§6. Examples of Abelian Coverings of Algebraic Varieties

Let us come back to the centre of the course, the theory of Abelian Coverings of Algebraic Varieties. We remind the definition:

"it means compact as complex analytic variety"

Def Let Y be a smooth complete algebraic variety over $\mathbb{C}$ and let G be a finite group.

"zero locus of polynom. in $\mathbb{C}^n$ or $\mathbb{P}^n$"

A Galois cover of Y is a finite morphism $\pi : X \to Y$

"finite fibres + π is top. proper in the Euclid. top of X and Y".

, with X normal, such that G acts faithfully on X

"not so bad singularities, roughly speaking the singular locus has codimension at least 2"

and π factors as the quotient map $X \to X/G$ and an isomorphism $X/G \xrightarrow{\sim} Y$:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d] & Y \\ X/G \arrow[ur, "\sim"'] & \end{tikzcd}$$

- We say that π is an abelian covering of Y if G is an abelian group.
- We say that π is a smooth Galois cover if X is smooth.

Now, we study deeply the examples presented in Lecture 1.

<u>Example 1 (Double covering)</u>

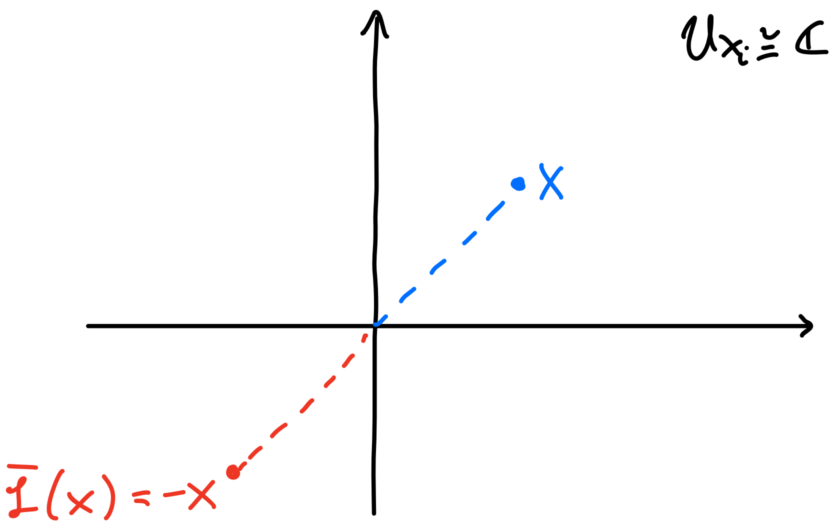
We take $X = \mathbb{P}^1(x_0, x_1)$ and $G = \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

$$\bar{0} := Id_X , \qquad \bar{1} : X \longrightarrow X$$
$$[x_0, x_1] \longmapsto [x_0, -x_1]$$

$U_{x_i} := \{x_i \neq 0\} \subseteq X$, then locally on U_{x_i} $\bar{1}$ is the opposite map:

$$\bar{1} : \mathcal{U}_{x_i} \cong \mathbb{C} \longrightarrow \mathcal{U}_{x_i} \cong \mathbb{C}$$
$$x \longmapsto -x \;, \text{ where } x := \frac{x_j}{x_i}$$

$\mathcal{U}_{x_i} \cong \mathbb{C}$



The action of G on $\mathbb{P}^1$ define the double quotient

$$\pi : X \longrightarrow Y := \mathbb{P}^1(z_0, z_1)$$
$$[x_0, x_1] \longmapsto [x_0^2, x_1^2]$$

From the picture is clear that the only points with <u>NO</u> trivial stabilizer are the origins of the two charts, namely the points $[1, 0]$ and $[0, 1]$ of X.

Let us study the ramification divisor of π:

$$d\pi_x = \frac{d}{dx}(x^2) = 2x = 0 \iff x = 0$$

So $\quad \mathrm{Ram}(\pi) = [1,0] + [0,1]$.

Remark

We obtained that the reduced ramification divisor of π consists of those points of X with <u>NO</u> trivial stabilizer.

We denote by R the reduced ramif. divisor of π. In this case, we have $R = \mathrm{Ram}(\pi)$.
We denote by $D := \pi(R) = [1,0] + [0,1]$.
the image of R. Notice that both $[1,0]$ and $[0,1]$ are fixed by $I : X \to X$; for this reason we denote ther sum as D_I; in this specific case we have

$$D = D_I$$

Let us consider now the sheaf $\pi_* \mathcal{O}_X$ on Y, we want to prove that it is a locally free sheaf of rank 2 on Y.

