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On the degree of the canonical map of surfaces of general type

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To my family,
thank you for always supporting me.
I love you.

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ABSTRACT. In this thesis, we study the degree of the canonical map of surfaces of general type. In particular, we give the first examples known in the literature of surfaces having degree $d = 10, 11, 13, 14, 15$, and 18 of the canonical map. They are presented in a self-contained and independent way from the rest of the thesis. We show also how we have discovered them. These surfaces are *product-quotient surfaces*. In this thesis, we study the theory of product-quotient surfaces giving also some new results and improvements. As a consequence of this, we have written and run a MAGMA script to produce a list of families of product-quotient surfaces having geometric genus three and a self-intersection of the canonical divisor large. After that, we study the canonical map of product-quotient surfaces and we apply the obtained results to the list of product-quotient surfaces just mentioned. In this way, we have discovered the examples of surfaces having degree $d = 10, 11, 14$, and 18 of the canonical map. The remaining ones with degrees 13 and 15 do not satisfy the assumptions to compute the degree of the canonical map directly. Hence we have had to compute the canonical degree of these two families of product-quotient surfaces in a very explicit way through the equations of the pair of curves defining them.

Another work of this thesis is the classification of all smooth surfaces of general type with geometric genus three which admits an action of a group G isomorphic to $\mathbb{Z}_2^k$ and such that the quotient is a projective plane. This classification is attained through the theory of *abelian covers*. We obtained in total eleven families of surfaces. We compute the canonical map of all of them, finding in particular a family of surfaces with a canonical map of degree 16 not in the literature. We discuss the quotients by all subgroups of G finding several K3 surfaces with symplectic involutions. In particular, we show that six families are families of triple K3 burgers in the sense of Laterveer.

Finally, in another work we study also the possible accumulation points for the slopes K^2/χ of unbounded sequences of minimal surfaces of general type having a degree d of the canonical map. As a new result, we construct unbounded families of minimal (product-quotient) surfaces of general type whose degree of the canonical map is 4 and such that the limits of the slopes K^2/χ assume countably many different values in the closed interval $[6 + \frac{2}{3}, 8]$.

Abstract. Questa tesi si concentra sullo studio e il calcolo del grado della mappa canonica di superfici di tipo generale In particolare, presentiamo i primi esempi noti in letteratura di superfici con grado $d = 10, 11, 13, 14, 15$, e 18 della mappa canonica. Per rendere questi esempi accessibili a un pubblico più ampio, li trattiamo in modo indipendente dal resto della tesi. Queste superfici sono superfici *prodotto-quoziente*.

Iniziamo approfondendo la teoria delle superfici prodotto-quoziente e fornendo anche nuovi risultati significativi.Utilizzando tali risultati, sviluppiamo uno script in MAGMA per produrre una lista di famiglie di superfici prodotto-quoziente aventi genere geometrico tre e un' alta auto-intersezione del divisore canonico. Successivamente, studiamo la mappa canonica delle superfici prodotto-quoziente e applichiamo i risultati ottenuti alle superfici presenti nella lista generata dallo script. In questo modo, scopriamo gli esempi di superfici con grado $d = 10, 11, 14$, e 18 della mappa canonica. Le restanti superfici con grado 13 e 15 non soddisfano le ipotesi necessarie per determinare direttamente il grado della mappa canonica, pertanto calcoliamo esplicitamente tali gradi attraverso le equazioni della coppia di curve che definiscono le due famiglie di superfici prodotto-quoziente.

Un altro contributo di questa tesi è la classificazione di tutte le superfici lisce di tipo generale con genere geometrico tre che ammettono un'azione di un gruppo G isomorfo a $\mathbb{Z}_2^k$ e tali che il quoziente sia un piano proiettivo. Per ottenere questa classificazione, utilizziamo la teoria dei rivestimenti abeliani. In totale, otteniamo undici famiglie di superfici, calcolando anche il grado della mappa canonica per ciascuna di esse. In particolare, troviamo una famiglia di superfici con una mappa canonica di grado 16 che non era presente in letteratura Discutiamo anche i quozienti di queste famiglie per tutti i sottogruppi di G, trovando diverse superfici K3 con involuzioni simplettiche. In particolare, dimostriamo che sei famiglie sono famiglie di triple K3 burger nel senso di Laterveer.

Infine, in un altro studio, esaminiamo i possibili punti di accumulazione per le pendenze K^2/χ di successioni illimitate di superfici minimali di tipo generale con un grado d della mappa canonica. Come risultato innovativo, costruiamo famiglie infinite di superfici minimali (prodotto-quoziente) di tipo generale, il cui grado della mappa canonica è 4, e i limiti delle pendenze K^2/χ assumono un insieme numerabile di valori nell'intervallo chiuso $[6 + \frac{2}{3}, 8]$.

