12.6.3 Quantum key distribution

Quantum key distribution (QKD) is a protocol which is *provably* secure, by which private key bits can be created between two parties over a *public* channel. The key bits can then be used to implement a classical private key cryptosystem, to enable the parties to communicate securely. The only requirement for the QKD protocol is that qubits can be communicated over the public channel with an error rate lower than a certain threshold. The security of the resulting key is guaranteed by the properties of quantum information, and thus is conditioned only on fundamental laws of physics being correct!

The basic idea behind QKD is the following fundamental observation: Eve cannot gain any information from the qubits transmitted from Alice to Bob without disturbing their state. First of all, by the no-cloning theorem (Box 12.1), Eve cannot clone Alice's qubit. Second, we have the following proposition:

Proposition 12.18: (**Information gain implies disturbance**) In any attempt to distinguish between two non-orthogonal quantum states, information gain is only possible at the expense of introducing disturbance to the signal.

Proof

Let $|\psi\rangle$ and $|\varphi\rangle$ be the non-orthogonal quantum states Eve is trying to obtain information about. By the results of Section 8.2, we may assume without loss of generality that the process she uses to obtain information is to unitarily interact the state ($|\psi\rangle$ or $|\varphi\rangle$) with an ancilla prepared in a standard state $|u\rangle$. Assuming that this process does not disturb the states, in the two cases one obtains

$$|\psi\rangle|u\rangle \rightarrow |\psi\rangle|v\rangle \tag{12.175}$$

$$|\varphi\rangle|u\rangle \rightarrow |\varphi\rangle|v'\rangle\,. \tag{12.176}$$

Eve would like $|v\rangle$ and $|v'\rangle$ to be different so that she can acquire information about

the identity of the state. However, since inner products are preserved under unitary transformations, it must be that

$$\langle v|v'\rangle \langle \psi|\varphi\rangle = \langle u|u\rangle \langle \psi|\varphi\rangle \tag{12.177}$$

$$\langle v|v'\rangle = \langle u|u\rangle = 1\,, \tag{12.178}$$

which implies that $|v\rangle$ and $|v'\rangle$ must be identical. Thus, distinguishing between $|\psi\rangle$ and $|\varphi\rangle$ must inevitably disturb at least one of these states. □

We make use of this idea by transmitting non-orthogonal qubit states between Alice and Bob. By checking for disturbance in their transmitted states, they establish an upper bound on any noise or eavesdropping occurring in their communication channel. These 'check' qubits are interspersed randomly among data qubits (from which key bits are later extracted), so that the upper bound applies to the data qubits as well. Alice and Bob then perform information reconciliation and privacy amplification to distill a shared secret key string. The threshold for the maximum tolerable error rate is thus determined by the efficacy of the best information reconciliation and privacy amplification protocols. Three different QKD protocols which work in this way are presented below.

The BB84 protocol

Alice begins with a and b, two strings each of $(4+\delta)n$ random classical bits. She then encodes these strings as a block of $(4+\delta)n$ qubits,

$$|\psi\rangle = \bigotimes_{k=1}^{(4+\delta)n} |\psi_{a_k b_k}\rangle\,, \tag{12.179}$$

where a_k is the k^{th} bit of a (and similarly for b), and each qubit is one of the four states

$$|\psi_{00}\rangle = |0\rangle \tag{12.180}$$

$$|\psi_{10}\rangle = |1\rangle \tag{12.181}$$

$$|\psi_{01}\rangle = |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \tag{12.182}$$

$$|\psi_{11}\rangle = |-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}\,. \tag{12.183}$$

The effect of this procedure is to encode a in the basis X or Z, as determined by b. Note that the four states are not all mutually orthogonal, and therefore no measurement can distinguish between (all of) them with certainty. Alice then sends $|\psi\rangle$ to Bob, over their public quantum communication channel.

Bob receives $\mathcal{E}(|\psi\rangle\langle\psi|)$, where $\mathcal{E}$ describes the quantum operation due to the combined effect of the channel and Eve's actions. He then publicly announces this fact. At this point, Alice, Bob, and Eve each have their own states described by separate density matrices. Note also that at this point, since Alice hasn't revealed b, Eve has no knowledge of what basis she should have measured in to eavesdrop on the communication; at best, she can only guess, and if her guess was wrong, then she would have disturbed the state received by Bob. Moreover, whereas in reality the noise $\mathcal{E}$ may be partially due to the environment (a poor channel) in addition to Eve's eavesdropping, it doesn't help Eve to have complete control over the channel, so that she is entirely responsible for $\mathcal{E}$.