We choose the coordinate charts $\mathcal{U}_{z_0}$ and $\mathcal{U}_{z_1}$ on $\mathcal{Y}$:

$$\pi_* \mathcal{O}_X(\mathcal{U}_{z_i}) = \mathcal{O}_X(\pi^{-1}(\mathcal{U}_{z_i})) = \mathcal{O}_X(\mathcal{U}_{x_i}) = \mathbb{C}[x],$$

where $x := \frac{x_J}{x_i}$.

By construction, $G = \mathbb{Z}_2$ acts naturally on $\pi_* \mathcal{O}_X(\mathcal{U}_{z_i})$ sending $x \mapsto \bar{1} \cdot x = -x$.

Thus, we have a representation of G on the space $\mathbb{C}[x]$. Let us determine its isotypic components W^η, $\eta \in \mathrm{Irr}(G)$.

Let us consider $p \in \mathbb{C}[x]$; we apply the Reynold Operator of character $\mathbb{1}$ to determine W^0:

$$\pi_0(p) = \frac{1}{2}\left(\overline{\mathbb{1}(\bar{0})} \cdot p(\bar{0}^{-1} x) + \overline{\mathbb{1}(\bar{1})} \cdot p(\bar{1}^{-1} x)\right)$$

$$= \frac{1}{2}(p(x) + p(-x)) \in \mathbb{C}[x^2]$$

Conversely, x^2 is fixed by G, so $\mathbb{C}[x^2]$ is fixed too. We have proved

$$W^0 = \mathbb{C}[x^2] \underset{\substack{\downarrow \\ x \xmapsto{\pi} x^2 = z}}{\cong} \mathbb{C}[z] = \mathcal{O}_{\mathcal{Y}}(\mathcal{U}_{z_i})$$

Instead, $\pi_{\bar{1}}(p) = \frac{1}{2}(p(x) - p(-x)) \in x \cdot \mathbb{C}[x^2]$

Conversely, each function in $X\cdot\mathbb{C}[X^2]$ is invariant with character $\mathbb{I}$ (namely $T\cdot f = -f$), so $W^{\mathbb{I}} = X\cdot\mathbb{C}[X^2]$.
Thus, we have that any $p\in\mathbb{C}[X]$ decomposes as
$$p(X) = \tfrac{1}{2}(p(X)+p(-X)) + \tfrac{1}{2}(p(X)-p(-X)).$$
We have proved that

$$(1)\qquad \pi_* \mathcal{O}_X(\mathcal{U}_{z_i}) = \mathbb{C}[X^2]\cdot 1 \oplus \mathbb{C}[X^2]\cdot X$$
$$\cong \mathcal{O}_Y(\mathcal{U}_{z_i})\cdot 1 \oplus \mathcal{O}_Y(\mathcal{U}_{z_i})\cdot X$$
as a $\mathcal{O}_Y(\mathcal{U}_{z_i})$-module!

Remark We have shown that the representation of G on $\pi_*\mathcal{O}_X(\mathcal{U}_{z_i})$ as a $\mathcal{O}_Y(\mathcal{U}_{z_i})$-module is the regular repres.

Since the decomposition (1) holds for any open coordinate chart $\mathcal{U}_{z_i}$ of Y, and they cover Y, then $\pi_*\mathcal{O}_Y$ is a locally free sheaf of Y of rank 2.

WARNING: To be precise, we should prove that
$$\pi_*\mathcal{O}_X|_{\mathcal{U}_{z_i}} \cong \mathcal{O}_Y|_{\mathcal{U}_{z_i}} \oplus \mathcal{O}_Y|_{\mathcal{U}_{z_i}} \quad \text{as sheaves.}$$
This holds by a similar decomp. as (1)!

