Assignment I - Bhabha scattering

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The solution to the first two problems is presented in the attached sheets at the end of the paper. The γ matrices properites used are from [1].

The code used for this project can be found here:
`https://github.com/FedericoNardi/ParticlePhysics.git`

1 Differential Cross Section

Starting from the expression for the matrix element

$$< |M_{fi}|^2 >= 4e^4\Bigg[2\frac{(p_1\cdot p_4)(p_2\cdot p_3)+(p_1\cdot p_3)(p_2\cdot p_4)}{s^2}+2\frac{(p_3\cdot p_4)(p_1\cdot p_2)+(p_1\cdot p_4)(p_3\cdot p_2)}{t^2}+ \\ -\frac{(p_1\cdot p_4)(p_3\cdot p_2)}{st}\Bigg]$$

$$\text{where}\quad s=(p_1+p_2)^2=(p_3+p_4)^2\quad\text{and}\quad t=(p_1-p_3)^2=(p_2-p_4)^2$$

and inserting the expressions for momenta in the **center-mass frame**

$$\begin{aligned}
p_1 &= (E,0,0,E)\\
p_2 &= (E,0,0,-E)\\
p_3 &= (E,0,E\sin\theta,E\cos\theta)\\
p_4 &= (E,0,-E\sin\theta,-E\cos\theta)
\end{aligned}$$

we get the identities

$$\begin{aligned}
&p_2\cdot p_3 = E^2(1-\cos\theta) = p_1\cdot p_4\\
&p_1\cdot p_3 = E^2(1-\cos\theta) = p_2\cdot p_4\\
&p_3\cdot p_4 = 2E^2 = p_1\cdot p_2\\
&s = 4E^2\\
&t = 2E^2(1-\cos\theta)
\end{aligned}$$

and therefore the expression for the matrix element, with $e^2=4\pi\alpha$ becomes

$$< |M_{fi}|^2 >= (4\pi\alpha)^2\Bigg[\underbrace{\frac{(1+\cos\theta)^2+(1-\cos\theta)^2}{2}}_{\text{Annihilation contribution}}+\underbrace{\frac{8+2(1+\cos\theta)^2}{(1-\cos\theta)^2}}_{\text{scattering contribution}}-\underbrace{\frac{(1+\cos\theta)^2}{1-\cos\theta}}_{\text{cross term contr.}}\Bigg].$$

The expression for the Lorentz-invariant differential cross section is

$$\frac{d\sigma}{dt} = \frac{1}{64\pi\, s|\vec{p_1}|^2} < |M_{fi}|^2 >$$

we can get an expression for our reference system by substituting the differential $dt = 2|\vec{p}_1|\,|\vec{p}_3|d(\cos\theta)$:

$$\frac{d\sigma}{d(\cos\theta)} = \frac{1}{32\pi\, s} < |M_{fi}|^2 > \tag{1}$$

since in our frame $|\vec{p_1}| = |\vec{p_3}| = E$.

Using the values $E = \frac{\sqrt{s}}{2} = 7\text{GeV}$, $e^2 = 4\pi\alpha = \frac{4\pi}{137}$, equation 1 has been implemented on a Python code evaluating it for various values of $\cos\theta$. Figure 1 shows the differential cross section from equation 1 evaluated at different angles, while in figure 2 each component of the total matrix element contribution is isolated.

