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Research paper

Path generation with dwells in the optimum dimensional synthesis of Stephenson III six-bar mechanisms

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ABSTRACT

The optimal dimensional synthesis for Stephenson III six-bar linkages, planar and spherical, is presented. The requirement for dwells of the output link and the path generation task are also addressed. We propose a formulation for the synthesis in which the velocity of the output link is calculated by using dual functions. The condition for dwell is obtained through the minimization of the angular velocity, using the evolutionary algorithm known as Differential Evolution. We perform a numerical comparison between the use of the dual numbers and finite differences for computing the velocity and acceleration coefficients of the output link for the spherical case. The results exhibit the importance of having a precise method for computing derivatives.

1. Introduction

Dwells are generally produced by cams. A six-bar mechanism is utilized as an alternative to produce dwells and to generate complex trajectories of the coupler point. Furthermore, this mechanism is able to meet certain specific requirements that might not be satisfied by a four-bar mechanism. Solving the synthesis of planar dwell mechanisms is a very difficult task. One of the early reports about the synthesis of six-bar mechanisms to design function generators was conducted by McLarnan in 1963 [1]. He formulated the loop equations for three types of six-bar linkages: the Watt II, and the Stephenson II and III types. Kim et al. [2] derived the synthesis equations for Watt I and Stephenson I, II and III path generators.

The synthesis of six-bar linkages has been recently addressed by optimization techniques, especially using metaheuristic algorithms. For instance, the problem of the optimum synthesis of six-bar linkages for dwell mechanisms with prescribed timing and transmission angle constraints was considered by Shiakolas et al. [3]. The synthesis was accomplished by combining the evolutionary Differential Evolution (DE) algorithm [4–7] and the so called technique of Geometric Centroid of Precision Points. The planar six-bar mechanism synthesis presented in [3] was also addressed and slightly improved[1] by Cabrera et al. [8] by using MUMSA, a DE-based algorithm, and in [9] by using the so called IOA^{s-at} method, also based on DE and using the concept of self-adaptive control parameters [10,11]. A new metaheuristic algorithm, known as Cuckoo

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[1] The synthesized mechanisms were, for practical purposes, the same.

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Search [12,13], was used by Bulatović et al., for the optimal dimensional synthesis of a double dwell Stephenson III six-bar linkage [14].

Unlike previous literature, where the dwell mechanisms were synthesized using the approach of function generation, Jagannath and Bandyopadhyay [15] directly used the concept of velocity (instantaneous kinematics) to formulate an optimization problem, thus synthesizing a six-bar double dwell mechanism. To meet the condition for dwells, the velocity coefficients of the output link were set approximately to zero over two prescribed intervals of the input crank motion. This formulation was implemented in the MatLab computing language. Jagannath [16] used the Non-dominated Sorting Genetic Algorithm II (NSGA-II) for solving the optimal synthesis of planar double dwell mechanisms based on six-bar linkages with only rotary joints. Agarwal and Bandyopadhyay [17] conducted the optimum synthesis of the planar Stephenson III mechanism fulfilling dwells, using a multiobjective optimization problem by defining the so called dual-order structural error. This approach combines the zero (angular positions) and first (angular velocities) order function generation errors.

To the best of the authors' knowledge, there are not reports addressing the synthesis of Stephenson III six-bar dwell mechanisms for the spherical case. There are few reports that studied the spherical six-bar case which did not include dwells. In contrast, the spherical four-bar linkage has been broadly studied (some well known references are [18–22], as well as some recent studies [23–26]). For example, the kinematic analysis of spherical mechanisms with multi-loops is studied in [27,28] and the dimensional synthesis is studied in [29–33]. More specifically, in [27] the loop equations are formulated and solved for all the indecomposable spherical structures up to three loops. Formulations using both rotational matrices and quaternions were considered. The solution method is a modification of Sylvester's elimination method, leading directly to a numerical calculation via standard eigenvalue routines. In [28], analytical methods are used to obtain a closed-form input–output equation of the spherical Stephenson-III six-bar mechanism. The displacement analysis constraint equations were derived by using spherical analytical theory. Bezout's elimination method for the forward analysis and Sylvester's resultant elimination method for the inverse kinematics were applied to solve the constraint equations. Regarding the dimensional synthesis, the design of spherical Stephenson mechanisms for a gearless robotic pitch-roll wrist is discussed in [29]. The authors derived the input–output equations of the spherical Stephenson III six-bar mechanism by using the approach proposed by Wampler [27]. A geometric synthesis procedure for spherical (6, 7) linkages (the notation (n, j) refers to a linkage with n links and j hinged or sliding joints), often known as spherical six-bar linkages, was presented in [30]. The synthesis and analysis procedure was demonstrated by the design of a Stephenson IIb linkage to drive the leg movement of a walking machine. A design procedure was presented in [31] for a one degree of freedom spherical (8,10) eight-bar linkage that guides its end effector through five arbitrary task poses. A dimensional synthesis procedure was used in [32] to constrain links of a 3-RRR spherical parallel manipulator using two spherical RR chains to obtain a spherical ten-bar linkage. The result was a constrained spherical parallel manipulator that moves through five task positions with one degree of freedom. The spherical Watt I linkage for the eight orientations synthesis was recently solved using the SU(2) group in [33].

In the present study we propose a new method for the optimal synthesis of Stephenson III six-bar dwell mechanisms, which implements the idea introduced in [15]. The optimization process is conducted by using DE, and the calculation of the velocities by using dual numbers [34,35]. The effectiveness of DE as a good optimization algorithm has been already proved in many studies. A review of the method can be found in [36], and specifically for the area of mechanisms, some recent studies can be found in [37–42]. However, even though the algebra of dual numbers was developed by Clifford in the late nineteenth century [34], the use of dual numbers to calculate velocity and acceleration is not very common. Some studies related to this topic can be found in [42–45]. The proposed method is tested by synthesizing the Stephenson III six-bar dwell mechanisms in its planar and spherical versions. Regarding the planar case, we use the mechanism synthesized in [3]. For the spherical case, we use the coupler point curve of the Spherical 4R mechanism presented in [46]. In addition to the path generation task, we set dwells for the output link of an Stephenson III six-bar mechanism. We also present a comparison between the finite differences method and the use of dual numbers for computing the velocity and the acceleration for the spherical case. From this comparison, we can see that the computation of derivatives can produce considerable errors when using finite differences.

