Comparison between LQR, LQG controller and PI controller for a Pitch Control System of an Aircraft

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Abstract

This project is meant to find an optimal controller for a pitch control system. We proposed a solution using a Linear Quadratic Controller (LQR), analyzing its performance with respect to a desired pitch angle. The result is compared with a PI controller. Moreover it is proposed a design of a LQG controller, which is than compared with the previous one controllers.

1 Mathematical model of the system

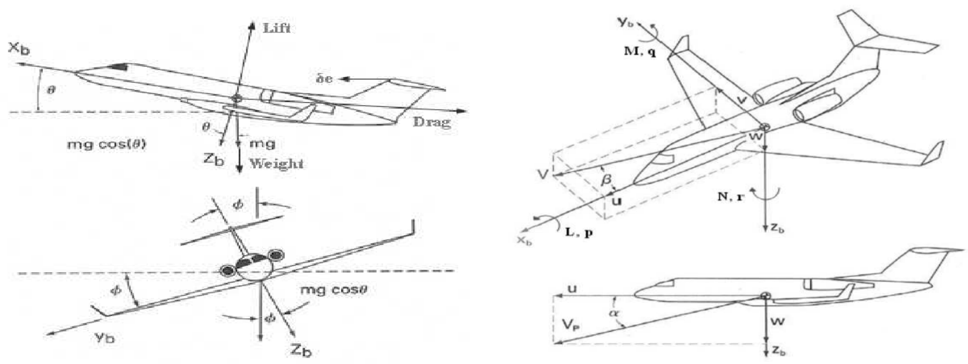
In order to develop a controller, it is necessary to study the dynamics of the system and find a possible mathematical model.

1.1 Dynamics of the system

The forces acting on the system are longitudinal and lateral, from now on the longitudinal ones will be considered which are: the lift force, the drag force, the thrust force and the weight force. The aerodynamics force components are indicated by X_b, Y_b, Z_b. These forces create moments, which components are indicated by L, M and N. The orientation of the aircraft is given by three angles:

- θ, pitch angle;
- ϕ, roll angle;
- δ_e, elevator deflection angle.

The angular rates are indicated by p, q and r and the velocity components are indicated by u, v and w. All these terms are shown in the two following pictures:



(a) Forces and angles (b) velocities and moments

Figure 1: airplane parameters

Moreover in Figure 1.b are shown two important angles: α and β. The first one is called the angle of attack which is the angle between the body's reference line and the vector representing the relative motion. The second one, β , is the angle of sideslip.
The elevator deflection angle is the angle that allows to control the pitch of the aircraft. Indeed the pitch angle is controlled by the rear part of the tail plane's horizontal stabilizer. The tailplane (also known as horizontal stabilizer) is characterized by: a fixed stabilizer and movable elevator surfaces (these are shown in Figure 2). By moving the elevator up, the downwards force on the horizontal stabilizer is increased. So the nose is pitched up and the lift force is increased. By moving the elevator down, the result is opposite than the latter.

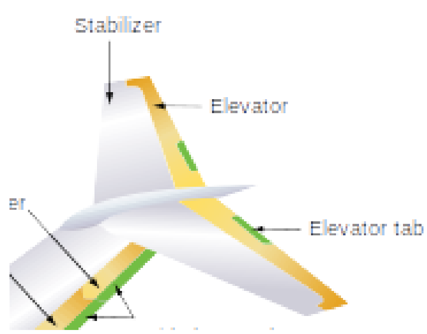


Figure 2: tailplain

Source: www.wikipedia.org

1.2 Dynamic equations

In general, the dynamic equations of the motion of an aircraft are six nonlinear coupled differential equations. But, as said before in 1.1 paragraph, the aircraft pitch is determined by the longitudinal dynamics. So it is possible to decouple the six equations, and take into account just three of them (the longitudinal ones). These nonlinear equations are very complicated, but working under some assumptions will simplify things. The assumptions are the following:

- The aircraft is steady-state cruise at constant altitude and velocity;
- the change in pitch angle does not change the speed of the aircraft.

