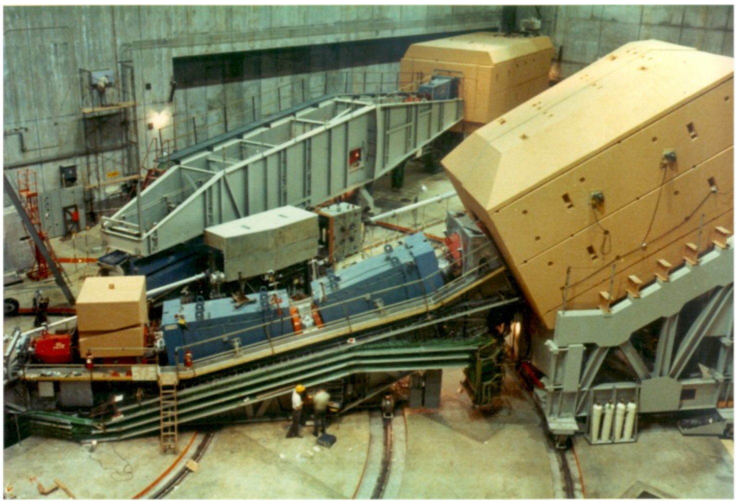
Particle Physics

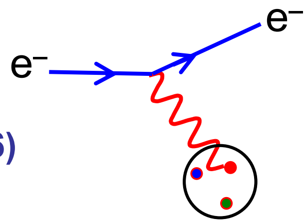
FYS4555 – UiO – 2018
Based on Prof Mark Thomson's slides



Handout 5 : Electron-Proton Elastic Scattering

Electron-Proton Scattering

- In this handout aiming towards a study of electron-proton scattering as a probe of the structure of the proton

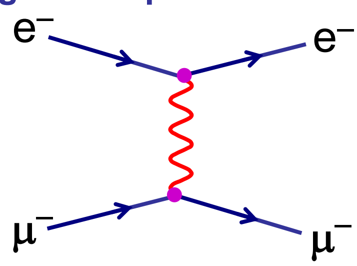


- Two main topics:
  - $e^-p \to e^-p$ elastic scattering (this handout)
  - $e^-p \to e^-X$ deep inelastic scattering (handout 6)
- But first consider scattering from a point-like particle e.g.

  $e^-\mu^- \to e^-\mu^-$

  i.e. the QED part of

  $(e^-q \to e^-q)$



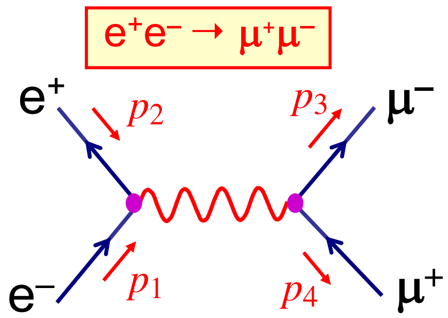
- Two ways to proceed:
  - perform QED calculation from scratch (Problem 6.7)

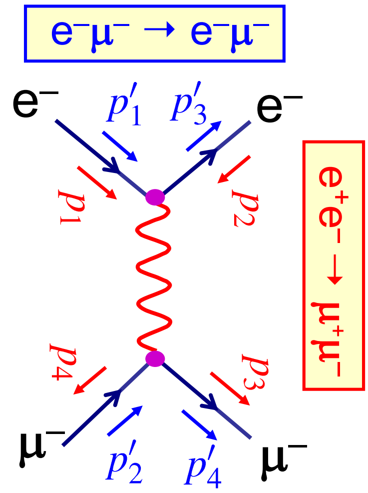
$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3)\right] \qquad (1)$$

  - take results from $e^+e^- \to \mu^+\mu^-$ and use "Crossing Symmetry" to obtain the matrix element for $e^-\mu^- \to e^-\mu^-$ (Extra I)

Extra I : Crossing Symmetry

★ Having derived the Lorentz invariant matrix element for $e^+e^- \to \mu^+\mu^-$ "rotate" the diagram to correspond to $e^-\mu^- \to e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !





★ The transformation:

$$p_1 \to p'_1;\; p_2 \to -p'_3;\; p_3 \to p'_4;\; p_4 \to -p'_2$$

Changes the spin averaged matrix element for

$e^-e^+ \to \mu^-\mu^+$ ($p_1\; p_2 \quad p_3\; p_4$) → $e^-\mu^- \to e^-\mu^-$ ($p'_1\; p'_2 \quad p'_3\; p'_4$)

