Lecture 29 _ LA

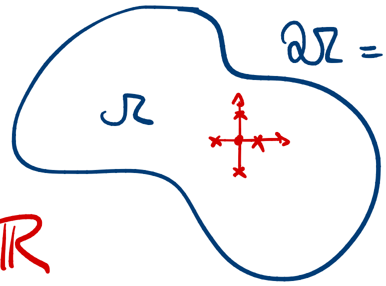
$\boxed{\text{FEM / FDM 2D}}$

14.01.2021

$N=2$

1) $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

$\partial\Omega = \Gamma$

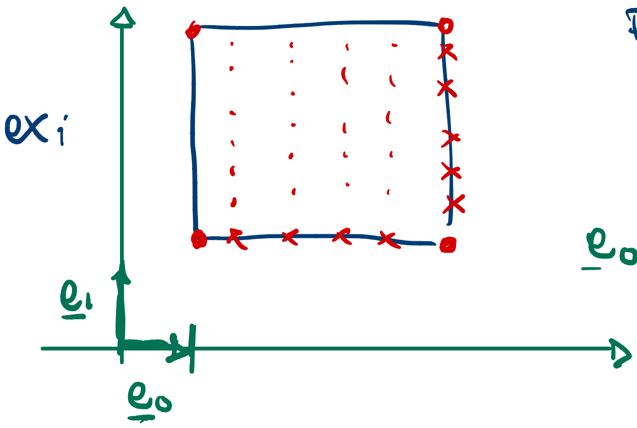


$u : \Omega \longrightarrow \mathbb{R}$

$$-\Delta u := -\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} = \sum_{d=0}^{N-1} -\frac{\partial^2 u}{\partial x_d^2}$$

sometimes: $-\Delta u + u = f$
2) $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$

For FD Ω has to be "simple" and "regular"



Fix a grid of points $X_{ij} \in \Omega \subset \mathbb{R}^2$

$\Omega = [a,b] \times [c,d]$

$e_0 := (1,0) \quad e_1 = (0,1)$

$\underline{x} = \underline{e}_0 x^0 + \underline{e}_1 x^1 = \underline{e}_i x^i$

FD: along e_0: choose n points as $\{\varphi_i\}_{i=0}^{n-1} := [a, a+\dots, \dots\dots, b]$

along e_1: choose m points as $\{\psi_\alpha\}_{\alpha=0}^{m-1} := [c, \dots, \dots, d]$

construct $\underline{X}_{\alpha i} \in \mathbb{R}^2$ ($n \times m$ points) $\quad \alpha \in [0, \dots, m-1]$

$(X_{\alpha i})_d \to d$ comp. of $X_{\alpha i}$ $\quad i \in [0, \dots, n-1]$

$\underline{X}_{\alpha i} := (\varphi_i, \psi_\alpha)$

$\{X_{\alpha i d}\}_{\substack{\alpha=0,\dots,m-1 \\ i=0,\dots,n-1}}$ as $n \times m$ points in dimension 2

$d = 0,1$

$\text{flat}(X) \in \mathbb{R}^{m \cdot n \cdot 2}$

$$X_{\alpha i d} = \begin{cases} \varphi_i & \text{if } d = 0 \\ \psi_\alpha & \text{if } d = 1 \end{cases}$$

$$X \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^N \qquad \text{shape}(X) = (m,n,N)$$

$$X_{\alpha j} \in \mathbb{R}^N \qquad \underline{x}$$

$$X_{\alpha j d} \in \mathbb{R} \qquad x_i$$

$$X_\alpha \in \mathbb{R}^{n \times N}$$

Now we assume we know u on X only (on all $n \cdot m$ points $X_{\alpha i} \in \Omega$)

$$u_{\alpha i} := u(X_{\alpha i})$$

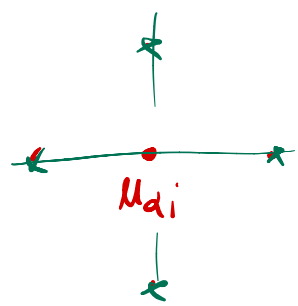
We apply CFD to $u_{\alpha i}$ on both directions:

CFD^2-1D: (along $i \rightsquigarrow \underline{e_0}$)

$$-v_i'' \sim \frac{-v_{i-1} + 2v_i - v_{i+1}}{h^2}$$

CFD^2-1D along $\alpha \rightsquigarrow \underline{e_1}$

$$-w_\alpha'' \sim \frac{-w_{\alpha-1} + 2w_\alpha - w_{\alpha+1}}{h^2}$$



$v_i := u_{\alpha i}$

$w_\alpha := u_{\alpha i}$

$$-\Delta u_{\alpha i} \simeq \frac{1}{h^2}\left[4u_{\alpha i} - u_{\alpha-1\, i} - u_{\alpha+1\, i} - u_{\alpha\, i-1} - u_{\alpha\, i+1}\right]$$

$$-\tilde{\Delta} u_{\alpha i} = f_{\alpha i} = f(X_{\alpha i}) \qquad A \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$$

$$Au = f$$

Construct

$$A_{\alpha i \, \beta J} \quad \text{s.t.} \quad \sum_{\beta, J} A_{\alpha i \, \beta J} \, u_{\beta J} = -\tilde{\Delta} u_{\alpha i}$$

$$A_{\alpha i \, \beta J} = \begin{cases} 4 & \beta = \alpha \quad J = i \\ -1 & \beta = \alpha - 1 \quad J = i \\ -1 & \beta = \alpha + 1 \quad J = i \\ -1 & \beta = \alpha \quad J = i - 1 \\ -1 & \beta = \alpha \quad J = i + 1 \\ 0 & \text{all other cases} \end{cases}$$

$$A_{\alpha i \, \beta J} \, u_{\beta J} = f_{\alpha i}$$

$$\Downarrow \text{ reshape}$$

$$\sum_J \tilde{A}_{IJ} \, \tilde{u}_J = \tilde{f}_J$$

$$I, J \in [0, \ldots, m \cdot n - 1]$$

$$\tilde{u}_J = \sum_i \left(\tilde{A}^{-1}\right)_{JI} \tilde{f}_I$$