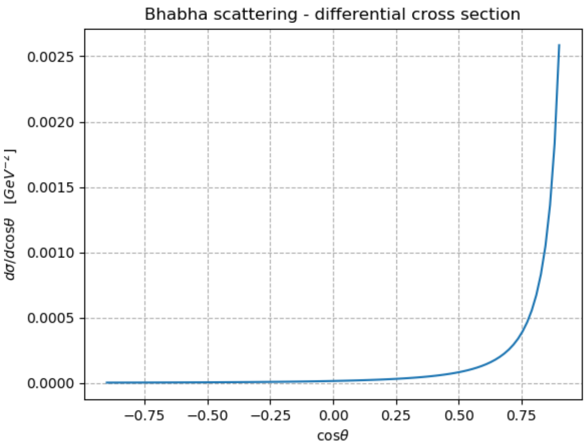


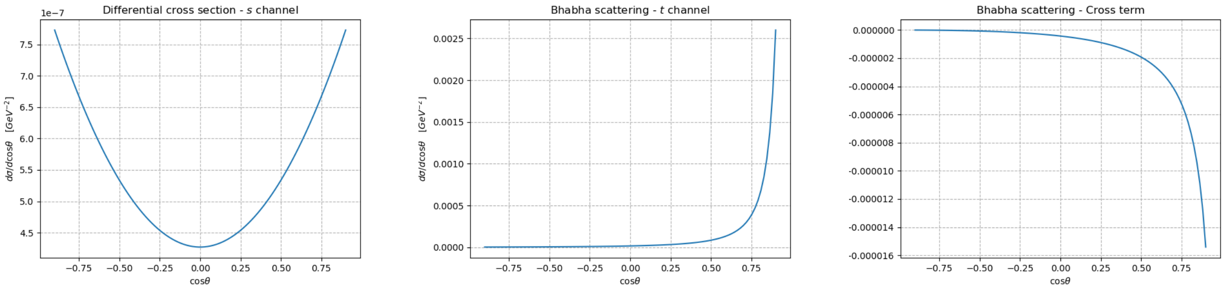
Figure 1: Angular dependence of differential cross section for electron-positron scattering

It is clear that the dominant contribution to the differential cross section for the process comes from the t-channel (scattering), while the interference cross term has a soft damping effect only for small angles $\cos\theta \sim 1$.

Using the result derived in the lectures for $e^+e^- \to \mu^+\mu^-$

$$\frac{d\sigma}{d(\cos\theta)} = 2\pi\frac{d\sigma}{d\Omega} = \frac{(4\pi\alpha^2)}{64\pi\, s}\left[(1 + cos^2\theta) + (1 - \cos^2\theta)\right]$$

we can note that it is the same expression obtained by combining equation 1 with the first term of the matrix element $< |M_{fi}|^2 >$. As expected, in the ultrarelativistic limit $m_{e^\pm} \sim m_{\mu^\pm} << E$ the masses of the leptons can be neglected and the s-channel processes themselves can produce muons-antimuons and electrons-antielectrons with the same yield.



(a) s-channel (b) t-channel (c) cross term

Figure 2: Different contribution for the differential cross section.

2 Total Bhabha Cross Section

The total cross section for the process is obtained as

$$\sigma = \int_{-1}^{1} d(\cos\theta) \frac{d\sigma}{d(\cos\theta)}.$$

Since the integral diverges for $cos\theta \to 1$, a cut-off ϵ has been introduced in the integration domain that becomes $[-1, 1-\epsilon]$. The integral has been evaluated numerically for $\epsilon = 0.001$ and for different values of center mass energies, and the results are shown in figure 3.

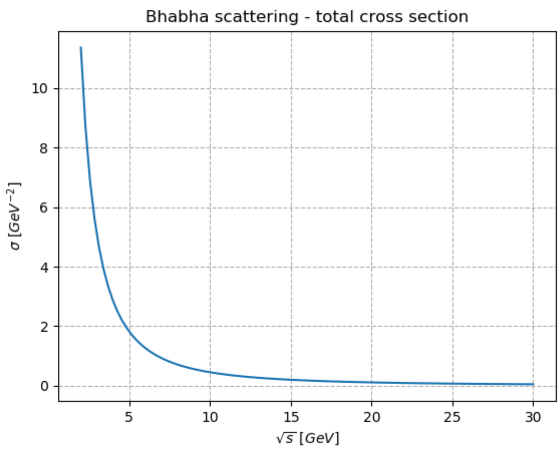


Figure 3: Energy dependence of total cross section for electron-positron scattering

The behaviour of the total cross section is as expected $\propto E^{-2}$, since the only energy dependence is in the prefactor of equation 1, that goes as $1/s$.
Due to the fact that the differential cross section diverges for small angles, the result is strongly dependent on the chosen cut off, i.e. the effective (small) scat-

tering angles that an hypothetical detector is sensitive to.

For $\sqrt{s} = 14$GeV the total cross section is

$$\sigma = 0.232\,\mathrm{GeV}^{-2} = 90.3\,\mu\mathrm{b}$$

being $1\mathrm{GeV}^{-2} = 0.3894\,\mathrm{pb}$. For a beam with integrated luminosity $\mathcal{L} = 10\,\mathrm{pb}^{-1}$ and a 50% acceptance efficiency, the expected number of events detected is

$$N = \sigma\,\mathcal{L}\,\times \text{efficiency} = 4.5 \times 10^8 \text{ events}$$

assuming that our setup detects events up to very small scattering angles.