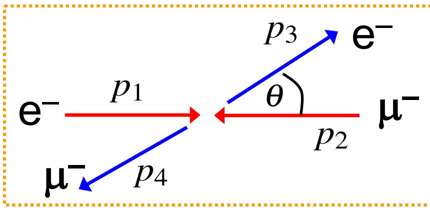
•Take ME for $e^+e^- \to \mu^+\mu^-$ (Handout 4) and apply crossing symmetry:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2} \quad \rightarrow \quad \boxed{\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p'_1.p'_4)^2 + (p'_1.p'_2)^2}{(p'_1.p'_3)^2}} \quad (1)$$

$$\equiv 2e^4 \left(\frac{t^2+u^2}{s^2} \right) \quad \rightarrow \quad \equiv 2e^4 \left(\frac{s^2+u^2}{t^2} \right)$$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2} \quad \text{(2)}$$

$$\equiv 2e^4 \left(\frac{s^2+u^2}{t^2} \right)$$



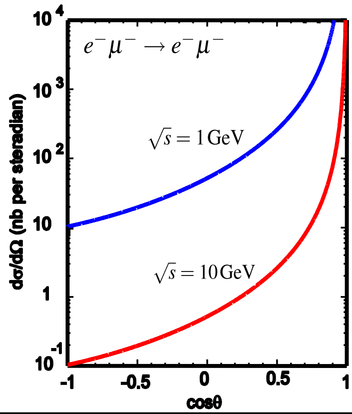
- Work in the C.o.M:

$$p_1 = (E,0,0,E) \qquad p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

giving $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{E^4(1+\cos\theta)^2 + 4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle |M_{fi}|^2 \rangle = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$



- The denominator arises from the propagator $-ig_{\mu\nu}/q^2$

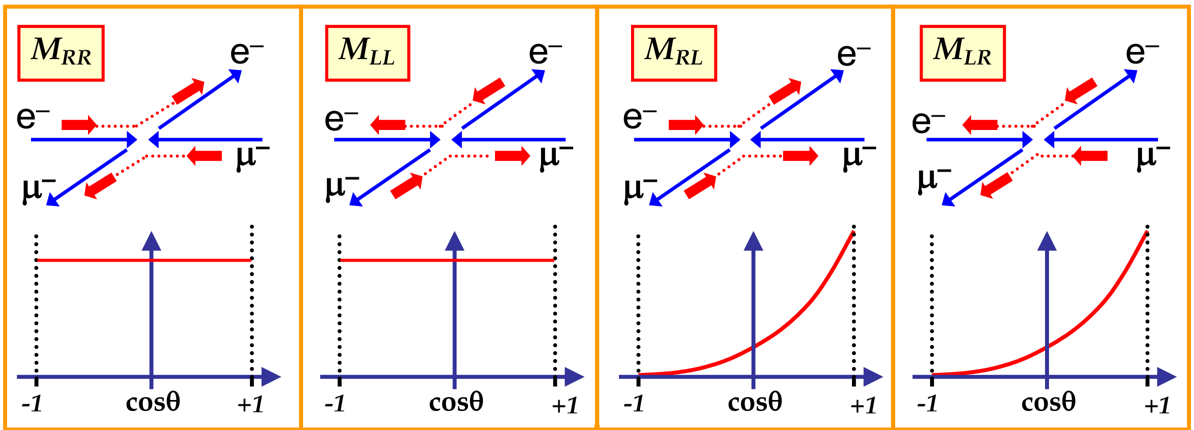
here $q^2 = (p_1 - p_3)^2 = E^2(1-\cos\theta)$

as $q^2 \to 0$ the cross section tends to infinity.

- What about the angular dependence of the numerator ?

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

- The factor $1+\frac{1}{4}(1+\cos\theta)^2$ reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



$S_z = 0$

$$\frac{d\sigma}{d\Omega} \propto 1$$

i.e. no preferred polar angle

$S_z = +1 \qquad S_z = -1$

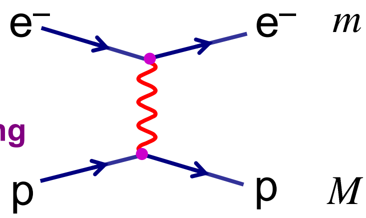
$$\frac{d\sigma}{d\Omega} \propto \frac{1}{4}(1+\cos\theta)^2$$

spin 1 rotation again

- The cross section calculated above is appropriate for the scattering of two spin half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2}$$

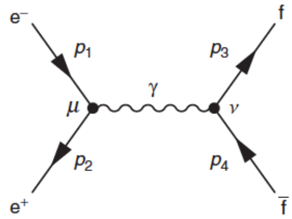
- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton (handout 6).
- Before doing so we will consider the scattering of electrons from the composite proton - **i.e. how do we know the proton isn't fundamental "point-like" particle ?**
- In this discussion we will not be able to use the relativistic limit and require the general expression for the matrix element (refer to the calculation made using completeness relations and traces formalism, where we did not neglect M):



$$\boxed{\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2\right]} \quad (3)$$

