1 Empty Set as a Semigroup

1.1 Claim

Let S be the empty set. Let f be the binary operator on S. Then (S, f) forms a semigroup.

1.2 Proof

As $S = \emptyset$ the domain and codomain of f are empty. So it is vacuously true that f is associative regarding S. $\square$

2 Groups are Semigroups

2.1 Claim

Let (G, f) be a group. Then (G, f) is also a semi-group.

2.2 Proof

By definition of a group, f must be a closed, associative, binary operator. Thus the pair forms a semim-group. $\square$

3 Constant Map constitutes a Semigroup

3.1 Claim

Let S be any non-empty set. Let the binary operator $f : S \times S \to S$ be a constant map. Then (S, f) forms a semigroup.

3.2 Proof

Since f is a constant map, it's output is fixed. Denote that output c. When f acts on any pair in S it always yields c. For any fixed, but arbitrary $x, y \in S$, even $f(x, c) = f(c, y) = c$. So f is trivially associtative (and commutative). $\square$

4 Singletons as Semigroups

4.1 Claim

Let S be a singleton $\{x\}$. Let f be the function $f : S \times S \to S$. Then (S, f) form a semigroup.

4.2 Proof

Since S only has one elment, f must be a constant map. Thus, from the prior exercise, (S, f) forms a semigroup. $\square$

5 Symmetric Semigroup

5.1 Claim

Let X be any set. Let $\mathbf{Symm}(X)$ be the set of all injections from X to itself. With function composition as the binary operator, $\mathbf{Symm}(X)$ forms a semigroup.

5.2 Proof

Each $f, g \in \mathbf{Symm}(X)$ have the same image and domain - both of which are equal to X. So each f is able to compose with any g and vice versa. Thus one can freely associate with any elements in the $\mathbf{Symm}(X)$ - it's function composition, trivially associative as it inherits it from the set theoretic relationship $\times$ operator.