The total cross section for the muon pair production process $e^+e^- \to \mu^+\mu^-$ is shown in figure 4. The behaviour is the same as for Bhabha scattering for the same argument as before, however for this process the total cross section is much smaller: the differential cross section does not diverge for any angle (see figure 2a) and there was no need to introduce a cut-off in the integral. This implies a much smaller contribution to the total cross section for small angles.

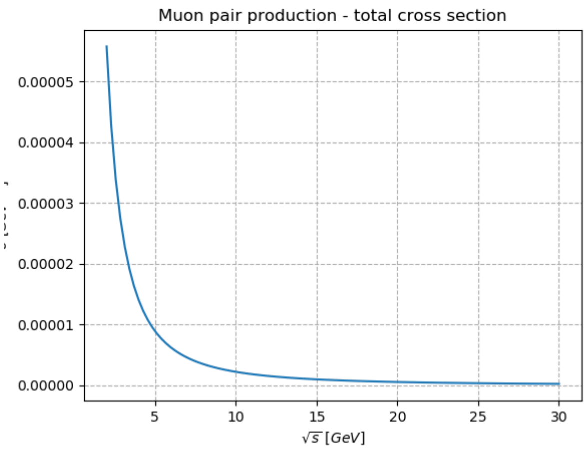


Figure 4: Energy dependence of total cross section for electron-positron scattering

In fact, for this process the total cross section at $\sqrt{s} = 14$GeV is

$$\sigma = 1.14 \times 10^{-6}\,\mathrm{GeV}^{-2} = 443\,\mathrm{pb}$$

3 Comparison with experimental results

A comparison with the number of events for Bhabha scattering and muon pair production has been made from [2].
In the article is considered a center mass energy of 29 GeV with integrated luminosity $\mathcal{L} = 19.6\mathrm{pb}^{-1}$ with angular acceptance $|\cos\theta| < 0.55$ and $0.75 < |\cos\theta| < 0.85$. With a 99% events sensitivity and a 12% dead time, a 88%

acceptance sensitivity has been considered. The expected number of events is therefore:

$$\begin{aligned} 10^5\,\text{events} \qquad & \text{for } e^+e^- \to e^+e^- \\ 200\,\text{events} \qquad & \text{for } e^+e^- \to \mu^+\mu^- \end{aligned}$$

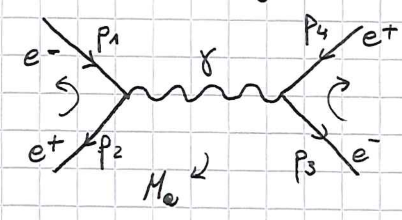
The sample in the article consists in 8915 e^+e^- events and 811 $\mu^+\mu^-$ events. While the first one can be compatible with our estimation, the latter is 4 times higher. This may be due to a too shallow analysis of the detector system from my part.

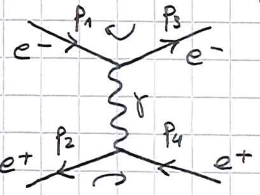
References

[1] `http://bolvan.ph.utexas.edu/~vadim/classes/2008f.homeworks/traceology.pdf`

[2] D. Bender et al., *Tests of QED at 29 GeV center-of-mass energy*, Phys. Rev. D 30, 515 – Published 1 August 1984

1. Lowest order diagrams of process $e^+e^- \to e^+e^-$ & matrix elements





$\to M_s$

$$M_a = -\frac{e^2}{q^2}[\bar{v}(p_2)\gamma^\mu u(p_1)]g_{\mu\nu}[\bar{u}(p_3)\gamma^\nu v(p_4)]$$

$$= -\frac{e^2}{q^2}[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\mu v(p_4)]$$

$$M_s = -\frac{e^2}{q^2}[\bar{u}(p_3)\gamma^\mu u(p_1)][\bar{v}(p_4)\gamma_\mu v(p_2)]$$

2.

