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CLASSICAL SEVENTH-, SIXTH-, AND FIFTH-ORDER RUNGE-KUTTA-NYSTRÖM FORMULAS WITH STEPSIZE CONTROL FOR GENERAL SECOND-ORDER DIFFERENTIAL EQUATIONS

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16. ABSTRACT

Runge-Kutta-Nyström formulas of the seventh, sixth, and fifth order are derived for the general second-order (vector) differential equation $\ddot{x} = f(t, x, \dot{x})$. These formulas include a stepsize control procedure, based on a complete coverage of the leading term of the local truncation error in x. The formulas require no more evaluations per step than our earlier Runge-Kutta formulas for $\dot{x} = f(t,x)$. Since we have not to convert the second-order differential equations into twice as many first-order differential equations, the new formulas can be expected to be time-saving compared with the Runge-Kutta formulas for first-order differential equations. Two examples are presented. With results being of the same accuracy, in these examples the new Runge-Kutta-Nyström formulas save from 25- to 60-percent computer time compared with our earlier Runge-Kutta formulas for first-order differential equations.

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CLASSICAL SEVENTH-, SIXTH-, AND FIFTH-ORDER RUNGE-KUTTA-NYSTRÖM FORMULAS WITH STEPSIZE CONTROL FOR GENERAL SECOND-ORDER DIFFERENTIAL EQUATIONS

INTRODUCTION

1. In two earlier reports [1], [2] this author derived Runge-Kutta-Nyström formulas for a special class of second-order (vector) differential equations

$$\ddot{x} = f(t, x) \tag{1}$$

which do not contain the first derivative $\dot{x}$ on the right-hand side. In this report we will derive Runge-Kutta-Nyström formulas for general second-order (vector) differential equations:

$$\ddot{x} = f(t, x, \dot{x}) \tag{2}$$

2. Similar to the Runge-Kutta-Nyström formulas of report [1] and the Runge-Kutta formulas of report [3], the formulas of this report include an automatic stepsize control based on a complete coverage of the leading term of the local truncation error in x. This coverage is achieved by one additional evaluation of the differential equations. Each of our Runge-Kutta-Nyström formulas represents a pair of integration formulas for x which differ from one another by the one additional evaluation of the differential equations. The orders of these two formulas differ by 1. Therefore, the difference of the formulas represents an approximation of the leading term of the local truncation error in x for the lower-order formula. By requiring that this difference remain between preset limits, an automatic stepsize control for the lower-order formula can be established.

3. The formulas for $\dot{x}$ are of the same order as the lower-order formulas for x. There is no automatic error control with respect to the formulas for $\dot{x}$. Such a control would require a considerable additional effort, and such formulas would require more evaluations per integration step than our Runge-Kutta formulas of report [3] for first-order differential equations.

However, when deriving the Runge-Kutta-Nyström formulas of this report, we have also considered the error terms in $\dot{x}$ and we have selected the coefficients of our formulas in such a way as to keep the error terms in $\dot{x}$ as small as possible.

SECTION I. THE EQUATIONS OF CONDITIONS FOR RUNGE-KUTTA-NYSTRÖM FORMULAS

4. The derivation of the equations of condition for our Runge-Kutta-Nyström formulas, as explained in this section, is based on a procedure of D. Sarafyan [4]. Sarafyan's method is extended to second-order differential equations to yield the equations of condition for the coefficients of Runge-Kutta-Nyström formulas.

5. In the following we explain in detail the procedure for a fourth-order Runge-Kutta-Nyström formula.

Let the evaluations for the Runge-Kutta-Nyström formula for (2) be

$$\left.\begin{aligned}
f_0 &= f(t_0, x_0, \dot{x}_0) \\
f_1 &= f\left(t_0 + \alpha_1 h,\ x_0 + \dot{x}_0\alpha_1 h + \frac{1}{2} f_0\alpha_1^2 h^2,\ \dot{x}_0 + f_0\alpha_1 h\right) \\
f_2 &= f\Big[t_0 + \alpha_2 h,\ x_0 + \dot{x}_0\alpha_2 h + \frac{1}{2} f_0\alpha_2^2 h^2 + \gamma_{21}(f_1 - f_0)h^2, \\
&\qquad \dot{x}_0 + f_0\alpha_2 h + \beta_{21}(f_1 - f_0)h\Big] \\
f_3 &= f\Big[t_0 + \alpha_3 h,\ x_0 + \dot{x}_0\alpha_3 h + \frac{1}{2} f_0\alpha_3^2 h^2 + \gamma_{31}(f_1 - f_0)h^2 + \gamma_{32}(f_2 - f_0)h^2, \\
&\qquad \dot{x}_0 + f_0\alpha_3 h + \beta_{31}(f_1 - f_0)h + \beta_{32}(f_2 - f_0)h\Big] \\
&\ \ \vdots
\end{aligned}\right\} \qquad (3)$$

with $t_0, x_0, \dot{x}_0$ being the initial conditions for the integration step, h the stepsize and the α's, β's and γ's the Runge-Kutta-Nyström coefficients that we want to find.

The evaluations (3) lead to the Runge-Kutta-Nyström formulas:

$$\left.\begin{aligned} x(t_0 + h) &= x_0 + \dot{x}_0 h + (c_0 f_0 + c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4 + c_5 f_5) h^2 + 0(h^6) \\ \dot{x}(t_0 + h) &= \dot{x}_0 + (\dot{c}_0 f_0 + \dot{c}_1 f_1 + \dot{c}_2 f_2 + \dot{c}_3 f_3 + \dot{c}_4 f_4 + \dot{c}_5 f_5) h + 0(h^5) \end{aligned}\right\} \quad (4)$$

with weight factors c_κ and $\dot{c}_\kappa$ that we also want to find.

6. To determine these unknown coefficients, we expand the solution $x(t)$ of (2) into a Taylor series at $t = t_0$:

$$\left.\begin{aligned} x(t) &= x_0 + \dot{x}_0 (t - t_0) + A_2 (t - t_0)^2 + A_3 (t - t_0)^3 + A_4 (t - t_0)^4 \\ &\quad + A_5 (t - t_0)^5 + 0(t - t_0)^6 \end{aligned}\right\} \quad (5)$$

with the abbreviation:

$$A_\kappa = \frac{1}{\kappa!}\left(\frac{d^\kappa x}{dt^\kappa}\right)_0 = \frac{1}{\kappa!} x_0^{(\kappa)} \qquad (\kappa = 2, 3, \ldots) \quad . \qquad (6)$$

Inserting (5) into (2) and setting $t - t_0 = h$ yields:

$$\left.\begin{aligned} & f_0 + 6A_3 h + 12 A_4 h^2 + 20 A_5 h^3 + \ldots \\ & \quad = f(t_0 + h,\ x_0 + \dot{x}_0 h + A_2 h^2 + A_3 h^3 + A_4 h^4 + A_5 h^5 + \ldots, \\ & \qquad \dot{x}_0 + 2A_2 h + 3A_3 h^2 + 4A_4 h^3 + 5A_5 h^4 + \ldots) = R \end{aligned}\right\} \quad . \qquad (7)$$

For the sake of briefness, we denoted the right-hand side of (7) by R. Taylor expansion of R at

$$(0) \ : \ (t_0 + h,\ x_0 + \dot{x}_0 h,\ \dot{x}_0 + 2A_2 h) \qquad (8)$$

leads to

$$R = f_{(0)} + \left(\frac{\partial f}{\partial x}\right)_{(0)} \left(A_2 h^2 + A_3 h^3\right)$$
$$+ \left(\frac{\partial f}{\partial \dot{x}}\right)_{(0)} \left(3A_3 h^2 + 4A_4 h^3\right) + 0(h^4) \quad . \tag{9}$$

An expansion at

$$0 \ : \ (t_0, x_0, \dot{x}_0) \tag{10}$$

leads to

$$\left.\begin{aligned} \left(\frac{\partial f}{\partial x}\right)_{(0)} &= \left(\frac{\partial f}{\partial x}\right)_0 + \left[D\left(\frac{\partial f}{\partial x}\right)\right]_0 h + 0(h^2) \\ \left(\frac{\partial f}{\partial \dot{x}}\right)_{(0)} &= \left(\frac{\partial f}{\partial \dot{x}}\right)_0 + \left[D\left(\frac{\partial f}{\partial \dot{x}}\right)\right]_0 h + 0(h^2) \end{aligned}\right\} \quad , \tag{11}$$

introducing the operator:

$$[D(\ )]_0 = \left[\frac{\partial}{\partial t}(\ ) + \frac{\partial}{\partial x}(\ )\dot{x} + \frac{\partial}{\partial \dot{x}}(\ )f\right]_0 \quad . \tag{12}$$

We insert (9) and (11) into (7) and replace h by $\alpha_\kappa h$:

$$\left.\begin{aligned} & f(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0\alpha_\kappa h,\ \dot{x}_0 + f_0\alpha_\kappa h) = f_0 + 6A_3\alpha_\kappa h + 12A_4\alpha_\kappa^2 h^2 + 20A_5\alpha_\kappa^3 h^3 \\ & -\left(\frac{\partial f}{\partial x}\right)_0 \left(A_2\alpha_\kappa^2 h^2 + A_3\alpha_\kappa^3 h^3\right) - \left[D\left(\frac{\partial f}{\partial x}\right)\right]_0 A_2\alpha_\kappa^3 h^3 \\ & -\left(\frac{\partial f}{\partial \dot{x}}\right)_0 \left(3A_3\alpha_\kappa^2 h^2 + 4A_4\alpha_\kappa^3 h^3\right) - 3\left[D\left(\frac{\partial f}{\partial \dot{x}}\right)\right]_0 A_3\alpha_\kappa^3 h^3 \end{aligned}\right\} \tag{13}$$

Because of (4) the functions f_κ in (3) must be correct up to the third power in h. We set

$$\left.\begin{aligned} f_\kappa = f\Big[t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0\alpha_\kappa h + (q_{0\kappa} + q_{1\kappa}h)h^2, \\ \dot{x}_0 + f_0\alpha_\kappa h + (p_{0\kappa} + p_{1\kappa}h)h^2\Big] \end{aligned}\right\} \tag{14}$$

with constants $q_{0\kappa}$, $q_{1\kappa}$, $p_{0\kappa}$, $p_{1\kappa}$ that will be determined as follows somewhat later.

By Taylor expansion of f_κ at

$$\langle 0\rangle \ :\ (t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0\alpha_\kappa h,\ \dot{x}_0 + f_0\alpha_\kappa h) \tag{15}$$

we obtain

$$\left.\begin{aligned} f_\kappa = (f_\kappa)_{\langle 0\rangle} + \left(\frac{\partial f_\kappa}{\partial x}\right)_{\langle 0\rangle}\left(q_{0\kappa}h^2 + q_{1\kappa}h^3\right) \\ + \left(\frac{\partial f_\kappa}{\partial \dot{x}}\right)_{\langle 0\rangle}\left(p_{0\kappa}h^2 + p_{1\kappa}h^3\right) + 0(h^4) \end{aligned}\right\}. \tag{16}$$

We now insert (13) and (11) into (16) after having replaced h in (11) by $\alpha_\kappa h$:

$$\left.\begin{aligned} f_\kappa = f_0 + 6A_3\alpha_\kappa h + 12A_4\alpha_\kappa^{\,2}h^2 + 20A_5\alpha_\kappa^{\,3}h^3 \\ + \left(\frac{\partial f}{\partial x}\right)_0\left(q_{0\kappa} - A_2\alpha_\kappa^2\right)h^2 + \left(\frac{\partial f}{\partial x}\right)_0\left(q_{1\kappa} - A_3\alpha_\kappa^3\right)h^3 \\ + \left[D\left(\frac{\partial f}{\partial x}\right)\right]_0\alpha_\kappa\left(q_{0\kappa} - A_2\alpha_\kappa^2\right)h^3 \end{aligned}\right\} \tag{17}$$

$$
\left.\begin{aligned}
&+\left(\frac{\partial f}{\partial \dot{x}}\right)_0 \left(p_{0\kappa} - 3A_3\alpha_\kappa^2\right)h^2 + \left(\frac{\partial f}{\partial \dot{x}}\right)_0 \left(p_{1\kappa} - 4A_4\alpha_\kappa^3\right)h^3 \\
&+\left[D\left(\frac{\partial f}{\partial \dot{x}}\right)\right]_0 \alpha_\kappa \left(p_{0\kappa} - 3A_3\alpha_\kappa^2\right)h^3 \\
&+ 0(h^4)
\end{aligned}\right\} \qquad \begin{matrix}(17)\\(\text{con.})\end{matrix}
$$

Equation (17) is called the generating formula since the introduction of (17) into (4) generates the equations of condition for the Runge-Kutta-Nyström coefficients.

7. For $\kappa = 1$ we find from (14) and the second equation (3)

$$
q_{01} = \frac{1}{2} f_0\alpha_1^2 , \quad q_{11} = 0 , \quad p_{01} = 0 , \quad p_{11} = 0 \tag{18}
$$

and from (17)

$$
\left.\begin{aligned}
f_1 = {} & f_0 + 6A_3\alpha_1 h + 12A_4\alpha_1^2 h^2 + 20A_5\alpha_1^3 h^3 - \left(\frac{\partial f}{\partial x}\right)_0 A_3\alpha_1^3 h^3 \\
& - 3\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_3\alpha_1^2 h^2 - 4\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_4\alpha_1^3 h^3 - 3\left[D\left(\frac{\partial f}{\partial \dot{x}}\right)\right]_0 A_3\alpha_1^3 h^3 \\
& + 0(h^4)
\end{aligned}\right\} \tag{19}
$$

For $\kappa = 2$ we find

$$
\left.\begin{aligned}
q_{02} &= \frac{1}{2} f_0\alpha_2^2, \; q_{12} = 6A_3\gamma_{21}\alpha_1 \\
p_{02} &= 6A_3\beta_{21}\alpha_1, \; p_{12} = 12A_4\beta_{21}\alpha_1^2 - 3\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_3\beta_{21}\alpha_1^2
\end{aligned}\right\} \tag{20}
$$

We introduce (20) into (17) and omit all terms that have already corresponding terms in (19):

$$f_2 = \ldots + 6\left(\frac{\partial f}{\partial x}\right)_0 A_3\gamma_{21}\alpha_1 h^3 + 6\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_3\beta_{21}\alpha_1 h^2$$

$$+ 12\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_4\beta_{21}\alpha_1^2 h^3 - 3\left(\frac{\partial f}{\partial \dot{x}}\right)_0^2 A_3\beta_{21}\alpha_1^2 h^3 \qquad (21)$$

$$+ 6\left[D\left(\frac{\partial f}{\partial \dot{x}}\right)\right]_0 A_3\alpha_2\beta_{21}\alpha_1 h^3 + 0(h^4) \quad .$$

Continuing, we find for $\kappa = 3$:

$$q_{03} = \frac{1}{2} f_0\alpha_3^2 \; , \quad q_{13} = 6A_3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) \; , \quad p_{03} = 6A_3(\beta_{31}\alpha_1 + \beta_{32}\alpha_2)$$

$$p_{13} = 12A_4(\beta_{31}\alpha_1^2 + \beta_{32}\alpha_2^2) - 3\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_3(\beta_{31}\alpha_1^2 + \beta_{32}\alpha_2^2) + 6\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_3\beta_{32}\beta_{21}\alpha_1 \qquad (22)$$

and again omitting all terms that have already corresponding terms in (19) or (21):

$$f_3 = \ldots + 6\left(\frac{\partial f}{\partial \dot{x}}\right)_0^2 A_3\beta_{32}\beta_{21}\alpha_1 h^3 + 0(h^4) \quad . \qquad (23)$$

No more additional terms are obtained for $\kappa = 4$ and $\kappa = 5$.

8. The equations of condition for the Runge-Kutta-Nyström coefficients are obtained by inserting the expansions (5) and (19), (21), (23), ... into the Runge-Kutta-Nyström formulas (4).

From the first equation (4) we find

$$
\left.
\begin{aligned}
& x_0 + \dot{x}_0 h + A_2 h^2 + A_3 h^3 + A_4 h^4 + A_5 h^5 \\
& = x_0 + \dot{x}_0 h + (c_0 + \ldots + c_5) f_0 h^2 + 6A_3(c_1\alpha_1 + \ldots + c_5\alpha_5) h^3 \\
& + 12A_4(c_1\alpha_1^2 + \ldots + c_5\alpha_5^2) h^4 + 20A_5(c_1\alpha_1^3 + \ldots + c_5\alpha_5^3) h^5 \\
& - \left(\frac{\partial f}{\partial x}\right)_0 A_3(c_1\alpha_1^3 + \ldots + c_5\alpha_5^3) h^5 - 3\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_3(c_1\alpha_1^2 + \ldots + c_5\alpha_5^2) h^4 \\
& - 4\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_4(c_1\alpha_1^3 + \ldots + c_5\alpha_5^3) h^5 - 3\left[D\left(\frac{\partial f}{\partial \dot{x}}\right)\right]_0 A_3(c_1\alpha_1^3 + \ldots + c_5\alpha_5^3) h^5 \\
& + 6\left(\frac{\partial f}{\partial x}\right)_0 A_3[c_2\gamma_{21}\alpha_1 + \ldots + c_5(\gamma_{51}\alpha_1 + \gamma_{52}\alpha_2 + \gamma_{53}\alpha_3 + \gamma_{54}\alpha_4)] \cdot h^5 \\
& + 6\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_3[c_2\beta_{21}\alpha_1 + \ldots + c_5(\beta_{51}\alpha_1 + \beta_{52}\alpha_2 + \beta_{53}\alpha_3 + \beta_{54}\alpha_4)] \cdot h^4 \\
& + 12\left(\frac{\partial f}{\partial \dot{x}}\right)_0 A_4[c_2 \cdot \beta_{21}\alpha_1^2 + \ldots + c_5(\beta_{51}\alpha_1^2 + \beta_{52}\alpha_2^2 + \beta_{53}\alpha_3^2 + \beta_{54}\alpha_4^2)] h^5 \\
& - 3\left(\frac{\partial f}{\partial \dot{x}}\right)_0^2 A_3[c_2 \cdot \beta_{21}\alpha_1^2 + \ldots + c_5(\beta_{51}\alpha_1^2 + \beta_{52}\alpha_2^2 + \beta_{53}\alpha_3^2 + \beta_{54}\alpha_4^2)] h^5 \\
& + 6\left[D\left(\frac{\partial f}{\partial \dot{x}}\right)\right]_0 A_3[c_2\alpha_2 \cdot \beta_{21}\alpha_1 + \ldots + c_5\alpha_5(\beta_{51}\alpha_1 + \beta_{52}\alpha_2 + \beta_{53}\alpha_3 + \beta_{54}\alpha_4)] h^5 \\
& + 6\left(\frac{\partial f}{\partial \dot{x}}\right)_0^2 A_3\{c_3 \cdot \beta_{32} \cdot \beta_{21}\alpha_1 + \ldots + c_5[\beta_{52} \cdot \beta_{21}\alpha_1 + \beta_{53}(\beta_{31}\alpha_1 + \beta_{32}\alpha_2) + \beta_{54}(\beta_{41}\alpha_1 + \beta_{42}\alpha_2 + \beta_{43}\alpha_3)]\} h^5 \quad .
\end{aligned}
\right\} \qquad (24)
$$

Equating corresponding terms in (24) yields

$$
\left.
\begin{aligned}
& c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = \frac{1}{2} \\
& c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5 = \frac{1}{6} \\
& c_1\alpha_1^2 + c_2\alpha_2^2 + c_3\alpha_3^2 + c_4\alpha_4^2 + c_5\alpha_5^2 = \frac{1}{12} \\
& c_1\alpha_1^3 + c_2\alpha_2^3 + c_3\alpha_3^3 + c_4\alpha_4^3 + c_5\alpha_5^3 = \frac{1}{20}
\end{aligned}
\right\} \qquad (25)
$$

and

$$
\left.
\begin{aligned}
& c_2\gamma_{21}\alpha_1 + c_3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) + c_4(\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3) + c_5(\gamma_{51}\alpha_1 + \gamma_{52}\alpha_2 + \gamma_{53}\alpha_3 + \gamma_{54}\alpha_4) = \frac{1}{120} \\
& c_2\beta_{21}\alpha_1 + c_3(\beta_{31}\alpha_1 + \beta_{32}\alpha_2) + c_4(\beta_{41}\alpha_1 + \beta_{42}\alpha_2 + \beta_{43}\alpha_3) + c_5(\beta_{51}\alpha_1 + \beta_{52}\alpha_2 + \beta_{53}\alpha_3 + \beta_{54}\alpha_4) = \frac{1}{24} \\
& c_2\beta_{21}\alpha_1^2 + c_3(\beta_{31}\alpha_1^2 + \beta_{32}\alpha_2^2) + c_4(\beta_{41}\alpha_1^2 + \beta_{42}\alpha_2^2 + \beta_{43}\alpha_3^2) + c_5(\beta_{51}\alpha_1^2 + \beta_{52}\alpha_2^2 + \beta_{53}\alpha_3^2 + \beta_{54}\alpha_4^2) = \frac{1}{60} \\
& c_2\alpha_2 \cdot \beta_{21}\alpha_1 + c_3\alpha_3(\beta_{31}\alpha_1 + \beta_{32}\alpha_2) + c_4\alpha_4(\beta_{41}\alpha_1 + \beta_{42}\alpha_2 + \beta_{43}\alpha_3) + c_5\alpha_5(\beta_{51}\alpha_1 + \beta_{52}\alpha_2 + \beta_{53}\alpha_3 + \beta_{54}\alpha_4) = \frac{1}{40} \\
& c_3\beta_{32}\beta_{21}\alpha_1 + c_4[\beta_{42} \cdot \beta_{21}\alpha_1 + \beta_{43}(\beta_{31}\alpha_1 + \beta_{32}\alpha_2)] \\
& \qquad + c_5[\beta_{52} \cdot \beta_{21}\alpha_1 + \beta_{53}(\beta_{31}\alpha_1 + \beta_{32}\alpha_2) + \beta_{54}(\beta_{41}\alpha_1 + \beta_{42}\alpha_2 + \beta_{43}\alpha_3)] = \frac{1}{120}
\end{aligned}
\right\} \qquad (26)
$$

Similar equations of condition are obtained from the second equation (4). Equations (25) and (26) are listed as the first nine equations of Table 1.[1] The left-hand part of Table 1 represents the equations of condition for x; the right-hand part represents those for $\dot{x}$. For the right-hand part the weight factors c_κ have to be replaced by $\dot{c}_\kappa$ as indicated at the top of the table.

9. The procedure described in this section can be extended to cover higher-order terms in the Taylor expansions. The resulting equations of condition up to the ninth order for x and the eighth order for $\dot{x}$ are listed in Table 1. Their derivation is naturally somewhat more involved than in the case of a fourth-order formula, but it follows along the same lines and is rather straightforward.

To shorten the equations we introduced in Table 1, the abbreviations

$$\left.\begin{aligned} \beta_{\kappa 1}\alpha_1^\lambda + \beta_{\kappa 2}\alpha_2^\lambda + \dots + \beta_{\kappa,\kappa-1}\alpha_{\kappa-1}^\lambda &= P_{\kappa\lambda} \\ \gamma_{\kappa 1}\alpha_1^\lambda + \gamma_{\kappa 2}\alpha_2^\lambda + \dots + \gamma_{\kappa,\kappa-1}\alpha_{\kappa-1}^\lambda &= Q_{\kappa\lambda} \end{aligned}\right\} . \tag{27}$$

SECTION II. SEVENTH-ORDER FORMULA RKN-G-7(8)[2]

10. We shall present in the following a seventh-order formula based on thirteen evaluations, a fourteenth evaluation being taken over as first evaluation for the next step.

