Combinatorics IEP HW

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1. how many integer numbers from 1 to 10000 are not squares of integers or cubes of integers?

$100^2 = 10000$, so 1…100 have the squares from 1 to 10000.

$21^3 = 9261$, $22^3 = 10648$, so 1…21 have cubes from 1 to 10000.

$4^6 = 4096$, $5^6 = 15625$, so 1…4 have the 6th power which is between 1 and 10000.*

The total number is (10000 - 100 - 21 + 4) = 9883

*Prove:

a*a=b*b*b. Let h be a prime factor of a, then on the left side of equation there is a*a = h*h*q. If we write all the prime factor of a and b, then the number of the same prime factors on the right side must be equal to the number of prime factors on the left side. So the number of each must be multiple of 2 and 3 at the same time. It means that it must be multiple of (3,2) = 6, i.e. the total number of prime factors is multiple of 6, so number is cube and a square, only when it is a 6th power at the same time.

2. How many permutations of 1, 2, 3, ……, 9 have at least one odd number in its natural position?

$i_1 = 1$ or $i_3 = 3$ or $i_5 = 5$ or $i_7 = 7$ or $i_9 = 9$.

To have one of the numbers to be in its natural position, we have C_5^1 ways to do it, and for other numbers there are 8! ways to put to the remaining 8 positions.

C_5^2 ways to have two of the numbers to be in their natural position, and for other numbers there are 7! ways to put to the remaining 7 positions.

Same for three, four and five numbers to be in their natural position:

$C_5^1 * 8! + C_5^2 * 7! + C_5^3 * 6! + C_5^4 * 5! + C_5^5 * 4! = 157824$

3. $x_1 + x_2 + x_3 + x_4 = 20, where\ 1 \le x_1 \le 6, 0 \le x_2 \le 7, 4 \le x_3 \le 8, 2 \le x_4 \le 6$
 please calculate the number of integral solutions.

Let:

$y_1 = x_1 - 1$, $y_2 = x_2$, $y_3 = x_3 - 4$, $y_4 = x_4 - 2$

Then:

$y_1 + y_2 + y_3 + y_4 = 13$

Without limitation, $C_{13+4-1}^{13} = C_{16}^3$ nonnegative solutions.

Let A1 be the solution where $y1 \ge 6, z_1 + 6 + y_2 + y_3 + y_4 = 13; |A1| = C_{7+4-1}^7 = C_{10}^3$

Let A2 be the solution where $y2 \ge 8, y_1 + z_2 + 8 + y_3 + y_4 = 13; |A2| = C_{5+4-1}^5 = C_8^3$

Let A3 be the solution where $y3 \ge 5, y_1 + y_2 + z_3 + 5 + y_4 = 13; |A3| = C_{8+4-1}^8 = C_{11}^3$

Let A4 be the solution where $y4 \ge 5, y_1 + y_2 + y_3 + z_4 + 5 = 13; |A1| = C_{8+4-1}^8 = C_{11}^3$

$A1 \cap A2 : z_1 + 6 + z_2 + 8 + y_3 + y_4 = 13, |A1 \cap A2| = 0$

$A1 \cap A3 : z_1 + 6 + y_2 + z_3 + 5 + y_4 = 13, |A1 \cap A3| = C_{2+4-1}^2 = C_5^3$

$A1 \cap A4 : z_1 + 6 + y_2 + y_3 + z_4 + 5 = 13, |A1 \cap A4| = C_{2+4-1}^2 = C_5^3$

$A2 \cap A3 : y_1 + z_2 + 8 + z_3 + 5 + y_4 = 13, |A2 \cap A3| = C_{0+4-1}^0 = 1$

$A2 \cap A4 : y_1 + z_2 + 8 + y_3 + z_4 + 5 = 13, |A2 \cap A4| = C_{0+4-1}^0 = 1$

$A3 \cap A4 : y_1 + y_2 + z_3 + 5 + z_4 + 5 = 13, |A3 \cap A4| = C_{3+4-1}^3 = C_6^3$

$A1 \cap A2 \cap A3 : z_1 + 6 + z_2 + 8 + z_3 + 5 + y_4 = 13, |A1 \cap A2 \cap A3| = 0$

$A1 \cap A2 \cap A4 : z_1 + 6 + z_2 + 8 + y_3 + z_4 + 5 = 13, |A1 \cap A2 \cap A4| = 0$

$A2 \cap A3 \cap A4 : y_1 + z_2 + 8 + z_3 + 5 + z_4 + 5 = 13, |A2 \cap A3 \cap A4| = 0$
$A1 \cap A2 \cap A3 : z_1 + 6 + z_2 + 8 + z_3 + 5 + y_4 = 13, |A1 \cap A2 \cap A3| = 0$
$A1 \cap A2 \cap A3 \cap A4 : z_1 + 6 + z_2 + 8 + z_3 + 5 + z_4 + 5 = 13, |A1 \cap A2 \cap A3 \cap A4| = 0$

The answer is $C_{16}^3 - C_{10}^3 - C_8^3 - C_{11}^3 - C_{11}^3 + 0 + C_5^3 + C_5^3 + 1 + 1 + C_6^3 = 560 - 120 - 56 - 165 - 165 + 10 + 10 + 2 + 20 = 96$

4. For the permutation P=P1 P2 P3 P4 of {1,2,3,4}, how many feasible permutations are there if we constrain that P1≠2， P2≠2、3， P3≠3、4， P4≠4? (4 points)

Let’s interpret it as rooks on the field 4x4. Put the forbidden positions

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r1 = 6
r2 = 1 + 1 + 3 * 3 = 11
r3 = 1*3 + 3*1 = 6
r4 = 0
Answer = 4! - 6 * 3! + 11 * 2! - 6 * 1! = 4