2. Method

2.1. Error functions

The path generation task with dwells can be conducted by solving an optimization problem. Concerning the path generation task, a mechanism with the closest coupler point curve to a set of points $\{\mathbf{p}_1, \mathbf{p}_2, \dots \mathbf{p}_n\}$ is targeted. For this purpose, an error function is constructed as follows

$$f_{\text{path}}(\mathbf{v}) = \sum_{i=1}^{n} \|\mathbf{p}_i - \mathbf{r}_{\text{gen}}(\mathbf{v})\|^2, \tag{1}$$

with $\mathbf{v}$ being the design variable vector and $\mathbf{r}_{\text{gen}}$ the parametric trajectory of the coupler point curve. Therefore, an optimization of Eq. (1) will produce a mechanism whose coupler point curve is the nearest (in the sense of the Euclidean norm $\|\cdot\|$) to the targeted path.

Most of the literature reports a function generation approach to obtain a mechanism providing dwells. This approach requires a constant angle for the link that will have the dwells (in the present paper, this link will be the output link).

However, it seems more natural to demand an angular velocity of zero for the output link.[2] This idea has been already implemented in [15]. We will define an error function for the angular velocities of the output link rather than constructing an error function for function generation.

The mechanisms considered in our study are mechanisms of one degree of freedom. We will call θ the input angle; it will be the independent variable (in addition to time), while θ_{out} denotes the angle of the output link chosen to have the dwells. Therefore, the angular velocity ω of the output link will be

$$\omega = \frac{\partial \theta_{\text{out}}}{\partial \theta}\dot{\theta} \tag{2}$$

The derivative with respect to time is represented by an overdot. Since $\dot{\theta}$ is in general arbitrary, a minimization of $\partial\theta_{\text{out}}/\partial\theta$ implies a minimization of ω. In complete analogy to the path generation task, the error function for dwells will be

$$f_{\text{dwells}}(\mathbf{v}) = \sum_j \left(\frac{\partial \theta^j_{\text{out}}}{\partial \theta}\right)^2, \tag{3}$$

where θ^j_{out} is the output angle when passing the point j where a dwell is desired.

The total objective function is constructed with the weighted sum of Eqs. (1) and (3)

$$f(\mathbf{v}) = k\, f_{\text{path}} + f_{\text{dwells}}, \tag{4}$$

where k is a quantity with units of the inverse of length squared $(1/[L]^2)$.

In some cases, the calculation of the involved velocities and accelerations to perform the synthesis process is not trivial. For the planar Stephenson III six-bar mechanism, a direct calculation of the derivative for the purpose of obtaining ω would produce an expression difficult to handle. The problem gets more complicated in the spherical case, where a direct calculation of the involved derivatives is impractical; and perhaps this is one reason why there are no reports addressing the synthesis of mechanisms with dwells other than the planar ones. A possible solution is to numerically calculate the involved derivatives using finite differences. However, relatively big truncations and/or cancellation errors will appear. Those errors are a serious problem in the optimum synthesis of mechanisms since in some cases they are bigger than the optimal value of the function to optimize. Another solution is to use dual numbers to obtain the involved derivatives. Below we present the essential ideas for calculating derivatives with dual numbers. For further reading, refer to [42], where some applications to the optimal synthesis of mechanisms calculating derivatives with dual numbers are presented.

2.2. Dual functions

A dual number $\hat{r}$ is a quantity of the form

$$\hat{r} = a + b\epsilon \tag{5}$$

with a and b real numbers, and ϵ is the dual unit ($\epsilon^2 = 0$). These numbers form a commutative and associative ring whose algebra allows the computation of derivatives.

Taking the special case $\hat{x} = x + \epsilon$, the dual version of an analytic function $f(x)$ will be

$$\hat{f}(\hat{x}) = f(x) + f'(x)\epsilon \tag{6}$$

$$\hat{f}(\hat{x}) = f_0 + f_1\epsilon. \tag{7}$$

This can be proved by substituting $\hat{x} = x + \epsilon$ in the Taylor expansion of f, which exists since f is analytic.

The generalization to a function $\hat{f}(\hat{g})$ is

$$\hat{f}(\hat{g}) = f_0(g_0) + f_1(g_0)g_1\epsilon. \tag{8}$$

Thus it is possible to calculate first order derivatives. The extension to second order derivatives can be done by defining the number

$$\tilde{r} = a + b\epsilon_1 + c\epsilon_2 \tag{9}$$

where a, b and c are real numbers, and ϵ_1 and ϵ_2 having the following multiplication table

$$\begin{array}{c|ccc} & 1 & \epsilon_1 & \epsilon_2 \\ \hline 1 & 1 & \epsilon_1 & \epsilon_2 \\ \epsilon_1 & \epsilon_1 & 2\epsilon_2 & 0 \\ \epsilon_2 & \epsilon_2 & 0 & 0. \end{array} \tag{10}$$

[2] Most of the time, in the function generation task, the goal of having a constant angle for a given link is only fulfilled approximately. However, small displacements do not necessarily imply small velocities.

As before, taking the special case $\tilde{x} = x + \epsilon_1 + 0\,\epsilon_2$, the dual version of a function $f(x)$ will be

$$\tilde{f}(\tilde{x}) = f(x) + f'(x)\,\epsilon_1 + f''(x)\,\epsilon_2 \tag{11}$$

$$\tilde{f}(\tilde{x}) = f_0 + f_1\,\epsilon_1 + f_2\,\epsilon_2 \tag{12}$$

The generalization to a function $\tilde{f}(\tilde{g})$ is

$$\tilde{f}(\tilde{g}) = f_0(g_0) + f_1(g_0)g_1\,\epsilon_1 + \left(f_2(g_0)g_1^2 + f_1(g_0)g_2\right)\epsilon_2. \tag{13}$$

The central idea for calculating the derivatives of $\theta_{\mathrm{out}}(\theta)$ is to change all the involved functions and operations, needed to obtain $\theta_{\mathrm{out}}(\theta)$, into their dual versions. For instance, assuming that $\widetilde{\mathrm{atan2}}$ is the dual version of the atan2 function, and that $\tilde{\mathbf{r}}(\tilde{\theta}_{\mathrm{out}})$ is the dual version of a two dimensional vector $\mathbf{r}(\theta_{\mathrm{out}})$, then the values of $[\theta_{\mathrm{out}}, \partial\theta_{\mathrm{out}}/\partial\theta, \partial^2\theta_{\mathrm{out}}/\partial\theta^2]$ are calculated by

$$\widetilde{\theta}_{\mathrm{out}}(\widetilde{\theta}) = \widetilde{\mathrm{atan2}}(\tilde{\mathbf{r}}_y(\widetilde{\theta}), \tilde{\mathbf{r}}_x(\widetilde{\theta})), \tag{14}$$

where

$$\widetilde{\theta} = \theta + \epsilon_1 \tag{15}$$

is the dual version of θ.