Let the evaluations be

$$\left.\begin{aligned} f_0 &= f(t_0, x_0, \dot{x}_0) \\ f_\kappa &= f\left(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0\alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa=1} \gamma_{\kappa\lambda} \cdot f_\lambda,\ \dot{x}_0 + h \cdot \sum_{\lambda=0}^{\kappa-1} \beta_{\kappa\lambda} \cdot f_\lambda\right) \\ &\quad (\kappa = 1, 2, 3, \dots, 13) \end{aligned}\right\} \tag{28}$$

1. All tables are at the end of this report.
2. We insert in the names of the formulas of this report the letter G, to indicate that these formulas hold for the general differential equation (2) in contrast to the formulas without G of [1] and [2], which hold for the special differential equation (1).

and the Runge-Kutta-Nyström formulas

$$
\left.
\begin{aligned}
x &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{12} c_\kappa f_\kappa + 0(h^8) \\
\hat{x} &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{13} \hat{c}_\kappa f_\kappa + 0(h^9) \\
\dot{x} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{12} \dot{c}_\kappa f_\kappa + 0(h^8) \quad .
\end{aligned}
\right\} \qquad (29)
$$

The first formula (29) is a seventh-order formula for x; the second formula is an eighth-order formula for x. The difference $x - \hat{x}$ will represent a first approximation of the local truncation error for x and will be used as stepsize control. The third formula (29) is a seventh-order formula for $\dot{x}$.

11. Similar to our previous reports [1], [2], [3], we make a number of assumptions for the Runge-Kutta-Nyström coefficients that will reduce the number of equations of Table 1 to such an extent that we can handle the remaining problem with relative ease.

Let us assume

$$
\left.
\begin{aligned}
&\hat{c}_1 = c_1 = 0, \ \ldots, \ \hat{c}_6 = c_6 = 0, \ \hat{c}_7 = c_7, \ \ldots, \ \hat{c}_{11} = c_{11}, \ \hat{c}_{12} = 0, \ \hat{c}_{13} = c_{12} \\
&\dot{c}_1 = 0, \ \ldots, \ \dot{c}_6 = 0 \ ; \quad \alpha_{12} = \alpha_{13} = 1
\end{aligned}
\right\} \qquad (30)
$$

$$
\left.
\begin{aligned}
&\beta_{31} = \beta_{41} = \ldots = \beta_{131} = 0 && \gamma_{31} = \gamma_{41} = \ldots = \gamma_{131} = 0 \\
&\beta_{52} = \beta_{62} = \ldots = \beta_{132} = 0 && \gamma_{62} = \gamma_{72} = \ldots = \gamma_{132} = 0 \\
&\beta_{73} = \beta_{83} = \ldots = \beta_{133} = 0 && \beta_{13,\lambda} = \dot{c}_\lambda \ (\lambda = 0, 1, 2, \ldots, 12) \\
&\beta_{74} = \beta_{84} = \ldots = \beta_{134} = 0 && \gamma_{13,\lambda} = c_\lambda \ (\lambda = 0, 1, 1, \ldots, 12)
\end{aligned}
\right\} \qquad (31)
$$

.

The assumptions (30) mean that only the last two weight factors of the first two formulas (29) differ. The last two weight factors in these two formulas are simply exchanged: c_{12}, 0 is replaced by 0, c_{12}. Therefore, we have to compute only one set of weight factors for these two formulas (29).

The assumptions (31) are necessary in connection with the assumptions of No. 12 and No. 13 to reduce the equations of Table 1.

12. Let us further assume

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_{\lambda} = \frac{1}{2}\alpha_{\kappa}^{2} \quad (\kappa = 2, 3, \ldots, 13) \quad . \tag{32}$$

As one can easily verify, assumptions (32) eliminate a large number of equations of Table 1 by converting them into other equations of this table.

In the following we list the equations of Table 1 that are eliminated by the assumptions (32)

IV:2
V:2,5
VI:2,7,8,9,10,13
VII:2,7,9,11,12,13,14,17,20,25,26,27,28,29,30,31,34
VIII:2,7,9,11,14,17,19,20,21,22,25,28,33,34,35,40,41,46,48,
50,51,52,53,54,55,56,57,58,59,60,61,62,65,66,67,70,71,
72,75,78,83,84,85,86,88,90,91,92

Since the ninth-order equations (IX) of Table 1 enter the computation only as eighth-order equations for $\dot{x}$, we will consider these evaluations when dealing with the local truncation error terms in $\dot{x}$.

We next assume

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_{\lambda}^{2} = \frac{1}{3}\alpha_{\kappa}^{3} \quad (\kappa = 2, 3, \ldots, 13) \quad , \tag{33}$$

thereby eliminating the following equations of Table 1:

V:4
VI:4, 12
VII:4, 16, 19, 22, 33
VIII:4, 15, 18, 24, 27, 30, 37, 43,
64, 69, 74, 77, 80, 94

The assumptions

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_{\lambda}^{3} = \frac{1}{4}\alpha_{\kappa}^{4} \quad (\kappa = 5, 6, \ldots, 13) \tag{34}$$

eliminate the following equations:

VI:6
VII:6, 24
VIII:6, 32, 45

and the assumptions

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_{\lambda}^{4} = \frac{1}{5}\alpha_{\kappa}^{5} \quad (\kappa = 7, 8, \ldots, 13) \tag{35}$$

the equations:

VII:10
VIII:10

Finally we assume

$$\left.\begin{aligned} c_7\beta_{75} + c_8\beta_{85} + c_9\beta_{95} + c_{10}\beta_{105} + c_{11}\beta_{115} + c_{12}\beta_{135} &= 0 \\ \dot{c}_7\beta_{75} + \dot{c}_8\beta_{85} + \dot{c}_9\beta_{95} + \dot{c}_{10}\beta_{105} + \dot{c}_{11}\beta_{115} + \dot{c}_{12}\beta_{125} &= 0 \end{aligned}\right\} \tag{36}$$

and

$$\left.\begin{aligned} c_7\beta_{76} + c_8\beta_{86} + c_9\beta_{96} + c_{10}\beta_{106} + c_{11}\beta_{116} + c_{12}\beta_{136} = 0 \\ \dot{c}_7\beta_{76} + \dot{c}_8\beta_{86} + \dot{c}_9\beta_{96} + \dot{c}_{10}\beta_{106} + \dot{c}_{11}\beta_{116} + \dot{c}_{12}\beta_{126} = 0 \end{aligned}\right\} , \qquad (37)$$

thereby eliminating equations VIII:49, 82, 95 from Table 1.

13. We make similar assumptions for the coefficients $\gamma_{\kappa\lambda}$:

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_{\lambda} = \frac{1}{6}\,\alpha_{\kappa}^{3} \qquad (\kappa = 2, 3, \ldots, 13) \qquad (38)$$

eliminating

V:3
VI:3, 11
VII:3, 15, 18, 21, 32
VIII:3, 13, 23, 26, 29, 36, 42,
63, 68, 73, 76, 79, 93 ;

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_{\lambda}^{2} = \frac{1}{12}\,\alpha_{\kappa}^{4} \qquad (\kappa = 5, 6, \ldots, 13) \qquad (39)$$

eliminating

VI:5
VII:5, 23
VIII:5, 31, 44 ;

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_{\lambda}^{3} = \frac{1}{20}\,\alpha_{\kappa}^{5} \qquad (\kappa = 5, 6, \ldots, 13) \qquad (40)$$

eliminating

VII:8
VIII:8, 47, 89

and

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda^4 = \frac{1}{30}\,\alpha_\kappa^6 \qquad (\kappa = 7, 8, \ldots, 13) \tag{41}$$

that eliminates VIII:12 from Table 1.

Finally we assume

$$\left.\begin{aligned} c_7\gamma_{73} + c_8\gamma_{83} + c_9\gamma_{93} + c_{10}\gamma_{103} + c_{11}\gamma_{113} + c_{12}\gamma_{133} &= 0 \\ \dot{c}_7\gamma_{73} + \dot{c}_8\gamma_{83} + \dot{c}_9\gamma_{93} + \dot{c}_{10}\gamma_{103} + \dot{c}_{11}\gamma_{113} + \dot{c}_{12}\gamma_{123} &= 0 \end{aligned}\right\} \tag{42}$$

and

$$\left.\begin{aligned} c_7\gamma_{74} + c_8\gamma_{84} + c_9\gamma_{94} + c_{10}\gamma_{104} + c_{11}\gamma_{114} + c_{12}\gamma_{134} &= 0 \\ \dot{c}_7\gamma_{74} + \dot{c}_8\gamma_{84} + \dot{c}_9\gamma_{94} + \dot{c}_{10}\gamma_{104} + \dot{c}_{11}\gamma_{114} + \dot{c}_{12}\gamma_{124} &= 0 \end{aligned}\right\} \tag{43}$$

thereby eliminating VIII:38, 39, 81, 87 from Table 1.

14. The assumptions of No. 11, 12, and 13 reduce the eighth- and lower-order equations of Table 1 to the following equations:

(II, 1), (III, 1), (IV, 1), (V, 1), (VI, 1), (VII, 1), (VIII, 1), (VIII, 16)

These equations have now to be solved together with the assumptions (30) through (43).

From equations

$$\left.\begin{aligned} &(\mathrm{III},1) \qquad c_7\alpha_7 + c_8\alpha_8 + c_9\alpha_9 + c_{10}\alpha_{10} + c_{11}\alpha_{11} + c_{12} = \frac{1}{6} \\ &(\mathrm{IV},1) \qquad c_7\alpha_7^2 + c_8\alpha_8^2 + c_9\alpha_9^2 + c_{10}\alpha_{10}^2 + c_{11}\alpha_{11}^2 + c_{12} = \frac{1}{12} \\ &(\mathrm{V},1) \qquad c_7\alpha_7^3 + c_8\alpha_8^3 + c_9\alpha_9^3 + c_{10}\alpha_{10}^3 + c_{11}\alpha_{11}^3 + c_{12} = \frac{1}{20} \end{aligned}\right\} \tag{44}$$

$$
\left.\begin{array}{ll}
(\mathrm{VI},1) & c_7\alpha_7^4 + c_8\alpha_8^4 + c_9\alpha_9^4 + c_{10}\alpha_{10}^4 + c_{11}\alpha_{11}^4 + c_{12} = \frac{1}{30} \\[2ex]
(\mathrm{VII},1) & c_7\alpha_7^5 + c_8\alpha_8^5 + c_9\alpha_9^5 + c_{10}\alpha_{10}^5 + c_{11}\alpha_{11}^5 + c_{12} = \frac{1}{42} \\[2ex]
(\mathrm{VIII},1) & c_7\alpha_7^6 + c_8\alpha_8^6 + c_9\alpha_9^6 + c_{10}\alpha_{10}^6 + c_{11}\alpha_{11}^6 + c_{12} = \frac{1}{56}
\end{array}\right\} \quad \begin{array}{c}(44)\\(\text{con.})\end{array}
$$

we find the weight factors c_7, c_8, c_9, c_{10}, c_{11}, c_{12} as functions of α_7, α_8, α_9, α_{10}, α_{11}.

From the right-hand sides of Table 1 we obtain for the corresponding equations

$$
\left.\begin{array}{ll}
(\mathrm{III},1)^{\bullet} & \dot{c}_7\alpha_7 + \dot{c}_8\alpha_8 + \dot{c}_9\alpha_9 + \dot{c}_{10}\alpha_{10} + \dot{c}_{11}\alpha_{11} + \dot{c}_{12} = \frac{1}{2} \\[2ex]
(\mathrm{IV},1)^{\bullet} & \dot{c}_7\alpha_7^2 + \dot{c}_8\alpha_8^2 + \dot{c}_9\alpha_9^2 + \dot{c}_{10}\alpha_{10}^2 + \dot{c}_{11}\alpha_{11}^2 + \dot{c}_{12} = \frac{1}{3} \\[2ex]
(\mathrm{V},1)^{\bullet} & \dot{c}_7\alpha_7^3 + \dot{c}_8\alpha_8^3 + \dot{c}_9\alpha_9^3 + \dot{c}_{10}\alpha_{10}^3 + \dot{c}_{11}\alpha_{11}^3 + \dot{c}_{12} = \frac{1}{4} \\[2ex]
(\mathrm{VI},1)^{\bullet} & \dot{c}_7\alpha_7^4 + \dot{c}_8\alpha_8^4 + \dot{c}_9\alpha_9^4 + \dot{c}_{10}\alpha_{10}^4 + \dot{c}_{11}\alpha_{11}^4 + \dot{c}_{12} = \frac{1}{5} \\[2ex]
(\mathrm{VII},1)^{\bullet} & \dot{c}_7\alpha_7^5 + \dot{c}_8\alpha_8^5 + \dot{c}_9\alpha_9^5 + \dot{c}_{10}\alpha_{10}^5 + \dot{c}_{11}\alpha_{11}^5 + \dot{c}_{12} = \frac{1}{6} \\[2ex]
(\mathrm{VIII},1)^{\bullet} & \dot{c}_7\alpha_7^6 + \dot{c}_8\alpha_8^6 + \dot{c}_9\alpha_9^6 + \dot{c}_{10}\alpha_{10}^6 + \dot{c}_{11}\alpha_{11}^6 + \dot{c}_{12} = \frac{1}{7}
\end{array}\right\} \quad (45)
$$

the weight factors $\dot{c}_7$, $\dot{c}_8$, $\dot{c}_9$, $\dot{c}_{10}$, $\dot{c}_{11}$, $\dot{c}_{12}$ as functions of α_7, α_8, α_9, α_{10}, α_{11}. Equation (II, 1) yields c_0, or $\dot{c}_0$ when written as $(\mathrm{II},1)^{\bullet}$.

15. We still have to satisfy equation (VIII, 16) and the assumptions of No. 12 and No. 13.

From $(32)_{\kappa=2}$ and $(33)_{\kappa=2}$ we obtain

$$
\beta_{21} = \frac{3}{4}\alpha_2 \tag{46}
$$

and as restrictive condition

$$\alpha_1 = \frac{2}{3}\,\alpha_2 \quad . \tag{47}$$

In the same way we obtain from $(32)_{\kappa=3}$ and $(33)_{\kappa=3}$

$$\beta_{32} = \frac{3}{4}\,\alpha_3 \quad . \tag{48}$$

and

$$\alpha_2 = \frac{2}{3}\,\alpha_3 \quad . \tag{49}$$

Equations $(32)_{\kappa=4}$ and $(33)_{\kappa=4}$ yield

$$\beta_{42} = \frac{1}{6}\,\alpha_4^2\,\frac{3\alpha_3 - 2\alpha_4}{\alpha_2(\alpha_3 - \alpha_2)} \tag{50}$$

and a corresponding formula for β_{43}, obtained from β_{42} by exchanging α_2 and α_3.

Equations $(32)_{\kappa=5}$, $(33)_{\kappa=5}$, $(34)_{\kappa=5}$ yield

$$\beta_{53} = \frac{1}{6}\,\alpha_5^2\,\frac{3\alpha_4 - 2\alpha_5}{\alpha_3(\alpha_4 - \alpha_3)} \quad , \tag{51}$$

a corresponding formula for β_{54} and the restrictive condition

$$\alpha_3 = \frac{1}{2}\,\alpha_5\,\frac{4\alpha_4 - 3\alpha_5}{3\alpha_4 - 2\alpha_5} \quad . \tag{52}$$

Equations $(32)_{\kappa=6}$, $(33)_{\kappa=6}$, $(34)_{\kappa=6}$ give

$$\beta_{63} = \frac{1}{12}\,\alpha_6^2\,\frac{6\alpha_4\alpha_5 - 4(\alpha_4 + \alpha_5)\alpha_6 + 3\alpha_6^2}{\alpha_3(\alpha_4 - \alpha_3)(\alpha_5 - \alpha_3)} \tag{53}$$

and corresponding formulas for β_{64} and β_{65}.

Equations $(32)_{\kappa=7}$, $(33)_{\kappa=7}$, $(34)_{\kappa=7}$, and $(35)_{\kappa=7}$ represent four linear equations for the two coefficients β_{75}, β_{76} which lead to

$$\beta_{75} = \frac{1}{6} \alpha_7^2 \frac{3\alpha_6 - 2\alpha_7}{\alpha_5(\alpha_6 - \alpha_5)} \quad , \tag{54}$$

a corresponding formula for β_{76} and to the two restrictive conditions

$$\alpha_5 = \frac{1}{10}(6 - \sqrt{6})\alpha_7 \quad , \quad \alpha_6 = \frac{1}{10}(6 + \sqrt{6})\alpha_7 \quad . \tag{55}$$

Equations $(32)_{\kappa=8}$, $(33)_{\kappa=8}$, $(34)_{\kappa=8}$, and $(35)_{\kappa=8}$ lead to

$$\beta_{85} = \frac{1}{12} \alpha_8^2 \frac{6\alpha_6\alpha_7 - 4(\alpha_6 + \alpha_7)\alpha_8 + 3\alpha_8^2}{\alpha_5(\alpha_6 - \alpha_5)(\alpha_7 - \alpha_5)} \quad , \tag{56}$$

corresponding formulas for β_{86}, β_{87} and to the restrictive condition

$$\alpha_8 = \frac{3}{4}\alpha_7 \quad . \tag{57}$$

The four equations $(32)_{\kappa=9}$, $(33)_{\kappa=9}$, $(34)_{\kappa=9}$, and $(35)_{\kappa=9}$ yield

$$\beta_{95} = \frac{1}{60} \alpha_9^2 \frac{30\alpha_6\alpha_7\alpha_8 - 20(\alpha_6\alpha_7 + \alpha_6\alpha_8 + \alpha_7\alpha_8)\alpha_9 + 15(\alpha_6 + \alpha_7 + \alpha_8)\alpha_9^2 - 12\alpha_9^3}{\alpha_5(\alpha_6 - \alpha_5)(\alpha_7 - \alpha_5)(\alpha_8 - \alpha_5)} \tag{58}$$

and corresponding formulas for β_{96}, β_{97}, and β_{98} .

Putting

$$\beta_{105} = 0 \quad , \tag{59}$$

we obtain from $(32)_{\kappa=10}$, $(33)_{\kappa=10}$, $(34)_{\kappa=10}$, and $(35)_{\kappa=10}$:

$$\beta_{106} = \frac{1}{60}\,\alpha_{10}^2\,\frac{30\alpha_7\alpha_8\alpha_9 - 20(\alpha_7\alpha_8 + \alpha_7\alpha_9 + \alpha_8\alpha_9)\alpha_{10} + 15(\alpha_7 + \alpha_8 + \alpha_9)\alpha_{10}^2 - 12\alpha_{10}^3}{\alpha_6(\alpha_7 - \alpha_6)(\alpha_8 - \alpha_6)(\alpha_9 - \alpha_6)} \tag{60}$$

and corresponding formulas for β_{107}, β_{108}, β_{109}.

From the first equation (36) and the first equation (37) we obtain

$$\left.\begin{aligned} \beta_{115} &= -\frac{1}{c_{11}}\,(c_7\beta_{75} + c_8\beta_{85} + c_9\beta_{95}) \\ \beta_{116} &= -\frac{1}{c_{11}}\,(c_7\beta_{76} + c_8\beta_{86} + c_9\beta_{96} + c_{10}\beta_{106}) \end{aligned}\right\} \tag{61}$$

and from equations $(32)_{\kappa=11}$, $(33)_{\kappa=11}$, $(34)_{\kappa=11}$, and $(35)_{\kappa=11}$:

$$\left.\begin{aligned} \beta_{117} = {} & \frac{1}{60}\,\alpha_{11}^2\,\frac{30\alpha_8\alpha_9\alpha_{10} - 20(\alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})\alpha_{11} + 15(\alpha_8 + \alpha_9 + \alpha_{10})\alpha_{11}^2 - 12\alpha_{11}^3}{\alpha_7(\alpha_8 - \alpha_7)(\alpha_9 - \alpha_7)(\alpha_{10} - \alpha_7)} \\ & - \beta_{115}\cdot\frac{\alpha_5(\alpha_8 - \alpha_5)(\alpha_9 - \alpha_5)(\alpha_{10} - \alpha_5)}{\alpha_7(\alpha_8 - \alpha_7)(\alpha_9 - \alpha_7)(\alpha_{10} - \alpha_7)} - \beta_{116}\cdot\frac{\alpha_6(\alpha_8 - \alpha_6)(\alpha_9 - \alpha_6)(\alpha_{10} - \alpha_6)}{\alpha_7(\alpha_8 - \alpha_7)(\alpha_9 - \alpha_7)(\alpha_{10} - \alpha_7)} \end{aligned}\right\} \tag{62}$$

and corresponding formulas for β_{118}, β_{119}, and β_{1110}.

