Lecture Notes:
Actuarial Risk Theory
Summer Term 2013

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31st October 2013

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1 Preface

These are lecture notes of a course on Actuarial Risk Theory which took place at Technische Universität München in summer 2013. No claim on completeness is made.
I would like to thank Vicky Fasen and Claudia Klüppelberg for leaving me their notes of earlier courses. The responsibility for errors is mine. If you spot an error, don't hesitate to drop me a line:
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2 The risk process

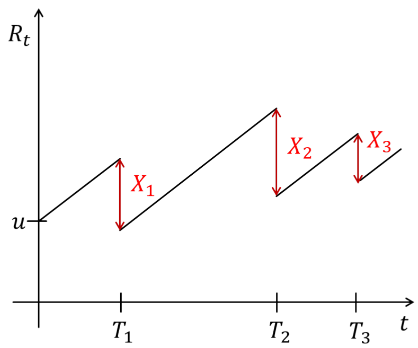
Definition 2.1. A **risk** is a non-negative rv (random variable).

A simple model for the wealth of an insurance company is the **risk process** $(R_t)_{t\geq 0}$ given by $R_t = u + P_t - S_t$, $t \geq 0$, where

(a) $R_0 = u > 0$ is the initial capital.

(b) $(P_t)_{t\geq 0}$ is the premium income process.

(c) $(S_t)_{t\geq 0}$ is the claim amount process.



$$P_t = ct$$

$$S_t = \sum_{k=1}^{N_t} X_k$$

This means:

1. Claims happen at the (random) times T_k satisfying $0 \leq T_1 \leq T_2 \leq \dots$. We call the T_k **claim arrival times** or **claim times**.

2. The k-th claim arriving at time T_k has the size X_k.

3. The claim arrival process is $N_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leq t\}} = \#\{n \in \mathbb{N} : T_n \leq t\}$ (where $\mathbb{1}$ is the indicator function). It counts the number of claims which arrived by time t.

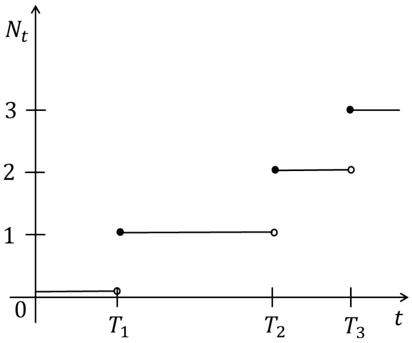
4. The **claim amount process** is $S_t = \sum_{k=1}^{N_t} X_k$. It counts the amount of all claims which arrived by time t.

2.1 The renewal counting process

Definition 2.2. A stochastic process $(N_t)_{t\geq 0}$ is a **counting process** if the following holds:

(a) $N_0 = 0$ *a.s.* (almost surely).

(b) The sample paths of $(N_t)_{t\geq 0}$ are a.s. increasing, right-continuous step functions with jumps of size 1.



Let $W_i = T_i - T_{i-1},\ T_0 := 0$. The sequence $(W_k)_{k\in\mathbb{N}}$ is called the sequence of **interarrival times**. Then $T_n = \sum_{k=1}^{n} W_k$ and $N_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leq t\}}$.

Definition 2.3. Let $(W_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. (independent and identically distributed) rv's with distribution function G and $G(0) = 0$. Define $T_n = \sum_{k=1}^{n} W_k$. Then the process (T_n) is called **renewal sequence** and the counting process $N_t = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leq t\}}$ is the **renewal process**.
The function $m(t) = \mathbb{E}[N_t]$ ($\Rightarrow\ m(t) = \sum_{n=1}^{\infty} \mathbb{P}(T_n \leq t) = \sum_{n=1}^{\infty} G^{*n}(t)$) is the **renewal function** ($G^{*n}(t)$ = distribution function of T_n = n-fold convolution of G). $m(t)$ is the **expected number of claim arrivals** until time t.

Remarks 2.4. The following relations hold:

1. $N_t = \#\{k \in \mathbb{N} : T_k \leq t\}$.
2. $T_m = \inf\{t \geq 0 : N_t = m\}$.
3. $\{N_t \geq n\} = \{T_n \leq t\}$.
4. $\{N_t = n\} = \{T_n \leq t\} \setminus \{T_{n+1} \leq t\} = \{T_n \leq t,\ T_{n+1} > t\}$.
5. $\mathbb{P}(N_t = n) = \mathbb{P}(T_n \leq t) - \mathbb{P}(T_{n+1} \leq t) = G^{*n}(t) - G^{*(n+1)}(t)$.
6. $\mathbb{P}(N_t = 0) = \mathbb{P}(T_1 > t) = 1 - G(t) =: \overline{G}(t)$ ($\overline{G}(t)$ is called **tail distribution function**).

Proof. Easy $\square$

Theorem 2.5. *Let $(N_t)_{t\geq 0}$ be a renewal process as given above and $\frac{1}{\lambda} := \mathbb{E}[W_1] < \infty$. Then:*

(a) $\frac{N_t}{t} \underset{t\to\infty}{\longrightarrow} \lambda$ *a.s. (strong law of large numbers).*

(b) Let $\sigma^2 = Var(W_1) < \infty$. Then $\frac{N_t - \lambda t}{\sqrt{\sigma^2\lambda^3 t}} \xrightarrow{d} Y$, where $Y \stackrel{d}{=} N(0,1)$, i.e. the laws of $\frac{N_t-\lambda t}{\sqrt{\sigma^2\lambda^3 t}}$ converge weakly to $N(0,1)$ (central limit theorem).

An example where the theorem can be verified by elementary calculations is $W_1, W_2, \ldots$ i.i.d. $Exp(\lambda)$ (exponential distribution with parameter λ). For the proof we will need:

Lemma 2.6. *Let $(Z_n)_{n\in}$ be a sequence of rv's and $(N_t)_{t\geq 0}$ be a stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Z be a rv s.t. $\lim_{n\to\infty} Z_n = Z$ a.s. and assume that $N_t \to \infty$ a.s. Then,*

$$\lim_{t\to\infty} Z_{N_t} = Z \ a.s.$$

Proof. Let $\Omega_1 = \{\omega \in \Omega : \lim_{t\to\infty} N_t = \infty\}$ and $\Omega_2 = \{\omega \in \Omega : \lim_{n\to\infty} Z_n(\omega) = Z(\omega)\}$. By assumption, $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = 1$, hence $\mathbb{P}(\Omega_1 \cap \Omega_2) = 1$.
$\Rightarrow \mathbb{P}(\{\omega \in \Omega : \lim_{t\to\infty} Z_{N_t}(\omega) = Z(\omega)\}) \geq \mathbb{P}(\Omega_1 \cap \Omega_2) = 1$. $\square$

Proof of Theorem 1.5.
(a) $(W_k)_{k\in\mathbb{N}}$ are i.i.d. with $\mathbb{E}[|W_1|] < \infty$.

$$\Longrightarrow \frac{T_n}{n} = \frac{1}{n}\sum_{k=1}^n W_k \xrightarrow{a.s.} \mathbb{E}[W_1] = \frac{1}{\lambda} \ a.s. \tag{2.1}$$

Further, $N_t \to \infty$ $a.s.$ (see exercises).

$$\begin{aligned}
&\underset{(2.1)+\text{Lemma } 2.6}{\Longrightarrow} && \frac{T_{N_t}}{N_t} \underset{t\to\infty}{\longrightarrow} \frac{1}{\lambda} \ a.s. \\
&\Longrightarrow && \underbrace{\frac{N_t}{N_t+1}}_{\to 1 \ a.s.} \underbrace{\frac{T_{N_t}}{N_t}}_{\to\frac{1}{\lambda} \ a.s.} \leq \frac{t}{N_t+1} \leq \underbrace{\frac{T_{N_t+1}}{N_t+1}}_{\to\frac{1}{\lambda} \ a.s.} \\
&\Longrightarrow && \frac{t}{N_t+1} \to \frac{1}{\lambda} \ a.s. \\
&\Longrightarrow && \frac{N_t}{t} \underset{t\to\infty}{\longrightarrow} \lambda \ a.s.
\end{aligned}$$

(b) Let $x \in \mathbb{R}$. Write $\lfloor x \rfloor := \sup\{m \in \mathbb{Z} : m \leq x\}$. By the CLT (central limit theorem) we have

$$\mathbb{P}\left(\frac{T_n - \frac{1}{\lambda}n}{\sigma\sqrt{n}} \leq x\right) = \mathbb{P}\left(\frac{\sum_{i=1}^n (W_i - \mathbb{E}[W_i])}{\sqrt{n\mathrm{Var}(W_1)}} \leq x\right) \underset{n\to\infty}{\longrightarrow} \Phi(x), \tag{2.2}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{u^2}{2}}\,du$. Define $h(t) = \lfloor x\sqrt{\sigma^2\lambda^3 t} + \lambda t\rfloor$. Then

$$\begin{aligned}
\mathbb{P}\left(\frac{N_t - \lambda t}{\sqrt{\sigma^2\lambda^3 t}} \leq x\right) &= \mathbb{P}(N_t \leq h(t)) \stackrel{Rem 2.4}{=} \mathbb{P}\left(T_{h(t)} \geq t\right) \\
&= \mathbb{P}\left(\frac{T_{h(t)} - \frac{1}{\lambda}h(t)}{\sigma\sqrt{h(t)}} \geq \underbrace{\frac{t - \frac{1}{\lambda}h(t)}{\sigma\sqrt{h(t)}}}_{=:g(t)}\right).
\end{aligned} \tag{2.3}$$

Suppose
$$h(t) \to \infty \qquad \text{and} \qquad g(t) \to -x. \tag{2.4}$$
Then (2.2)-(2.4) imply that $\mathbb{P}\left(\frac{N_t-\lambda t}{\sqrt{\sigma^2\lambda^3 t}} \le x\right) \to 1-\Phi(-x) = \Phi(x)$. Thus, it suffices to show that $h(t) \sim \lambda t$ for $t \to \infty$ and $g(t) \xrightarrow[t\to\infty]{} -x$ $\left(a_t \sim b_t \text{ if } \frac{a_t}{b_t} \xrightarrow[t\to\infty]{} 1\right)$. Note that $h(t) = x\sqrt{\sigma^2\lambda^3 t} + \lambda t + \varepsilon(t)$ where $|\varepsilon(t)| \le 1$.

$$\Rightarrow \frac{h(t)}{t} \xrightarrow[t\to\infty]{} \lambda.$$

$$\Rightarrow \quad g(t) = \frac{t - \frac{1}{\lambda}h(t)}{\sigma\sqrt{h(t)}} = \frac{t - \frac{1}{\lambda}\left(x\sqrt{\sigma^2\lambda^3 t} + \lambda t + \varepsilon(t)\right)}{\sigma\underbrace{\sqrt{h(t)}}_{\sim\sqrt{\lambda t}}} \sim \frac{-\frac{1}{\lambda}x\sqrt{\sigma^2\lambda^3 t}}{\sigma\sqrt{\lambda t}} \xrightarrow[t\to\infty]{} -x.$$

□

Remarks 2.7.

1. We can estimate λ by $\frac{N_t}{t}$: $\widehat{\lambda}_t = \frac{N_t}{t}$ is a consistent estimator and it is also asymptotically normal.

2. We have $\frac{\mathbb{E}[N_t]}{t} \xrightarrow[t\to\infty]{} \lambda$ (see exercises), and, under the assumptions of (b), $\frac{\mathrm{Var}(N_t)}{t} \xrightarrow[t\to\infty]{} \sigma^2\lambda^3$. Hence $\mathbb{P}\left(\frac{N_t-\mathbb{E}[N_t]}{\sqrt{\mathrm{Var}(N_t)}} \le x\right) \xrightarrow[t\to\infty]{} \Phi(x)$.

2.2 The Poisson process

Recall that for $X \sim Poi(\lambda)$ (Poisson with intensity $\lambda > 0$):

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!}e^{-\lambda} \qquad k = 0, 1, 2, \ldots.$$

Definition 2.8. A stochastic process $(N_t)_{t\ge 0}$ is a Poisson process with intensity $\lambda \in (0,\infty)$ if and only if:

(a) $N_0 = 0$ a.s.

(b) $(N_t)_{t\ge 0}$ has independent increments: $\forall 0 = t_0 < t_1 < \ldots < t_m$
$N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, \ldots, N_{t_m} - N_{t_{m-1}}$ are independent.

(c) For $s, t \ge 0$: $N_{t+s} - N_t \stackrel{d}{=} N_s \sim Poi(\lambda s)$.

(d) With probability 1, the sample paths $t \mapsto N_t(\omega)$ are right-continuous and have left limits (= càdlàg = continue à droite, limite à gauche).

Remark 2.9. Poisson processes and Brownian Motion belong to the large class of Lévy processes which satisfy (a), (b), (d) and have stationary increments.

Theorem 2.10. *Let $(N_t)_{t\geq 0}$ be a counting process with arrival times $0 \leq T_1 \leq T_2 \leq \ldots$ and interarrival times $(W_k)_{k\in\mathbb{N}}$. Then the following conditions are equivalent:*

(a) $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ.

(b) $(W_k)_{k\in\mathbb{N}}$ are i.i.d. $Exp(\lambda)$ distributed.

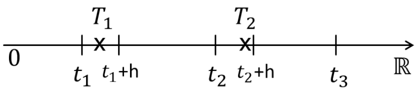
(c) $(N_t)_{t\geq 0}$ has independent and stationary increments, $N_0 = 0$ a.s. and for $t \geq 0$:

$$\begin{aligned} &\mathbb{P}(N_{t+h} - N_t = 1) = \lambda h + o(h) \qquad \textit{for } h \to 0 \\ &\mathbb{P}(N_{t+h} - N_t > 1) = o(h). \end{aligned}$$

Proof. (**a**) $\Rightarrow$ (**b**) : First, we show that $(T_1, \ldots, T_k)$ has density

$$\tilde{f}_k(t_1, \ldots, t_k) = \lambda^k e^{-\lambda t_k} \mathbb{1}_{\{t_1 < \ldots < t_k\}}. \qquad (\star)$$

Let $t_1 < t_2 < \ldots < t_k$ and $h_1, \ldots, h_k > 0$ be small enough such that $(t_i, t_{i+h_i}]$, $i = 1, \ldots, k$ are disjoint.



$$\begin{aligned} &\mathbb{P}\left(T_1 \in (t_1, t_1 + h_1], \ldots, T_k \in (t_k, t_k + h_k)\right) \\ &= \mathbb{P}\left(N_{t_1} = 0, N_{t_1+h_1} - N_{t_1} = 1, \ldots, N_{t_k} - N_{t_{k-1}+h_{k-1}} = 0, N_{t_k+h_k} - N_{t_k} = 1\right) \\ &\overset{ind.}{\underset{incr.}{=}} \mathbb{P}\left(N_{t_1} = 0\right) \mathbb{P}\left(N_{t_1+h_1} - N_{t_1} = 1\right) \cdot \ldots \cdot \mathbb{P}\left(N_{t_k} - N_{t_{k-1}+h_{k-1}} = 0\right) \mathbb{P}\left(N_{t_k+h_k} - N_{t_k} = 1\right) \\ &= \left(e^{-\lambda t_1}\right)\left(e^{-\lambda h_1}\lambda h_1\right) \cdot \ldots \cdot \left(e^{-\lambda(t_k - t_{k-1} - h_{k-1})}\right)\left(e^{-\lambda h_k}\lambda h_k\right) \\ &= e^{-\lambda(t_k+h_k)}\lambda^k \prod_{i=1}^k h_i. \end{aligned}$$

$$\Rightarrow \tilde{f}_k(t_1, \ldots, t_k) = \lim_{\max h_i \to 0} \frac{e^{-\lambda(t_k+h_k)}\lambda^k \prod_{i=1}^k h_i}{\prod_{i=1}^k h_i} = \lambda^k e^{-\lambda t_k},$$

$$\left(\text{since } \frac{F_{T_1}(t_1 + h_1) - F_{T_1}(t_1)}{h_1} \underset{h_1 \to 0}{\longrightarrow} F'_{T_1}(t_1)\right).$$

This proves ($\star$). The interarrival times can be obtained as $(W_1, \ldots, W_k) = g(T_1, \ldots, T_k)$ with $g : \mathbb{R}_+^k \to \mathbb{R}_+^k$, $g(t_1, \ldots, t_k) = (t_1, t_2 - t_1, \ldots, t_k - t_{k-1})$. g has inverse $g^{-1}(w_1, \ldots, w_k) = (w_1, w_1 + w_2, \ldots, w_1 + \ldots + w_k)$ with Jacobi determinant 1. The density transformation theorem gives for $(W_1, W_2, \ldots, W_k)$ the density

$$f_k(w_1, \ldots, w_k) = \tilde{f}_k\left(g^{-1}(w_1, \ldots, w_k)\right) = \lambda^k e^{-\lambda(w_1 + \ldots + w_k)} = \prod_{i=1}^k \lambda e^{-\lambda w_k}.$$

$$\Rightarrow W_1, \ldots, W_k \text{ are i.i.d. } Exp(\lambda) \text{ distributed.}$$

(**b**) $\Rightarrow$ (**a**) :
Step 1: Since $W_1 > 0$ a.s., we have $N_0 = 0$ a.s. and $(N_t)_{t\geq 0}$ is càdlàg.
Step 2: $N_t \sim Poi(\lambda t)$ (Exercise 1.3).