The dual number of Eq. (9) can be conveniently implemented in any programming language supporting function overloading. For instance, it can be defined in Fortran as

```
type, public :: dual2
 real(8) :: f0, f1, f2
end type dual2
```

By doing this, we can overload all the necessary functions and operators, thus avoiding cumbersome notation for coding dual functions and operators (we will however, still use a tilde for clarity in this article). For instance, we can use `sin(x)` to refer to both the real sine function if `x` is real or the dual version of the sine function if `x` is dual. As additional material to this article [47], we have coded the dual version of all the involved functions. Appendix shows an example of use.

At this point, we have an efficient and precise method for calculating Eq. (3) and, therefore, for minimizing Eq. (4). This minimization is conducted by using the evolutionary DE algorithm outlined below.

2.3. DE Algorithm

The DE algorithm has basically four operators and emulates natural evolution. An initial population of m individuals is subject to mutation, crossover, and selection. Let $\mathbf{X_0}$, $\mathbf{V}$, $\mathbf{U}$, and $\mathbf{S}$ be the population initialization operator, the mutation operator, the crossover operator, and the selection operator, respectively. The DE algorithm can be stated as:

Algorithm 1 DE Method.

$X \leftarrow \mathbf{X}_0(m, \mathcal{V})$
while (the stopping criterion has not been met) **do**
  $V \leftarrow \mathbf{V}(X, F)$
  $U \leftarrow \mathbf{U}(X, V, C_r)$
  $X \leftarrow \mathbf{S}(X, U, f)$
end while
$\mathbf{x}_{best} \leftarrow \mathbf{B}(X, f)$.

The last line in Algorithm 1 introduces the $\mathbf{B}$ operator which, after the complete evolutionary process, selects from X the individual ($\mathbf{x}_{best}$) that optimizes the objective function f. The mutation scale factor F and the crossover probability C_r are user-defined parameters. The search space is denoted by $\mathcal{V}$. The auxiliary variables X, V, U, $\mathbf{x}_{best}$ store the results of the defined operators. The explicit form of the DE operators can be found elsewhere in the literature, for instance in [48].

3. Planar Stephenson III six-bar mechanism

In this section we study the example presented in [3]. This example refers to the synthesis of a six-bar mechanism for 18 precision points, which is capable of generating dwells in $160^\circ \leqslant \theta_2 \leqslant 220^\circ$ and $-15^\circ \leqslant \theta_2 \leqslant 15^\circ$ ranges, with prescribed timing. The notation used is illustrated in Fig. 1, where the design variable vector, including the input angle θ_2, is defined as

$$\mathbf{v} = [\theta_2, x_0, y_0, r_1, r_2, r_3, r_4, r_5, r_6, r_{cx}, r_{cy}, r_1', \theta_0, \theta_1']. \tag{16}$$

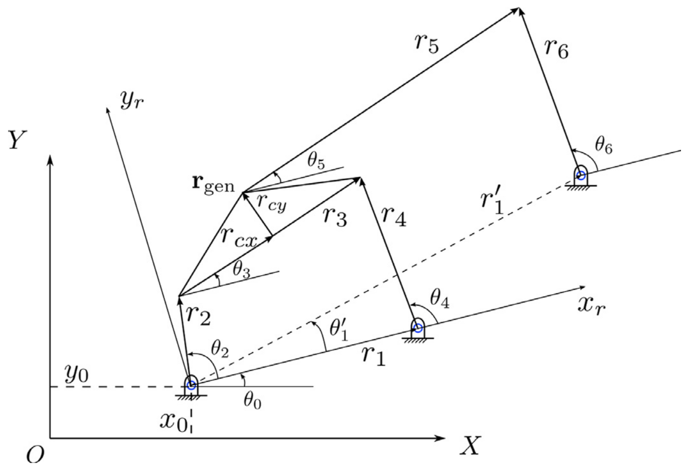

Fig. 1. Planar Stephenson III six-bar mechanism. Design variables.

Table 1
Precision points and dwells for the synthesis of the planar Stephenson III six-bar mechanism.

Precision Point	x-Coordinate	y-Coordinate	Prescribed timing $\delta\theta^i = (\theta_2^i - \theta_2^0)$	$\partial\theta_6/\partial\theta_2$
$\mathbf{p}_1$	−0.5424	2.3708	0°	0
$\mathbf{p}_2$	0.2202	2.9871	15°	0
$\mathbf{p}_3$	0.9761	3.4633	40°	–
$\mathbf{p}_4$	1.0618	3.6380	60°	–
$\mathbf{p}_5$	0.8835	3.7226	80°	–
$\mathbf{p}_6$	0.5629	3.7156	100°	–
$\mathbf{p}_7$	0.1744	3.6128	120°	–
$\mathbf{p}_8$	−0.2338	3.4206	140°	–
$\mathbf{p}_9$	−0.6315	3.1536	160°	0
$\mathbf{p}_{10}$	−1.0000	2.8284	180°	0
$\mathbf{p}_{11}$	−1.3251	2.4600	200°	0
$\mathbf{p}_{12}$	−1.5922	2.0622	220°	0
$\mathbf{p}_{13}$	−1.7844	1.6539	240°	–
$\mathbf{p}_{14}$	−1.8872	1.2654	260°	–
$\mathbf{p}_{15}$	−1.8942	0.9448	280°	–
$\mathbf{p}_{16}$	−1.8096	0.7665	300°	–
$\mathbf{p}_{17}$	−1.6349	0.8522	320°	–
$\mathbf{p}_{18}$	−1.1587	1.6081	345°	0

Table 1 summarizes the example. We have used $\theta_2 \to \theta_2^i$ to denote the input angle corresponding to point i. For instance, the input angle when passing trough point 7 will be $\theta_2^7 = \theta_2^0 + 120^\circ$, where θ_2^0 is the assembly input angle.