Equations $(32)_{\kappa=11}$, $(33)_{\kappa=11}$, $(34)_{\kappa=11}$, $(35)_{\kappa=11}$, and (VIII, 16) can be considered as five linear equations for the four coefficients β_{117}, β_{118}, β_{119}, and β_{1110}. Therefore, a restrictive condition for the α's can be derived from these five linear equations:

$$\alpha_{11} = \frac{N(\alpha_{11})}{D(\alpha_{11})} \tag{63}$$

with

$$N(\alpha_{11}) = 70\alpha_7\alpha_8\alpha_9\alpha_{10} - 42(\alpha_7\alpha_8\alpha_9 + \alpha_7\alpha_8\alpha_{10} + \alpha_7\alpha_9\alpha_{10} + \alpha_8\alpha_9\alpha_{10})$$

$$+ 28(\alpha_7\alpha_8 + \alpha_7\alpha_9 + \alpha_7\alpha_{10} + \alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})$$

$$- 20(\alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) + 15$$

$$- 840c_{12}(1 - \alpha_7)(1 - \alpha_8)(1 - \alpha_9)(1 - \alpha_{10})$$

$$D(\alpha_{11}) = 140\alpha_7\alpha_8\alpha_9\alpha_{10} - 70(\alpha_7\alpha_8\alpha_9 + \alpha_7\alpha_8\alpha_{10} + \alpha_7\alpha_9\alpha_{10} + \alpha_8\alpha_9\alpha_{10})$$

$$+ 42(\alpha_7\alpha_8 + \alpha_7\alpha_9 + \alpha_7\alpha_{10} + \alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})$$

$$- 28(\alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) + 20$$

$$- 840c_{12}(1 - \alpha_7)(1 - \alpha_8)(1 - \alpha_9)(1 - \alpha_{10})$$

and

$$c_{12} = \frac{1}{28} \cdot \frac{N(c_{12})}{D(c_{12})} \tag{64}$$

with

$$N(c_{12}) = 70\alpha_7\alpha_8\alpha_9\alpha_{10} - 28(\alpha_7\alpha_8\alpha_9 + \alpha_7\alpha_8\alpha_{10} + \alpha_7\alpha_9\alpha_{10} + \alpha_8\alpha_9\alpha_{10})$$

$$+ 14(\alpha_7\alpha_8 + \alpha_7\alpha_9 + \alpha_7\alpha_{10} + \alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})$$

$$- 8(\alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) + 5$$

$$D(c_{12}) = 30\alpha_7\alpha_8\alpha_9\alpha_{10} - 20(\alpha_7\alpha_8\alpha_9 + \alpha_7\alpha_8\alpha_{10} + \alpha_7\alpha_9\alpha_{10} + \alpha_8\alpha_9\alpha_{10})$$

$$+ 15(\alpha_7\alpha_8 + \alpha_7\alpha_9 + \alpha_7\alpha_{10} + \alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})$$

$$- 12(\alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) + 10 \quad .$$

The second equations (36) and (37) yield

$$\left.\begin{aligned}\beta_{125} &= -\frac{1}{\dot{c}_{12}}\left(\dot{c}_7\beta_{75} + \dot{c}_8\beta_{85} + \dot{c}_9\beta_{95} + \dot{c}_{11}\beta_{115}\right)\\ \beta_{126} &= -\frac{1}{\dot{c}_{12}}\left(\dot{c}_7\beta_{76} + \dot{c}_8\beta_{86} + \dot{c}_9\beta_{96} + \dot{c}_{10}\beta_{106} + \dot{c}_{11}\beta_{116}\right)\end{aligned}\right\} . \tag{65}$$

We then obtain from $(32)_{\kappa=12}$, $(33)_{\kappa=12}$, $(34)_{\kappa=12}$, $(35)_{\kappa=12}$, and (VIII, 16)$^{\cdot}$:

$$\beta_{127} = \frac{N(\beta_{127})}{\alpha_7(\alpha_8 - \alpha_7)(\alpha_9 - \alpha_7)(\alpha_{10} - \alpha_7)(\alpha_{11} - \alpha_7)} \tag{66}$$

with

$$\begin{aligned}N(\beta_{127}) = \frac{1}{60}\ &[30\alpha_8\alpha_9\alpha_{10}\alpha_{11} - 20(\alpha_8\alpha_9\alpha_{10} + \alpha_8\alpha_9\alpha_{11} + \alpha_8\alpha_{10}\alpha_{11}\\ &+ \alpha_9\alpha_{10}\alpha_{11}) + 15(\alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_8\alpha_{11} + \alpha_9\alpha_{10}\\ &+ \alpha_9\alpha_{11} + \alpha_{10}\alpha_{11}) - 12(\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) + 10]\\ &- \beta_{125}\alpha_5(\alpha_8 - \alpha_5)(\alpha_9 - \alpha_5)(\alpha_{10} - \alpha_5)(\alpha_{11} - \alpha_5)\\ &- \beta_{126}\alpha_6(\alpha_8 - \alpha_6((\alpha_9 - \alpha_6)(\alpha_{10} - \alpha_6)(\alpha_{11} - \alpha_6) - R\end{aligned}$$

and corresponding formulas for β_{128}, β_{129}, β_{1210}, and β_{1211}.

The abbreviation R stands for

$$\left.\begin{aligned}R = \frac{1}{6} - (\beta_{125}\alpha_5^5 + \beta_{126}\alpha_6^5) - \frac{1}{\dot{c}_{12}}\Big[\frac{1}{42} - \dot{c}_8\beta_{87}\alpha_7^5 - \dot{c}_9(\beta_{97}\alpha_7^5 + \beta_{98}\alpha_8^5) - \dot{c}_{10}(\beta_{107}\alpha_7^5\\ + \beta_{108}\alpha_8^5 + \beta_{109}\alpha_9^5) - \dot{c}_{11}(\beta_{117}\alpha_7^5 + \beta_{118}\alpha_8^5 + \beta_{119}\alpha_9^5 + \beta_{1110}\alpha_{10}^5)\Big]\end{aligned}\right\} . \tag{67}$$

This concludes the computation of the coefficients $\beta_{\kappa\lambda}$.

16. The computation of the coefficients $\gamma_{\kappa\lambda}$ proceeds in a similar way. From $(38)_{\kappa=2}$ we obtain

$$\gamma_{21} = \frac{1}{4}\,\alpha_2^2 \quad , \tag{68}$$

and from $(38)_{\kappa=3}$:

$$\gamma_{32} = \frac{1}{4}\,\alpha_3^2 \quad . \tag{69}$$

Putting

$$\gamma_{42} = 0 \quad , \tag{70}$$

equation $(38)_{\kappa=4}$ yields:

$$\gamma_{43} = \frac{1}{6}\cdot\frac{\alpha_4^3}{\alpha_3} \quad . \tag{71}$$

Equations $(38)_{\kappa=5}$, $(39)_{\kappa=5}$, $(40)_{\kappa=5}$ yield

$$\gamma_{52} = \frac{1}{60}\,\alpha_5^3\,\frac{10\alpha_3\alpha_4 - 5(\alpha_3+\alpha_4)\alpha_5 + 3\alpha_5^2}{\alpha_2(\alpha_3-\alpha_2)(\alpha_4-\alpha_2)} \tag{72}$$

and corresponding formulas for γ_{53}, γ_{54}.

In the same way, we obtain from $(38)_{\kappa=6}$, $(39)_{\kappa=6}$, and $(40)_{\kappa=6}$:

$$\gamma_{63} = \frac{1}{60}\,\alpha_6^3\,\frac{10\alpha_4\alpha_5 - 5(\alpha_4+\alpha_5)\alpha_6 + 3\alpha_6^2}{\alpha_3(\alpha_4-\alpha_3)(\alpha_5-\alpha_3)} \tag{73}$$

and corresponding formulas for γ_{64}, γ_{65}, and from $(38)_{\kappa=7}$, $(39)_{\kappa=7}$, $(40)_{\kappa=7}$, and $(41)_{\kappa=7}$:

$$\gamma_{73} = \frac{1}{60}\alpha_7^3 \frac{10\alpha_4\alpha_5\alpha_6 - 5(\alpha_4\alpha_5 + \alpha_4\alpha_6 + \alpha_5\alpha_6)\alpha_7 + 3(\alpha_4 + \alpha_5 + \alpha_6)\alpha_7^2 - 2\alpha_7^3}{\alpha_3(\alpha_4 - \alpha_3)(\alpha_5 - \alpha_3)(\alpha_6 - \alpha_3)} \tag{74}$$

and corresponding formulas for γ_{74}, γ_{75}, γ_{76}.

Putting

$$\gamma_{83} = 0 \quad , \tag{75}$$

equations $(38)_{\kappa=8}$, $(39)_{\kappa=8}$, $(40)_{\kappa=8}$, $(41)_{\kappa=8}$ yield

$$\gamma_{84} = \frac{1}{60}\alpha_8^3 \frac{10\alpha_5\alpha_6\alpha_7 - 5(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7)\alpha_8 + 3(\alpha_5 + \alpha_6 + \alpha_7)\alpha_8^2 - 2\alpha_8^3}{\alpha_4(\alpha_5 - \alpha_4)(\alpha_6 - \alpha_4)(\alpha_7 - \alpha_4)} \tag{76}$$

and corresponding formulas for γ_{85}, γ_{86}, and γ_{87}.

Putting

$$\gamma_{93} = \gamma_{94} = 0 \quad , \tag{77}$$

equations $(38)_{\kappa=9}$, $(39)_{\kappa=9}$, $(40)_{\kappa=9}$, and $(41)_{\kappa=9}$ lead to

$$\gamma_{95} = \frac{1}{60}\alpha_9^3 \frac{10\alpha_6\alpha_7\alpha_8 - 5(\alpha_6\alpha_7 + \alpha_6\alpha_8 + \alpha_7\alpha_8)\alpha_9 + 3(\alpha_6 + \alpha_7 + \alpha_8)\alpha_9^2 - 2\alpha_9^3}{\alpha_5(\alpha_6 - \alpha_5)(\alpha_7 - \alpha_5)(\alpha_8 - \alpha_5)} \tag{78}$$

and corresponding formulas for γ_{96}, γ_{97}, and γ_{98}.

With

$$\gamma_{103} = \gamma_{104} = \gamma_{105} = 0 \quad , \tag{79}$$

equations $(38)_{\kappa=10}$, $(39)_{\kappa=10}$, $(40)_{\kappa=10}$, and $(41)_{\kappa=10}$ give

$$\gamma_{106} = \frac{1}{60}\alpha_{10}^3 \frac{10\alpha_7\alpha_8\alpha_9 - 5(\alpha_7\alpha_8 + \alpha_7\alpha_9 + \alpha_8\alpha_9)\alpha_{10} + 3(\alpha_7 + \alpha_8 + \alpha_9)\alpha_{10}^2 - 2\alpha_{10}^3}{\alpha_6(\alpha_7 - \alpha_6)(\alpha_8 - \alpha_6)(\alpha_9 - \alpha_6)} \tag{80}$$

and correspondingly γ_{107}, γ_{108}, and γ_{109}.

The first equations (42) and (43) yield

$$\gamma_{113} = -\frac{c_7}{c_{11}} \cdot \gamma_{73} \quad , \quad \gamma_{114} = -\frac{c_7}{c_{11}}\gamma_{74} - \frac{c_8}{c_{11}}\gamma_{84} \quad . \tag{81}$$

From equations $(38)_{\kappa=11}$, $(39)_{\kappa=11}$, $(40)_{\kappa=11}$, and $(41)_{\kappa=11}$ we can now obtain

$$\gamma_{117} = \frac{N(\gamma_{117})}{\alpha_7(\alpha_8 - \alpha_7)(\alpha_9 - \alpha_7)(\alpha_{10} - \alpha_7)} \tag{82}$$

with

$$N(\gamma_{117}) = \frac{1}{60}\alpha_{11}^3 [10\alpha_8\alpha_9\alpha_{10} - 5(\alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_9\alpha_{10})\alpha_{11} + 3(\alpha_8 + \alpha_9 + \alpha_{10})\alpha_{11}^2 - 2\alpha_{11}^3]$$
$$- \gamma_{113}\alpha_3(\alpha_8 - \alpha_3)(\alpha_9 - \alpha_3)(\alpha_{10} - \alpha_3) - \gamma_{114}\alpha_4(\alpha_8 - \alpha_4)(\alpha_9 - \alpha_4)(\alpha_{10} - \alpha_4)$$

and corresponding formulas for γ_{118}, γ_{119}, and γ_{1110}.

From the second equations (42) and (43) we find

$$\gamma_{123} = -\frac{1}{\dot{c}_{12}}(\dot{c}_7\gamma_{73} + \dot{c}_{11}\gamma_{113}) \quad , \quad \gamma_{124} = -\frac{1}{\dot{c}_{12}}(\dot{c}_7\gamma_{74} + \dot{c}_8\gamma_{84} + \dot{c}_{11}\gamma_{114}) \tag{83}$$

For the computation of γ_{127}, γ_{128}, γ_{129}, γ_{1210}, and γ_{1211} we make use of $(38)_{\kappa=12}$, $(39)_{\kappa=12}$, $(40)_{\kappa=12}$, $(41)_{\kappa=12}$ and of

$$\dot{c}_7Q_{75} + \dot{c}_8Q_{85} + \dot{c}_9Q_{95} + \dot{c}_{10}Q_{105} + \dot{c}_{11}Q_{115} + \dot{c}_{12}Q_{125} = \frac{1}{336} \quad .$$

The last equation is equation (IX, 20)' of Table 1. If this last equation is satisfied, one of the error coefficients in $\dot{x}$ becomes zero, thereby reducing the error in $\dot{x}$.

The above five equations yield

$$\gamma_{127} = \frac{N(\gamma_{127})}{\alpha_7(\alpha_8 - \alpha_7)(\alpha_9 - \alpha_7)(\alpha_{10} - \alpha_7)(\alpha_{11} - \alpha_7)} \tag{84}$$

with

$$\begin{aligned} N(\gamma_{127}) = \; & \frac{1}{420}\, [70\alpha_8\alpha_9\alpha_{10}\alpha_{11} - 35(\alpha_8\alpha_9\alpha_{10} + \alpha_8\alpha_9\alpha_{11} + \alpha_8\alpha_{10}\alpha_{11} + \alpha_9\alpha_{10}\alpha_{11}) \\ & + 21(\alpha_8\alpha_9 + \alpha_8\alpha_{10} + \alpha_8\alpha_{11} + \alpha_9\alpha_{10} + \alpha_9\alpha_{11} + \alpha_{10}\alpha_{11}) \\ & - 14(\alpha_8 + \alpha_9 + \alpha_{10} + \alpha_{11}) + 10] \\ & - \gamma_{123}\alpha_3(\alpha_8 - \alpha_3)(\alpha_9 - \alpha_3)(\alpha_{10} - \alpha_3)(\alpha_{11} - \alpha_3) \\ & - \gamma_{124}\alpha_4(\alpha_8 - \alpha_4)(\alpha_9 - \alpha_4)(\alpha_{10} - \alpha_4)(\alpha_{11} - \alpha_4) \\ & - \frac{1}{42} + \frac{1}{\dot{c}_{12}}\left(\frac{1}{336} - \dot{c}_7 Q_{75} - \dot{c}_8 Q_{85} - \dot{c}_9 Q_{95} - \dot{c}_{10} Q_{105} - \dot{c}_{11} Q_{115}\right) \end{aligned}$$

and corresponding formulas for γ_{128}, γ_{129}, γ_{1210}, and γ_{1211}.

17. The coefficients $\beta_{\kappa 0}$ and $\gamma_{\kappa 0}$ can not be found from the equations of condition since they do not enter these equations.

However, comparing (3) with (28), we find immediately

$$\left.\begin{aligned} \beta_{\kappa 0} &= \alpha_\kappa - (\beta_{\kappa 1} + \beta_{\kappa 2} + \dots + \beta_{\kappa, \kappa-1}) \\ \gamma_{\kappa 0} &= \frac{1}{2}\alpha_\kappa^2 - (\gamma_{\kappa 1} + \gamma_{\kappa 2}, \dots + \gamma_{\kappa, \kappa-1}) \end{aligned}\right\} (\kappa = 1, 2, 3 \dots, 13) \;, \tag{85}$$

with the parentheses in (85) being omitted for $\kappa = 1$.

18. In No. 15 and No. 16 we have expressed the coefficients $\beta_{\kappa\lambda}$ and $\gamma_{\kappa\lambda}$ by the coefficients α_κ. As explained in No. 15, there are relations between the α_κ. These relations leave us with only four independent α_κ: α_4, α_7, α_9, and α_{10}. Except for trivial restrictions, these four α_κ can be chosen arbitrarily and lead to a Runge-Kutta-Nyström formula RKN-G-7(8)-13 based on thirteen evaluations per step of the differential equation (2).

19. There remains the problem of how to select the four independent coefficients α_4, α_7, α_9, and α_{10}. Naturally, one would like to have a Runge-Kutta-Nyström formula with small truncation errors for whatever the problem (2) might be. Unfortunately, the truncation errors also depend on the problem (2). A formula that might be very efficient for a certain problem can prove to be relatively poor for another problem.

Therefore, the only reasonable way to select the independent parameters α_4, α_7, α_9, α_{10} seems to be to find in Table 1 the error coefficients for the eighth-order terms in x and $\dot{x}$. However, one has to keep in mind that these error coefficients have to be multiplied with certain expressions in the partial derivatives of (2), summed up, and multiplied with h^8 to represent an approximation of the local truncation error in x or $\dot{x}$.

It can be assumed, however, that the local truncation error becomes small if the error coefficients are sufficiently small since the local truncation error obviously tends to zero if all error coefficients go to zero.

From Table 1 we find the following 10 error coefficients in x which are different from each other:

$$T_{16},\ T_{38},\ T_{39},\ T_{49},\ T_{67},\ T_{78},\ T_{81},\ T_{82},\ T_{87},\ T_{95}\quad ,$$

the suffix indicating the number of the eighth-order equation of condition in Table 1. For instance

$$T_{16} = \sum_{\kappa=7}^{12} c_\kappa P_{\kappa 5} - \frac{1}{336} \quad .$$

There are more error coefficients for x in Table 1. These additional error coefficients, however, differ by a constant factor only from those listed above.

From the eighth-order equations of condition for $\dot{x}$ in Table 1 we find the following 28 error coefficients.[3]

$\dot{T}_1$, $\dot{T}_{16}$, $\dot{T}_{26}$, $\dot{T}_{48}$, $\dot{T}_{49}$, $\dot{T}_{59}$, $\dot{T}_{76}$, $\dot{T}_{89}$, $\dot{T}_{93}$, $\dot{T}_{143}$

$\dot{T}_{159}$, $\dot{T}_{175}$, $\dot{T}_{182}$, $\dot{T}_{189}$, $\dot{T}_{192}$, $\dot{T}_{194}$, $\dot{T}_{213}$, $\dot{T}_{220}$, $\dot{T}_{222}$, $\dot{T}_{223}$

$\dot{T}_{249}$, $\dot{T}_{255}$, $\dot{T}_{256}$, $\dot{T}_{258}$, $\dot{T}_{260}$, $\dot{T}_{262}$, $\dot{T}_{263}$, $\dot{T}_{266}$

It is, for instance

$$\dot{T}_1 = \sum_{\kappa=7}^{12} \dot{c}_\kappa \alpha_\kappa^7 - \frac{1}{8} \quad .$$

Again, there are more error coefficients for $\dot{x}$ in Table 1 which differ by a constant factor from those listed above.

Since there are considerably more error terms contributing to the truncation error in $\dot{x}$ than there are for the truncation error in x, the error control should be based on the truncation error in $\dot{x}$. However, since this seems to be impossible without an unreasonable increase in the computational effort, we have to resort to an error control in x and try to keep the errors in $\dot{x}$ as small as possible. Since these errors in $\dot{x}$ propagate directly through the differential equation (2), their influence is likely to be more serious than in the case of the differential equation (1).

20. We computed the above listed error coefficients in x and $\dot{x}$ for a large variety of combinations of the parameters α_4, α_7, α_9, α_{10} and finally decided on a combination for which the error coefficients in x as well as the ratio of the error coefficients in $\dot{x}$ to the error coefficients in x were reasonably small.

3. Because of the choice of $\gamma_{127}, \ldots, \gamma_{1211}$ in No. 16 the error coefficient $\dot{T}_{20}$ is zero.

We selected for our seventh-order Runge-Kutta-Nyström formula RKN-G-7(8)-13 the following combination

$$\alpha_4 = \frac{1}{2}, \quad \alpha_7 = \frac{3}{4}, \quad \alpha_9 = \frac{1}{8}, \quad \alpha_{10} = \frac{3}{8} \quad . \tag{86}$$

The coefficients for this formula are listed in Table 3. Since the last evaluation (28) is supposed to be taken over as first evaluation for the next step, the coefficients $\beta_{130}, \beta_{131}, \ldots, \beta_{1312}$ and $\gamma_{130}, \gamma_{131}, \ldots, \gamma_{1312}$ in Table 3 have to be equal to the weight factors $\dot{c}_0, \dot{c}_1, \ldots, \dot{c}_{12}$ and $c_0, c_1, \ldots, c_{12}$ of (29).

The computation of the coefficients for our seventh-order formula and also for the sixth-order formula, listed later, was performed in 40-digit arithmetic.

Table 2 shows the pattern of our seventh-order formula. All coefficients different from 0 and 1 are marked by an asterisk.

SECTION III. SIXTH-ORDER FORMULA RKN-G-6(7)

21. The derivation of a sixth-order formula is similar to the derivation of the seventh-order formula in Section II.

We base the sixth-order formula on ten evaluations with an eleventh evaluation which is taken over as first evaluation for the next step.

Similar to Section II we make the following assumptions for the coefficients of our sixth-order formula:

$$\left.\begin{array}{l} \hat{c}_1 = c_1 = 0, \ \ldots, \ \hat{c}_4 = c_4 = 0, \ \hat{c}_5 = c_5, \ \ldots, \ \hat{c}_8 = c_8, \ \hat{c}_9 = 0, \ \hat{c}_{10} = c_9 \\ \dot{c}_1 = 0, \ \ldots, \ \dot{c}_4 = 0 \ ; \ \alpha_9 = \alpha_{10} = 1 \end{array}\right\} \tag{87}$$

$$\left.\begin{array}{ll} \beta_{31} = \beta_{41} = \ldots = \beta_{101} = 0 & \gamma_{31} = \gamma_{41} = \ldots = \gamma_{101} = 0 \\ \beta_{52} = \beta_{62} = \ldots = \beta_{102} = 0 & \beta_{10,\lambda} = \dot{c}_\lambda (\lambda = 0,1,2, \ldots, 9) \\ & \gamma_{10,\lambda} = c_\lambda (\lambda = 0,1,2, \ldots, 9) \end{array}\right\} \tag{88}$$

and

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda = \frac{1}{2}\alpha_\kappa^2 \quad (\kappa = 2,3, \ldots, 10) \tag{89}$$

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda^2 = \frac{1}{3}\alpha_\kappa^3 \quad (\kappa = 2,3, \ldots, 10) \tag{90}$$

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda^3 = \frac{1}{4}\alpha_\kappa^4 \quad (\kappa = 5,6, \ldots, 10) \tag{91}$$

$$\left.\begin{aligned} c_5\beta_{53} + c_6\beta_{63} + c_7\beta_{73} + c_8\beta_{83} + c_9\beta_{103} &= 0 \\ \dot{c}_5\beta_{53} + \dot{c}_6\beta_{63} + \dot{c}_7\beta_{73} + \dot{c}_8\beta_{83} + \dot{c}_9\beta_{93} &= 0 \end{aligned}\right\} \tag{92}$$

$$\left.\begin{aligned} c_5\beta_{54} + c_6\beta_{64} + c_7\beta_{74} + c_8\beta_{84} + c_9\beta_{104} &= 0 \\ \dot{c}_5\beta_{54} + \dot{c}_6\beta_{64} + \dot{c}_7\beta_{74} + \dot{c}_8\beta_{84} + \dot{c}_9\beta_{94} &= 0 \end{aligned}\right\} \tag{93}$$

and

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda = \frac{1}{6}\alpha_\kappa^3 \quad (\kappa = 2,3, \ldots, 10) \qquad (94$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda^2 = \frac{1}{12}\alpha_\kappa^4 \quad (\kappa = 5,6, \ldots, 10) \tag{95}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda^3 = \frac{1}{20}\alpha_\kappa^5 \quad (\kappa = 5,6, \ldots, 10) \tag{96}$$

$$\left.\begin{aligned} &c_5\gamma_{52} + c_6\gamma_{62} + c_7\gamma_{72} + c_8\gamma_{82} + c_9\gamma_{102} = 0 \\ &\dot{c}_5\gamma_{52} + \dot{c}_6\gamma_{62} + \dot{c}_7\gamma_{72} + \dot{c}_8\gamma_{82} + \dot{c}_9\gamma_{92} = 0 \end{aligned}\right\} \qquad (97)$$

22. The assumptions of No. 21 reduce the seventh- and lower-order equations of Table 1 to

(II, 1), (III, 1), (IV, 1), (V, 1), (VI, 1), (VII, 1), (VII, 10) .

The first six of these equations yield, in the same way as in Section II, the weight factors c_0, c_5, c_6, c_7, c_8, c_9 and $\dot{c}_0$, $\dot{c}_5$, $\dot{c}_6$, $\dot{c}_7$, $\dot{c}_8$, $\dot{c}_9$. Equation (VII, 10) has to be solved together with the assumptions of No. 21. The resulting values for β_{21}, β_{32}, β_{42}, β_{43}, β_{53}, β_{54}, β_{63}, β_{64}, β_{65} and the restrictions for α_1, α_2, α_3 are the same as in Section II.. They are given by equations (46) through (53).