Step 3: We show that $(N_t)_{t\geq 0}$ has independent and stationary increments. We consider only the two-dimensional case, i.e., we show for $t, h > 0$:

$$\begin{aligned}\mathbb{P}\left(N_t = k, N_{t+h} - N_t = l\right) &= \mathbb{P}\left(N_t = k\right)\mathbb{P}\left(N_{t+h} - N_t = l\right) \\ &= e^{-\lambda(t+h)}\frac{(\lambda t)^k}{k!}\frac{(\lambda h)^l}{l!} \qquad \forall k, l \in \mathbb{N}_0.\end{aligned}$$

- $k = l = 0: \mathbb{P}\left(N_t = 0, N_{t+h} - N_t = 0\right) = \mathbb{P}\left(N_{t+h} = 0\right) = e^{-\lambda(t+h)}$
- $l = 0, k \in \mathbb{N}:$

$$\begin{aligned}\mathbb{P}\left(N_t = k, N_{t+h} - N_t = l\right) &= \mathbb{P}\left(N_t = k, N_{t+h} = k\right) = \mathbb{P}\left(T_k \leq t, T_{k+1} > t + h\right) \\ &= \mathbb{P}\left(T_k \leq t, t + h < T_k + W_k\right) \\ &= \mathbb{E}\left[\mathbb{P}\left(T_k \leq t, t + h < T_k + W_k | T_k\right)\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\{T_k \leq t\}}\mathbb{P}\left(t + h < T_k + W_k | T_k\right)\right] \\ &= \int_0^t \mathbb{P}\left(t + h < z + W_k\right)\, F_{T_k}(dz) \\ &= \int_0^t \underbrace{e^{-\lambda(t+h-z)}}_{W_k \sim Exp(\lambda)} \underbrace{\frac{\lambda^k z^{k-1}}{(k-1)!}e^{-\lambda z}}_{T_k \sim Gamma(k,\lambda)}\, dz \\ &= e^{-\lambda(t+h)}\frac{\lambda^k}{(k-1)!}\int_0^t z^{k-1}\, dz \\ &= e^{-\lambda(t+h)}\frac{(\lambda t)^k}{k!}.\end{aligned}$$

- $l, k \in \mathbb{N}:$

$$\begin{aligned}&\mathbb{P}\left(N_t = k, N_{t+h} - N_t = l\right) = \mathbb{P}\left(T_k \leq t < T_{k+1}, T_{k+l} \leq t + h < T_{k+l+1}\right) \\ &= \mathbb{E}[\mathbb{P}(\underbrace{T_k \leq t < T_{k+1} \leq t + h}_{\substack{\mathbb{1}_{\{T_k \leq t < T_{k+1} \leq t+h\}}, \\ T_{k+1} = T_k + W_{k+1}}}, \underbrace{T_{k+l} - T_{k+1} \leq t + h - T_{k+1} < T_{k+l+1} - T_{k+1} | T_k, T_{k+1}}_{\substack{\mathbb{P}(N'_{t+h-T_{k+1}} = l-1 | T_k, T_{k+1}) = e^{-\lambda(t+h-T_{k+1})}\frac{(\lambda(t+h-T_{k+1}))^{l-1}}{(l-1)!} \\ \text{with } (N'_t)_{t\geq 0} \text{ an independent copy of } (N_t)_{t\geq 0} \text{ and } T_{k+1} = T_k + W_{k+1}. \\ \text{(Note that } (T_{k+l} - T_{k+1}, T_{k+l+1} - T_{k+1}) \text{ is independent of } (T_k, T_{k+1}))}})] \\ &= \int_0^t \underbrace{\frac{\lambda^k}{(k-1)!}x^{k-1}e^{-\lambda x}}_{\text{density of } T_k}\int_{t-x}^{t+h-x} \underbrace{\lambda e^{-\lambda u}}_{\text{density of } W_{k+1}}\, e^{-\lambda(t+h-x-u)}\frac{(\lambda(t+h-x-u))^{l-1}}{(l-1)!}\, du\, dx \\ &= \ldots = e^{-\lambda(t+h)}\frac{(\lambda t)^k}{k!}\frac{(\lambda h)^l}{l!}.\end{aligned}$$

(**a**) $\Rightarrow$ (**c**) : Using Taylor expansion, we see $e^{-x} = 1 - x + o(x)$ for $x \to 0$. For $t \geq 0$ this gives

$$\begin{aligned}
\mathbb{P}\left(N_{t+h}-N_t=1\right) &= \mathbb{P}\left(N_h=1\right)=e^{-\lambda h}\lambda h=\lambda h(1-\lambda h+o(h))\\
&= \lambda h+o(h)\\
\mathbb{P}\left(N_{t+h}-N_t>1\right) &= \mathbb{P}\left(N_h>1\right)=1-\mathbb{P}\left(N_h=1\right)-\mathbb{P}\left(N_h=0\right)\\
&= 1-\lambda h+o(h)-\underbrace{e^{-\lambda h}}_{=1-\lambda h+o(h)}\\
&= o(h).
\end{aligned}$$

(c) $\Rightarrow$ (a) : Let $w_i(t):=\mathbb{P}\left(N_t=i\right)$, we need to show $w_i(t)=\frac{(\lambda t)^i}{i!}e^{-\lambda t}$.
Case $i=0$:

$$\begin{aligned}
w_0(t+h) &= \mathbb{P}\left(N_{t+h}=0\right)=\mathbb{P}\left(N_t=0,N_{t+h}-N_t=0\right)=\mathbb{P}\left(N_t=0\right)\mathbb{P}\left(N_h=0\right)\\
&= w_0(t)w_0(h),
\end{aligned}$$

we have

$$w_0(h)=1-\mathbb{P}\left(N_h=1\right)-\mathbb{P}\left(N_h>1\right)=1-\lambda h+o(h).$$
$$\Rightarrow \frac{w_0(t+h)-w_0(t)}{h}=w_0(t)\frac{w_0(h)-1}{h}=w_0\left(-\lambda+\frac{o(h)}{h}\right)\underset{h\to 0}{\longrightarrow}-\lambda w_0(t).$$

Thus, w_0 solves the differential equation

$$w_0'(t)=-\lambda w_0(t) \qquad \text{with} \qquad w_0(0)=\mathbb{P}\left(N_0=0\right)=1.$$

This has the unique solution $w_0(t)=e^{-\lambda t}$.
For general i, solving a system of ODEs gives $w_i(t)=\frac{(\lambda t)^i}{i!}e^{-\lambda t}$. $\square$

Remark 2.11. This means that a Poisson process is a renewal process with $Exp(\lambda)$ distributed waiting times. By Theorem 2.5, we have

$$\lim_{t\to\infty}\frac{N_t}{t}=\lambda \ \ a.s. \qquad \text{and} \qquad \frac{N_t-\lambda t}{\sqrt{\lambda t}}\underset{t\to\infty}{\xrightarrow{d}}N(0,1).$$

(Here, $\mathbb{E}\left[N_t\right]=\lambda t=\operatorname{Var}\left(N_t\right)$, since $N_t\sim Poi(\lambda t)$).

2.3 The claim amount process

Let $(N_t)_{t\geq 0}$ with $N_t=\sum_{k=1}^{\infty}\mathbb{1}_{\{T_k\leq t\}}$ be the claim number process which counts the number of claims which occurred until time t and let $(X_k)_{k\in\mathbb{N}}$ be the claim sizes, where the claim of size X_k arrives at time T_k. Then $S_t=\sum_{k=1}^{N_t}X_k$, $t\geq 0$ is the **claim amount process** describing the aggregated claims until time t.
We will assume the following:

1. The claim sizes are an i.i.d. sequence with distribution function F and $F(0)=0$.
2. $(X_k)_{k\in\mathbb{N}}$ and $(T_k)_{k\in\mathbb{N}}$ are independent ($\Rightarrow$ $(X_k)_{k\in\mathbb{N}}$ and $(N_t)_{t\geq 0}$ are independent).

We will investigate the amount of claims in a fixed time interval, e.g. 1 year, and study the properties of the annual amount of claims. We write shortly $S:=S_{1\text{ year}}$, $N:=N_{1\text{ year}}$ such that $S=\sum_{k=1}^{N}X_k$.

Remarks 2.12.

1. N is often modelled by a Binomial, Poisson (mostly used), negative binomial or logarithmic distribution.
   Z has a **negative binomial distribution** with parameters p, v ($p \in (0,1), v > 0$) if
   $$\mathbb{P}(Z = k) = \binom{v+k-1}{k} p^v (1-p)^k, \ k \in \mathbb{N}_0,$$
   where $\binom{r}{k} = \frac{r(r-1)\cdot\ldots\cdot(r-k+1)}{k!}$ for $r > 0$, $k \in \mathbb{N}_0$. For $v \in \mathbb{N}$, Z has the distribution of the waiting time for the v-th success in coin tosses with parameter p.
   Z has a **logarithmic distribution** with parameter $p \in (0,1)$ if
   $$\mathbb{P}(Z = k) = \frac{\frac{1}{k} p^k}{-\log(1-p)}, \ k \in \mathbb{N}.$$

2. The claim size distribution F is often modelled by a Gamma, log-Gamma, inverse Gaussian or log-Gaussian distribution.
   The **Gamma distribution** with parameters r, λ ($r, \lambda > 0$) has the density
   $$f_{(r,\lambda)}(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \mathbb{1}_{\{x \geq 0\}},$$
   with $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}\, dx$. If $r = n \in \mathbb{N}$, the Gamma distribution with parameters n, λ is also denoted as **Erlang distribution** and is the law of the sum of n i.i.d. rv's with distribution $Exp(\lambda)$.
   The **log-Gamma distribution** with parameters α, β ($\alpha, \beta > 0$) has the density
   $$f_{(\alpha,\beta)}(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\log x)^{\beta-1} x^{-(\alpha+1)} \mathbb{1}_{\{x \geq 1\}}.$$
   The **inverse Gaussian distribution** with parameters μ, λ ($\mu, \lambda > 0$) has the density
   $$f_{(\mu,\lambda)}(x) = \sqrt{\frac{\lambda}{2\pi x^3}} e^{-\frac{\lambda(x-\mu)^2}{2\mu x}} \mathbb{1}_{\{x > 0\}}.$$
   The **log-Gaussian distribution** or **log-Normal distribution** with parameters μ, σ ($\mu \in \mathbb{R}, \sigma > 0$) has the density
   $$f_{(\mu,\sigma)}(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}} \mathbb{1}_{\{x > 0\}}.$$
   If Y has a $N(\mu, \sigma^2)$ distribution, then e^Y has a log-Normal distribution with parameters μ, σ^2.

To derive the properties of S, we will use moment generating functions (mgf's) or Laplace transforms:

Definition 2.13. Let X be a real-valued rv with distribution functions F. Then, the function
$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx}\, F(dx)$$
is the **moment generating function** of X (or **Laplace transform** if $t \leq 0$). It is possible that the mgf is only finite for $t = 0$ or for $t \in (-s_1, s_2)$, where $s_1, s_2 > 0$.

Proposition 2.14.

(i) 0 is in the interval where M_X is finite and $M_X(0) = 1$.

(ii) Let $X \geq 0$ almost surely. Then $M_X(t) \leq 1,\ \forall t \leq 0$.

(iii) Assume M_X is finite on $(-s_1, s_2)$. Then M_X is infinitely often differentiable in $(-s_1, s_2)$ and
$$M_X^{(k)}(t) = \int_{-\infty}^{\infty} x^k e^{tx}\, F(dx) = \mathbb{E}\left[X^k e^{tX}\right].$$
Furthermore, all moments of X exist with
$$M_X^{(k)}(0) = \mathbb{E}\left[X^k\right] \qquad \textit{and} \qquad M_X(t) = \sum_{n=0}^{\infty} \frac{\mathbb{E}\left[X^n\right]}{n!} t^n,\ t \in (-s_1, s_2).$$

(iv) Convolution: Let $X_1, \ldots, X_n$ be independent rv's with mgf's $M_{X_1}, \ldots, M_{X_n}$ which are finite on $(-s_1, s_2)$ $(s_1, s_2 > 0)$. Then,
$$M_{X_1 + \ldots + X_n}(t) = \prod_{k=1}^{n} M_{X_k}(t) \textit{ for } t \in (-s_1, s_2).$$

(v) Uniqueness: Let $X \geq 0$, $Y \geq 0$ and $t_0 \leq 0$. Then
$$X \stackrel{d}{=} Y \Leftrightarrow M_X(t) = M_Y(t),\ \forall t \leq t_0.$$

Proof.
(i) $M_X(0) = \mathbb{E}\left[e^0\right] = 1$.
(ii) $X \geq 0$, $t \leq 0 \Rightarrow e^{tX} \leq 1$.
(iii) Proof for $k = 1$:
Let $t \in (-s_1, s_2)$, $\delta > 0$ s.t. $-s_1 < t - \delta < t + \delta < s_2$, h s.t. $|h| \leq \delta$. Then
$$\frac{1}{h}\left(M_X(t+h) - M_X(t)\right) = \int_{-\infty}^{\infty} \frac{e^{hx} - 1}{h} e^{tx}\, F(dx).$$
But
$$\left|\frac{e^{hx} - 1}{h}\right| = \left|\sum_{k=1}^{\infty} \frac{x^k h^{k-1}}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|x|^k \delta^{k-1}}{k!} = \frac{1}{\delta}\left(e^{|x|\delta} - 1\right) \leq \frac{1}{\delta} e^{|x|\delta}$$
$$< \frac{1}{\delta}\left(e^{x\delta} + e^{-x\delta}\right) \qquad \text{and} \qquad \int_{-\infty}^{\infty} \left(e^{(t+\delta)x} + e^{(t-\delta)x}\right) F(dx) < \infty.$$

Thus, dominated convergence yields

$$M_X'(t) = \int_{-\infty}^{\infty} \lim_{h\to 0} \frac{e^{hx}-1}{h} e^{tx}\, F(dx) = \int_{-\infty}^{\infty} x e^{tx}\, F(dx).$$
$$\Rightarrow M_X'(0) = \mathbb{E}[X].$$

The proof for $k \geq 2$ uses dominated convergence as well.
The last statement is the Taylor expansion of M_X around 0.
(iv) We have

$$M_{X_1+\ldots+X_n}(t) = \mathbb{E}\left[e^{(X_1+\ldots+X_n)t}\right] \underset{indep.}{=} \prod_{k=1}^{n} \mathbb{E}\left[e^{tX_k}\right] = \prod_{k=1}^{n} M_{X_k}(t).$$

(v) See [2, Theorem 22.2]. $\square$

Examples 2.15.

1. $N \stackrel{d}{=} Poi(\lambda)$. Then

$$M_N(t) = \sum_{k=0}^{\infty} e^{tk} \mathbb{P}(N=k) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{\lambda\left(e^t-1\right)},\ t \in \mathbb{R}.$$

2. $X \stackrel{d}{=} Geom(p)$, i.e. $\mathbb{P}(X=k) = p(1-p)^{k-1},\ k \in \mathbb{N}$ (also used: $\mathbb{P}(X=k) = p(1-p)^k,\ k \in \mathbb{N}_0$). Then

$$M_X(t) = \sum_{k=1}^{\infty} e^{tk} \underbrace{p(1-p)^{k-1}}_{\mathbb{P}(X=k)} = pe^t \sum_{k=1}^{\infty} \left(e^t(1-p)\right)^{k-1} = \frac{pe^t}{1-e^t(1-p)},$$

if $e^t(1-p) < 1$, i.e. $t < \log\left(\frac{1}{1-p}\right)$, $M_X(t) = \infty$ else.

Lemma 2.16. *Consider the annual claim amount S and assume that the following moments are finite. Then:*

(a) $M_S(t) = M_N(\log M_X(t))$.