Introduction

In this thesis, we study the canonical map of surfaces of general type.
It is a well-known fact that the canonical map of a curve C of genus at least two is either an embedding or of degree 2. The latter happens if and only if C is hyperelliptic. For a smooth surface S of general type the situation is more difficult: suppose that the canonical map $\Phi_{K_S} \colon S \dashrightarrow \mathbb{P}^{p_g-1}$ is not composed with a pencil. Then Beauville proved that the degree d of the canonical map is bounded by

$$d := \deg\left(\Phi_{K_S}\right) \leq 9 + \frac{27-9q}{p_g-2} \leq 36.$$

Note that the bound $d \leq 36$ was shown first by Persson [Per78, Prop. 5.7]. In particular, $28 \leq d$ is only possible if $q = 0$, and $p_g = 3$. Motivated by this observation, the construction of regular surfaces with $p_g = 3$ and canonical map of degree d for every value $2 \leq d \leq 36$ is an interesting, but still, a wide-open problem, posed also by M. M. Lopes and R. Pardini in a recent survey, [MLP23, Question 5.2].

For a long time, the only examples with $10 \leq d$ were the surfaces of Persson [Per78], with canonical map of degree 16, and Tan [Tan03], with degree 12.

At the moment, the main result in this direction of my thesis is the following

Theorem 0.0.1. *There exist surfaces S of general type with $p_g = 3$, $q = 0$, and canonical map of degree $d = 10, 11, 13, 14, 15,$ and 18.*

These surfaces are the first examples known in the literature with that degree of the canonical map.
In recent years, this problem attracted the attention of many authors, putting an increased effort into the construction of new examples. Two are the main methods of construction found in the literature, *generating pairs* and *abelian covers*. As a result, together with the surfaces obtained in my thesis, we have now examples in the literature for all degrees $2 \leq d \leq 16$ and $d = 18, 20, 24, 27, 32$ and 36 (see [MLP23, Rit15, Rit17a, Rit17b, Rit22, LY21, GPR22, Bin19, Bin21a, Fal22, FG23, Bin23] for more details).

One of the purposes of this thesis is to provide an overview of the topic of the canonical map and to explain how we obtained the examples of Theorem 0.0.1. These are the so-called *product-quotient* surfaces. We decided to focus our attention on product-quotient surfaces since they are easy to describe and we have seen that sometimes their canonical map is accessible to be studied.

Product-quotient surfaces are studied for the first time by Catanese in [Cat00]. Apart from other works, that mainly deal with irregular surfaces, we want to mention the complete classification of surfaces isogenous to a product with $p_g = q = 0$ [BCG08] and the classification for $p_g = 1$, and $q = 0$, under the assumption that the action is diagonal [Gle15], the rigid but not infinitesimally rigid manifolds [BP21] of Bauer and Pignatelli that gave a negative answer to a question of Kodaira and Morrow [MK71, p.45], and also the infinite series of n-dimensional infinitesimally rigid manifolds of general type with non-contractible universal cover for each $n \geq 3$, provided by Frapporti and Gleissner[FG23].

The setting is the following: consider an action of a finite group G on two curves C_1 and C_2. Let S be the minimal resolution of singularities of the product $C_1 \times C_2$ modulo the induced (diagonal) action of G on it. S is called *product-quotient surface* of *quotient model* $(C_1 \times C_2)/G$. Assume S is a regular surface of general type. This implies that the genera of C_1 and C_2 are at least two, and the quotients $C_i/G \cong \mathbb{P}^1$. In other words, C_1 and C_2 are G-coverings of the projective line.
We first consider the problem of determine all families of regular product-quotient surfaces given by a pair of G-coverings of the projective line, up to isomorphism. A complete answer to this problem is Theorem 4.5.8. This answer has been implemented in MAGMA as an algorithm that takes as input two G-coverings C_1, and C_2 of the projective line. It iterates through the list of (diagonal) actions of G on $C_1 \times C_2$, and returns only those actions that yield distinct families of product-quotient surfaces of quotient model $(C_1 \times C_2)/G$. In other words, we are able to move on from a database of G-coverings of the line[1], to a database of families of product-quotient surfaces. As a consequence of this, we have produced a MAGMA script[2] which gives in input a pair of natural numbers (K^2, χ) and returns all regular surfaces S of general type with $K_S^2 = K^2$ and $\chi(S) = \chi$, which are product-quotient surfaces.

However, we recall that surfaces with a high degree of the canonical map are regular and have p_g equal to three. For this reason, we are running the script mentioned above for χ equal to four and K^2 large.

The script is producing several hundreds of families of product-quotient

[1] as for example the database: `https://mate.unipv.it/ghigi/tipitopo/`.

[2] Most of the script is a modification of the script of [BP12] to any χ, and we give some other improvements.

surfaces with p_g equal to three. As a sample we give in this thesis a list for the maximal possible value of K^2.

