WI1421LR

Calculus I

Feeblebridges

January 30, 2025

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Lecture 1: Dot Product

1.1. Theory

For vectors, only the magnitude and direction matter
Notation: $\vec{u}$, $\mathbf{u}$, $\underline{u}$
Vector $\vec{v}$ in $\mathbb{R}^3$ has components v_1, v_2, v_3 and can be expressed as $\vec{v} = \langle v_1, v_2, v_3 \rangle$
Standard basis in $\mathbb{R}^3$ is $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$
$\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$
Length of $\vec{v}$ is $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$
Unit vector of $\vec{v}$ is $\hat{v} = \frac{\vec{v}}{|v|}$

Theorem 1.1.1 Angle between vectors

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta, \qquad 0 \leq \theta \leq \pi \tag{1.1}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}, \qquad \vec{u} \neq \vec{0} \wedge \vec{v} \neq \vec{0} \tag{1.2}$$

$$\vec{u} \cdot \vec{v} = 0 \implies \vec{u} \perp \vec{v} \qquad \vec{u} \neq \vec{0} \wedge \vec{v} \neq \vec{0} \tag{1.3}$$

The orthogonal projection of $\vec{u}$ onto $\vec{v} \neq 0$ is:

$$\vec{u}_v = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$$

1.2. Practice

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \qquad \vec{F} = \begin{pmatrix} -4 \\ -4 \\ -3 \end{pmatrix} \tag{1.4}$$

$$W = \overrightarrow{AB} \cdot \vec{F} \tag{1.5}$$

$$= 12\, J \tag{1.6}$$

. . . .

$$A = (2, -2, 0), \qquad B = (3, 1, -3) \tag{1.7}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \tag{1.8}$$

$$|\overrightarrow{AB}| = \sqrt{1^2 + 3^2 + (-3)^2} \tag{1.9}$$

$$= \sqrt{19} \tag{1.10}$$

. . . .

$$\vec{v} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \tag{1.11}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \tag{1.12}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} \tag{1.13}$$

$$\hat{v} = \begin{pmatrix} -\frac{\sqrt{5}}{5} \\ 0 \\ -\frac{2\sqrt{5}}{5} \end{pmatrix} \tag{1.14}$$

....

$$\vec{u} = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \tag{1.15}$$

$$\text{proj}_v \vec{u} = \vec{u}_v = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \tag{1.16}$$

$$= \frac{-5}{6} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \tag{1.17}$$

$$\vec{v} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \qquad \vec{w} = \begin{pmatrix} -2 \\ -3 \\ h \end{pmatrix} \tag{1.18}$$

$$\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0 \tag{1.19}$$

$$\vec{v} \cdot \vec{w} = 4 + 6 - 2h = 0 \tag{1.20}$$

$$2h = 10 \tag{1.21}$$

$$h = 5 \tag{1.22}$$

....

$$\vec{r} = 2\vec{u} - 2\vec{v}, \qquad |\vec{u}| = 3, \tag{1.23}$$

$$|\vec{v}| = 2, \qquad \vec{u} \cdot \vec{v} = -3 \tag{1.24}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \tag{1.25}$$

$$= -\frac{3}{2 \cdot 3} \tag{1.26}$$

$$= -\frac{1}{2} \tag{1.27}$$

$$\theta = \frac{\pi}{2} + \frac{\pi}{6} \tag{1.28}$$

$$= \frac{2\pi}{3} \tag{1.29}$$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \tag{1.30}$$

$$|\vec{r}| = \sqrt{(2|\vec{u}| - 2|\vec{v}| \cos\theta)^2 + (2|\vec{v}| \sin\theta)^2} \tag{1.31}$$

$$= \sqrt{\left(2 \cdot 3 - 2 \cdot 2 \cos \frac{2\pi}{3}\right)^2 + \left(2 \cdot 2 \sin \frac{2\pi}{3}\right)^2} \tag{1.32}$$

$$= \sqrt{(6+2)^2 + \left(4 \cdot \frac{\sqrt{3}}{2}\right)^2} \tag{1.33}$$

$$= \sqrt{64+12} = \sqrt{76} \tag{1.34}$$

$$|\vec{r}| = 2\sqrt{19} \tag{1.35}$$

. . . .

$$\vec{r} = 3\vec{u} - 2\vec{v} + \vec{w} \tag{1.36}$$

$$\vec{u}, \vec{v}, \vec{w} \text{ are unit vectors that are mutually perpendicular} \tag{1.37}$$

$$|\vec{r}| = \sqrt{\vec{r} \cdot \vec{r}} \tag{1.38}$$

$$= \sqrt{(3\vec{u} - 2\vec{v} + \vec{w}) \cdot (3\vec{u} - 2\vec{v} + \vec{w})} \tag{1.39}$$

$$= \sqrt{3^2 + (-2)^2 + 1^2} \tag{1.40}$$

$$= \sqrt{14} \tag{1.41}$$

. . . .

$$\vec{v} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \tag{1.42}$$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} \tag{1.43}$$

$$= \frac{-2+2+1}{\sqrt{(-2)^2+(-2)^2+(-1)^2} \cdot \sqrt{1^2+(-1)^2+(-1)^2}} \tag{1.44}$$

$$= \frac{1}{\sqrt{9} \cdot \sqrt{3}} \tag{1.45}$$

$$= \frac{1}{3\sqrt{3}} \tag{1.46}$$

$$= \frac{\sqrt{3}}{9} \tag{1.47}$$

$$= \arccos \frac{\sqrt{3}}{9} \tag{1.48}$$

. . . .

Consider the triangle ABC with $A = (1,0,0)$, $B = (0,4,0)$ and $C = (0,0,3)$. Find the cosines of the angles.

$$A: \quad \cos\theta_A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} \tag{1.49}$$

$$= \frac{\begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}}{\sqrt{17} \cdot \sqrt{9}} \tag{1.50}$$

$$= \frac{1}{\sqrt{17 \cdot 10}} \tag{1.51}$$

$$= \frac{1}{\sqrt{170}} \tag{1.52}$$

$$B: \quad \cos\theta_B = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \tag{1.53}$$
$$= \frac{\begin{pmatrix}1\\-4\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-4\\3\end{pmatrix}}{\sqrt{17}\cdot\sqrt{25}} \tag{1.54}$$
$$= \frac{16}{5\sqrt{17}} \tag{1.55}$$
$$C: \quad \cos\theta_C = \frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} \tag{1.56}$$
$$= \frac{\begin{pmatrix}1\\0\\-3\end{pmatrix}\cdot\begin{pmatrix}0\\4\\-3\end{pmatrix}}{\sqrt{10}\cdot\sqrt{25}} \tag{1.57}$$
$$= \frac{9}{5\sqrt{10}} \tag{1.58}$$

. . . .

Consider $A = (-3, 1, 1)$ and the line ℓ through the origin in the direction of $\vec{v} = \langle 1, 3, -3\rangle$. Let B be the orthogonal projection of A onto line ℓ. Find $|\overrightarrow{OB}|$.

$$\overrightarrow{OB} = \overrightarrow{OA}_v = \frac{\overrightarrow{OA}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} \tag{1.59}$$
$$= \frac{\begin{pmatrix}-3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-3\end{pmatrix}}{\begin{pmatrix}1\\3\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\3\\-3\end{pmatrix}}\begin{pmatrix}1\\3\\-3\end{pmatrix} \tag{1.60}$$
$$= \frac{-3}{19}\begin{pmatrix}1\\3\\-3\end{pmatrix} \tag{1.61}$$
$$|\overrightarrow{OB}| = \frac{3}{19}\sqrt{19} \tag{1.62}$$
$$B = \left(-\frac{3}{19}, -\frac{9}{19}, \frac{9}{19}\right) \tag{1.63}$$
$$\overrightarrow{AB} = \begin{pmatrix}-\frac{3}{19}+3\\-\frac{9}{19}-1\\\frac{9}{19}-1\end{pmatrix} \tag{1.64}$$
$$\boxed{|\overrightarrow{AB}| = \sqrt{\left(3-\frac{3}{19}\right)^2+\left(-1-\frac{9}{19}\right)^2+\left(\frac{9}{19}-1\right)^2}} \tag{1.65}$$

Take the points of a tetrahedon: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$ with the centroid at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Find the bond angle.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \tag{1.66}$$

$$= \frac{\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} \cdot \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}} \tag{1.67}$$

$$= \frac{-\frac{1}{4} - \frac{1}{4} + \frac{1}{4}}{\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{3}{4}}} \tag{1.68}$$

$$= \frac{-\frac{1}{4}}{\frac{3}{4}} \tag{1.69}$$

$$\boxed{\cos\theta = -\frac{1}{3}} \tag{1.70}$$

....

$$\frac{|\overrightarrow{AP}|}{|\overrightarrow{AM}|} = \frac{2}{7} \tag{1.71}$$

$$|\overrightarrow{AP}| = \frac{2}{7}|\overrightarrow{AM}| \tag{1.72}$$

$$\overrightarrow{AP} \parallel \overrightarrow{AM} \tag{1.73}$$

$$\overrightarrow{AM} = \frac{\vec{c} - \vec{b}}{2} - \vec{a} \tag{1.74}$$

$$\vec{p} = \vec{a} + \frac{|\overrightarrow{AP}|}{|\overrightarrow{AM}|}\overrightarrow{AM} \tag{1.75}$$

$$= \vec{a} + \frac{2}{7}\left(\frac{\vec{b} + \vec{c}}{2} - a\right) \tag{1.76}$$

$$\vec{p} = \frac{5}{7}\vec{a} + \frac{\vec{b}}{7} + \frac{\vec{c}}{7} \tag{1.77}$$

2

Lecture 2: Cross Product

2.1. Theory

Definition 2.1.1: Cross product

For any vectors $\vec{u}$ and $\vec{v}$ <u>in $\mathbb{R}^3$</u>, the cross product $\vec{u} \times \vec{v}$ is the <u>unique</u> vector satisfying the following three conditions:

1. $(\vec{u} \times \vec{v}) \perp \vec{u}$ and $\vec{u} \times \vec{v} \perp \vec{v}$
2. $\vec{u}$, $\vec{v}$, and $\vec{u} \times \vec{v}$ are positively oriented
3. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

Definition 2.1.2: Determinant

2×2 matrix:

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1 \tag{2.1}$$

3×3 matrix:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \tag{2.2}$$

The cross product of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \tag{2.3}$$

The cross product can also be written as the following determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{i}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{2.4}$$

The area of the parallelogram spanned by vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ is

$$\text{Area} = |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta = \left\| \begin{matrix} u_1 & u_2 \\ v_1 & v_2 \end{matrix} \right\| = \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{i}} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{matrix} \right\| \tag{2.5}$$

The volume of the parallelepiped spanned by vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ is

$$\text{Vol} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = \left\| \begin{matrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{matrix} \right\| \tag{2.6}$$

2.2. Practice

$$\tag{2.7}$$

$$A = (4, 1, -1), \quad B = (3, 2, 0), \quad C = (2, -1, 2) \tag{2.8}$$

$$\text{Area} = |\vec{a} \times \vec{b}| \tag{2.9}$$

$$= \left| \begin{pmatrix} 0-2 \\ -3-0 \\ 8-3 \end{pmatrix} \right| \tag{2.10}$$

$$= \left| \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} \right| \tag{2.11}$$

$$= \sqrt{(-2)^2 + (-3)^2 + 5^2} \tag{2.12}$$

$$= \sqrt{38} \tag{2.13}$$

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \overrightarrow{AC} = \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} \tag{2.14}$$

$$\text{Area} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| \tag{2.15}$$

$$= \frac{1}{2} \left| \begin{pmatrix} -1+2 \\ -2-1 \\ 2+2 \end{pmatrix} \right| \tag{2.16}$$

$$= \frac{1}{2}\sqrt{1^2 + (-3)^2 + 4^2} \tag{2.17}$$

$$\text{Area} = \frac{\sqrt{26}}{2} \tag{2.18}$$

....

$$q = 1, \quad \vec{v} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \tag{2.19}$$

$$\vec{F} = q(\vec{v} \times \vec{B}) = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \tag{2.20}$$

$$= 1 \begin{pmatrix} 12-4 \\ 8-12 \\ 6-12 \end{pmatrix} \tag{2.21}$$

$$\boxed{\vec{F} = \begin{pmatrix} 8 \\ -4 \\ -6 \end{pmatrix}} \tag{2.22}$$

....

$$M = \begin{pmatrix} -1 & -1 \\ -3 & 0 \end{pmatrix} \tag{2.23}$$

$$\det(M) = |M| = \begin{vmatrix} -1 & -1 \\ -3 & 0 \end{vmatrix} \tag{2.24}$$

$$= -1 \cdot 0 - (-1) \cdot (-3) \tag{2.25}$$

$$= -3 \tag{2.26}$$

$$\vec{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{2.27}$$

$$\text{Area} = |\vec{a} \times \vec{b}| \tag{2.28}$$

$$= \left| \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} \right| \tag{2.29}$$

$$= |2 - 3| \tag{2.30}$$

$$\text{Area} = 1 \tag{2.31}$$

$$\vec{a} = \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{2.32}$$

$$\text{Vol} = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \begin{pmatrix} -3 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2+1 \\ 1 \\ 2 \end{pmatrix} \right| \tag{2.33}$$

$$= |-3 \cdot -1 - 4| \tag{2.34}$$

$$= 1 \tag{2.35}$$

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} -2 \\ 0 \\ h \end{pmatrix} \tag{2.36}$$

$$coplanar \iff (\text{Vol} = 0) \tag{2.37}$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) = 0 \tag{2.38}$$

$$= \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -h \\ -4 \\ -2 \end{pmatrix} \tag{2.39}$$

$$= -3h - 8 + 4 \tag{2.40}$$

$$3h = -4 \tag{2.41}$$

$$h = -\frac{4}{3} \tag{2.42}$$

$$M = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 2 & 2 \end{pmatrix} \tag{2.43}$$

$$\det M = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \left(\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right) \tag{2.44}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} \tag{2.45}$$

$$= 4 \tag{2.46}$$

. . . .

$$N = \begin{pmatrix} a_1 & b_1 & 4a_1 \\ a_2 & b_2 & 4a_2 \\ a_3 & b_3 & 4a_3 \end{pmatrix} \tag{2.47}$$

$$\det N = a_1 \begin{vmatrix} b_2 & 4a_2 \\ b_3 & 4a_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & 4a_2 \\ a_3 & 4a_3 \end{vmatrix} + 4a_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{2.48}$$

$$= a_1(4a_3b_2 - 4a_2b_3) - b_1(4a_2a_3 - 4a_2a_3) + 4a_1(a_2b_3 - a_3b_2) \tag{2.49}$$

$$= \cancel{4a_1a_3b_2} - \cancel{4a_1a_2b_3} - \cancel{4a_2a_3b_1} + \cancel{4a_2a_3b_1} + \cancel{4a_1a_2b_3} - \cancel{4a_1a_3b_2} \tag{2.50}$$

$$= 0 \tag{2.51}$$

. . . .

$$A(x) = \begin{pmatrix} x^2+1 & -x & -11 \\ x^2 & 1-x & -7 \\ x^2 & -x & -10 \end{pmatrix}, \quad G(x) = \det(A(x)) \tag{2.52}$$

$$G(x) = \begin{pmatrix} x^2+1 \\ -x \\ -11 \end{pmatrix} \cdot \left(\begin{pmatrix} x^2 \\ 1-x \\ -7 \end{pmatrix} \times \begin{pmatrix} x^2 \\ -x \\ -10 \end{pmatrix} \right) \tag{2.53}$$

$$= \ldots \cdot \begin{pmatrix} 10x - 10 - 7x \\ -7x^2 + 10x^2 \\ -x^3 + x^3 - x^2 \end{pmatrix} \tag{2.54}$$

$$= \begin{pmatrix} x^2+1 \\ -x \\ -11 \end{pmatrix} \cdot \begin{pmatrix} 3x - 10 \\ 3x^2 \\ -x^2 \end{pmatrix} \tag{2.55}$$

$$= 3x^3 - 10x^2 + 3x - 10 - 4x^3 + 11x^2 \tag{2.56}$$

$$= -x^3 + x^2 + 3x - 10 \tag{2.57}$$

$$\tag{2.58}$$

$$\begin{pmatrix} x^2+1 & -x & -11 \\ x^2 & 1-x & -7 \\ x^2 & -x & -10 \end{pmatrix} \tag{2.59}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ x^2 & -x & -10 \end{pmatrix} \tag{2.60}$$

$$\det A = 0 = -10 + 3x - x^2 \tag{2.61}$$

$$0 = x^2 - 3x + 10 \tag{2.62}$$

. . . .

$$\vec{u} = \langle 0, 3, 0 \rangle, \quad \vec{u} \times \vec{v} = \langle -3, -3, 10 \rangle \tag{2.63}$$

$$\vec{u} \times \vec{v} = \begin{pmatrix} 3v_3 \\ 0 \\ \ldots \end{pmatrix} \implies None \tag{2.64}$$

. . . .

$$\vec{u} = \langle 6, 6, 5 \rangle, \quad \vec{u} \times \vec{v} = \vec{0} \tag{2.65}$$

$$\vec{u} \times \vec{v} = \begin{pmatrix} 6v_3 - 5v_2 \\ 5v_1 - 6v_3 \\ 6v_2 - 6v_1 \end{pmatrix} \tag{2.66}$$

$$= \begin{pmatrix} 6v_3 - 5v_2 \\ 5v_1 - 6v_3 \\ 6v_2 - 6v_1 \end{pmatrix} \tag{2.67}$$

$$\implies v_1 = v_2 \tag{2.68}$$

$$\implies 6v_3 - 5v_2 = 0 \tag{2.69}$$

$$6v_3 = 5v_2 = 5v_1 \tag{2.70}$$

$$v_3 = \frac{5}{6} v_2 \tag{2.71}$$

$$\vec{v} = v \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix} \tag{2.72}$$

$$R = \frac{1}{2} |\overrightarrow{OA} \times \overrightarrow{AB}| + \frac{1}{2} |\overrightarrow{OB} \times \overrightarrow{BC}| \tag{2.73}$$

3

Lecture 3: Functions and Invertibility

Sets: $A, B, \ldots$
Element: $x \in A$
Subset: $B \subset A$
Intersection: $A \cap B$
Union: $A \cup B$

$$y = \log_a x \implies a^y = x \tag{3.1}$$
$$\tag{3.2}$$

$$\arcsin\left(-\frac{1}{2}\sqrt{2}\right) = -\frac{\pi}{4} \tag{3.3}$$

$$\arccos\left(-\frac{1}{2}\sqrt{2}\right) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} \tag{3.4}$$