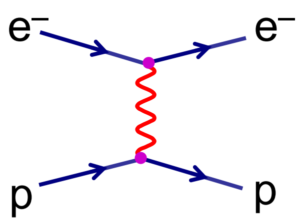
- Neglecting m (not M=m_f) for $e^+e^- \rightarrow$ f fbar

$$\langle |\mathcal{M}_{fi}|^2 \rangle = 2\frac{Q_f^2 e^4}{(p_1 \cdot p_2)^2}\left[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m_f^2 (p_1 \cdot p_2)\right]$$



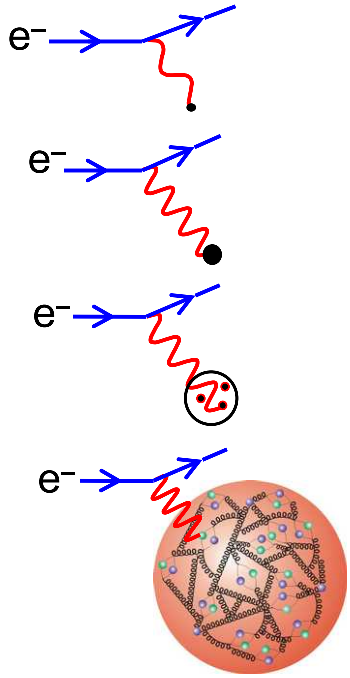
➔ Scattering not neglecting m and M and using crossing symmetry

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2\right]$$



Probing the Structure of the Proton

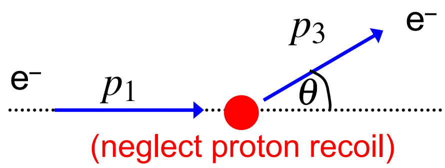
★ In e⁻p → e⁻p scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength



- At **very low** electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a "point-like" **spin-less object**
- At **low** electron energies $\lambda \sim r_p$: the scattering is equivalent to that from an extended charged object
- At **high** electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- At **very high** electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.

Rutherford Scattering Revisited

★ Rutherford scattering is the **low energy limit** where the recoil of the proton can be neglected and the **electron is non-relativistic**



- Start from RH and LH Helicity particle spinors

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad \begin{aligned} N &= \sqrt{E+m}; \\ s &= \sin(\theta/2); \quad c = \cos(\theta/2) \end{aligned}$$

- Now write in terms of: $$\alpha = \frac{|\vec{p}|}{E+m_e}$$

Non-relativistic limit: $\alpha \to 0$
Ultra-relativistic limit: $\alpha \to 1$

$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \alpha c \\ \alpha e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \alpha s \\ -\alpha e^{i\phi}c \end{pmatrix}$$

and the possible initial and final state electron spinors are:

$$u_\uparrow(p_1) = N_e\begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e\begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e\begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

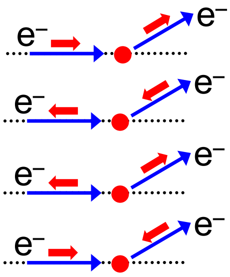
- **Make use of results in handout 4**

$$
\begin{aligned}
\overline{\psi}\gamma^0\phi &= \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\overline{\psi}\gamma^1\phi &= \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\overline{\psi}\gamma^2\phi &= \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\overline{\psi}\gamma^3\phi &= \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}
$$

- To calculate the electron and proton currents for each of the 4 helicity combinations per current

+2iαs

- **Consider all four possible electron currents, i.e. Helicities R→R, L→L, L→R, R→L**



$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad \textbf{(4)}$$

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad \textbf{(5)}$$

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \quad \textbf{(6)}$$

$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \quad \textbf{(7)}$$

- **In the relativistic limit ($\alpha = 1$), i.e. $E \gg m$**
  (6) and (7) are identically zero; only R→R and L→L combinations non-zero
- **In the non-relativistic limit, $|\vec{p}| \ll E$ we have $\alpha = 0$**

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = \overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2m_e)[c, 0, 0, 0]$$

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = -\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (2m_e)[s, 0, 0, 0]$$

$$\boxed{\textbf{All four electron helicity combinations have non-zero Matrix Element}}$$

i.e. Helicity eigenstates ≠ Chirality eigenstates

- The initial and final state **proton** spinors (assuming no recoil) are:

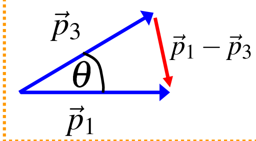
$$u_\uparrow(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_\downarrow(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Solutions of Dirac equation for a particle at rest

giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p(1,0,0,0)$$
$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

- The spin-averaged ME summing over the 8 allowed helicity states



$$\langle|M_{fi}^2|\rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2 m_e^2)\underline{(4c^2+4s^2)} = \frac{16M_p^2 m_e^2 e^4}{q^4}$$

where $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2 \sin^2(\theta/2)$

$$\langle|M_{fi}^2|\rangle = \frac{M_p^2 m_e^2 e^4}{|\vec{p}|^4 \sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- The formula for the differential cross-section in the lab. frame was derived in handout 1:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad \textbf{(8)}$$

- Here the electron is non-relativistic so $E \sim m_e \ll M_p$ and we can neglect E_1 in the denominator of equation (8)

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

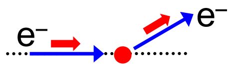
- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4\theta/2}} \qquad \textbf{(9)}$$

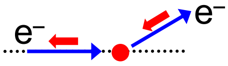
★ This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$, **without any consideration of the interaction due to the intrinsic magnetic moments** of the electron or proton. From this we can conclude, that in this non-relativistic limit only the interaction between the electric charges of the particles matters.