Theorem 0.0.2. *Let S be a product-quotient surface of general type with $p_g = 3$, $q = 0$, and $K_S^2 = 32$ of quotient model $(C_1 \times C_2)/G$. Assume that both the topological types of the G-action on C_1 and C_2 are in the database* [CGP23, 11^{th} of June 2023]*. Then S realizes one of the 213 families of surfaces of general type described in tables 4.1 and 4.2.*

The second part of this thesis studies the canonical map of product-quotient surfaces. Let S be a surface of general type. Assume that the canonical map of S is not composed with a pencil, hence its image Σ has dimension two. The degree of the canonical map of S is computable via elementary intersection theory once we know the degree of the canonical image Σ in $\mathbb{P}^{p_g-1}$ and the (schematic) base locus of the canonical system $|K_S|$.

If p_g is equal to three, which is the case most interesting for us, the degree of Σ is one, hence we have only to compute the base locus of $|K_S|$.

If S is a product-quotient surface, it is simpler to compute the degree of the composition of the canonical map of S with the map $C_1 \times C_2 \dashrightarrow S$ and divide the result by $|G|$. So we have to compute the base locus of the pull-back of the canonical system of $|K_S|$ to $C_1 \times C_2$ that is a linear subsytem $|K_{C_1\times C_2}|^G$ of $|K_{C_1\times C_2}|$.

We are able to determine the base locus of $|K_{C_1\times C_2}|^G$ if certain[3] irreducible characters of G have degree 1. We remark that this assumption always holds for G abelian, and it is sometimes satisfied also when G is not abelian as for example for the first family in Table 6.2 with canonical map of degree 18. Indeed, we prove in Theorem 5.2.8 and Corollary 5.2.9 that under the above mentioned assumption $|K_{C_1\times C_2}|^G$ is spanned by p_g divisors that decompose as union of fibres for the natural projections $C_1 \times C_2 \to C_i$.

Since two fibres either do not intersect or they intersect transversally at a point, this makes the base locus of $|K_{C_1\times C_2}|^G$ explicit. To finish the computation of the degree, when $p_g = 3$, we proved the following Correction Term formula that seems of independent interest.

Theorem 0.0.3 (Correction Term formula)**.** *Let M be a (not necessarily complete) two-dimensional linear system on a surface spanned by three divisors D_1, D_2, and D_3 of the form*

$$D_1 = aH, \quad D_2 = bK \quad \text{and} \quad D_3 = cH + dK$$

locally around a smooth point p of the surface. Here H and K are reduced, smooth, and intersect transversally at p and a, b, c, d are non-negative integers. Take a minimal sequence of blow ups such that the strict transform $\widehat{M}$

[3]These are at most p_g irreducible characters determined by the construction, see Subsection 5.2.1.

of M has no base point points on the preimage of p. Then

$$\widehat{M}^2 = M^2 - \min\{ab, ad + bc\}.$$

In some examples (see sections 6.2, and 6.3) even if our assumption on the characters fails, we compute the degree of the canonical map using a very explicit description of the surface.

We have also tried to construct examples of surfaces with high degree of the canonical map in a different way.

Many of the known examples with a high degree of the canonical map are obtained as Galois covers of rational surfaces with Galois group isomorphic to $\mathbb{Z}_2^k$ (see [MLP23]). Inspired by that we classified smooth Galois covers of the plane with group $\mathbb{Z}_2^k$ and p_g equal to three. We call them smooth k-double covers of the plane. We have the following

Theorem 0.0.4. *All smooth k-double covers S of the plane with geometric genus 3 are regular surfaces with ample canonical class.*

The canonical map Φ_{K_S} is a morphism of degree K_S^2 on $\mathbb{P}^2$ unless S of type E3, in which case the canonical map is a rational map of degree $K_S^2 - 4 = 4$ undefined at 4 points.

Each family is unirational. The modular dimension of each family, that is the dimension of its image in the Gieseker moduli space of the surfaces of general type, equals $4 + 2^{6-k}$ with one exception, the family B2, whose dimension is 19.

The values of K_S^2, of $\deg \varphi_{K_S}$ and of the modular dimension of each family are listed in the following table:

Family	A1	A2	A3	A4	B2	C3	C4	D3	D4	D5	E3
K_S^2	2	4	8	16	9	8	16	2	4	8	8
$\deg \varphi_{K_S}$	2	4	8	16	9	8	16	2	4	8	4
mod. dim.	36	20	12	8	19	12	8	12	8	6	12

These surfaces are natural candidates to be triple K3 burgers in the sense of [Lat19], which are important in relation to a conjecture of C. Voisin. We determined which of these surfaces are triple K3 burgers, finding that they are exactly the surfaces in the families B2, C3, D3, D4, D5 and E3.

Our last result concerns unbounded families of minimal surfaces of general type with canonical map of fixed degree, say d. In fact by [Bea79] and [Xia86] we know that if there is such a family then $d \leq 8$. Examples are known only for d even. Focusing on the case $d = 4$ we noticed that the only unbounded families in literature have slope $\mu(S) := K_S^2/\chi(\mathcal{O}_S)$ tending to either 4 or 8. We proved

Theorem 0.0.5. *There is an unbounded sequence S_n of surfaces that have canonical map of degree 4 such that*

$$\lim_{n \to \infty} \mu(S_n) = 8\left(1 - \frac{1}{m}\right)$$

for all positive integers $m \geq 6$ that are not prime numbers.