The remaining coefficients $\beta_{7\lambda}$, $\beta_{8\lambda}$, etc. are different from those of Section II.

Putting

$$\beta_{73} = 0 \quad , \qquad (98)$$

we obtain from $(89)_{\kappa=7}$, $(90)_{\kappa=7}$, $(91)_{\kappa=7}$

$$\beta_{74} = \frac{1}{12}\,\alpha_7^2 \cdot \frac{6\alpha_5\alpha_6 - 4(\alpha_5 + \alpha_6)\alpha_7 + 3\alpha_7^2}{\alpha_4(\alpha_5 - \alpha_4)(\alpha_6 - \alpha_4)} \qquad (99)$$

and corresponding formulas for β_{75} and β_{76}.

The first equations (92) and (93) yield

$$\left.\begin{aligned} \beta_{83} &= -\frac{1}{c_8}\,(c_5\beta_{53} + c_6\beta_{63}) \\ \beta_{84} &= -\frac{1}{c_8}\,(c_5\beta_{54} + c_6\beta_{64} + c_7\beta_{74}) \quad . \end{aligned}\right\} \qquad (100)$$

Equations $(89)_{\kappa=8}$, $(90)_{\kappa=8}$, $(91)_{\kappa=8}$, and (VII, 10) can then be considered as four linear equations for the three coefficients β_{85}, β_{86}, β_{87}. They lead to a restriction for α_8:

$$\alpha_8 = \frac{N(\alpha_8)}{D(\alpha_8)} \tag{101}$$

with

$$\begin{aligned}
N(\alpha_8) = {} & 30(1-\alpha_5)(1-\alpha_6)(1-\alpha_7)\,[35\alpha_5\alpha_6\alpha_7 - 14(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 7(\alpha_5 + \alpha_6 + \alpha_7) - 4] \\
& + [30\alpha_5\alpha_6\alpha_7 - 20(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 15(\alpha_5 + \alpha_6 + \alpha_7) - 12] \cdot \\
& [35\alpha_5\alpha_6\alpha_7 - 21(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 14(\alpha_5 + \alpha_6 + \alpha_7) - 10] \\
D(\alpha_8) = {} & 30(1-\alpha_5)(1-\alpha_6)(1-\alpha_7)\,[35\alpha_5\alpha_6\alpha_7 - 14(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 7(\alpha_5 + \alpha_6 + \alpha_7) - 4] \\
& + [30\alpha_5\alpha_6\alpha_7 - 20(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 15(\alpha_5 + \alpha_6 + \alpha_7) - 12] \cdot \\
& [70\alpha_5\alpha_6\alpha_7 - 35(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 21(\alpha_5 + \alpha_6 + \alpha_7) - 14]
\end{aligned}$$

and to

$$\beta_{85} = \frac{\frac{1}{12}\alpha_8^2[6\alpha_6\alpha_7 - 4(\alpha_6 + \alpha_7)\alpha_8 + 3\alpha_8^2] - \beta_{83}\alpha_3(\alpha_6 - \alpha_3)(\alpha_7 - \alpha_3) - \beta_{84}\alpha_4(\alpha_6 - \alpha_4)(\alpha_7 - \alpha_4)}{\alpha_5(\alpha_6 - \alpha_5)(\alpha_7 - \alpha_5)} \tag{102}$$

and corresponding formulas for β_{86} and β_{87}.

The second equations (92) and (93) give

$$\left.\begin{aligned}
\beta_{93} &= -\frac{1}{\dot{c}_9}\left(\dot{c}_5\beta_{53} + \dot{c}_6\beta_{63} + \dot{c}_8\beta_{83}\right) \\
\beta_{94} &= -\frac{1}{\dot{c}_9}\left(\dot{c}_5\beta_{54} + \dot{c}_6\beta_{64} + \dot{c}_7\beta_{74} + \dot{c}_8\beta_{84}\right)
\end{aligned}\right\} \tag{103}$$

Equations $(89)_{\kappa=9}$, $(90)_{\kappa=9}$, $(91)_{\kappa=9}$, and $(VII,10)^{\cdot}$ then represent four equations for the four coefficients β_{95}, β_{96}, β_{97}, β_{98}. Their solution is

$$\beta_{98} = \frac{1}{60} \frac{10\alpha_5\alpha_6\alpha_7 - 5(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 3(\alpha_5 + \alpha_6 + \alpha_7) - 2}{\dot{c}_9\alpha_8(\alpha_7 - \alpha_8)(\alpha_6 - \alpha_8)(\alpha_5 - \alpha_8)} \qquad (104)$$

$$\beta_{95} = \frac{\frac{1}{12}\,[6\alpha_6\alpha_7 - 4(\alpha_6 + \alpha_7) + 3] - \beta_{93}\alpha_3(\alpha_6 - \alpha_3)(\alpha_7 - \alpha_3) - \beta_{94}\alpha_4(\alpha_6 - \alpha_4)(\alpha_7 - \alpha_4) - \beta_{98}\alpha_8(\alpha_6 - \alpha_8)(\alpha_7 - \alpha_8)}{\alpha_5(\alpha_6 - \alpha_5)(\alpha_7 - \alpha_5)} \qquad (105)$$

and expressions β_{96}, β_{97} that correspond to β_{95}.

23. We still have to determine the coefficients $\gamma_{\kappa\lambda}$. The coefficients γ_{21}, γ_{32}, γ_{42}, γ_{43}, γ_{52}, γ_{53}, γ_{54}, γ_{62}, γ_{63}, γ_{64}, γ_{65} are the same as the corresponding coefficients in Section II and are given by equations (68) through (73).

Setting

$$\gamma_{72} = \gamma_{73} = 0 \quad , \qquad (106)$$

we find from $(94)_{\kappa=7}$, $(95)_{\kappa=7}$, $(96)_{\kappa=7}$

$$\gamma_{74} = \frac{1}{60}\,\alpha_7^3\,\frac{10\alpha_5\alpha_6 - 5(\alpha_5 + \alpha_6)\alpha_7 + 3\alpha_7^2}{\alpha_4(\alpha_5 - \alpha_4)(\alpha_6 - \alpha_4)} \qquad (107)$$

and corresponding expressions for γ_{75} and γ_{76}.

The first equation (97) yields

$$\gamma_{82} = -\frac{c_5}{c_8}\,\gamma_{52} \quad . \qquad (108)$$

Setting

$$\gamma_{83} = \gamma_{84} = 0 \qquad (109)$$

equations $(94)_{\kappa=8}$, $(95)_{\kappa=8}$, $(96)_{\kappa=8}$ give:

$$\gamma_{85} = \frac{\frac{1}{60}\alpha_8^3[10\alpha_6\alpha_7 - 5(\alpha_6 + \alpha_7)\alpha_8 + 3\alpha_8^2] - \gamma_{82}\alpha_2(\alpha_6 - \alpha_2)(\alpha_7 - \alpha_2)}{\alpha_5(\alpha_6 - \alpha_5)(\alpha_7 - \alpha_5)} \tag{110}$$

and corresponding expressions for γ_{86} and γ_{87}.

From the second equation (97), we find

$$\gamma_{92} = -\frac{1}{\dot{c}_9}(\dot{c}_5\gamma_{52} + \dot{c}_8\gamma_{82}) \quad . \tag{111}$$

With

$$\gamma_{93} = \gamma_{94} = \gamma_{95} = 0 \tag{112}$$

we find from $(94)_{\kappa=9}$, $(95)_{\kappa=9}$, $(96)_{\kappa=9}$

$$\gamma_{96} = \frac{\frac{1}{60}[10\alpha_7\alpha_8 - 5(\alpha_7 + \alpha_8) + 3] - \gamma_{92}\alpha_2(\alpha_7 - \alpha_2)(\alpha_8 - \alpha_2)}{\alpha_6(\alpha_7 - \alpha_6)(\alpha_8 - \alpha_6)} \tag{113}$$

and corresponding expressions for γ_{97} and γ_{98}.

This concludes the computation of the coefficients $\beta_{\kappa\lambda}$ and $\gamma_{\kappa\lambda}$, since $\beta_{\kappa 0}$ and $\gamma_{\kappa 0}$ are again obtained from (85) with $\kappa = 1, 2, 3, \ldots, 10$.

24. The expressions for $\beta_{\kappa\lambda}$ and $\gamma_{\kappa\lambda}$ in No. 22 and No. 23 contain four parameters α_4, α_5, α_6, and α_7 which we can choose arbitrarily.

As in Section II, we choose these parameters so that the error coefficients in x and the ratio of the error coefficients in $\dot{x}$ and in x become reasonably small.

In the case of a sixth-order Runge-Kutta-Nyström formula there are five different error coefficients in x:

$$T_{10}, T_{23}, T_{24}, T_{28}, T_{34}$$

and eighteen different error coefficients in $\dot{x}$:

$$\dot{T}_1,\ \dot{T}_{10},\ \dot{T}_{12},\ \dot{T}_{16},\ \dot{T}_{31},\ \dot{T}_{32},\ \dot{T}_{38},\ \dot{T}_{39},\ \dot{T}_{47},\ \dot{T}_{49}$$

$$\dot{T}_{81},\ \dot{T}_{82},\ \dot{T}_{86},\ \dot{T}_{87},\ \dot{T}_{89},\ \dot{T}_{91},\ \dot{T}_{92},\ \dot{T}_{95}$$

In our final sixth-order formula we chose for the free parameters

$$\alpha_4 = \frac{5}{8}\ ,\quad \alpha_5 = \frac{3}{8}\ ,\quad \alpha_6 = \frac{2}{3}\ ,\quad \alpha_7 = \frac{1}{6}\quad . \tag{114}$$

The coefficients of the sixth-order formula with the parameter values (114) are listed in Table 5.

The pattern of our sixth-order formula is shown in Table 4.

SECTION IV. FIFTH-ORDER FORMULA RKN-G-5(6)

25. Although it is possible to construct fifth-order formulas based on seven evaluations per step, we prefer to use eight evaluations per step, since we then obtain formulas with smaller local truncation error terms. In spite of the one additional evaluation per step such formulas proved to be more economical. They allow a larger stepsize because of their smaller truncation errors.

We make the following assumptions for the coefficients of our fifth-order formula:

$$\left.\begin{aligned} &\hat{c}_1 = c_1 = 0\ ,\quad \hat{c}_2 = c_2 = 0\ ,\quad \hat{c}_3 = c_3 = 0\ ,\\ &\hat{c}_4 = c_4\ ,\quad \hat{c}_5 = c_5\ ,\quad \hat{c}_6 = c_6\ ,\quad \hat{c}_7 = 0\ ,\quad \hat{c}_8 = c_7\\ &\dot{c}_1 = 0\ ,\quad \dot{c}_2 = 0\ ,\quad \dot{c}_3 = 0\ ,\quad \alpha_7 = \alpha_8 = 1 \end{aligned}\right\} \tag{115}$$

$$\left.\begin{aligned}
&\beta_{31} = \beta_{41} = \beta_{51} = \beta_{61} = \beta_{71} = \beta_{81} = 0 \\
&\gamma_{41} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = 0 \\
&\beta_{8\lambda} = \dot{c}_{\lambda}\,(\lambda = 0,1,2, \ldots, 7) \\
&\gamma_{8\lambda} = c_{\lambda}\,(\lambda = 0,1,2, \ldots, 7)
\end{aligned}\right\} \tag{116}$$

and

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_{\lambda} = \frac{1}{2}\alpha_{\kappa}^{2} \quad (\kappa = 2,3, \ldots, 8) \tag{117}$$

$$\sum_{\lambda=1}^{\kappa-1} \beta_{\kappa\lambda}\alpha_{\lambda}^{2} = \frac{1}{3}\alpha_{\kappa}^{3} \quad (\kappa = 2,3, \ldots, 8) \tag{118}$$

$$\left.\begin{aligned}
&c_4\beta_{42} + c_5\beta_{52} + c_6\beta_{62} + c_7\beta_{82} = 0 \\
&\dot{c}_4\beta_{42} + \dot{c}_5\beta_{52} + \dot{c}_6\beta_{62} + \dot{c}_7\beta_{72} = 0
\end{aligned}\right\} \tag{119}$$

and

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_{\lambda} = \frac{1}{6}\alpha_{\kappa}^{3} \quad (\kappa = 2,3, \ldots, 8) \tag{120}$$

$$\sum_{\lambda=1}^{\kappa-1} \gamma_{\kappa\lambda}\alpha_{\lambda}^{2} = \frac{1}{12}\alpha_{\kappa}^{4} \quad (\kappa = 2,3, \ldots, 8) \tag{121}$$

$$\dot{c}_4\gamma_{42} + \dot{c}_5\gamma_{52} + \dot{c}_6\gamma_{62} + \dot{c}_7\gamma_{72} = 0 \quad . \tag{122}$$

26. The assumptions of No. 25 reduce the sixth- and lower-order equations of condition of Table 1 to

(II,1), (III,1), (IV,1), (V,1), (VI,1), (VI,6)

The first five of these equations yield, as in Section II and III, the weight factors c_0, c_4, c_5, c_6, c_7 and $\dot{c}_0$, $\dot{c}_4$, $\dot{c}_5$, $\dot{c}_6$, $\dot{c}_7$. Equation (VI,6) has to be solved together with the assumptions of No. 25. The resulting values for β_{21}, β_{32} and the restrictions for α_1, α_2 are the same as in Sections II and III and are given by equations (46) through (49).

Equations (119) can be satisfied by

$$\beta_{42} = \beta_{52} = \beta_{62} = \beta_{72} = \beta_{82} = 0 \quad . \tag{123}$$

From $(117)_{\kappa=4}$ and $(118)_{\kappa=4}$ we then find

$$\alpha_3 = \frac{2}{3}\alpha_4 \tag{124}$$

and

$$\beta_{43} = \frac{3}{4}\alpha_4 \quad . \tag{125}$$

Equations $(117)_{\kappa=5}$ and $(118)_{\kappa=5}$ yield

$$\beta_{53} = \frac{1}{6}\alpha_5^2 \frac{3\alpha_4 - 2\alpha_5}{\alpha_3(\alpha_4 - \alpha_3)} \quad , \quad \beta_{54} = \frac{1}{6}\alpha_5^2 \frac{3\alpha_3 - 2\alpha_5}{\alpha_4(\alpha_3 - \alpha_4)} \quad . \tag{126}$$

Equations $(117)_{\kappa=6}$, $(118)_{\kappa=6}$ and (VI,6) result in

$$\beta_{63} = \frac{1}{12}\alpha_6^2 \frac{6\alpha_4\alpha_5 - 4(\alpha_4 + \alpha_5)\alpha_6 + 3\alpha_6^2 R_1}{\alpha_3(\alpha_4 - \alpha_3)(\alpha_5 - \alpha_3)} \tag{127}$$

and corresponding expressions for β_{64}, β_{65}. The abbreviation R_1 in (127) stands for

$$R_1 = \frac{4}{c_6\alpha_6^4}\left\{\frac{1}{120} - c_4\beta_{43}\alpha_3^3 - c_5(\beta_{53}\alpha_3^3 + \beta_{54}\alpha_4^3) - \frac{1}{4}c_7\right\} \quad .$$

We compute the coefficients β_{73}, β_{74}, β_{75}, β_{76} from $(117)_{\kappa=6}$, $(118)_{\kappa=6}$, $(VI,6)^{\cdot}$ and from the additional condition $\dot{T}_{10} = 0$ (see Table 1). The last condition helps to reduce the local truncation error in $\dot{x}$. The above equations lead to

$$\beta_{73} = \frac{1}{60} \frac{30\alpha_4\alpha_5\alpha_6 - 20(\alpha_4\alpha_5 + \alpha_4\alpha_6 + \alpha_5\alpha_6) + 15(\alpha_4 + \alpha_5 + \alpha_6) \cdot R_2 - 12R_3}{\alpha_3(\alpha_4 - \alpha_3)(\alpha_5 - \alpha_3)(\alpha_6 - \alpha_3)} \tag{128}$$

and corresponding expressions for β_{74}, β_{75}, β_{76}. Here we have used the abbreviations

$$R_2 = \frac{4}{\dot{c}_7} \left\{ \frac{1}{20} - \dot{c}_4\beta_{43}\alpha_3^3 - \dot{c}_5(\beta_{53}\alpha_3^3 + \beta_{54}\alpha_4^3) - \dot{c}_6(\beta_{63}\alpha_3^3 + \beta_{64}\alpha_4^3 + \beta_{65}\alpha_5^3) \right\}$$

$$R_3 = \frac{5}{\dot{c}_7} \left\{ \frac{1}{30} - \dot{c}_4\beta_{43}\alpha_3^4 - \dot{c}_5(\beta_{53}\alpha_3^4 + \beta_{54}\alpha_4^4) - \dot{c}_6(\beta_{63}\alpha_3^4 + \beta_{64}\alpha_4^4 + \beta_{65}\alpha_5^4) \right\} \quad .$$

27. We now compute the coefficients $\gamma_{\kappa\lambda}$. The coefficient γ_{21} is again given by (68). The coefficients γ_{31} and γ_{32} are obtained from equations $(120)_{\kappa=3}$ and $(121)_{\kappa=3}$:

$$\gamma_{31} = \frac{1}{12}\,\alpha_3^3\,\frac{2\alpha_2 - \alpha_3}{\alpha_1(\alpha_2 - \alpha_1)} \quad , \quad \gamma_{32} = \frac{1}{12}\,\alpha_3^3\,\frac{2\alpha_1 - \alpha_3}{\alpha_2(\alpha_1 - \alpha_2)} \tag{129}$$

With $\gamma_{41} = 0$ we find from $(120)_{\kappa=4}$ and $(121)_{\kappa=4}$:

$$\gamma_{42} = \frac{1}{12}\,\alpha_4^3\,\frac{2\alpha_3 - \alpha_4}{\alpha_2(\alpha_3 - \alpha_2)} \quad , \quad \gamma_{43} = \frac{1}{12}\,\alpha_4^3\,\frac{2\alpha_2 - \alpha_4}{\alpha_3(\alpha_2 - \alpha_3)} \tag{130}$$

Setting

$$\gamma_{62} = \gamma_{72} = 0 \quad , \tag{131}$$

we find from (122):

$$\gamma_{52} = -\frac{\dot{c}_4}{\dot{c}_5}\,\gamma_{42} \tag{132}$$

Equations $(120)_{\kappa=5}$ and $(121)_{\kappa=5}$ then yield

$$\left.\begin{aligned} \gamma_{53} &= \frac{1}{12}\,\alpha_5^3\,\frac{2\alpha_4-\alpha_5}{\alpha_3(\alpha_4-\alpha_3)} - \gamma_{52}\,\frac{\alpha_2(\alpha_4-\alpha_2)}{\alpha_3(\alpha_4-\alpha_3)} \\ \gamma_{54} &= \frac{1}{12}\,\alpha_5^3\,\frac{2\alpha_3-\alpha_5}{\alpha_4(\alpha_3-\alpha_4)} - \gamma_{52}\,\frac{\alpha_2(\alpha_3-\alpha_2)}{\alpha_4(\alpha_3-\alpha_4)} \end{aligned}\right\} \tag{133}$$

With

$$\gamma_{63} = \gamma_{73} = 0 \tag{134}$$

we find from $(120)_{\kappa=6}$ and $(121)_{\kappa=6}$:

$$\gamma_{64} = \frac{1}{12}\,\alpha_6^3\,\frac{2\alpha_5-\alpha_6}{\alpha_4(\alpha_5-\alpha_4)}\;,\quad \gamma_{65} = \frac{1}{12}\,\alpha_6^3\,\frac{2\alpha_4-\alpha_6}{\alpha_5(\alpha_4-\alpha_5)} \tag{135}$$

and from $(120)_{\kappa=7}$, $(121)_{\kappa=7}$ and from $\dot{T}_8 = 0$ (see Table 1):

$$\gamma_{74} = \frac{1}{60}\,\frac{10\alpha_5\alpha_6 - 5(\alpha_5+\alpha_6) + 3R_4}{\alpha_4(\alpha_5-\alpha_4)(\alpha_6-\alpha_4)} \tag{136}$$

and two corresponding formulas for γ_{75} and γ_{76}. Here we have used the abbreviation

$$R_4 = \frac{20}{\dot{c}_7}\left\{\frac{1}{120} - \dot{c}_4\gamma_{43}\alpha_3^3 - \dot{c}_5(\gamma_{53}\alpha_3^3 + \gamma_{54}\alpha_4^3) - \dot{c}_6(\gamma_{63}\alpha_3^3 + \gamma_{64}\alpha_4^3 + \gamma_{65}\alpha_5^3)\right\}\;. \tag{137}$$

This concludes the computation of the coefficients $\beta_{\kappa\lambda}$ and $\gamma_{\kappa\lambda}$, since $\beta_{\kappa 0}$ and $\gamma_{\kappa 0}$ are again computed from (85) with $\kappa = 1,2,3,\ldots,8$.

28. The expressions for $\beta_{\kappa\lambda}$ and $\gamma_{\kappa\lambda}$ in No. 26 and No. 27 contain three free parameters: α_4, α_5, and α_6. By a proper choice of these parameters, we might try to obtain formulas with reasonable small error coefficients in x as well as in $\dot{x}$.

In the case of our fifth-order formula we have only one error coefficient in x:

$$T_6$$

and four error terms in $\dot{x}$:

$$\dot{T}_1, \dot{T}_6, \dot{T}_{24}, \dot{T}_{34} \quad .$$

29. It is interesting to notice that by adding one more condition $(\mathrm{VII}, 1)^{\cdot}$ to $(\mathrm{II}, 1)^{\cdot}$, $(\mathrm{III}, 1)^{\cdot}$, $(\mathrm{IV}, 1)^{\cdot}$, $(\mathrm{V}, 1)^{\cdot}$, and $(\mathrm{VI}, 1)^{\cdot}$, we obtain a restrictive condition for the α's:

$$\alpha_6 = \frac{5\alpha_4\alpha_5 - 3(\alpha_4 + \alpha_5) + 2}{10\,\alpha_4\alpha_5 - 5(\alpha_4 + \alpha_5) + 3} \tag{138}$$

that makes all our error coefficients T_6, $\dot{T}_1$, $\dot{T}_6$, $\dot{T}_{24}$, $\dot{T}_{34}$ zero. Naturally, such a choice of α_6 is not suitable for our fifth-order formula, since our stepsize control would break down in this case.

However, by choosing α_6 close to the value of (138), we can obtain sufficiently small error coefficients in x and $\dot{x}$ that lead to efficient fifth-order formulas.

Such a formula is obtained for

$$\alpha_4 = \frac{9}{10} \; , \quad \alpha_5 = \frac{3}{4} \; , \quad \alpha_6 = \frac{2}{7} \quad . \tag{139}$$

For the values α_4 and α_5 of (139) the condition (138) would result in $\alpha_6 = 17/60$ (≈ 0.2833), which is reasonably close to $\alpha_6 = 2/7$ (≈ 0.2857).

The coefficients for our fifth-order formula based on the values (139) for the free parameters α_4, α_5, α_6 are listed in Table 7. Since these coefficients have relatively simple values, we have listed them in fraction form.