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \tag{3.5}$$

$$f(x) = 4\ln x + 5 \tag{3.6}$$
$$\frac{y-5}{4} = \ln x \tag{3.7}$$
$$x = e^{\frac{y-5}{4}} \tag{3.8}$$
$$f^{-1} = y = e^{\frac{x-5}{4}} \tag{3.9}$$

$$f(x) = e^{\sqrt{3+x}} \tag{3.10}$$
$$y = e^{\sqrt{3+x}} \tag{3.11}$$
$$\sqrt{3+x} = \ln y \tag{3.12}$$
$$3 + x = (\ln y)^2 \tag{3.13}$$
$$x = (\ln y)^2 - 3 \tag{3.14}$$
$$f^{-1}(x) = y = (\ln x)^2 - 3 \tag{3.15}$$

$$f(x) = \sqrt{\frac{x}{x-16}} \tag{3.16}$$
$$D_f = (-\infty, 0] \cup (16, \infty) \tag{3.17}$$

$$f(x) = \ln\left(\frac{x}{x-4}\right) \tag{3.18}$$
$$D_f = (-\infty, 0) \cup (4, \infty) \tag{3.19}$$

$$\sin\left(\arctan\left(-\frac{9}{8}\right)\right) = -\frac{9}{\sqrt{145}} \tag{3.20}$$
$$\cos\left(\arctan\left(-\frac{2}{5}\right)\right) = \frac{5}{\sqrt{29}} \tag{3.21}$$

$$f(x) = \frac{x+5}{x-4} \tag{3.22}$$
$$D_f = (-\infty, 4) \cup (4, \infty) \tag{3.23}$$
$$y = \frac{x+5}{x-4} \tag{3.24}$$
$$y(x-4) = x+5 \tag{3.25}$$
$$yx - 4y - x - 5 = 0 \tag{3.26}$$
$$x(y-1) = 4y+5 \tag{3.27}$$
$$x = \frac{4y+5}{y-1} \tag{3.28}$$
$$f^{-1}(x) = \frac{4x+5}{x-1} \tag{3.29}$$
$$R_f = D_{f^{-1}} = (-\infty, 1) \cup (1, \infty) \tag{3.30}$$

$$f(x) = \frac{1}{x^2} | x \in [1, \infty] \tag{3.31}$$
$$f(u) = x \tag{3.32}$$
$$\frac{1}{u^2} = x \tag{3.33}$$
$$\frac{1}{x} = u^2 \tag{3.34}$$
$$\frac{1}{\sqrt{x}} = u \tag{3.35}$$

4

Lecture 4: Limits

4.1. Theory

$$f(x) = \frac{1}{x+2}, \text{ prove that } \lim_{x\to 0} f(x) = \frac{1}{2} \tag{4.1}$$
$$\epsilon > 0, \left| f(x) - \frac{1}{2} \right| < \epsilon \tag{4.2}$$
$$\left| f(x) - \frac{1}{2} \right| = \left| \frac{2}{2x+4} - \frac{x+2}{2x+4} \right| = \left| \frac{x}{2(x+2)} \right| \tag{4.3}$$

Definition 4.1.1: Epsilon-Delta definition

$$\lim_{x\to a} f(x) = L \text{ means} \tag{4.4}$$
$$\forall\, \epsilon > 0,\ \exists\, \delta > 0, \text{ s.t.} \tag{4.5}$$
$$0 < |x-a| < \delta \implies |f(x) - L| < \epsilon \tag{4.6}$$

$$f(x) = \frac{x^2+x-6}{x-2}, \text{ prove that } \lim_{x\to 2} f(x) = 5 \tag{4.7}$$
$$f(x) = \frac{(x-2)(x+3)}{x-2} = x + 3 \forall x \neq 2 \tag{4.8}$$
$$\text{Given } \epsilon > 0 \tag{4.9}$$
$$\text{Choose } \delta = \epsilon \tag{4.10}$$
$$\text{Suppose } 0 < |x-2| < \delta \tag{4.11}$$
$$\text{Check: } |f(x) - 5| < \epsilon \tag{4.12}$$
$$= |x+3-5| < \epsilon \tag{4.13}$$
$$= |x-2| < \epsilon = \delta \tag{4.14}$$
$$0 < |x-2| < \delta \iff |f(x) - 5| < \epsilon \tag{4.15}$$

$$f(x) = \frac{1}{x+2}, \text{ prove that } \lim_{x\to 0} f(x) = \frac{1}{2} \tag{4.16}$$
$$\text{Given } \epsilon > 0 \tag{4.17}$$
$$\text{Choose } \delta = \tag{4.18}$$
$$\text{Suppose } 0 < |x-0| < \delta \tag{4.19}$$
$$\text{Check: } \left| f(x) - \frac{1}{2} \right| < \epsilon \tag{4.20}$$
$$\left| \frac{1}{x+2} - \frac{1}{2} \right| < \epsilon \tag{4.21}$$
$$\left| \frac{2}{2(x+2)} - \frac{x+2}{2(x+2)} \right| < \epsilon \tag{4.22}$$

$$\left|\frac{x}{2(x+2)}\right| < \epsilon \tag{4.23}$$

$$\left|\frac{x}{2(x+2)}\right| \leq \left|\frac{x}{2}\right| < \epsilon = \frac{\delta}{2} \tag{4.24}$$

$$0 < |x-0| < \delta \iff |idkman| \tag{4.25}$$

4.2. Practice

$$\lim_{x\to 3} \frac{x^2-11x+24}{x^2-12x+27} = \lim_{x\to 3} \frac{(x-3)(x-8)}{(x-3)(x-9)} \tag{4.26}$$

$$= \lim_{x\to 3} \frac{x-8}{x-9} \tag{4.27}$$

$$= \frac{3-8}{3-9} \tag{4.28}$$

$$= \frac{-5}{-6} \tag{4.29}$$

$$= \frac{5}{6} \tag{4.30}$$

....

$$\lim_{x\to -4} \frac{\sqrt{-x-3}-1}{x+4} = \lim_{x\to -4} \frac{(\sqrt{-x-3}-1)(\sqrt{-x-3}+1)}{(x+4)(\sqrt{-x-3}+1)} \tag{4.31}$$

$$= \lim_{x\to -4} \frac{-x-4}{(x+4)(\sqrt{-x-3}+1)} \tag{4.32}$$

$$= \lim_{x\to -4} \frac{-(x+4)}{(x+4)(\sqrt{-x-3}+1)} \tag{4.33}$$

$$= \lim_{x\to -4} \frac{1}{\sqrt{-x-3}+1} \tag{4.34}$$

$$= -\frac{1}{2} \tag{4.35}$$

....

$$\lim_{x\to\infty} \frac{5x-2}{x^2-5x-8} = \lim_{x\to\infty} \frac{\frac{5}{x}-\frac{2}{x^2}}{1-\frac{5}{x}-\frac{8}{x^2}} = 0 \tag{4.36}$$

$$\lim_{x\to -3} \frac{1-\frac{9}{x^2}}{x-3} = 0 \tag{4.37}$$

$$\lim_{x\to 3} \frac{1-\frac{9}{x^2}}{x-3} = \lim_{x\to 3} \frac{x^2-9}{x^2(x-3)} \tag{4.38}$$

$$= \lim_{x\to 3} \frac{(x+3)(x-3)}{x^2(x-3)} \tag{4.39}$$

$$= \lim_{x\to 3} \frac{x+3}{x^2} \tag{4.40}$$

$$= \frac{6}{9} = \frac{2}{3} \tag{4.41}$$

....

$$\lim_{x\uparrow 7} \frac{x^2-10x+21}{|x-7|} = \lim_{x\uparrow 7} \frac{x^2-10x+21}{-(x-7)} \tag{4.42}$$

$$= \lim_{x\uparrow 7} \frac{(x-3)(x-7)}{-(x-7)} \tag{4.43}$$

$$= \lim_{x\uparrow 7} (3-x) \tag{4.44}$$

$$= -4 \tag{4.45}$$

4.3. Restart lol

$$\lim_{x\to 2} \frac{x^2-3x+2}{x^2-6x+8} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)}{(x-4)\cancel{(x-2)}} \tag{4.46}$$

$$= \lim_{x\to 2} \frac{x-1}{x-4} \tag{4.47}$$

$$\boxed{= -\frac{1}{2}} \tag{4.48}$$

$$\lim_{x\to -4} \frac{\sqrt{-x-3}-1}{x+4} = \lim_{x\to -4} \frac{\sqrt{-x-3}-1}{x+4} \cdot \frac{\sqrt{-x-3}+1}{\sqrt{-x-3}+1} \tag{4.49}$$

$$= \lim_{x\to -4} \frac{-x-4}{(x+4)(\sqrt{-x-3}+1)} \tag{4.50}$$

$$= \lim_{x\to -4} -\frac{x+4}{(x+4)(\sqrt{-x-3}+1)} \tag{4.51}$$

$$= \lim_{x\to -4} -\frac{1}{\sqrt{-x-3}+1} \tag{4.52}$$

$$= -\frac{1}{2} \tag{4.53}$$

$$\lim_{x\to\infty} \frac{5x^2+8x+9}{-x^2-10x-9} = \lim_{x\to\infty} \frac{5+\frac{8}{x}+\frac{9}{x^2}}{-1-\frac{10}{x}-\frac{9}{x^2}} \tag{4.54}$$

$$= -5 \tag{4.55}$$

$$\lim_{x\to\infty} \frac{5x+8}{-x^2-10x-9} = \lim_{x\to\infty} \frac{5+\frac{8}{x}}{-x-10-\frac{9}{x}} \tag{4.56}$$

$$= -0 \tag{4.57}$$

$$\lim_{x\to\infty} \frac{5x^3+8x^2+9x+7}{-x^2-10x-9} = \lim_{x\to\infty} \frac{5x+8+\frac{9}{x}+\frac{7}{x^2}}{-1-\frac{10}{x}-\frac{9}{x^2}} \tag{4.58}$$

$$= -\infty \tag{4.59}$$

$$\lim_{x\to 5} \frac{1-\frac{25}{x^2}}{x-5} = \lim_{x\to 5} \frac{x^2-25}{x^3-5x^2} \tag{4.60}$$

$$= \lim_{x\to 5} \frac{(x-5)(x+5)}{x^2(x-5)} \tag{4.61}$$

$$= \lim_{x\to 5} \frac{x+5}{x^2} \tag{4.62}$$
$$= \frac{5+5}{5^2} \tag{4.63}$$
$$\boxed{= \frac{2}{5}} \tag{4.64}$$

....

$$\lim_{x\uparrow 7} \frac{x^2-12x+35}{|x-7|} = \lim_{x\uparrow 7} \frac{(x-5)(x-7)}{-(x-7)} \tag{4.65}$$
$$= \lim_{x\uparrow 7} 5-x \tag{4.66}$$
$$= -2 \tag{4.67}$$

....

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} e^{n\ln\left(1+\frac{a}{n}\right)} \tag{4.68}$$
$$= \lim_{n\to\infty} e^a \tag{4.69}$$
$$= e^a \tag{4.70}$$

....

$$h(x) \leq 5e^x \leq \frac{2h(x)}{x} \tag{4.71}$$
$$\frac{h(x)}{e^x} \leq 5 \leq \frac{2h(x)}{xe^x} \tag{4.72}$$
$$\frac{xh(x)}{2e^x} \leq \frac{5x}{2} \leq \frac{h(x)}{e^x} \tag{4.73}$$
$$\frac{5x}{2} \leq \frac{h(x)}{e^x} \leq 5 \tag{4.74}$$

....

$$\lim_{x\to 0} \frac{\sin x}{\arcsin x} = \lim_{x\to 0} \frac{\sin x}{\arcsin x} \cdot \frac{x}{x} \tag{4.75}$$
$$= \lim_{x\to 0} \frac{x}{\arcsin x} \cdot \frac{\sin x}{x} \tag{4.76}$$
$$= 1 \cdot 1 \tag{4.77}$$
$$= 1 \tag{4.78}$$

5

Lecture 5: Continuity

$$f_1(2) = f_2(2) \tag{5.1}$$
$$f_1(2) = 2c = 2^3 = f_2(2) \tag{5.2}$$
$$c = 2^2 \tag{5.3}$$
$$\boxed{c = 4} \tag{5.4}$$

6

Lecture 6: Differentiation

6.1. Lecture

$$
\begin{aligned}
\mathscr{C} : x^2 + y^2 &= 25 && (6.1)\\
y^2 &= 25 - x^2 && (6.2)\\
y &= \sqrt{25 - x^2} = (25 - x^2)^{1/2} && (6.3)\\
\frac{dy}{dx} &= \frac{1}{2}(25 - x^2)^{-1/2}(-2x) && (6.4)\\
&= -\frac{x}{\sqrt{25 - x^2}} && (6.5)\\
\left.\frac{dy}{dx}\right|_{x=3} &= -\frac{3}{4} && (6.6)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{C} : x^2 + y^2 &= 25 && (6.7)\\
2x + 2y\frac{dy}{dx} &= 0 && (6.8)\\
\frac{dy}{dx} &= -\frac{x}{y} && (6.9)\\
\left.\frac{dy}{dx}\right|_{x=3, y=4} &= -\frac{3}{4} && (6.10)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{C} : y^2 &= x^3 + 2 && (6.11)\\
2y\frac{dy}{dx} &= 3x^2 && (6.12)\\
\frac{dy}{dx} &= \frac{3x^2}{2y} && (6.13)\\
\frac{dy}{dx}(1, \sqrt{3}) &= \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} && (6.14)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{C} : y^2 &= x^3 + 2 && (6.15)\\
2y &= 3x^2\frac{dx}{dy} && (6.16)\\
\frac{dx}{dy} &= \frac{2y}{3x^2} && (6.17)\\
\text{Vertical at } 2y = 0 &\implies y = 0 && (6.18)
\end{aligned}
$$

6.2. Homework

$$f(x) = \ln\left(\frac{1}{\cos x}\right) \tag{6.19}$$
$$f'(x) = \frac{1}{\frac{1}{\cos x}} \cdot \frac{\sin x}{\cos^2 x} \tag{6.20}$$
$$= \frac{\cancel{\cos x}\sin x}{\cos^{\cancel{2}} x} \tag{6.21}$$
$$= \frac{\sin x}{\cos x} \tag{6.22}$$
$$= \tan x \tag{6.23}$$

. . . .

$$f(x) = \sin(x\ln(x)) \tag{6.24}$$
$$\frac{d}{dx}(x\ln x) = \ln x + 1 \tag{6.25}$$
$$f'(x) = \cos(x\ln x)(\ln x + 1) \tag{6.26}$$

. . . .

$$f(x) = \cos e^{3x} \tag{6.27}$$
$$f'(x) = -\sin(e^{3x})3e^{3x} \tag{6.28}$$

. . . .

$$\text{Find } \frac{d}{dt}(\sin^2(f(t)) + \cos^2(f(t))) \tag{6.29}$$
$$= 2\sin(f(t))\cos(f(t))f'(t) + 2\cos(f(t))(-\sin)\ldots \tag{6.30}$$
$$\text{it cancels} \tag{6.31}$$
$$= 0 \tag{6.32}$$

. . . .

$$f(x) = \frac{3}{\sin(x^2+1)} \tag{6.33}$$
$$f'(x) = \frac{-3\cos(x^2+1)(2x)}{\sin(x^2+1)^2} \tag{6.34}$$

. . . .

$$f(x) = \ln(\ln x) \tag{6.35}$$
$$f'(x) = \frac{1}{\ln x}\frac{1}{x} \tag{6.36}$$
$$= \frac{1}{x\ln x} \tag{6.37}$$

. . . .

$$\lim_{h\to 0}\frac{\tan(x+h) - \tan x}{h} = \frac{d}{dx}\tan x \tag{6.38}$$

$$= \frac{d}{dx}\frac{\sin x}{\cos x} \tag{6.39}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \tag{6.40}$$
$$= \frac{1}{\cos^2 x} \tag{6.41}$$

$$\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{d}{dx}\frac{1}{\sqrt{x}} \tag{6.42}$$
$$= \frac{d}{dx}x^{-1/2} \tag{6.43}$$
$$= -\frac{1}{2}x^{-3/2} \tag{6.44}$$

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h} \tag{6.45}$$
$$= \frac{d}{dx}\sqrt{x}\bigg|_{x=4} \tag{6.46}$$
$$= \frac{1}{2\sqrt{x}}\bigg|_{x=4} \tag{6.47}$$
$$= \frac{1}{4} \tag{6.48}$$

$$\lim_{h\to 0}\frac{(2+h)^3-8}{h} = \frac{d}{dx}(x^3)\bigg|_{x=2} \tag{6.49}$$
$$= 3\cdot 2^2 \tag{6.50}$$
$$= 12 \tag{6.51}$$

$$f(x) = \ln\left(\frac{1+\sin x}{\cos x}\right) \tag{6.52}$$
$$f'(x) = \frac{\cos x}{1+\sin x}\cdot\frac{\cos x\cos x - (1+\sin x)(-\sin x)}{\cos^2 x} \tag{6.53}$$
$$= \frac{\cos x}{1+\sin x}\cdot\frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x} \tag{6.54}$$
$$= \frac{\cos x}{1+\sin x}\cdot\frac{1+\sin x}{\cos^2 x} \tag{6.55}$$
$$= \frac{\cancel{(1+\sin x)}\,\cancel{\cos x}}{\cancel{(1+\sin x)}\,\cos^{\cancel{2}} x} \tag{6.56}$$
$$= \frac{1}{\cos x} \tag{6.57}$$

$$\mathscr{C} : x^2y - 3x + y^2 = 2x - y - 4 \tag{6.58}$$

$$2xy + x^2\frac{dy}{dx} - 3 + 2y\frac{dy}{dx} = 2 - \frac{dy}{dx} \tag{6.59}$$
$$(x^2 + 2y + 1)\frac{dy}{dx} = 2 - 2xy + 3 \tag{6.60}$$
$$\frac{dy}{dx} = \frac{5 - 2xy}{x^2 + 2y + 1} \tag{6.61}$$
$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{5 - 2 \cdot 2 \cdot 1}{2^2 + 2 \cdot 1 + 1} \tag{6.62}$$
$$= \frac{1}{7} \tag{6.63}$$

. . . .

$$f(x) = \frac{x}{x^2 + 1} \tag{6.64}$$
$$f'(2) = \lim_{h\to 0} g(h) \tag{6.65}$$
$$= \lim_{h\to 0} \frac{f(2+h) - f(h)}{h} \tag{6.66}$$
$$= \lim_{h\to 0} \frac{\frac{2+h}{(2+h)^2+1} - \frac{2}{2^2+1}}{h} \tag{6.67}$$
$$= \lim_{h\to 0} \frac{\frac{(2+h)}{h^2+4h+5} - \frac{2}{5}}{h} \tag{6.68}$$
$$= \lim_{h\to 0} \frac{5(2+h) - 2(h^2 + 4h + 5)}{5h^3 + 20h^2 + 25h} \tag{6.69}$$
$$= \lim_{h\to 0} \frac{\cancel{10} + 5h - 2h^2 - 8h - \cancel{10}}{5h(h^2 + 4h + 5)} \tag{6.70}$$
$$= \lim_{h\to 0} \frac{-2h^2 - 3h}{5h((h+2)^2 + 1)} \tag{6.71}$$
$$= \lim_{h\to 0} \frac{-\cancel{h}(h+3)}{5\cancel{h}((h+2)^2 + 1)} \tag{6.72}$$
$$= -\frac{1}{5}\lim_{h\to 0} \frac{h+3}{(h+2)^2 + 1} \tag{6.73}$$
$$= -\frac{1}{5}\frac{3}{5} \tag{6.74}$$
$$= -\frac{3}{25} \tag{6.75}$$