The first assumption causes that thrust and drag forces balance each other, weight and lift forces balance each other too. Under these assumptions the following three equations are found [1]:

$$X - mgsin(\theta) = m(\dot{u} + qv - rv)$$

$$Z + mgcos(\theta)cos(\phi) = m(\dot{w} + pv - qu)$$

$$M = I_y\dot{q} + rq(I_x - I_z) + I_{xz}(p^2 - r^2)$$

Using the small disturbance theory, these three equations can be linearized. In order to do that, we consider some of the terms like composed by a reference value plus a perturbation:

$$\begin{array}{lll} u = u_0 + \Delta u & v = v_0 + \Delta v & w = w_0 + \Delta w \\ p = p_0 + \Delta p & q = q_0 + \Delta q & r = r_0 + \Delta r \\ X = X_0 + \Delta X & M = M_0 + \Delta M & Z = Z_0 + \Delta Z \\ \delta = \delta_0 + \Delta\delta & \theta = \theta_0 + \Delta\theta & \phi = \phi_0 + \Delta\phi \end{array}$$

We assume that the reference flight condition is symmetric and the propulsive forces are constant, this implies that $v_0 = p_0 = q_0 = r_0 = \phi_0 = w_0 = 0$. After linearization and using the small disturbance theory the following equations are found [1]:

$$(\frac{d}{dt} - X_u)\Delta u - X_w\Delta w + (gcos(\theta_0))\Delta\theta = X\delta_e\Delta\delta_e$$

$$-Z_u\Delta u + [(1 - Z_w)\frac{d}{dt} - Z_w]\Delta w - [(u_0 - Z_q)\frac{d}{dt} - gsin(\theta_0)]\Delta\theta = Z_{\delta_e}\Delta\delta_e$$

$$-M_u\Delta u - (M_w\frac{d}{dt} + M_w)\Delta w + (\frac{d^2}{dt^2} - M_q\frac{d}{dt})\Delta\theta = M_{\delta_e}\Delta\delta_e$$

1.3 Transfer function and state-space system

Before finding the transfer function of the system, it is necessary to define some parameters. These parameters are called longitudinal stability derivatives, they measure how much change occurs in a force or moment acting on the aircraft when there is a small change in a flight condition parameter (for example the angle of attack). The parameters used here are shown in Figure 3:

Longitudinal Derivatives	Components		
	X-Force (S^{-1})	Z-Force (F^{-1})	Pitching Moment (FT^{-1})
Rolling velocities	$X_u = -0.045$	$Z_u = -0.369$	$M_u = 0$
Yawing velocities	$X_w = 0.036$	$Z_w = -2.02$	$M_w = -0.05$
	$X_{\dot{w}} = 0$	$Z_{\dot{w}} = 0$	$M_{\dot{w}} = -0.051$
Angle of attack	$X_{\alpha} = 0$	$Z_{\alpha} = -355.42$	$M_{\alpha} = -8.8$
	$X_{\dot{\alpha}} = 0$	$Z_{\dot{\alpha}} = 0$	$M_{\dot{\alpha}} = -0.8976$
Pitching rate	$X_q = 0$	$Z_q = 0$	$M_q = -2.05$
Elevator deflection	$X_{\delta e} = 0$	$Z_{\delta e} = -28.15$	$M_{\delta e} = -11.874$

Figure 3: Longitudinal derivative stability parameters[1]

The transfer function of a system which has as input the elevator deflection angle and as output the pitch angle can be found. To do that, the last three equations of paragraph 1.3 are considered and transformed in the Laplace domain using the Laplace transform. Moreover they are written depending on the longitudinal derivatives defined above. At the end the following transfer function is found [1]:

$$\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = \frac{1}{s}\frac{-(M_{\delta_e} + M_{\dot{\alpha}} Z_{\delta_e}/u_0)s - (M_\alpha Z_{\delta_e}/u_0 - M_{\delta_e} Z_\alpha/u_0)}{s^2 - (M_q + M_{\dot{\alpha}} + Z_\alpha/u_0)s + (Z_\alpha M_q/u_0 - M_\alpha})$$