The pattern of our fifth-order formula is shown in Table 6.

SECTION V. APPLICATION TO TWO NUMERICAL EXAMPLES

30. In this section we apply the Runge-Kutta-Nyström formulas of this report and the Runge-Kutta formulas of [3] to two problems; one of them is linear in the first derivatives and the other one is nonlinear. As the linear problem we choose an orbit of the restricted problem of three bodies. The same orbit has already been integrated in an earlier paper of ours [5], using a power series expansion technique. The orbit is pictured in Figure 1.

Table 8 shows the differential equations and the initial conditions for the problem. Since the problem has no solution in closed form, we integrated the problem by the above mentioned power series expansion technique using thirty decimal digits.

Truncating the series after 12th-order, 16th-order, or 20th-order terms, the results (for $t = 6$) for x, y, $\dot{x}$, $\dot{y}$ agreed to about twenty decimal places. Rounding these results to 16 decimal places, we found

$$t = 6 : \left\{ \begin{array}{lll} x & = & 0.1167\ 0361\ 7342\ 5520 \cdot 10^{+1} \\ y & = & 0.1966\ 6280\ 9565\ 9560 \\ \dot{x} & = & 0.3341\ 2960\ 6037\ 2482 \\ \dot{y} & = & -0.9745\ 3805\ 7977\ 8027 \end{array} \right\} \qquad (140)$$

We substituted these values for the solution of our problem. The errors Δx, Δy, $\Delta \dot{x}$, $\Delta \dot{y}$ in Table 8 are the deviations of the solution obtained by our Runge-Kutta-Nyström or Runge-Kutta formulas from the above values (140).

31. Table 9 shows the differential equations and the initial condition for a problem that is nonlinear in the first derivatives. Since this problem has a solution in closed form, the errors of our numerical solutions could easily be established.

32. All calculations in Tables 8 and 9 were executed on an IBM-7094 computer in double precision (16 decimal places). The computer was equipped with an electronic clock to measure the execution time for the various formulas.

The stepsize control for our Runge-Kutta-Nyström and our Runge-Kutta formulas is described in No. 26 of our earlier report [1].

33. Tables 8 and 9 show the results of the various formulas applied to Problem I and II. Comparing the results of our new Runge-Kutta-Nyström formulas with those of our Runge-Kutta formulas of [3], we notice that, in these examples, we save from 25 percent to 60 percent of the execution time by using the new formulas. In most cases, our new formulas are also slightly more accurate. When relaxing the tolerance from $0.1 \cdot 10^{-16}$ to $0.1 \cdot 10^{-15}$, the relative savings in execution time for our new formulas do not change very much.

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REFERENCES

[1] Fehlberg, E.: Classical Eighth- and Lower-Order Runge-Kutta-Nyström Formulas with Stepsize Control for Special Second-Order Differential Equations. NASA TR R-381, March 1972.

[2] Fehlberg, E.: Classical Eighth- and Lower-Order Runge-Kutta-Nyström Formulas with a New Stepsize Control Procedure for Special Second-Order Differential Equations. NASA TR R-410, June 1973.

[3] Fehlberg, E.: Classical Fifth-, Sixth-, Seventh- and Eighth-Order Runge-Kutta Formulas with Stepsize Control. NASA TR R-287, October 1968.

[4] Sarafyan, D.: Improvements in the Derivation of Runge-Kutta Formulas and Computer Implementation. Louisiana State University in New Orleans, Department of Mathematics, Technical Report No. 4, August 1968.

[5] Fehlberg, E.: Numerical Integration of Differential Equations by Power Series Expansions, illustrated by Physical Examples. NASA TN D-2356, October 1964 (also published in ZAMM vol. 44 (1964), pp. 83-88).

TABLE 1. EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA-NYSTRÖM COEFFICIENTS

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$			
II	1	$\frac{1}{2} = \sum_{0} c_\kappa$	=	1	I
III	1	$\frac{1}{6} = \sum_{1} c_\kappa \alpha_\kappa$	=	$\frac{1}{2}$	II
IV	1	$\frac{1}{12} = \sum_{1} c_\kappa \alpha_\kappa^2$	=	$\frac{1}{3}$	III
	2	$\frac{1}{24} = \sum_{2} c_\kappa P_{\kappa 1}$	=	$\frac{1}{6}$	
V	1	$\frac{1}{20} = \sum_{1} c_\kappa \alpha_\kappa^3$	=	$\frac{1}{4}$	IV
	2	$\frac{1}{40} = \sum_{2} c_\kappa \alpha_\kappa P_{\kappa 1}$	=	$\frac{1}{8}$	
	3	$\frac{1}{120} = \sum_{2} c_\kappa Q_{\kappa 1}$	=	$\frac{1}{24}$	
	4	$\frac{1}{60} = \sum_{2} c_\kappa P_{\kappa 2}$	=	$\frac{1}{12}$	
	5	$\frac{1}{120} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	=	$\frac{1}{24}$	
VI	1	$\frac{1}{30} = \sum_{1} c_\kappa \alpha_\kappa^4$	=	$\frac{1}{5}$	V
	2	$\frac{1}{60} = \sum_{2} c_\kappa \alpha_\kappa^2 P_{\kappa 1}$	=	$\frac{1}{10}$	
	3	$\frac{1}{180} = \sum_{2} c_\kappa \alpha_\kappa Q_{\kappa 1}$	=	$\frac{1}{30}$	
	4	$\frac{1}{90} = \sum_{2} c_\kappa \alpha_\kappa P_{\kappa 2}$	=	$\frac{1}{15}$	
	5	$\frac{1}{360} = \sum_{2} c_\kappa Q_{\kappa 2}$	=	$\frac{1}{60}$	
	6	$\frac{1}{120} = \sum_{2} c_\kappa P_{\kappa 3}$	=	$\frac{1}{20}$	
	7	$\frac{1}{120} = \sum_{2} c_\kappa P_{\kappa 1}^2$	=	$\frac{1}{20}$	
	8	$\frac{1}{180} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	=	$\frac{1}{30}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa$	$\longrightarrow \dot{x}$	
VI	9	$\frac{1}{720} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{120}$	V
	10	$\frac{1}{240} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{40}$	
	11	$\frac{1}{720} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{120}$	
	12	$\frac{1}{360} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{60}$	
	13	$\frac{1}{720} = \sum_{4} c_\kappa \left[\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1} \right) \right]$	$= \frac{1}{120}$	
VII	1	$\frac{1}{42} = \sum_{1} c_\kappa \alpha_\kappa^5$	$= \frac{1}{6}$	VI
	2	$\frac{1}{84} = \sum_{2} c_\kappa \alpha_\kappa^3 P_{\kappa 1}$	$= \frac{1}{12}$	
	3	$\frac{1}{252} = \sum_{2} c_\kappa \alpha_\kappa^2 Q_{\kappa 1}$	$= \frac{1}{36}$	
	4	$\frac{1}{126} = \sum_{2} c_\kappa \alpha_\kappa^2 P_{\kappa 2}$	$= \frac{1}{18}$	
	5	$\frac{1}{504} = \sum_{2} c_\kappa \alpha_\kappa Q_{\kappa 2}$	$= \frac{1}{72}$	
	6	$\frac{1}{168} = \sum_{2} c_\kappa \alpha_\kappa P_{\kappa 3}$	$= \frac{1}{24}$	
	7	$\frac{1}{168} = \sum_{2} c_\kappa \alpha_\kappa P_{\kappa 1}^2$	$= \frac{1}{24}$	
	8	$\frac{1}{840} = \sum_{2} c_\kappa Q_{\kappa 3}$	$= \frac{1}{120}$	
	9	$\frac{1}{504} = \sum_{2} c_\kappa Q_{\kappa 1} P_{\kappa 1}$	$= \frac{1}{72}$	
	10	$\frac{1}{210} = \sum_{2} c_\kappa P_{\kappa 4}$	$= \frac{1}{30}$	
	11	$\frac{1}{252} = \sum_{2} c_\kappa P_{\kappa 2} P_{\kappa 1}$	$= \frac{1}{36}$	
	12	$\frac{1}{252} = \sum_{3} c_\kappa \alpha_\kappa^2 \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{36}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
VII	13	$\frac{1}{1008} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{144}$	VI
	14	$\frac{1}{336} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{48}$	
	15	$\frac{1}{1008} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{144}$	
	16	$\frac{1}{504} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{72}$	
	17	$\frac{1}{1680} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{240}$	
	18	$\frac{1}{5040} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{720}$	
	19	$\frac{1}{2520} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{360}$	
	20	$\frac{1}{420} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 1} \right)$	$= \frac{1}{60}$	
	21	$\frac{1}{1260} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 1} \right)$	$= \frac{1}{180}$	
	22	$\frac{1}{630} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 2} \right)$	$= \frac{1}{90}$	
	23	$\frac{1}{2520} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 2} \right)$	$= \frac{1}{360}$	
	24	$\frac{1}{840} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 3} \right)$	$= \frac{1}{120}$	
	25	$\frac{1}{840} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}^2 \right)$	$= \frac{1}{120}$	
	26	$\frac{1}{504} = \sum_{3} c_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{72}$	
	27	$\frac{1}{1008} = \sum_{4} c_\kappa \alpha_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1} \right) \right]$	$= \frac{1}{144}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$	
VII	28	$\frac{1}{5040} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{720}$	VI
	29	$\frac{1}{1260} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{180}$	
	30	$\frac{1}{5040} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{720}$	
	31	$\frac{1}{1680} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{240}$	
	32	$\frac{1}{5040} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{720}$	
	33	$\frac{1}{2520} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{360}$	
	34	$\frac{1}{5040} = \sum_5 c_\kappa \left\{\sum_4^{\kappa-1} \beta_{\kappa\lambda}\left[\sum_3^{\lambda-1} \beta_{\lambda\mu}\left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{720}$	
VIII	1	$\frac{1}{56} = \sum_1 c_\kappa \alpha_\kappa^6 = \frac{1}{7}$	VII
	2	$\frac{1}{112} = \sum_2 c_\kappa \alpha_\kappa^4 P_{\kappa 1} = \frac{1}{14}$	
	3	$\frac{1}{336} = \sum_2 c_\kappa \alpha_\kappa^3 Q_{\kappa 1} = \frac{1}{42}$	
	4	$\frac{1}{168} = \sum_2 c_\kappa \alpha_\kappa^3 P_{\kappa 2} = \frac{1}{21}$	
	5	$\frac{1}{672} = \sum_2 c_\kappa \alpha_\kappa^2 Q_{\kappa 2} = \frac{1}{84}$	
	6	$\frac{1}{224} = \sum_2 c_\kappa \alpha_\kappa^2 P_{\kappa 3} = \frac{1}{28}$	
	7	$\frac{1}{224} = \sum_2 c_\kappa \alpha_\kappa^2 P_{\kappa 1}^2 = \frac{1}{28}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
VIII	8	$\frac{1}{1120} = \sum_2 c_\kappa \alpha_\kappa Q_{\kappa 3}$	$= \frac{1}{140}$	VII
	9	$\frac{1}{672} = \sum_2 c_\kappa \alpha_\kappa Q_{\kappa 1} P_{\kappa 1}$	$= \frac{1}{84}$	
	10	$\frac{1}{280} = \sum_2 c_\kappa \alpha_\kappa P_{\kappa 4}$	$= \frac{1}{35}$	
	11	$\frac{1}{336} = \sum_2 c_\kappa \alpha_\kappa P_{\kappa 2} P_{\kappa 1}$	$= \frac{1}{42}$	
	12	$\frac{1}{1680} = \sum_2 c_\kappa Q_{\kappa 4}$	$= \frac{1}{210}$	
	13	$\frac{1}{2016} = \sum_2 c_\kappa Q^2_{\kappa 1}$	$= \frac{1}{252}$	
	14	$\frac{1}{1344} = \sum_2 c_\kappa Q_{\kappa 2} P_{\kappa 1}$	$= \frac{1}{168}$	
	15	$\frac{1}{1008} = \sum_2 c_\kappa Q_{\kappa 1} P_{\kappa 2}$	$= \frac{1}{126}$	
	16	$\frac{1}{336} = \sum_2 c_\kappa P_{\kappa 5}$	$= \frac{1}{42}$	
	17	$\frac{1}{448} = \sum_2 c_\kappa P_{\kappa 3} P_{\kappa 1}$	$= \frac{1}{56}$	
	18	$\frac{1}{504} = \sum_2 c_\kappa P^2_{\kappa 2}$	$= \frac{1}{63}$	
	19	$\frac{1}{448} = \sum_2 c_\kappa P^3_{\kappa 1}$	$= \frac{1}{56}$	
	20	$\frac{1}{336} = \sum_3 c_\kappa \alpha^3_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{42}$	
	21	$\frac{1}{1344} = \sum_3 c_\kappa \alpha^2_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{168}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa$	$\longrightarrow \dot{x}$	
VIII	22	$\frac{1}{448} = \sum_{3} c_\kappa \alpha_\kappa^2 \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{56}$	VII
	23	$\frac{1}{1344} = \sum_{3} c_\kappa \alpha_\kappa^2 \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{168}$	
	24	$\frac{1}{672} = \sum_{3} c_\kappa \alpha_\kappa^2 \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{84}$	
	25	$\frac{1}{2240} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{280}$	
	26	$\frac{1}{6720} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{840}$	
	27	$\frac{1}{3360} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{420}$	
	28	$\frac{1}{560} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 1} \right)$	$= \frac{1}{70}$	
	29	$\frac{1}{1680} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 1} \right)$	$= \frac{1}{210}$	
	30	$\frac{1}{840} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 2} \right)$	$= \frac{1}{105}$	
	31	$\frac{1}{3360} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 2} \right)$	$= \frac{1}{420}$	
	32	$\frac{1}{1120} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 3} \right)$	$= \frac{1}{140}$	
	33	$\frac{1}{1120} = \sum_{3} c_\kappa \alpha_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}^2 \right)$	$= \frac{1}{140}$	
	34	$\frac{1}{672} = \sum_{3} c_\kappa \alpha_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{84}$	
	35	$\frac{1}{3360} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 1} \right)$	$= \frac{1}{420}$	

TABLE 1. (Continued)

		x ← c_κ, $\dot{c}_\kappa$ → $\dot{x}$		
VIII	36	$\frac{1}{10\,080} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda Q_{\lambda 1} \right)$	$= \frac{1}{1260}$	VII
	37	$\frac{1}{5040} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 2} \right)$	$= \frac{1}{630}$	
	38	$\frac{1}{20\,160} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 2} \right)$	$= \frac{1}{2520}$	
	39	$\frac{1}{6720} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 3} \right)$	$= \frac{1}{840}$	
	40	$\frac{1}{6720} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P^2_{\lambda 1} \right)$	$= \frac{1}{840}$	
	41	$\frac{1}{672} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha^3_\lambda P_{\lambda 1} \right)$	$= \frac{1}{84}$	
	42	$\frac{1}{2016} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha^2_\lambda Q_{\lambda 1} \right)$	$= \frac{1}{252}$	
	43	$\frac{1}{1008} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha^2_\lambda P_{\lambda 2} \right)$	$= \frac{1}{126}$	
	44	$\frac{1}{4032} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 2} \right)$	$= \frac{1}{504}$	
	45	$\frac{1}{1344} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 3} \right)$	$= \frac{1}{168}$	
	46	$\frac{1}{1344} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P^2_{\lambda 1} \right)$	$= \frac{1}{168}$	
	47	$\frac{1}{6720} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 3} \right)$	$= \frac{1}{840}$	
	48	$\frac{1}{4032} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} P_{\lambda 1} \right)$	$= \frac{1}{504}$	
	49	$\frac{1}{1680} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 4} \right)$	$= \frac{1}{210}$	
	50	$\frac{1}{2016} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} P_{\lambda 1} \right)$	$= \frac{1}{252}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$	
VIII	51	$\frac{1}{2016} = \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right)^2 = \frac{1}{252}$	VII
	52	$\frac{1}{2016} = \sum_{3} c_\kappa Q_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{252}$	
	53	$\frac{1}{1008} = \sum_{3} c_\kappa P_{\kappa 2} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{126}$	
	54	$\frac{1}{2688} = \sum_{3} c_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{336}$	
	55	$\frac{1}{896} = \sum_{3} c_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1}\right) = \frac{1}{112}$	
	56	$\frac{1}{2688} = \sum_{3} c_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1}\right) = \frac{1}{336}$	
	57	$\frac{1}{1344} = \sum_{3} c_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2}\right) = \frac{1}{168}$	
	58	$\frac{1}{1344} = \sum_{4} c_\kappa \alpha_\kappa^2 \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{168}$	
	59	$\frac{1}{6720} = \sum_{4} c_\kappa \alpha_\kappa \left[\sum_{3}^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{840}$	
	60	$\frac{1}{1680} = \sum_{4} c_\kappa \alpha_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{210}$	
	61	$\frac{1}{6720} = \sum_{4} c_\kappa \alpha_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{840}$	
	62	$\frac{1}{2240} = \sum_{4} c_\kappa \alpha_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{280}$	
	63	$\frac{1}{6720} = \sum_{4} c_\kappa \alpha_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{840}$	
	64	$\frac{1}{3360} = \sum_{4} c_\kappa \alpha_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{420}$	
	65	$\frac{1}{10\,080} = \sum_{4} c_\kappa \left[\sum_{3}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{1260}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa, \dot{c}_\kappa \longrightarrow \dot{x}$	
VIII	66	$\frac{1}{40\,320} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{5040}$	VII
	67	$\frac{1}{13\,440} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{1680}$	
	68	$\frac{1}{40\,320} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{5040}$	
	69	$\frac{1}{20\,160} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{2520}$	
	70	$\frac{1}{2016} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda^2\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{252}$	
	71	$\frac{1}{8064} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{1008}$	
	72	$\frac{1}{2688} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{336}$	
	73	$\frac{1}{8064} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{1008}$	
	74	$\frac{1}{4032} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{504}$	
	75	$\frac{1}{13\,440} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{1680}$	
	76	$\frac{1}{40\,320} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{5040}$	
	77	$\frac{1}{20\,160} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{2520}$	
	78	$\frac{1}{3360} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu^2 P_{\mu 1}\right)\right] = \frac{1}{420}$	
	79	$\frac{1}{10\,080} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu Q_{\mu 1}\right)\right] = \frac{1}{1260}$	
	80	$\frac{1}{5040} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 2}\right)\right] = \frac{1}{630}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$	
VIII	81	$\frac{1}{20\,160} = \sum_{4} c_\kappa \sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \sum_{2}^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 2} = \frac{1}{2520}$	VII
	82	$\frac{1}{6720} = \sum_{4} c_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 3}\right)\right] = \frac{1}{840}$	
	83	$\frac{1}{6720} = \sum_{4} c_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P^2_{\mu 1}\right)\right] = \frac{1}{840}$	
	84	$\frac{1}{4032} = \sum_{4} c_\kappa \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{504}$	
	85	$\frac{1}{2688} = \sum_{4} c_\kappa P_{\kappa 1} \left[\sum_{3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{336}$	
	86	$\frac{1}{6720} = \sum_{5} c_\kappa \alpha_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{840}$	
	87	$\frac{1}{40\,320} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{5040}$	
	88	$\frac{1}{8064} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{1008}$	
	89	$\frac{1}{40\,320} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{5040}$	
	90	$\frac{1}{10\,080} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{1260}$	
	91	$\frac{1}{40\,320} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \gamma_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{5040}$	
	92	$\frac{1}{13\,440} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} \alpha_\nu P_{\nu 1}\right)\right]\right\} = \frac{1}{1680}$	
	93	$\frac{1}{40\,320} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} Q_{\nu 1}\right)\right]\right\} = \frac{1}{5040}$	
	94	$\frac{1}{20\,160} = \sum_{5} c_\kappa \left\{\sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 2}\right)\right]\right\} = \frac{1}{2520}$	
	95	$\frac{1}{40\,320} = \sum_{6} c_\kappa \left\langle\sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{\sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1}\right)\right]\right\}\right\rangle = \frac{1}{5040}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa$	$\longrightarrow \dot{x}$	
IX	1	$\frac{1}{72} = \sum_{1} c_\kappa \alpha_\kappa^7$	$= \frac{1}{8}$	VIII
	2	$\frac{1}{144} = \sum_{2} c_\kappa \alpha_\kappa^5 P_{\kappa 1}$	$= \frac{1}{16}$	
	3	$\frac{1}{432} = \sum_{2} c_\kappa \alpha_\kappa^4 Q_{\kappa 1}$	$= \frac{1}{48}$	
	4	$\frac{1}{216} = \sum_{2} c_\kappa \alpha_\kappa^4 P_{\kappa 2}$	$= \frac{1}{24}$	
	5	$\frac{1}{864} = \sum_{2} c_\kappa \alpha_\kappa^3 Q_{\kappa 2}$	$= \frac{1}{96}$	
	6	$\frac{1}{288} = \sum_{2} c_\kappa \alpha_\kappa^3 P_{\kappa 3}$	$= \frac{1}{32}$	
	7	$\frac{1}{288} = \sum_{2} c_\kappa \alpha_\kappa^3 P_{\kappa 1}^2$	$= \frac{1}{32}$	
	8	$\frac{1}{1440} = \sum_{2} c_\kappa \alpha_\kappa^2 Q_{\kappa 3}$	$= \frac{1}{160}$	
	9	$\frac{1}{864} = \sum_{2} c_\kappa \alpha_\kappa^2 Q_{\kappa 1} P_{\kappa 1}$	$= \frac{1}{96}$	
	10	$\frac{1}{360} = \sum_{2} c_\kappa \alpha_\kappa^2 P_{\kappa 4}$	$= \frac{1}{40}$	
	11	$\frac{1}{432} = \sum_{2} c_\kappa \alpha_\kappa^2 P_{\kappa 2} P_{\kappa 1}$	$= \frac{1}{48}$	
	12	$\frac{1}{2160} = \sum_{2} c_\kappa \alpha_\kappa Q_{\kappa 4}$	$= \frac{1}{240}$	
	13	$\frac{1}{1728} = \sum_{2} c_\kappa \alpha_\kappa Q_{\kappa 2} P_{\kappa 1}$	$= \frac{1}{192}$	
	14	$\frac{1}{2592} = \sum_{2} c_\kappa \alpha_\kappa Q_{\kappa 1}^2$	$= \frac{1}{288}$	
	15	$\frac{1}{1296} = \sum_{2} c_\kappa \alpha_\kappa Q_{\kappa 1} P_{\kappa 2}$	$= \frac{1}{144}$	
	16	$\frac{1}{432} = \sum_{2} c_\kappa \alpha_\kappa P_{\kappa 5}$	$= \frac{1}{48}$	
	17	$\frac{1}{576} = \sum_{2} c_\kappa \alpha_\kappa P_{\kappa 3} P_{\kappa 1}$	$= \frac{1}{64}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
IX	18	$\frac{1}{648} = \sum_2 c_\kappa \alpha_\kappa P^2_{\kappa 2}$	$= \frac{1}{72}$	VIII
	19	$\frac{1}{576} = \sum_2 c_\kappa \alpha_\kappa P^3_{\kappa 1}$	$= \frac{1}{64}$	
	20	$\frac{1}{3024} = \sum_2 c_\kappa Q_{\kappa 5}$	$= \frac{1}{336}$	
	21	$\frac{1}{2880} = \sum_2 c_\kappa Q_{\kappa 3} P_{\kappa 1}$	$= \frac{1}{320}$	
	22	$\frac{1}{5184} = \sum_2 c_\kappa Q_{\kappa 2} Q_{\kappa 1}$	$= \frac{1}{576}$	
	23	$\frac{1}{2592} = \sum_2 c_\kappa Q_{\kappa 2} P_{\kappa 2}$	$= \frac{1}{288}$	
	24	$\frac{1}{1728} = \sum_2 c_\kappa Q_{\kappa 1} P_{\kappa 3}$	$= \frac{1}{192}$	
	25	$\frac{1}{1728} = \sum_2 c_\kappa Q_{\kappa 1} P^2_{\kappa 1}$	$= \frac{1}{192}$	
	26	$\frac{1}{504} = \sum_2 c_\kappa P_{\kappa 6}$	$= \frac{1}{56}$	
	27	$\frac{1}{720} = \sum_2 c_\kappa P_{\kappa 4} P_{\kappa 1}$	$= \frac{1}{80}$	
	28	$\frac{1}{864} = \sum_2 c_\kappa P_{\kappa 3} P_{\kappa 2}$	$= \frac{1}{96}$	
	29	$\frac{1}{864} = \sum_2 c_\kappa P_{\kappa 2} P^2_{\kappa 1}$	$= \frac{1}{96}$	
	30	$\frac{1}{432} = \sum_3 c_\kappa \alpha^4_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{48}$	
	31	$\frac{1}{1728} = \sum_3 c_\kappa \alpha^3_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{192}$	
	32	$\frac{1}{576} = \sum_3 c_\kappa \alpha^3_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{64}$	
	33	$\frac{1}{1728} = \sum_3 c_\kappa \alpha^3_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{192}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
IX	34	$\frac{1}{864} = \sum_3 c_\kappa \alpha_\kappa^3 \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2}\right)$	$= \frac{1}{96}$	VIII
	35	$\frac{1}{2880} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 1}\right)$	$= \frac{1}{320}$	
	36	$\frac{1}{8640} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 1}\right)$	$= \frac{1}{960}$	
	37	$\frac{1}{4320} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 2}\right)$	$= \frac{1}{480}$	
	38	$\frac{1}{720} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 1}\right)$	$= \frac{1}{80}$	
	39	$\frac{1}{2160} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 1}\right)$	$= \frac{1}{240}$	
	40	$\frac{1}{1080} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 2}\right)$	$= \frac{1}{120}$	
	41	$\frac{1}{4320} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 2}\right)$	$= \frac{1}{480}$	
	42	$\frac{1}{1440} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 3}\right)$	$= \frac{1}{160}$	
	43	$\frac{1}{1440} = \sum_3 c_\kappa \alpha_\kappa^2 \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}^2\right)$	$= \frac{1}{160}$	
	44	$\frac{1}{864} = \sum_3 c_\kappa \alpha_\kappa^2 P_{\kappa 1} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right)$	$= \frac{1}{96}$	
	45	$\frac{1}{4320} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 1}\right)$	$= \frac{1}{480}$	
	46	$\frac{1}{12\,960} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda Q_{\lambda 1}\right)$	$= \frac{1}{1440}$	
	47	$\frac{1}{6480} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 2}\right)$	$= \frac{1}{720}$	
	48	$\frac{1}{25\,920} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 2}\right)$	$= \frac{1}{2880}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$			
IX	49	$\frac{1}{8640} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 3} \right)$	=	$\frac{1}{960}$	VIII
	50	$\frac{1}{8640} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P^2_{\lambda 1} \right)$	=	$\frac{1}{960}$	
	51	$\frac{1}{864} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha^3_\lambda P_{\lambda 1} \right)$	=	$\frac{1}{96}$	
	52	$\frac{1}{2592} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha^2_\lambda Q_{\lambda 1} \right)$	=	$\frac{1}{288}$	
	53	$\frac{1}{1296} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha^2_\lambda P_{\lambda 2} \right)$	=	$\frac{1}{144}$	
	54	$\frac{1}{5184} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 2} \right)$	=	$\frac{1}{576}$	
	55	$\frac{1}{1728} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 3} \right)$	=	$\frac{1}{192}$	
	56	$\frac{1}{1728} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P^2_{\lambda 1} \right)$	=	$\frac{1}{192}$	
	57	$\frac{1}{8640} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 3} \right)$	=	$\frac{1}{960}$	
	58	$\frac{1}{5184} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} P_{\lambda 1} \right)$	=	$\frac{1}{576}$	
	59	$\frac{1}{2160} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 4} \right)$	=	$\frac{1}{240}$	
	60	$\frac{1}{2592} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} P_{\lambda 1} \right)$	=	$\frac{1}{288}$	
	61	$\frac{1}{2592} = \sum_3 c_\kappa \alpha_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)^2$	=	$\frac{1}{288}$	
	62	$\frac{1}{2592} = \sum_3 c_\kappa \alpha_\kappa Q_{\kappa 1} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	=	$\frac{1}{288}$	
	63	$\frac{1}{1296} = \sum_3 c_\kappa \alpha_\kappa P_{\kappa 2} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	=	$\frac{1}{144}$	
	64	$\frac{1}{3456} = \sum_3 c_\kappa \alpha_\kappa P_{\kappa 1} \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right)$	=	$\frac{1}{384}$	