. . . .

$$\mathscr{C} : x^5y^3 + 2x^4 - y^4 = 6 \tag{6.76}$$
$$5x^4\frac{dx}{dy}y^3 + 3x^5y^2 + 8x^3\frac{dx}{dy} - 4y^3 = 0 \tag{6.77}$$
$$(5x^4y^3 + 8x^3)\frac{dx}{dy} = 4y^3 - 3x^5y^2 \tag{6.78}$$
$$\frac{dx}{dy} = \frac{4y^3 - 3x^5y^2}{5x^4y^3 + 8x^3} \tag{6.79}$$

. . . .

$$f(x) = \arccos(\sin x) \tag{6.80}$$
$$f'(x) = -\frac{\cos x}{\sqrt{1 - \sin^2 x}} \tag{6.81}$$

$$\sin x = 1 \vee \sin x = -1 \tag{6.82}$$

$$x = \frac{\pi}{2} + k\pi \tag{6.83}$$

7

Lecture 7: Linear Approximations and Differentials

7.1. Theory

Linearization!!

$$y = f(a) + f'(a)(x - a) \tag{7.1}$$

$$\begin{aligned} L(x) &= \sqrt{4} + \frac{d}{dx}(\sqrt{x})\bigg|_{x=4}(3.94 - 4) && (7.2)\\ &= 2 + \frac{1}{2 \cdot 2}(3.94 - 4) && (7.3)\\ &= 2 - \frac{1}{4}(0.06) && (7.4)\\ &= 1.985 && (7.5) \end{aligned}$$

Shorthand

$$\begin{aligned} df(x_0) = f'(x_0)dx \iff L(x) - f(x_0) = f'(x_0)(x - x_0) && (7.6)\\ df = f'dx && (7.7) \end{aligned}$$

$$df(x^2) = f'(x^2)2x\,dx \tag{7.8}$$

Error formula or (Lagrange[1]) Remainer:

$$R(x) = \frac{1}{2}f''(s)(x - x_0)^2 \text{ for some } s \text{ between } x_0 \text{ and } x \tag{7.9}$$

7.2. Practice

$$\begin{aligned} y(u) &= 21u^{-1} && (7.10)\\ y'(u) &= -21u^{-2} && (7.11)\\ dy &= y'(8)du && (7.12)\\ &= -\frac{21}{8^2} \cdot -1 && (7.13)\\ &= \frac{21}{64} && (7.14) \end{aligned}$$

....

$$y = 3x^2 - 4x \tag{7.15}$$

[1]This is actually a reference to ZZ Top

$$\Delta y = (3\cdot(1+0.2)^2 - 4\cdot(1+0.2)) - (3\cdot 1^2 - 4\cdot 1) \tag{7.16}$$
$$= 3\cdot 1.2^2 - 4\cdot 1.2 - 3 + 4 \tag{7.17}$$
$$= \text{dont like calculating} \tag{7.18}$$

$$\frac{dy}{dx} = 6x - 4 \tag{7.19}$$
$$dy = \left.\frac{dy}{dx}\right|_1 dx \tag{7.20}$$
$$= (6-4)0.2 \tag{7.21}$$
$$= \frac{2}{5} \tag{7.22}$$

. . . .

$$f(x) = \sqrt{4-x} \tag{7.23}$$
$$f'(x) = -\frac{1}{2\sqrt{4-x}} \tag{7.24}$$
$$L(x) = f(a) + f'(a)dx \tag{7.25}$$
$$= \sqrt{4-3} + \frac{3-x}{2\sqrt{4-3}} \tag{7.26}$$

. . . .

$$f(x) = \arctan 6x \tag{7.27}$$
$$f'(x) = \frac{6}{1+36x^2} \tag{7.28}$$
$$\tag{7.29}$$

. . . .

$$y = x^{1/3} \tag{7.30}$$
$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \tag{7.31}$$
$$L_8(x) = 2 + \frac{1}{3}8^{-2/3}(x-8) \tag{7.32}$$
$$\tag{7.33}$$

. . . .

$$y = Kx^n \tag{7.34}$$
$$dy = Knx^{n-1}dx \tag{7.35}$$
$$\frac{dy}{y} = \frac{Knx^{n-1}dx}{Kx^n} \tag{7.36}$$
$$\frac{dy}{y} = \frac{nx^{-1}dx\cdot x}{x} \tag{7.37}$$
$$\frac{dy}{y} = \frac{ndx}{x} \tag{7.38}$$

. . . .

$$V = \frac{4}{3}\pi r^3 \tag{7.39}$$
$$\frac{dV}{dr} = 4\pi r^2 \tag{7.40}$$
$$\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} \tag{7.41}$$
$$= \frac{3dr}{r} \tag{7.42}$$

....

$$R = C\frac{L}{r^4} \tag{7.43}$$
$$\frac{R}{R_0} = \frac{\frac{1}{r^4}}{\frac{1}{r_0^4}} \tag{7.44}$$
$$= \frac{r_0^4}{r^4} \tag{7.45}$$
$$= \frac{1}{1.012^4} \tag{7.46}$$
$$dR = -4CLr^{-5}dr \tag{7.47}$$
$$\frac{dR}{R_0} = -\frac{4CLr^{-5}dr}{CLr^{-4}} \tag{7.48}$$
$$= -4r^{-1}dr \tag{7.49}$$
$$= -\frac{4dr}{r} \tag{7.50}$$

....

$$f(x) = \sqrt{x} \tag{7.51}$$
$$L_{36}(x) = f(36) + f'(36)(x-36) \tag{7.52}$$
$$= 6 + \frac{1}{12}(x-36) \tag{7.53}$$
$$L_{36}(35.6) = 6 + \frac{35.6-36}{12} \tag{7.54}$$
$$= \frac{179}{30} \tag{7.55}$$

....

$$f(x) = \sin x \tag{7.56}$$
$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) \tag{7.57}$$
$$R(x) = \frac{1}{2}f''(s)(x-x_0)^2 \tag{7.58}$$
$$x_0 = \frac{\pi}{6} \tag{7.59}$$
$$s = \frac{\pi}{6}?\text{ or } s = x_{f(x)_{max}}? \tag{7.60}$$
$$f''(x) = -\sin x \tag{7.61}$$
$$f''(x)_{max} = -\frac{1}{2} \tag{7.62}$$

$$|R_{max}(x)| = \left| -\frac{1}{2} \cdot \frac{1}{2} \left(x - \frac{\pi}{6} \right)^2 \right| \tag{7.63}$$

$$= \frac{1}{4} \left(\frac{29}{180}\pi - \frac{\pi}{6} \right)^2 \tag{7.64}$$

. . . .

$$(x^2 + y^2)^2 = 3x^2 - 2y^2 \tag{7.65}$$

$$2(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) = 6x - 4y \frac{dy}{dx} \tag{7.66}$$

$$4x(x^2 + y^2) + 4y(x^2 + y^2) \frac{dy}{dx} = 6x - 4y \frac{dy}{dx} \tag{7.67}$$

$$(4y(x^2 + y^2) + 4y) \frac{dy}{dx} = 6x - 4x(x^2 + y^2) \tag{7.68}$$

$$\frac{dy}{dx} = \frac{6x - 4x(x^2 + y^2)}{4y(x^2 + y^2) + 4y} \tag{7.69}$$

$$\left. \frac{dy}{dx} \right|_{\left(\frac{3}{2}, \frac{1}{2}\right)} = -\frac{6}{7} \tag{7.70}$$

$$L(x) = \frac{1}{2} - \frac{6}{7} \left(x - \frac{3}{2} \right) \tag{7.71}$$

. . . .

$$x^y = y^x \tag{7.72}$$

$$y \ln x = x \ln y \tag{7.73}$$

$$\ln x \frac{dy}{dx} + \frac{y}{x} = \ln y + \frac{x}{y} \frac{dy}{dx} \tag{7.74}$$

$$\frac{dy}{dx} \left(\ln x - \frac{x}{y} \right) = \ln y - \frac{y}{x} \tag{7.75}$$

$$\frac{dy}{dx} = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} \tag{7.76}$$

$$\left. \frac{dy}{dx} \right|_{(2,4)} = \frac{\ln 4 - \frac{4}{2}}{\ln 2 - \frac{2}{4}} \tag{7.77}$$

$$L(x) = 4 + \frac{\ln 4 - \frac{4}{2}}{\ln 2 - \frac{2}{4}} (x - 2) \tag{7.78}$$

8

Lecture 8: Fundamental Theory of Calculus + Substitution Rule

8.1. Lecture

$$\int e^{x^3+5x}(3x^2+5)\,dx = * \tag{8.1}$$
$$u = x^3+5x \tag{8.2}$$
$$du = (3x^2+5)dx \tag{8.3}$$
$$* = \int e^u\,du \tag{8.4}$$
$$= e^u + C \tag{8.5}$$
$$= e^{x^3+5x} + C \tag{8.6}$$

$$\int \tan x\,dx = * \tag{8.7}$$
$$u = \cos x \tag{8.8}$$
$$du = -\sin x\,dx \tag{8.9}$$
$$* = \int \frac{\sin x}{\cos x}\,dx \tag{8.10}$$
$$= -\int \frac{1}{u}\,du \tag{8.11}$$
$$= -\ln u + C \tag{8.12}$$
$$= -\cos x + C \tag{8.13}$$

8.2. Practice

$$F(t) = \int_0^t x^2\sin(5x^4)\,dx \tag{8.14}$$
$$F'(t) = t^2\sin(5t^4) \tag{8.15}$$

....

$$\int \tan 3x\,dx \text{ for } -\frac{\pi}{6} < x < \frac{\pi}{6} \tag{8.16}$$
$$\int \tan 3x\,dx = \int \frac{\sin 3x}{\cos 3x}\,dx \tag{8.17}$$
$$u = \cos 3x \tag{8.18}$$
$$du = -3\sin 3x\,dx \tag{8.19}$$

$$\int \frac{\sin 3x}{\cos 3x}\,dx = -\int \frac{1}{3u}\,du \tag{8.20}$$
$$= -\frac{1}{3}\ln u + C \tag{8.21}$$
$$= -\frac{1}{3}\ln(\cos(3x)) + C \tag{8.22}$$

....

$$\int_2^5 x^4 e^{x^5}\,dx = * \tag{8.23}$$
$$u = x^5 \tag{8.24}$$
$$du = 5x^4 dx \tag{8.25}$$
$$* = \int_{x=2}^{x=5} \frac{e^u}{5}\,du \tag{8.26}$$
$$= \frac{1}{5}\left[e^{x^5}\right]_2^5 \tag{8.27}$$
$$= \frac{1}{5}\left(e^{5^5} - e^{2^5}\right) \tag{8.28}$$

....

$$\int \tan 4x\,dx = * \tag{8.29}$$
$$u = \cos 4x \tag{8.30}$$
$$du = -4\sin 4x\,dx \tag{8.31}$$
$$* = \int \frac{1}{-4u}\,du \tag{8.32}$$
$$= -\frac{1}{4}\int \frac{1}{u}\,du \tag{8.33}$$
$$= -\frac{1}{4}\ln(\cos(4x)) + C \tag{8.34}$$

....

$$\int_0^4 x^4 e^{x^5}\,dx = * \tag{8.35}$$
$$u = x^5 \tag{8.36}$$
$$du = 5x^4 du \tag{8.37}$$
$$* = \int_{x=0}^{4} \frac{1}{5} e^u\,du \tag{8.38}$$
$$= \frac{1}{5}\int_{x=0}^{4} e^u\,du \tag{8.39}$$
$$= \frac{1}{5}\left[e^{x^5}\right]_0^4 \tag{8.40}$$
$$= \frac{1}{5}(e^{4^5} - 1) \tag{8.41}$$

....

$$\int \frac{s^3}{\sqrt{s^2+4}}\,ds = * \tag{8.42}$$
$$u = s^2 + 4 \tag{8.43}$$

$$du = 2s\,ds \tag{8.44}$$
$$* = \int \frac{u-4}{2\sqrt{u}}\,du \tag{8.45}$$
$$= \int \frac{u}{2\sqrt{u}}\,du + \int \frac{-4}{2\sqrt{u}}\,du \tag{8.46}$$
$$= \frac{1}{2}\int u^{1/2}\,du - 2\int u^{-1/2}\,du \tag{8.47}$$
$$= \frac{1}{2}\frac{2}{3}u^{3/2} - 2\cdot 2\sqrt{u} + C \tag{8.48}$$
$$= \frac{u^{3/2}}{3} - 4\sqrt{u} + C \tag{8.49}$$
$$= \frac{(s^2+4)^{3/2}}{3} - 4\sqrt{s^2+4} + C \tag{8.50}$$

....

$$\int \frac{1}{x^2+3}\,dx = \frac{1}{3}\int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}\,dx = * \tag{8.51}$$
$$u = \frac{x}{\sqrt{3}} \tag{8.52}$$
$$du = \frac{1}{\sqrt{3}}dx \tag{8.53}$$
$$* = \frac{1}{3}\int \frac{\sqrt{3}}{u^2+1}\,du \tag{8.54}$$
$$= \frac{\sqrt{3}}{3}\arctan u + C \tag{8.55}$$
$$= \frac{\sqrt{3}}{3}\arctan \frac{x}{\sqrt{3}} + C \tag{8.56}$$

....

$$\int_{\frac{1}{6}}^{e^4/6} \frac{(\ln 6x)^3}{x}\,dx = * \tag{8.57}$$
$$u = \ln 6x \tag{8.58}$$
$$du = \frac{1}{x}dx \tag{8.59}$$
$$* = \int_{x=\frac{1}{6}}^{e^4/6} u^3\,du \tag{8.60}$$
$$= \left[(\ln 6x)^3\right]_{\frac{1}{6}}^{\frac{e^4}{6}} \tag{8.61}$$
$$= \ln(e^4)^3 - \ln(1)^3 \tag{8.62}$$
$$= 64 \tag{8.63}$$

....

$$\int_0^{\pi/10} \sin^4(5x)\cos(5x)\,dx = * \tag{8.64}$$
$$u = \sin 5x \tag{8.65}$$
$$du = 5\cos 5x\,dx \tag{8.66}$$
$$* = \int_{x=0}^{\frac{\pi}{10}} \frac{u^4}{5}\,du \tag{8.67}$$

$$= \frac{1}{5}\left[\frac{u^5}{5}\right]_{x=0}^{\frac{\pi}{10}} \tag{8.68}$$
$$= \frac{1}{25}\left[\sin^5(5x)\right]_0^{\frac{\pi}{10}} \tag{8.69}$$
$$= \frac{1}{25}\sin\left(\frac{\pi}{2}\right)^5 \tag{8.70}$$
$$= \frac{1}{25} \tag{8.71}$$

....

$$\int_0^{\frac{1}{6}} \frac{\arctan(6x)}{1+36x^2}\,dx = * \tag{8.72}$$
$$u = \arctan(6x) \tag{8.73}$$
$$du = \frac{6}{1+36x^2}\,dx \tag{8.74}$$
$$* = \int_0^{\frac{1}{6}} \frac{u}{6}\,du \tag{8.75}$$
$$= \frac{1}{12}\left[u^2\right]_{x=0}^{\frac{1}{6}} \tag{8.76}$$
$$= \frac{1}{12}(\arctan^2(1) - \arctan^2(0)) \tag{8.77}$$
$$= \frac{1}{12}\left(\frac{\pi}{4}\right)^2 \tag{8.78}$$

....

$$\int_0^5 x^3\sqrt{x^4+1}\,dx = * \tag{8.79}$$
$$u = x^4 + 1 \tag{8.80}$$
$$du = 4x^3\,dx \tag{8.81}$$
$$\frac{du}{4} = x^3\,dx \tag{8.82}$$
$$* = \int_{x=0}^5 \frac{\sqrt{u}}{4}\,du \tag{8.83}$$
$$= \frac{1}{4}\frac{2}{3}\left[u^{3/2}\right]_{x=0}^5 \tag{8.84}$$
$$= \frac{1}{6}\left[(x^4+1)^{\frac{3}{2}}\right]_0^5 \tag{8.85}$$
$$= \frac{1}{6}\left((5^4+1)^{\frac{3}{2}} - 1\right) \tag{8.86}$$

....

$$\int_0^{\sqrt{3}} \frac{2x^3}{\sqrt{x^2+2}}\,dx = * \tag{8.87}$$
$$u = x^2 + 2 \tag{8.88}$$
$$du = 2x\,dx \tag{8.89}$$
$$u(\sqrt{3}) = 5 \tag{8.90}$$
$$u(0) = 2 \tag{8.91}$$
$$* = \int_{x=0}^{\sqrt{3}} \frac{u-2}{\sqrt{u}}\,du \tag{8.92}$$

$$= \int_{x=0}^{\sqrt{3}} u^{\frac{1}{2}}\, du - \int_{x=0}^{\sqrt{3}} 2u^{-\frac{1}{2}}\, du \tag{8.93}$$
$$= \frac{2}{3}\left[u^{3/2}\right]_{x=0}^{\sqrt{3}} - 4\left[\sqrt{u}\right]_{x=0}^{\sqrt{3}} \tag{8.94}$$
$$= \frac{2}{3}\left[u^{\frac{3}{2}}\right]_2^5 - 4\left[\sqrt{u}\right]_2^5 \tag{8.95}$$
$$= \frac{2}{3}\left(5^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) - 4(\sqrt{5} - \sqrt{2}) \tag{8.96}$$

....

$$F(x) = \int_0^{\sqrt{x}} \sin(3u^4)\, du \tag{8.97}$$
$$F'(x) = f(x) \cdot \frac{d}{dx}\sqrt{x} \tag{8.98}$$
$$= \sin(3x^2)\frac{1}{2\sqrt{x}} \tag{8.99}$$
$$= \frac{\sin(3 \cdot 5^2)}{2\sqrt{5}} \tag{8.100}$$

....

$$\int_0^1 x^{\sin x}(t+x)^3\, dt = x^{\sin x}\int_0^1 (t+x)^3\, dt \tag{8.101}$$
$$= x^{\sin x}\left[\frac{(t+x)^4}{4}\right]_0^1 \tag{8.102}$$
$$= \frac{x^{\sin x}}{4}((1+x)^4 - x^4) \tag{8.103}$$

....

$$\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2 + 1}\, dx = * \tag{8.104}$$
$$u = \frac{x}{2} \tag{8.105}$$
$$du = \frac{1}{2}\, dx \tag{8.106}$$
$$2du = dx \tag{8.107}$$
$$= \frac{1}{4}\int \frac{1}{u^2+1}\, 2du \tag{8.108}$$
$$= \frac{1}{2}\arctan\frac{x}{2} + C \tag{8.109}$$

....