The thesis is divided in 8 chapters and an appendix. The appendix contains only classical results on surfaces, so most of the readers will not need to look at it.

The first three chapters of this thesis are purely compilative. In the first one we collect some classical results and relevant examples on the canonical map of surfaces of general type. The second chapter is dedicated to the standard theory of the Galois covers of the projective line. The third chapter discusses cyclic quotient quotient singularities of surfaces.

Chapter 4 discusses product-quotient surfaces. The first four sections are the now standard tools for product-quotient surfaces: we follow [Fra12] here.

Section 4.5 contains statement and proof of the first original result of this thesis, the above mentioned Theorem 4.5.8, classifying all families of product-quotient surfaces given by a pair of topological types of group actions on curves.

In the last sections, following [BP12], we describe an algorithm to classify all families of regular product-quotient surfaces with fixed K^2 and χ. The main improvement respect to other algorithms in literature as the one in [BP12] is the use of Theorem 4.5.8 and of [CGP23]. More precisely, we use a script developed in [CGP23] (or the database produced by them if possible) to compute all possible topological types of group actions on curves of our interest and then use Theorem 4.5.8 to deduce from it a complete list of families of product-quotient surfaces. The last Section 4.9 contains the Theorem 0.0.2.

Chapter 5 studies the canonical map of product-quotient surfaces. The main results of this chapter are the above mentioned Theorem 5.2.8 and Corollary 5.2.9, and the Correction Term formula 0.0.3. Here the Correction Term formula is slightly differently formulated, see Theorem 5.4.3.

Chapter 6 gives explicit computations of the degree of the canonical map of certain product-quotient surfaces with p_g equal to three, surfaces found with the program in Chapter 4. Theorem 0.0.1 is immediate consequence of the results of Section 6.1 and 6.2, that are respectively in [FG23] and [Fal22].

Chapter 7 is devoted to Theorem 0.0.5. This chapter's content is essentially [FP21].

Finally, Chapter 8 discusses the classification of Theorem 0.0.4 This chapter's content is [FP23].

Contents

Notation

Let S be a smooth projective surface over the field $\mathbb{C}$ of complex numbers.
An algebraic surface S is a *canonical model* if it has at most rational double points as singularities and ample canonical divisor. Recall that each surface of general type is birational to a unique canonical model. In particular the minimal resolution of the singularities of S is its minimal model;
$H^{p,q}(S) := H^q(S, \Omega^p_S)$;
$q(S)$ or $q = \dim H^{0,1}(S) = \dim H^{1,0}(S)$ is the *irregularity.*
$p_g(S)$ or $p_g = \dim H^{0,2}(S) = \dim H^{2,0}(S)$ is the *geometric genus.*
K_S or K is a *canonical divisor* of S, so a divisor such that $\mathcal{O}_S(K) \cong \Omega^2_S$.
Ω^2_S is sometimes denoted also by ω_S.
By abuse of notation, for any divisor D, we will denote by D its class in the Picard group.
$|T|$ is the (complete) linear system defined by a divisor T of S.
The (schematic) base locus of a linear system $|T|$ is denoted by $Bs(|T|)$.
Φ_T denotes the (a priori rational) map of S induced by a linear system $|T|$ of S.
Hence Φ_{K_S} is the canonical map of S.
Given an irreducible surface $\Sigma \subseteq \mathbb{P}^n$, then $\deg(\Sigma)$ denotes the degree of Σ in $\mathbb{P}^n$, namely, given a resolution $\eta \colon X \to \Sigma$, and denoted by $|H|$ the inverse image of the system of hyperplanes of $\mathbb{P}^n$, then $\deg(\Sigma) = H^2$.
The symbol $\equiv$ is the linear equivalence of divisors, while $\equiv_{num}$ denotes the numerical equivalence;
Given a divisor D, then $O_S(D)$ is its invertible sheaf, or equivalently, its line bundle.
Given an invertible sheaf θ, then $H^0(S, \theta)$ is the space of global sections of θ, and $h^0(S, \theta) := \dim H^0(S, \theta)$. Therefore, for example, if $\theta = \omega_S$, then $h^0(S, \omega_S)$ and p_g denote the same number. We will use one of the two notations by the context.
$Spec(f)$ is the spectrum of an endomorphism $f \colon V \to V$.
$Crit(\lambda)$ is the set of branch points of the morphism $\lambda \colon C \to C'$, with C and C' smooth curves.
Given a group G, then g^h denotes hgh^{-1}, for any $h, g \in G$.
Given $h \in G$, then $inn_h \colon G \to G$ is the inner automorphism $g \mapsto hgh^{-1}$.
The subgroup $Inn(G) \trianglelefteq \mathrm{Aut}(G)$ is the group of inner automorphisms of G.

