HW w9-1
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1. Color the edges of the equilateral triangle with red, blue and green, how many solutions are there?

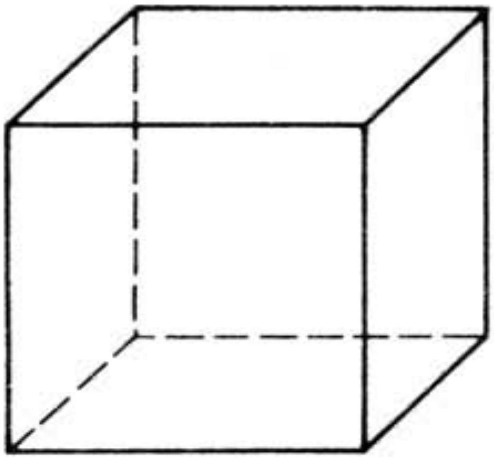
Let G is rotation and symmetry group (symmetry works as rotation in 3d space).
G consists of 6 different elements:
First one leaves all the edges in initial position: <1,1,1>
Second and third are the rotation around center by 60 and 120 degrees: <3>
Fourth, fifth and sixth are the rotation around central perpendiculars,: <1,2>
The number of solutions is: $(1 * 3^3 + 2 * 3^1 + 3 * 3^2)/6 = 10$
*PS x inside <x> means the number of edges to be in a permutation cycle.

2. Embed 2 red beads and 2 blue beads onto the 4 vertices of a cube, how many possible ways are there?

Let G be a cube's rotation group.
It consists of 24 permutations.
Their types:
1 permutation of type <1,1,1,1,1,1,1,1>
6 permutations of type <4,4>
9 permutations of type <2,2,2,2>
8 permutations of type <1,1,3,3>

For the 1st type, we can choose 2 vertices to put 2 blue beads onto and 2 of the rest 6 vertices to put red beads onto.
$C_8^2 * C_6^2 = 28 * 15 = 420$ different options;
For the 2nd type, there are no solutions, because at least 4 vertices need to be of the same color (in other words, you can't put 2 blue and 2 red beads on the cycle of length 4, because after the permutation you get the new sequence (or the figure) anyway).
For the 3rd type, we can choose one of the four cycles to put red beads onto the 2 vertices, and one of the rest 3 cycles to put blue beads onto.
$9 * 4 * 3 = 108$ different options;
For the 4th type, there are no solutions, because for the cycles of length 3, we need at least 3 beads of the same color to leave the sequence (or the figure) the same. There are only 2 cycles of length 1, which is not enough to use 4 different beads.
Total number of solutions is: $(420 + 108)/24 = 22$

3. Use 12 identical matches to construct a cube, how many possible solutions?



One match can be rotated by 180 degrees and the cub becomes different from how it looked before. In other words, the side of the match describes the binary state of an edge, same as painting the edge with one of the two colours.

Enumerate cube's edges from 1 to 12, we split the rotation group of the cube to the five classes:

1st class describes cube staying in initial position. Other words, it's permutation combined of 12 cycles of length 1: (1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12). Can have 2^{12} different colourings.

2nd class is rotation by 90 degrees around the centres of opposite sides. 3 cycles of length 4; 6 different elements in the class because 3 pairs of sides, can rotate to the left and to the right. Can have $6 * 2^3$ different colourings, because the cycles must be of the same colour.

3rd class is rotation by 180 degrees around the centres of opposite sides. 6 cycles of length 2; 3 different elements in the class because 3 pairs of sides. $3 * 2^6$ different colourings.

4th class is the rotation by 120 degrees around the axis connecting two opposite vertices. 4 cycles of length 3; 8 different elements in the class because for 4 pairs of vertices we can rotate the cube to the right or to the left. $8 * 2^4$ colourings.

5th class is rotation by 180 degrees connecting the centres of two opposite edges. Have two cycles of length 1 and 5 cycles of length 2; 6 different elements in the class because have 6 pairs of opposite edges. $6 * 2^7$ colourings, because the 7 cycles can be coloured in 2^7 ways.

The total number of different solutions is:

$$\frac{2^{12} + 6 * 2^3 + 3 * 2^6 + 8 * 2^4 + 6 * 2^7}{24} = \frac{4096 + 48 + 192 + 128 + 768}{24} = 218$$