$$\int_{\frac{1}{3}}^{\frac{e^3}{3}} \frac{\ln^3 3x}{x}\, dx = * \tag{8.110}$$
$$u = \ln 3x \tag{8.111}$$
$$du = \frac{dx}{x} \tag{8.112}$$
$$* = \int_{x=\frac{1}{3}}^{\frac{e^3}{3}} u^3\, du \tag{8.113}$$

$$= \frac{1}{4}\left[u^4\right]_{x=\frac{1}{3}}^{\frac{e^3}{3}} \tag{8.114}$$
$$= \frac{1}{4}\left[\ln^4 3x\right]_{\frac{1}{3}}^{\frac{e^3}{3}} \tag{8.115}$$
$$= \frac{1}{4}(\ln^4 e^3 - \ln^4 1) \tag{8.116}$$

. . . .

$$\int_0^{\frac{\pi}{6}} \sin^5 3x \cos 3x \, dx = * \tag{8.117}$$
$$u = \sin 3x \tag{8.118}$$
$$\frac{du}{3} = \cos 3x \, dx \tag{8.119}$$
$$* = \int_{x=0}^{\frac{\pi}{6}} u^5 \, \frac{du}{3} \tag{8.120}$$
$$= \frac{1}{18}\left[u^6\right]_{x=0}^{\frac{\pi}{6}} \tag{8.121}$$
$$= \frac{1}{18}\left[\sin^6 3x\right]_0^{\frac{\pi}{6}} \tag{8.122}$$
$$= \frac{1}{18}\left(\sin^6 \frac{\pi}{2}\right) \tag{8.123}$$

. . . .

$$\int_0^{1/3} \frac{\arctan 3x}{1+9x^2} \, dx = * \tag{8.124}$$
$$u = \arctan 3x \tag{8.125}$$
$$\frac{du}{3} = \frac{dx}{1+(3x)^2} \tag{8.126}$$
$$* = \int_{x=0}^{1/3} u \, \frac{du}{3} \tag{8.127}$$
$$= \frac{1}{3}\int_{x=0}^{1/3} u \, du \tag{8.128}$$
$$= \frac{1}{3}\frac{1}{2}\left[\arctan^2 3x\right]_0^{1/3} \tag{8.129}$$
$$= \frac{1}{6}\arctan^2 1 \tag{8.130}$$
$$= \frac{1}{6}\left(\frac{\pi}{4}\right)^2 \tag{8.131}$$

. . . .

$$\int_0^4 x^2\sqrt{x^3+1} \, dx = * \tag{8.132}$$
$$u = x^3 + 1 \tag{8.133}$$
$$u(0) = 1 \tag{8.134}$$
$$u(4) = 65 \tag{8.135}$$
$$\frac{du}{3} = x^2 \, dx \tag{8.136}$$
$$* = \int_1^{65} \sqrt{u} \, \frac{du}{3} \tag{8.137}$$

$$= \frac{1}{3}\int_1^{65} \sqrt{u}\, du \tag{8.138}$$

$$= \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^{65} \tag{8.139}$$

$$= \frac{2}{9}\left(65^{\frac{3}{2}} - 1\right) \tag{8.140}$$

. . . .

$$\int_0^2 \frac{2x^3}{\sqrt{x^2+2}}\, dx = * \tag{8.141}$$

$$u = x^2 + 2 \tag{8.142}$$

$$u(0) = 2 \tag{8.143}$$

$$u(2) = 6 \tag{8.144}$$

$$du = 2x\, dx \tag{8.145}$$

$$* = \int_2^6 \frac{u-2}{\sqrt{u}}\, du \tag{8.146}$$

$$= \int_2^6 u^{1/2}\, du - \int_2^6 2u^{-1/2}\, du \tag{8.147}$$

$$= \frac{2}{3}\left[u^{3/2}\right]_2^6 - 4\left[u^{1/2}\right]_2^6 \tag{8.148}$$

$$= \frac{2}{3}\left(6^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) - 4(\sqrt{6} - \sqrt{2}) \tag{8.149}$$

. . . .

$$F(x) = \int_0^{\sqrt{x}} \sin 2u^4\, du \tag{8.150}$$

$$F'(x) = \frac{d}{dx}(\sqrt{x}) \cdot f(x) \tag{8.151}$$

$$F'(4) = \frac{1}{2\sqrt{4}} \sin(2 \cdot 4^2) \tag{8.152}$$

. . . .

$$\int_0^1 x^{\sin x}(t+x)^2\, dt = \tag{8.153}$$

$$= x^{\sin x}\int_0^1 (t+x)^2\, dt \tag{8.154}$$

$$= x^{\sin x}\left[\frac{1}{3}(t+x)^3\right]_0^1 \tag{8.155}$$

$$= \frac{x^{\sin x}}{3}((1+x)^3 - x^3) \tag{8.156}$$

. . . .

$$\frac{d}{dx}\int_{x^2}^{x^3} \sin(t^2)\, dt = f(b) \cdot \frac{d}{dx}x^3 - f(a) \cdot \frac{d}{dx}x^2 \tag{8.157}$$

$$= 3x^2 \sin(x^6) - 2x \sin(x^4) \tag{8.158}$$

. . . .

$$\int_0^\pi xf(\sin x)\,dx = * \tag{8.159}$$
$$u = \pi - x \tag{8.160}$$
$$u(0) = \pi \tag{8.161}$$
$$u(\pi) = 0 \tag{8.162}$$
$$x = \pi - u \tag{8.163}$$
$$dx = -du \tag{8.164}$$
$$* = \int_\pi^0 -(\pi - u)f(\sin u)\,du \tag{8.165}$$
$$= \int_0^\pi (\pi - u)f(\sin u)\,du \tag{8.166}$$
$$\int_0^\pi xf(\sin x)\,dx = \int_0^\pi (\pi - x)f(\sin x)\,dx \tag{8.167}$$
$$2\int_0^\pi xf(\sin x)\,dx = \int_0^\pi \pi f(\sin x)\,dx \tag{8.168}$$
$$\int_0^\pi xf(\sin x)\,dx = \boxed{\frac{\pi}{2}}\int_0^\pi f(\sin x)\,dx \tag{8.169}$$

. . . .

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx = * \tag{8.170}$$
$$u = \cos x \tag{8.171}$$
$$u(0) = 1 \tag{8.172}$$
$$u(\pi) = -1 \tag{8.173}$$
$$-du = \sin x\,dx \tag{8.174}$$
$$* = -\frac{\pi}{2}\int_0^\pi \frac{1}{1+u^2}\,du \tag{8.175}$$
$$= -\frac{\pi}{2}[\arctan u]_1^{-1} \tag{8.176}$$
$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) \tag{8.177}$$
$$= \frac{\pi^2}{4} \tag{8.178}$$

9

Integration by Parts + Partial Fraction Decomposition

9.1. Theory

Theorem 9.1.1 Integration by Parts
If f and g are continuously differentiable functions, then integration by parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{9.1}$$

Note:-
Let $u = f(x)$ and $v = g(x)$. Then

$$\int u\,dv = uv - \int v\,du \tag{9.2}$$

9.2. Practice

$$\int xe^{9x}\,dx = * \tag{9.3}$$

$$f(x) = x \implies f'(x) = 1 \tag{9.4}$$

$$g'(x) = e^{9x} \implies g(x) = \frac{1}{9}e^{9x} \tag{9.5}$$

$$* = \int f(x)g'(x)\,dx \tag{9.6}$$

$$= \frac{1}{9}xe^{9x} - \int 1 \cdot \frac{1}{9}e^{9x}\,dx \tag{9.7}$$

$$= \frac{1}{9}xe^{9x} - \frac{1}{81}e^{9x} + C \tag{9.8}$$

. . . .

$$\int t\sin 5t\,dt = * \tag{9.9}$$

$$f(t) = t \implies f'(t) = 1 \tag{9.10}$$

$$g'(t) = \sin 5t \implies g(t) = -\frac{1}{5}\cos 5t \tag{9.11}$$

$$* = f(t)g(t) - \int f'(t)g(t)\,dt \tag{9.12}$$

$$= -\frac{t}{5}\cos 5t - \int -\frac{1}{5}\cos 5t\,dt \tag{9.13}$$

$$= -\frac{t}{5}\cos 5t + \frac{1}{5}\cdot\frac{1}{5}\cdot\sin 5t \tag{9.14}$$

$$= -\frac{t}{5}\cos 5t + \frac{1}{25}\sin 5t + C \tag{9.15}$$

$$= \frac{\sin 5t - 5t\cos 5t}{25} + C \tag{9.16}$$

....

$$\text{Given } \ln(n!) = \int_1^n \ln x\, dx \tag{9.17}$$

$$\text{Find } E = \frac{\ln(n!) - (n\ln(n) - n)}{\ln(n!)} \text{ for n=15} \tag{9.18}$$

$$= \frac{\int_1^n \ln x\, dx - (n\ln(n) - n)}{\int_1^n \ln x\, dx} = *_1 \tag{9.19}$$

$$\int_1^n \ln x\, dx = *_2 \tag{9.20}$$

$$f(x) = \ln x \implies f'(x) = \frac{1}{x} \tag{9.21}$$

$$g'(x) = 1 \implies g(x) = x \tag{9.22}$$

$$*_2 = [x\ln x]_1^n - \int_1^n \frac{1}{x} x\, dx \tag{9.23}$$

$$= n\ln n - (n-1) \tag{9.24}$$

$$= n\ln n - n + 1 \tag{9.25}$$

$$*_1 = \frac{\cancel{n\ln n} - \cancel{n} + 1 - \cancel{n\ln n} + \cancel{n}}{n\ln n - n + 1} \tag{9.26}$$

$$= \frac{1}{n\ln n - n + 1} \tag{9.27}$$

$$= \frac{1}{15\ln 15 - 14} \tag{9.28}$$

$$= \boxed{???????} \tag{9.29}$$

....

$$f(x) = \frac{-4}{x^2 + 5 - 6x} \tag{9.30}$$

$$= \frac{-1}{x-5} + \frac{1}{x-1} \tag{9.31}$$

$$= \frac{-1(x-1)}{(x-5)(x-1)} + \frac{x-5}{(x-5)(x-1)} \tag{9.32}$$

$$= \frac{1-x}{x^2 - 6x + 5} + \frac{x-5}{x^2 - 6x + 5} \tag{9.33}$$

9.3. Intermission: how the fuck does partial fraction decomposition work?

The top doesn't matter **IF** the degree on the top is smaller than the one on the bottom.
Always make sure that the degree on the top is ONE less than the one on the bottom.

Example 9.3.1

$$\frac{2x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \tag{9.34}$$

Example 9.3.2

$$\frac{2x+1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2} \tag{9.35}$$

Example 9.3.3

$$\begin{aligned}\frac{2x+1}{(x+1)x^2} &= \frac{A}{x+1} + \frac{Bx+C}{x^2} && (9.36)\\ &= \frac{A}{x+1} + \frac{Bx}{x^2} + \frac{C}{x^2} && (9.37)\\ &= \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2} && (9.38)\end{aligned}$$

Example 9.3.4

$$\frac{2x+1}{(x+1)x^4} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} \tag{9.39}$$

Example 9.3.5

$$\begin{aligned}\frac{1}{x^3-2x^2+x} &= \frac{1}{(x-1)^2x} && (9.40)\\ &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x} && (9.41)\end{aligned}$$

9.4. Practice!!!

$$\begin{aligned}f(x) &= \frac{-4}{x^2-6x+5} && (9.42)\\ &= \frac{-4}{(x-5)(x-1)} && (9.43)\\ &= \frac{A}{x-5} + \frac{B}{x-1} && (9.44)\\ \frac{-4}{x-1} &= A + \frac{B}{x-1}(x-5) && (9.45)\\ \frac{-4}{5-1} &= A + \frac{B}{5-1}\cancelto{0}{(5-5)} && (9.46)\\ &\boxed{A=-1} && (9.47)\\ \frac{-4}{x-5} &= \frac{A}{x-5}(x-1) + B && (9.48)\\ \frac{-4}{1-5} &= \frac{A}{1-5}\cancelto{0}{(1-1)} + B && (9.49)\\ &\boxed{B=1} && (9.50)\\ \frac{-4}{x^2-6x+5} &= \frac{-1}{x-5} + \frac{1}{x-1} && (9.51)\end{aligned}$$

$$\int \frac{1}{(x-5)(x-6)}\,dx = * \tag{9.52}$$
$$\frac{1}{(x-5)(x-6)} = \frac{A}{x-5} + \frac{B}{x-6} \tag{9.53}$$
$$A = \frac{1}{5-6} = -1 \tag{9.54}$$
$$B = \frac{1}{6-5} = 1 \tag{9.55}$$
$$* = -\int \frac{1}{x-5}\,dx + \int \frac{1}{x-6}\,dx \tag{9.56}$$
$$= -\ln(x-5) + \ln(x-6) + C \tag{9.57}$$

....

$$f(y) = \frac{1}{y(y+2)} \tag{9.58}$$
$$\frac{1}{y(y+2)} = \frac{A}{y} + \frac{B}{y+2} \tag{9.59}$$
$$\frac{1}{y+2} = A + B\frac{y}{y+2} \tag{9.60}$$
$$A = \frac{1}{0+2} = \frac{1}{2} \tag{9.61}$$
$$\frac{1}{y} = A\frac{y+2}{y} + B \tag{9.62}$$
$$B = \frac{1}{-2} \tag{9.63}$$
$$f(y) = \frac{\frac{1}{2}}{y} + \frac{-\frac{1}{2}}{y+2} \tag{9.64}$$
$$\int f(y)\,dy = \int \frac{\frac{1}{2}}{y}\,dy + \int \frac{-\frac{1}{2}}{y+2}\,dy \tag{9.65}$$
$$= \frac{1}{2}\ln(y) - \frac{1}{2}\ln(y+2) + C \tag{9.66}$$

....

$$1: \quad dv = \sin 4t\,dt \tag{9.67}$$
$$v = -\frac{1}{4}\cos 4t \tag{9.68}$$
$$2: \quad dv = e^{3x}dx \tag{9.69}$$
$$v = \frac{1}{3}e^{3x} \tag{9.70}$$
$$3: \quad dv = x^3\,dx \tag{9.71}$$
$$v = \frac{1}{4}x^4 \tag{9.72}$$
$$4: \quad dv = \frac{1}{x}\,dx \tag{9.73}$$
$$v = \ln x \tag{9.74}$$

....

$$\int_{-\sqrt{3}}^{\frac{\sqrt{3}}{3}} \arctan s\,ds = *_1 \tag{9.75}$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{9.76}$$

$$f(s) = \arctan s \implies f'(s) = \frac{1}{1+s^2} \tag{9.77}$$

$$g'(s) = 1 \implies g(s) = s \tag{9.78}$$

$$*_1 = [s \arctan s]_{-\sqrt{3}}^{\frac{\sqrt{3}}{3}} - \int_{-\sqrt{3}}^{\frac{\sqrt{3}}{3}} \frac{s}{1+s^2}\,ds = *_2 \tag{9.79}$$

$$u = 1 + s^2 \tag{9.80}$$

$$du = 2s\,ds \implies \frac{du}{2s} = ds \tag{9.81}$$

$$*_2 = [s \arctan s]_{-\sqrt{3}}^{\frac{\sqrt{3}}{3}} - \frac{1}{2}\int_{s=-\sqrt{3}}^{\frac{\sqrt{3}}{3}} \frac{1}{u}\,du \tag{9.82}$$

$$= [s \arctan s]_{-\sqrt{3}}^{\frac{\sqrt{3}}{3}} - \frac{1}{2}[\ln u]_{s=-\sqrt{3}}^{\frac{\sqrt{3}}{3}} \tag{9.83}$$

$$= [s \arctan s]_{-\sqrt{3}}^{\frac{\sqrt{3}}{3}} - \frac{1}{2}[\ln(1+s^2)]_{\sqrt{3}}^{\frac{\sqrt{3}}{3}} \tag{9.84}$$

$$= \left(\frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\right) - \sqrt{3}\arctan(\sqrt{3})\right) - \frac{1}{2}\left[\ln\left(1+\left(\frac{\sqrt{3}}{3}\right)^2\right) - \ln(1+3)\right] \tag{9.85}$$

....

$$\int \ln 4t\,dt = * \tag{9.86}$$

$$f(t) = \ln 4t \implies f'(t) = \frac{1}{t} \tag{9.87}$$

$$g'(t) = 1 \implies g(t) = t \tag{9.88}$$

$$* = f(t)g(t) - \int f'(t)g(t)\,dt \tag{9.89}$$

$$= t\ln(4t) - \int \frac{\cancel{t}}{\cancel{t}}\,dt \tag{9.90}$$

$$= t\ln 4t - t + C \tag{9.91}$$

....

$$\int \arctan x \cdot x^{-2}\,dx = *_1 \tag{9.92}$$

$$f(x) = \arctan x \implies f'(x) = \frac{1}{1+x^2} \tag{9.93}$$

$$g'(x) = x^{-2} \implies g(x) = -x^{-1} \tag{9.94}$$

$$*_1 = -x^{-1}\arctan x - \int -x^{-1}\frac{1}{1+x^2}\,dx \tag{9.95}$$

$$= -\frac{\arctan x}{x} + \int \frac{1}{x(1+x^2)}\,dx = *_2 \tag{9.96}$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \tag{9.97}$$

$$\frac{1}{1+x^2} = A + \frac{Bx+C}{1+x^2}x \tag{9.98}$$

$$1 = A \tag{9.99}$$

$$\frac{1}{x(1+x^2)} = \frac{1}{x} + \frac{Bx+C}{1+x^2} \tag{9.100}$$
$$1 = 1 + x^2 + Bx^2 + Cx \tag{9.101}$$
$$0x^2 + 0x + 1 = (1+B)x^2 + Cx + 1 \tag{9.102}$$
$$B = -1,\ C = 0 \tag{9.103}$$
$$\frac{1}{x(1+x^2)} = \frac{1}{x} + \frac{-x}{1+x^2} \tag{9.104}$$
$$*_2 = -\frac{\arctan x}{x} + \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx \tag{9.105}$$
$$= -\frac{\arctan x}{x} + \ln x - \frac{1}{2}\ln(1+x^2) + C \tag{9.106}$$

....

$$\int \arctan\sqrt{x}\,dx = * \tag{9.107}$$
$$f(x) = \arctan\sqrt{x} \implies f'(x) = \frac{1}{x+1}\left(\frac{1}{2\sqrt{x}}\right) \tag{9.108}$$
$$g'(x) = 1 \implies g(x) = x \tag{9.109}$$
$$* = f(x)g(x) - \int f'(x)g(x)\,dx \tag{9.110}$$
$$= x\arctan\sqrt{x} - \int \frac{x}{(x+1)(2\sqrt{x})}\,dx \tag{9.111}$$
$$= x\arctan\sqrt{x} - \int \frac{\sqrt{x}}{2x+2}\,dx \tag{9.112}$$
$$\tag{9.113}$$

....