$Irr(G)$ is the set of irreducible characters of a finite group G.
$CF(G)$ denotes the class group function space of G, so the space of functions of G constant on conjugacy classes.
Let $\langle\cdot\rangle$ be the classical Hermitian product defined on $CF(G)$;
Given a representation $\phi_\eta : G \to GL(V)$ afforded by a character η, and an irreducible character χ, then V^χ or V_χ denotes the isotypic component of V of character χ. Hence $\dim(V^\chi) = \langle \eta, \chi \rangle \cdot \chi(1)$.
$\mathrm{GL}_m(k)$ denotes the group of invertible matrices of size m over the field k.
For each real number z, let $\lceil z \rceil$ be the smallest integer greater or equal than z.
Given a positive number n, then $0 \leq [a]_n < n$ denotes the rest of the division of a by n.
When we write $\sqrt[n]{\lambda}$ we mean the first root of the complex number λ, i.e. if $\lambda = |\lambda| \cdot e^{i\theta}$, then $\sqrt[n]{\lambda} = \sqrt[n]{|\lambda|} \cdot e^{i\frac{\theta}{n}}$.

Chapter 1

The history of the problem and main results in the literature

It is a well-known fact that a smooth projective variety X of dimension n admits always an invertible sheaf, the sheaf $\omega_X = \Omega^n_X$ of the sections of the n-alternating power of the cotangent bundle of X, denoted as $\Lambda^n T^*X$. Since any X is canonically equipped by this object, then ω_X takes the name of *canonical sheaf*, and $\Lambda^n T^*X$ of *canonical bundle*.
Moreover, from the correspondence among invertible sheaves and classes of $\mathrm{Pic}(X)$, then ω_X corresponds to a class of divisors in $\mathrm{Pic}(X)$, usually denoted by K_X, which takes the name of *canonical divisor*, as one can expect.
Whenever the corresponding class divisor in $\mathrm{Pic}(X)$ of an invertible sheaf θ is effective, namely $h^0(X, \theta) > 0$, then θ induces a (rational) map $X \dashrightarrow \mathbb{P}^{h^0(X,\theta)-1}$, up to projective transformations. Let me spend some words more about it. Firstly, this map is usually denoted by Φ_θ, or equivalently by Φ_Θ, where $\Theta \in \mathrm{Pic}(X)$ is the class divisor whose θ corresponds to: $\theta = O_X(\Theta)$. To be consistent with the notation fixed in this thesis, I will adopt the second kind of notation. The map is defined by sending a point $p \in X$ to the functional class $[ev_p] \in \mathbb{P}(H^0(X, \theta)^*)$, where $ev_p(s)$ is the evaluation at p of a global section s, for a fixed trivialization of θ at p. Although ev_p depends on the choice of the trivialization, its class $[ev_p]$ is not. In fact, ev_p defined in different charts would be different just by a multiplicative constant, which is the co-cycle of θ in those charts, evaluated at the point p.
From the geometrical point of view, one can see Φ_Θ as the map sending each point p to the hyperplane $\ker(ev_p) \subseteq H^0(X, \theta)$ of the effective divisors on X in the class Θ, and passing through the point p.

However, some points $p \in X$ might be problematic: any section of θ could vanish at p, which translates in Φ_Θ to be not defined at p. For this

reason, Φ_Θ is a priori only a rational map.
Sometimes it is useful to write Φ_Θ also in projective coordinates, by fixing a basis $s_0, \dots, s_{h^0(X,\theta)-1}$ of $H^0(X,\theta)$ (and hence fixing also its dual basis). In this case, each element $[f] \in \mathbb{P}(H^0(X,\theta)^*)$ is uniquely identified by its coordinates $[f(s_0) : \cdots : f(s_{h^0(X,\theta)-1})]$, and so $\Phi_\Theta \colon X \dashrightarrow \mathbb{P}^{h^0(X,\theta)-1}$ sends $p \mapsto [s_0(p) : \cdots : s_{h^0(X,\theta)-1}(p)]$.

Definition 1.0.1. The *geometric genus* of X is $p_g := h^0(X, O_X(K_X))$. Assume $p_g > 0$, namely K_X is a an effective divisor.
The (rational) map $\Phi_{K_X} \colon X \dashrightarrow \mathbb{P}^{p_g-1}$ given by the *canonical sheaf* ω_X, or equivalently by the *canonical divisor* K_X, is called *canonical map*.

Instead, the *plurigenus* of X is $P_n := h^0(X, O_X(nK_X))$, $n \geq 2$. Assume $P_n > 0$, namely nK_X is effective.
The *pluricanonical map* $\Phi_{nK_X} \colon X \dashrightarrow \mathbb{P}^{P_n-1}$ is the (rational) map given by the class divisor nK_X.

Since these maps are always attached on a smooth projective variety X, then the geometry of X can be studied through them. In particular, the canonical map plays an important role in the theory of algebraic curves, and the pluricanonical maps for the classification of algebraic surfaces.