TABLE 1. (Continued)

		$x \longleftarrow$	$c_\kappa,\ \dot{c}_\kappa$	$\longrightarrow \dot{x}$	
IX	65	$\frac{1}{1152}$	$= \sum_{3} c_\kappa \alpha_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{128}$	VIII
	66	$\frac{1}{3456}$	$= \sum_{3} c_\kappa \alpha_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{384}$	
	67	$\frac{1}{1728}$	$= \sum_{3} c_\kappa \alpha_\kappa P_{\kappa 1} \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{192}$	
	68	$\frac{1}{6048}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda^3 P_{\lambda 1} \right)$	$= \frac{1}{672}$	
	69	$\frac{1}{18\,144}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda^2 Q_{\lambda 1} \right)$	$= \frac{1}{2016}$	
	70	$\frac{1}{9072}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 2} \right)$	$= \frac{1}{1008}$	
	71	$\frac{1}{36\,288}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda Q_{\lambda 2} \right)$	$= \frac{1}{4032}$	
	72	$\frac{1}{12\,096}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 3} \right)$	$= \frac{1}{1344}$	
	73	$\frac{1}{12\,096}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 1}^2 \right)$	$= \frac{1}{1344}$	
	74	$\frac{1}{60\,480}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 3} \right)$	$= \frac{1}{6720}$	
	75	$\frac{1}{36\,288}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 1} P_{\lambda 1} \right)$	$= \frac{1}{4032}$	
	76	$\frac{1}{15\,120}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 4} \right)$	$= \frac{1}{1680}$	
	77	$\frac{1}{18\,144}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 2} P_{\lambda 1} \right)$	$= \frac{1}{2016}$	
	78	$\frac{1}{10\,368}$	$= \sum_{3} c_\kappa \left[\left(\sum_{2}^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right) \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right) \right]$	$= \frac{1}{1152}$	
	79	$\frac{1}{1008}$	$= \sum_{3} c_\kappa \left(\sum_{2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^4 P_{\lambda 1} \right)$	$= \frac{1}{112}$	

TABLE 1. (Continued)

		$x \longleftarrow$ c_κ, $\dot{c}_\kappa$ $\longrightarrow \dot{x}$		
IX	80	$\frac{1}{3024} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^3 Q_{\lambda 1} \right)$	$= \frac{1}{336}$	VIII
	81	$\frac{1}{1512} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^3 P_{\lambda 2} \right)$	$= \frac{1}{168}$	
	82	$\frac{1}{6048} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 Q_{\lambda 2} \right)$	$= \frac{1}{672}$	
	83	$\frac{1}{2016} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 3} \right)$	$= \frac{1}{224}$	
	84	$\frac{1}{2016} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 P_{\lambda 1}^2 \right)$	$= \frac{1}{224}$	
	85	$\frac{1}{10\,080} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 3} \right)$	$= \frac{1}{1120}$	
	86	$\frac{1}{6048} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 1} P_{\lambda 1} \right)$	$= \frac{1}{672}$	
	87	$\frac{1}{2520} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 4} \right)$	$= \frac{1}{280}$	
	88	$\frac{1}{3024} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 2} P_{\lambda 1} \right)$	$= \frac{1}{336}$	
	89	$\frac{1}{15\,120} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 4} \right)$	$= \frac{1}{1680}$	
	90	$\frac{1}{12\,096} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 2} P_{\lambda 1} \right)$	$= \frac{1}{1344}$	
	91	$\frac{1}{18\,144} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1}^2 \right)$	$= \frac{1}{2016}$	
	92	$\frac{1}{9072} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} P_{\lambda 2} \right)$	$= \frac{1}{1008}$	
	93	$\frac{1}{3024} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 5} \right)$	$= \frac{1}{336}$	
	94	$\frac{1}{4032} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 3} P_{\lambda 1} \right)$	$= \frac{1}{448}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow$	$\dot{x}$	
IX	95	$\frac{1}{4536} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P^2_{\lambda 2} \right)$	$= \frac{1}{504}$	VIII
	96	$\frac{1}{4032} = \sum_3 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P^3_{\lambda 1} \right)$	$= \frac{1}{448}$	
	97	$\frac{1}{10\,368} = \sum_3 c_\kappa \left[\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right) \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right) \right]$	$= \frac{1}{1152}$	
	98	$\frac{1}{3456} = \sum_3 c_\kappa \left[\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right) \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right) \right]$	$= \frac{1}{384}$	
	99	$\frac{1}{5184} = \sum_3 c_\kappa \left[\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right) \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right) \right]$	$= \frac{1}{576}$	
	100	$\frac{1}{5184} = \sum_3 c_\kappa Q_{\kappa 2} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{576}$	
	101	$\frac{1}{10\,368} = \sum_3 c_\kappa Q_{\kappa 1} \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{1152}$	
	102	$\frac{1}{3456} = \sum_3 c_\kappa Q_{\kappa 1} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{384}$	
	103	$\frac{1}{10\,368} = \sum_3 c_\kappa Q_{\kappa 1} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{1152}$	
	104	$\frac{1}{5184} = \sum_3 c_\kappa Q_{\kappa 1} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{576}$	
	105	$\frac{1}{1728} = \sum_3 c_\kappa P_{\kappa 3} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{192}$	
	106	$\frac{1}{5184} = \sum_3 c_\kappa P_{\kappa 2} \left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \right)$	$= \frac{1}{576}$	
	107	$\frac{1}{1728} = \sum_3 c_\kappa P_{\kappa 2} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right)$	$= \frac{1}{192}$	
	108	$\frac{1}{5184} = \sum_3 c_\kappa P_{\kappa 2} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1} \right)$	$= \frac{1}{576}$	
	109	$\frac{1}{2592} = \sum_3 c_\kappa P_{\kappa 2} \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right)$	$= \frac{1}{288}$	

TABLE 1. (Continued)

		x	$\longleftarrow c_\kappa, \dot{c}_\kappa \longrightarrow$	$\dot{x}$	
IX	110	$\frac{1}{1728}$	$= \sum_3 c_\kappa P^2_{\kappa 1}\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right) =$	$\frac{1}{192}$	VIII
	111	$\frac{1}{5760}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda P_{\lambda 1}\right) =$	$\frac{1}{640}$	
	112	$\frac{1}{17\,280}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} Q_{\lambda 1}\right) =$	$\frac{1}{1920}$	
	113	$\frac{1}{8640}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 2}\right) =$	$\frac{1}{960}$	
	114	$\frac{1}{1440}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha^2_\lambda P_{\lambda 1}\right) =$	$\frac{1}{160}$	
	115	$\frac{1}{4320}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda Q_{\lambda 1}\right) =$	$\frac{1}{480}$	
	116	$\frac{1}{2160}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 2}\right) =$	$\frac{1}{240}$	
	117	$\frac{1}{8640}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 2}\right) =$	$\frac{1}{960}$	
	118	$\frac{1}{2880}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 3}\right) =$	$\frac{1}{320}$	
	119	$\frac{1}{2880}$	$= \sum_3 c_\kappa P_{\kappa 1}\left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P^2_{\lambda 1}\right) =$	$\frac{1}{320}$	
	120	$\frac{1}{1728}$	$= \sum_4 c_\kappa \alpha^3_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] =$	$\frac{1}{192}$	
	121	$\frac{1}{8640}$	$= \sum_4 c_\kappa \alpha^2_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] =$	$\frac{1}{960}$	
	122	$\frac{1}{2160}$	$= \sum_4 c_\kappa \alpha^2_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] =$	$\frac{1}{240}$	
	123	$\frac{1}{8640}$	$= \sum_4 c_\kappa \alpha^2_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] =$	$\frac{1}{960}$	
	124	$\frac{1}{2880}$	$= \sum_4 c_\kappa \alpha^2_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 1}\right)\right] =$	$\frac{1}{320}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa, \dot{c}_\kappa \longrightarrow \dot{x}$	
IX	125	$\frac{1}{8640} = \sum_4 c_\kappa \alpha_\kappa^2 \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{960}$	VIII
	126	$\frac{1}{4320} = \sum_4 c_\kappa \alpha_\kappa^2 \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{480}$	
	127	$\frac{1}{12\,960} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{1440}$	
	128	$\frac{1}{51\,840} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{5760}$	
	129	$\frac{1}{17\,280} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{1920}$	
	130	$\frac{1}{51\,840} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{5760}$	
	131	$\frac{1}{25\,920} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{2880}$	
	132	$\frac{1}{2592} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda^2\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{288}$	
	133	$\frac{1}{10\,368} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{1152}$	
	134	$\frac{1}{3456} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{384}$	
	135	$\frac{1}{10\,368} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{1152}$	
	136	$\frac{1}{5184} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{576}$	
	137	$\frac{1}{17\,280} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{1920}$	
	138	$\frac{1}{51\,840} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{5760}$	
	139	$\frac{1}{25\,920} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{2880}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$	
IX	140	$\frac{1}{4320} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu^2 P_{\mu 1}\right)\right] = \frac{1}{480}$	VIII
	141	$\frac{1}{12\,960} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu Q_{\mu 1}\right)\right] = \frac{1}{1440}$	
	142	$\frac{1}{6480} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 2}\right)\right] = \frac{1}{720}$	
	143	$\frac{1}{25\,920} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 2}\right)\right] = \frac{1}{2880}$	
	144	$\frac{1}{8640} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 3}\right)\right] = \frac{1}{960}$	
	145	$\frac{1}{8640} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}^2\right)\right] = \frac{1}{960}$	
	146	$\frac{1}{5184} = \sum_4 c_\kappa \alpha_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{576}$	
	147	$\frac{1}{3456} = \sum_4 c_\kappa \alpha_\kappa P_{\kappa 1} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{384}$	
	148	$\frac{1}{18\,144} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda^2\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{2016}$	
	149	$\frac{1}{72\,576} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{8064}$	
	150	$\frac{1}{24\,192} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{2688}$	
	151	$\frac{1}{72\,576} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{8064}$	
	152	$\frac{1}{36\,288} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\alpha_\lambda\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{4032}$	
	153	$\frac{1}{120\,960} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{13\,440}$	
	154	$\frac{1}{362\,880} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{40\,320}$	

TABLE 1. (Continued)

		$x \longleftarrow$ $c_\kappa,\ \dot{c}_\kappa$ $\longrightarrow \dot{x}$	
IX	155	$\frac{1}{181\,440} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{20\,160}$	VIII
	156	$\frac{1}{30\,240} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu^2 P_{\mu 1}\right)\right] = \frac{1}{3360}$	
	157	$\frac{1}{90\,720} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu Q_{\mu 1}\right)\right] = \frac{1}{10\,080}$	
	158	$\frac{1}{45\,360} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 2}\right)\right] = \frac{1}{5040}$	
	159	$\frac{1}{181\,440} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 2}\right)\right] = \frac{1}{20\,160}$	
	160	$\frac{1}{60\,480} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 3}\right)\right] = \frac{1}{6720}$	
	161	$\frac{1}{60\,480} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}^2\right)\right] = \frac{1}{6720}$	
	162	$\frac{1}{36\,288} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} P_{\lambda 1} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{4032}$	
	163	$\frac{1}{10\,368} = \sum_4 c_\kappa Q_{\kappa 1} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{1152}$	
	164	$\frac{1}{3024} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^3 \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{336}$	
	165	$\frac{1}{12\,096} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{1344}$	
	166	$\frac{1}{4032} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{448}$	
	167	$\frac{1}{12\,096} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{1344}$	
	168	$\frac{1}{6048} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{672}$	
	169	$\frac{1}{20\,160} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} \alpha_\mu P_{\mu 1}\right)\right] = \frac{1}{2240}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$	
IX	170	$\frac{1}{60\,480} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} Q_{\mu 1}\right)\right] = \frac{1}{6720}$	VIII
	171	$\frac{1}{30\,240} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{3360}$	
	172	$\frac{1}{5040} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu^2 P_{\mu 1}\right)\right] = \frac{1}{560}$	
	173	$\frac{1}{15\,120} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu Q_{\mu 1}\right)\right] = \frac{1}{1680}$	
	174	$\frac{1}{7560} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 2}\right)\right] = \frac{1}{840}$	
	175	$\frac{1}{30\,240} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 2}\right)\right] = \frac{1}{3360}$	
	176	$\frac{1}{10\,080} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 3}\right)\right] = \frac{1}{1120}$	
	177	$\frac{1}{10\,080} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}^2\right)\right] = \frac{1}{1120}$	
	178	$\frac{1}{6048} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda P_{\lambda 1} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{672}$	
	179	$\frac{1}{30\,240} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} \alpha_\mu^2 P_{\mu 1}\right)\right] = \frac{1}{3360}$	
	180	$\frac{1}{90\,720} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} \alpha_\mu Q_{\mu 1}\right)\right] = \frac{1}{10\,080}$	
	181	$\frac{1}{45\,360} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} \alpha_\mu P_{\mu 2}\right)\right] = \frac{1}{5040}$	
	182	$\frac{1}{181\,440} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} Q_{\mu 2}\right)\right] = \frac{1}{20\,160}$	
	183	$\frac{1}{60\,480} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 3}\right)\right] = \frac{1}{6720}$	
	184	$\frac{1}{60\,480} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}^2\right)\right] = \frac{1}{6720}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa, \dot{c}_\kappa \longrightarrow \dot{x}$		
IX	185	$\frac{1}{18\,144} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} Q_{\lambda 1}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{2016}$	
	186	$\frac{1}{6048} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu^3 P_{\mu 1}\right)\right]$	$= \frac{1}{672}$	
	187	$\frac{1}{18\,144} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu^2 Q_{\mu 1}\right)\right]$	$= \frac{1}{2016}$	
	188	$\frac{1}{9072} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu^2 P_{\mu 2}\right)\right]$	$= \frac{1}{1008}$	
	189	$\frac{1}{36\,288} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu Q_{\mu 2}\right)\right]$	$= \frac{1}{4032}$	
	190	$\frac{1}{12\,096} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 3}\right)\right]$	$= \frac{1}{1344}$	
	191	$\frac{1}{12\,096} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 1}^2\right)\right]$	$= \frac{1}{1344}$	
	192	$\frac{1}{60\,480} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 3}\right)\right]$	$= \frac{1}{6720}$	
	193	$\frac{1}{36\,288} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1} P_{\mu 1}\right)\right]$	$= \frac{1}{4032}$	
	194	$\frac{1}{15\,120} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 4}\right)\right]$	$= \frac{1}{1680}$	
	195	$\frac{1}{18\,144} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2} P_{\mu 1}\right)\right]$	$= \frac{1}{2016}$	
	196	$\frac{1}{18\,144} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)^2\right]$	$= \frac{1}{2016}$	
	197	$\frac{1}{9072} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{1008}$	
	198	$\frac{1}{24\,192} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{2688}$	
	199	$\frac{1}{8064} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 1}\right)\right]$	$= \frac{1}{896}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
IX	200	$\frac{1}{24\,192} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right]$	$= \frac{1}{2688}$	VIII
	201	$\frac{1}{12\,096} = \sum_4 c_\kappa \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right]$	$= \frac{1}{1344}$	
	202	$\frac{1}{5184} = \sum_4 c_\kappa P_{\kappa 2} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{576}$	
	203	$\frac{1}{17\,280} = \sum_4 c_\kappa P_{\kappa 1} \left[\sum_3^{\kappa-1} \gamma_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{1920}$	
	204	$\frac{1}{4320} = \sum_4 c_\kappa P_{\kappa 1} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{480}$	
	205	$\frac{1}{17\,280} = \sum_4 c_\kappa P_{\kappa 1} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \gamma_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{1920}$	
	206	$\frac{1}{5760} = \sum_4 c_\kappa P_{\kappa 1} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu P_{\mu 1}\right)\right]$	$= \frac{1}{640}$	
	207	$\frac{1}{17\,280} = \sum_4 c_\kappa P_{\kappa 1} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} Q_{\mu 1}\right)\right]$	$= \frac{1}{1920}$	
	208	$\frac{1}{8640} = \sum_4 c_\kappa P_{\kappa 1} \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2}\right)\right]$	$= \frac{1}{960}$	
	209	$\frac{1}{10\,368} = \sum_4 c_\kappa \left(\sum_2^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right) \left[\sum_3^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_2^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right]$	$= \frac{1}{1152}$	
	210	$\frac{1}{8640} = \sum_5 c_\kappa \alpha_\kappa^2 \left\{\sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\}$	$= \frac{1}{960}$	
	211	$\frac{1}{51\,840} = \sum_5 c_\kappa \alpha_\kappa \left\{\sum_4^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\}$	$= \frac{1}{5760}$	
	212	$\frac{1}{10\,368} = \sum_5 c_\kappa \alpha_\kappa \left\{\sum_4^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\}$	$= \frac{1}{1152}$	
	213	$\frac{1}{51\,840} = \sum_5 c_\kappa \alpha_\kappa \left\{\sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\}$	$= \frac{1}{5760}$	
	214	$\frac{1}{12\,960} = \sum_5 c_\kappa \alpha_\kappa \left\{\sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1}\right)\right]\right\}$	$= \frac{1}{1440}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$	
IX	215	$\frac{1}{51\,840} = \sum_{5} c_\kappa \alpha_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \gamma_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{5760}$	VIII
	216	$\frac{1}{17\,280} = \sum_{5} c_\kappa \alpha_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} \alpha_\nu P_{\nu 1} \right) \right] \right\} = \frac{1}{1920}$	
	217	$\frac{1}{51\,840} = \sum_{5} c_\kappa \alpha_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} Q_{\nu 1} \right) \right] \right\} = \frac{1}{5760}$	
	218	$\frac{1}{25\,920} = \sum_{5} c_\kappa \alpha_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 2} \right) \right] \right\} = \frac{1}{2880}$	
	219	$\frac{1}{72\,576} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \alpha_\lambda \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{8064}$	
	220	$\frac{1}{362\,880} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{40\,320}$	
	221	$\frac{1}{90\,720} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{10\,080}$	
	222	$\frac{1}{362\,880} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \gamma_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{40\,320}$	
	223	$\frac{1}{120\,960} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} \alpha_\nu P_{\nu 1} \right) \right] \right\} = \frac{1}{13\,440}$	
	224	$\frac{1}{362\,880} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} Q_{\nu 1} \right) \right] \right\} = \frac{1}{40\,320}$	
	225	$\frac{1}{181\,440} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \gamma_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 2} \right) \right] \right\} = \frac{1}{20\,160}$	
	226	$\frac{1}{12\,096} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda^2 \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{1344}$	
	227	$\frac{1}{60\,480} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left[\sum_{3}^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{6720}$	
	228	$\frac{1}{15\,120} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{1680}$	
	229	$\frac{1}{60\,480} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \gamma_{\mu\nu} P_{\nu 1} \right) \right] \right\} = \frac{1}{6720}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow x$		
IX	230	$\frac{1}{20\,160} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu}\alpha_\nu P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{2240}$	VIII
	231	$\frac{1}{60\,480} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} Q_{\nu 1} \right) \right] \right\}$	$= \frac{1}{6720}$	
	232	$\frac{1}{30\,240} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 2} \right) \right] \right\}$	$= \frac{1}{3360}$	
	233	$\frac{1}{90\,720} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \gamma_{\lambda\mu}\alpha_\mu \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{10\,080}$	
	234	$\frac{1}{362\,880} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_2^{\mu-1} \gamma_{\mu\nu} P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{40\,320}$	
	235	$\frac{1}{120\,960} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu}\alpha_\nu P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{13\,440}$	
	236	$\frac{1}{362\,880} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} Q_{\nu 1} \right) \right] \right\}$	$= \frac{1}{40\,320}$	
	237	$\frac{1}{181\,440} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \gamma_{\lambda\mu} \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 2} \right) \right] \right\}$	$= \frac{1}{20\,160}$	
	238	$\frac{1}{18\,144} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu^2 \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{2016}$	
	239	$\frac{1}{72\,576} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu \left(\sum_2^{\mu-1} \gamma_{\mu\nu} P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{8064}$	
	240	$\frac{1}{24\,192} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu \left(\sum_2^{\mu-1} \beta_{\mu\nu}\alpha_\nu P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{2688}$	
	241	$\frac{1}{72\,576} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu \left(\sum_2^{\mu-1} \beta_{\mu\nu} Q_{\nu 1} \right) \right] \right\}$	$= \frac{1}{8064}$	
	242	$\frac{1}{36\,288} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu \left(\sum_2^{\mu-1} \beta_{\mu\nu} P_{\nu 2} \right) \right] \right\}$	$= \frac{1}{4032}$	
	243	$\frac{1}{120\,960} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \gamma_{\mu\nu}\alpha_\nu P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{13\,440}$	
	244	$\frac{1}{362\,880} = \sum_5 c_\kappa \left\{ \sum_4^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_3^{\lambda-1} \beta_{\lambda\mu} \left(\sum_2^{\mu-1} \gamma_{\mu\nu} Q_{\nu 1} \right) \right] \right\}$	$= \frac{1}{40\,320}$	