$$\int e^{x^{1/3}}\,dx = *_1 \tag{9.114}$$
$$u = x^{\frac{1}{3}} \tag{9.115}$$
$$du = \frac{1}{3}x^{-\frac{2}{3}}\,dx \tag{9.116}$$
$$3u^2\,du = dx \tag{9.117}$$
$$*_1 = \int e^u 3u^2\,du \tag{9.118}$$
$$= 3\int e^u u^2\,du = \times \tag{9.119}$$
$$f(x) = u^2 \implies f'(x) = 2u \tag{9.120}$$
$$g'(x) = e^u \implies g(x) = e^u \tag{9.121}$$
$$\times = 3\left(u^2e^u - \int 2ue^u\,du\right) \tag{9.122}$$
$$= 3u^2e^u - 6\int ue^u\,du = *_2 \tag{9.123}$$
$$f(x) = u \implies f'(x) = 1 \tag{9.124}$$
$$g'(x) = e^u \implies g(x) = e^u \tag{9.125}$$
$$*_2 = 3u^2e^u - 6\left(ue^u - \int e^u\,du\right) \tag{9.126}$$
$$= 3u^2e^u - 6ue^u + 6e^u \tag{9.127}$$

$$= e^u(3u^2 - 6u + 6) \tag{9.128}$$
$$= e^{x^{\frac{1}{3}}}(3x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 6) \tag{9.129}$$

....

$$\int_2^3 x^3 e^{-4x}\,dx = *_1 \tag{9.130}$$
$$f(x) = x^3 \implies f'(x) = 3x^2 \tag{9.131}$$
$$g'(x) = e^{-4x} \implies g(x) = -\frac{1}{4}e^{-4x} \tag{9.132}$$
$$*_1 = -\frac{1}{4}x^3e^{-4x} + \frac{3}{4}\int x^2e^{-4x}\,dx = *_2 \tag{9.133}$$
$$f(x) = x^2 \implies f'(x) = 2x \tag{9.134}$$
$$g'(x) = e^{-4x} \implies g(x) = -\frac{1}{4}e^{-4x} \tag{9.135}$$
$$*_2 = -\frac{1}{4}x^3e^{-4x} + \frac{3}{4}\left(-\frac{1}{2}x^2e^{-4x} + \frac{1}{2}\int xe^{-4x}\,dx\right) \tag{9.136}$$
$$= -\frac{1}{4}x^3e^{-4x} - \frac{3}{8}x^2e^{-4x} + \frac{3}{8}\int xe^{-4x}\,dx = *_3 \tag{9.137}$$
$$f(x) = x \implies f'(x) = 1 \tag{9.138}$$
$$g'(x) = e^{-4x} \implies g(x) = -\frac{1}{4}e^{-4x} \tag{9.139}$$
$$*_3 = -\frac{1}{4}x^3e^{-4x} - \frac{3}{8}x^2e^{-4x} + \frac{3}{8}\left(-\frac{1}{4}xe^{-4x} + \frac{1}{4}\int e^{-4x}\,dx\right) \tag{9.140}$$
$$= -\frac{1}{4}x^3e^{-4x} - \frac{3}{8}x^2e^{-4x} - \frac{3}{32}xe^{-4x} + \frac{3}{32}\left(\frac{1}{-4}\right)e^{-4x} \tag{9.141}$$
$$\text{lots of arithmetic that i dont feel like doing} \tag{9.142}$$

....

$$\int \frac{x^2}{(x-2)(x-3)}\,dx = *_1 \tag{9.143}$$
$$\frac{x^2}{(x-2)(x-3)} = \frac{x^2 - 5x + 6 + x^2}{(x-2)(x-3)} \tag{9.144}$$
$$= 1 + \frac{5x-6}{(x-2)(x-3)} \tag{9.145}$$
$$= 1 + \frac{A}{x-2} + \frac{B}{x-3} \tag{9.146}$$
$$A = -4,\ B = 9 \tag{9.147}$$
$$\frac{x^2}{(x-2)(x-3)} = 1 + \frac{-4}{x-2} + \frac{9}{x-3} \tag{9.148}$$
$$*_1 = \int 1 - \frac{4}{x-2} + \frac{9}{x-3}\,dx \tag{9.149}$$
$$= x - 4\ln(x-2) + 9\ln(x-3) + C \tag{9.150}$$
$$\int_4^5 \frac{x^2}{(x-2)(x-3)}\,dx = 1 - 4\ln\frac{3}{2} + 9\ln 2 \tag{9.151}$$

....

$$I_n = \int x^n e^x\,dx \tag{9.152}$$

$$
\begin{aligned}
I_0 &= e^x + C && (9.153)\\
f(x) = x^n &\implies f'(x) = nx^{n-1} && (9.154)\\
g'(x) = e^x &\implies g(x) = e^x && (9.155)\\
I_n &= x^n e^x - \int nx^{n-1} e^x \, d && (9.156)\\
&= x^n e^x - nI_{n-1} && (9.157)\\
I_4 &= x^4 e^x - 4I_3 && (9.158)\\
&= \dots - 4x^3 e^x + 12I_2 && (9.159)\\
&= \dots 12x^2 e^x - 12I_1 && (9.160)\\
&= \dots - 12xe^x + 12I_0 && (9.161)\\
I_4 &= e^x(x^4 - 4x^3 - 12x^2 + 12x) && (9.162)
\end{aligned}
$$

. . . .

$$
\begin{aligned}
\int_{e^3}^{e^5} (\ln x)^2 \, dx &= *_1 && (9.163)\\
f(x) = \ln^2 x &\implies f'(x) = \frac{2}{x}(\ln x) && (9.164)\\
g'(x) = 1 &\implies g(x) = x && (9.165)\\
*_1 &= x\ln^2 x - \int 2\ln x \, d && (9.166)\\
&= x\ln^2 x - 2\int \ln x \, dx = *_2 && (9.167)\\
f(x) = \ln x &\implies f'(x) = \frac{1}{x} && (9.168)\\
g'(x) = 1 &\implies g(x) = x && (9.169)\\
*_2 &= x\ln^2 x - 2x\ln x + 2\int 1 \, dx && (9.170)\\
&= x\ln^2 x - 2x\ln x + 2x + C && (9.171)\\
\int_{e^3}^{e^5} (\ln x)^2 \, dx &= \left[x\ln^2 x - 2x\ln x + 2x\right]_{e^3}^{e^5} && (9.172)\\
&= (25e^5 - 10e^5 + 2e^5) - (9e^3 - 6e^3 + 2e^3) && (9.173)\\
&= 17e^5 - 5e^3 && (9.174)
\end{aligned}
$$

. . . .

$$
\begin{aligned}
\int_3^5 \frac{\ln x}{\sqrt{x}} \, dx &= *_1 && (9.175)\\
f(x) = \ln x &\implies f'(x) = \frac{1}{x} && (9.176)\\
g'(x) = x^{-1/2} &\implies g(x) = 2\sqrt{x} && (9.177)\\
*_1 &= 2\sqrt{x}\ln x - \int \frac{1}{x} 2\sqrt{x} \, dx && (9.178)\\
&= 2\sqrt{x}\ln x - 2\int x^{-1/2} \, dx && (9.179)\\
&= 2\sqrt{x}\ln x - 4\sqrt{x} + C && (9.180)\\
\int_3^5 \frac{\ln x}{\sqrt{x}} \, dx &= (2\sqrt{5}\ln 5 - 4\sqrt{5}) - (2\sqrt{3}\ln 3 - 4\sqrt{3}) && (9.181)
\end{aligned}
$$

$$\int \frac{1}{(x+5)\sqrt{11-x}}\,dx = *_1 \tag{9.182}$$
$$u = \sqrt{11-x} \tag{9.183}$$
$$u^2 = 11 - x \tag{9.184}$$
$$-u^2 + 16 = x - 11 + 16 = x + 5 \tag{9.185}$$
$$du = \frac{-1}{2\sqrt{11-x}}\,dx \tag{9.186}$$
$$-2du = \frac{dx}{\sqrt{11-x}} \tag{9.187}$$
$$*_1 = \int \frac{-2}{-u^2+16}\,du \tag{9.188}$$
$$= 2\int \frac{1}{u^2-16}\,du \tag{9.189}$$
$$= 2\int \frac{1}{(u+4)(u-4)}\,du = *_2 \tag{9.190}$$
$$\frac{1}{(u+4)(u-4)} = \frac{A}{u+4} + \frac{B}{u-4} \tag{9.191}$$
$$A = -\frac{1}{8},\ B = \frac{1}{8} \tag{9.192}$$
$$*_2 = -\frac{1}{4}\int \frac{1}{u+4}\,du + \frac{1}{4}\int \frac{1}{u-4}\,du \tag{9.193}$$
$$= -\frac{1}{4}\ln(u+4) + \frac{1}{4}\ln(u-4) \tag{9.194}$$
$$= -\frac{1}{4}\ln(\sqrt{11-x}+4) + \frac{1}{4}\ln(\sqrt{11-x}-4) + C \tag{9.195}$$

$$\int e^{6x}\cos(5x)\,dx = *_1 \tag{9.196}$$
$$f(x) = \cos 5x \implies f'(x) \implies -5\sin 5x \tag{9.197}$$
$$g'(x) = e^{6x} \implies g(x) = \frac{1}{6}e^{6x} \tag{9.198}$$
$$*_1 = \frac{1}{6}e^{6x}\cos 5x + \frac{5}{6}\int e^{6x}\sin 5x\,dx = *_2 \tag{9.199}$$
$$f(x) = \sin 5x \implies f'(x) = 5\cos 5x \tag{9.200}$$
$$g'(x) = e^{6x} \implies g(x) = \frac{1}{6}e^{6x} \tag{9.201}$$
$$*_2 = \frac{1}{6}e^{6x}\cos 5x + \frac{5}{6}\frac{1}{6}e^{6x}\sin 5x - \frac{5}{6}\frac{5}{6}\int e^{6x}\cos 5x\,dx \tag{9.202}$$
$$\int e^{6x}\cos 5x\,dx = \frac{1}{6}e^{6x}\cos 5x + \frac{5}{36}e^{6x}\sin 5x - \frac{25}{36}\int e^{6x}\cos 5x\,dx \tag{9.203}$$
$$\frac{61}{36}\int e^{6x}\cos 5x\,dx = \frac{1}{6}e^{6x}\cos 5x + \frac{5}{36}e^{6x}\sin 5x \tag{9.204}$$
$$\int e^{6x}\cos 5x\,dx = \frac{36}{61}e^{6x}\left(\frac{1}{6}\cos 5x + \frac{5}{36}\sin 5x\right) \tag{9.205}$$

10

Lecture 10: Improper Integrals

10.1. Theory

Definition 10.1.1: p-integral

$$p\text{-integral of type I: } \int_a^\infty \frac{1}{x^p}\,dx \text{ converges for } p > 1 \tag{10.1}$$

$$p\text{-integral of type II: } \int_0^a \frac{1}{x^p}\,dx \text{ converges for } p < 1 \tag{10.2}$$

10.2. Practice

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = [\arcsin x]_0^1 \tag{10.3}$$

$$\int_{\sqrt{3}}^\infty \frac{1}{x^2+1}\,dx = \arctan x \Big|_{\sqrt{3}}^\infty \tag{10.4}$$

$$= \lim_{x\to\infty} \arctan x \Big|_{\sqrt{3}}^\infty \tag{10.5}$$

$$= \frac{\pi}{2} - \frac{\pi}{3} \tag{10.6}$$

$$= \frac{\pi}{6} \tag{10.7}$$

$$\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx = *_1 \tag{10.8}$$

$$u = \ln x \tag{10.9}$$

$$du = \frac{1}{x}\,dx \tag{10.10}$$

$$u(1) = 0 \tag{10.11}$$

$$u(e) = 1 \tag{10.12}$$

$$*_1 = \int_0^1 \frac{1}{\sqrt{u}}\,du \tag{10.13}$$

$$= [2\sqrt{u}]_0^1 \tag{10.14}$$

$$= 2 \tag{10.15}$$

$$\int_{-2}^5 \frac{1}{(x+2)^{3/2}}\,dx = \int_{-2}^5 (x+2)^{-3/2}\,dx \tag{10.16}$$

$$= \left[\frac{1}{-\frac{1}{2}}(x+2)^{-1/2}\right]_{-2}^{5} \tag{10.17}$$

$$= -2 \lim_{x\downarrow -2}\left[\frac{1}{\sqrt{x+2}}\right]_{x}^{5} \tag{10.18}$$

$$= -2 \lim_{x\downarrow -2}\left(\frac{1}{\sqrt{7}} - \frac{1}{\sqrt{x+2}}\right) \tag{10.19}$$

$$= -2(-\infty) \tag{10.20}$$

$$= \infty \tag{10.21}$$

....

$$\int_e^\infty \frac{1}{x \ln^9 x}\, dx = *_1 \tag{10.22}$$

$$u = \ln x \tag{10.23}$$

$$du = \frac{1}{x} \tag{10.24}$$

$$u(e) = 1 \tag{10.25}$$

$$u(\infty) = \infty \tag{10.26}$$

$$*_1 = \int_1^\infty u^{-9}\, du \tag{10.27}$$

$$= -\frac{1}{8}\left[u^{-8}\right]_1^\infty \tag{10.28}$$

$$= -\frac{1}{8}\lim_{u\to\infty}\left(\frac{1}{u^8} - 1\right) \tag{10.29}$$

$$= \frac{1}{8} \tag{10.30}$$

$$\int_0^1 \frac{1}{\sqrt{x}(1+x)}\, dx = *_1 \tag{10.31}$$

$$u = \sqrt{x} \tag{10.32}$$

$$du = \frac{1}{2\sqrt{x}} \implies 2du = \frac{1}{\sqrt{x}} \tag{10.33}$$

$$*_1 = \int_0^1 \frac{2}{1+u^2}\, du \tag{10.34}$$

$$= 2\int_0^1 \frac{1}{1+u^2}\, du \tag{10.35}$$

$$= 2 \arctan u \,\Big|_0^1 \tag{10.36}$$

$$= \frac{\pi}{2} \tag{10.37}$$

....

$$\int_0^7 \frac{1}{x-3}\, dx = \int_0^3 \frac{1}{x-3}\, dx + \int_3^7 \frac{1}{x-3}\, dx \tag{10.38}$$

$$= \lim_{a\uparrow 3}\left[\ln(x-3)\right]_0^a + \lim_{b\downarrow 3}\left[\ln(x-3)\right]_b^7 \tag{10.39}$$

$$= \ln(-0) - \ln(-3) + \ln 4 - \ln(+0) \tag{10.40}$$

$$= DIV \tag{10.41}$$

$$\int_0^\infty e^{-5x}\,dx = -\frac{1}{5}\lim_{R\to\infty}\left[e^{-5x}\right]_0^R \tag{10.42}$$
$$= -\frac{1}{5}(0-1) \tag{10.43}$$
$$= \frac{1}{5} \tag{10.44}$$

$$\int_{-\infty}^\infty \frac{1}{x^2+2}\,dx = \int_{-\infty}^\infty \frac{\frac{1}{2}}{\frac{x^2}{2}+1}\,dx \tag{10.45}$$
$$= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\,dx = *_1 \tag{10.46}$$
$$u = \frac{x}{\sqrt{2}} \tag{10.47}$$
$$du = \frac{1}{\sqrt{2}}\,dx \implies \sqrt{2}du = dx \tag{10.48}$$
$$*_1 = \frac{\sqrt{2}}{2}\lim_{R\to\infty}\left(\int_{-R}^0 \frac{1}{u^2+1}\,du + \int_0^R \frac{1}{u^2+1}\,du\right) \tag{10.49}$$
$$= \frac{\sqrt{2}}{2}\lim_{R\to\infty}\left([\arctan u]_{-R}^0 + [\arctan u]_0^R\right) \tag{10.50}$$
$$= \frac{\sqrt{2}}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) \tag{10.51}$$
$$= \frac{\sqrt{2}\pi}{2} \tag{10.52}$$
$$= \frac{\pi}{\sqrt{2}} \tag{10.53}$$

$$\int_0^1 \ln x\,dx = \lim_{R\downarrow 0}\,[x\ln x - x]_R^1 \tag{10.54}$$
$$= (\ln 1 - 1) - 0 \tag{10.55}$$
$$= -1 \tag{10.56}$$

$$\int_0^\infty \frac{\arctan x}{1+x^2}\,dx = *_1 \tag{10.57}$$
$$f(x) = \arctan x \implies f'(x) = \frac{1}{1+x^2} \tag{10.58}$$
$$g'(x) = \frac{1}{1+x^2} \implies g(x) = \arctan x \tag{10.59}$$
$$*_1 = \int_0^\infty \frac{\arctan x}{1+x^2}\,dx = \arctan^2 x - \int_0^\infty \frac{\arctan x}{1+x^2}\,dx \tag{10.60}$$
$$2\int_0^\infty \frac{\arctan x}{1+x^2}\,dx = \arctan^2 x \tag{10.61}$$
$$\int_0^\infty \frac{\arctan x}{1+x^2}\,dx = \frac{1}{2}\lim_{R\to\infty}\left[\arctan^2 x\right]_0^R \tag{10.62}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 \tag{10.63}$$

$$= \frac{\pi^2}{8} \tag{10.64}$$

. . . .

$$\int_0^1 \ln^2 x \, dx = *_1 \tag{10.65}$$

$$f(x) = \ln^2 x \implies f'(x) = \frac{2}{x}\ln x \tag{10.66}$$

$$g'(x) = 1 \implies g(x) = x \tag{10.67}$$

$$*_1 = x\ln^2 x - \int_0^1 \frac{2}{x}\ln x \cdot x \, dx \tag{10.68}$$

$$= x\ln^2 x - 2(x\ln x - x) \tag{10.69}$$

$$= \lim_{R\downarrow 0}(\left[x\ln^2 x\right]_R^1 - 2\left[x\ln x - x\right]_R^1) \tag{10.70}$$

$$= (1\ln^2(1) - 0) + 2 \tag{10.71}$$

$$= 2 \tag{10.72}$$

11

Lecture 11: Intro to Differential Equations

11.1. Practice

Which of the following functions is a solution to the differential equation $y'' + 2y' + y = x$?

$$
\begin{aligned}
y &= 3xe^{-x} && (11.1)\\
y' &= 3e^{-x} - 3xe^{-x} && (11.2)\\
y'' &= -3e^{-x} - 3e^{-x} + 3xe^{-x} && (11.3)\\
y'' + 2y' + y &= -6e^{-x} + 3xe^{-x} + 6e^{-x} - 6xe^{-x} + 3xe^{-x} && (11.4)\\
&= 0 \implies \text{doesn't work} && (11.5)\\
y &= e^{-x} + xe^{-x} + x - 2 && (11.6)\\
y' &= -\cancel{e^{-x}} + \cancel{e^{-x}} - xe^{-x} + 1 && (11.7)\\
y'' &= xe^{-x} - e^{-x} && (11.8)\\
y'' + 2y' + y &= \cancel{e^{-x}} + \cancel{xe^{-x}} + x - \cancel{2} - \cancel{2xe^{-x}} + \cancel{2} + \cancel{xe^{-x}} - \cancel{e^{-x}} && (11.9)\\
y'' + 2y' + y &= x && (11.10)
\end{aligned}
$$

....