+2iαs

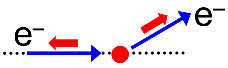
- Consider all four possible electron currents, i.e. Helicities **R→R, L→L, L→R, R→L**



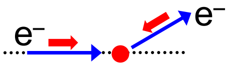
$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (4)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (5)$$



$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \quad (6)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \quad (7)$$

- In the relativistic limit ($\alpha = 1$), i.e. $E \gg m$
  (6) and (7) are identically zero; only R→R and L→L combinations non-zero

- The initial state proton

$$u_\uparrow(p_2) = \sqrt{2m_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \equiv u_1(p_2) \quad \text{and} \quad u_\downarrow(p_2) = \sqrt{2m_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \equiv u_2(p_2)$$

Solutions of Dirac equation for a particle at rest

- The final state proton

$$u_\uparrow(p_4) \approx \sqrt{2m_p}\begin{pmatrix}c_\eta\\-s_\eta\\0\\0\end{pmatrix} \quad \text{and} \quad u_\downarrow(p_4) \approx \sqrt{2m_p}\begin{pmatrix}-s_\eta\\-c_\eta\\0\\0\end{pmatrix}$$

- The proton currents are thus:

$$j_{p\uparrow\uparrow} = -j_{p\downarrow\downarrow} = 2m_p\left[c_\eta, 0, 0, 0\right] \quad \text{and} \quad j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = -2m_p\left[s_\eta, 0, 0, 0\right]$$

- The QED matrix element $\mathcal{M}_{fi} = \frac{e^2}{q^2} j_e \cdot j_p$ leads to

- the spin-averaged ME over all allowed helicity states

$$\begin{aligned}\langle|\mathcal{M}_{fi}^2|\rangle &= \frac{1}{4}\sum|\mathcal{M}_{fi}^2| \\ &= \frac{1}{4}\frac{e^4}{q^4} \times 4m_p^2(E+m_e)^2 \cdot \left[c_\eta^2 + s_\eta^2\right] \cdot \left[4(1+\kappa^2)^2c^2 + 4(1-\kappa^2)^2s^2\right] \\ &= \frac{4m_p^2m_e^2e^4(\gamma_e+1)^2}{q^4}\left[(1-\kappa^2)^2 + 4\kappa^2c^2\right],\end{aligned}$$

$$\kappa = \frac{\beta_e\gamma_e}{\gamma_e+1} \quad \text{and} \quad (1-\beta_e^2)\gamma_e^2 = 1 \quad \Rightarrow \quad \boxed{\langle|\mathcal{M}_{fi}^2|\rangle = \frac{16m_p^2m_e^2e^4}{q^4}\left[1 + \beta_e^2\gamma_e^2\cos^2\frac{\theta}{2}\right]}$$

- **Elastic scattering process**

For the elastic scattering process where the recoil of the proton can be neglected, the energies and momenta of the initial- and final-state electrons are $E_1 = E_3 = E$ and $\mathrm{p}_1 = \mathrm{p}_3 = \mathrm{p}$, and hence

$$q^2 = (0, \mathbf{p}_1 - \mathbf{p}_3)^2 = -2\mathrm{p}^2(1-\cos\theta) = -4\mathrm{p}^2\sin^2(\theta/2).$$

Substituting this expression for q^2 into (7.8) gives

$$\langle|\mathcal{M}_{fi}^2|\rangle = \frac{m_\mathrm{p}^2 m_\mathrm{e}^2 e^4}{\mathrm{p}^4\sin^4(\theta/2)}\left[1+\beta_\mathrm{e}^2\gamma_\mathrm{e}^2\cos^2\frac{\theta}{2}\right]. \tag{7.9}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2|M_{fi}|^2$$

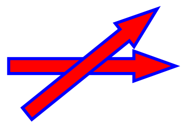
The Mott Scattering Cross Section

- For Rutherford scattering we are in the limit where the target recoil is neglected and the scattered particle is non-relativistic $E_K \ll m_e$
- The limit where the target recoil is neglected and the scattered particle is relativistic (i.e. just neglect the electron mass) is called Mott Scattering

Relativistic ➡ Electron "helicity conserved"

- Cross section:

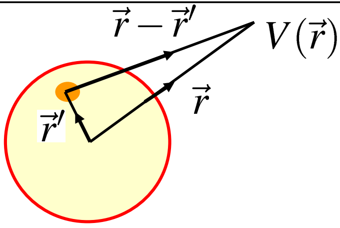
$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}} = \underbrace{\frac{\alpha^2}{4E^2\sin^4\theta/2}}_{\text{Rutherford formula with } E_K = E\ (E\gg m_e)}\underbrace{\cos^2\frac{\theta}{2}}_{\text{Overlap between initial/final state electron wave-functions. Just QM of spin ½}} \quad (10)$$



★ NOTE: we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$. The interaction is ELECTRIC rather than magnetic (spin-spin) in nature.

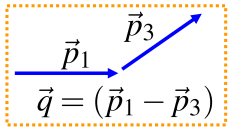
★ Still haven't taken into account the charge distribution of the proton…..

Form Factors



- Consider the scattering of an electron in the static potential due to an extended charge distribution.
- The potential at $\vec{r}$ from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r})\mathrm{d}^3\vec{r} = 1$$



- In first order perturbation theory the matrix element is given by:

$$M_{fi} = \langle\psi_f|V(\vec{r})|\psi_i\rangle = \int e^{-i\vec{p}_3.\vec{r}} V(\vec{r}) e^{i\vec{p}_1.\vec{r}} \mathrm{d}^3\vec{r}$$

$$= \int\int e^{i\vec{q}.\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r} = \int\int e^{i\vec{q}.(\vec{r}-\vec{r}')} e^{i\vec{q}.\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r}$$

- Fix $\vec{r}'$ and integrate over $\mathrm{d}^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

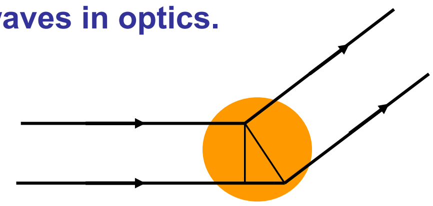
$$M_{fi} = \int e^{i\vec{q}.\vec{R}} \frac{Q}{4\pi|\vec{R}|} \mathrm{d}^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}.\vec{r}'} \mathrm{d}^3\vec{r}' = (M_{fi})_{point} F(\vec{q}^2)$$

★ The resulting matrix element is equivalent to the matrix element for scattering from a **point source** multiplied by the **form factor**

$$\boxed{F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} \mathrm{d}^3\vec{r}}$$

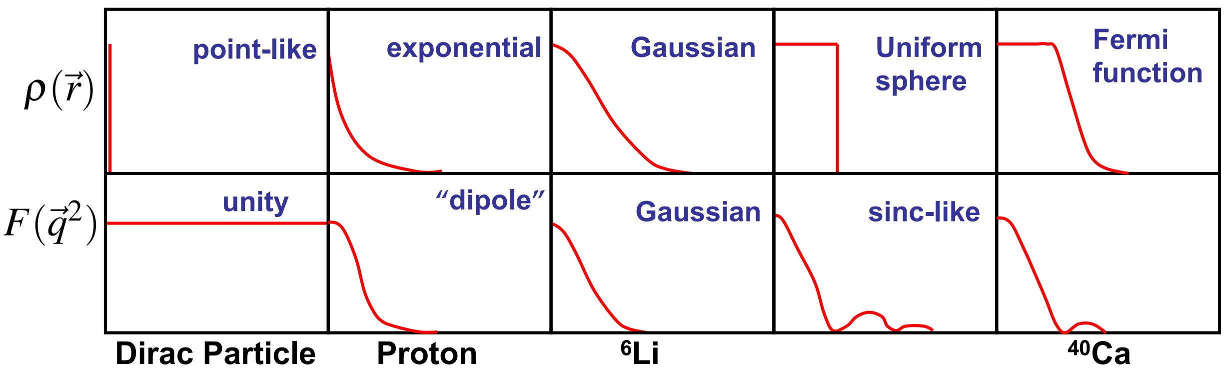
$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}|F(\vec{q}^2)|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.



- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space". If wavelength is long compared to size all waves in phase and $F(\vec{q}^2) = 1$

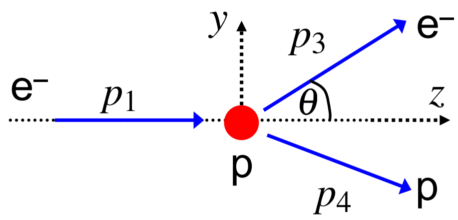
For example:



- **NOTE that for a point charge the form factor is unity.**

Point-like Electron-Proton Elastic Scattering

- So far have only considered the case where the proton does not recoil...
  For $E_1 \gg m_e$ the general case is



$$\begin{aligned} p_1 &= (E_1, 0, 0, E_1) \\ p_2 &= (M, 0, 0, 0) \\ \boxed{p_3} &\boxed{= (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)} \\ p_4 &= (E_4, \vec{p}_4) \end{aligned}$$