Using the values of the Figure 3:

$$\frac{\Delta\theta(s)}{\Delta\delta_e(s)} = \frac{11.7304s + 22.578}{s^3 + 4.9676s^2 + 12.941s}$$

This transfer function can be represented in the state-space form.
The state-space form is given by:

$$\begin{bmatrix} \Delta\dot{\alpha} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} -2.02 & 1 & 0 \\ -6.9868 & -2.9476 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix} + \begin{bmatrix} 0.16 \\ 11.7304 \\ 0 \end{bmatrix} \Delta\delta_e$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\alpha \\ \Delta q \\ \Delta\theta \end{bmatrix} + [0]$$

2 Design of a controller

In this section it will be described how to design a controller for a pitch angle of 0.2 radiant (11.5 degree) with the following requirements:

- Rising time between 2 and 3 seconds;
- settling time less than 5 s;

- overshot less than 10%;
- steady-state error less than 2%;

Two different cotrollers are proposed. The first one is a Linear Quadratic Regulator (LQR), the second one is a PI controller.

2.1 LQR Controller

A Linear Quadratic Regulator (LQR) is a full-state feedback controller. The control scheme of this type of control is shown below:

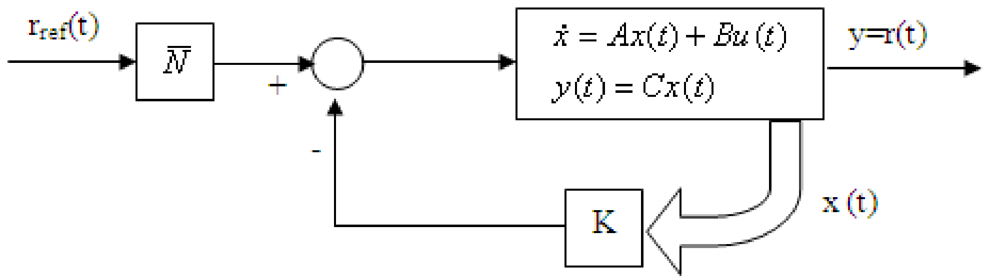


Figure 4: Full-state feedback controller

So the state-feedback law is $u = -Kx$ which is such that the quadratic cost function J(u) is minimized, where J(u) is:

$$J(u) = \int_0^\infty (x^T Q x + u^T R u + 2x^T N u) dt$$

The matrix K is given by:

$$K = R^{-1}(B^T P + N^T)$$

Where P is the solution of the continuous time Riccati equation:

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0$$

For the cost function the following matrices are chosen:

$$Q = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 500 \end{bmatrix} \qquad R = \begin{bmatrix} 1 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

These matrices were chosen in such a way that the requirements has been respected and a smooth and reasonable response is achieved.
The K gain can be found using the Matlab code:

```
% LQR parameters
R=1;
Q=[100 0 0; 0 200 0; 0 0 500];
%Riccati equation
[P,L,K] = care(A,B,Q);
```

Where R has been omitted, because it is just 1.
The care function gives the following result:

$$P = \begin{bmatrix} 25.7757 & -0.3116 & -12.3700 \\ -0.3116 & 1.1991 & 2.0749 \\ -12.3700 & 2.0749 & 331.9008 \end{bmatrix} \quad L = \begin{bmatrix} -165.8766 \\ -2.1995 \\ -1.3837 \end{bmatrix} \quad K = \begin{bmatrix} 0.4691 & 14.0163 & 22.3607 \end{bmatrix}$$

Where the elements of the vector L are the eigenvalues of the closed loop system: $L = eig(A - BK, I)$.
In order to eliminate the steady-state error to a step reference, a constant gain was computed and called $\bar{N}$. This gain is added after the reference and in this case its value is $\bar{N} = 22.360$. The step response of the closed loop system with this controller respects the design requirements and it is:

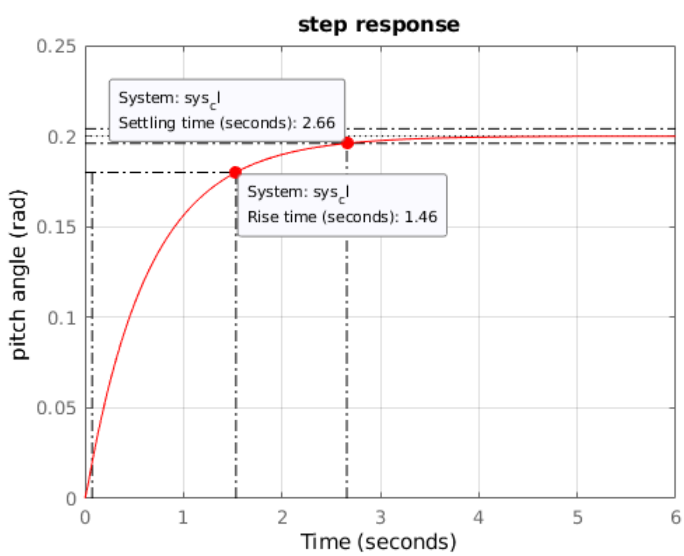


Figure 5: Step response of the closed-loop system, with LQR

2.2 PID Controller

For comparison reasons, a PID controller is proposed. The PID controller used is a controller with proportianal and integral action, a PI (it will be called C_pi):

$$C\_pi = Kp + Ki \cdot \frac{1}{s}$$

The gain of the proportional part (Kp) and of the integral part (Ki) can be computed by using a tuning function of Matlab:

```
%tuning of the PI controller
[C_pi,info] = pidtune(sys,'PI')
```

Which gives the following datas: $Kp = 0.812, Ki = 0.834$.
The step response of the closed loop system is:

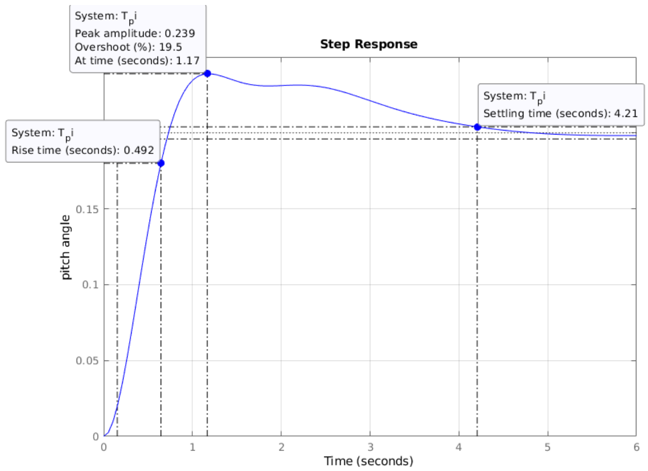


Figure 6: Step response of the closed loop system, with PI controller

This response does not respect the requirements, indeed the overshoot is more than 10%. Different values for Kp and Ki are chosen using the tuner tool of Matlab: $Kp = 0.82088, Ki = 0.028734$. With this value of PI controller the step response of the closed loop system is:

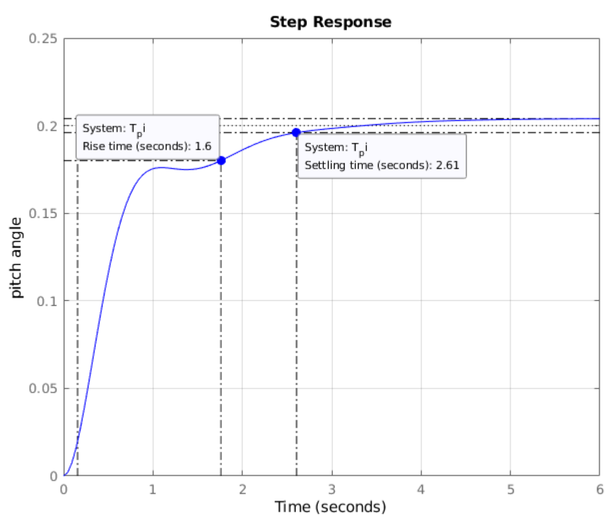


Figure 7: Step response of the closed loop system, with PI controller

In this case all the requirements are respected.

2.3 LQG

The Linear Quadratic Regulator that has been developed in section 2.1 does not take into account the possibility of disturbances and noises. However this system can be affected by different disturbances, like for example the wind or measurements noises and so on. Moreover in section 2.1 the states are considered directly accessible. In this section they will be estimated by using the measurements of the output and disturbances are going to be considered.

Two disturbances are introduced, one which affects the states of the system and the other one which affects the measurement of the output. Both are supposed to be gaussian noises, in particular white noises. The white process noise is indicated by w and the white measurement noise by v. So now the state space model of the system is given by:

$$\dot{x} = Ax + Bu + Gw$$
$$y = Cx + Du + Hw + v$$

In this case, an optimal control scheme within the class of linear control laws is given by the Linear-Quadratic-Gaussian controller. This control law is unique and it is a combination of a Kalman filter with a linear quadratic regulator (LQR). The state estimator and the state feedback can be designed independently thanks to the separation principle.

The cost function considered is the one already used in section 2.1 for the LQR design, so the gain of the regulator is going to be the same as before:

$$K = \begin{bmatrix} 0.4691 & 14.0163 & 22.3607 \end{bmatrix}$$

But now this matrix is going to multiply the estimation of the states, not the states. So the control law is:

$$u = -K\hat{x}$$

Where $\hat{x}$ is the estimation of the state given by the Kalman filter.

Before starting with the design of the Kalman filter estimator, it is necessary to define the matrices G and H. These matrices have to have a physical meaning, in what follows they will be defined as:

$$G = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Which were chosen just for simplicity, for later studies they should be well defined.

The Kalman filter allows us to design a filter which minimizes the steady-state error covariance: $\lim_{t\to\infty} E(\{x-\hat{x}\}\{x-\hat{x}\}^T)$.

The optimal estimation of the state is given by the following equation:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du)$$

Where L is the filter gain given by:

$$L = (PC^T + \bar{N})\bar{R}^{-1}$$

Where $\bar{R}$ and $\bar{N}$ are:

$$\bar{R} = R + HN + N^T H^T + HQH^T$$
$$\bar{N} = G(QH^T + N)$$

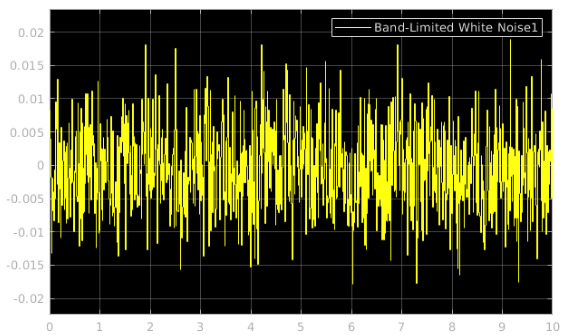
And P is the symmetric positive definite matrix that is the solution of the corresponding algebraic Riccati equation:

$$0 = AP + PA^T + Qn - PC^T Rn^{-1} CP$$

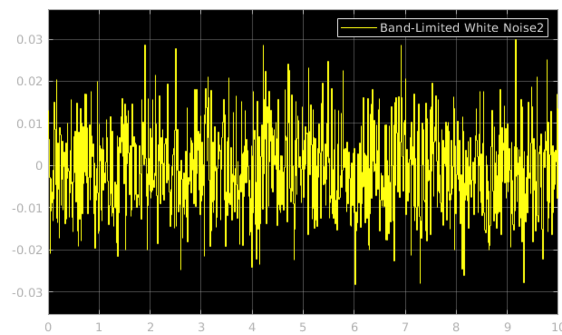
Where the matrices Qn and Rn are the covariance matrices of the two noises:

$$Qn = E\{ww^T\} \qquad\qquad Rn = E\{vv^T\}$$

In what follows two white noises are chosen with different covariance. The first one has covariance $\sigma_1^2 = 0.00004$ and it is the disturbance of the state, the second one has covariance equal to $\sigma_2^2 = 0.0001$ and it is the measurement noise. The two noises are shown below:



(a) disturbance on the state (w), $\sigma_1 = 0.0063$



(b) measurement noise (v), $\sigma_2 = 0.01$

These two noises were chosen taking into account that the step input is equal to 0.2. Now that everything is defined it is possible to find the gain matrix of the filter using Matlab, the following command can be used:

```
%Kalman State Estimator
sys=ss(A,[B ones(3,1)],C,D);%transfer function of the system
Qn=0.00004;
Rn=0.0001;
[kalmf,L,P] = kalman(sys,Qn,Rn);
```

Which gives the following result for the gain matrix of the filter:

$$L = \begin{bmatrix} 0.1172 \\ -0.0909 \\ 0.4672 \end{bmatrix}$$

At this point all the matrices needed for the Linear Quadratica Gaussian regulator are found and it is possible to implement it. A possible way to implement it is using Simulink, and it is the method used here. Below is shown how it can be designed in Simulink:

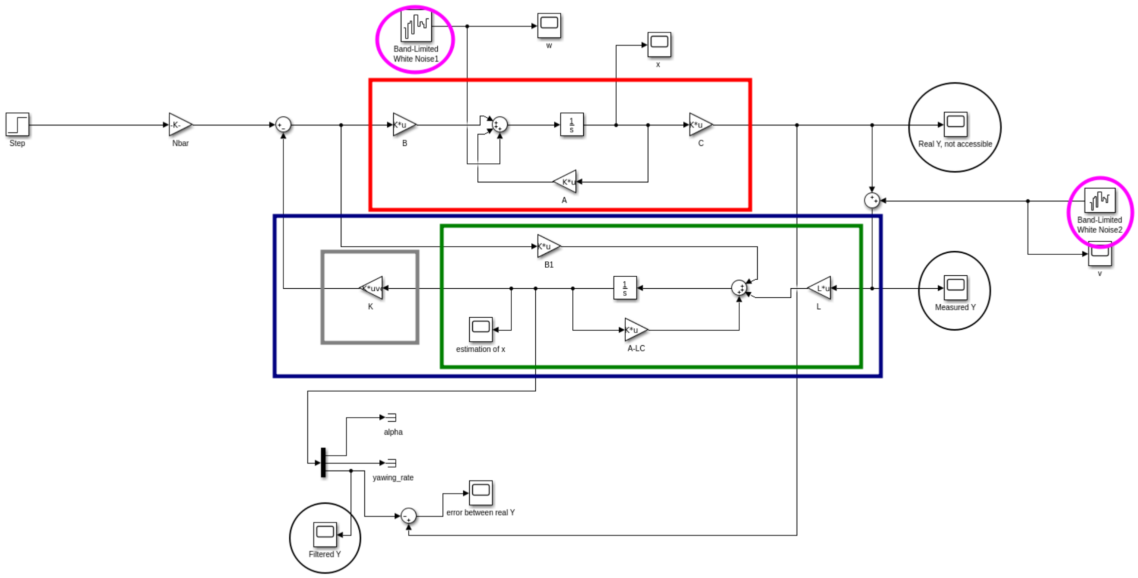
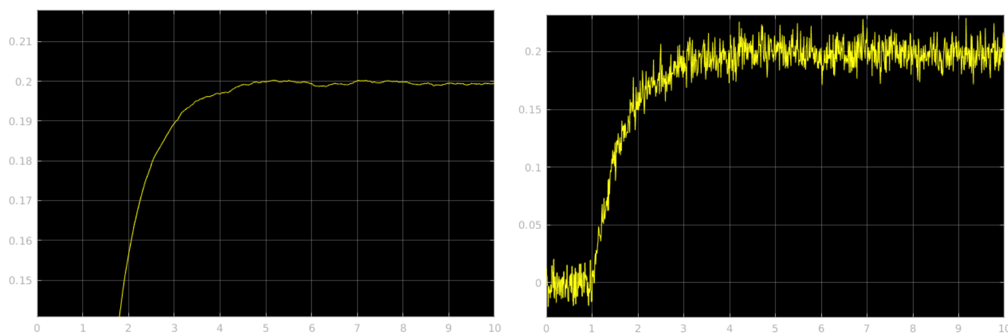