$$
\begin{aligned}
y &= \frac{1}{4 + k \ln x} && (11.11)\\
xy' &= y^2 && (11.12)\\
y' &= -(4 + k \ln x)^{-2} \left(\frac{k}{x}\right) && (11.13)\\
y' &= -\frac{k}{x} y^2 && (11.14)\\
xy' &= -ky^2 && (11.15)\\
k &= -1 && (11.16)
\end{aligned}
$$

$$
\begin{aligned}
y &= e^{kt}(\sin t + 2\cos t) && (11.17)\\
y' &= ke^{kt}(\sin t + 2\cos t) + e^{kt}(\cos t - 2\sin t) && (11.18)\\
y'' &= k^2 e^{kt}(\sin t + 2\cos t) + ke^{kt}(\cos t - 2\sin t) + e^{kt}(-\sin t - 2\cos t) && (11.19)\\
y'' - 6y' + 10y = 0 &= e^{kt}(\sin t + 2\cos t) && (11.20)\\
&\quad - 6ke^{kt}(\sin t + 2\cos t) + 6e^{kt}(\cos t - 2\sin t)\\
&\quad + 10k^2 e^{kt}(\sin t + 2\cos t) + 10ke^{kt}(\cos t - 2\sin t) + 10e^{kt}(-\sin t - 2\cos t)\\
&= e^{kt}\left[k^2(10\sin t + 20\cos t)\right. && (11.21)\\
&\quad + k(10\cos t - 20\sin t - 6\sin t - 12\cos t)\\
&\quad \left.+(-10\sin t - 20\cos t + 6\cos t - 12\sin t + \sin t + 2\cos t)\right] && (11.22)
\end{aligned}
$$

$$= e^{kt}\left[k^2(10\sin t + 20\cos t) + k(-26\sin t - 2\cos t) + (-21\sin t - 12\cos t)\right] \tag{11.23}$$

....

$$k = -1,\ y_0 = 9 \tag{11.24}$$

$$\sqrt{y} = \frac{k}{2}t + C = C - \frac{t}{2} \tag{11.25}$$

$$y = \left(C - \frac{t}{2}\right)^2 \tag{11.26}$$

$$y = \left(3 - \frac{t}{2}\right)^2 \tag{11.27}$$

$$y(4) = 1 \tag{11.28}$$

12

Lecture 12: 1st Order Differential Equations

12.1. Practice

$$\frac{dy}{dt} = + = 4ty + 2t \tag{12.1}$$
$$y' - 4ty = 2t \tag{12.2}$$
$$I(t) = e^{-2t^2} \tag{12.3}$$
$$\underbrace{e^{-2t^2}y' + e^{-2t^2}(-4ty)}_{(I(t)(y(t))'} = e^{-2t^2}2t \tag{12.4}$$
$$I(t)y(t) = \int e^{-2t^2}2t\,dt \tag{12.5}$$
$$= 2\int e^{-2t^2}t\,dt = *_1 \tag{12.6}$$
$$u = -2t^2 \tag{12.7}$$
$$du = -4t\ dt \implies -\frac{du}{4} = t\ dt \tag{12.8}$$
$$*_1 = -\frac{1}{2}\int e^u\,du \tag{12.9}$$
$$= -\frac{1}{2}e^u + C \tag{12.10}$$
$$I(t)y(t) = -\frac{1}{2}e^{-2t^2} + C \tag{12.11}$$
$$y(t) = \frac{-\frac{1}{2}e^{-2t^2}}{e^{-2t^2}} + \frac{C}{e^{-2t^2}} \tag{12.12}$$
$$= -\frac{1}{2} + Ce^{2t^2} \tag{12.13}$$
$$y(0) = 2 = -\frac{1}{2} + C \tag{12.14}$$
$$C = \frac{5}{2} \tag{12.15}$$
$$\boxed{y(t) = \frac{5}{2}e^{2t^2} - \frac{1}{2}} \tag{12.16}$$

....

$$y' = (t-1)y(3y-4) \tag{12.17}$$
$$y = 0 \vee 3y - 4 = 0 \tag{12.18}$$
$$y = 0 \vee y = \frac{4}{3} \tag{12.19}$$

....

$$\frac{dy}{dt} = t^2y^2 - 2y^2, \quad y(1) = 3 \tag{12.20}$$
$$\frac{dy}{dt} = (t^2 - 2)y^2 \tag{12.21}$$
$$\frac{dy}{y^2} = (t^2 - 2)\,dt \tag{12.22}$$
$$\int y^{-2}\,dy = \int t^2 - 2\,dt \tag{12.23}$$
$$-\frac{1}{y} = \frac{t^3}{3} - 2t + C \tag{12.24}$$
$$-\frac{1}{3} = \frac{1^3}{3} - 2\cdot 1 + C \tag{12.25}$$
$$C = -\frac{1}{3} - \frac{1}{3} + 2 = \frac{4}{3} \tag{12.26}$$
$$-\frac{1}{y} = \frac{t^3}{3} - 2t + \frac{4}{3} \tag{12.27}$$
$$\boxed{y = -\frac{1}{\frac{t^3}{3} - 2t + \frac{4}{3}}} \tag{12.28}$$

. . . .

$$y' = 5 - \frac{y}{10}, \quad y(0) = 0 \tag{12.29}$$
$$y' + \frac{1}{10}y = 5 \tag{12.30}$$
$$I(t) = e^{\frac{t}{10}} \tag{12.31}$$
$$y'e^{\frac{t}{10}} + \frac{1}{10}e^{\frac{t}{10}}y = 5e^{\frac{t}{10}} \tag{12.32}$$
$$e^{\frac{t}{10}}y = \int 5e^{\frac{t}{10}}\,dt \tag{12.33}$$
$$e^{\frac{t}{10}}y = 50e^{\frac{t}{10}} + C \tag{12.34}$$
$$y = 50 + \frac{C}{e^{\frac{t}{10}}} \tag{12.35}$$
$$y(0) = 0 = 50 + \frac{C}{e^{\frac{0}{10}}} \tag{12.36}$$
$$C = -50 \tag{12.37}$$
$$y(t) = 50 - 50e^{-\frac{t}{10}} \tag{12.38}$$
$$c(t) = \frac{y(t)}{V(t)} \tag{12.39}$$
$$V' = 0 \implies c(t) = \frac{50 - 50e^{-\frac{t}{10}}}{100} \tag{12.40}$$
$$c(5) = \frac{50 - 50e^{-\frac{5}{10}}}{100} \tag{12.41}$$

. . . .

$$y' = \ln(4t + 2), \quad y(0) = 2 \tag{12.42}$$
$$dy = \ln(4t + 2)\,dt \tag{12.43}$$
$$y = \int \ln(4t + 2)\,dt = *_1 \tag{12.44}$$

$$
\begin{aligned}
u &= 4t + 2 && (12.45)\\
\frac{du}{4} &= dt && (12.46)\\
*_1 &= \frac{1}{4}\int \ln u \, du && (12.47)\\
&= \frac{1}{4}(u \ln u - u) + C && (12.48)\\
y(t) &= \frac{4t+2}{4}[\ln(4t+2) - 1] + C && (12.49)\\
y(0) = 2 &= \frac{1}{2}\ln 2 - \frac{1}{2} + C && (12.50)\\
C &= 2 + \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{5 - \ln 2}{2} && (12.51)\\
y(t) &= \frac{4t+2}{4}[\ln(4t+2) - 1] + \frac{5 - \ln 2}{2} && (12.52)
\end{aligned}
$$

. . . .

$$
\begin{aligned}
L\frac{dI}{dt} + RI &= E(t) && (12.53)\\
R = 15,\ L = 5,\ E(t) &= 50,\ I(0) = 0 && (12.54)\\
\frac{dI}{dt} + \frac{R}{L}I &= \frac{E(t)}{L} && (12.55)\\
\frac{dI}{dt} + 3I &= 10 && (12.56)\\
e^{3t}I' + 3e^{3t}I &= 10e^{3t} && (12.57)\\
e^{3t}I &= \int 10e^{3t}\, dt && (12.58)\\
e^{3t}I(t) &= \frac{10}{3}e^{3t} + C && (12.59)\\
I(t) &= \frac{10}{3} + Ce^{-3t} && (12.60)\\
I(0) = 0 &= \frac{10}{3} + C && (12.61)\\
C &= -\frac{10}{3} && (12.62)\\
I(t) &= \frac{10}{3} - \frac{10}{3}e^{-3t} && (12.63)
\end{aligned}
$$

$$
\begin{aligned}
I' + \frac{R}{L}I &= \frac{E(t)}{L} && (12.64)\\
e^{\int \frac{R}{L}dt}I' + e^{\int \frac{R}{L}dt}\frac{R}{L}I &= e^{\int \frac{R}{L}dt}\frac{E(t)}{L} && (12.65)\\
e^{\int \frac{R}{L}dt}I &= \int e^{\int \frac{R}{L}dt}\frac{E(t)}{L}\, dt && (12.66)\\
I(t) &= \frac{\int e^{\int \frac{R}{L}dt}\frac{E(t)}{L}\, dt}{e^{\int \frac{R}{L}dt}} && (12.67)\\
&= \frac{\int \cancel{e^{3t}}\frac{40\cancel{e^{-3t}}\sin(5t)}{4}\, dt}{e^{3t}} && (12.68)\\
&= \frac{10\int \sin 5t\, dt}{e^{3t}} && (12.69)\\
&= \frac{-2\cos 5t + C}{e^{3t}} && (12.70)
\end{aligned}
$$

$$I(0) = 0 = \frac{-2\cos 0 + C}{1} \tag{12.71}$$
$$C = 2 \tag{12.72}$$
$$I(t) = \frac{2 - 2\cos 5t}{e^{3t}} \tag{12.73}$$

. . . .

$$y' = \frac{e^{4x}}{3y^2+1}, \; y(0) = 2 \tag{12.74}$$
$$\frac{dy}{dx}(3y^2+1) = e^{4x} \tag{12.75}$$
$$(3y^2+1)\,dy = e^{4x}\,dx \tag{12.76}$$
$$y^3 + y = \frac{e^{4x}}{4} + C \tag{12.77}$$
$$2^3 + 2 = \frac{e^0}{4} + C \tag{12.78}$$
$$C = 10 - \frac{1}{4} = \frac{39}{4} \tag{12.79}$$
$$y^3 + y = \frac{e^{4x}}{4} + \frac{39}{4} \tag{12.80}$$

. . . .

$$\frac{dy}{dx} = (1+y^2)\cos 4x, \quad y(0) = \sqrt{3} \tag{12.81}$$
$$\frac{dy}{1+y^2} = \cos 4x\,dx \tag{12.82}$$
$$\arctan y = \frac{1}{4}\sin 4x + C \tag{12.83}$$
$$y(x) = \tan\left(\frac{1}{4}\sin 4x + C\right) \tag{12.84}$$
$$y(0) = \sqrt{3} = \tan(C) \tag{12.85}$$
$$C = \frac{\pi}{3} \tag{12.86}$$
$$y(x) = \tan\left(\frac{1}{4}\sin(4x) + \frac{\pi}{3}\right) \tag{12.87}$$

. . . .

$$\frac{dy}{dt} = -k\sqrt{y} \tag{12.88}$$
$$y(0) = H, \; y(T) = 0 \tag{12.89}$$
$$\frac{dy}{\sqrt{y}} = -k\,dt \tag{12.90}$$
$$\frac{1}{\frac{1}{2}}\sqrt{y} = -kt + C \tag{12.91}$$
$$\sqrt{y} = \frac{-kt + C}{2} \tag{12.92}$$
$$y(t) = \left(\frac{C - kt}{2}\right)^2 \tag{12.93}$$
$$y(0) = H = \left(\frac{C}{2}\right)^2 \tag{12.94}$$

$$C = 2\sqrt{H} \tag{12.95}$$
$$y(T) = 0 = \left(\frac{2\sqrt{H} - kT}{2}\right)^2 \tag{12.96}$$
$$2\sqrt{H} = kT \tag{12.97}$$
$$k = \frac{2\sqrt{H}}{T} \tag{12.98}$$
$$y(t) = \left(\frac{2\sqrt{H} - 2\frac{\sqrt{H}}{T}t}{2}\right)^2 \tag{12.99}$$

....

$$\frac{dx}{dt} = k(a-x)(b-x) \tag{12.100}$$
$$a = 5,\ b = 2,\ k = \frac{1}{20} \tag{12.101}$$
$$\frac{dx}{(a-x)(b-x)} = k\,dt = *_1 \tag{12.102}$$
$$\frac{A}{a-x} + \frac{B}{b-x} = \frac{1}{(a-x)(b-x)} \tag{12.103}$$
$$A = \frac{1}{b-a} \tag{12.104}$$
$$B = \frac{1}{a-b} \tag{12.105}$$
$$*_1 = \left(\frac{\frac{1}{b-a}}{a-x} + \frac{\frac{1}{a-b}}{b-x}\right) dx = k\,dt \tag{12.106}$$
$$\int \frac{1}{b-a}\frac{1}{a-x} + \frac{1}{a-b}\frac{1}{b-x}\,dx = \int k\,dt \tag{12.107}$$
$$\frac{1}{b-a}\ln(a-x) + \frac{1}{a-b}\ln(b-x) = kt + C \tag{12.108}$$
$$-\frac{1}{a-b}\ln(a-x) + \frac{1}{a-b}\ln(b-x) = kt + C \tag{12.109}$$
$$\frac{1}{a-b}(\ln(b-x) - \ln(a-x)) = kt + C \tag{12.110}$$
$$\frac{1}{a-b}\ln\frac{b-x}{a-x} = kt + C \tag{12.111}$$
$$\ln\frac{b-x}{a-x} = (a-b)(kt + C) \tag{12.112}$$

$$\frac{b-x}{a-x} = e^{(a-b)(kt+C)} \tag{12.113}$$
$$\frac{b-x}{a-x} = e^{(a-b)C}e^{(a-b)kt} \tag{12.114}$$
$$b - x = (a-x)e^{(a-b)C}e^{(a-b)kt} \tag{12.115}$$
$$x = b - (a-x)e^{(a-b)C}e^{(a-b)kt} \tag{12.116}$$
$$x = b - ae^{(a-b)C}e^{(a-b)kt} + xe^{(a-b)C}e^{(a-b)kt} \tag{12.117}$$
$$x - xe^{(a-b)C}e^{(a-b)kt} = b - ae^{(a-b)C}e^{(a-b)kt} \tag{12.118}$$
$$x(1 - e^{(a-b)C}e^{(a-b)kt}) = b - ae^{(a-b)C}e^{(a-b)kt} \tag{12.119}$$
$$x(t) = \frac{b - ae^{(a-b)C}e^{(a-b)kt}}{1 - e^{(a-b)C}e^{(a-b)kt}} \tag{12.120}$$

$$x(0) = 0 = \frac{b - ae^{(a-b)C}}{1 - e^{(a-b)C}} \tag{12.121}$$
$$b = ae^{(a-b)C} \tag{12.122}$$
$$e^{(a-b)C} = \frac{b}{a} \tag{12.123}$$
$$x(t) = \frac{b - \cancel{a} \cdot \frac{b}{\cancel{a}} \cdot e^{(a-b)kt}}{1 - \frac{b}{a}e^{(a-b)kt}} \tag{12.124}$$
$$= \frac{b(1 - e^{(a-b)kt})}{1 - \frac{b}{a}e^{(a-b)kt}} \tag{12.125}$$
$$x(t) = \frac{ab(1 - e^{(a-b)kt})}{a - be^{(a-b)kt}} \tag{12.126}$$

. . . .

$$V'(t) = 10 - 15 = -5 \tag{12.127}$$
$$V(0) = 100 \tag{12.128}$$
$$y' = 5 - 15c(t) = 5 - 15\frac{y}{100 - 5t} \tag{12.129}$$
$$y' + \frac{15}{100 - 5t}y = 5 \tag{12.130}$$
$$15\int \frac{1}{100 - 5t}\,dt = 15 \ln|100 - 5t| \frac{1}{-5} = -3\ln(100 - 5t) \tag{12.131}$$
$$I(t) = e^{-3\ln(100-5t)} = (100 - 5t)^{-3} \tag{12.132}$$
$$I(t)y' + I(t)\frac{15}{100 - 5t}y = 5I(t) \tag{12.133}$$
$$I(t)y = \int 5I(t)\,dt \tag{12.134}$$
$$(100 - 5t)^{-3}y = 5\int (100 - 5t)^{-3}\,dt \tag{12.135}$$
$$(100 - 5t)^{-3}y = (5)\left(\frac{1}{-2}\right)(100 - 5t)^{-2}\left(\frac{1}{-5}\right) + C \tag{12.136}$$
$$(100 - 5t)^{-3}y = \frac{1}{2}(100 - 5t)^{-2} + C \tag{12.137}$$
$$y = \frac{\frac{1}{2}(100 - 5t)^{-2} + C}{(100 - 5t)^{-3}} \tag{12.138}$$
$$y = \frac{100 - 5t}{2} + C(100 - 5t)^3 \tag{12.139}$$

$$y(0) = 0 = \frac{100 - 5 \cdot 0}{2} + C(100)^3 \tag{12.140}$$
$$100^3 C = -50 \tag{12.141}$$
$$C = -\frac{50}{100^3} \tag{12.142}$$
$$y(t) = \frac{100 - 5t}{2} - \frac{50}{100^3}(100 - 5t)^3 \tag{12.143}$$
$$c(t) = \frac{y(t)}{V(t)} = \frac{\frac{100-5t}{2} - \frac{50}{100^3}(100 - 5t)^3}{100 - 5t} \tag{12.144}$$
$$c(5) = \frac{\frac{100-25}{2} - \frac{50}{100^3}(100 - 25)^3}{100 - 25} \tag{12.145}$$
$$= \frac{7}{32} \tag{12.146}$$

13

Lecture 13: Complex Numbers I

13.1. Theory? Perhaps

$$\text{Modulus } |z| = |a+ib| = \sqrt{a^2+b^2} \tag{13.1}$$

$$\text{Argument: } \tan(\arg(z)) = \frac{b}{a} \text{ if } a \neq 0 \tag{13.2}$$

$$\text{Complex conjugate: } z^* = \bar{z} = \overline{a+ib} = a - ib \tag{13.3}$$

13.2. Practice

$$z = 1+i,\ w = -1+\sqrt{3}i \tag{13.4}$$

$$r = z^3 w^5 \tag{13.5}$$

$$|r| = |z|^3 |w|^5 \tag{13.6}$$

$$= \sqrt{1^2+1^2}^3 \sqrt{(-1)^2+3}^5 \tag{13.7}$$

$$= 2\sqrt{2}\cdot 2^5 \tag{13.8}$$

$$= 2^6\sqrt{2} \tag{13.9}$$

$$= 64\sqrt{2} \tag{13.10}$$

$$\arg(r) = 3\arg(z) + 5\arg(w) \tag{13.11}$$

$$= 3\arctan\left(\frac{1}{1}\right) + 5\arctan\left(\frac{\sqrt{3}}{-1}\right) \tag{13.12}$$

$$= \frac{3\pi}{4} + 5\cdot\frac{2\pi}{3} \tag{13.13}$$

$$= \pi\left(\frac{3}{4} + \frac{10}{3}\right) \tag{13.14}$$

$$= \pi\left(\frac{9+40}{12}\right) = \frac{49}{12}\pi \tag{13.15}$$

$$= \frac{\pi}{12} \mod 2\pi \tag{13.16}$$

....