- From Eqn. (2) with $m = m_e = 0$ the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \quad \text{(11)}$$

- Experimentally observe scattered electron so eliminate p_4
- The scalar products not involving p_4 are:

$$p_1.p_2 = E_1 M \qquad p_1.p_3 = E_1 E_3 (1 - \cos\theta) \qquad p_2.p_3 = E_3 M$$

- From momentum conservation can eliminate p_4 : $\quad p_4 = p_1 + p_2 - p_3$

$$p_3.p_4 = p_3.p_1 + p_3.p_2 - \cancel{p_3.p_3} = E_1 E_3 (1 - \cos\theta) + E_3 M$$
$$p_1.p_4 = \boxed{\cancel{p_1.p_1}} + p_1.p_2 - p_1.p_3 = E_1 M - E_1 E_3 (1 - \cos\theta)$$

$$\boxed{p_1.p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0}$$ i.e. neglect m_e

- Substituting these scalar products in Eqn. (11) gives

$$\begin{aligned} \langle |M_{fi}|^2 \rangle &= \frac{8e^4}{(p_1 - p_3)^4} M E_1 E_3 \left[(E_1 - E_3)(1 - \cos\theta) + M(1 + \cos\theta) \right] \\ &= \frac{8e^4}{(p_1 - p_3)^4} 2 M E_1 E_3 \left[(E_1 - E_3) \sin^2(\theta/2) + M \cos^2(\theta/2) \right] \quad \text{(12)} \end{aligned}$$

- Now obtain expressions for $q^4 = (p_1 - p_3)^4$ and $(E_1 - E_3)$

$$q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = -2E_1 E_3 (1 - \cos\theta) \quad \text{(13)}$$
$$= -4E_1 E_3 \sin^2\theta/2 \quad \text{(14)}$$

NOTE: $q^2 < 0$ $\boxed{\text{Space-like}}$

- For $(E_1 - E_3)$ start from

$$q.p_2 = (p_1 - p_3).p_2 = M(E_1 - E_3)$$

and use $\qquad q = (p_1 - p_3) = (p_4 - p_2)$

$$\begin{aligned} (q + p_2)^2 &= p_4^2 \\ q^2 + p_2^2 + 2q.p_2 &= p_4^2 \\ q^2 + M^2 + 2q.p_2 &= M^2 \\ \Rightarrow \quad q.p_2 &= -q^2/2 \end{aligned}$$

- Hence the energy transferred to the proton:

$$\boxed{E_1 - E_3 = -\frac{q^2}{2M}} \qquad \textbf{(15)}$$

Because q^2 is always negative $E_1 - E_3 > 0$ and the scattered electron is always lower in energy than the incoming electron

- Combining equations (11), (13) and (14):

$$\begin{aligned}\langle |M_{fi}|^2 \rangle &= \frac{8e^4}{16E_1^2E_3^2\sin^4\theta/2} 2ME_1E_3\left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2\right] \\ &= \frac{M^2e^4}{E_1E_3\sin^4\theta/2}\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right]\end{aligned}$$

- For $E \gg m_e$ we have (see handout 1)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \qquad \alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)} \qquad \textbf{(16)}$$

Interpretation

- So far have derived the differential cross-section for e⁻p → e⁻p **elastic** scattering assuming point-like Dirac spin ½ particles. How should we interpret the equation?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

- Compare with $$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}$$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin ½ electrons in a fixed **electro-static** potential. Here the term E_3/E_1 is due to the proton recoil.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \underbrace{\frac{q^2}{2M^2}\sin^2\theta/2}\right)$$

- the new term: $\propto \sin^2\frac{\theta}{2}$ ⟷ Magnetic interaction : due to the spin-spin interaction

- The above differential cross-section depends on a single parameter. For an electron scattering angle θ, both q^2 and the energy, E_3, are fixed by kinematics

• Equating (13) and (15) $-2M(E_1-E_3) = -2E_1E_3(1-\cos\theta)$ $\Rightarrow \frac{E_3}{E_1} = \frac{M}{M+E_1(1-\cos\theta)}$	• Substituting back into (13): $\Rightarrow q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M+E_1(1-\cos\theta)}$

- e.g. $e^-p \to e^-p$ at E_{beam}= 529.5 MeV, look at scattered electrons at θ= 75°

For elastic scattering expect:

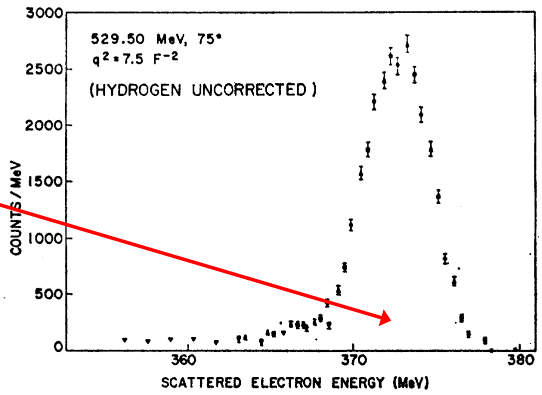
$$E_3 = \frac{ME_1}{M+E_1(1-\cos\theta)}$$

$$E_3 = \frac{938\times 529}{938+529(1-\cos 75^\circ)} = 373\text{ MeV}$$

The energy identifies the scatter as elastic.
Also know squared four-momentum transfer

$$|q^2| = \frac{2\times 938\times 529^2(1-\cos 75^\circ)}{938+529(1-\cos 75^\circ)} = 294\,\text{MeV}^2$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



Elastic Scattering from a Finite Size Proton

★ **In general** the finite size of the proton can be accounted for by introducing **two structure functions**. One related to the **charge distribution** in the proton, $G_E(q^2)$ and the other related to the distribution of the **magnetic moment** of the proton, $G_M(q^2)$

- It can be shown that equation (16) generalizes to the **ROSENBLUTH FORMULA.**

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right)}$$

with the Lorentz Invariant quantity: $\boxed{\tau = -\frac{q^2}{4M^2} > 0}$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions.

But $q^2 = (E_1-E_3)^2 - \vec{q}^2$ and from eq (15) obtain

$$\Rightarrow -\vec{q}^2 = q^2\left[1-\left(\frac{q}{2M}\right)^2\right]$$

So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

- Hence in the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}}\rho(\vec{r})\mathrm{d}^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}}\mu(\vec{r})\mathrm{d}^3\vec{r}$$

- Note in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M}\vec{S}$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79\frac{e}{M}\vec{S}$$

So for the proton expect

$$G_E(0) = \int \rho(\vec{r})\mathrm{d}^3\vec{r} = 1 \qquad G_M(0) = \int \mu(\vec{r})\mathrm{d}^3\vec{r} = \mu_p = +2.79$$

- Of course the anomalous magnetic moment of the proton is already evidence that it is not point-like !

Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where $$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

i.e. the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton.

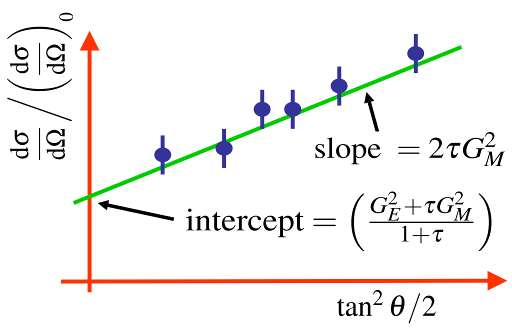
- At very low q^2: $\tau = -q^2/4M^2 \approx 0$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \Big/ \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0 \approx G_E^2(q^2)$$

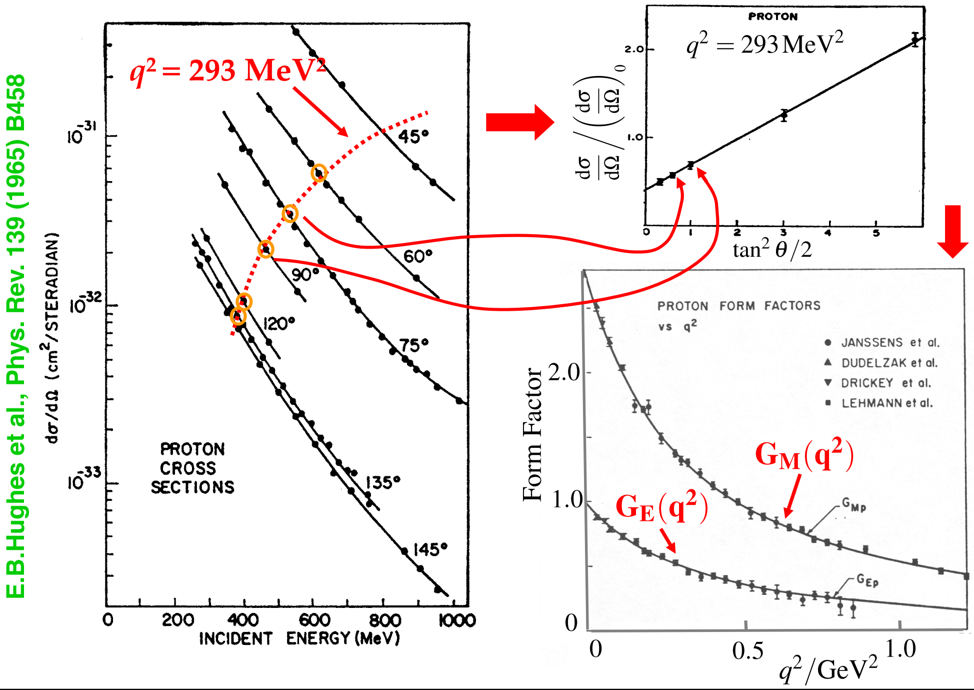
- At high q^2: $\tau \gg 1$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \Big/ \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0 \approx \left(1 + 2\tau\tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$