Let me give a brief panoramic of what is known in the literature, which is the state of the art at the moment, and which are in my opinion the most interesting questions not yet solved. We remind also to the nice survey [BCP06] for more details to the topic.

In the case of the curves, everything is known. In particular, the canonical map of a curve C of genus ≥ 2 is either an embedding or of degree 2. The latter happens if and only if C is hyperelliptic.
Instead, in the case of complex surfaces, the pluricanonical maps are extensively studied by Enriques [Enr49], Kodaira, and Bombieri [Bom73]. In particular, they attacked the problem to establish when the pluricanonical map Φ_{nK_S} of a surface of general type (see the Def. A.2.5) is birational (onto its image), for n sufficiently large. The question is naturally raised since one can observe that the plurigenus P_n grows very quickly, and so nK_S may have enough independent global sections to ensure Φ_{nK_S} is birational. Let S be the canonical model of a surface of general type, and let K^2 be the self-intersection of a canonical divisor. Bombieri obtained the fascinating result that Φ_{nK_S} is an isomorphism, for $n \geq 5$. The maps Φ_{4K_S} and Φ_{3K_S} are birational with few well-understood exceptions.
Nowadays, the result reached by Bombieri, together with several other results obtained later on (see the footnotes), can be summarized as follows:

Theorem 1.0.2. *(Modern version of Bombieri theorem on the pluricanonical maps) Let S be the canonical model of a surface of general type, and let K^2 be the self-intersection of its canonical bundle. We have:*

(i) For $n \geq 5$, Φ_{nK_S} is an embedding;

(ii) Φ_{4K_S} a morphism, and

a. if $K^2 \geq 2$, then it is an embedding;

b. if $K^2 = 1$, then it is a birational morphism, with one exception [1]*:*
$K^2 = 1$, $p_g = 2$, where $\Phi_{4K_S}(S)$ is a quadric cone $Q \subset \mathbb{P}^3$ embedded in $\mathbb{P}^8$ by means of the linear system of quadrics of $\mathbb{P}^3$, and the degree of Φ_{4K_S} is 2;

(iii) Φ_{3K_S} is

a. an embedding, if $K^2 \geq 3$;

b. a morphism, if $K^2 = 2$. Moreover, Φ_{3K_S} is birational, with the exception[2]*:*
$K^2 = 2$, $p_g = 3$, where $\Phi_{3K_S}(S) = \mathbb{P}^2$ embedded in $\mathbb{P}^9$ by means of the linear system of plane cubics, and the degree of Φ_{3K_S} is 2;

c. a birational map, if $K^2 = 1$, with the exception[3]*:*
$K^2 = 1$, $p_g = 2$, where $\Phi_{3K_S}(S)$ is a quadric cone embedded in $\mathbb{P}^5$ as a rational ruled surface of degree 4, and the degree of Φ_{3K_S} is 2;

(iv) Φ_{2K_S}[4] *is*

a. a birational morphism, if $K^2 \geq 10$, except when S has a structure of a genus 2 fibration, in which case the bicanonical map is generically finite of degree 2 over a rational or ruled surface;

b. a generically finite morphism, if $K^2 \geq 5$, or $p_g \geq 1$;

[1] Bombieri's original formulation had another possible exception for surfaces with $K^2 = 1$, $p_g = 0$, called *numerical Godeaux surfaces*, that was later excluded by Miyaoka in [Miy76]

[2] Bombieri's original formulation had another possible exception for surfaces with $K^2 = 2$, $p_g = 0$, called *numerical Campedelli surfaces*, later on excluded in [BC78].

[3] Here Bombieri had the same possible exception for numerical Godeaux surfaces as above, excluded in [Miy76]

[4] We sketch here the improvements respect to the original results of Bombieri on the bicanonical map. Xiao studied the bicanonical map and he proved in [Xia85a] that it is generically finite if and only if S is not numerical Godeaux: $K^2 = 1$, and $p_g = q = 0$. In this case, note that $P_2 = 2$, so that Φ_{2K_S} is a rational fibration. At the moment, only examples of genus 4 fibrations are known, although the genus may be 3, or 4 (see [CP06]). Moreover, Persson proved in [Per78, Prop. 5.5] that if Φ_{2K_S} is generically finite, then its degree is at most 8 (sharp, [Per78, Ex. 5.6]). Finally, as a consequence of Reider results [Rei88, Prop. 3], then Φ_{nK_S} $(n \geq 3)$ is a morphism if $(n-1)^2K^2 \geq 5$, and it is an embedding if $(n-1)^2K^2 \geq 10$. Furthermore, if $K^2 \geq 5$, or $p_g \geq 1$, then Φ_{2K_S} is a morphism (see [CP06, Thm. 6]), and if $K^2 \geq 10$, then either it is a birational morphism, or it admits a pencil of curves of genus 2.

c. *a generically finite rational map, of degree at most* 8 *(sharp), except for* S *with* $K^2 = 1$, $p_g = q = 0$. *In this case,* Φ_{2K_S} *is a fibration with fibres of genus* 3, *or* 4, *and base of genus* 0.