$$z = -1-i,\ w = 1+\sqrt{3}i \tag{13.17}$$

$$r = \left|\frac{z^3}{w^5}\right| = \frac{|z|^3}{|w|^5} \tag{13.18}$$

$$= \frac{\sqrt{(-1)^2+(-1)^2}^3}{\sqrt{1^2+3}^5} \tag{13.19}$$

$$= \frac{2\sqrt{2}}{2^5} \tag{13.20}$$

$$= \frac{2^{3/2}}{2^5} \tag{13.21}$$

$$= 2^{-7/2} \tag{13.22}$$
$$= \frac{1}{8\sqrt{2}} \tag{13.23}$$

$$\theta = 3\arg(z) - 5\arg(w) \tag{13.24}$$
$$= 3\left(\frac{5\pi}{4}\right) - 5\left(\frac{\pi}{3}\right) \tag{13.25}$$
$$= \frac{15\pi}{4} - \frac{5\pi}{3} \tag{13.26}$$
$$= \frac{45-20}{12}\pi \tag{13.27}$$
$$= \frac{25}{12}\pi \tag{13.28}$$
$$\frac{25}{12}\pi = \frac{\pi}{12} \mod 2\pi \tag{13.29}$$

....

$$v = 2 + i,\ w = 3 + 2i \tag{13.30}$$
$$z = \frac{v^3}{w^2} \tag{13.31}$$
$$|z| = \frac{|v|^3}{|w|^2} \tag{13.32}$$
$$= \frac{\sqrt{2^2+1^2}^3}{\sqrt{3^2+2^2}^2} \tag{13.33}$$
$$= \frac{5\sqrt{5}}{13} \tag{13.34}$$
$$\arg(z) = 3\arg(v) - 2\arg(w) \tag{13.35}$$
$$= 3\arctan\left(\frac{1}{2}\right) - 2\arctan\left(\frac{2}{3}\right) \tag{13.36}$$

....

$$z = 4 + 2i,\ w = 2 - 3i \tag{13.37}$$
$$zw = 8 - 12i + 4i + 6 \tag{13.38}$$
$$= 14 - 8i \tag{13.39}$$
$$zz^* = 16 + 4 = 20 \tag{13.40}$$
$$\frac{z}{w} = \frac{4+2i}{2-3i} \tag{13.41}$$
$$= \frac{(4+2i)(2+3i)}{4+9} \tag{13.42}$$
$$= \frac{8+12i+4i-6}{13} \tag{13.43}$$
$$= \frac{2+16i}{13} \tag{13.44}$$

....

$$\arg(z_k) = \frac{2k\pi}{n} \tag{13.45}$$
$$z_k = |z_k|(\cos(\arg z_k) + i\sin(\arg z_k)) \tag{13.46}$$

$$\prod_{i=1}^{n}\left(\cos\left(\frac{2k\pi}{n}\right)+i\sin\left(\frac{2k\pi}{n}\right)\right) \tag{13.47}$$

14

Lecture 14: Complex Number II

14.1. Theory?

Definition 14.1.1: Polar Form of Complex Number

$$re^{i\theta} = \cos\theta + i\sin\theta \tag{14.1}$$

Theorem 14.1.1 Euler's Formula

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \tag{14.2}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{14.3}$$

Theorem 14.1.2 De Moivre's Theorem

$$(r\cos\theta + ir\sin\theta)^n = r^n\cos n\theta + ir^n\sin n\theta \tag{14.4}$$

14.2. Practice

$$f(t) = e^{(-3-2i)t} \tag{14.5}$$

$$\frac{d}{dt}f(t) = (-3-2i)e^{(-3-2i)t} \tag{14.6}$$

$$g(t) = e^{-4t}(\cos 3t + i\sin 3t) \tag{14.7}$$

$$= e^{-4t}e^{3it} \tag{14.8}$$

$$\frac{d}{dt}g(t) = -4e^{-4t}e^{3it} + 3ie^{-4t}e^{3it} \tag{14.9}$$

$$= (3-4)e^{-4t}e^{3it} \tag{14.10}$$

$$= (3-4)e^{(3i-4)t} \tag{14.11}$$

....

$$z = re^{i\theta},\ r = 8,\ \theta = -\frac{\pi}{6} \tag{14.12}$$

$$x = r\cos\theta \tag{14.13}$$

$$= 8\cos -\frac{\pi}{6} \tag{14.14}$$

$$= 8 \cdot \frac{\sqrt{3}}{2} \tag{14.15}$$

$$= 4\sqrt{3} \tag{14.16}$$

$$\begin{aligned} y &= r\sin\theta && (14.17)\\ &= 8\sin -\frac{\pi}{6} && (14.18)\\ &= 8\cdot -\frac{1}{2} && (14.19)\\ &= -4 && (14.20) \end{aligned}$$

. . . .

$$\begin{aligned} 3+1i &= r_1 e^{\theta_1 i} && (14.21)\\ r_1 &= \sqrt{3^2+1^2} = \sqrt{10} && (14.22)\\ \theta_1 &= \arctan\frac{1}{3} && (14.23)\\ -1+2i &= r_2 e^{\theta_2 i} && (14.24)\\ r_2 &= \sqrt{(-1)^2+2^2} = \sqrt{5} && (14.25)\\ \theta_2 &= \pi - \arctan 2 && (14.26)\\ -2-4i &= r_3 e^{\theta_3 i} && (14.27)\\ r_3 &= \sqrt{20} = 2\sqrt{5} && (14.28)\\ \theta_3 &= -\pi + \arctan 2 && (14.29)\\ 1-5i &= r_4 e^{\theta_4 i} && (14.30)\\ r_4 &= \sqrt{26} && (14.31)\\ \theta_4 &= -\arctan 5 && (14.32) \end{aligned}$$

. . . .

$$\begin{aligned} z &= e^{4+2i} && (14.33)\\ &= e^4 e^{2i} && (14.34)\\ &= e^4(\cos 2 + i\sin 2) && (14.35)\\ \mathrm{Re}(z) &= e^4\cos 2 && (14.36)\\ \mathrm{Im}(z) &= e^4\sin 2 && (14.37) \end{aligned}$$

. . . .

$$\begin{aligned} f(x,t) &= e^{i(2x-3t)} && (14.38)\\ Re(f) &= \cos(2x-3t) && (14.39)\\ \mathrm{Im}(f) &= \sin(2x-3t) && (14.40) \end{aligned}$$

. . . .

$$\begin{aligned} p(z) &= z^4 - 6z^3 + 6z^2 + 24z - 40 && (14.41)\\ &= (z-2)(z+2)\underbrace{(z-z_3)(z-z_4)}_{q(z)} && (14.42)\\ &= (z^2-4)q(z) && (14.43)\\ q(z) &= \frac{z^4 - 6z^3 + 6z^2 + 24z - 40}{z^2-4} && (14.44) \end{aligned}$$

$$\text{Total: } z^2 - 6z + 10 \qquad (14.45)$$

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$$z^2 - 4 \mid z^4 - 6z^3 + 6z^2 + 24z - 40 \tag{14.46}$$
$$-z^2 \tag{14.47}$$
$$z^2 - 4 \mid \quad -6z^3 + 10z^2 + 24z - 40 \tag{14.48}$$
$$+6z \tag{14.49}$$
$$z^2 - 4 \mid \quad 10z^2 - 40 \tag{14.50}$$
$$-10 \tag{14.51}$$

$$q(z) = z^2 - 6z + 10 \tag{14.52}$$
$$0 = (z-3)(z-3) + 1 \tag{14.53}$$
$$-1 = (z-3)^2 \tag{14.54}$$
$$z = 3 \pm i \tag{14.55}$$

....

$$16z^2 + 64z + 69 = 0 \tag{14.56}$$
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{14.57}$$
$$= \frac{-64 \pm \sqrt{64^2 - 4 \cdot 16 \cdot 69}}{2 \cdot 16} \tag{14.58}$$
$$= -2 \pm \frac{\sqrt{4096 - 4416}}{32} \tag{14.59}$$
$$= -2 \pm \frac{\sqrt{-320}}{32} \tag{14.60}$$
$$= -2 \pm \frac{8\sqrt{-5}}{32} \tag{14.61}$$
$$= -2 \pm \frac{\sqrt{5}}{4} i \tag{14.62}$$

....

$$z^2 - 10z + 29 = 0 \tag{14.63}$$
$$(z-5)^2 + 4 = 0 \tag{14.64}$$
$$(z-5)^2 = -4 \tag{14.65}$$
$$z - 5 = \pm 2i \tag{14.66}$$
$$z = 5 \pm 2i \tag{14.67}$$

....

$$(-\sqrt{3} + i)^7 = \left(\sqrt{(-\sqrt{3})^2 + 1^2} \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) \right)^7 \tag{14.68}$$
$$= 2^7 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)^7 \tag{14.69}$$
$$\text{Wrong lol} \tag{14.70}$$
$$\theta = 7 \cdot \frac{5\pi}{6} \tag{14.71}$$
$$= \frac{35\pi}{6} \tag{14.72}$$

$$= -\frac{\pi}{6} \mod 2\pi \tag{14.73}$$
$$\implies 2^7 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right) \tag{14.74}$$

....

$$(1-i)^9 = re^{i\theta} \tag{14.75}$$
$$r = \left(2^{\frac{1}{2}}\right) 9 = 2^{4\frac{1}{2}} \tag{14.76}$$
$$= 16\sqrt{2} \tag{14.77}$$
$$\theta = 9 \cdot -\frac{\pi}{4} \tag{14.78}$$
$$= -\frac{9\pi}{4} \tag{14.79}$$
$$= -\frac{\pi}{4} \mod 2\pi \tag{14.80}$$
$$\implies 16\sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right) \tag{14.81}$$

....

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^5 = re^{i\theta} \tag{14.82}$$
$$r = \left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)^5 \tag{14.83}$$
$$= \left(\sqrt{\frac{1}{4} + \frac{3}{4}}\right)^5 \tag{14.84}$$
$$= 1 \tag{14.85}$$
$$\theta = 5 \arctan \sqrt{3} \tag{14.86}$$
$$= \frac{5\pi}{3} \implies \qquad \cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) \tag{14.87}$$
$$\tag{14.88}$$

....

$$z = re^{it},\ w = ue^{iv} \tag{14.89}$$
$$|\bar{z}| = r \tag{14.90}$$
$$\frac{1}{z} = \frac{1}{re^{it}} = \frac{1}{r} e^{-it} \tag{14.91}$$

....

$$f(z) = z^3 + 7z^2 + 21z + 27 \tag{14.92}$$
$$f(-3) = 0 = z^3 + 7z^2 + 21z + 27 \tag{14.93}$$
$$z^2 + 4z + 9 \tag{14.94}$$
$$z + 3 \mid \quad z^3 + 7z^2 + 21z + 27 \tag{14.95}$$
$$4z^2 + 21z + 27 \tag{14.96}$$
$$9z + 27 \tag{14.97}$$
$$\implies 0 = z^2 + 4z + 9 \tag{14.98}$$

$$(z+2)^2 + 5 = 0 \tag{14.99}$$
$$(z+2)^2 = -5 \tag{14.100}$$
$$z+2 = \pm\sqrt{5}i \tag{14.101}$$
$$z = -2 \pm \sqrt{5}i \tag{14.102}$$

....

$$e^z = -\pi \tag{14.103}$$
$$e^{\text{Re}(z)} e^{\text{Im}(z)i} = -\pi \tag{14.104}$$
$$\text{Re}(z) = \ln|-\pi| \tag{14.105}$$
$$\text{Im}(z) = \pi \tag{14.106}$$

....

$$z^3 + 8i = 0 \tag{14.107}$$
$$z^3 = -8i \tag{14.108}$$
$$z^3 = 8i^3 \tag{14.109}$$
$$= 2i \tag{14.110}$$

....

$$p(z) = z^4 + 16z^2 + 100 \tag{14.111}$$
$$z_1 = -1 - 3i \tag{14.112}$$
$$z_2 = z_1^* = -1 + 3i \tag{14.113}$$
$$z_3 = -z_1 = 1 + 3i \tag{14.114}$$
$$z_4 = z_3^* = 1 - 3i \tag{14.115}$$

15

Exam Practice

15.1. Dot Product

$$\vec{v} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}, \ \vec{w} = \begin{pmatrix} -1 \\ -2 \\ h \end{pmatrix} \tag{15.1}$$

$$\perp \iff \frac{\vec{v}\vec{w}}{|\vec{v}||\vec{w}|} = 0 \tag{15.2}$$

$$3 + 2 + h = 0 \tag{15.3}$$

$$h = -5 \tag{15.4}$$

....

$$\vec{r} = 2\vec{u} - 2\vec{v}, \ |\vec{u}| = 3 \tag{15.5}$$

$$|\vec{v}| = 2, \ \vec{u} \cdot \vec{v} = -3 \tag{15.6}$$

$$\cos\theta = \frac{-3}{6} \tag{15.7}$$

$$\theta = \pi \pm \frac{\pi}{6} + 2k\pi \tag{15.8}$$

$$|\vec{r}_{\parallel \vec{u}}| = 2|\vec{u}| - 2|\vec{v}|\cos\theta \tag{15.9}$$

$$|\vec{r}_{\perp \vec{u}}| = 2|\vec{v}|\sin\theta \tag{15.10}$$

$$|\vec{r}| = \sqrt{|\vec{r}_{\parallel \vec{u}}|^2 + |\vec{r}_{\perp \vec{u}}|^2} \tag{15.11}$$

$$= \sqrt{\left(2 \cdot 3 - 2 \cdot 2\cos\frac{2\pi}{3}\right)^2 + \left(2 \cdot 2\sin\frac{2\pi}{3}\right)^2} \tag{15.12}$$

....

$$\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \ \vec{w} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \tag{15.13}$$

$$\theta = \arccos\frac{2 - 2 - 1}{3\sqrt{3}} \tag{15.14}$$

$$= \arccos\left(-\frac{1}{3\sqrt{3}}\right) \tag{15.15}$$

....

$$\cos\theta_A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} \tag{15.16}$$

$$= \frac{\begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}}{\sqrt{13}\sqrt{20}} \tag{15.17}$$

$$= \frac{4}{\sqrt{260}} \tag{15.18}$$

. . . .

$$A = (-2, 1, -2) \tag{15.19}$$
$$\ell = \lambda \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{15.20}$$
$$\overrightarrow{OB} = \overrightarrow{OA}_\ell = \frac{\overrightarrow{OA}\,\vec{\ell}}{\vec{\ell}\,\vec{\ell}} \cdot \vec{\ell} \tag{15.21}$$
$$= \frac{-4+3-4}{4+9+4} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{15.22}$$
$$= -\frac{5}{17} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{15.23}$$
$$|\overrightarrow{OB}| = \frac{5}{17}\sqrt{2^2 + 3^2 + 2^2} \tag{15.24}$$
$$= \frac{5}{17}\sqrt{17} \tag{15.25}$$

$$\hat{OB} = \frac{\overrightarrow{OB}}{|\overrightarrow{OB}|} \tag{15.26}$$
$$= \frac{-\frac{5}{17}\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}}{\frac{5}{17}\sqrt{17}} \tag{15.27}$$
$$= -\frac{1}{\sqrt{17}} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \tag{15.28}$$
$$|\overrightarrow{AB}| = \sqrt{\left(-2 + \frac{2}{\sqrt{17}}\right)^2 + \left(1 + \frac{3}{\sqrt{17}}\right)^2 + \left(-2 + \frac{2}{\sqrt{17}}\right)^2} \tag{15.29}$$
$$\text{whoops, used unit vector } \hat{OB} \text{ instead of point B} \tag{15.30}$$

15.2. Cross Product

$$\vec{u} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}, \ \vec{u} \times \vec{v} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \tag{15.31}$$
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{15.32}$$

. . . .

15.3. Functions and Invertibility

$$f(x) = 2\ln x + 3 \tag{15.33}$$
$$\frac{y-3}{2} = \ln x \tag{15.34}$$

$$x = e^{\frac{y-3}{2}} \tag{15.35}$$
$$y = e^{\frac{x-3}{2}} \tag{15.36}$$

....

$$f(x) = e^{\sqrt{3+x}} \tag{15.37}$$
$$D_f : [-3, \infty) \tag{15.38}$$
$$y = e^{\sqrt{3+x}} \tag{15.39}$$
$$\ln y = \sqrt{3+x} \tag{15.40}$$
$$\ln^2 y = 3 + x \tag{15.41}$$
$$x = \ln^2 y - 3 \tag{15.42}$$
$$y = \ln^2 x - 3 \tag{15.43}$$

....

$$f(x) = \sqrt{\frac{x}{x-9}} \tag{15.44}$$
$$x - 9 = 0 \implies (-\infty, 0] \cup (9, \infty) \tag{15.45}$$

....

$$f(x) = \frac{x+1}{x-2} \tag{15.46}$$
$$D : x \in \mathbb{R}/\{2\} \tag{15.47}$$
$$y = \frac{x+1}{x-2} \tag{15.48}$$
$$y(x-2) = x+1 \tag{15.49}$$
$$yx - 2y = x + 1 \tag{15.50}$$
$$yx - x = 2y + 1 \tag{15.51}$$
$$x(y-1) = 2y + 1 \tag{15.52}$$
$$x = \frac{2y+1}{y-1} \tag{15.53}$$
$$y = \frac{2x+1}{x-1} \tag{15.54}$$

....

$$f\left(\frac{1}{x^2}\right) = x \tag{15.55}$$
$$y = \frac{1}{x^2} \tag{15.56}$$
$$x^2 = \frac{1}{y} \tag{15.57}$$
$$x = \pm\frac{1}{\sqrt{y}} \tag{15.58}$$
$$y = \frac{1}{\sqrt{x}} \tag{15.59}$$

....

$$\arcsin 0.89 + \arccos 0.89 = \frac{\pi}{2} \tag{15.60}$$

15.4. Limits

$$\lim_{x\to 3} \frac{x^2-6x+9}{x^2-9x+18} = \tag{15.61}$$

$$\lim_{x\to 3} \frac{(x-3)\cancel{(x-3)}}{(x-6)\cancel{(x-3)}} = \tag{15.62}$$

$$\frac{0}{9} = 0 \tag{15.63}$$

....

$$\lim_{x\to -4} \frac{\sqrt{-x-3}-1}{x+4} = \tag{15.64}$$

$$= \frac{\sqrt{-x-3}-1}{x+4} \frac{\sqrt{-x-3}+1}{\sqrt{-x-3}+1} \tag{15.65}$$

$$= \frac{-x-3-1}{(x+4)(\sqrt{-x-3}+1)} \tag{15.66}$$

$$= \frac{-\cancel{(x+4)}}{\cancel{(x+4)}(\sqrt{-x-3}+1)} \tag{15.67}$$

$$= \lim_{x\to -4} \frac{-1}{\sqrt{-x-3}+1} \tag{15.68}$$

$$= -\frac{1}{2} \tag{15.69}$$

....