Of course, Bob also finds $\mathcal{E}(|\psi\rangle\langle\psi|)$ uninformative at this point, because he does not know anything about b. Nevertheless, he goes ahead and measures each qubit in basis X or Z, as determined by a random $(4+\delta)n$ bit string b' which he creates on his own. Let

Bob's measurement result be a'. After this, Alice publicly announces b, and by discussion over a public channel, Bob and Alice discard all bits in $\{a', a\}$ except those for which corresponding bits of b' and b are equal. Their remaining bits satisfy $a' = a$, since for these bits Bob measured in the same basis Alice prepared in. Note that b reveals nothing about either a, or the bits a' resulting from Bob's measurement, but it is important that Alice not publish b until after Bob announces reception of Alice's qubits. For simplicity in the following explanation, let Alice and Bob keep just $2n$ bits of their result; δ can be chosen sufficiently large so that this can be done with exponentially high probability.

Now Alice and Bob perform some tests to determine how much noise or eavesdropping happened during their communication. Alice selects n bits (of their $2n$ bits) at random, and publicly announces the selection. Bob and Alice then publish and compare the values of these check bits. If more than t bits disagree, then they abort and re-try the protocol from the start. t is selected such that if the test passes, then they can apply information reconciliation and privacy amplification algorithms to obtain m acceptably secret shared key bits from the remaining n bits.

This protocol, known as BB84 after its inventors (see the end of chapter 'History and further reading'), is summarized in Figure 12.13, and an experimental realization is described in Box 12.7. Related versions of this protocol, such as using fewer check bits, are also known by the same name.

The BB84 QKD protocol

1: Alice chooses $(4 + \delta)n$ random data bits.
2: Alice chooses a random $(4 + \delta)n$-bit string b. She encodes each data bit as $\{|0\rangle, |1\rangle\}$ if the corresponding bit of b is 0 or $\{|+\rangle, |-\rangle\}$ if b is 1.
3: Alice sends the resulting state to Bob.
4: Bob receives the $(4 + \delta)n$ qubits, announces this fact, and measures each qubit in the X or Z basis at random.
5: Alice announces b.
6: Alice and Bob discard any bits where Bob measured a different basis than Alice prepared. With high probability, there are at least $2n$ bits left (if not, abort the protocol). They keep $2n$ bits.
7: Alice selects a subset of n bits that will to serve as a check on Eve's interference, and tells Bob which bits she selected.
8: Alice and Bob announce and compare the values of the n check bits. If more than an acceptable number disagree, they abort the protocol.
9: Alice and Bob perform information reconciliation and privacy amplification on the remaining n bits to obtain m shared key bits.

Figure 12.13. The four state quantum key distribution protocol known as BB84.

Exercise 12.26: Let a'_k be Bob's measurement result of qubit $|\psi_{a_k b_k}\rangle$, assuming a noiseless channel with no eavesdropping. Show that when $b'_k \neq b_k$, a'_k is random and completely uncorrelated with a_k. But when $b'_k = b_k$, $a'_k = a_k$.

The B92 protocol

The BB84 protocol can be generalized to use other states and bases, and similar conclusions hold. In fact, a particularly simple protocol exists in which only two states are used. For simplicity, it is sufficient to consider what happens to a single bit at a time; the description easily generalizes to block tests just as is done in BB84.

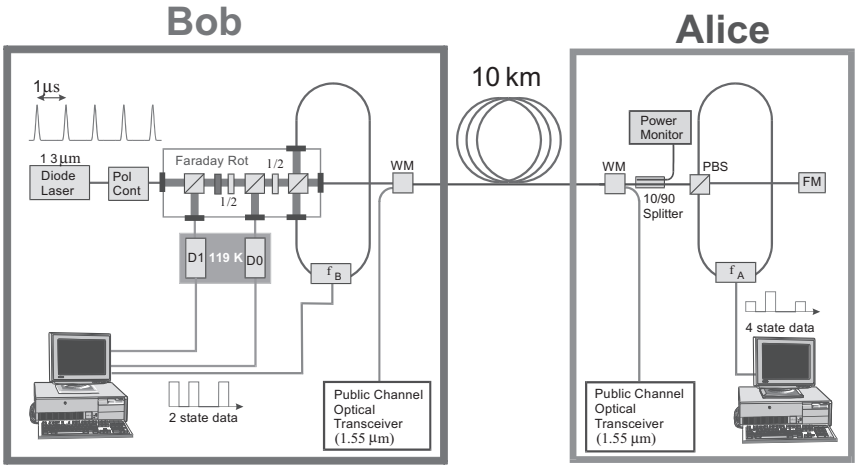
Suppose Alice prepares one random classical bit a, and, depending on the result, sends Bob

$$|\psi\rangle = \begin{cases} |0\rangle & \text{if } a = 0 \\ \dfrac{|0\rangle + |1\rangle}{\sqrt{2}} & \text{if } a = 1. \end{cases} \tag{12.188}$$

Depending on a random classical bit a' which he generates, Bob subsequently measures the qubit he receives from Alice in either the Z basis $|0\rangle, |1\rangle$ (if $a' = 0$), or in the X basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ (if $a' = 1$). From his measurement, he obtains the result b, which is 0 or 1, corresponding to the -1 and $+1$ eigenstates of X and Z. Bob then publicly announces b (but keeps a' secret), and Alice and Bob conduct a public discussion keeping