See [CP06, Thm. 8] for details.

At the end of the paper, Bombieri leads the point on the most interesting problems still open: which is the structure of the canonical map Φ_{K_S}, and the famous conjecture $K^2 \leq 9\chi(\mathcal{O}_\mathcal{S})$, nowadays known as *Bogomolov-Miyaoka-Yau inequality*, proved independently five years later by Miyaoka (see [Miy77]), and Yau (see [Yau77], [Yau78]).

The starting point of this thesis comes with the epochal paper [Bea79] published by Beauville in 1979, where he takes up the open question of the structure of the canonical map, and gives, together with Persson in [Per78, Prop. 5.7], the first bound (see Thm. 1.1.5) of the degree of the canonical map, by also using the just proved *Bogomolov-Miyaoka-Yau inequality* (see the proof of the Thm. 1.1.5).
He brought to light the great variety of the behaviour of Φ_{K_S}, with respect to the more rigid pluricanonical maps. Let me exhibit his principal results:

1. If the canonical map is composed with a pencil, and $p_g > 863$, then the fibres of the pencil have genus $2 \leq g \leq 4$, and the pencil has not fixed points (see [Che17]);

2. If the canonical map is not composed by a pencil, the image of the canonical map is a surface having $p_g = 0$, or a *canonical surface* (see Thm. 1.0.3). In particular, the degree of Φ_{K_S} is ≤ 9, if $\chi(\mathcal{O}_S) \geq 31$, for the first case, [5] and ≤ 3, if $\chi(\mathcal{O}_S) \geq 14$, for the second case.
   Moreover, if $K^2 < 3p_g - 7$, then Φ_{K_S} is a rational map of degree 2 over a ruled surface.

Later on, one can find some refinements of these results in the literature, (see for instance [Xia85b], [Xia86]).
Recently, M.M. Lopes and R. Pardini published a self-contained survey [MLP23] about point 2 of the statement, of great inspiration to me. Let us suppose that $\Sigma := \Phi_{K_S}(S)$ is a surface, and let d be the degree of Φ_{K_S}. In this survey, they explain very well the topic and exhibit the known possibilities for Σ and d, reached by Beauville. Moreover, they investigate the question of producing concrete examples for such possibilities, presenting the two main methods (*abelian covers* and *generating pairs*) of construction used by several authors in the literature. In the end, the authors lead on

[5] He gives also an example of an unbounded family of surfaces (in the sense that $\chi(\mathcal{O}_S)$ increases arbitrarily) such that the degree of Φ_{K_S} is 8. This implies that the inequality $\deg(\Phi_{K_S}) \leq 9$ is close to be sharp.

several open questions. One focused my attention particularly, which became one of the central topics of this thesis:

[MLP23, **Question 5.2**]: For every $2 \leq d \leq 36$, does there exists any surface S with $p_g = 3$, and canonical map of degree d?

To make sense of this question, and then try to attack it, I would like to take a step back. First, I am going to show the proof of the point 2 due to Beauville, and then determine the bounds of the canonical map leading to the natural formulation of the above question.

Let S be a surface of general type with $p_g := p_g(S) \geq 3$. Let $\Phi_{K_S} \colon S \dashrightarrow \mathbb{P}^{p_g-1}$ be the canonical map of S; we assume that the image $\Sigma := \Phi_{K_S}(S)$ is a surface, and we set d be the degree of Φ_{K_S}. Sometimes Σ could be singular; we choose a resolution $\eta \colon X \to \Sigma$ of the singularities of Σ. We define $p_g(\Sigma) := p_g(X)$, $q(\Sigma) := q(X)$ (note that they do not depend on the choice of η), and we say that Σ is of general type if X is of general type.

Theorem 1.0.3. *(*[Bea79, Thm. 3.1]*) There are the following possibilities:*

(A) $p_g(\Sigma) = 0$,

(B) $p_g(\Sigma) = p_g(S)$, *and* Σ *is a canonical surface (in the sense that the canonical map of a resolution of* Σ *is the resolution itself, or equivalently,* Σ *is embedded in* $\mathbb{P}^{p_g-1}$ *by its canonical map).*

Proof. Denote by $p_g := p_g(S)$. Let $\eta \colon X \to \Sigma$ be a resolution of Σ. One can apply the classical result [Bea96, Theorem II.7] to the rational map $\eta^{-1} \circ \Phi_{K_S} \colon S \dashrightarrow X$ to get a (surjective) morphism $\pi = (\eta^{-1} \circ \Phi_{K_S}) \circ \epsilon$ from $\widehat{S}$ to X, where ϵ is a composition of a finite number of blow-ups. Moreover, the canonical map of $\widehat{S}$ is $\Phi_{K_S} \circ \epsilon$, by construction. The situation is therefore the following:

$$\begin{tikzcd}
\widehat{S} \arrow[rr, "\epsilon"] \arrow[d, "\pi"'] \arrow[drr, "\Phi_{K_{\widehat{S}}}"] & & S \arrow[d, dashed, "\Phi_{K_S}"] \\
X \arrow[rr, "\eta"'] & & \Sigma \subseteq \mathbb{P}^{p_g-1}.
\end{tikzcd}$$