TABLE 1. (Continued)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
IX	245	$\frac{1}{181\,440} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \gamma_{\mu\nu} P_{\nu 2} \right) \right] \right\}$	$= \frac{1}{20\,160}$	VIII
	246	$\frac{1}{30\,240} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} \alpha_\nu^2 P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{3360}$	
	247	$\frac{1}{90\,720} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} \alpha_\nu Q_{\nu 1} \right) \right] \right\}$	$= \frac{1}{10\,080}$	
	248	$\frac{1}{45\,360} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} \alpha_\nu P_{\nu 2} \right) \right] \right\}$	$= \frac{1}{5040}$	
	249	$\frac{1}{181\,440} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} Q_{\nu 2} \right) \right] \right\}$	$= \frac{1}{20\,160}$	
	250	$\frac{1}{60\,480} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 3} \right) \right] \right\}$	$= \frac{1}{6720}$	
	251	$\frac{1}{60\,480} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P^2_{\nu 1} \right) \right] \right\}$	$= \frac{1}{6720}$	
	252	$\frac{1}{36\,288} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{4032}$	
	253	$\frac{1}{24\,192} = \sum_{5} c_\kappa \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{2688}$	
	254	$\frac{1}{17\,280} = \sum_{5} c_\kappa P_{\kappa 1} \left\{ \sum_{4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\}$	$= \frac{1}{1920}$	
	255	$\frac{1}{51\,840} = \sum_{6} c_\kappa \alpha_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle$	$= \frac{1}{5760}$	
	256	$\frac{1}{362\,880} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \gamma_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle$	$= \frac{1}{40\,320}$	
	257	$\frac{1}{60\,480} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle$	$= \frac{1}{6720}$	
	258	$\frac{1}{362\,880} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \gamma_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle$	$= \frac{1}{40\,320}$	
	259	$\frac{1}{72\,576} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle$	$= \frac{1}{8064}$	

TABLE 1. (Concluded)

		$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$	
IX	260	$\frac{1}{362\ 880} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \gamma_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle = \frac{1}{40\ 320}$	VIII
	261	$\frac{1}{90\ 720} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \alpha_\nu \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle = \frac{1}{10\ 080}$	
	262	$\frac{1}{362\ 880} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \gamma_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle = \frac{1}{40\ 320}$	
	263	$\frac{1}{120\ 960} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} \alpha_\rho P_{\rho 1} \right) \right] \right\} \right\rangle = \frac{1}{13\ 440}$	
	264	$\frac{1}{362\ 880} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} Q_{\rho 1} \right) \right] \right\} \right\rangle = \frac{1}{40\ 320}$	
	265	$\frac{1}{181\ 440} = \sum_{6} c_\kappa \left\langle \sum_{5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum_{4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum_{3}^{\mu-1} \beta_{\mu\nu} \left(\sum_{2}^{\nu-1} \beta_{\nu\rho} P_{\rho 2} \right) \right] \right\} \right\rangle = \frac{1}{20\ 160}$	
	266	$\frac{1}{362\ 880} = \sum_{7} c_\kappa \left(\sum_{6}^{\kappa-1} \beta_{\kappa\lambda} \left\langle \sum_{5}^{\lambda-1} \beta_{\lambda\mu} \left\{ \sum_{4}^{\mu-1} \beta_{\mu\nu} \left[\sum_{3}^{\nu-1} \beta_{\nu\rho} \left(\sum_{2}^{\rho-1} \beta_{\rho\sigma} P_{\sigma 1} \right) \right] \right\} \right\rangle \right) = \frac{1}{40\ 320}$	

TABLE 2. PATTERN FOR RKN-G-7(8)-13

$\kappa \backslash \lambda$	α_κ	$\beta_{\kappa\lambda}$													$\gamma_{\kappa\lambda}$												
		0	1	2	3	4	5	6	7	8	9	10	11	12	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0													0												
1	*	*													*												
2	*	*	*												*	*											
3	*	*	0	*											*	0	*										
4	*	*	0	*	*										*	0	0	*									
5	*	*	0	0	*	*									*	0	*	*	*								
6	*	*	0	0	*	*	*								*	0	0	*	*	*							
7	*	*	0	0	0	0	*	*							*	0	0	*	*	*	*						
8	*	*	0	0	0	0	*	*	*						*	0	0	0	*	*	*	*					
9	*	*	0	0	0	0	*	*	*	*					*	0	0	0	0	*	*	*	*				
10	*	*	0	0	0	0	0	*	*	*	*				*	0	0	0	0	0	*	*	*	*			
11	*	*	0	0	0	0	*	*	*	*	*	*			*	0	0	*	*	0	0	*	*	*	*		
12	1	*	0	0	0	0	*	*	*	*	*	*	*		*	0	0	*	*	0	0	*	*	*	*	*	
13	1	*	0	0	0	0	0	0	*	*	*	*	*	*	*	0	0	0	0	0	0	*	*	*	*	*	*

Local truncation error in x: TE = $\gamma_{1312}\,(f_{12} - f_{13})\,h^2$

TABLE 3. COEFFICIENTS FOR RKN-G-7(8)-13

α_1	=	0.7347	0804	8410	6438	3606	1035	6687	4183	$\cdot\ 10^{-1}$
α_2	=	0.1102	0620	7261	5965	7540	9155	3503	1127	
α_3	=	0.1653	0931	0892	3948	6311	3733	0254	6691	
α_4	=	0.5								
α_5	=	0.2662	8826	9291	2616	4263	5203	6943	9706	
α_6	=	0.6337	1173	0708	7383	5736	4796	3056	0294	
α_7	=	0.75								
α_8	=	0.5625								
α_9	=	0.125								
α_{10}	=	0.375								
α_{11}	=	0.9665	2805	1602	3570	0834	4516	4211	9099	
α_{12}	=	1								
α_{13}	=	1								

β_{10}	=	0.7347	0804	8410	6438	3606	1035	6687	4183	$\cdot\ 10^{-1}$
β_{20}	=	0.2755	1551	8153	9914	3852	2888	3757	7819	$\cdot\ 10^{-1}$
β_{21}	=	0.8265	4655	4461	9743	1556	8665	1273	3456	$\cdot\ 10^{-1}$
β_{30}	=	0.4132	7327	7230	9871	5778	4332	5636	6728	$\cdot\ 10^{-1}$
β_{31}	=	0								
β_{32}	=	0.1239	8198	3169	2961	4733	5299	7691	0018	
β_{40}	=	0.8967	0558	7637	9578	2658	3783	2538	9875	
β_{41}	=	0								
β_{42}	=	-0.3458	5915	2683	1433	6799	4110	9332	7799	$\cdot\ 10^{+1}$
β_{43}	=	0.3061	8859	3919	3475	8533	5732	6078	8811	$\cdot\ 10^{+1}$
β_{50}	=	0.5705	3369	4239	6532	8229	3636	4664	4660	$\cdot\ 10^{-1}$
β_{51}	=	0								
β_{52}	=	0								
β_{53}	=	0.2066	4670	6954	9824	5793	1750	7585	7561	
β_{54}	=	0.2588	1929	1231	3856	4740	8928	9176	7848	$\cdot\ 10^{-2}$
β_{60}	=	0.2213	0953	4027	3273	8534	2980	5428	5043	$\cdot\ 10^{-1}$
β_{61}	=	0								
β_{62}	=	0								
β_{63}	=	0.3666	6901	8421	5938	0713	9353	1564	8106	
β_{64}	=	0.3207	5560	5327	6707	8702	4980	3819	9180	
β_{65}	=	-0.7584	3846	4432	5897	5333	8352	8715	4967	$\cdot\ 10^{-1}$
β_{70}	=	0.8333	3333	3333	3333	3333	3333	3333	3333	$\cdot\ 10^{-1}$
β_{71}	=	0								
β_{72}	=	0								
β_{73}	=	0								
β_{74}	=	0								

TABLE 3. (Continued)

β_{75} =	0.3843	6436	9641	3162	1037	9110	0848	8971	
β_{76} =	0.2823	0229	7025	3504	5628	7556	5817	7696	
β_{80} =	0.8349	6093	7500	0000	0000	0000	0000	0000	$\cdot 10^{-1}$
β_{81} =	0								
β_{82} =	0								
β_{83} =	0								
β_{84} =	0								
β_{85} =	0.3830	6747	4793	9740	8845	8700	6356	1136	
β_{86} =	0.1354	8721	2706	0259	1154	1299	3643	8864	
β_{87} =	-0.3955	0781	2500	0000	0000	0000	0000	0000	$\cdot 10^{-1}$
β_{90} =	0.7342	0353	2235	9396	4334	7050	7544	5816	$\cdot 10^{-1}$
β_{91} =	0								
β_{92} =	0								
β_{93} =	0								
β_{94} =	0								
β_{95} =	0.9880	8964	9160	2291	6024	2057	1033	6420	$\cdot 10^{-1}$
β_{96} =	0.2415	3311	3273	2774	9549	8428	0345	1955	
β_{97} =	-0.4870	7561	7283	9506	1728	3950	6172	8395	$\cdot 10^{-1}$
β_{98} =	-0.2400	5486	9684	4993	1412	8943	7585	7339	
β_{100} =	0.8137	8441	1270	6706	4904	0410	5640	4207	$\cdot 10^{-2}$
β_{101} =	0								
β_{102} =	0								
β_{103} =	0								
β_{104} =	0								
β_{105} =	0								
β_{106} =	-0.3626	6091	1746	4713	4384	0315	3205	8792	
β_{107} =	0.6972	6880	5971	2792	8317	2726	0984	7243	$\cdot 10^{-1}$
β_{108} =	0.3779	7780	6207	6339	2161	1543	4150	9711	
β_{109} =	0.2818	1838	0829	0027	8742	1095	1900	0315	
β_{110} =	-0.1404	2538	9224	8283	8913	2800	3122	5476	$\cdot 10^{+1}$
β_{111} =	0								
β_{112} =	0								
β_{113} =	0								
β_{114} =	0								
β_{115} =	-0.1355	5559	0294	0495	7528	3041	1334	2361	$\cdot 10^{+2}$
β_{116} =	-0.1502	1472	8248	4805	0961	7213	3096	9968	$\cdot 10^{+1}$
β_{117} =	0.1476	7543	2841	6794	9686	2336	0684	1588	$\cdot 10^{+1}$
β_{118} =	-0.2170	7681	9651	3368	8432	5773	7360	7995	$\cdot 10^{+1}$
β_{119} =	0.6614	9759	5026	7655	8681	0392	0283	3030	$\cdot 10^{+1}$

TABLE 3. (Continued)

β_{1110} =	0.1150	7526	1735	6932	1530	6792	2237	6434	$\cdot 10^{+2}$
β_{120} =	-0.5270	8651	8158	0131	5268	1768	8218	7497	$\cdot 10^{+1}$
β_{121} =	0								
β_{122} =	0								
β_{123} =	0								
β_{124} =	0								
β_{125} =	-0.4996	5599	5536	5683	3001	0459	2152	9105	$\cdot 10^{+2}$
β_{126} =	-0.5030	2228	9286	5823	1516	1351	2481	2231	$\cdot 10^{+1}$
β_{127} =	0.4454	8269	0452	9876	0506	5182	3862	2704	$\cdot 10^{+1}$
β_{128} =	-0.8607	1533	1240	3384	1312	4067	4298	9148	$\cdot 10^{+1}$
β_{129} =	0.2384	0410	0463	7228	7590	0786	7645	6468	$\cdot 10^{+2}$
β_{1210} =	0.4171	1581	4660	2838	8124	0696	6716	4840	$\cdot 10^{+2}$
β_{1211} =	-0.1329	7747	6424	3799	5408	2370	9555	8512	
β_{130} =	0.3509	9303	0565	8188	3152	6601	7368	1744	$\cdot 10^{-1}$
β_{131} =	0								
β_{132} =	0								
β_{133} =	0								
β_{134} =	0								
β_{135} =	0								
β_{136} =	0								
β_{137} =	0.2522	3475	2766	3160	6400	6388	5341	7712	
β_{138} =	0.1184	0033	3068	7654	9234	1625	1536	4336	
β_{139} =	0.2025	8133	6112	5092	9893	1878	9987	1888	
β_{1310} =	0.2675	7025	2594	2014	0796	3933	2927	2621	
β_{1311} =	0.1658	6384	5106	2987	3791	2680	9815	0965	
β_{1312} =	-0.4174	9822	7046	7288	4309	1671	3445	6960	$\cdot 10^{-1}$
γ_{10} =	0.2698	9795	8199	6884	8329	9949	7050	8715	$\cdot 10^{-2}$
γ_{20} =	0.3036	3520	2974	6495	4371	2443	4182	2304	$\cdot 10^{-2}$
γ_{21} =	0.3036	3520	2974	6495	4371	2443	4182	2304	$\cdot 10^{-2}$
γ_{30} =	0.6831	7920	6692	9614	7335	2997	6910	0184	$\cdot 10^{-2}$
γ_{31} =	0								
γ_{32} =	0.6831	7920	6692	9614	7335	2997	6910	0184	$\cdot 10^{-2}$
γ_{40} =	-0.1026	3757	7319	7788	8994	3108	7282	4217	$\cdot 10^{-2}$
γ_{41} =	0								
γ_{42} =	0								
γ_{43} =	0.1260	2637	5773	1977	8889	9431	0872	8242	
γ_{50} =	0.9890	9903	8431	0741	7913	3134	9906	4241	$\cdot 10^{-2}$
γ_{51} =	0								
γ_{52} =	0.2040	1758	7591	1134	9514	1705	1849	8571	$\cdot 10^{-1}$
γ_{53} =	0.5026	5147	7133	2870	3261	8251	0473	5338	$\cdot 10^{-2}$

TABLE 3. (Continued)

γ_{54} =	0.1354	5726	6312	7775	5415	7283	6001	4730	$\cdot 10^{-3}$
γ_{60} =	0.3677	2464	6953	1772	1429	7415	7246	2201	$\cdot 10^{-1}$
γ_{61} =	0								
γ_{62} =	0								
γ_{63} =	0.8213	2294	7785	2178	5827	7217	4140	7693	$\cdot 10^{-1}$
γ_{64} =	0.3008	7165	4090	9896	3036	8709	1811	9641	$\cdot 10^{-1}$
γ_{65} =	0.5180	3353	9359	9379	0519	8241	0553	1789	$\cdot 10^{-1}$
γ_{70} =	0.4123	3049	0882	7287	3123	2210	0402	1091	$\cdot 10^{-1}$
γ_{71} =	0								
γ_{72} =	0								
γ_{73} =	0.1133	5100	2930	6181	9105	3287	9807	8376	
γ_{74} =	0.5672	2148	5922	3766	8841	3016	7743	6715	$\cdot 10^{-1}$
γ_{75} =	0.5745	6202	0649	5452	5469	3769	2473	6474	$\cdot 10^{-1}$
γ_{76} =	0.1248	7597	3239	1674	1512	8124	1302	1961	$\cdot 10^{-1}$
γ_{80} =	0.4214	6301	2695	3125	0000	0000	0000	0000	$\cdot 10^{-1}$
γ_{81} =	0								
γ_{82} =	0								
γ_{83} =	0								
γ_{84} =	-0.7808	8073	7304	6875	0000	0000	0000	0000	$\cdot 10^{-1}$
γ_{85} =	0.1410	4682	1029	2877	2004	3970	8553	6135	
γ_{86} =	0.7460	3813	7363	3727	9956	0291	4463	8648	$\cdot 10^{-1}$
γ_{87} =	-0.2150	5737	3046	8750	0000	0000	0000	0000	$\cdot 10^{-1}$
γ_{90} =	0.5524	3877	1719	2501	1431	1842	7069	0444	$\cdot 10^{-2}$
γ_{91} =	0								
γ_{92} =	0								
γ_{93} =	0								
γ_{94} =	0								
γ_{95} =	0.4591	3375	8935	0515	8838	0181	1180	7029	$\cdot 10^{-2}$
γ_{96} =	0.1200	9956	9922	6813	9808	9279	5562	3138	$\cdot 10^{-1}$
γ_{97} =	-0.2436	1818	4156	3786	0082	3045	2674	8971	$\cdot 10^{-2}$
γ_{98} =	-0.1187	7000	4572	4737	0827	6177	4119	7988	$\cdot 10^{-1}$
γ_{100} =	0.1239	6099	0923	0042	8073	8555	8121	1091	$\cdot 10^{-1}$
γ_{101} =	0								
γ_{102} =	0								
γ_{103} =	0								
γ_{104} =	0								
γ_{105} =	0								
γ_{106} =	-0.2314	8568	8348	8114	9606	8286	3748	4335	$\cdot 10^{-1}$
γ_{107} =	0.4405	7716	3385	9229	4670	5995	3820	0368	$\cdot 10^{-2}$
γ_{108} =	0.2416	4236	8703	9608	6181	8918	2892	7171	$\cdot 10^{-1}$

TABLE 3. (Concluded)

γ_{109}	=	0.5249	4961	2383	2540	5884	0212	7352	6037	$\cdot\ 10^{-1}$
γ_{110}	=	-0.1214	8292	3371	7236	6838	6923	7170	6654	
γ_{111}	=	0								
γ_{112}	=	0								
γ_{113}	=	-0.1594	8786	8094	6904	7245	6588	6876	3595	$\cdot\ 10^{+1}$
γ_{114}	=	0.7708	9844	4095	9035	4601	1435	8063	0038	$\cdot\ 10^{-1}$
γ_{115}	=	0								
γ_{116}	=	0								
γ_{117}	=	0.9884	4932	1354	4261	8048	6243	2844	1900	$\cdot\ 10^{-1}$
γ_{118}	=	-0.1851	7690	1776	5400	9760	1245	5930	3975	
γ_{119}	=	0.1666	5727	1178	0734	2381	8671	5463	0279	$\cdot\ 10^{+1}$
γ_{1110}	=	0.5261	1925	5036	5255	2568	0405	9902	2802	
γ_{120}	=	-0.4947	5846	7641	0233	2689	6037	0954	7782	
γ_{121}	=	0								
γ_{122}	=	0								
γ_{123}	=	-0.5651	3209	6413	6430	5307	6482	3207	0852	$\cdot\ 10^{+1}$
γ_{124}	=	0.4275	0028	7290	4367	7987	3893	2431	0306	
γ_{125}	=	0								
γ_{126}	=	0								
γ_{127}	=	0.3029	3416	7269	5682	8108	5679	5437	6506	
γ_{128}	=	-0.1028	0329	3795	0334	2611	6141	5109	1571	$\cdot\ 10^{+1}$
γ_{129}	=	0.5425	4171	2796	6918	2157	8547	6416	2220	$\cdot\ 10^{+1}$
γ_{1210}	=	0.1534	0242	6078	6703	1086	6711	9989	5920	$\cdot\ 10^{+1}$
γ_{1211}	=	-0.1576	3473	5858	3826	6589	8937	8096	2028	$\cdot\ 10^{-1}$
γ_{130}	=	0.3517	3987	5863	0671	3954	7258	1903	7907	$\cdot\ 10^{-1}$
γ_{131}	=	0								
γ_{132}	=	0								
γ_{133}	=	0								
γ_{134}	=	0								
γ_{135}	=	0								
γ_{136}	=	0								
γ_{137}	=	0.6385	8784	3542	5830	8506	8828	9280	0822	$\cdot\ 10^{-1}$
γ_{138}	=	0.5086	6724	9055	8144	8754	2910	0714	8603	$\cdot\ 10^{-1}$
γ_{139}	=	0.1770	3179	4727	6675	2427	0314	9422	6269	
γ_{1310}	=	0.1678	1715	6130	4150	9463	9110	6721	5393	
γ_{1311}	=	0.4538	5629	2579	4244	0722	3753	9295	0401	$\cdot\ 10^{-2}$
γ_{1312}	=	0.7129	8936	9976	6658	0243	7127	3010	1115	$\cdot\ 10^{-3}$

TABLE 4. PATTERN FOR RKN-G-6(7)-10

	α_κ	$\beta_{\kappa\lambda}$										$\gamma_{\kappa\lambda}$									
κ \ λ		0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9
0	0	0										0									
1	*	*										*									
2	*	*	*									*	*								
3	*	*	0	*								*	0	*							
4	*	*	0	*	*							*	0	0	*						
5	*	*	0	0	*	*						*	0	*	*	*					
6	*	*	0	0	*	*	*					*	0	0	*	*	*				
7	*	*	0	0	0	*	*	*				*	0	0	0	*	*	*			
8	*	*	0	0	*	*	*	*	*			*	0	*	0	0	*	*	*		
9	1	*	0	0	*	*	*	*	*	*		*	0	*	0	0	0	*	*	*	
10	1	*	0	0	0	0	*	*	*	*	*	*	0	0	0	0	*	*	*	*	*