$$\lim_{x\to\infty} \frac{x^2+9x+10}{5x^2-9x-1} \tag{15.70}$$

$$= \frac{1+\frac{9}{x}+\frac{10}{x^2}}{5-\frac{9}{x}-\frac{1}{x^2}} \tag{15.71}$$

$$= \frac{1}{5} \tag{15.72}$$

....

$$\lim_{x\to 4} f(x) = \lim_{x\to -4} \frac{1-16x^{-2}}{x-4} \tag{15.73}$$

$$= \lim_{x\to 4} \frac{x^{-2}\cancel{(x-4)}(x+4)}{\cancel{x-4}} \tag{15.74}$$

$$= \lim_{x\to 4} \frac{x+4}{x^2} \tag{15.75}$$

$$= \frac{8}{16} = 0 \tag{15.76}$$

....

$$\lim_{x\to\infty} \sqrt{x^2-9x+8} - \sqrt{x^2-3x+6} = \tag{15.77}$$

$$= \frac{\sqrt{x^2-9x+8}-\sqrt{x^2-3x+6}}{1} \frac{\sqrt{x^2-9x+8}+\sqrt{x^2-3x+6}}{\sqrt{x^2-9x+8}+\sqrt{x^2-3x+6}} \tag{15.78}$$

$$= \frac{x^2-9x+8-x^2+3x-6}{\sqrt{x^2-9x+8}+\sqrt{x^2-3x+6}} \tag{15.79}$$

....

$$\lim_{x\uparrow 6}\frac{x^2-10x+24}{|x-6|}= \tag{15.80}$$
$$=\frac{(x-4)(x-6)}{6-x} \tag{15.81}$$
$$=-(x-4)=4-x \tag{15.82}$$
$$=-2 \tag{15.83}$$

....

$$\lim_{x\to\infty}\frac{-8x}{\sqrt{4x^2-6x-9}}= \tag{15.84}$$
$$=\frac{-8\cancel{x}}{\cancel{x}\sqrt{4-\frac{6}{x}-\frac{9}{x^2}}} \tag{15.85}$$
$$=-\frac{8}{2}=-4 \tag{15.86}$$

....

$$\lim_{u\to 0}\frac{\ln(1+u)}{u}=1 \tag{15.87}$$
$$u=\frac{a}{n} \tag{15.88}$$
$$\lim_{n\to\infty} n\ln\left(1+\frac{a}{n}\right)= \tag{15.89}$$
$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{a}{n}\right)}{\frac{1}{n}}= \tag{15.90}$$
$$\lim_{n\to\infty} a\frac{\ln\left(1+\frac{a}{n}\right)}{\frac{a}{n}}= \tag{15.91}$$
$$a\lim_{u\to 0}\frac{\ln(1+u)}{u}=a \tag{15.92}$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n= \tag{15.93}$$
$$\lim_{n\to\infty} e^{n\ln\left(1+\frac{a}{n}\right)}= \tag{15.94}$$
$$\lim_{u\to 0} e^{a\frac{\ln(1+u)}{u}}= \tag{15.95}$$
$$=e^a \tag{15.96}$$

....

$$g(x)\le 3x\le\frac{g(x)}{\cos x} \tag{15.97}$$
$$\frac{g(x)}{x}\le 3\le\frac{g(x)}{x\cos x} \tag{15.98}$$

....

$$h(x)\le 5e^x\le\frac{2h(x)}{x} \tag{15.99}$$
$$\frac{h(x)}{e^x}\le 5 \tag{15.100}$$

$$\frac{5x}{2} \leq \frac{h(x)}{e^x} \tag{15.101}$$
$$\frac{5x}{2} \leq \frac{h(x)}{e^x} \leq 5 \tag{15.102}$$

15.5. Continuity

$$f(x) = x + c \tag{15.103}$$
$$g(x) = x^2 + 4 \tag{15.104}$$
$$f(c) = g(c) \tag{15.105}$$
$$2c = c^2 + 4 \tag{15.106}$$
$$c^2 - 2 + 4 = 0 \tag{15.107}$$
$$c = \frac{2 \pm \sqrt{4 - 16}}{2} \tag{15.108}$$
$$c = 1 \pm \sqrt{3}i \tag{15.109}$$

15.6. Exam from Q1

Question 1: 6

$$y' = \frac{2x-2}{x^2+1} + \frac{1}{x+1}y \tag{15.110}$$
$$y' - \frac{y}{x+1} = \frac{2x-2}{x^2+1} \tag{15.111}$$
$$I(x) = e^{-\int (x+1)^{-1}\,dx} \tag{15.112}$$
$$= e^{-\ln(x+1)} \tag{15.113}$$
$$= \frac{1}{x+1} \tag{15.114}$$
$$\frac{y}{x+1} = \int \frac{1}{x+1}\frac{2(x-1)}{x^2+1}\,dx \tag{15.115}$$
$$= \int \frac{2x-2}{(x^2+1)(x+1)}\,dx = *_1 \tag{15.116}$$
$$\frac{2x-2}{(x^2+1)(x+1)} = \frac{Ax}{x^2+1} + \frac{B}{x+1} \tag{15.117}$$
$$B = -\frac{4}{2} = -2 \tag{15.118}$$
$$\frac{2x-2}{(x^2+1)(x+1)} = \frac{Ax(x+1) - 2(x^2+1)}{(x^2+1)(x+1)} \tag{15.119}$$
$$2x - 2 = Ax^2 + Ax - 2x^2 - 2 \tag{15.120}$$
$$A = 2 \tag{15.121}$$
$$\frac{y}{x+1} = \int \frac{2x}{x^2+1}\,dx - \int \frac{2}{x+1}\,dx \tag{15.122}$$
$$= \ln(x^2+1) - 2\ln(x+1) + C \tag{15.123}$$
$$y = (x+1)(\ln(x^2+1) - 2\ln(x+1) + C) \tag{15.124}$$

Question 2: 4

$$\int x(3-x)^{1/2}\,dx = *_1 \tag{15.125}$$
$$u = 3 - x \implies u - 3 = -x \tag{15.126}$$
$$du = -dx \tag{15.127}$$
$$*_1 = \int (u-3)\sqrt{u}\,du \tag{15.128}$$
$$= \int u^{3/2} - 3\sqrt{u}\,du \tag{15.129}$$
$$= \frac{1}{\frac{5}{2}}u^{5/2} - \frac{3}{\frac{3}{2}}u^{3/2} + C \tag{15.130}$$
$$= \frac{2}{5}u^{5/2} - 2u^{3/2} + C \tag{15.131}$$
$$= \frac{2}{5}(3-x)^{5/2} - 2(3-x)^{3/2} + C \tag{15.132}$$

$$\int x^3\cos(x^2)\,dx = *_1 \tag{15.133}$$
$$u = x^2 \tag{15.134}$$
$$du = 2x\,du \tag{15.135}$$
$$*_1 = \frac{1}{2}\int u\cos u\,du = *_2 \tag{15.136}$$
$$f(x) = u \implies f'(x) = 1 \tag{15.137}$$
$$g'(x) = \cos u \implies g(x) = \sin u \tag{15.138}$$
$$*_2 = \frac{1}{2}\left(u\sin u - \int \sin u\,du\right) \tag{15.139}$$
$$= \frac{1}{2}(u\sin u + \cos u + C) \tag{15.140}$$
$$= \frac{1}{2}(x^2\sin x^2 + \cos x^2 + C) \tag{15.141}$$

$$\int x\sqrt{x+2}\,dx = *_1 \tag{15.142}$$
$$u = x + 2 \tag{15.143}$$
$$du = dx \tag{15.144}$$
$$*_1 = \int (u-2)\sqrt{u}\,du \tag{15.145}$$
$$= \int u^{3/2} - 2u^{1/2}\,du \tag{15.146}$$
$$= \frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + C \tag{15.147}$$
$$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \tag{15.148}$$
$$= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C \tag{15.149}$$

$$f(x) = -x^2 + 2x + 2 \tag{15.150}$$
$$-y = x^2 - 2x - 2 \tag{15.151}$$

$$x^2 - 2x - 2 + y = 0 \tag{15.152}$$

$$x = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot (y-2)}}{2} \tag{15.153}$$

$$= 1 + \sqrt{1 - (y-2)} \tag{15.154}$$

$$= 1 + \sqrt{3 - y} \tag{15.155}$$

$$f^{-1}(x) = 1 + \sqrt{3 - x} \tag{15.156}$$

16

Practice Exam 1

16.1. Short answer questions

Question 1

a)

$$\vec{a} \times \vec{b} = 0 \iff \vec{a} \parallel \vec{b} \quad (16.1)$$
$$\hat{a} = \pm\hat{b} \quad (16.2)$$
$$\boxed{\text{Statement 1 False}} \quad (16.3)$$

b)

$$(\vec{a} \cdot \vec{b}) = |\vec{a}||\vec{b}| \cos\theta \quad (16.4)$$
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \quad (16.5)$$
$$\frac{\vec{a} \cdot \vec{b}}{\cos\theta} = \frac{|\vec{a} \times \vec{b}|}{\sin\theta} \quad (16.6)$$
$$(\vec{a} \cdot \vec{b}) \tan\theta = |\vec{a} \times \vec{b}| \quad (16.7)$$
$$\boxed{\text{Statement 2 True}} \quad (16.8)$$

c)

$$(3\vec{a} \times 4\vec{b}) \cdot (\vec{a} - 6\vec{b}) = 0 \quad (16.9)$$

Question 2

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad (16.10)$$
$$2y = e^x - e^{-x} \quad (16.11)$$
$$e^{2x} - 2ye^x - 1 = 0 \quad (16.12)$$
$$a^2 - 2ya - 1 = 0 \quad (16.13)$$
$$a = \frac{2y \pm \sqrt{4y^2 + 4}}{2} \quad (16.14)$$
$$e^x = y + \sqrt{y^2 + 1} \quad (16.15)$$
$$x = \ln(y + \sqrt{y^2 + 1}) \quad (16.16)$$
$$\boxed{y = \ln(x + \sqrt{x^2 + 1})} \quad (16.17)$$

$$x + \sqrt{x^2 + 1} = 0 \tag{16.18}$$
$$x^2 + x^2 + 1 = 0 \tag{16.19}$$
$$2x^2 = -1 \tag{16.20}$$
$$x^2 = -\frac{1}{2} \tag{16.21}$$
$$\boxed{x \in \mathbb{R}} \tag{16.22}$$

Question 3

$$f(x) = \ln x \tag{16.23}$$
$$f'(x) = \frac{1}{x} \tag{16.24}$$
$$L_1(x) = 0 + \frac{1}{x}x = x \tag{16.25}$$
$$L_1(1.1) = 1.1 \tag{16.26}$$
$$f''(x) = -\frac{1}{x^2} \tag{16.27}$$
$$E(1.1) = \frac{1}{2}f''(1)(1.1 - 1.0)^2 \tag{16.28}$$
$$= \frac{1}{2}\left(-\frac{1}{1}\right)(0.1)^2 \tag{16.29}$$
$$= -\frac{1}{2}\frac{1}{100} \tag{16.30}$$
$$= -0.005 \tag{16.31}$$

Question 4

$$\int_1^e x^2 \ln^2 x\, dx = *_1 \tag{16.32}$$
$$f(x) = \ln^2 x \implies f'(x) = \frac{2}{x}(\ln x) \tag{16.33}$$
$$g'(x) = x^2 \implies g(x) = \frac{x^3}{3} \tag{16.34}$$
$$*_1 = \frac{x^3}{3}\ln^2 x - \int_1^e \frac{x^3}{3}\frac{2}{x}\ln x\, dx \tag{16.35}$$
$$= \frac{x^3}{3}\ln^2 x - \frac{2}{3}\int_1^e x^2 \ln x\, dx = *_2 \tag{16.36}$$
$$f(x) = \ln x \implies f'(x) = \frac{1}{x} \tag{16.37}$$
$$g'(x) = x^2 \implies g(x) = \frac{x^3}{3} \tag{16.38}$$
$$*_2 = \frac{x^3}{3}\ln^2 x - \frac{2}{3}\left(\ln x\frac{x^3}{3} - \int_1^e \frac{x^2}{3}\, dx\right) \tag{16.39}$$
$$= \left[\frac{1}{3}x^3 \ln^2 x\right]_1^e - \left[\frac{2}{9}x^3 \ln x\right]_1^e + \frac{2}{3}\left[\frac{x^3}{9}\right]_1^e \tag{16.40}$$
$$= \frac{1}{3}e^3 - \frac{2}{9}e^3 + \frac{2}{27}e^3 - \frac{2}{27} \tag{16.41}$$

$$= \frac{5}{27}e^3 - \frac{2}{27} \tag{16.42}$$

Question 5

Question 6

$$z = \frac{(1+\sqrt{3}i)^9}{(2\sqrt{3}-2i)^4} \tag{16.43}$$
$$= \frac{u^9}{w^4} \tag{16.44}$$
$$|u| = \sqrt{4} = 2 \tag{16.45}$$
$$\theta_u = \arctan\sqrt{3} = \frac{\pi}{3} \tag{16.46}$$
$$\theta_1 = 9\theta_u = \frac{9\pi}{3} = \pi \mod 2\pi \tag{16.47}$$
$$u^9 = 2^9 e^{\pi i} \tag{16.48}$$
$$|w| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4 \tag{16.49}$$
$$\theta_w = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6} \tag{16.50}$$
$$\theta_2 = 4\theta_w = \frac{2\pi}{3} \tag{16.51}$$
$$w^4 = 4^4 e^{\frac{2\pi}{3}i} \tag{16.52}$$
$$z = \frac{2^9}{4^4}\frac{e^{\pi i}}{e^{\frac{2}{3}\pi i}} \tag{16.53}$$
$$= 2e^{\frac{1}{3}\pi} \tag{16.54}$$
$$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \tag{16.55}$$
$$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \tag{16.56}$$
$$= 1 + \sqrt{3}i \tag{16.57}$$

Question 7

$$F(x) = \sin(x-a)\sin(x-b) + 2x \tag{16.58}$$
$$F(a) = 2a \tag{16.59}$$
$$F(b) = 2b \tag{16.60}$$
$$\text{Intermediate-value theorem} \tag{16.61}$$
$$\tag{16.62}$$

17

Practice Exam 2

17.1. Short Answer Questions

Question 1

$$\vec{a} \cdot \vec{b} = 4b_3 = 0 \tag{17.1}$$

$$\vec{b} = \begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix} \tag{17.2}$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix} \tag{17.3}$$

$$= \begin{pmatrix} -4b_2 \\ 0 \\ 3b_2 \end{pmatrix} \tag{17.4}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-4b_2)^2 + 0 + (3b_2)^2} = 15 \tag{17.5}$$

$$16b_2^2 + 9b_2^2 = 225 \tag{17.6}$$

$$25b^2 = 225 \tag{17.7}$$

$$b_2^2 = 9 \tag{17.8}$$

$$b_2^2 = 3 \tag{17.9}$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} -12 \\ 0 \\ 9 \end{pmatrix} \tag{17.10}$$

Question 2

Question 3

$$k(x) = e^{3x} + 2x + 1 \tag{17.11}$$

$$k'(x) = 3e^{3x} + 2 \tag{17.12}$$

$$3e^{3x} + 2 = 0 \tag{17.13}$$

$$e^{3x} = -\frac{2}{3} \tag{17.14}$$

$$3x = \ln\left(-\frac{2}{3}\right) \tag{17.15}$$

Question 4

$$x^3 - \cos\frac{2\pi x}{y} = yx^2 + 3 \tag{17.16}$$

$$3x^2 + \frac{2\pi}{y}\sin\left(\frac{2\pi x}{y}\right) + \sin\left(\frac{2\pi x}{y}\right)\left(-\frac{2\pi x}{y^2}\right)\frac{dy}{dx} = 2yx + x^2\frac{dy}{dx} \tag{17.17}$$

$$\frac{dy}{dx}\left(-\frac{2\pi x}{y^2} - x^2\right) = 2yx - 3x^2 - \frac{2\pi}{y}\sin\left(\frac{2\pi x}{y}\right) \tag{17.18}$$

$$\frac{dy}{dx} = \frac{2yx - 3x^2 - \frac{2\pi}{y}\sin\left(\frac{2\pi x}{y}\right)}{-\frac{2\pi x}{y^2} - x^2} \tag{17.19}$$

$$\frac{dy}{dx}(2,1) = \frac{4 - 12 - 2\pi\sin(4\pi)}{-4\pi - 4} \tag{17.20}$$

$$= \frac{-8}{-4\pi - 4} \tag{17.21}$$

$$= \frac{8}{4\pi - 4} \tag{17.22}$$

$$y = \frac{8(x-2)}{4\pi - 4} + 1 \tag{17.23}$$

Question 5

$$f(x) = \int \frac{1}{e^x + 3 + 2e^{-x}}\,dx \tag{17.24}$$

$$= \int \frac{e^x}{e^{2x} + 3e^x + 2}\,dx \tag{17.25}$$

$$= \int \frac{e^x}{(e^x+2)(e^x+1)}\,dx = *_1 \tag{17.26}$$

$$u = e^x \tag{17.27}$$

$$du = e^x\,dx \tag{17.28}$$

$$*_1 = \int \frac{1}{(u+2)(u+1)}\,du = *_2 \tag{17.29}$$

$$\frac{A}{u+2} + \frac{B}{u+1} = \frac{1}{(u+2)(u+1)} \tag{17.30}$$

$$A = -1 \tag{17.31}$$

$$B = 1 \tag{17.32}$$

$$*_2 = \int \frac{1}{u+1} - \frac{1}{u+2}\,du \tag{17.33}$$

$$= \int \frac{1}{u+1}\,du - \int \frac{1}{u+2}\,du \tag{17.34}$$

$$= \ln(u+1) - \ln(u+2) \tag{17.35}$$

$$= \ln\frac{e^x+1}{e^x+2} + C \tag{17.36}$$

Question 6

Question 7

$$x \cos\alpha = L \tag{17.37}$$
$$dx\ \cos\alpha - x \sin\alpha\, d\alpha = 0 \tag{17.38}$$
$$\cos\alpha\, dx = x \sin\alpha\, d\alpha \tag{17.39}$$
$$d\alpha = \frac{\cos\alpha\, dx}{x \sin\alpha} \tag{17.40}$$
$$d\alpha = \frac{dx}{x}\frac{1}{\tan\alpha} \tag{17.41}$$
$$= \frac{1}{10} \tag{17.42}$$

Question 8

$$V' = -320A\sqrt{V} \tag{17.43}$$
$$\frac{dV}{\sqrt{V}} = -320A\, dt \tag{17.44}$$
$$2\sqrt{V} = -320At + C \tag{17.45}$$
$$V = \left(\frac{C - 320At}{2}\right)^2 \tag{17.46}$$
$$V(0) = 100 = \frac{C^2}{4} \tag{17.47}$$
$$C = 20 \tag{17.48}$$
$$V(10) = 0 = \left(\frac{20 - 320A \cdot 10}{2}\right)^2 \tag{17.49}$$
$$20 - 3200A = 0 \tag{17.50}$$
$$A = \frac{20}{3200} \tag{17.51}$$
$$A = \frac{1}{160} \tag{17.52}$$

Question 9