(b) *Wald's identity:* $\mathbb{E}[S] = \mathbb{E}[N]\,\mathbb{E}[X]$,
$Var(S) = Var(N)\mathbb{E}[X]^2 + Var(X)\mathbb{E}[N]$.

(c) *The distribution function of the annual claim amount S is given by*

$$\mathbb{P}(S \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\, F^{*n}(x),\ x \in \mathbb{R}.$$

Proof.

(a) $M_S(t) = \mathbb{E}\left[e^{t\sum_{k=1}^{N} X_k}\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[e^{t\sum_{k=1}^{N} X_k} \mid N = n\right] \mathbb{P}\left(N = n\right)$

$$= \sum_{n=0}^{\infty} \underbrace{\mathbb{E}\left[e^{t\sum_{k=1}^{n} X_k}\right]}_{M_{\sum_{k=1}^{n} X_k}(t) \underset{Prop. 2.14}{=} \prod_{k=1}^{n} M_{X_k}(t) = M_X(t)^n} \mathbb{P}(N = n).$$

Hence,

$$M_S(t) = \sum_{n=0}^{\infty} \underbrace{M_X(t)^n}_{e^{n \log M_X(t)}} \mathbb{P}(N = n) = \mathbb{E}\left[e^{N \log M_X(t)}\right]$$
$$= M_N\left(\log M_X(t)\right).$$

(b) Proposition 2.14(iii) and (a) imply that

$$M_S'(t) = M_N'\left(\log M_X(t)\right) \frac{M_X'(t)}{M_X(t)} \qquad \text{and}$$
$$M_S''(t) = M_N''\left(\log M_X(t)\right) \left(\frac{M_X'(t)}{M_X(t)}\right)^2 + M_N'\left(\log M_X(t)\right) \left(\frac{M_X''(t)}{M_X(t)} - \left(\frac{M_X'(t)}{M_X(t)}\right)^2\right).$$

$$\Rightarrow \mathbb{E}[S] = M_S'(0) = M_N'(0) M_X'(0) = \mathbb{E}[N]\mathbb{E}[X] \qquad \text{and}$$

$$\mathbb{E}\left[S^2\right] = M_N''(0) M_X'(0)^2 + M_N'(0)\Big(\underbrace{M_X''(0)}_{\mathbb{E}[X^2]} - \underbrace{M_X'(0)^2}_{\mathbb{E}[X]^2}\Big) = \mathbb{E}\left[N^2\right]\mathbb{E}\left[X\right]^2 + \mathbb{E}[N]\mathrm{Var}(X).$$

$$\Rightarrow \mathrm{Var}(S) = \mathbb{E}\left[S^2\right] - \mathbb{E}\left[S\right]^2 = \ldots = \mathbb{E}\left[X\right]^2 \mathrm{Var}(N) + \mathbb{E}[N]\mathrm{Var}(X).$$

(c) $\mathbb{P}(S \le x) = \sum_{n=0}^{\infty} \mathbb{P}(S \le x \mid N = n)\mathbb{P}(N = n) = \sum_{n=0}^{\infty} \underbrace{\mathbb{P}\left(\sum_{k=0}^{n} X_k \le x\right)}_{F^{*n}(x)} \mathbb{P}(N = n).$

□

Theorem 2.17. *Suppose* $W_1 \overset{d}{=} Exp(\lambda)$. *Then* $N_t \overset{d}{=} Poi(\lambda t)$. *Consider* $S = S_t = \sum_{k=1}^{N_t} X_k$. *Assume as always that* $X, X_1, X_2, \ldots$ *are i.i.d.*

(a) Let $M_X(v) < \infty$ *for some* $v \in \mathbb{R}$. *Then* $M_S(v) = e^{\lambda t (M_X(v) - 1)}$.

(b) Let $\mathbb{E}[|X|] < \infty$. *Then* $\mathbb{E}[S] = \lambda t \mathbb{E}[X]$.
Let $\mathbb{E}\left[X^2\right] < \infty$. *Then* $Var(S) = \lambda t \mathbb{E}\left[X^2\right]$.

(c) $\mathbb{P}(S \le x) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} F^{*n}(x)$, $x \in \mathbb{R}$.

Proof. See exercises. □

Remarks 2.18. Let $(N_t)_{t\geq 0}$ be a renewal process with $\mathbb{E}\left[W_1^2\right] < \infty$. Then, see Remarks 2.7 and exercises,

$$\begin{aligned}\mathbb{E}\left[S_t\right] &\sim \lambda t \mathbb{E}\left[X\right] \text{ for } t \to \infty,\\ \mathrm{Var}(S_t) &\sim t\left(\lambda \mathrm{Var}(X) + \lambda^3 \mathrm{Var}(W_1)\mathbb{E}[X]^2\right) \text{ for } t \to \infty\end{aligned}$$

$\left(\text{where we write } a_t \sim b_t \text{ for } t \to \infty \text{ if } \frac{a_t}{b_t} \xrightarrow[t\to\infty]{} 1\right)$.

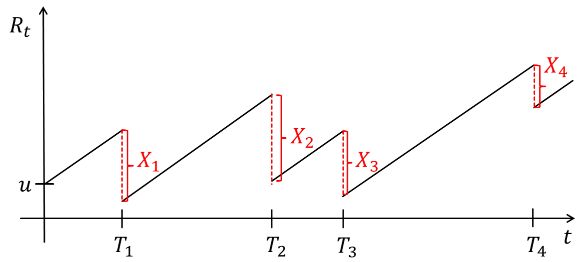
2.4 The classical risk process

All rv's of this chapter are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.19 (Cramér-Lundberg model)**.** Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. claims with distribution function F, $F(0) = 0$, $\mathbb{E}\left[X_1\right] = \mu < \infty$, and $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda \in (0,\infty)$, which is independent of $(X_k)_{k\in\mathbb{N}}$.

(a) The claim amount process $S_t = \sum_{k=1}^{N_t} X_k$, $t \geq 0$, is the **claim amount process in the Cramér-Lundberg model**.

(b) Let $u > 0$, $c > 0$. The risk process $R_t = u + ct - \sum_{k=1}^{N_t} X_k$, $t \geq 0$, is the **classical risk process** and $Y_t = R_t - u = ct - \sum_{k=1}^{N_t} X_k$, $t \geq 0$, is the **surplus process**.

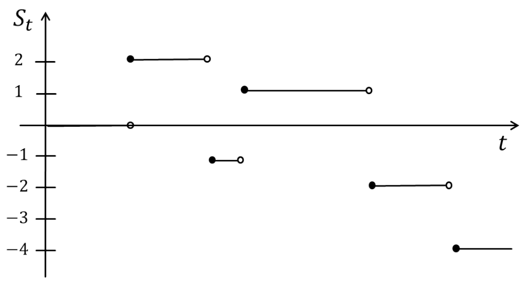


Since $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, we know that interarrival times $(T_k)_{k\in\mathbb{N}}$ of the claims have the representation $T_n = \sum_{k=1}^{n} W_k$, where $(W_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. $Exp(\lambda)$-distributed rv's.

Definition 2.20. Let $(N_t)_{t\geq 0}$ be a Poisson process and $(X_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. rv's independent of $(N_t)_{t\geq 0}$ (the X_k are not necessarily non-negative). Then $S_t = \sum_{k=1}^{N_t} X_k$, $t \geq 0$, is a **compound Poisson process**.

An example for a compound Poisson process with

$$X_1 = \begin{cases} 2, \\ -2, \\ -3. \end{cases}$$



Remarks 2.21.

1. A compound Poisson process is a **Lévy process**, i.e. a process with independent and stationary increments and càdlàg sample paths.

2. The claim amount process of the Cramér-Lundberg model is a compound Poisson process.

Theorem 2.22. *Consider the Cramér-Lundberg model with claim amount process $(S_t)_{t\geq 0}$. Suppose that $\mathbb{E}[X^2] < \infty$. Then*

$$\frac{S_t - \mathbb{E}[S_t]}{\sqrt{Var(S_t)}} = \frac{S_t - \lambda t \mathbb{E}[X]}{\sqrt{\lambda t \mathbb{E}[X^2]}} \xrightarrow{d} Z, \text{ where } Z \stackrel{d}{=} N(0,1).$$

Proof. For the case $M_X(v) < \infty$ for some $v > 0$:
By Theorem 2.17, $\mathbb{E}[S_t] = \lambda t \mathbb{E}[X]$, $\mathrm{Var}(S_t) = \lambda t \mathbb{E}[X^2]$ and

$$M_{S_t}(v) = e^{\lambda t (M_X(v)-1)}. \tag{1}$$

Let $Z_t = \frac{S_t - \mathbb{E}[S_t]}{\sqrt{\mathrm{Var}(S_t)}}$. Then

$$M_{Z_t}(v) = \mathbb{E}\left[e^{vZ_t}\right] = e^{-v\frac{\mathbb{E}[S_t]}{\sqrt{\mathrm{Var}(S_t)}}} \underbrace{\mathbb{E}\left[e^{\frac{v}{\sqrt{\mathrm{Var}(S_t)}} S_t}\right]}_{M_{S_t}\left(\frac{v}{\sqrt{\mathrm{Var}(S_t)}}\right) \stackrel{(1)}{=} \exp\left(\lambda t \left(M_X\left(\frac{v}{\sqrt{\mathrm{Var}(S_t)}}\right)-1\right)\right)}.$$

Hence,

$$\mathbb{E}\left[e^{vZ_t}\right] = e^{-v\frac{\mathbb{E}[S_t]}{\sqrt{\mathrm{Var}(S_t)}}} \exp\left(\lambda t \left(M_X\left(\frac{v}{\sqrt{\mathrm{Var}(S_t)}}\right) - 1\right)\right). \tag{2}$$

Let s be close enough to 0. Then Proposition 2.14 gives

$$M_X(s) = \sum_{n=0}^{\infty} \frac{\mathbb{E}[X^n]}{n!} s^n.$$

$$\Rightarrow M_X(s) = 1 + \mathbb{E}[X]s + \mathbb{E}\left[X^2\right]\frac{1}{2}s^2 + o(s^2) \text{ for } s \to 0.$$

$$\begin{aligned}
\Rightarrow M_X\left(\frac{v}{\sqrt{\mathrm{Var}(S_t)}}\right) - 1 &= M_X\left(\frac{v}{\sqrt{\lambda t \mathbb{E}[X^2]}}\right) - 1 \\
= 1 + \mathbb{E}[X]\frac{v}{\sqrt{\lambda t \mathbb{E}[X^2]}} &+ \frac{\mathbb{E}[X^2]}{2}\frac{v^2}{\lambda t \mathbb{E}[X^2]} + o\left(\frac{1}{t}\right) - 1 \text{ for } t \to \infty \\
&= \frac{\mathbb{E}[X]}{\sqrt{\mathbb{E}[X^2]}}\frac{v}{\sqrt{\lambda t}} + \frac{1}{2}\frac{v^2}{\lambda t} + o\left(\frac{1}{t}\right).
\end{aligned} \tag{3}$$

Further,

$$\frac{v\mathbb{E}[S_t]}{\sqrt{\mathrm{Var}(S_t)}} = \frac{v\lambda t \mathbb{E}[X]}{\sqrt{\lambda t \mathbb{E}[X^2]}} = \frac{v\mathbb{E}[X]\sqrt{\lambda t}}{\sqrt{\mathbb{E}[X^2]}}. \tag{4}$$

(2)-(4) imply that $M_{Z_t}(v) \underset{t\to\infty}{\longrightarrow} e^{\frac{v^2}{2}} = M_Z(v)$.
This suffices since we can use:

Theorem 2.23. *„Convergence of mgf's implies convergence of the corresponding distributions", cf. [7, Theorem 9.5].*
Let $Z_1, Z_2, \dots$ be rv's such that $|\, M_{Z_i}(s)\, | < \infty$, $\forall |s| < h$, $\forall i \in \mathbb{N}$, for some $h > 0$, and Z a rv with $|\, M_Z(s)\, | < \infty$ for $|s| < h$. Assume that $M_{Z_n}(s) \underset{s\to\infty}{\longrightarrow} M_Z(s)$ for $|s| < h$. Then $Z_n \xrightarrow{d} Z$.

□

We saw that $\frac{S_t - \mathbb{E}[S_t]}{\sqrt{\mathrm{Var}(S_t)}} \xrightarrow{d} Z$ for $t \to \infty$, where $Z \overset{d}{=} N(0,1)$.

Remark 2.24. Hence, we can approximate the claim amount distribution by

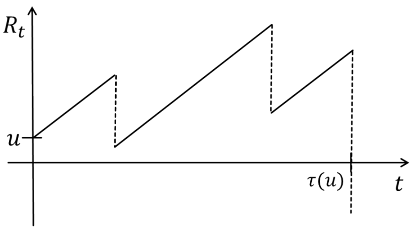
$$\mathbb{P}(S_t \le x) = \mathbb{P}\left(\frac{S_t - \lambda t \mathbb{E}[X]}{\sqrt{\lambda t \mathbb{E}[X^2]}} \le \frac{x - \lambda t \mathbb{E}[X]}{\sqrt{\lambda t \mathbb{E}[X^2]}}\right) \approx \Phi\left(\frac{x - \lambda t \mathbb{E}[X]}{\sqrt{\lambda t \mathbb{E}[X^2]}}\right).$$

Problems:

- t is often not large enough.
- Φ is a distribution function on $\mathbb{R}$ but S_t takes only positive values.
- The distribution of S_t is usually skewed (not symmetric).

Definition 2.25. Let $(R_t)_{t\ge 0}$ be a risk process with initial reserve $u := R_0 > 0$. Then

- $\tau(u) := \inf\{t > 0 : R_t < 0\}$ is the **ruin time**.
- $\psi(u) := \mathbb{P}(\tau(u) < \infty)$ is the **ruin probability**.
- $\theta(u) := 1 - \psi(u)$ is the **survival probability**.



Remarks 2.26.

(a) $\tau(u) = \inf\{t > 0 : u + Y_t < 0\} = \inf\{t > 0 : Y_t < -u\}$ is an $\mathcal{F}_t$-stopping time, where $\mathcal{F}_t = \sigma(Y_s, s \le t)$.

(b) $$\begin{aligned}\psi(u) &= \mathbb{P}(\tau(u) < \infty) = \mathbb{P}(\inf\{t > 0 : Y_t < -u\} < \infty) = \mathbb{P}\left(\inf_{t\ge 0} Y_t < -u\right) \\ &= \mathbb{P}\left(\inf_{t\ge 0}\left(ct - \sum_{k=1}^{N_t} X_k\right) < -u\right) = \mathbb{P}\left(\inf_{n\in\mathbb{N}}\left(cT_n - \sum_{k=1}^{n} X_k\right) < -u\right) \\ &= \mathbb{P}\left(\inf_{n\in\mathbb{N}} -\widetilde{Y}_n < -u\right) \qquad \text{with} \qquad \widetilde{Y}_n = \sum_{k=1}^{n} X_k - cT_n \\ &= \mathbb{P}\left(\sup_{n\in\mathbb{N}} \widetilde{Y}_n > u\right),\end{aligned}$$

and $\theta(u) = 1 - \psi(u) = \mathbb{P}\left(\sup_{n\in\mathbb{N}} \widetilde{Y}_n \le u\right)$.

But $\widetilde{Y}_n = \sum_{k=1}^n X_k - cT_n = \sum_{k=1}^n (X_k - cW_k)$, where $(X_k - cW_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence. $\Rightarrow \widetilde{Y}_n$ is a **random walk** (= sum of i.i.d. rv's).
The strong law of large numbers implies that

$$\frac{1}{n}\widetilde{Y}_n = \frac{1}{n}\sum_{k=1}^{n}(X_k - cW_k) \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[X_1 - cW_1\right] \text{ a.s.} \tag{$\star$}$$

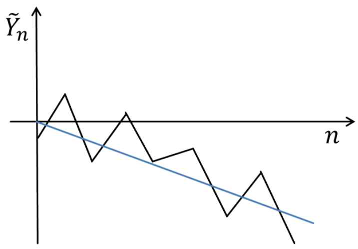
$$\mathbb{E}\left[X_1 - cW_1\right] = \mathbb{E}\left[X_1\right] - c\mathbb{E}\left[W_1\right] = \mu - \frac{c}{\lambda} = \frac{c}{\lambda}\Big(\underbrace{\frac{\lambda\mu}{c}}_{=:\rho} - 1\Big).$$

Definition 2.27. Consider a classical risk process with premium rate $c > 0$, Poisson intensity $\lambda \in (0,\infty)$ and $\mathbb{E}\left[X_1\right] = \mu < \infty$. Then $r := \frac{c}{\lambda\mu} - 1$ is called **safety loading** and $\rho = \frac{\lambda\mu}{c}$.