Local truncation error in x: $TE = \gamma_{109}\,(f_9 - f_{10})\,h^2$

TABLE 5. COEFFICIENTS FOR RKN-G-6(7)-10

α_1	=	0.1018	5185	1851	8518	5185	1851	8518	5185
α_2	=	0.1527	7777	7777	7777	7777	7777	7777	1778
α_3	=	0.2291	6666	6666	6666	6666	6666	6666	6667
α_4	=	0.625							
α_5	=	0.375							
α_6	=	0.6666	6666	6666	6666	6666	6666	6666	6667
α_7	=	0.1666	6666	6666	6666	6666	6666	6666	6667
α_8	=	0.9703	7314	8321	7701	1063	1914	4946	5592
α_9	=	1							
α_{10}	=	1							

β_{10}	=	0.1018	5185	1851	8518	5185	1851	8518	5185	
β_{20}	=	0.3819	4444	4444	4444	4444	4444	4444	4444	$\cdot\ 10^{-1}$
β_{21}	=	0.1145	8333	3333	3333	3333	3333	3333	3333	
β_{30}	=	0.5729	1666	6666	6666	6666	6666	6666	6667	$\cdot\ 10^{-1}$
β_{31}	=	0								
β_{32}	=	0.1718	75							
β_{40}	=	0.8186	9834	7107	4380	1652	8925	6198	3471	
β_{41}	=	0								
β_{42}	=	-0.3137	9132	2314	0495	8677	6859	5041	3223	$\cdot\ 10^{+1}$
β_{43}	=	0.2944	2148	7603	3057	8512	3966	9421	4876	$\cdot\ 10^{+1}$
β_{50}	=	0.7840	9090	9090	9090	9090	9090	9090	9091	$\cdot\ 10^{-1}$
β_{51}	=	0								
β_{52}	=	0								
β_{53}	=	0.2906	6985	6459	3301	4354	0669	8564	5933	
β_{54}	=	0.5921	0526	3157	8947	3684	2105	2631	5789	$\cdot\ 10^{-2}$
β_{60}	=	0.8963	7111	8593	3408	1556	3037	7852	6001	$\cdot\ 10^{-1}$
β_{61}	=	0								
β_{62}	=	0								
β_{63}	=	0.2041	4673	0462	5199	3620	4146	7304	6252	
β_{64}	=	0.1424	3014	9447	6933	0734	2430	1494	4769	
β_{65}	=	0.2304	5267	4897	1193	4156	3786	0082	3045	
β_{70}	=	0.1018	0041	1522	6337	4485	5967	0781	8930	
β_{71}	=	0								
β_{72}	=	0								
β_{73}	=	0								
β_{74}	=	-0.3160	4938	2716	0493	8271	6049	3827	1605	
β_{75}	=	0.1457	9659	0241	0346	8547	9129	9235	7437	
β_{76}	=	0.2351	1904	7619	0476	1904	7619	0476	1905	
β_{80}	=	-0.1674	6873	1305	8845	5834	9712	6386	9845	$\cdot\ 10^{+1}$
β_{81}	=	0								
β_{82}	=	0								

TABLE 5. (Continued)

β_{83}	=	-0.7871	6193	2682	9395	6927	2722	4851	2526	$\cdot\ 10^{+1}$
β_{84}	=	0.5392	4880	7631	6058	6316	9737	2744	6928	$\cdot\ 10^{+1}$
β_{85}	=	-0.1544	8282	1050	0843	0450	1130	2169	1446	$\cdot\ 10^{+1}$
β_{86}	=	-0.3255	5460	0653	6995	8589	4405	8687	3101	$\cdot\ 10^{+1}$
β_{87}	=	0.9924	5659	2893	1791	6591	1425	3844	6550	$\cdot\ 10^{+1}$
β_{90}	=	-0.3470	3198	5621	8999	1428	6348	1503	3844	$\cdot\ 10^{+1}$
β_{91}	=	0								
β_{92}	=	0								
β_{93}	=	-0.1593	8792	7828	8467	3788	8236	8550	9672	$\cdot\ 10^{+2}$
β_{94}	=	0.1140	4863	8019	8694	0109	1083	4461	7441	$\cdot\ 10^{+2}$
β_{95}	=	-0.3469	8562	8685	7847	2962	3361	5811	0987	$\cdot\ 10^{+1}$
β_{96}	=	-0.7404	3819	3468	9437	2049	1200	7960	7603	$\cdot\ 10^{+1}$
β_{97}	=	0.1994	1980	7723	6203	6997	8978	8502	5319	$\cdot\ 10^{+2}$
β_{98}	=	-0.6349	3713	6980	1967	4173	4388	5784	4779	$\cdot\ 10^{-1}$
β_{100}	=	0.5371	0681	6961	8894	2565	7541	5995	7059	$\cdot\ 10^{-1}$
β_{101}	=	0								
β_{102}	=	0								
β_{103}	=	0								
β_{104}	=	0								
β_{105}	=	0.2151	6847	1682	4671	9445	4911	4739	2647	
β_{106}	=	0.3420	7435	3923	4998	3829	7121	6887	6606	
β_{107}	=	0.2265	0729	0897	0719	7573	5894	5444	6475	
β_{108}	=	0.3047	2907	5218	4935	6793	3659	9050	3756	
β_{109}	=	-0.1421	8987	3417	7215	1898	7341	7721	5190	

γ_{10}	=	0.5186	8998	6282	5788	7517	1467	7640	6036	$\cdot\ 10^{-2}$
γ_{20}	=	0.5835	2623	4567	9012	3456	7901	2345	6790	$\cdot\ 10^{-2}$
γ_{21}	=	0.5835	2623	4567	9012	3456	7901	2345	6790	$\cdot\ 10^{-2}$
γ_{30}	=	0.1312	9340	2777	7777	7777	7777	7777	7778	$\cdot\ 10^{-1}$
γ_{31}	=	0								
γ_{32}	=	0.1312	9340	2777	7777	7777	7777	7777	7778	$\cdot\ 10^{-1}$
γ_{40}	=	0.1775	5681	8181	8181	8181	8181	8181	8182	$\cdot\ 10^{-1}$
γ_{41}	=	0								
γ_{42}	=	0								
γ_{43}	=	0.1775	5681	8181	8181	8181	8181	8181	8182	
γ_{50}	=	0.1925	7489	6694	2148	7603	3057	8512	3967	$\cdot\ 10^{-1}$
γ_{51}	=	0								
γ_{52}	=	0.4028	5268	2912	0077	7831	7938	7457	4623	$\cdot\ 10^{-1}$
γ_{53}	=	0.1034	9608	5254	4584	6020	0086	9943	4537	$\cdot\ 10^{-1}$
γ_{54}	=	0.4201	3351	3931	8885	4489	1640	8668	7307	$\cdot\ 10^{-3}$
γ_{60}	=	0.5015	0891	6323	7311	3854	5953	3607	6818	$\cdot\ 10^{-1}$

TABLE 5. (Concluded)

γ_{61} =	0								
γ_{62} =	0								
γ_{63} =	0.1283	2080	2005	0125	3132	8320	8020	0501	
γ_{64} =	0.1427	7669	4823	4784	4920	9443	3614	9015	$\cdot 10^{-1}$
γ_{65} =	0.2947	2859	1024	8873	2118	3617	4799	1378	$\cdot 10^{-1}$
γ_{70} =	0.1008	4019	2043	8957	4759	9451	3031	5501	$\cdot 10^{-1}$
γ_{71} =	0								
γ_{72} =	0								
γ_{73} =	0								
γ_{74} =	-0.2032	9218	1069	9588	4773	6625	5144	0329	$\cdot 10^{-1}$
γ_{75} =	0.8955	5163	6292	3770	3311	7773	8585	1460	$\cdot 10^{-2}$
γ_{76} =	0.1517	8571	4285	7142	8571	4285	7142	8571	$\cdot 10^{-1}$
γ_{80} =	0.5878	8608	9282	4074	8733	2252	3459	2929	$\cdot 10^{-1}$
γ_{81} =	0								
γ_{82} =	-0.6816	7418	1961	6552	8792	8230	9588	5550	
γ_{83} =	0								
γ_{84} =	0								
γ_{85} =	0.3904	9081	2480	6691	0635	4602	0166	2578	$\cdot 10^{-1}$
γ_{86} =	0.1320	2858	1017	7748	3299	4568	2086	5647	
γ_{87} =	0.9226	1993	4259	5248	2280	2944	4534	3764	
γ_{90} =	0.8435	6399	1262	8663	2857	5687	5609	5560	$\cdot 10^{-1}$
γ_{91} =	0								
γ_{92} =	-0.1399	9437	4195	1129	4662	3845	3271	0268	$\cdot 10^{+1}$
γ_{93} =	0								
γ_{94} =	0								
γ_{95} =	0								
γ_{96} =	0.1518	2217	0689	4949	2865	2410	0270	4702	
γ_{97} =	0.1661	2294	3949	3108	2916	6194	1317	3874	$\cdot 10^{+1}$
γ_{98} =	0.2535	7326	4223	9613	0665	3317	0496	8283	$\cdot 10^{-2}$
γ_{100} =	0.5359	5659	8420	3665	3630	8565	2940	7254	$\cdot 10^{-1}$
γ_{101} =	0								
γ_{102} =	0								
γ_{103} =	0								
γ_{104} =	0								
γ_{105} =	0.1339	3181	3529	5796	4677	4132	9304	7098	
γ_{106} =	0.1144	9729	3051	8163	2233	0537	4835	6721	
γ_{107} =	0.1891	5603	3529	7940	7729	1841	4167	3934	
γ_{108} =	0.7915	0409	1476	6066	6995	7558	1949	0274	$\cdot 10^{-2}$
γ_{109} =	0.9041	5913	2007	2332	7305	6057	8661	8445	$\cdot 10^{-3}$

TABLE 6. PATTERN FOR RKN-G-5(6)-8

	α_κ	$\beta_{\kappa\lambda}$								$\gamma_{\kappa\lambda}$							
κ \ λ		0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7
0	0																
1	*	*								*							
2	*	*	*							*	*						
3	*	*	0	*						*	*	*					
4	*	*	0	0	*					*	0	*	*				
5	*	*	0	0	*	*				*	0	*	*	*			
6	*	*	0	0	*	*	*			*	0	0	0	*	*		
7	1	*	0	0	*	*	*	*		*	0	0	0	*	*	*	
8	1	*	0	0	0	*	*	*	*	*	0	0	0	0	*	*	*

Local truncation error in x: $TE = \gamma_{87}\,(f_7 - f_8)\,h^2$

TABLE 7. COEFFICIENTS FOR RKN-G-5(6)-8

$$\alpha_1 = \frac{4}{15}, \alpha_2 = \frac{2}{5}, \alpha_3 = \frac{3}{5}, \alpha_4 = \frac{9}{10}, \alpha_5 = \frac{3}{4}, \alpha_6 = \frac{2}{7}, \alpha_7 = \alpha_8 = 1$$

$$\beta_{10} = \frac{4}{15}, \beta_{20} = \frac{1}{10}, \beta_{21} = \frac{3}{10}, \beta_{30} = \frac{3}{20}, \beta_{31} = 0, \beta_{32} = \frac{9}{20},$$

$$\beta_{40} = \frac{9}{40}, \beta_{41} = \beta_{42} = 0, \beta_{43} = \frac{27}{40}, \beta_{50} = \frac{11}{48}, \beta_{51} = \beta_{52} = 0,$$

$$\beta_{53} = \frac{5}{8}, \beta_{54} = -\frac{5}{48}, \beta_{60} = \frac{27\ 112}{194\ 481}, \beta_{61} = \beta_{62} = 0,$$

$$\beta_{63} = \frac{56\ 450}{64\ 827}, \beta_{64} = \frac{80\ 000}{194\ 481}, \beta_{65} = -\frac{24\ 544}{21\ 609},$$

$$\beta_{70} = -\frac{26\ 033}{41\ 796}, \beta_{71} = \beta_{72} = 0, \beta_{73} = -\frac{236\ 575}{38\ 313},$$

$$\beta_{74} = -\frac{14\ 500}{10\ 449}, \beta_{75} = \frac{275\ 936}{45\ 279} . \beta_{76} = \frac{228\ 095}{73\ 788},$$

$$\beta_{80} = \frac{7}{81}, \beta_{81} = \beta_{82} = \beta_{83} = 0, \beta_{84} = -\frac{250}{3483}, \beta_{85} = \frac{160}{351},$$

$$\beta_{86} = \frac{2401}{5590}, \beta_{87} = \frac{1}{10}.$$

$$\gamma_{10} = \frac{8}{225}, \gamma_{20} = \frac{1}{25}, \gamma_{21} = \frac{1}{25}, \gamma_{30} = \frac{9}{160}, \gamma_{31} = \frac{81}{800}, \gamma_{32} = \frac{9}{400},$$

$$\gamma_{40} = \frac{81}{640}, \gamma_{41} = 0, \gamma_{42} = \frac{729}{3200}, \gamma_{43} = \frac{81}{1600}, \gamma_{50} = \frac{11\ 283}{88\ 064},$$

$$\gamma_{51} = 0, \gamma_{52} = \frac{3159}{88\ 064}, \gamma_{53} = \frac{7275}{44\ 032}, \gamma_{54} = -\frac{33}{688}, \gamma_{60} = \frac{6250}{194\ 481},$$

$$\gamma_{61} = \gamma_{62} = \gamma_{63} = 0, \gamma_{64} = -\frac{3400}{194\ 481}, \gamma_{65} = \frac{1696}{64\ 827}, \gamma_{70} = -\frac{6706}{45\ 279},$$

$$\gamma_{71} = \gamma_{72} = \gamma_{73} = 0, \gamma_{74} = \frac{1\ 047\ 925}{1\ 946\ 997}, \gamma_{75} = -\frac{147\ 544}{196\ 209},$$

$$\gamma_{76} = \frac{1\ 615\ 873}{1\ 874\ 886}, \gamma_{80} = \frac{31}{360}, \gamma_{81} = \gamma_{82} = \gamma_{83} = 0, \gamma_{84} = 0,$$

$$\gamma_{85} = \frac{64}{585}, \gamma_{86} = \frac{2401}{7800}, \gamma_{87} = -\frac{1}{300}.$$

TABLE 8. APPLICATION OF THE VARIOUS FORMULAS TO PROBLEM I

Problem I:

$$\left\{\begin{aligned} \ddot{x} &= 2\dot{y} + x - (1-\mu)\frac{x+\mu}{\left(\sqrt{(x+\mu)^2+y^2}\right)^3} - \mu\cdot\frac{x-1+\mu}{\left(\sqrt{(x-1+\mu)^2+y^2}\right)^3} \\ \ddot{y} &= -2\dot{x} + y - (1-\mu)\frac{y}{\left(\sqrt{(x+\mu)^2+y^2}\right)^3} - \mu\cdot\frac{y}{\left(\sqrt{(x-1+\mu)^2+y^2}\right)^3} \end{aligned}\right\}$$

Initial Values: $t_0 = 0$ $\left\{\begin{aligned} & x_0 = 1.2 \; ; \; \dot{x}_0 = 0 \\ & y_0 = 0 \; ; \; \dot{y}_0 = -1.049\,357\,509\,830\,320 \end{aligned}\right.$

$(\mu = 0.1212\ 8562\ 7653\ 1231 \cdot 10^{-1})$

Results for $t = 6$ ($\mathrm{TOL} = 0.1 \cdot 10^{-16}$)

Formula	Δx	Δy	$\Delta \dot{x}$	$\Delta \dot{y}$	Number of Steps	7094 Time (min)
RK5(6)-8	$+0.220 \cdot 10^{-12}$	$+0.333 \cdot 10^{-12}$	$-0.274 \cdot 10^{-12}$	$-0.242 \cdot 10^{-12}$	7642	1.55
RKN-G-5(6)-8	$+0.354 \cdot 10^{-12}$	$+0.222 \cdot 10^{-13}$	$-0.159 \cdot 10^{-12}$	$-0.349 \cdot 10^{-12}$	4572	0.86
RK6(7)-10	$+0.115 \cdot 10^{-12}$	$+0.192 \cdot 10^{-12}$	$-0.191 \cdot 10^{-12}$	$-0.132 \cdot 10^{-12}$	4212	1.07
RKN-G-6(7)-10	$+0.457 \cdot 10^{-13}$	$+0.986 \cdot 10^{-13}$	$-0.113 \cdot 10^{-12}$	$-0.586 \cdot 10^{-13}$	2258	0.55
RK7(8)-13	$+0.590 \cdot 10^{-13}$	$-0.202 \cdot 10^{-13}$	$+0.105 \cdot 10^{-12}$	$-0.491 \cdot 10^{-13}$	1662	0.56
RKN-G-7(8)-13	$+0.130 \cdot 10^{-13}$	$+0.750 \cdot 10^{-13}$	$-0.971 \cdot 10^{-13}$	$-0.227 \cdot 10^{-13}$	1101	0.36

TABLE 9. APPLICATION OF THE VARIOUS FORMULAS TO PROBLEM II

Problem II
$$\left\{\begin{aligned} \ddot{x} &= -4t^2x + \frac{2\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}\cdot\sqrt{x^2+y^2}} \\ \ddot{y} &= -4t^2y + \frac{2\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}\cdot\sqrt{x^2+y^2}} \end{aligned}\right\}$$

Initial Values: $t_0 = \sqrt{\frac{\pi}{2}}$ $\left\{\begin{aligned} x_0 = 0 ,\ \dot{x}_0 = -\sqrt{2\pi} \\ y_0 = 1 ,\ \dot{y}_0 = 0 \end{aligned}\right\}$ Solution: $\left\{\begin{aligned} x &= \cos(t^2) \\ y &= \sin(t^2) \end{aligned}\right.$

Results for $t = 10$ ($TOL = 0.1 \cdot 10^{-16}$)

Formula	Δx	Δy	$\Delta \dot{x}$	$\Delta \dot{y}$	Number of Steps	7094 Time (min)
RK5(6)-8	$-0.378 \cdot 10^{-12}$	$-0.553 \cdot 10^{-12}$	$+0.110 \cdot 10^{-10}$	$-0.746 \cdot 10^{-11}$	27 275	4.17
RKN-G-5(6)-8	$-0.118 \cdot 10^{-12}$	$-0.201 \cdot 10^{-12}$	$+0.405 \cdot 10^{-11}$	$-0.236 \cdot 10^{-11}$	10 371	1.79
RK6(7)-10	$-0.114 \cdot 10^{-12}$	$-0.199 \cdot 10^{-12}$	$+0.405 \cdot 10^{-11}$	$-0.230 \cdot 10^{-11}$	10 873	2.10
RKN-G-6(7)-10	$-0.553 \cdot 10^{-13}$	$-0.110 \cdot 10^{-12}$	$+0.216 \cdot 10^{-11}$	$-0.112 \cdot 10^{-11}$	4 285	0.83
RK7(8)-13	$-0.778 \cdot 10^{-13}$	$-0.133 \cdot 10^{-12}$	$+0.264 \cdot 10^{-11}$	$-0.158 \cdot 10^{-11}$	3 873	1.05
RKN-G-7(8)-13	$-0.510 \cdot 10^{-13}$	$-0.863 \cdot 10^{-13}$	$+0.169 \cdot 10^{-11}$	$-0.106 \cdot 10^{-11}$	3 007	0.78

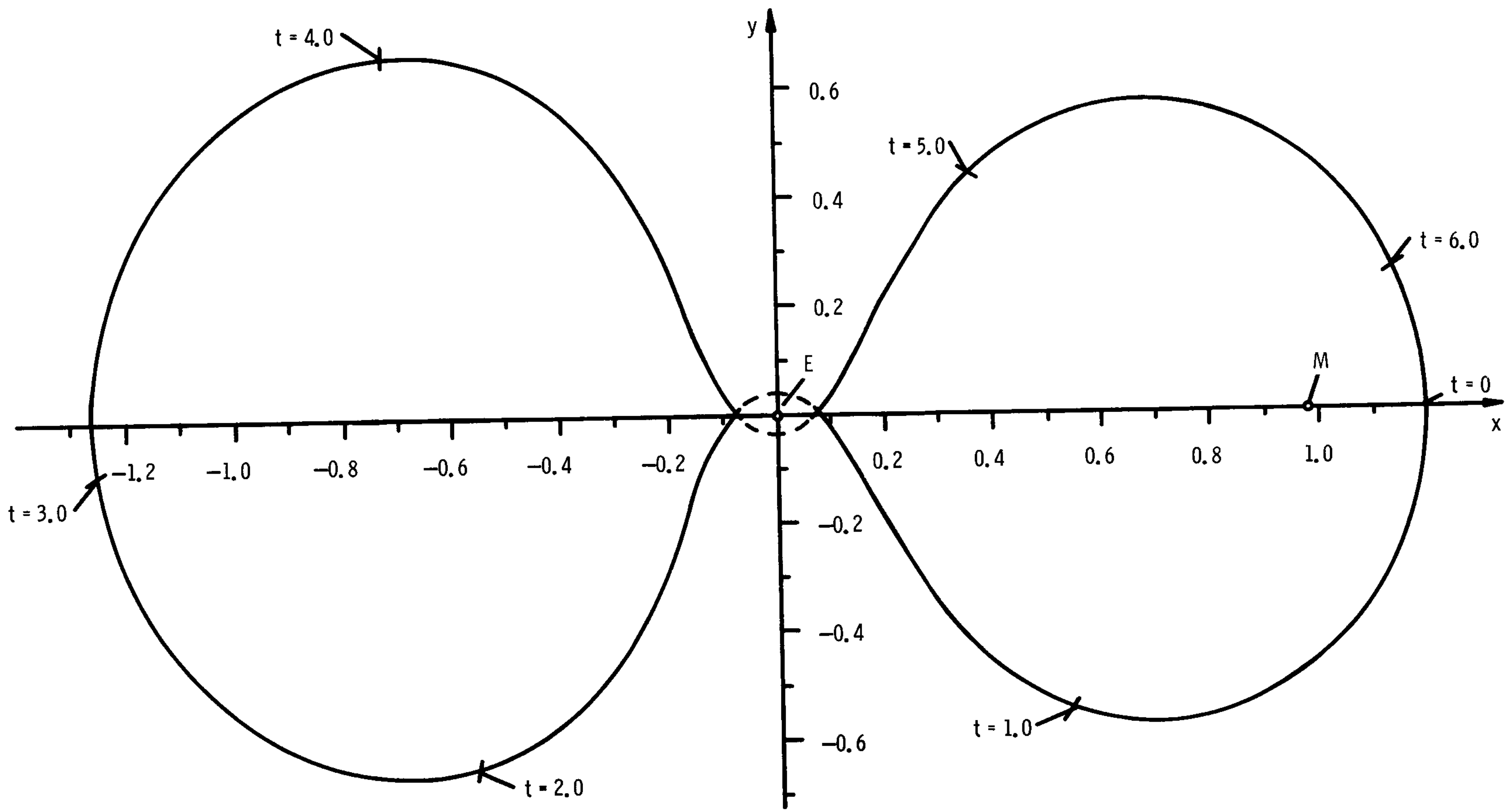

Figure 1. Orbit for the restricted problem of three bodies.

NASA-Langley, 1974