Defining

$$\begin{aligned} l_1 &= r_1/r_2, \\ l_2 &= r_1/r_3, \\ l_3 &= \left(r_4^2 - r_1^2 - r_2^2 - r_3^2\right)/(2\,r_2\,r_3), \\ k_a &= \cos\theta_2 - l_1 + l_2\,\cos\theta_2 + l_3, \\ k_b &= -2\,\sin\theta_2, \\ k_c &= l_1 + (l_2 - 1)\,\cos\theta_2 + l_3, \end{aligned} \tag{17}$$

the coordinates of the coupler point curve

$$\mathbf{r}_{\text{gen}}(\theta_2, x_0, y_0, r_1, r_2, r_3, r_4, r_{cx}, r_{cy}, \theta_0) = [P_x, P_y], \tag{18}$$

are given by

$$\begin{aligned} P_x &= x_0 + r_2\,\cos(\theta_2 + \theta_0) + r_{cx}\,\cos(\theta_3 + \theta_0) - r_{cy}\,\sin(\theta_3 + \theta_0) \\ P_y &= y_0 + r_2\,\sin(\theta_2 + \theta_0) + r_{cx}\,\sin(\theta_3 + \theta_0) + r_{cy}\,\cos(\theta_3 + \theta_0), \end{aligned} \tag{19}$$

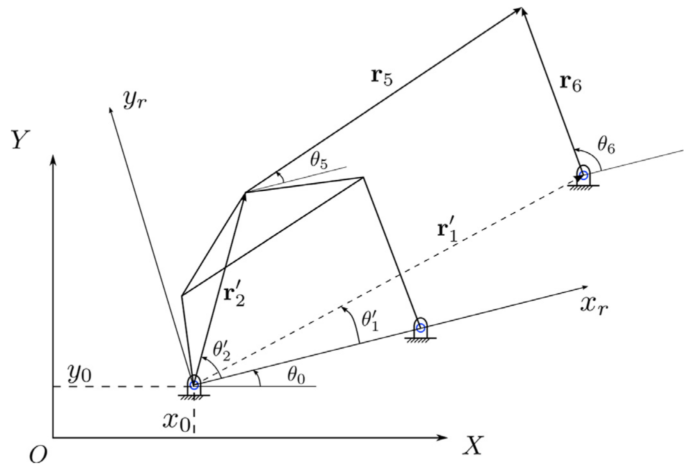

Fig. 2. Planar Stephenson III six-bar mechanism. Variables used to obtain the output θ_6.

where θ_3 is

$$\theta_3 = 2\,\text{atan2}\left(-k_b \pm \sqrt{k_b^2 - 4\ \ k_a\, k_c}, 2\,ka\right). \tag{20}$$

Therefore, we can write Eq. (1), for this particular case as:

$$f_{\text{path}} = \sum_{i=1}^{18} \left\| \mathbf{p}_i - \mathbf{r}_{\text{gen}}(\theta_2^i, x_0, y_0, r_1, r_2, r_3, r_4, r_{cx}, r_{cy}, \theta_0) \right\|^2. \tag{21}$$

The calculation of the output angle θ_6 is needed to construct Eq. (3). This angle can be determined once θ_5 is known. However, θ_5 can be calculated in an analogous way to θ_3 of the common four-bar mechanism—see Fig. 2, which illustrates a four-bar mechanism ($\mathbf{r}_1'$, $\mathbf{r}_2'$, $\mathbf{r}_5$, $\mathbf{r}_6$).

We can write all the vectors in the (x_r, y_r) reference system to calculate θ_6. Therefore, the $\mathbf{r}_2'$ vector can be constructed as

$$\mathbf{r}_2' = \mathbf{r}_{\text{gen}}(\theta_2, 0, 0, r_1, r_2, r_3, r_4, r_{cx}, r_{cy}, 0) \tag{22}$$

from where

$$\theta_2' = \text{atan2}\left(\mathbf{r}_{2y}',\ \mathbf{r}_{2x}'\right). \tag{23}$$

The norm of $\mathbf{r}_2'$ ($r_2' = \left\|\mathbf{r}_2'\right\|$) is a variable quantity. Defining

$$\begin{aligned}
L_1 &= r_1'/r_2' \\
L_2 &= r_1'/r_5 \\
L_3 &= \left(r_6^2 - r_1'^2 - r_2'^2 - r_5^2\right)/(2\,r_2'\,r_5) \\
K_A &= \cos(\theta_2' - \theta_1') - L_1 + L_2\cos(\theta_2' - \theta_1') + L_3 \\
K_B &= -2\sin(\theta_2' - \theta_1') \\
K_C &= L_1 + (L_2 - 1)\cos(\theta_2' - \theta_1') + L_3.
\end{aligned} \tag{24}$$

The angle θ_5 is

$$\theta_5 = 2\,\text{atan2}\left(-K_B \pm \sqrt{K_B^2 - 4\,K_A\,K_C},\ \ 2K_A\right) + \theta_1'. \tag{25}$$

From this we can construct the vectors:

$$\mathbf{r}_1' = [r_1'\cos\theta_1',\ \ r_1'\sin\theta_1'] \tag{26}$$

$$\mathbf{r}_5 = [r_5\cos\theta_5,\ \ r_5\sin\theta_5] \tag{27}$$

$$\mathbf{r}_6 = \mathbf{r}_2' + \mathbf{r}_5 - \mathbf{r}_1', \tag{28}$$

Table 2
Results and comparison with [3,9] for the Planar Stephenson III six-bar Mechanism.

Design vector	This work	Ref. [3]	Ref. [9]
θ_2^0	0.00000	−0.0079	6.280025
x_0	0.00000	0.0415	0.013322
y_0	0.00064	−0.0377	0.002008
r_1	1.82756	1.8065	1.810744
r_2	0.99984	0.9826	0.991753
r_3	1.99822	2.0177	1.987899
r_4	1.99772	2.0009	1.982751
r_5	4.68461	5.7769	5.825138
r_6	5.78898	2.5296	2.540682
r_{cx}	1.99852	2.0336	1.989389
r_{cy}	2.00066	2.0268	2.012227
r_1'	7.59627	5.2817	5.394427
θ_0	0.00016	0.0116	0.001105
θ_1'	0.41577	0.0054	0.009139
fob (Eq. (31) with $k=1/[L]^2$)	0.00025	0.0079	0.0086

from which

$$\theta_6(\theta_2, r_1, r_2, r_3, r_4, r_5, r_6, r_{cx}, r_{cy}, r_1', \theta_1') = \text{atan2}\big(\mathbf{r}_{6y}, \mathbf{r}_{6x}\big). \tag{29}$$

Therefore, Eq. (3) reads ($\theta_{\text{out}} = \theta_6$)

$$f_{\text{dwells}} = \sum_{j=\text{indices}} \left[\frac{\partial \theta_6}{\partial \theta_2}\left(\theta_2^j, r_1, r_2, r_3, r_4, r_5, r_6, r_{cx}, r_{cy}, r_1', \theta_1'\right) \right]^2, \tag{30}$$

where the indices are {18, 1, 2, 9, 10, 11, 12}.