Proposition 2.28. *Consider a classical risk process $(R_t)_{t\ge 0}$. Then $u \to \theta(u)$ is increasing and*

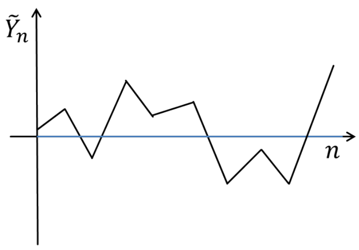
(a) Let $\rho < 1$. Then

- $\lim\limits_{n\to\infty} \widetilde{Y}_n = -\infty$ *a.s.*
- $\mathbb{P}\left(\sup\limits_{n\in\mathbb{N}} \widetilde{Y}_n \le 0\right) > 0.$
- $\lim\limits_{u\to\infty} \theta(u) = \lim\limits_{u\to\infty} \mathbb{P}\left(\sup\limits_{n\in\mathbb{N}} \widetilde{Y}_n \le u\right) = 1.$



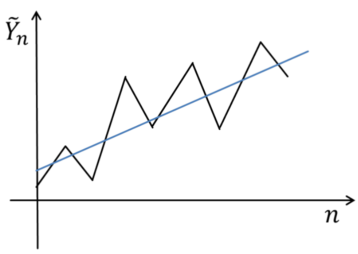
(b) Let $\rho = 1$. Then

- $\limsup\limits_{n\to\infty} \widetilde{Y}_n = \infty$ *a.s.*
- $\liminf\limits_{n\to\infty} \widetilde{Y}_n = -\infty$ *a.s.*
- $\theta(u) = 0$ *for* $u \ge 0$.



(c) Let $\rho > 1$. Then

- $\lim\limits_{n\to\infty} \widetilde{Y}_n = \infty$ *a.s.*
- $\mathbb{P}\left(\sup\limits_{n\in\mathbb{N}} \widetilde{Y}_n \ge 0\right) = 1.$
- $\theta(u) = 0$ *for* $u \ge 0$.



Proof.
(a) follows from ($\star$).
(b) **Lemma:** Let $M_n = \sum_{k=1}^{n} Z_k$, where $Z_1, Z_2, \dots$ i.i.d. with $\mathbb{E}[Z_i] = 0$ and $\mathrm{Var}(Z_1) > 0$. Then

$$\limsup_{n\to\infty} M_n = \infty \text{ a.s.}, \qquad \liminf_{n\to\infty} M_n = -\infty \text{ a.s.}$$

Proof. See [1]. □

(c) follows from ($\star$) as (a). □

Definition 2.29 (Net profit condition)**.** The classical risk process $(R_t)_{t\geq 0}$ with premium rate $c > 0$, Poisson intensity $\lambda \in (0,\infty)$ and $\mathbb{E}[X_1] = \mu$ satisfies the **net profit condition** (NPC) if $\rho = \frac{\lambda\mu}{c} < 1$, which is equivalent to $r = \frac{c}{\lambda\mu} - 1 > 0$. The net profit condition is necessary for $\theta(u) > 0$ for some u.

3 Renewal Theory and the ruin probability

Definition 3.1 (Renewal equation)**.** Let $v : \mathbb{R}_+ \to \mathbb{R}$ be a function and ν a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ with $\nu(\mathbb{R}_+) \leq 1$. Then

$$V(t) = v(t) + \int_0^t V(t-s)\,\nu(ds),\ t \geq 0,$$

is called **renewal equation**. If $\nu(\mathbb{R}_+) < 1$, the renewal equation is **defect**, if $\nu(\mathbb{R}_+) = 1$, it is called **proper**.

Theorem 3.2 (Smith's key renewal lemma)**.** *Let $(N_t)_{t\geq 0}$ be a renewal process with interarrival times $(W_k)_{k\in\mathbb{N}}$, where $\frac{1}{\lambda} := \mathbb{E}[W_1]$. Suppose the distribution function G of W_1 has a density and $U(t) := m(t) + 1 = \mathbb{E}[N_t] + 1$. Let v be bounded on any finite interval. Then*

$$V(t) = \int_0^t v(t-s)\,U(ds),\ t \geq 0,$$

is the unique solution of the renewal equation

$$V(t) = v(t) + \int_0^t V(t-s)\,G(ds),\ t \geq 0.$$

If v is the difference of two non-increasing, non-negative Riemann-integrable functions, then

$$\lim_{t\to\infty} V(t) = \lambda \int_0^\infty v(t)\,dt.$$

Example 3.3. The renewal function m ($m(t) = \mathbb{E}[N_t]$) satisfies the renewal equation

$$m(t) = G(t) + \int_0^t m(t-s)\,G(ds),\ t \geq 0.$$

Proof.

1. Theorem 3.2, with $v(t) \equiv 1$. Then $V(t) = U(t) = m(t) + 1$. Then
$$V(t) = m(t) + 1 = 1 + \int_0^t (m(t-s)+1)\, G(ds) = 1 + \int_0^t m(t-s)\, G(ds) + G(t).$$
Hence,
$$m(t) = G(t) + \int_0^t m(t-s)\, G(ds).$$

2. $$\begin{aligned} m(t) &= \sum_{n=1}^{\infty} G^{*n}(t) = \sum_{n=1}^{\infty} \left(G^{*(n-1)} * G\right)(t) \\ &= \sum_{n=1}^{\infty} \int_0^t G^{*(n-1)}(t-s)\, G(ds) = G(t) + \sum_{n=1}^{\infty} \int_0^t G^{*n}(t-s)\, G(ds) \\ &= G(t) + \int_0^t m(t-s)\, G(ds). \end{aligned}$$

$\square$

Example: $G(t) = 1 - e^{-\lambda t}$, $m(t) = \lambda t$.

Theorem 3.4 (Blackwell's renewal theorem)**.** *Under the assumptions of Theorem 3.2,*
$$\lim_{t\to\infty} (U(t) - U(t-a)) = \lambda a.$$

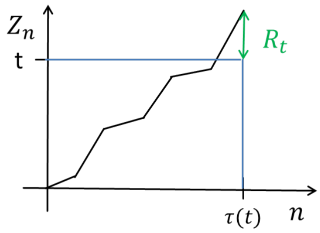
Example: If $W_1, W_2, \ldots$ are i.i.d. with law $Exp(\lambda)$, then $N_t \overset{d}{=} Poi(\lambda t)$ and $U(t) = m(t) + 1 = \lambda t + 1 \Rightarrow U(t) - U(t-a) = \lambda a$.

Proof. Theorem 3.4 follows from Theorem 3.2 by taking $v(t) = \mathbb{1}_{\{t \le a\}}$. Then
$$\begin{aligned} V(t) &= \int_0^t v(t-s) U(ds) = \int_0^t \mathbb{1}_{\{t-s\le a\}}\, U(ds) = \int_0^t \mathbb{1}_{\{s \ge t-a\}}\, U(ds) \\ &= U(t) - U(t-a) \end{aligned}$$
and
$$\lim_{t\to\infty} V(t) = \lambda \int_0^\infty v(t)\, dt = \lambda a.$$

$\square$

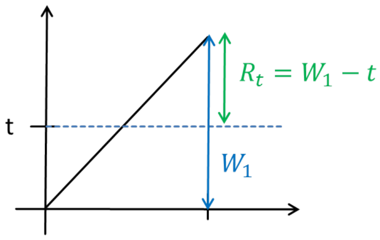
The proof of Theorem 3.2 relies on first proving Theorem 3.4. See [1].

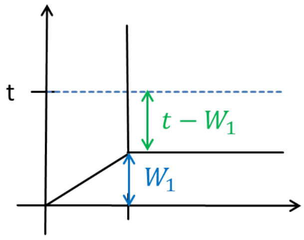
Example 3.5 (Distribution of the overshoot)**.**



Let $W_1, W_2, \dots$ be i.i.d. with distribution function G, assume $G(0) = 0$, and G has a density and $\mathbb{E}[W_1] = \frac{1}{\lambda} < \infty$. Let $Z_n = \sum_{k=1}^{n} W_k$ and let $\tau(t) = \inf\{n \geq 0 : Z_n \geq t\}$. $R_t := Z_{\tau(t)} - t$, $t \geq 0$, is the **overshoot** at time t.

Then $R_t = \mathbb{1}_{\{W_1 > t\}}(W_1 - t) + \mathbb{1}_{\{W_1 \leq t\}} R_{t-W_1}$.





Fix $u \in (0, \infty)$. Then, with G as the distribution function of W_1,

$$\underbrace{\mathbb{P}(R_t > u)}_{V(t)} = \underbrace{\mathbb{P}(W_1 > t + u)}_{v(t)} + \int_0^t \underbrace{\mathbb{P}(R_{t-s} > u)}_{V(t-s)} \, G(ds).$$

The function $t \mapsto \mathbb{P}(W_1 > t + u)\mathbb{1}_{[0,\infty)}(t)$ **is** Riemann-integrable. Hence

$$\lim_{t \to \infty} \mathbb{P}(R_t > u) = \lambda \int_0^\infty \mathbb{P}(W_1 > t + u) \, dt = 1 - F(u),$$

where $F(u) = 1 - \lambda \int_0^\infty \mathbb{P}(W_1 > t + u) \, dt$ **is** a distribution function and $u \mapsto F(u)$ continuous (exercise). In particular, R_t converges in law for $t \to \infty$!

For a distribution function F on $[0, \infty)$ with $\mu = \int_0^\infty x \, F(dx) < \infty$, let $\overline{F}(y) := 1 - F(y)$ and $F_I(x) = \frac{1}{\mu} \int_0^x \overline{F}(y) \, dy$. F_I is the **integrated tail distribution function**.
Exercise: F_I **is** a distribution function.

Theorem 3.6. *Let $(R_t)_{t \geq 0}$ be a classical risk process which satisfies the NPC $\left(\rho = \frac{\lambda\mu}{c} < 1\right)$. Then the survival probability θ and the ruin probability ψ have the following representation:*

(a) Renewal equation: $\theta(u) = \theta(0) + \rho \underbrace{(F_I * \theta)(u)}_{\int_0^\infty \theta(u-x) \, F_I(dx)}$

$\theta(u) = \theta(0) + \frac{\lambda}{c} \int_0^u \theta(u - y)\overline{F}(y) \, dy, \;\; u \geq 0.$

(b) Pollaczek-Khintchine formula: $\theta(u) = (1 - \rho) \sum_{n=0}^{\infty} \rho^n F_I^{*n}(u), \;\; u \geq 0,$ *where*

$$F^{*0}(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

$$\psi(u) = (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{F_I^{*n}}(u), \ \ u \geq 0.$$

Proof. In the classical risk model, the claim interarrival times $W_1, W_2, \dots$ are i.i.d. with law $Exp(\lambda)$. Define $\widetilde{Y}_n := -R_{T_n} = \sum_{k=1}^{n}(X_k - cW_k)$, $n \in \mathbb{N}$. Let $u \geq 0$. We have

$$\begin{aligned}
\theta(u) &\overset{\text{Remark 2.26 (b)}}{=} \mathbb{P}\left(\sup_{n\in\mathbb{N}} \widetilde{Y}_n \leq u\right) = \mathbb{P}\left(\widetilde{Y}_1 \leq u, \ \sup_{n\geq 2}\widetilde{Y}_n \leq u\right) \\
&= \mathbb{P}\left(\widetilde{Y}_1 \leq u, \ \sup_{n\geq 2}\left(\widetilde{Y}_n - \widetilde{Y}_1\right) \leq u - \widetilde{Y}_1\right) \\
&\underset{\widetilde{Y}_n=\sum_{k=1}^n X_k - cW_k}{=} \mathbb{P}\left(X_1 \leq u + cW_1, \ \sup_{n\geq 2}\sum_{k=2}^{n}(X_k - cW_k) \leq u - (X_1 - cW_1)\right) \\
&\overset{\text{Integrating over the values of } X_1, W_1}{=} \int_{w=0}^{\infty}\int_{x=0}^{u+cw} \mathbb{P}\left(\sup_{n\geq 2}\sum_{k=2}^{n}(X_k - cW_k) \leq u - (x - cw)\right) F(dx)\, \underbrace{F_{W_1}(dw)}_{\lambda e^{-\lambda w} dw} \\
&= \int_{w=0}^{\infty}\int_{x=0}^{u+cw} \theta(u - (x - cw))\, F(dx) \lambda e^{-\lambda w} dw \\
&= \int_{s=u}^{\infty}\int_{x=0}^{s} \theta(s - x)\, F(dx) \frac{1}{c}\lambda e^{-\lambda \frac{s}{c}} e^{\lambda \frac{u}{c}} ds \\
&\left(\text{Substitute: } s = u + cw \Rightarrow w = \frac{s-u}{c} \Rightarrow ds = c\, dw\right) \\
&= e^{\frac{\lambda u}{c}} \frac{\lambda}{c}\int_{s=u}^{\infty} e^{-\lambda \frac{s}{c}} \int_{x=0}^{s} \theta(s-x)\, F(dx)\, ds.
\end{aligned}$$

Hence, the measure corresponding to $\theta(\cdot)$ ($u \mapsto \theta(u)$ is increasing and continuous) is absolutely continuous w.r.t. the Lebesgue measure on $(0, \infty)$ with density

$$\begin{aligned}
f_\theta(u)(=\theta'(u)) &= \frac{\lambda}{c}\theta(u) - \frac{\lambda}{c}\int_0^u \theta(u-x)\, F(dx) = \frac{\lambda}{c}\theta(u) - \frac{\lambda}{c}(F * \theta)(u) \\
&= \frac{\lambda}{c}\left(\overline{F} * \theta\right)(u).
\end{aligned}$$

Remark 3.7. Let F, G be distribution functions on $(0, \infty]$. Then $F * G = G * F$ is a distribution function. For h measurable and bounded we define

$$(h * G)(u) = \int_0^u h(u-s)\, G(ds).$$

Then $G - G * F = \overline{F} * G$.

Proof. See exercises. □

We derived: $f_\theta(u) = \frac{\lambda}{c}\left(\overline{F} * \theta\right)(u)$. Integration on both sides yields

$$\begin{aligned}
\theta(u) - \theta(0) &= \int_0^u f_\theta(t)\,dt = \frac{\lambda}{c}\int_0^u \left(\overline{F} * \theta\right)(t)\,dt = \frac{\lambda}{c}\int_0^u \int_0^t \overline{F}(t-y)\,\theta(dy)\,dt \\
&\overset{Fubini}{=} \frac{\lambda}{c}\int_0^u \int_y^u \overline{F}(t-y)\,dt\,\theta(dy) \underset{ds=dt}{\overset{s=t-y}{=}} \frac{\lambda}{c}\int_0^u \int_0^{u-y} \overline{F}(s)\,ds\,\theta(dy) \\
&= \frac{\lambda\mu}{c}\int_0^u F_I(u-y)\,\theta(dy) \\
&= \rho\left(F_I * \theta\right)(u).
\end{aligned}$$

Hence

$$\theta(u) = \theta(0) + \rho\left(F_I * \theta\right)(u), \;\; u \geq 0 \tag{3.1}$$

and (a) is proved.