Let us determine the cocycles of $\pi_* \mathcal{O}_X$ by using the correspondence of locally-free sheaves and vector bundles.

$$\bigoplus_{i=1}^{2} \mathcal{O}_Y(U_{z_0} \cap U_{z_1}) \xrightarrow{\phi_0^{-1}} \pi_* \mathcal{O}_X(U_{z_0} \cap U_{z_1}) \xrightarrow{\quad \phi_1 \quad} \bigoplus_{i=1}^{2} \mathcal{O}_Y(U_{z_0} \cap U_{z_1})$$

$$(f_1, f_2) \longmapsto f_1 \cdot 1 + f_2 \cdot x = f_1 \cdot 1 + \frac{x_1^2}{x_0^2} \cdot f_2 \cdot \frac{x_0}{x_1} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & \frac{z_1}{z_0} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$x = \frac{x_1}{x_0}$, $\quad \frac{x_1^2}{x_0^2} = \frac{z_1}{z_0}$

$\Rightarrow$ the cocycles of $\pi_* \mathcal{O}_X$ are $g_{10} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{z_1}{z_0} \end{pmatrix}$.

We have proved that

$$\pi_* \mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-1) =: \mathcal{L}_0 \oplus \mathcal{L}_{\bar{1}}^{-1}$$

We use this notation to remind that they correspond to the anti-invariant fcts of X.

We observe that from the construction we obtained a global section of $\pi^* \mathcal{L}_{\bar{1}}$ on X:

$$s = \left\{ \left(U_{x_0}, \frac{x_1}{x_0}\right), \left(U_{x_1}, \frac{x_0}{x_1}\right) \right\}.$$

Furthermore, $\mathcal{L}_{\bar{1}}^{\otimes 2}$ has cocycles $g_{10} = \frac{z_0^2}{z_1^2}$ and a global section is $s^2 = \left\{ \left(U_{z_0}, \frac{z_1}{z_0}\right), \left(U_{z_1}, \frac{z_0}{z_1}\right) \right\}$

The divisor associated to such section is $[1,0]+[0,1]$, so we obtained the linear equivalence relation:

$$(*) \qquad 2\cdot L_I \equiv [1,0]+[0,1] = D_I$$

class divisor associated to the line bundle $\mathcal{L}_I$.

The above equation is called Pardini Equation of the double cover π.

Let us consider the line bundle of Y, $V(L_I) \xrightarrow{\pi'} Y$ with local coordinates (z, y^i_I) on $(\pi')^{-1}(U_{z_i})$, $z := \frac{z_j}{z_i}$. Thus,

$$y^j_I = g_{ji}\, y^i_I = \frac{z_i}{z_j}\cdot y^i_I .$$

The group $G=\mathbb{Z}_2$ is naturally acting on $V(L_I)$ by sending $I : (z, y^i_I) \mapsto (z, -y^i_I)$.

Pardini Equation $(*)$ suggests to consider the subvariety of $V(L_I)$:

$$X' \cap (\pi')^{-1}(U_{z_i}) := \{(z, y^i_I) \mid (y^i_I)^2 = z\} \subseteq V(L_I)$$

By construction, X' is invariant by the action of G.

Thus, $\pi' : X' \to Y$ is a Galois cover
$$(z, y_{\bar{I}}^{i}) \mapsto z$$
of Y with group $G = \mathbb{Z}_2$.

Notice that this cover only depends on the line bundle $L_{\bar{I}}$ and the divisor $D_{\bar{I}}$ of Y and not on the double cover $\pi : X \to Y$.

Finally, X and X' are isomorphic using the section s defining $\pi^* L_{\bar{I}}$:

$$\Psi : X \longrightarrow X'$$
$$p \longmapsto (\pi(p), s(p))$$

$$\begin{tikzcd} X \arrow[r, "\Psi"] \arrow[dr, "\pi"'] & X' \arrow[d, "\pi'"] \\ & Y \end{tikzcd}$$

Remark It is not so difficult to show that in our case $V(L_{\bar{I}}) = \mathbb{P}^3(z_0, z_1, y_{\bar{I}}) \setminus \{z_0 = z_1 = 0\}$, that $X' = \{(z_0, z_1, y_{\bar{I}}) \in \mathbb{P}^3 \mid y_{\bar{I}}^2 = z_0 z_1\}$ with an action of $G = \mathbb{Z}_2$ sending $(z_0, z_1, y_{\bar{I}}) \mapsto (z_0, z_1, -y_{\bar{I}})$, and that the isomorphism Ψ is the Veronese embedding of degree 2 $(x_0, x_1) \overset{\Psi}{\longmapsto} (x_0^2, x_1^2, x_0 x_1)$.