Box 12.7: Experimental quantum cryptography

Quantum key distribution is particularly interesting and astonishing because it is easily experimentally realized. Here is a schematic diagram of one system employing commercial fiber-optic components to deliver key bits over a ten kilometer distance, which has been built at IBM:



Bob initially generates strong coherent states $|\alpha\rangle$ using a diode laser emitting light at a wavelength of 1.3 μm, and transmits them to Alice, who attenuates them to (approximately) generate a single photon. She also polarizes the photon in one of the four states of the BB84 protocol, using as $|0\rangle$ and $|1\rangle$ states horizontal and vertical polarization. She then returns the photon to Bob, who measures it using a polarization analyzer, in a random basis. By using this special configuration in which the photon traverses the same path twice, the apparatus can be made to autocompensate for imperfections (such as slowly fluctuating path lengths and polarization shifts) along the fiber link. Alice and Bob then select the subset of results in which they used the same basis, reconcile their information, and perform privacy amplification, communicating over a public channel with photons (over the same fiber) of 1.55 μm wavelength. Key bits can be exchanged at the rate of a few hundred per second. Ultimately, improvements in the light source and detector should allow the rate to be improved by a few orders of magnitude. Quantum key distribution over distances exceeding 40 kilometers, and also in installed telecommunication fiber (under Lake Geneva) has been demonstrated.

only those pairs $\{a, a'\}$ for which $b = 1$. Note that when $a = a'$, then $b = 0$ always. Only if $a' = 1 - a$ will Bob obtain $b = 1$, and that occurs with probability $1/2$. The final key is a for Alice, and $1 - a'$ for Bob.

This protocol, known as B92 (see the end of chapter 'History and further reading'), highlights how the impossibility of perfect distinction between non-orthogonal states lies at the heart of quantum cryptography. As in BB84, because it is impossible for any eavesdropper to distinguish between Alice's states without disrupting the correlation between the bits Alice and Bob finally keep, this protocol allows Alice and Bob to created shared

key bits while also placing an upper bound on the noise and eavesdropping during their communication. They can then apply information reconciliation and privacy amplification to extract secret bits from their resulting correlated random bit strings.

Exercise 12.28: Show that when $b = 1$, then a and a' are perfectly correlated with each other.

Exercise 12.29: Give a protocol using six states, the eigenstates of X, Y, and Z, and argue why it is also secure. Discuss the sensitivity of this protocol to noise and eavesdropping, in comparison with that of BB84 and B92.

The EPR protocol

The key bits generated in the BB84 and B92 protocols may appear to have been originated by Alice. However, it turns out that the key can be seen to arise from a fundamentally random process involving the properties of entanglement. This is illustrated by the following protocol.

Suppose Alice and Bob share a set of n entangled pairs of qubits in the state

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{12.189}$$

These states are known as EPR pairs. Obtaining these states could have come about in many different ways; for example, Alice could prepare the pairs and then send half of each to Bob, or vice versa. Alternatively, a third party could prepare the pairs and send halves to Alice and Bob. Or they could have met a long time ago and shared them, storing them until the present. Alice and Bob then select a random subset of the EPR pairs, and test to see if they violate Bell's inequality (Equation (2.225), on page 115 in Section 2.6), or some other appropriate test of fidelity. Passing the test certifies that they continue to hold sufficiently pure, entangled quantum states, placing a lower bound on the fidelity of the remaining EPR pairs (and thus any noise or eavesdropping). And when they measure these in jointly determined random bases, Alice and Bob obtain correlated classical bit strings from which they can obtain secret key bits as in the B92 and BB84 protocols. Using an argument based on Holevo's bound, the fidelity of their EPR pairs can be used to establish an upper bound on Eve's accessible information about the key bits.

Where do the key bits come from in this EPR protocol? Since it is symmetric – Alice and Bob perform identical tasks on their qubits, even possibly simultaneously – it cannot be said that either Alice or Bob generates the key. Rather, the key is truly random. In fact the same applies to the BB84 protocol, since it can be reduced to an instance of a generalized version of the EPR protocol. Suppose that Alice prepares a random classical bit b, and according to it, measures her half of the EPR pair in either the $|0\rangle, |1\rangle$ basis, or in the basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, obtaining a. Let Bob do identically, measuring in (his randomly chosen) basis b' and obtaining a'. Now they communicate b and b' over a public classical channel, and keep as their key only those $\{a, a'\}$ for which $b = b'$. Note that this key is *undetermined* until Alice or Bob performs a measurement on their EPR pair half. Similar observations can be made about the B92 protocol. For this reason, quantum cryptography is sometimes thought of not as secret key exchange or transfer, but rather as secret key *generation*, since fundamentally neither Alice nor Bob can pre-determine the key they will ultimately end up with upon completion of the protocol.