- In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at **FIXED** q^2



- **EXAMPLE:** $e^-p \rightarrow e^-p$ at E_{beam}= 529.5 MeV
  - Electron beam energies chosen to give certain values of q^2
  - Cross sections measured to 2-3 %



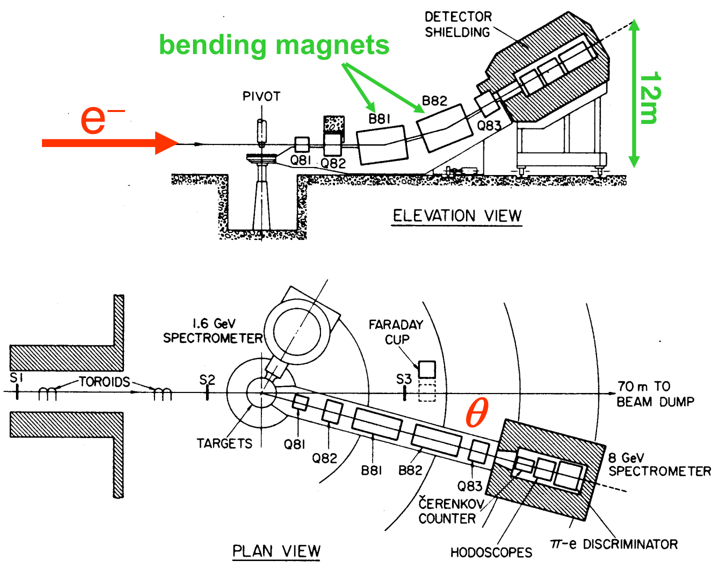
NOTE

Experimentally find $G_M(q^2) = 2.79 G_E(q^2)$, i.e. the electric and and magnetic form factors have same distribution

Higher Energy Electron-Proton Scattering

★ Use electron beam from SLAC LINAC: 5 < E_{beam} < 20 GeV

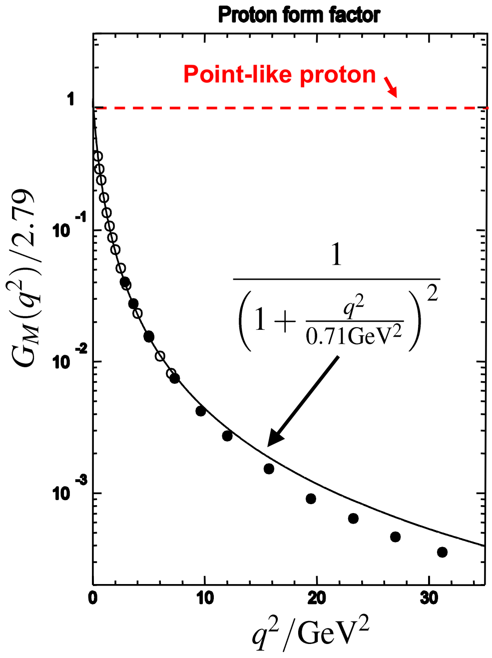
- Detect scattered electrons using the "8 GeV Spectrometer"



High q^2 → Measure $G_M(q^2)$

P.N.Kirk et al., Phys Rev D8 (1973) 63

High q^2 Results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671
A.F.Sill et al., Phys. Rev. D48 (1993) 29

★ Form factor falls rapidly with q^2
- Proton is not point-like
- Good fit to the data with "dipole form":

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2}$$

★ Taking FT find spatial charge and magnetic moment distribution

$$\boxed{\rho(r) \approx \rho_0 e^{-r/a}}$$

with $a \approx 0.24$ fm

- Corresponds to a rms charge radius

$$r_{rms} \approx 0.8 \text{ fm}$$

★ Although suggestive, does not imply proton is composite !

★ **Note:** so far have only considered ELASTIC scattering; Inelastic scattering is the subject of next handout

Summary: Elastic Scattering

★ For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2\sin^4\theta/2}}_{\text{Rutherford}}\underbrace{\frac{E_3}{E_1}}_{\text{Proton recoil}}\left(\underbrace{\cos^2\frac{\theta}{2}}_{\text{Electric/Magnetic scattering}} - \underbrace{\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}}_{\text{Magnetic term due to spin}}\right)$$

★ For elastic scattering of relativistic electrons from an extended proton:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right)}$$

Rosenbluth Formula

★ Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of ~0.8 fm

Solve

- **Problems 7.4, 7.7, 7.8**