1 Thomas algorithm

Consider the *tridiagonal* matrix

$$\mathrm{A} = \begin{bmatrix} a_1 & c_1 & & 0 \\ e_2 & a_2 & \ddots & \\ & \ddots & & c_{n-1} \\ 0 & & e_n & a_n \end{bmatrix}$$

If the $\mathbb{LU}$ decomposition exists, then the factors $\mathbb{L}$ and $\mathbb{U}$ are *bidiagonal*

$$\mathrm{L} = \begin{bmatrix} 1 & & 0 \\ \beta_2 & 1 & \\ & \ddots & \ddots & \\ 0 & & \beta_n & 1 \end{bmatrix}, \quad \mathrm{U} = \begin{bmatrix} \alpha_1 & c_1 & & 0 \\ & \alpha_2 & \ddots & \\ & & \ddots & c_{n-1} \\ 0 & & & \alpha_n \end{bmatrix}$$

The unknown coefficients α_i and β_i can be determined by requiring that the equality $\mathbb{LU} = \mathbb{A}$ holds

$$\alpha_1 = a_1, \quad \beta_i = \frac{e_i}{\alpha_{i-1}}, \quad \alpha_i = a_i - \beta_i c_{i-1}, \quad i = 2, \ldots, n.$$

Moreover, due to the bidiagonal structure of $\mathbb{L}$ and $\mathbb{U}$, a special version of the substitution algorithms can be applied:

$$(\mathbf{Ly} = \mathbf{b}) \quad y_1 = b_1, \quad y_i = b_i - \beta_i y_{i-1}, \quad i = 2, \ldots, n,$$

$$(\mathbf{Ux} = \mathbf{y}) \quad x_n = \frac{y_n}{\alpha_n}, \quad x_i = (y_i - c_i x_{i+1}) / \alpha_i, \quad i = n-1, \ldots, 1$$

Exercise 6.1. Consider the tridiagonal matrix $\mathbb{A} \in \mathbb{R}^{10\times 10}$ defined as

$$\mathbb{A} = \begin{bmatrix} 1 & 11 & & & & \\ 102 & 2 & 12 & & & \\ & 103 & 3 & 13 & & \\ & \cdots & \cdots & \cdots & & \\ & & 109 & 9 & 19 \\ & & & 110 & 10 \end{bmatrix}.$$

Consider also the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ such that $\boldsymbol{x} = [1, 1, \ldots, 1]^T$. Implement the Thomas algorithm and solve the linear system.

2 Condition number

Exercise 6.2. Consider the linear system $\mathbb{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where $0 < \varepsilon \ll 1$.

a. Compute on paper $\mathcal{K}_p(\mathbb{A})$ for $p = 1, 2, \infty$.

b. Consider the perturbation term $\boldsymbol{\delta b} = [0\ 0\ \alpha]^T,\ |\alpha| \ll 1$. What is the perturbation on the solution $\boldsymbol{x}$?

c. Consider instead $\boldsymbol{\delta b} = [\alpha\ 0\ 0]^T,\ |\alpha| \ll 1$. What is now the perturbation on the solution?

d. Verify the obtained results for the case $p = \infty$ with $\varepsilon = 10^{-6}$ and $\alpha = 10^{-12}, 10^{-6}$.

3 Fill-in phenomenon

Exercise 6.3. Consider the following block matrix

$$\mathbb{A}_4 = \left[\begin{array}{c|c|c|c} B_4 & I_4 & 0 & 0 \\ \hline I_4 & B_4 & I_4 & 0 \\ \hline 0 & I_4 & B_4 & I_4 \\ \hline 0 & 0 & I_4 & B_4 \end{array}\right],$$

where

$$\mathbb{B}_4 = \begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix} \qquad \text{and} \qquad \mathbb{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

a. Compute the $\mathbb{L}\mathbb{U}$ decomposition of $\mathbb{A}_4$.

b. Compare the sparsity plots of $\mathbb{A}_4$, $\mathbb{L}$ and $\mathbb{U}$. What can you observe? Which is the consequence of this when you want to solve a system with matrix $\mathbb{A}_4$ with a direct method?

4 Iterative methods: stationary methods

Exercise 6.4. Consider the linear systems $\mathbb{A}_i\boldsymbol{x} = \boldsymbol{b}_i,\ i = 1, \ldots, 4$ with

$$\mathbb{A}_1 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad \mathbb{A}_2 = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\mathbb{A}_3 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad \mathbb{A}_4 = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

with $\boldsymbol{x} = [1\ 2\ 3]^T$ and $\boldsymbol{b}_i = \mathbb{A}_i\boldsymbol{x}$. Study (on paper and in python) the convergence for Jacobi and Gauss-Seidel methods.

Exercise 6.5. Consider the linear system $\mathbb{A}_3\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbb{A}_3 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b}_3 = \mathbb{A}_3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

a. Apply Jacobi method to compute the solution with a tolerance of 10^{-5} and 10^{-8}. What do you observe?

b. Compute the iteration matrix $\mathbb{B}_J$. Evaluate its spectral radius $\rho(\mathbb{B}_J)$ and $(\mathbb{B}_J)^3$. Relying on the results, motivate what you found at the previous point.

Exercise 6.6. Consider the linear system obtained with the following instructions:

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>> A = diag(8*ones(8,1)) + diag(2*ones(7,1),1) ...
>>  + diag(2*ones(7,1),-1);
>> b = A*ones(8,1);
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a. Analyze the convergence of Jacobi and Gauss-Seidel methods.

b. How many iterations are needed to obtain the solution with tolerance 10^{-12} starting with $\boldsymbol{x}^{(0)} = \mathbf{0}$ with the two methods?

c. Write a function that implements Richardson method.

d. Experimentally verify the answers to points a and b using the implemented function.