(b) θ solves the renewal equation (3.1). On the one hand, we have by Proposition 2.28 (a), $\theta(\infty) = 1$. On the other hand, by (3.1) and monotone convergence

$$\begin{aligned}
1 = \theta(\infty) = \theta(0) + \rho\left(F_I * \theta\right)(\infty) &= \theta(0) + \rho. \\
\Rightarrow \theta(0) &= 1 - \rho. \qquad (3.2)
\end{aligned}$$

Due to (3.1), we have

$$\begin{aligned}
1 - \theta(u) &= 1 - \theta(0) - \rho\left(F_I * \theta\right)(u), \;\; u \geq 0 \\
&\overset{(3.2)}{=} \rho - \rho\left(F_I * \theta\right)(u), \;\; u \geq 0 \\
&= \rho\left(1 - \left(F_I * \theta\right)(u)\right), \;\; u \geq 0.
\end{aligned}$$

Let $s \leq 0$. Then

$$s\int_0^\infty e^{su}(1-\theta(u))du = \rho s \int_0^\infty e^{su}\left(1 - \left(F_I * \theta\right)(u)\right)du. \tag{3.3}$$

Let M_θ and M_{F_I} be the mgfs of θ and F_I, i.e.

$$M_\theta(s) = \int_0^\infty e^{su}\,\theta(du), \qquad M_{F_I}(s) = \int_0^\infty e^{su}\,F_I(du).$$

Now, integration by parts gives

$$\begin{aligned}
s\int_0^\infty e^{su}(1-\theta(u))\,du &= e^{su}(1-\theta(u))\Big|_{u=0}^{\infty} + \int_0^\infty e^{su}\,\underbrace{f_\theta(u)\,du}_{\theta(du)} \\
&= -1 + \theta(0) + \int_0^\infty e^{su} f_\theta(u)\,du \\
&= -1 + M_\theta(s), \qquad (3.4)
\end{aligned}$$

because
$$M_\theta(s) = \int_0^\infty e^{su}\,\theta(du) = \theta(0) + \int_0^\infty e^{su} f_\theta(u)\,du.$$
Similarly,
$$\begin{aligned}
\rho s \int_0^\infty e^{su}\left(1-(F_I*\theta)(u)\right)du &= \rho e^{su}\left(1-(F_I*\theta)(u)\right)\Big|_0^\infty + \rho\int_0^\infty e^{su}\,(F_I*\theta)(du) \\
&= -\rho\left(1-(F_I*\theta)(0)\right) + \rho\, \underbrace{M_{F_I*\theta}(s)}_{M_{F_I}(s)M_\theta(s)} \\
&= -\rho + \rho M_{F_I}(s) M_\theta(s). \qquad (3.5)
\end{aligned}$$
(3.3)-(3.5) imply
$$\begin{aligned}
-1 + M_\theta(s) &= -\rho + \rho M_{F_I}(s) M_\theta(s). \\
\Rightarrow M_\theta(s) &= \frac{1-\rho}{1-\rho M_{F_I}(s)}.
\end{aligned}$$
Since $s \le 0$, we have $M_{F_I}(s) \le 1$ and $0 \le \rho M_{F_I}(s) < 1$.
$$\begin{aligned}
\Rightarrow M_\theta(s) &= (1-\rho)\sum_{n=0}^\infty \left(\rho M_{F_I}(s)\right)^n = \sum_{n=0}^\infty \underbrace{(1-\rho)\rho^n}_{\mathbb{P}(N=n)} e^{n\log M_{F_I}(s)} \\
&= M_N\left(\log M_{F_I}(s)\right),
\end{aligned}$$
where $N \overset{d}{=} \mathrm{Geom}(\rho)$, independent of everything. Hence
$$M_\theta(s) = M_{(1-\rho)\sum_{n=0}^\infty \rho^n F_I^{*n}}(s),$$
for $s \le 0$, see Lemma 2.16. From Proposition 2.14 (v)
$$\begin{aligned}
\Rightarrow \theta(u) &= (1-\rho)\sum_{n=0}^\infty \rho^n F_I^{*n}(u),\ u \ge 0. \\
\Rightarrow \psi(u) &= 1-\theta(u) = \frac{1-\rho}{1-\rho} - (1-\rho)\sum_{n=0}^\infty \rho^n F_I^{*n}(u) \\
&= (1-\rho)\sum_{n=0}^\infty \rho^n - (1-\rho)\sum_{n=0}^\infty \rho^n F_I^{*n}(u) = (1-\rho)\sum_{n=1}^\infty \rho^n\left(1-F_I^{*n}(u)\right) \\
&= (1-\rho)\sum_{n=1}^\infty \rho^n \overline{F_I^{*n}}(u).
\end{aligned}$$

□

Corollary 3.8. *Let $(R_t)_{t\ge 0}$ be a classical risk process which satisfies the NPC, i.e. $\rho < 1$. Then* *(a)* $\theta(\infty) = 1$.

(b) $\theta(0) = 1 - \rho$.

(c) $\psi(0) = \rho$.

Hence for the initial capital $u = 0$*, the ruin probability depends only on* $\mu = \mathbb{E}[X_1]$ *and not on the whole distribution* F.

4 Asymptotic behaviour of the ruin probability

4.1 Small claim case

4.1.1 The Lundberg coefficient

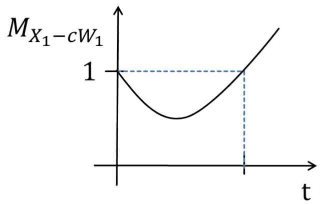
Definition 4.1. Consider the Cramér-Lundberg model. Suppose that for some $h_0 > 0$, M_X is finite in the neighbourhood $(-h_0, h_0)$ of the origin. If there exists a constant $K > 0$, such that

$$M_{X_1 - cW_1}(K) = \mathbb{E}\left[e^{K(X_1 - cW_1)}\right] = 1,$$

then K is called **Lundberg coefficient**.

Remarks 4.2.

(i) Suppose that the NPC is satisfied and the Lundberg coefficient exists. Then, the Lundberg coefficient K is unique. Reason: $t \mapsto M_{X_1 - cW_1}(t)$ is convex and continuous (convexity follows from the convexity of the function $t \mapsto e^t$).



$$M_{X_1 - cW_1}(0) = 1$$
$$M'_{X_1 - cW_1}(0) = \mathbb{E}[X_1 - cW_1] = \mu - \frac{c}{\lambda} \underset{\frac{\lambda\mu}{c}<1}{<} 0$$

(ii) The Lundberg coefficient is only explicitly known in few examples. However, there are methods to approximate it.

Lemma 4.3. *Consider the Cramér-Lundberg model. Then, with* $M_F = M_{X_1}$,

$$K \text{ Lundberg coefficient } \Leftrightarrow M_F(K) = \frac{\lambda + cK}{\lambda} \Leftrightarrow M_{F_I}(K) = \frac{1}{\rho}.$$

Proof. Since

$$\begin{aligned} M_{X_1 - cW_1}(v) &= \mathbb{E}\left[e^{v(X_1 - cW_1)}\right] \underset{\text{of } X_1 \text{ and } W_1}{\overset{\text{independence}}{=}} M_{X_1}(v)\mathbb{E}\left[e^{-vcW_1}\right] \\ &= M_{X_1}(v) \int_0^\infty \lambda e^{-vcs} e^{-\lambda s}\, ds \\ &= M_{X_1}(v) \frac{\lambda}{\lambda + vc}. \end{aligned}$$

Hence

$$M_{X_1 - cW_1}(K) = 1 \;\Leftrightarrow\; M_{X_1}(K) = \frac{\lambda + cK}{\lambda}. \tag{4.1}$$

On the other hand,

$$M_{F_I}(v) = \frac{M_F(v) - 1}{\mu v} \tag{4.2}$$

Proof of (4.2): See exercises.
Hence

$$\begin{aligned} M_{F_I}(K) &= \frac{\frac{\lambda + Kc}{\lambda} - 1}{\mu K} = \frac{c}{\mu\lambda} = \frac{1}{\rho} \\ \Leftrightarrow\; M_F(K) &= 1 + \frac{cK}{\lambda} = \frac{\lambda + cK}{\lambda}. \end{aligned}$$

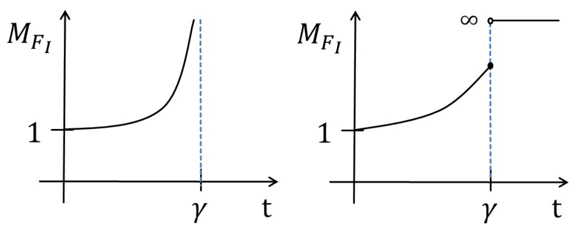
□

Lemma 4.4. *Assume that* $\lim_{x\to\infty} \frac{-\log \overline{F}(x)}{x} = \gamma > 0$. *Then*

(a) $M_{F_I}(t) < \infty,\ \forall t < \gamma$ *and* $M_{F_I}(t) = \infty$ *if* $t > \gamma$.

(b) If $M_{F_I}(\gamma) = \infty$, *then the Lundberg coefficient exists.*

(c) Let F *be the distribution function of* $Exp\left(\frac{1}{\mu}\right)$. *Then* F_I *is the distribution function of* $Exp\left(\frac{1}{\mu}\right)$ *and the Lundberg coefficient is* $K = \frac{1-\rho}{\mu}$.



Proof.
(a) Let $t \in (0, \gamma)$ and choose $\varepsilon \in \left(0, \frac{\gamma - t}{2}\right)$. Then there is some $x_0 > 0$ such that

$$\left| \frac{-\log \overline{F}(x)}{x} - \gamma \right| \le \varepsilon < \frac{\gamma - t}{2} \quad \forall x \ge x_0.$$

Thus

$$\begin{aligned} \int_{x_0}^{\infty} e^{tx}\overline{F}(x)\,dx &= \int_{x_0}^{\infty} e^{-(\gamma - t)x - \left(\frac{-\log \overline{F}(x)}{x} - \gamma\right)x}\,dx \le \int_{x_0}^{\infty} e^{-(\gamma - t)x + \varepsilon x} \\ &\le \int_{x_0}^{\infty} e^{-\frac{\gamma - t}{2}x} dx < \infty. \end{aligned}$$

Similarly, one shows that $M_{F_I}(t) = \infty$ for $t > \gamma$.
(b) Since $M_{F_I}(\gamma) = \infty$ and $M_{F_I}(\cdot)$ is continuous on $(-\infty, \gamma)$ (see Proposition 2.14) we have $\lim_{s \nearrow \gamma} M_{F_I}(s) = \infty$. Further,

$$M_{F_I}(0) = 1 \;\Rightarrow\; \exists K > 0 \text{ s.t. } M_{F_I}(K) = \frac{1}{\rho} > 1.$$

(c) follows from (4.2) and Lemma 4.3 □

4.1.2 The ruin probability

Theorem 4.5. *Consider the Cramér-Lundberg model. Assume that the NPC is satisfied and that the Lundberg coefficient $K > 0$ exists. Then*

$$\psi(u) \leq e^{-Ku}, \quad \forall u \geq 0.$$

Proof. Let $t_0 < \infty$ be fixed and $\mathcal{F}_t = \sigma\left(\{Y_s,\ s \leq t\}\right)$, where $S_t = \sum_{k=1}^{N_t} X_k$ and $Y_t = ct - S_t$. Then $\tau^\star(u) = \min\left(t_0, \tau(u)\right)$ is a bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping time. Let $v \in \mathbb{R}$ with $M_F(v) < \infty$. Then, by Theorem 2.17 (a)

$$\mathbb{E}\left[e^{-vY_t}\right] = \mathbb{E}\left[e^{vS_t - vct}\right] = e^{\lambda t(M_X(v)-1)-vct}.$$

Thus, define

$$\varphi_t(v) := \lambda t\left(M_X(v) - 1\right) - vct, \qquad \widetilde{M}_t := e^{-vR_t - \varphi_t(v)} = e^{-vu - vY_t - \varphi_t(v)}.$$

Then $(\widetilde{M}_t)_{t\geq 0}$ is a martingale w.r.t. $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$ with $\widetilde{M}_0 = e^{-vu}$. Note that if $\tau(u) < \infty$ then $R_{\tau(u)} = u + Y_{\tau(u)} < 0$.

$$\Rightarrow\; \widetilde{M}_{\tau(u)} = e^{-vR_{\tau(u)} - \varphi_{\tau(u)}(v)} \geq e^{-\varphi_{\tau(u)}(v)} \;\text{ on }\; \{\tau(u) < \infty\}. \tag{4.3}$$

By the optional sampling theorem, we have

$$\begin{aligned}
e^{-vu} &= \mathbb{E}\left[\widetilde{M}_0\right] = \mathbb{E}\left[\widetilde{M}_{\tau^\star(u)}\right] \\
&= \mathbb{E}\left[\widetilde{M}_{\tau^\star(u)} \mid \tau(u) < t_0\right] \mathbb{P}(\tau(u) < t_0) + \mathbb{E}\left[\widetilde{M}_{\tau^\star(u)} \mid \tau(u) \geq t_0\right] \mathbb{P}(\tau(u) \geq t_0) \\
&\geq \mathbb{E}\left[\widetilde{M}_{\tau^\star(u)} \mid \tau(u) < t_0\right] \mathbb{P}(\tau(u) < t_0) \qquad (4.4) \\
&\underset{(4.3)}{\geq} \mathbb{E}\left[e^{-\varphi_{\tau(u)}(v)} \mid \tau(u) < t_0\right] \mathbb{P}(\tau(u) < t_0) \\
&\geq \mathbb{E}\left[\inf_{0\leq t\leq t_0} e^{-\varphi_t(v)}\right] \mathbb{P}(\tau(u) < t_0) \\
&= \inf_{0\leq t\leq t_0} e^{-\varphi_t(v)} \mathbb{P}(\tau(u) < t_0). \\
\Rightarrow\; \mathbb{P}(\tau(u) < t_0) &\leq \frac{e^{-vu}}{\inf_{0\leq t\leq t_0} e^{-\varphi_t(v)}} = e^{-vu} \sup_{0\leq t\leq t_0} e^{\varphi_t(v)} \leq e^{-vu} \sup_{t\geq 0} e^{\varphi_t(v)}. \\
\Rightarrow\; \psi(u) &= \lim_{t_0\to\infty} \mathbb{P}(\tau(u) < t_0) \leq e^{-vu} \sup_{t\geq 0} e^{\varphi_t(v)}. \qquad (4.5)
\end{aligned}$$

The aim is to make the right hand side of (4.5) as small as possible, i.e. v should be ≥ 0, and large, under the constraint $\sup_{t\geq 0} e^{\varphi_t(v)} \leq 1$. Thus, define $v_0 = \sup\{v \geq 0 : \sup_{t\geq 0} \varphi_t(v) \leq 0\}$. Then $\psi(u) \leq e^{-v_0 u}$ for $u \geq 0$.
We now show that $v_0 = K$. In fact,

$$\begin{aligned}
v_0 &= \sup\left\{v \geq 0 : \sup_{t\geq 0} \varphi_t(v) \leq 0\right\} \\
&= \sup\left\{v \geq 0 : \lambda t\left(M_F(v) - 1\right) - ctv \leq 0,\ \forall t \geq 0\right\} \\
&= \sup\left\{v \geq 0 : \frac{M_F(v) - 1}{\mu} - \frac{cv}{\mu\lambda} \leq 0\right\} \\
&= \sup\left\{v \geq 0 : \frac{M_F(v) - 1}{\mu v} \leq \underbrace{\frac{c}{\mu\lambda}}_{\frac{1}{\rho}}\right\} \\
&= \sup\left\{v \geq 0 : M_{F_I}(v) \leq \frac{1}{\rho}\right\} \\
&\overset{\text{Lemma 4.3}}{=} K.
\end{aligned}$$

□

Theorem 4.6 (Cramér-Lundberg approximation)**.** *Consider the classical risk process $(R_t)_{t\geq 0}$, suppose that the NPC is satisfied and that the Lundberg coefficient $K > 0$ exists. Assume further that $M_F(K + \varepsilon) < \infty$ for some $\varepsilon > 0$. Define*

$$\mu^\star = \frac{\lambda}{c}\int_0^\infty x e^{Kx}\, \overline{F}(x) dx.$$

Then $\mu^\star < \infty$ and

$$\psi(u) \sim \frac{1-\rho}{K\mu^\star} e^{-Ku} \quad \textit{for} \quad u \to \infty.$$

Proof. Since the Lundberg coefficient $K > 0$ exists, we have by Lemma 4.3

$$\rho \int_0^\infty e^{Kx}\, F_I(dx) = \rho M_{F_I}(K) = 1.$$

$$\Rightarrow\ F_I^{(K)}(x) = \rho \int_0^x e^{Ky}\, F_I(dy),\ x \geq 0 \text{ is a distribution function.}$$

$F_I^{(K)}$ is called **Esscher transformation** of F_I. Further, $\mu^\star = \int_0^\infty x\, F_I^{(K)}(dx) < \infty$, see exercises. The renewal equation of Theorem 3.6 (a) says

$$\theta(u) = \theta(0) + \rho(F_I * \theta)(u).$$

Hence

$$\begin{aligned}
\psi(u) = 1 - \theta(u) &\underset{1-\theta(0) \overset{\text{Cor. 3.8}}{=} \rho}{=} \rho - \rho(F_I * \theta)(u) = \rho(1 - (F_I * (1 - \psi))(u)) \\
&= \rho(1 - F_I(u) + (F_I * \psi)(u)) \\
&= \rho\overline{F_I}(u) + (\psi * \rho F_I)(u),\ u \geq 0. \qquad (4.6)
\end{aligned}$$