I. $\sum_{spins} |M_a|^2 = Tr[(\not{p}_2 - m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu] Tr[(\not{p}_3 + m)\gamma_\mu(\not{p}_4 - m)\gamma_\nu]\frac{e^4}{q^4}$

↳ Since we are interested in $\sqrt{s} = 14\,GeV$, $m \ll E$, then we can neglect masses of e^- and e^+ (ultrarelativistic limit)

$$\simeq Tr[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu] Tr[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu]$$

$$Tr[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu] = p_{2\alpha}p_{1\beta} Tr[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu]$$

$$= p_{2\alpha}p_{1\beta}\, 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta})$$

$$= 4[p_2^\mu p_1^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\nu p_1^\mu]$$

$$Tr[\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu] = p_{3}^{\alpha}p_{4}^{\beta} Tr[\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu]$$

$$= 4[p_{3\mu}p_{4\nu} - g_{\mu\nu}(p_3\cdot p_4) + p_3^\nu p_4^\mu]$$

$$\Rightarrow \sum_{spins}|M_a|^2 = 16[(p_2\cdot p_3)(p_1\cdot p_4) - \cancel{(p_1\cdot p_2)(p_3\cdot p_4)} + (p_1\cdot p_3)(p_2\cdot p_4) + - \cancel{(p_3\cdot p_4)(p_1\cdot p_2)} + 4\cancel{(p_1\cdot p_2)(p_3\cdot p_4)} - \cancel{(p_3\cdot p_4)(p_1\cdot p_2)} + + (p_2\cdot p_4)(p_1\cdot p_3) - \cancel{(p_1\cdot p_2)(p_3\cdot p_4)} + (p_2\cdot p_3)(p_1\cdot p_4)]\frac{e^4}{q^4}$$

$$= 32[(p_2\cdot p_3)(p_1\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4)]\frac{e^4}{q^4}$$

$$\langle|M_a|^2\rangle = \frac{1}{4}\sum_{spins}|M_a|^2 = 8[(p_2\cdot p_3)(p_1\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4)]\frac{e^4}{q^4}$$

$$= \frac{8e^4}{s^2}[(p_2\cdot p_3)(p_1\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4)]$$

II. $\sum_{\text{spin}} |M_s|^2 = \mathrm{Tr}\left[(\not{p}_3 + m)\gamma^\mu(\not{p}_1 + m)\gamma^\nu\right] \mathrm{Tr}\left[(\not{p}_4 - m)\gamma_\mu(\not{p}_2 - m)\gamma_\nu\right] \frac{e^4}{q^4}$

$\leftarrow$ ultrarelativistic limit $m \ll E$

$$= \mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\right]\mathrm{Tr}\left[\not{p}_4\gamma_\mu\not{p}_2\gamma_\nu\right]\frac{e^4}{q^4}$$

$$\mathrm{Tr}\left[\not{p}_3\gamma^\mu\not{p}_1\gamma^\nu\right] = 4\left[p_3^\mu p_1^\nu - g^{\mu\nu}(p_1\cdot p_3) + p_3^\nu p_1^\mu\right]$$

$$\mathrm{Tr}\left[\not{p}_4\gamma_\mu\not{p}_2\gamma_\nu\right] = 4\left[p_{4\mu}p_{2\nu} - g^{\mu\nu}(p_4\cdot p_2) + p_{4\nu}p_{2\mu}\right]$$

$$\Rightarrow \sum_{\text{spin}}|M_s|^2 = 16\Big[(p_3\cdot p_4)(p_1\cdot p_2) - \cancel{(p_3\cdot p_1)(p_4\cdot p_2)} + (p_1\cdot p_4)(p_3\cdot p_2) +$$
$$- \cancel{(p_1\cdot p_3)(p_4\cdot p_2)} + 4\cancel{(p_1\cdot p_3)(p_4\cdot p_2)} - \cancel{(p_1\cdot p_3)(p_4\cdot p_2)} +$$
$$+ (p_3\cdot p_2)(p_4\cdot p_1) - \cancel{(p_3\cdot p_1)(p_4\cdot p_2)} + (p_4\cdot p_3)(p_1\cdot p_2)\Big]\frac{e^4}{q^4}$$
$$= 32\left[(p_3\cdot p_4)(p_1\cdot p_2) + (p_1\cdot p_4)(p_3\cdot p_2)\right]\frac{e^4}{q^4} \quad \rightarrow q^2 = t$$

III. $\langle M_a^* M_s\rangle =$

$$= \frac{e^4}{4st}\sum_{rspt}\left[\bar{v}^r(p_2)\gamma^\mu u^s(p_1)\right]^\dagger\left[\bar{u}^p(p_3)\gamma_\mu v^t(p_4)\right]^\dagger\left[\bar{u}^p(p_3)\gamma^\nu u^s(p_1)\right]\left[\bar{v}^r(p_2)\gamma_\nu v^t(p_4)\right]$$