Writing Eq. (29) in its dual form $\tilde{\theta}_6$, the derivative $\partial\theta_6/\partial\theta_2$ will be the second component of $\tilde{\theta}_6$. In general, writing any quantity in its dual form is enough to obtain its derivatives, being the main advantage of the dual number approach. Using Eqs. (21) and (30), the objective function, Eq. (4) becomes

$$f(\theta_2^0, x_0, y_0, r_1, r_2, r_3, r_4, r_5, r_6, r_{cx}, r_{cy}, r_1', \theta_0, \theta_1') = k\, f_{\text{path}} + f_{\text{dwells}}. \tag{31}$$

An optimization of Eq. (31) using the DE algorithm produces the parameters of the design variable vector illustrated in Table 2, for which an optimal value of 2.5×10^{-4} for the objective function was obtained ($f_{\text{path}} = 1.195 \times 10^{-6}$, $f_{\text{dwells}} = 2.490 \times 10^{-4}$). This mechanism was obtained by using $k = 1/[L]^2$ (no preference between the f_{path} and the f_{dwells} objective functions), a mutation scale factor of 1.0, a crossover probability of 0.9, a population of 60 individuals, and 5000 generations. In an Intel processor (i7-7700) CPU @ 3.6 GHz, 8 GB of memory and using the Intel Fortran compiler, the running time was of 3 seconds. Two optimizations were used: the first optimization served to refine the search space. For this first case, we used the search space

$$\mathbf{b}_L = [-\pi, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\pi, -\pi] \tag{32}$$

$$\mathbf{b}_U = [\pi, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, \pi, \pi], \tag{33}$$

with $\mathbf{b}_{L(U)}$ being the lower (upper) bound of the search space. Then, from the result of this first optimization, the search space is refined to

$$\mathbf{b}_L = \mathbf{0} \tag{34}$$

$$\mathbf{b}_U = [0.01, 0.01, 0.01, 6, 6, 6, 6, 6, 6, 6, 6, 8, 0.01, 0.5]. \tag{35}$$

The individuals whose values for the coupler point curve and the output angle are complex (not real) numbers were penalized. This was done for both the planar and spherical cases. It is interesting to note that the comparison with previous studies (Refs. [3,9]), shows practically the same Mechanism for the 4R closed chain (parameters r_1, r_2, r_3, r_4). Nevertheless, the parameters r_6 and r_1' are significantly different.

4. Spherical Stephenson III six-bar mechanism

There is a great similarity between the planar and spherical Stephenson III six-bar mechanisms. Defining the design parameters illustrated in Fig. 3, the similarity is manifest. In the spherical case, the angles are measured between the geodesics associated to the links. Moreover, it is common to take a sphere of radius one, so the lengths of the links, denoted by

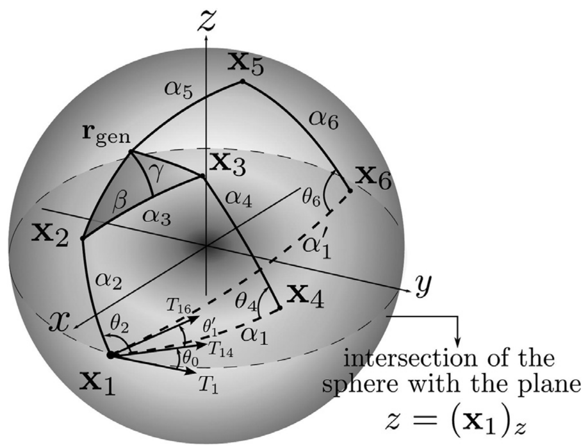

Fig. 3. Spherical Stephenson III six-bar mechanism. Design variables.

$\alpha_1, \ldots \alpha_6, \alpha'_1$, are given by the angles between the position vectors of their corresponding joints (denoted by $\mathbf{x}_1, \ldots, \mathbf{x}_6$). However, in our study, these lengths are chosen as parameters. The design variable vector, including the input angle θ_2, is given by

$$\mathbf{v} = [\theta_2, \eta_1, \phi_1, \theta_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha'_1, \beta, \gamma, \theta'_1], \tag{36}$$

where η_1 and ϕ_1 are the polar and azimuthal angles of the point $\mathbf{x}_1$. The other parameters are illustrated in Fig. 3. Therefore the parametric equation for the coupler point curve can be written as

$$\mathbf{r}_{\text{gen}}(\theta_2, \eta_1, \phi_1, \theta_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta, \gamma) = \mathbf{R}(\pi/2, \mathbf{r}_\beta)\,\mathbf{r}_{\beta\gamma}, \tag{37}$$

where $\mathbf{R}(\theta, \mathbf{u})$ is the active rotation matrix of angle θ about the unit vector $\mathbf{u}$, and

$$\begin{aligned}
\mathbf{r}_{\beta\gamma} &= \mathbf{R}(\beta+\gamma, \mathbf{n}_{23})\,\mathbf{r}_2\\
\mathbf{r}_\beta &= \mathbf{R}(\beta, \mathbf{n}_{23})\,\mathbf{r}_2\\
\mathbf{r}_2 &= \mathbf{R}(\theta_2, \mathbf{x}_1)\,\mathbf{r}_1\\
\mathbf{r}_1 &= \mathbf{R}(\alpha_2, \mathbf{n}_{14})\,\mathbf{x}_1\\
\mathbf{x}_1 &= \cos\phi_1\,\sin\eta_1\,\mathbf{i} + \sin\phi_1\,\sin\eta_1\,\mathbf{j} + \cos\eta_1\,\mathbf{k}\\
\mathbf{n}_{14} &= \frac{\mathbf{x}_1 \times \mathbf{T}_{14}}{\|\mathbf{x}_1 \times \mathbf{T}_{14}\|}\\
\mathbf{T}_{14} &= \mathbf{R}(\theta_0, \mathbf{x}_1)\,\mathbf{T}_1
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}_1 &= \mathbf{Gtangent}(\mathbf{x}_1, \mathbf{x}_{1\text{T}})\\
\mathbf{x}_{1\text{T}} &= \mathbf{R}(\sigma, \mathbf{k})\,\mathbf{x}_1\\
\mathbf{n}_{23} &= \frac{\mathbf{r}_2 \times \mathbf{r}_3}{\|\mathbf{r}_2 \times \mathbf{r}_3\|}\\
\mathbf{r}_3 &= \mathbf{R}(\theta_4, -\mathbf{x}_4)\,\mathbf{R}(\alpha_4, -\mathbf{n}_{14})\,\mathbf{x}_4\\
\mathbf{x}_4 &= \mathbf{R}(\alpha_1, \mathbf{n}_{14})\,\mathbf{x}_1,
\end{aligned}$$

with **i, j, k**, the unit vectors along the axis *x*, *y*, and *z*, respectively. The parameter σ corresponds to an arbitrary angle in (0, π). The function **Gtangent**(**v, w**) gives the tangent vector in **v** to the spherical geodesic from **v** to **w**. Its functional form is:

$$\mathbf{Gtangent}(\mathbf{v}, \mathbf{w}) = \frac{\mathbf{w} - (\mathbf{w}\cdot\mathbf{v})\mathbf{v}}{\|\mathbf{w} - (\mathbf{w}\cdot\mathbf{v})\mathbf{v}\|}. \tag{38}$$