Problem: Since $\rho F_I(\infty) = \rho < 1$, we can not apply the Smith key renewal theorem. We multiply both sides of (4.6) with e^{Ku} which yields

$$\begin{aligned}
\underbrace{e^{Ku}\psi(u)}_{\psi^\star(u)} &= \underbrace{\rho e^{Ku}\overline{F_I}(u)}_{\beta^\star(u)} + e^{Ku}\int_0^u \psi(u-y)\rho\, F_I(dy) \\
&= \beta^\star(u) + \int_0^u \underbrace{e^{K(u-y)}\psi(u-y)}_{\psi^\star(u-y)}\, \underbrace{e^{Ky}\rho\, F_I(dy)}_{F_I^{(K)}(dy)} \\
&= \beta^\star(u) + \left(\psi^\star * F_I^{(K)}\right)(u).
\end{aligned}$$

Now, we are allowed to apply Theorem 3.2, since by integration by parts

$$\begin{aligned}
\overline{F_I^{(K)}}(u) = \int_u^\infty e^{Ky}\rho\, F_I(dy) &= -\rho e^{Kx}\overline{F_I}(x)\Big|_{x=u}^{\infty} + K\int_u^\infty e^{Kx}\overline{F_I}(x)\, dx \\
&= \beta^\star(u) + K\int_u^\infty e^{Kx}\overline{F_I}(x)\, dx.
\end{aligned}$$

Hence, $\beta^\star$ is the difference of two positive, Riemann-integrable functions. Therefore, Theorem 3.2 yields

$$\begin{aligned}
\lim_{u\to\infty}\psi^\star(u) &= \frac{1}{\mu^\star}\int_0^\infty \beta^\star(y)\, dy = \frac{1}{\mu^\star}\int_0^\infty \rho e^{Kx}\overline{F_I}(x)\, dx \\
&= \frac{\rho}{\mu^\star}\Bigg(\frac{e^{Kx}}{K}\overline{F_I}(x)\Big|_0^\infty + \frac{1}{K}\underbrace{\int_0^\infty e^{Kx}\, F_I(dx)}_{M_{F_I}(K)}\Bigg) = \frac{\rho}{\mu^\star}\left(0 - \frac{1}{K} + \frac{1}{K\rho}\right) \\
&= \frac{1}{\mu^\star K}(-\rho + 1). \\
&\Rightarrow\ \psi(u) \sim e^{-Ku}\frac{1-\rho}{K\mu^\star} \quad \text{for} \quad u\to\infty.
\end{aligned}$$

□

4.2 Large claim case

4.2.1 Subexponential distributions

Definition 4.7. Let F be a distribution function on $(0,\infty)$, where $F(x) < 1$ for $x > 0$. F is the distribution function of a **subexponential distribution** if

$$\lim_{x\to\infty}\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n \quad \text{for all } n \geq 2. \tag{4.7}$$

We write $F \in \mathcal{S}$.

Remarks 4.8.

(i) If $X_1, X_2, \ldots$ are i.i.d. with distribution function F, (4.7) can be written as

$$\lim_{x\to\infty} \frac{\mathbb{P}(S_n > x)}{\mathbb{P}(X_1 > x)} = n,$$

where $S_n = \sum_{i=1}^n X_i$.
Example: Let $X_1 \overset{d}{=} N(0,1)$. Then $\mathbb{P}(X_1 > x) \sim \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}}$, $S_n = \sum_{i=1}^n X_i \overset{d}{=} N(0,n)$ and $\mathbb{P}(S_n > x) \sim \frac{\sqrt{n}}{\sqrt{2\pi}x} e^{-\frac{x^2}{2n}}$. Hence

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \sim \sqrt{n}\, e^{-\frac{x^2}{2}\overbrace{\left(\frac{1}{n}-1\right)}^{<0}} \to \infty.$$

Therefore $N(0,1) \notin \mathcal{S}$.

(ii) One can show that if (4.7) holds for **some** $n \geq 2$, then it holds for **all** $n \geq 2$.

Lemma 4.9 (Basic properties of subexponential distributions)**.**

(i) If $F \in \mathcal{S}$ then, $\forall y > 0$,

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1. \tag{4.8}$$

(ii) If (4.8) holds, then for all $\varepsilon > 0$,

$$e^{\varepsilon x}\overline{F}(x) \underset{x\to\infty}{\longrightarrow} \infty.$$

Remark: *The name ,,subexponential" refers to (ii): no (positive) exponential moments exist, i.e. $M_F(\varepsilon) = \infty$, $\forall \varepsilon > 0$. In other words, if X has a subexponential distribution, $\mathbb{E}\left[e^{\varepsilon X}\right] = \infty$ $\forall \varepsilon > 0$. $\mathcal{S}$ is also called the class of **heavy-tailed** distributions. If $F \in \mathcal{S}$, no Lundberg coefficient exists.*

Proof of Lemma 4.9.
(i) Write $G(x) = \mathbb{P}(X_1 + X_2 \leq x)$, i.e. $G = F * F = F^{*2}$. For $x \geq y > 0$,

$$\frac{\overline{G}(x)}{\overline{F}(x)} = 1 + \int_0^y \frac{\overline{F}(x-t)}{\overline{F}(x)}\, F(dt) + \int_y^x \frac{\overline{F}(x-t)}{\overline{F}(x)}\, F(dt),$$

since $\overline{G}(x) = \int_0^\infty \overline{F}(x-t)\, F(dt)$.
Hence,

$$\frac{\overline{G}(x)}{\overline{F}(x)} \geq 1 + F(y) + \frac{\overline{F}(x-y)}{\overline{F}(x)}\left(F(x) - F(y)\right).$$

Take x large enough s.t. $F(x) - F(y) > 0$, then

$$1 \leq \frac{\overline{F}(x-y)}{\overline{F}(x)} \leq \frac{\frac{\overline{G}(x)}{\overline{F}(x)} - 1 - F(y)}{F(x) - F(y)} \underset{x\to\infty}{\longrightarrow} 1$$

due to (4.7) since

$$\frac{\overline{G}(x)}{\overline{F}(x)} \underset{x\to\infty}{\longrightarrow} 2.$$

(ii) will be proved later. □

Theorem 4.10. *Let F be a distribution function on $(0,\infty)$ with $F(x) < 1$ for $x > 0$ and let $(X_k)_{k\in\mathbb{N}}$ be an i.i.d. sequence of rv's with distribution function F. Let $S_n = \sum_{k=1}^n X_k$ and $M_n = \max_{1\leq k\leq n} X_k$. Then, the following conditions are equivalent:*

(i) $F \in \mathcal{S}$.

(ii) $\lim\limits_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$ *for* ***all*** $n \geq 2$.

(iii) $\lim\limits_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$ *for* ***some*** $n \geq 2$.

(iv) $\lim\limits_{x\to\infty} \frac{\mathbb{P}(S_n\geq x)}{\mathbb{P}(M_n>x)} = 1$ *for* ***all*** $n \geq 2$.

(v) $\lim\limits_{x\to\infty} \frac{\mathbb{P}(S_n\geq x)}{\mathbb{P}(M_n>x)} = 1$ *for* ***some*** $n \geq 2$.

Proof.
(i)⇔(ii) by definition.
(ii)⇔(iii) see [5].
(ii)⇔(iv) and (iii)⇔(v): Note that

$$\begin{aligned}
\mathbb{P}(M_n > x) &= \mathbb{P}(X_1 > x) + \sum_{k=1}^{n-1} \mathbb{P}\left(\max_{1\leq j\leq k} X_j \leq x,\ X_{k+1} > x\right) \\
&\overset{(X_k)\text{ i.i.d.}}{=} \overline{F}(x) + \sum_{k=1}^{n-1} F(x)^k \overline{F}(x) \\
&= \overline{F}(x) \sum_{k=0}^{n-1} F(x)^k.
\end{aligned}$$

But $\sum_{k=0}^{n-1} F(x)^k \underset{x\to\infty}{\longrightarrow} n$, hence we obtain

$$\mathbb{P}(M_n > x) \sim n\overline{F}(x). \tag{4.9}$$

Thus, (ii)⇔(iv) and (iii)⇔(v).
For (iv)⇔(v) see [5]. □

Note that $F \in \mathcal{S}$ was **not** needed for (4.9).
If $F \in \mathcal{S}$, then ,,large aggregated claims are initiated by just a single very large claim".

Lemma 4.11.

(a) Let $F \in \mathcal{S}$ and $\varepsilon > 0$. Then there exists a $\beta > 0$ such that

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq \beta(1+\varepsilon)^n \qquad \forall x \geq 0,\ n \geq 1.$$

(b) Let $F \in \mathcal{S}$ and G be a distribution function such that $\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = C \in (0,\infty)$. Then $G \in \mathcal{S}$ and

$$\lim_{x\to\infty} \frac{\overline{G * F}(x)}{\overline{F}(x)} = 1 + C.$$

(c) Let $(p_n)_{n\in\mathbb{N}_0}$ be a sequence of weights defining a probability measure on $\mathbb{N}_0$, i.e. $p_n \geq 0$, $\forall n$ and $\sum_{n=0}^{\infty} p_n = 1$. Assume $p_l > 0$ for some $l \geq 2$. Finally, suppose $\sum_{n=0}^{\infty}(1+\varepsilon)^n p_n < \infty$ for some $\varepsilon > 0$. Define $G(x) = \sum_{n=0}^{\infty} p_n F^{*n}(x)$, $x \geq 0$. Then

$$F \in \mathcal{S} \ \Leftrightarrow\ \lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = \sum_{n=0}^{\infty} n p_n \ \Leftrightarrow G \in \mathcal{S}.$$

Example: *$p_n = \mathbb{P}(N_t = n)$. Then G is the distribution function of S_t.*

Proof.
(a) We proceed by induction:

$$\frac{\overline{F^{*(n+1)}}(x)}{\overline{F}(x)} = \frac{\overline{F}(x) + F(x) - F^{*(n+1)}(x)}{\overline{F}(x)} = 1 + \frac{F(x) - \int_0^x F^{*n}(x-y)\, F(dy)}{\overline{F}(x)}. \tag{4.10}$$

Define $\alpha_n := \sup_{x\geq 0} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)}$. Let $T > 0$. Then

$$\begin{aligned}
\alpha_{n+1} \underset{(4.10)}{\leq}\ & 1 + \sup_{0\leq x\leq T} \frac{\int_0^x \overline{F^{*n}}(x-y)\, F(dy)}{\overline{F}(x)} + \sup_{x\geq T} \int_0^x \frac{\overline{F^{*n}}(x-y)\overline{F}(x-y)}{\overline{F}(x-y)\overline{F}(x)}\, F(dy) \\
& \left(\text{as} \int_0^x \overline{F^{*n}}(x-y)\, F(dy) = \int_0^x \left(1 - F^{*n}(x-y)\right)\, F(dy) \right. \\
& \qquad\qquad \left. = \underbrace{\int_0^x 1\, F(dy)}_{F(x)} - \int_0^x F^{*n}(x-y)\, F(dy) \right) \\
\leq\ & 1 + \frac{1}{\overline{F}(T)} + \alpha_n \sup_{x\geq T} \underbrace{\int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)}\, F(dy)}_{\leq \frac{F(x)}{\overline{F}(x)} - \frac{\int_0^x F(x-y)\, F(dy)}{\overline{F}(x)} = \frac{F(x) - F*F(x)}{\overline{F}(x)}} \\
\leq\ & 1 + \frac{1}{\overline{F}(T)} + \alpha_n \sup_{x\geq T} \frac{F(x) - F*F(x)}{\overline{F}(x)}.
\end{aligned} \tag{4.11}$$

Since $F \in \mathcal{S}$, we have

$$\lim_{x\to\infty} \frac{F(x) - F * F(x)}{\overline{F}(x)} = \lim_{x\to\infty} \frac{\overline{F^{*2}}(x) - \overline{F}(x)}{\overline{F}(x)} = 1.$$

Thus, for any $\varepsilon > 0$, there exists a $T > 0$ such that

$$\sup_{x\geq T} \frac{F(x) - F * F(x)}{\overline{F}(x)} \leq 1 + \varepsilon. \tag{4.12}$$

(4.11) and (4.12) imply that $\alpha_{n+1} \leq 1 + \frac{1}{\overline{F}(T)} + \alpha_n(1+\varepsilon)$.

$$\begin{aligned} \Rightarrow\ \alpha_{n+1} &\leq \left(1 + \frac{1}{\overline{F}(T)}\right) \sum_{k=0}^{n-1} (1+\varepsilon)^k = \left(1 + \frac{1}{\overline{F}(T)}\right) \frac{(1+\varepsilon)^n - 1}{\varepsilon} \\ &\leq \frac{1}{\varepsilon}\left(1 + \frac{1}{\overline{F}(T)}\right)(1+\varepsilon)^n. \end{aligned}$$

and this implies (a).
(b) Let X, Y be i.i.d. rv's with distribution function G and $z > 0$, $v > 0$ s.t. $z > 2v$. We have

$$\begin{aligned} \{X + Y > z\} &= \{X \leq v,\ X + Y > z\} \cup \{Y \leq v,\ X + Y > z\} \\ &\quad \cup \{v < X \leq z - v,\ X + Y > z\} \cup \{Y > v,\ X > z - v\}. \end{aligned} \tag{4.13}$$

$$\Rightarrow\ \frac{\overline{G^{*2}}(z)}{\overline{G}(z)} = \underbrace{2\int_0^v \frac{\overline{G}(z-y)}{\overline{G}(z)}\, G(dy)}_{I_1(z,v)} + \underbrace{\int_v^{z-v} \frac{\overline{G}(z-y)}{\overline{G}(z)}\, G(dy)}_{I_2(z,v)} + \underbrace{\frac{\overline{G}(z-v)}{\overline{G}(z)}\overline{G}(v)}_{I_3(z,v)}. \tag{4.14}$$

Step 1: Since

$$\lim_{z\to\infty} \frac{\overline{G}(z-y)}{\overline{G}(z)} = \lim_{z\to\infty} \underbrace{\frac{\overline{G}(z-y)}{\overline{F}(z-y)}}_{\to C} \underbrace{\frac{\overline{F}(z-y)}{\overline{F}(z)}}_{\to 1} \underbrace{\frac{\overline{F}(z)}{\overline{G}(z)}}_{\to \frac{1}{C}} = 1,$$

we obtain

$$\lim_{v\to\infty} \lim_{z\to\infty} I_1(z,v) = 2 \lim_{v\to\infty} G(v) = 2 \tag{4.15}$$

and

$$\lim_{v\to\infty} \lim_{z\to\infty} I_3(z,v) = \lim_{v\to\infty} \overline{G}(v) = 0. \tag{4.16}$$

Step 2: For $\varepsilon > 0$ there exists a $z_0 = z_0(\varepsilon) > 0$ such that for $z \geq z_0$,

$$C - \varepsilon \leq \frac{\overline{G}(x)}{\overline{F}(x)} \leq C + \varepsilon.$$

Therefore, for v large and $v \le y \le z - v$, we have

$$\frac{\overline{G}(z-y)}{\overline{F}(z-y)} \underset{z-y>v}{\le} C + \varepsilon$$

and

$$\frac{\overline{F}(z)}{\overline{G}(z)} \underset{z>2v}{\le} \frac{1}{C-\varepsilon}.$$

$$\Rightarrow \; I_2(z,v) \le \frac{C+\varepsilon}{C-\varepsilon} \int_v^{z-v} \frac{\overline{F}(z-y)}{\overline{F}(z)} \, G(dy).$$

Now,

$$\int_v^{z-v} \overline{F}(z-y)\, G(dy) = \overline{F}(z-v)\overline{G}(v) - \overline{F}(v)\overline{G}(z-v) + \int_v^{z-v} \overline{G}(z-y)\, F(dy). \qquad (\star)$$

Proof of ($\star$): see exercises.
Hence,

$$\begin{aligned} I_2(z,v) &\le \frac{C+\varepsilon}{C-\varepsilon} \frac{1}{\overline{F}(z)} \left(\overline{F}(z-v)\overline{G}(v) - \overline{F}(v)\overline{G}(z-v) + \int_v^{z-v} \overline{G}(z-y)\, F(dy) \right) \\ &\le \frac{C+\varepsilon}{C-\varepsilon} \left(\frac{\overline{F}(z-v)}{\overline{F}(z)} \overline{G}(v) - \overline{F}(v) \frac{\overline{G}(z-v)}{\overline{F}(z-v)} \frac{\overline{F}(z-v)}{\overline{F}(z)} + \right. \\ &\qquad\qquad \left. + (C+\varepsilon) \int_v^{z-v} \frac{\overline{F}(z-y)}{\overline{F}(z)} \, F(dy) \right) \end{aligned}$$

On the one hand,

$$\lim_{z\to\infty} \left(\frac{\overline{F}(z-v)}{\overline{F}(z)} \overline{G}(v) - \overline{F}(v) \frac{\overline{G}(z-v)}{\overline{F}(z-v)} \frac{\overline{F}(z-v)}{\overline{F}(z)} \right) = \overline{G}(v) - \overline{F}(v) C \underset{v\to\infty}{\longrightarrow} 0.$$

On the other hand,

$$\lim_{v\to\infty} \lim_{z\to\infty} \int_v^{z-v} \frac{\overline{F}(z-y)}{\overline{F}(z)} \, F(dy) = 0$$

(otherwise there would be a contradiction to (4.14) with F instead of G, since we know $F \in \mathcal{S}$).
Hence

$$\lim_{v\to\infty} \lim_{z\to\infty} I_2(z,v) = 0. \qquad (4.17)$$

From (4.14)-(4.17) we conclude, that

$$\lim_{z\to\infty} \frac{\overline{G^{*2}}(z)}{\overline{G}(z)} = 2,$$

hence $G \in \mathcal{S}$.