$$= \frac{e^4}{4st}\sum_{rspt}\bar{u}^s(p_1)_a\gamma^\mu_{ab}v^r(p_2)_b\,\bar{v}^t(p_4)_i\gamma_{\mu_{ij}}u^p(p_3)_j\,\bar{u}^p(p_3)_k\gamma^\nu_{kl}u^s(p_1)_l\,\bar{v}^r(p_2)_m\gamma_{\nu_{mn}}v^t(p_4)_n$$

$$= \frac{e^4}{4st}\sum_r v^r_b\bar{v}^r_m(p_2)\sum_s u^s_l\bar{u}^s_a(p_1)\sum_p u_j(p_3)\bar{u}_k(p_3)\sum_t v^t_n\bar{v}^t_i(p_4)\,\gamma^\mu_{ab}\gamma_{\mu_{ij}}\gamma^\nu_{kl}\gamma_{\nu_{mn}}$$

$\leftarrow$ completeness relations: $\sum_s u^s\bar{u}^s(p) = \not{p} + m$, $\sum_s v^s\bar{v}^s = \not{p} - m$

$$= \frac{e^4}{4st}(\not{p}_2 - m)_{bm}(\not{p}_1 + m)_{la}(\not{p}_3 + m)_{jk}(\not{p}_4 - m)_{ni}\gamma^\mu_{ab}\gamma_{\mu_{ij}}\gamma^\nu_{kl}\gamma_{\nu_{mn}}$$

$$= \frac{e^4}{4st}(\not{p}_2 - m)_{bm}\gamma_{\nu_{mn}}(\not{p}_4 - m)_{ni}\gamma_{\mu_{ij}}(\not{p}_3 + m)_{jk}\gamma^\nu_{kl}(\not{p}_1 + m)_{la}\gamma^\mu_{ab}$$

$\leftarrow$ ultrarelativistic limit

$$= \frac{e^4}{4st}\mathrm{Tr}\left[\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu\not{p}_3\gamma^\nu\not{p}_1\gamma^\mu\right]$$

Permuting matrices, cyclically

$$= \frac{e^4}{4st}\mathrm{Tr}\Big[\not{p}_1\underbrace{\gamma^\mu\not{p}_2\gamma_\nu\not{p}_4\gamma_\mu}\not{p}_3\gamma^\nu\Big]$$

$$\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$$
$$\Rightarrow = -2\not{p}_4\gamma_\nu\not{p}_2$$

$$= -\frac{e^4}{2st}\mathrm{Tr}\left[\not{p}_1\not{p}_4\gamma_\nu\not{p}_2\not{p}_3\gamma^\nu\right] \overset{\text{non-cyclical perm.}}{=} \frac{e^4}{2st}\mathrm{Tr}\Big[\not{p}_1\not{p}_4\overbrace{\gamma^\nu\not{p}_2\not{p}_3\gamma_\nu}^{(p_2\cdot p_3)}\Big]$$

$$= \frac{e^4}{2st}(p_2\cdot p_3)\,\mathrm{Tr}(\not{p}_1\not{p}_4) = \frac{e^4}{2st}(p_2\cdot p_3)\underbrace{p_{1\alpha}\,p_{4\beta}\,\mathrm{Tr}(\gamma^\alpha\gamma^\beta)}_{4(p_1\cdot p_4)}, \quad \mathrm{Tr}(\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta}$$

$$= \frac{2e^4}{st}(p_2\cdot p_3)(p_1\cdot p_4)$$

Total matrix element:

$$|\mathcal{M}_{TOT}|^2 = |\mathcal{M}_a|^2 + |\mathcal{M}_s^2| \mp 2\,\mathrm{Re}\{\mathcal{M}_a^* \mathcal{M}_s\}$$

Average $\langle\ \rangle$ is linear:

$$\langle|\mathcal{M}_{TOT}|^2\rangle = \langle|\mathcal{M}_a|^2\rangle + \langle|\mathcal{M}_s|^2\rangle + 2\,\mathrm{Re}\{\langle\mathcal{M}_a^* \mathcal{M}_s\rangle\}$$

$$\overset{!}{=} 4e^4\left[\frac{2(p_2\cdot p_3)(p_1\cdot p_4) + 2(p_1\cdot p_3)(p_2\cdot p_4)}{s^2} + 2\,\frac{(p_3\cdot p_4)(p_1\cdot p_2) + (p_1\cdot p_4)(p_3\cdot p_2)}{t^2} + \mp \frac{(p_2\cdot p_3)(p_1\cdot p_4)}{st}\right]$$

where $s = (p_1 + p_2)^2 = (p_3 + p_4)^2$
$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$