The output angle θ_4 is

$$\theta_4 = 2\,\text{atan}\frac{-A \pm \sqrt{A^2 + B^2 - C^2}}{C - B}, \tag{39}$$

where

$$A = \sin\alpha_2\,\sin\alpha_4\,\sin\theta_2$$

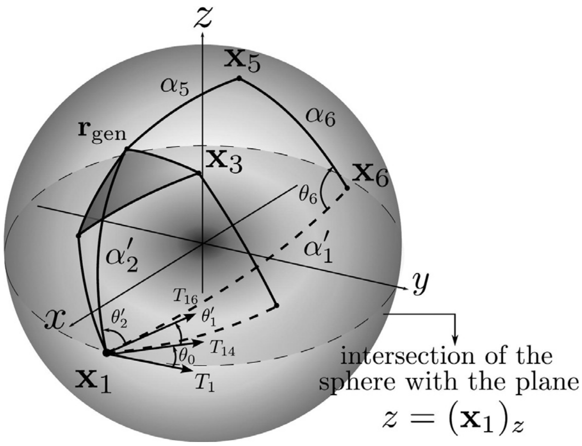

Fig. 4. Spherical Stephenson III six-bar mechanism. Variables used to obtain the Output θ_6.

$$B = \cos\alpha_2 \sin\alpha_4 \sin\alpha_1 - \sin\alpha_2 \sin\alpha_4 \cos\alpha_1 \cos\theta_2$$
$$C = \cos\alpha_2 \cos\alpha_4 \cos\alpha_1 + \sin\alpha_2 \cos\alpha_4 \sin\alpha_1 \cos\theta_2 - \cos\alpha_3.$$

Once we obtain the coupler point curve—Eq. (37), we can address the optimum path generation task by optimizing Eq. (1). An expression for the output angle θ_6 is required to address the dwells for the output link (α_6). From Fig. 4, and using an analogous procedure to the planar case, we have

$$\theta_6 = 2\,\mathrm{atan}\frac{-\mathscr{A} \pm \sqrt{\mathscr{A}^2 + \mathscr{B}^2 - \mathscr{C}^2}}{\mathscr{C} - \mathscr{B}}, \tag{40}$$

with

$$\mathscr{A} = \sin\alpha'_2 \sin\alpha_6 \sin\theta'_2$$
$$\mathscr{B} = \cos\alpha'_2 \sin\alpha_6 \sin\alpha'_1 - \sin\alpha'_2 \sin\alpha_6 \cos\alpha'_1 \cos\theta'_2$$
$$\mathscr{C} = \cos\alpha'_2 \cos\alpha_6 \cos\alpha'_1 + \sin\alpha'_2 \cos\alpha_6 \sin\alpha'_1 \cos\theta'_2 - \cos\alpha_5.$$

where

$$\cos\alpha'_2 = \mathbf{x}_1 \cdot \mathbf{r}_{\text{gen}}$$
$$\cos\theta'_2 = \mathbf{T}_{16} \cdot \mathbf{Gtangent}(\mathbf{x}_1, \mathbf{r}_{\text{gen}})$$
$$\mathbf{T}_{16} = \mathbf{R}(\theta_0 + \theta'_1, \mathbf{x}_1)\,\mathbf{T}_1.$$

5. Path generation of the spherical Stephenson III six-bar mechanism fulfilling dwells

It is interesting to apply the proposed method for synthesizing a spherical Stephenson III six-bar mechanism fulfilling dwells and addressing the path generation task. As an example, let us synthesize such a mechanism addressing the requirements of Table 3. These points were taken from [46] and correspond to the normalized (translated to a unit sphere) version of that presented in [49]. As can be seen, dwells for the mechanism ranging from the 21th to the 40th point are required from the 64 points of the path generation task.

Using Eq. (37) and the dual version of Eq. (40), which will give us θ_6, $\partial\theta_6/\partial\theta_2$ and $\partial^2\theta_6/\partial\theta_2^2$, we can construct Eq. (4). From this, we proceed to its optimization by using DE, thus obtaining the mechanism whose parameters are shown in Table 4. The objective function (the corresponding Eq. (4) for the spherical case) reaches a value of 2.251×10^{-3}. As before, the constant k has been taken as $1/[L]^2$. Figs. 5 and 6 illustrate the coupler point curve and the velocity coefficient of the output angle for the synthesized mechanism, respectively. The values for the components of the objective function were $f_{\text{path}} = 2.809 \times 10^{-4}$ and $f_{\text{dwells}} = 1.970 \times 10^{-3}$. These values were obtained in two steps. In the first step we used the search space:

$$\mathbf{b}_L = \mathbf{0} \tag{41}$$

$$\mathbf{b}_U = [2\pi, \pi, 2\pi, 2\pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi, 2\pi] \tag{42}$$

and then from the result of the first step, the search space was refined to

$$\mathbf{b}_L = [0.0, 1.0, 0.0, 0.0, 1.0, 0.0, 0.5, 1.0, 2.0, 1.0, 1.0, 0.1, 0.0, 4.5] \tag{43}$$

Table 3
Precision points and dwells for the synthesis of the spherical Stephenson III six-bar mechanism. The input angles are taken equally spaced ($|\theta_{i+1} - \theta_i| = 2\pi/64$).