(c) „⇒": $F \in \mathcal{S}$. (b) and dominated convergence give (using (a))

$$\lim_{x\to\infty} \frac{\overline{G}(x)}{\overline{F}(x)} = \lim_{x\to\infty} \sum_{n=0}^{\infty} p_n \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = \sum_{n=0}^{\infty} p_n \lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = \sum_{n=0}^{\infty} p_n n.$$

(b) now implies $G \in \mathcal{S}$.
„⇐": cf. [5, Theorem A. 320]. □

Corollary 4.12. *Consider the Cramér-Lundberg model with claim amount process S_t and suppose for the claim size distribution we have $F \in \mathcal{S}$. Then*

$$\mathbb{P}(S_t > x) \sim \lambda t \overline{F}(x) \qquad \text{for } x \to \infty \text{ and } S_t \in \mathcal{S}.$$

Proof. Follows from Lemma 4.11 (c) with

$$p_n = \mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \ n = 0, 1, 2, \ldots.$$
$$\Rightarrow \sum_{n=0}^{\infty} n p_n = \lambda t.$$

□

Definition 4.13.

(a) A measurable function $l : (0, \infty) \to (0, \infty)$ is **slowly varying** (s.l.) if

$$\lim_{t\to\infty} \frac{l(tx)}{l(t)} = 1 \qquad \forall x > 0. \tag{4.18}$$

We write $l \in \mathcal{R}_0$.

(b) A measurable function $f : (0, \infty) \to (0, \infty)$ is **regularly varying** with index α ($\alpha \in \mathbb{R}$) if

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^{\alpha} \qquad \forall x > 0. \tag{4.19}$$

We write $f \in \mathcal{R}_{\alpha}$.
We say that a rv X is regularly varying with index $-\alpha$ if $\overline{F} \in \mathcal{R}_{-\alpha}$ (for some $\alpha > 0$).

Examples 4.14. $l(x) = \log^+ x$, $l(x) = \log\log^+ x$, $l(x) = \log^+ x \log\log^+ x$ are s.l.
$f(x) = x^{\alpha}$, $f(x) = x^{\alpha}\log(17 + x)$, $f(x) = (x\log(1 + x))^{\alpha}$, $f(x) = x^{\alpha}\log(\log(37 + x)) + \sin x$ are in $\mathcal{R}_{\alpha}$.

Remarks 4.15.

1. $f \in \mathcal{R}_{-\alpha} \Leftrightarrow \exists l \in \mathcal{R}_0$ *s.t.* $f(x) = \frac{l(x)}{x^{\alpha}}$.

   Proof. See [3]. □

2. Assume $f \in \mathcal{R}_\alpha$ and $f(x) \sim g(x)$ for $x \to \infty$. Then

$$\frac{g(xt)}{g(t)} = \underbrace{\frac{g(xt)}{f(xt)}}_{\xrightarrow[t\to\infty]{} 1} \underbrace{\frac{f(xt)}{f(t)}}_{\xrightarrow[t\to\infty]{} x^\alpha} \underbrace{\frac{f(t)}{g(t)}}_{\xrightarrow[t\to\infty]{} 1} \xrightarrow[t\to\infty]{} x^\alpha, \qquad \forall x > 0.$$

$$\Rightarrow \; g \in \mathcal{R}_\alpha.$$

Lemma 4.16. *Assume X_1, X_2 are independent, regularly varying rv's with the same index $-\alpha$ (for some $\alpha > 0$), i.e. $\overline{F_I}(x) = \frac{l_i(x)}{x^\alpha}$, $\forall x > 0$, $i = 1, 2$ for some s.l. functions l_1, l_2. Then $X_1 + X_2 \in \mathcal{R}_{-\alpha}$. More precisely,*

$$\mathbb{P}(X_1 + X_2 > x) \sim \frac{l_1(x) + l_2(x)}{x^\alpha} = \overline{F_1}(x) + \overline{F_2}(x) \qquad \textit{for } x \to \infty. \tag{4.20}$$

Corollary 4.17. $\overline{F} \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 0 \;\Rightarrow\; F \in \mathcal{S}$.

Proof of Corollary 4.17. From (4.20), $\frac{\overline{F*F}(x)}{\overline{F}(x)} \xrightarrow[x\to\infty]{} 2$ $\left(\text{and by iterating, } \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \xrightarrow[x\to\infty]{} n, \; \forall n\right)$. □

Hence: lots of examples for distributions in $\mathcal{S}$!

Proof of Lemma 4.16. Note that $\{X_1 + X_2 > x\} \supseteq \{X_1 > x\} \cup \{X_2 > x\}$.

$$\Rightarrow \; \mathbb{P}(X_1 + X_2 > x) \geq \mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x) - \mathbb{P}(X_1 > x, \; X_2 > x). \tag{4.21}$$

Since

$$\frac{\mathbb{P}(X_1 > x, \; X_2 > x)}{\mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x)} = \frac{\mathbb{P}(X_1 > x)\mathbb{P}(X_2 > x)}{\mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x)} = \frac{1}{\frac{1}{\mathbb{P}(X_2 > x)} + \frac{1}{\mathbb{P}(X_1 > x)}} \xrightarrow[x\to\infty]{} 0, \tag{4.22}$$

we conclude from (4.21)

$$\liminf_{x\to\infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x)} \geq \liminf_{x\to\infty} \frac{\mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x)}{\mathbb{P}(X_1 > x) + \mathbb{P}(X_2 > x)} = 1. \tag{4.23}$$

On the other hand, let $0 < \delta < \frac{1}{2}$. Then

$$\{X_1 + X_2 > x\} \subseteq \{X_1 > (1-\delta)x\} \cup \{X_2 > (1-\delta)x\} \cup \{X_1 > \delta x, \; X_2 > \delta x\}.$$

$$\Rightarrow \; \mathbb{P}(X_1 + X_2 > x) \leq \overline{F_1}\left((1-\delta)x\right) + \overline{F_2}\left((1-\delta)x\right) + \overline{F_1}(\delta x)\overline{F_2}(\delta x). \tag{4.24}$$

For the first two terms on the right hand side of (4.24),

$$\begin{aligned} 1 &\leq \frac{\overline{F_1}((1-\delta)x) + \overline{F_2}((1-\delta)x)}{\overline{F_1}(x) + \overline{F_2}(x)} = 1 + \frac{\overline{F_1}((1-\delta)x) - \overline{F_1}(x)}{\overline{F_1}(x) + \overline{F_2}(x)} + \frac{\overline{F_2}((1-\delta)x) - \overline{F_2}(x)}{\overline{F_1}(x) + \overline{F_2}(x)} \\ &\leq 1 + \frac{\overline{F_1}((1-\delta)x)}{\overline{F_1}(x)} - 1 + \frac{\overline{F_2}((1-\delta)x)}{\overline{F_2}(x)} - 1 \xrightarrow[x\to\infty]{} 2\frac{1}{(1-\delta)^\alpha} - 1 \\ &\xrightarrow[\delta\searrow 0]{} 1. \end{aligned} \tag{4.25}$$

For the last term in (4.24),

$$\frac{\overline{F_1}(x)\overline{F_2}(\delta x)}{\overline{F_1}(x)+\overline{F_2}(x)} = \underbrace{\frac{\overline{F_1}(\delta x)\overline{F_2}(\delta x)}{\overline{F_1}(x)\overline{F_2}(x)}}_{\substack{\xrightarrow[x\to\infty]{} \frac{1}{\delta^\alpha}\frac{1}{\delta^\alpha}, \\ \text{since } \overline{F_1},\overline{F_2}\in\mathcal{R}_{-\alpha}.}} \underbrace{\frac{\overline{F_1}(x)\overline{F_2}(x)}{\overline{F_1}(x)+\overline{F_2}(x)}}_{\substack{\xrightarrow[x\to\infty]{} 0, \\ \text{see (4.22)}}}. \tag{4.26}$$

(4.24)-(4.26) imply that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(X_1+X_2>x)}{\mathbb{P}(X_1>x)+\mathbb{P}(X_2>x)} \le 1.$$

□

Remark 4.18. There are distributions $F\in\mathcal{S}$ such that there is no α with $\overline{F}\in\mathcal{R}_{-\alpha}$. An example is the **Weibull distribution**

$$\overline{F}(x) = e^{-cx^r}, \qquad x\ge 0,$$

where $c>0$, $0<r<1$.
(If $r>1$, $F\notin\mathcal{S}$: Why not?).

Proof of Lemma 4.9 (ii). We show that

$$\frac{\overline{F}(x-y)}{\overline{F}(x)} \xrightarrow[x\to\infty]{} 1 \;\; \forall y>0 \;\; (4.8) \;\Rightarrow e^{\varepsilon x}\overline{F}(x) \xrightarrow[x\to\infty]{} \infty \; \forall\varepsilon>0.$$

We use the following representation for s.l. functions:
If l is s.l. then $\exists x_0>0$, a positive function $c_0(\cdot)$ s.t. $c_0(t)\to\overline{c_0}$ for some constant $\overline{c_0}>0$, $\varepsilon(\cdot)$ s.t. $\varepsilon(t)\xrightarrow[t\to\infty]{} 0$ and

$$l(x) = c_0(x)e^{\int_{x_0}^x \frac{\varepsilon(t)}{t}\,dt}, \qquad x\ge x_0. \tag{4.27}$$

(4.27) implies that $\forall\delta>0$, $\lim_{x\to\infty}\frac{l(x)}{x\delta}=0$,

$$\lim_{x\to\infty} x^\delta l(x) = \infty, \tag{4.28}$$

i.e. „l is small compared to any power x^δ".

Proof of (4.27). See [3]. □

Now, if (4.8) holds, $l(y)=\overline{F}(\log y)$ is slowly varying. But now, using (4.28),

$$y^\varepsilon\overline{F}(\log y) \xrightarrow[y\to\infty]{} \infty.$$

Now, write $y=e^x$. □

4.2.2 The ruin probability

Theorem 4.19. *Consider the Cramér-Lundberg model. Suppose that the NPC holds. Then the following statements are equivalent:*

(a) $F_I \in \mathcal{S}$.

(b) $\theta \in \mathcal{S}$.

(c) $\lim\limits_{u\to\infty} \frac{\psi(u)}{\overline{F_I}(u)} = \frac{\rho}{1-\rho} = \frac{\mu}{\frac{c}{\lambda}-\mu}$.

Proof. Recall Lemma 4.11 (c): $G(x) := \sum_{n=0}^{\infty} p_n F^{*n}(x)$. Then

$$F \in \mathcal{S} \;\Leftrightarrow\; \frac{\overline{G}(x)}{\overline{F}(x)} \underset{x\to\infty}{\longrightarrow} \sum_{n=0}^{\infty} p_n n \;\Leftrightarrow\; G \in \mathcal{S}.$$

The Pollaczek-Khintchine formula says

$$\theta(u) = (1-\rho)\sum_{n=0}^{\infty} \rho^n F_I^{*n}(u) \qquad \text{and} \qquad \psi(u) = (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{F_I^{*n}}(u).$$

Further,

$$(1-\rho)\sum_{n=1}^{\infty} n\rho^n = \rho \underbrace{\sum_{n=1}^{\infty} n\rho^{n-1}(1-\rho)}_{=\mathbb{E}[Y],\ Y \overset{d}{=} Geo(1-\rho)} = \rho\,\frac{1}{1-\rho}.$$

With Lemma 4.11 (c), with $p_n = (1-\rho)\rho^n$, we conclude

$$\begin{aligned} F_I \in \mathcal{S} \;&\Leftrightarrow\; \frac{\psi(u)}{\overline{F_I}(u)} = (1-\rho)\sum_{n=1}^{\infty} \rho^n \frac{\overline{F_I^{*n}}(u)}{\overline{F_I}(u)} \overset{n\to\infty}{\longrightarrow} (1-\rho)\sum_{n=1}^{\infty} \rho^n n = \frac{\rho}{1-\rho} \\ &\Leftrightarrow\; \theta = 1-\psi \in \mathcal{S}. \end{aligned}$$

□

Possible generalizations:

- Replace the Poisson process by a general subordinator, i.e. replace the exponential law of the interarrival times with a different law.
- Investigate the ruin probability in finite time, i.e.

$$\psi_T(u) = \mathbb{P}\left(\inf_{t\in[0,T]} R_t < 0\right),$$

  instead of $\psi(u)$.

5 Risk measures

cf. [6].

5.1 Risk measures and their acceptance sets

Let Ω be a set of scenarios. A **financial position** is described by a mapping $X : \Omega \to \mathbb{R}$, where $X(\omega)$ is the value of the position of the scenario $\omega \in \Omega$.
Aim: Quantify the risk of X by some number $\rho(X)$, where X belongs to a given class $\mathcal{H}$ of financial positions, where $\mathcal{H}$ is a linear space of functions containing the constants.

Definition 5.1. A mapping $\rho : \mathcal{H} \to \mathbb{R}$ is a risk measure if it satisfies the following conditions for all $X, Y \in \mathcal{H}$:

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Translation invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

Translation invariance implies in particular

$$\rho(X + \rho(X)) = 0 \tag{5.1}$$

and

$$\rho(m) = \rho(0) - m, \qquad \forall m \in \mathbb{R}.$$

The risk measure is called **normalized** if $\rho(0) = 0$.

Examples:

1. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then $\rho(X) = -\mathbb{E}[X]$ is a risk measure on $\mathcal{H} = L^1$.
2. Let $\mathcal{H} \subseteq \{\text{all bounded functions on } \Omega\}$. Then $\rho_{\max}(X) = -\inf_{\omega \in \Omega} X(\omega)$ is called the **worst-case risk measure**.

Definition 5.2. A risk measure is convex if $\forall X, Y \in \mathcal{H}$, $\beta \in [0, 1]$

$$\rho\left(\beta X + (1 - \beta)\, Y\right) \leq \beta \rho(X) + (1 - \beta)\, \rho(Y).$$

,,Diversification should not increase the risk".

Definition 5.3.

1. A convex risk measure is called **coherent** if it satisfies **positive homogeneity**: If $\beta \geq 0$, then $\rho(\beta X) = \beta \rho(X)$.
2. A risk measure is **subadditive** if it satisfies $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Exercise: Let ρ be a normalized risk measure on $\mathcal{H}$. Show that any two of the following properties imply the third:

- Convexity.
- Positive homogeneity.
- Subadditivity.

Lemma 5.4. *Any risk measure ρ is Lipschitz-continuous w.r.t. the supremum norm $\|\cdot\|$:*

$$|\rho(X) - \rho(Y)| \leq \|X - Y\|.$$

Proof. Clearly, $X \leq Y + \|X - Y\|$ and so

$$\begin{aligned} \rho(Y) - \|X - Y\| &\leq \rho(X). \\ \Rightarrow \|X - Y\| &\geq \rho(Y) - \rho(X), \end{aligned}$$

by monotonicity and translation invariance. Changing the roles of X and Y yields the assertion. □

A risk measure induces the class

$$\mathcal{A}_\rho = \{X \in \mathcal{H} : \rho(X) \leq 0\} \tag{5.2}$$

of positions which are **acceptable**. $\mathcal{A}_\rho$ is the **acceptance set** of ρ.