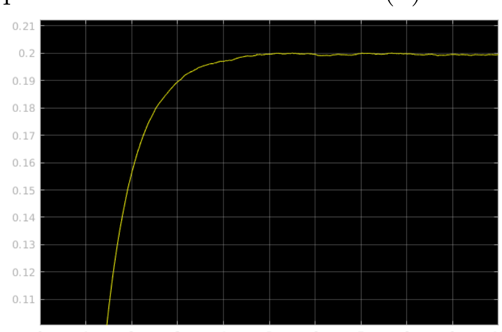
Figure 9: ■ open loop system; ■ LQG regulator; ■ Kalman filter; ■ LQR; ■ noise

The black circles are the scopes that display what follows:



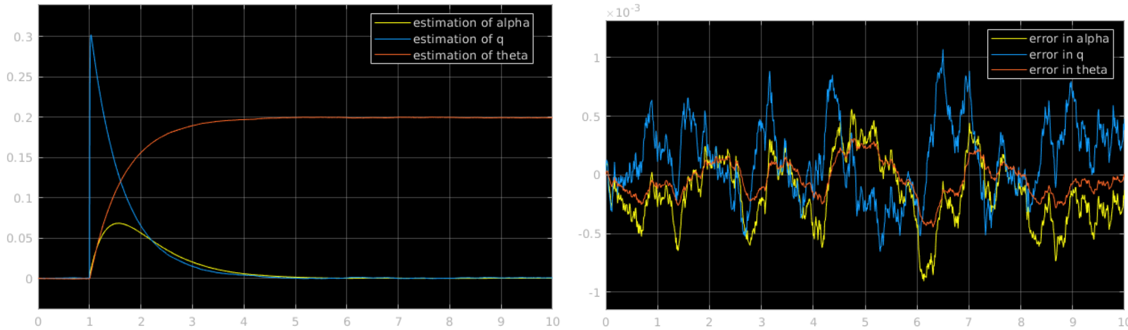
(a) real output, supposed not accessible

(b) measured output



(c) filtered output

The filtered output has been well filtered, indeed the error between the real output and the filtered one. In the next picture is shown the error between the real state of the system and the estimated one:



(a) estimation of the state (b) error in the estimation of the state

3 Final comments

The efficiency of the PI controller respect the LQR controller is quite similar but the step response of the closed loop system with the LQR regulator is smoother than the one with the PI controller. Moreover it converges to a zero error faster than the PI.
The LQR regulator is an ideal case since it doesn't take into account the possibility of errors due to disturbances. The LQG regulator instead is the real case, and it can allowed to develop a controller for real situations.
For further studies, the validation of this theoretical result should be done. Moreover for the LQG controller the matrices (H and G) and the disturbances should be chosen after a deep analysis of the physical situation.

References

[1] Nurbaiti Wahid, Mohd Rahmat, and Kamaruzaman Jusoff. Comparative assessment using lqr and fuzzy logic controller for a pitch control system. *European Journal of Scientific Research*, 42, 05 2010.