Point number	Point	$\partial\theta_6/\partial\theta_2$	Point number	Point	$\partial\theta_6/\partial\theta_2$
1	(0.85737, −0.18481, 0.48037)	–	33	(0.7887, −0.60370, 0.11578)	0
2	(0.82985, −0.20167, 0.52030)	–	34	(0.80152, −0.59270, 0.07900)	0
3	(0.80241, −0.21996, 0.55478)	–	35	(0.81378, −0.57959, 0.04311)	0
4	(0.77567, −0.23967, 0.58389)	–	36	(0.82552, −0.56433, 0.00841)	0
5	(0.75011, −0.26056, 0.60785)	–	37	(0.83678, −0.54700, −0.02478)	0
6	(0.72607, −0.28244, 0.62693)	–	38	(0.84759, −0.52763, −0.05611)	0
7	(0.70381, −0.30515, 0.64152)	–	39	(0.85807, −0.50641, −0.08530)	0
8	(0.68352, −0.32833, 0.65193)	–	40	(0.86819, −0.48344, −0.11200)	0
9	(0.66533, −0.35185, 0.65844)	–	41	(0.87804, −0.45889, −0.13596)	–
10	(0.64933, −0.37537, 0.66144)	–	42	(0.88763, −0.43304, −0.15689)	–
11	(0.63559, −0.39867, 0.66115)	–	43	(0.89704, −0.40611, −0.17448)	–
12	(0.62415, −0.42159, 0.65781)	–	44	(0.90626, −0.37837, −0.18848)	–
13	(0.61504, −0.44389, 0.65167)	–	45	(0.91537, −0.35022, −0.19867)	–
14	(0.60833, −0.46541, 0.64293)	–	46	(0.92433, −0.32193, −0.20481)	–
15	(0.60396, −0.48596, 0.63170)	–	47	(0.93322, −0.29396, −0.20667)	–
16	(0.60196, −0.50548, 0.61819)	–	48	(0.94196, −0.26667, −0.20400)	–
17	(0.60230, −0.52381, 0.60241)	–	49	(0.95052, −0.24048, −0.19667)	–
18	(0.60485, −0.54085, 0.58448)	–	50	(0.95885, −0.21581, −0.18441)	–
19	(0.60959, −0.55656, 0.56448)	–	51	(0.96685, −0.19315, −0.16707)	–
20	(0.61637, −0.57081, 0.54244)	–	52	(0.97430, −0.17289, −0.14452)	–
21	(0.62500, −0.58363, 0.51844)	0	52	(0.98096, −0.15541, −0.11656)	–
22	(0.63530, −0.59489, 0.49248)	0	54	(0.98652, −0.14104, −0.08315)	–
23	(0.64700, −0.60456, 0.46467)	0	55	(0.99052, −0.13007, −0.04430)	–
24	(0.65989, −0.61259, 0.43507)	0	56	(0.99244, −0.12263, −0.00019)	–
25	(0.67370, −0.61896, 0.40378)	0	57	(0.99174, −0.11870, 0.04874)	–
26	(0.68811, −0.62363, 0.37093)	0	58	(0.98774, −0.11822, 0.10185)	–
27	(0.70293, −0.62652, 0.33674)	0	59	(0.98000, −0.12085, 0.15807)	–
28	(0.71789, −0.62759, 0.30133)	0	60	(0.96819, −0.12626, 0.21604)	–
29	(0.73274, −0.62678, 0.26500)	0	61	(0.95226, −0.13415, 0.27426)	–
30	(0.74737, −0.62404, 0.22800)	0	62	(0.93252, −0.14411, 0.33115)	–
31	(0.76167, −0.61933, 0.19059)	0	63	(0.90956, −0.15600, 0.38515)	–
32	(0.77544, −0.61256, 0.15307)	0	64	(0.88422, −0.16959, 0.43519)	–

Table 4
Parameters of optimal spherical Stephenson III six-bar mechanism.

θ_2^0	η_1	ϕ_1	θ_0	α_1	α_2	α_3
0.696256	1.584366	0.044949	0.065562	1.102375	0.397082	0.922304
α_4	α_5	α_6	α_1'	β	γ	θ_1'
1.322563	2.420675	1.564758	1.191126	0.368934	0.438782	4.769355

$$\mathbf{b}_U = [1.0, 2.0, 1.0, 1.0, 2.0, 1.0, 2.0, 2.0, 2.5, 3.0, 2.0, 1.0, 1.0, 5.0]. \tag{44}$$

We have penalized individuals for which $\alpha_2 > \min(\alpha_1, \alpha_3, \alpha_4)$. In both cases, the used parameters for the DE algorithm were a mutation scale factor of 1.0, a crossover probability of 0.9, a population size of 50 individuals, and a number of generations of 10000. The running time was of 4.5 minutes for the aforementioned computing characteristics.

5.1. Numerical derivatives using finite differences

A traditional computation of the output angle derivatives with respect to θ_2, by using finite differences can yield wrong results, and obstaculize the dwell synthesis process. This can be seen in Table 5, where a numerical comparison between the finite difference method and the use of dual numbers has been made. For the first and second order derivatives approximations, we use the formulae