Proposition 5.5. $\mathcal{H} = \{$*all bounded functions on* $\Omega\}$.
Suppose ρ is a risk measure with acceptance set $\mathcal{A} = \mathcal{A}_\rho$. Then

(a) $\mathcal{A}$ is non-empty and closed in $\mathcal{H}$ (w.r.t. the supremum-norm $\|\cdot\|$) and satisfies the following two conditions:

$$\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty. \tag{5.3}$$

$$X \in \mathcal{A},\ Y \in \mathcal{H},\ Y \geq X \ \Rightarrow\ Y \in \mathcal{A}. \tag{5.4}$$

(b) ρ can be recovered from $\mathcal{A}$:

$$\rho(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}. \tag{5.5}$$

(c) ρ is a convex risk measure if and only if $\mathcal{A}$ is convex.

(d) ρ is positive homogeneous if and only if $\mathcal{A}$ is a cone.
In particular, ρ is coherent if and only if $\mathcal{A}$ is a convex cone.

Proof.
(a) (5.3) and (5.4) are straightforward. Closedness follows from Lemma 5.4.
(b) Translation invariance implies that for $X \in \mathcal{H}$,

$$\begin{aligned} \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}_\rho\} &= \inf\{m \in \mathbb{R} : \rho(m + X) \leq 0\} = \inf\{m \in \mathbb{R} : \rho(X) \leq m\} \\ &= \rho(X). \end{aligned}$$

(c) $\mathcal{A}$ is clearly convex if ρ is a convex measure of risk. The converse will follow from (5.6) and Proposition 5.6.
(d) Clearly, positive homogeneity of ρ implies that $\mathcal{A}$ is a cone. The converse will follow from (5.6) and Proposition 5.6. □

One could take a given class $\mathcal{A}$ of acceptable positions as the primary object. For a position $X \in \mathcal{H}$, we can define $\rho(X)$ as the minimal amount m for which $X + m$ becomes acceptable:

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}.$$

Then (5.5) can be written as

$$\rho_{\mathcal{A}_\rho} = \rho. \tag{5.6}$$

Proposition 5.6. *$\mathcal{H} = \{$bounded functions on $\Omega\}$.*
Assume that $\mathcal{A}$ is a non-empty subset of $\mathcal{H}$ which satisfies (5.3) and (5.4). Then the functional $\rho_{\mathcal{A}}$ has the following properties:

(a) $\rho_{\mathcal{A}}$ is a risk measure.

(b) If $\mathcal{A}$ is a convex set, then $\rho_{\mathcal{A}}$ is a convex risk measure.

(c) If $\mathcal{A}$ is a cone, then $\rho_{\mathcal{A}}$ is positive homogeneous.
In particular, $\rho_{\mathcal{A}}$ is a coherent risk measure if $\mathcal{A}$ is a convex cone.

(d) $\mathcal{A}$ is a subset of $\mathcal{A}_{\rho_{\mathcal{A}}}$ and $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$ holds if and only if $\mathcal{A}$ is $\|\cdot\|$-closed in $\mathcal{H}$.

Proof.
(a) $\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}$. Clearly, $\rho_{\mathcal{A}}$ is translation invariant and monotone. We have to show that $\rho_{\mathcal{A}}$ takes only finite values. Fix $Y \in \mathcal{A}$. For $X \in \mathcal{H}$ given, $\exists$ finite number $m \in \mathbb{R}$ s.t. $m + X > Y$ (because X and Y are bounded!). Then

$$\begin{aligned} \rho_{\mathcal{A}}(X) - m = \rho_{\mathcal{A}}(m + X) &\leq \rho_{\mathcal{A}}(Y) \leq 0 \\ \Rightarrow\ \rho_{\mathcal{A}}(X) &\leq m < \infty. \end{aligned}$$

Note that (5.3) is equivalent to $\rho_{\mathcal{A}}(0) > -\infty$. To show that $\rho_{\mathcal{A}}(X) > -\infty$ for any $X \in \mathcal{H}$, take $\widetilde{m}$ s.t. $X + \widetilde{m} \leq 0$ (possible since X is bounded!). Then we conclude by monotonicity and translation invariance that

$$\rho_{\mathcal{A}}(X) \geq \rho_{\mathcal{A}}(0) + \widetilde{m} > -\infty.$$

(b) Suppose $X_1, X_2 \in \mathcal{H}$ and take m_1, m_2 s.t. $m_i + X_i \in \mathcal{A}$. For $\beta \in [0,1]$, $\beta(m_1 + X_1) + (1-\beta)(m_2 + X_2) \in \mathcal{A}$ since $\mathcal{A}$ is convex

$$\begin{aligned} 0 \quad &\geq \quad \rho_{\mathcal{A}}\left(\beta(m_1 + X_1) + (1-\beta)(m_2 + X_2)\right) \\ &\underset{\substack{\text{translation} \\ \text{invariance}}}{\geq} \quad \rho_{\mathcal{A}}(\beta X_1 + (1-\beta)X_2) - (\beta m_1 + (1-\beta)m_2). \\ &\Rightarrow\ \rho_{\mathcal{A}} \text{ is convex.} \end{aligned}$$

(c) If $\mathcal{A}$ is a cone,

$$\begin{aligned} \rho_{\mathcal{A}}(\beta X) &= \inf\{m \in \mathbb{R} : m + \beta X \in \mathcal{A}\} = \beta \inf\left\{\frac{m}{\beta} \in \mathbb{R} : \beta\frac{m}{\beta} + \beta X \in \mathcal{A}\right\} \\ &= \beta \inf\{m \in \mathbb{R} : \beta(m + X) \in \mathcal{A}\} \underset{\mathcal{A}\text{ cone}}{=} \beta \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\} \\ &= \beta \rho_{\mathcal{A}}(X). \end{aligned}$$

(d) $A \subseteq \mathcal{A}_{\rho_\mathcal{A}}$. Proposition 5.5 (a) implies that $\mathcal{A}$ is closed w.r.t. $\|\cdot\|$ as soon as $\mathcal{A} = \mathcal{A}_{\rho_\mathcal{A}}$. Conversely, assume that $\mathcal{A}$ is closed w.r.t. $\|\cdot\|$. We have to show that $X \notin \mathcal{A}$ implies $\rho_\mathcal{A}(X) > 0$. Take $m > \|X\|$. Since $\mathcal{A}$ is closed and $X \notin \mathcal{A}$, $\exists \beta \in (0,1)$ such that $\beta m + (1-\beta)X \notin \mathcal{A}$. Thus,

$$0 \leq \rho_\mathcal{A}\left(\beta m + (1-\beta)\, X\right) = \rho_\mathcal{A}\left((1-\beta)\, X\right) - \beta m.$$

Since $\rho_\mathcal{A}$ is a risk measure, Lemma 5.4 yields

$$\left|\rho_\mathcal{A}\left((1-\beta)\, X\right) - \rho_\mathcal{A}(x)\right| \leq \beta\|X\|.$$

Hence

$$\rho_\mathcal{A}(X) \geq \rho_\mathcal{A}((1-\beta)X) - \beta\|X\| \geq \beta(m - \|X\|) > 0.$$

□

Remark: In fact, $\mathcal{A}_{\rho_\mathcal{A}}$ is the closure of $\mathcal{A}$ w.r.t. $\|\cdot\|$.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote $\mathcal{H}_0 = \{$all bounded measurable functions on $\Omega\}$ and $\mathcal{M}_1 = \mathcal{M}_1(\Omega, \mathcal{F}) = \{$probability measures on $(\Omega, \mathcal{F})\}$.

Example 5.7. $\mathcal{H} = \mathcal{H}_0$.
Consider the worst-case risk measure $\rho_{\max}$ defined by

$$\rho_{\max}(X) = -\inf_{\omega\in\Omega} X(\omega) \qquad (X \in \mathcal{H}_0).$$

The corresponding acceptance set is given by the convex cone of all non-negative functions in $\mathcal{H}_0$. Thus, $\rho_{\max}$ is a coherent risk measure. It is the most „conservative" risk measure in the sense that any normalized risk measure ρ on $\mathcal{H}_0$ satisfies

$$\rho(X) \leq \rho\left(\inf_{\omega\in\Omega} X(\omega)\right) = \rho_{\max}(X).$$

Note that $\rho_{\max}$ can be represented in the following form:

$$\rho_{\max}(X) = \sup_{Q\in\mathcal{M}_1} \mathbb{E}_Q[-X].$$

Example 5.8. Let $\mathcal{H} = \mathcal{H}_0$ and $\widetilde{\mathcal{M}}_1 \subseteq \mathcal{M}_1$. Let $\gamma : \widetilde{\mathcal{M}}_1 \to \mathbb{R}$ be a mapping with $\sup_{Q\in\widetilde{\mathcal{M}}_1} \gamma(Q) < \infty$.
Interpretation: γ specifies for each $Q \in \widetilde{\mathcal{M}}_1$ a „threshold" $\gamma(Q)$. Assume that a position X is acceptable if $E_Q[X] \geq \gamma(Q)$, $\forall Q \in \widetilde{\mathcal{M}}_1$ $\left(\mathbb{E}_Q[X] = \int_\Omega X(\omega)\, Q(d\omega)\right)$. The set $\mathcal{A}$ of such positions satisfies (5.3) and (5.4) and it is convex. Thus, the associated risk measure $\rho = \rho_\mathcal{A}$ is convex. It has the form

$$\rho(X) = \sup_{Q\in\widetilde{\mathcal{M}}_1} \left\{\gamma(Q) - \mathbb{E}_Q[X]\right\}.$$

Alternatively, we can write

$$\rho(X) = \sup_{Q \in \mathcal{M}_1} \left(\mathbb{E}_Q[-X] - \alpha(Q) \right),$$

where the penalty function $\alpha : \mathcal{M}_1 \to (-\infty, \infty]$ is given by

$$\alpha(Q) = \begin{cases} -\gamma(Q), & Q \in \widetilde{\mathcal{M}}_1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that ρ is a coherent risk measure if $\gamma(Q) = 0$, $\forall Q \in \widetilde{\mathcal{M}}_1$.

Example 5.9. Take a **utility function** u on $\mathbb{R}$, i.e. a function which is strictly concave and increasing (Example: $u(x) = a - e^{-bx}$) and a probability measure $Q \in \mathcal{M}_1$, and fix a threshold $c \in \mathbb{R}$. A position X is acceptable if

$$\mathbb{E}_Q[u(X)] \geq u(c).$$

The set $\mathcal{A} = \{X \in \mathcal{H}_0 : \mathbb{E}_Q[u(X)] \geq u(c)\}$ is non-empty and satisfies (5.3) and (5.4) and is convex. Thus $\rho_{\mathcal{A}}$ is a convex risk measure. As an extension we can define the acceptable positions in terms of a whole class $\widetilde{\mathcal{M}}_1 \subseteq \mathcal{M}_1$, i.e.

$$\mathcal{A} = \bigcap_{Q \in \widetilde{\mathcal{M}}_1} \{X \in \mathcal{H}_0 : \mathbb{E}_Q[u(X)] \geq u(c_Q)\},$$

with constants c_Q such that $\max_{Q \in \widetilde{\mathcal{M}}_1} c_Q < \infty$.

Example 5.10 (Value at risk)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A position X is often considered to be acceptable if the probability of a loss is bounded by a given level $\alpha \in (0,1)$, i.e. $\mathbb{P}(X < 0) \leq \alpha$. The corresponding risk measure $\mathrm{V@R}_\alpha$, defined by

$$V@R_\alpha(X) = \inf\{m \in \mathbb{R} : \mathbb{P}(m + X < 0) \leq \alpha\}$$

is called **value at risk at level $\boldsymbol{\alpha}$**. Note that $\mathrm{V@R}_\alpha$ is well defined in the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$ of all rv's on $(\Omega, \mathcal{F})$ which are $\mathbb{P}$-a.s. finite. Clearly, $\mathrm{V@R}_\alpha$ is positively homogeneous. In general it is not convex, as we will see in the next section.
Example: If X is a Gaussian rv with expectation μ and variance σ^2 and Φ^{-1} is the inverse of the distribution function Φ of $N(0,1)$, then

$$V@R_\alpha(X) = -\mu + \Phi^{-1}(1-\alpha)\sigma \qquad \text{(Check !).}$$

5.2 Value at risk and expected shortfall

Definition 5.11. For $\alpha \in (0,1)$, an **$\boldsymbol{\alpha}$-quantile** of a rv X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is any real number q with the property

$$\mathbb{P}(X \leq q) \geq \alpha \qquad \text{and} \qquad \mathbb{P}(X < q) \leq \alpha.$$

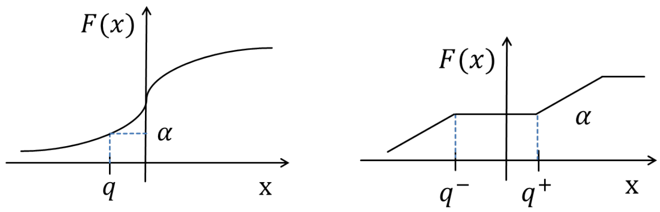
The set of all α-quantiles of X is an interval $[q_X^-(\alpha), q_X^+(\alpha)]$, where

$$q_X^-(t) = \sup\{z : \mathbb{P}(X < z) < t\} = \inf\{z : \mathbb{P}(X \leq z) \geq t\}$$

is the lower and

$$q_X^+(t) = \inf\{z : \mathbb{P}(X \leq z) > t\} = \sup\{z : \mathbb{P}(X < z) \leq t\}.$$

is the upper quantile function of X.



Definition 5.12. Let $\alpha \in (0,1)$, X rv on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then

$$V@R_\alpha(X) = -q_X^+(\alpha) = \inf\{m \in \mathbb{R} : \mathbb{P}(X + m < 0) \leq \alpha\}.$$

Then $\mathrm{V@R}_\alpha$ is a risk measure on L^0,

$$L^0 = \{X : X \text{ rv on } (\Omega, \mathcal{F}, \mathbb{P}),\ \mathbb{P}(|X| < \infty) = 1\}.$$

$\mathrm{V@R}_\alpha$ is positive homogeneous. The following example shows that $\mathrm{V@R}_\alpha$ is not a convex risk measure.

Example 5.13. Investment into two defaultable bonds, each with return $\widetilde{r} > r$, where r is the return of a riskless investment. The discounted net gain of an investment $z > 0$ in the i-th bond is given by

$$X_i = \begin{cases} -z, & \text{in case of default,} \\ \frac{z(\widetilde{r}-r)}{1+r}, & \text{otherwise.} \end{cases}$$

If a default of the first bond occurs with probability $p \leq \alpha$, then

$$\mathbb{P}\left(X_1 - \frac{z(\widetilde{r}-r)}{1+r} < 0\right) = \mathbb{P}(\text{first bond defaults}) = p \leq \alpha.$$

Hence,

$$V@R_\alpha(X_1) = -\frac{z(\widetilde{r}-r)}{1+r} < 0.$$

Diversifying the portfolio by investing the amount $\frac{z}{2}$ into each of the bonds leads to the position $Y = \frac{X_1+X_2}{2}$. Assume that the two bonds default independently of each other, each of them with probability p. If $\frac{\widetilde{r}-r}{1+r} < 1$, then

$$\mathbb{P}(Y < 0) = \mathbb{P}(\text{at least one of the bonds defaults}) = p(2-p).$$

Take for instance $p = 0,009$, $\alpha = 0,01$, then

$$p < \alpha < p(2-p).$$

Hence

$$V@R_\alpha(Y) = \frac{z}{2}\left(1 - \frac{\widetilde{r} - r}{1+r}\right) > 0.$$
$$\Rightarrow \; V@R_\alpha \text{ is } \textbf{not} \text{ a convex risk measure.}$$

The example shows that in fact, V@R$_\alpha$ may strongly discourage diversification, it penalizes drastically the probability that something goes wrong, without regarding the considerable reduction of the expected loss conditioned on the event of default. Thus (warning!) optimizing a portfolio w.r.t V@R$_\alpha$ may lead to a concentration of the portfolio in one single asset, which has a small default probability, but large losses in the case of default.

Definition 5.14. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The **expected shortfall** (or **average value at risk**) at level $\alpha \in (0, 1]$ of a position X is given by

$$ESF_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha V@R_\gamma(X)\, d\gamma, \qquad X \in L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Remark: $ESF_1(X) = -\int_0^1 q_X^+(t)\, dt = \mathbb{E}[-X]$.

Theorem 5.15. *For $\alpha \in (0, 1]$, ESF_α is a coherent risk measure. It has the representation*

$$ESF_\alpha(X) = \max_{Q \in \mathcal{M}_1(\alpha)} \mathbb{E}_Q[-X], \qquad X \in \mathcal{H}_0.$$

$\mathcal{M}_1(\alpha)$ is the set of probability measures on $(\Omega, \mathcal{F})$ with $Q \ll P$ and $\frac{dQ}{dP} \leq \frac{1}{\alpha}$.

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