamma^\mu [\not{p}_1 + m]\gamma^\nu\right) \times \mathrm{Tr}\left([\not{p}_3 + M]\gamma_\mu [\not{p}_4 - M]\gamma_\nu\right)$$

- m=m$_e$=0 (neglect electron mass), set M=m$_f$

$$\langle |\mathcal{M}_{fi}|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{Q_f^2 e^4}{4q^4} \mathrm{Tr}\left(\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right) \mathrm{Tr}\left([\not{p}_3 + m_f]\gamma_\mu [\not{p}_4 - m_f]\gamma_\nu\right)$$

$\not{p}_1 = \gamma^\sigma p_{1\sigma}$ and $\not{p}_2 = \gamma^\rho p_{2\rho}$ $\Rightarrow$

$$\begin{aligned} \mathrm{Tr}\left(\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right) &= p_{2\rho} p_{1\sigma} \mathrm{Tr}\left(\gamma^\rho \gamma^\mu \gamma^\sigma \gamma^\nu\right) \\ &= 4 p_{2\rho} p_{1\sigma} (g^{\rho\mu} g^{\sigma\nu} - g^{\rho\sigma} g^{\mu\nu} + g^{\rho\nu} g^{\mu\sigma}) \\ &= 4 p_2^\mu p_1^\nu - 4 g^{\mu\nu} (p_1 \cdot p_2) + 4 p_2^\nu p_1^\mu . \end{aligned}$$

- Trace of odd number of γ-matrices is zero
- Tr (A + B) = Tr (A) + Tr (B)

$\Rightarrow$

$$\begin{aligned} \mathrm{Tr}\left([\not{p}_3 + m_f]\gamma_\mu [\not{p}_4 - m_f]\gamma_\nu\right) &= \mathrm{Tr}\left(\not{p}_3 \gamma_\mu \not{p}_4 \gamma_\nu\right) - m_f^2 \mathrm{Tr}\left(\gamma_\mu \gamma_\nu\right) \qquad (6 \\ &= 4 p_{3\mu} p_{4\nu} - 4 g_{\mu\nu} (p_3 \cdot p_4) + 4 p_{3\nu} p_{4\mu} - 4 m_f^2 g_{\mu\nu} \end{aligned}$$

➔Spin-averaged matrix element squared

$$\begin{aligned} \langle |\mathcal{M}_{fi}|^2 \rangle = 16 \frac{Q_f^2 e^4}{4q^4} &\left[p_2^\mu p_1^\nu - g^{\mu\nu} (p_1 \cdot p_2) + p_2^\nu p_1^\mu \right] \\ \times &\left[p_{3\mu} p_{4\nu} - g_{\mu\nu} (p_3 \cdot p_4) + p_{3\nu} p_{4\mu} - m_f^2 g_{\mu\nu} \right] \end{aligned}$$

- simplified by contracting t

$g^{\mu\nu} g_{\mu\nu} = 4, \quad p_2^\mu p_1^\nu g_{\mu\nu} = (p_1 \cdot p_2)$

$p_2^\mu p_1^\nu p_{3\mu} p_{4\nu} = (p_2 \cdot p_3)(p_1 \cdot p_4)$ $\Rightarrow$

$$\begin{aligned} \langle |\mathcal{M}_{fi}|^2 \rangle = 4 \frac{Q_f^2 e^4}{q^4} \Big[&(p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) \\ &- (p_1 \cdot p_2)(p_3 \cdot p_4) + 4(p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) \\ &+ (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \\ &- m_f^2 (p_1 \cdot p_2) + 4 m_f^2 (p_1 \cdot p_2) - m_f^2 (p_1 \cdot p_2) \Big], \end{aligned}$$

- leading to

$$\langle |\mathcal{M}_{fi}|^2 \rangle = 4 \frac{Q_f^2 e^4}{q^4} \left[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2 m_f^2 (p_1 \cdot p_2) \right]$$

$$\langle|\mathcal{M}_{fi}|^2\rangle = 4\frac{Q_f^2 e^4}{q^4}\left[2(p_1\cdot p_3)(p_2\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) + 2m_f^2(p_1\cdot p_2)\right]$$

- $m_e=0 \rightarrow$ $$q^2 = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2(p_1\cdot p_2) \approx 2(p_1\cdot p_2)$$

$$\boxed{\langle|\mathcal{M}_{fi}|^2\rangle = 2\frac{Q_f^2 e^4}{(p_1\cdot p_2)^2}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_f^2(p_1\cdot p_2)\right]}$$

- $m_f=0 \rightarrow$ (6.25) obtained from helicity amplitudes $$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2}$$

- In above calculation, neither explicit form of the spinors nor the specific representation of the γ-matrices used.
- Spin-averaged matrix element squared determined from completeness relations for spinors and commutation and Hermiticity properties of the γ-matrices alone

e^+e^- annihilation close to threshold

$$\boxed{\langle|\mathcal{M}_{fi}|^2\rangle = 2\frac{Q_f^2 e^4}{(p_1\cdot p_2)^2}\left[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m_f^2(p_1\cdot p_2)\right]}$$

- $$\begin{aligned} p_1 &= (E,0,0,+E),\\ p_2 &= (E,0,0,-E),\\ p_3 &= (E,+\beta E\sin\theta,0,+\beta E\cos\theta),\\ p_4 &= (E,-\beta E\sin\theta,0,-\beta E\cos\theta), \end{aligned} \Rightarrow \begin{aligned} p_1\cdot p_3 = p_2\cdot p_4 &= E^2(1-\beta\cos\theta),\\ p_1\cdot p_4 = p_2\cdot p_3 &= E^2(1+\beta\cos\theta),\\ p_1\cdot p_2 &= 2E^2. \end{aligned}$$

$$\Rightarrow \begin{aligned}\langle|\mathcal{M}_{fi}|^2\rangle &= 2\frac{Q_f^2 e^4}{4E^4}\left[E^4(1-\beta\cos\theta)^2 + E^4(1+\beta\cos\theta)^2 + 2E^2m_f^2\right]\\ &= Q_f^2 e^4\left(1+\beta^2\cos^2\theta + \frac{E^2-\mathrm{p}^2}{E^2}\right)\\ &= Q_f^2 e^4\left(2+\beta^2\cos^2\theta-\beta^2\right).\end{aligned} \Rightarrow \boxed{\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}\frac{\mathrm{p}}{E}\langle|M_{fi}|^2\rangle\\ &= \frac{1}{4s}\beta Q_f^2\alpha^2\left(2+\beta^2\cos^2\theta-\beta^2\right)\end{aligned}}$$

$$\sigma(e^+e^- \to f\bar{f}) = \frac{4\pi\alpha^2 Q_f^2}{3s}\beta\left(\frac{3-\beta^2}{2}\right) \quad \text{with} \quad \beta^2 = \left(1-\frac{4m_f^2}{s}\right)$$

Tau-pair production at threshold

$$\sigma(e^+e^- \to f\bar{f}) = \frac{4\pi\alpha^2 Q_f^2}{3s}\beta\left(\frac{3-\beta^2}{2}\right) \quad \text{with} \quad \beta^2 = \left(1 - \frac{4m_f^2}{s}\right)$$

- Close to threshold, cross section approximately proportional to velocity of final state particles.
- measurements of the total $e^+e^- \to \tau^+\tau^-$ cross section at centre-of-mass energies just above threshold.
- Data in good agreement with prediction above.

- In relativistic limit $\beta \to 1$ total cross section reduces to expression we derived earlier

$$\sigma = \frac{4\pi\alpha^2}{3s}$$