$$f'(x) = \frac{f(x+h) - f(x)}{h} \tag{45}$$

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}, \tag{46}$$

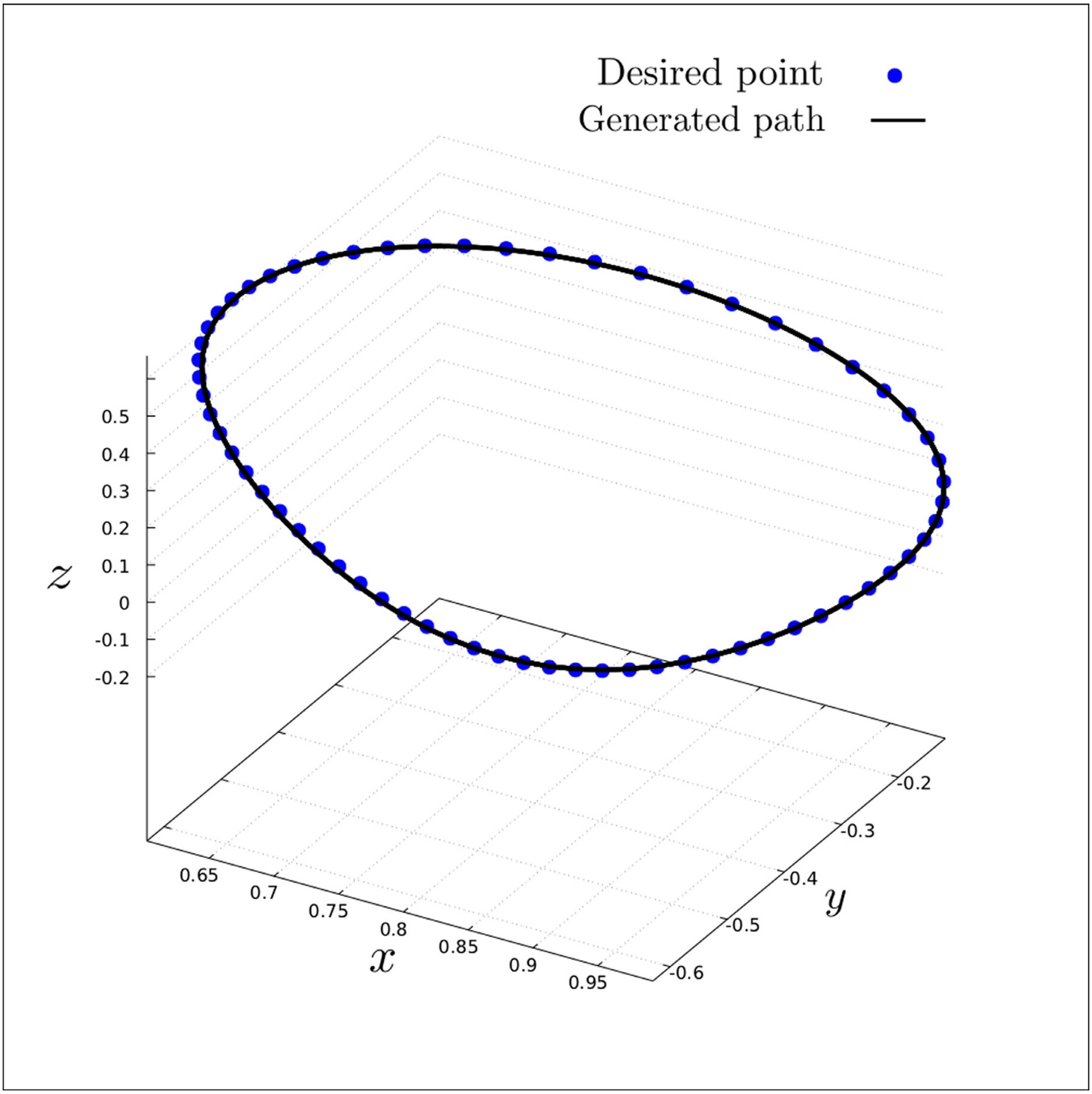

Fig. 5. Desired points and generated path.

Table 5
First and second order derivatives of the output angle θ_6 with respect to the input parameter θ_2. For finite differences we used $h = 1.0 \times 10^{-7}$.

θ_2	$\partial\theta_6/\partial\theta_2$	$\partial^2\theta_6/\partial\theta_2^2$	$\partial\theta_6/\partial\theta_2$ (differences)	$\partial^2\theta_6/\partial\theta_2^2$ (differences)
0.000000	0.267550	−0.683414	0.267550	−0.710543
0.628320	−0.179827	−0.606880	−0.179827	−0.532907
1.256640	−0.397260	−0.041020	−0.397260	0.044409
1.884960	−0.256329	0.388531	−0.256328	0.488498
2.513270	−0.047038	0.218298	−0.047038	0.222045

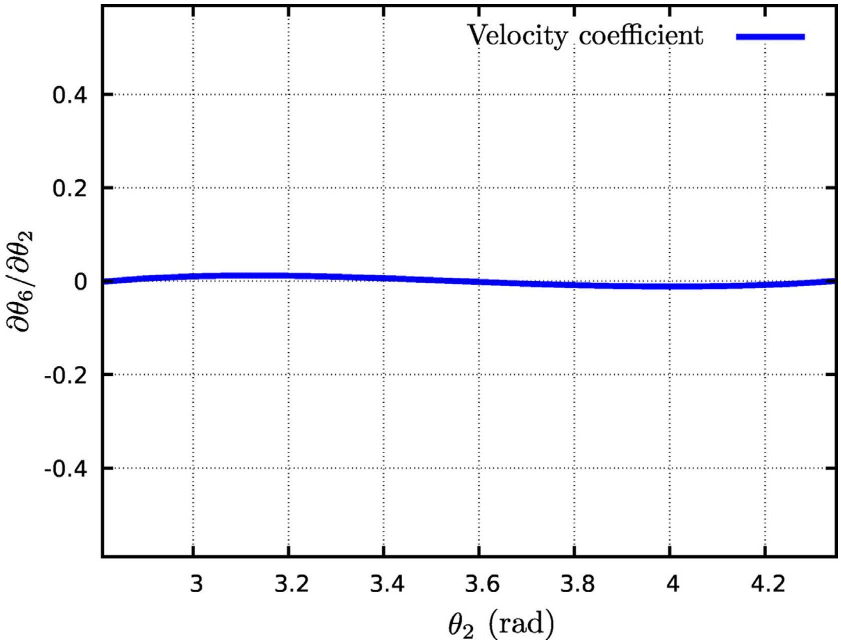

Fig. 6. Velocity coefficient for the output link for the Stepheson III 6R mechanism.

Table 6
First and second order derivatives of the output angle θ_6 with respect to the input parameter θ_2. For finite differences we used $h = 1.0 \times 10^{-8}$.

θ_2	$\partial\theta_6/\partial\theta_2$	$\partial^2\theta_6/\partial\theta_2^2$	$\partial\theta_6/\partial\theta_2$ (differences)	$\partial^2\theta_6/\partial\theta_2^2$ (differences)
0.000000	0.267550	−0.683414	0.267550	−4.440892
0.628320	−0.179827	−0.606880	−0.179827	−8.881784
1.256640	−0.397260	−0.041020	−0.397260	−4.440892
1.884960	−0.256329	0.388531	−0.256329	0.000000
2.513270	−0.047038	0.218298	−0.047038	−8.881784

where $h = 1.0 \times 10^{-7}$. A small value for h yields a cancellation of the numerator in Eq. (46) for some values of θ_2. For instance, taking $h = 1.0 \times 10^{-8}$, we obtain the values expressed in Table 6. As can be observed, the errors in computing the second order derivatives by using finite differences are considerable.

6. Conclusions

In this study, we have presented a method to solve the optimal dimensional synthesis for planar and spherical Stephenson III six-bar linkages. The path generation task and the requirement for dwells of the output link were also addressed. The method combines the Differential Evolution algorithm as optimization method, and the use of dual numbers for computing velocities and accelerations. With the dualization of the position synthesis equations, the higher order synthesis equations are automatically established. The difficulties of calculating the derivatives in an analytical or numerical way for complex mechanisms are solved by eliminating truncation and cancellation errors. The proposed method was successfully applied to synthesizing the Stephenson III six-bar dwell mechanisms in its planar and spherical versions. We consider that the proposed method is simple yet useful and can be a computationally effective tool for the higher-order kinematic synthesis, especially of space kinematic chains.

Appendix

Example of differentiation with dual numbers

The Fortran dual number implementation (file `dual2_mod.f90`) is available at [47]. The following operators and mathematical functions are overloaded:

(**), (*), (+), (-), (/), (==), (/=), `acos`, `acosh`, `asin`, `asinh`, `atan`, `atan2`, `atanh`, `sin`, `cos`, `cosh`, `erf`, `sinh`, `tan`, `tanh`, `exp`, `log`, `sqrt`, `abs`.

All the above functions and operators are coded as elemental functions, which means that they accept arrays as arguments (element-wise operations). Below a simple example of use.

```
include ''dual2_mod.f90''
program test
 use dual2_mod
 implicit none

 type(dual2) :: x, fx

 x = dual2(1.1d0,1.0d0,0.0d0)

 fx = sin(x*x)*x**3

 !derivatives: 0,1,2
 print*,''f(1.1)'',fx
 print*,''f'(1.1)'',fx
 print*,''f''(1.1)'',fx
end program test
```

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.inffus.2018.01.011.

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