Let us suppose there exists a no-zero holomorphic 2-form ω of X (hence, we are supposing that $p_g(\Sigma) \neq 0$, and we are going to show that we fall in the case (B) of the statement). Since $\pi^*\omega$ is a 2-form of $\widehat{S}$ and $\Phi_{K_{\widehat{S}}}$ factorizes through π, then $\mathrm{div}(\pi^*\omega) = \pi^*H + Z$, where H is the pullback in X of an hyperplane in $\mathbb{P}^{p_g-1}$, and Z is an effective divisor, the fixed part of $|K_{\widehat{S}}|$.

Let us denote by K_0 the divisor of ω. One can apply the classical Theorem A.1.6 to get also another equality (among *divisors*)

$$\pi^*H + Z = \mathrm{div}(\pi^*\omega) = \pi^*K_0 + \sum_{j=1}^{q}(e_j - 1)C_j + \sum_{i=1}^{p} r_i E_i, \tag{1.1}$$

where C_j are irreducible curves whose image with respect to π is a curve Γ_j, while E_i are the curves contracted by π. Here e_j is the ramification index of C_j, namely the coefficient of C_j in the divisor $\pi^*\Gamma_j$, and $r_i \geq 0$.

Let us consider now an irreducible curve Γ of X, and a curve C on $\widehat{S}$ whose image is $\pi(C) = \Gamma$. Denote by e the coefficient of C in the divisor $\pi^*\Gamma$, and by h and k the coefficients of Γ in the divisors H, and K_0 respectively. Then the coefficient of C in the divisor $\pi^*H + Z$ is at least he, whilst that of the right member of the identity (1.1) is exactly $ke + (e-1)$. We have got

$$he \leq ke + (e-1) \implies h \leq k.$$

In other words any irreducible curve Γ occurring in H (with coefficient h) has to occur also in K_0 (with coefficient $k \geq h$), so that $K_0 - H$ is an effective divisor, denoted as E: $K_0 = H + E$.
Thus $h^0(X, O_X(H)) \leq p_g(X)$, by the classical Theorem A.1.1. The thesis follows once one observe the following facts:

- A subsystem of the linear system $|H|$ on X induces the map η, whose image is Σ, a surface not contained in an hyperplane, since Φ_{K_S} is non-degenerate. This means that $h^0(X, O_X(H)) \geq p_g = p_g(S)$.

- By the classical result A.1.7 applied to the map π, then the pullback $H^0(X, O_X(K_X)) \xrightarrow{\pi^*} H^0(\widehat{S}, O_{\widehat{S}}(K_{\widehat{S}}))$ is injective, so that $p_g(X) \leq p_g(\widehat{S}) = p_g$.

Putting together the obtained inequalities, we get

$$p_g \leq h^0(X, O_X(H)) \leq p_g(X) \leq p_g.$$

Hence all inequalities are equalities, and we get $|K_X| = |H| + E$, by the Theorem A.1.1. This means that E is the fixed part of $|K_X|$, and η is the canonical map of X, always by the Theorem A.1.1. We have fallen in case (B) of the statement. $\square$

1.1 Bounds on the degree of the canonical map

Here we discuss the possibilities of Σ and d. Let us write $K_S \equiv M + Z$, where Z is the fixed part of the canonical linear system $|K_S|$. Let $\Phi_M \colon S \dashrightarrow \Sigma \subseteq \mathbb{P}^{p_g - 1}$ be the canonical map of S. If $|M|$ is not base-point free, or equivalently

Φ_M is not a morphism, we can apply the classical result [Bea96, Theorem II.7] to get a morphism $(\Phi_M \circ \epsilon) : \widehat{S} \to \Sigma \subseteq \mathbb{P}^{p_g-1}$, where $\epsilon \colon \widehat{S} \to S$ is a composition of blow-ups.

Lemma 1.1.1. *We have*

$$M^2 \geq d \deg \Sigma,$$

and the equality holds if and only if the mobile system $|M|$ of $|K_S|$ is base-point free.

Proof. The mobile system $|M|$ would be base-point free if and only if its induced map Φ_M is a morphism, and so if and only if $M^2 = (\Phi_M^* H)^2 = d \deg \Sigma$, by the projection formula. However, in the case in which the mobile system is not base-point free, then we consider $\Phi_M \circ \epsilon$. By construction of ϵ, which is a composition of blow-ups, we get

$$M^2 \geq ((\Phi_M \circ \epsilon)^* H)^2 = d \deg \Sigma.$$

□

Lemma 1.1.2. *The following inequalities holds*

(A) $\deg \Sigma \geq p_g - 2$*, if S falls in the case (A) of the Beauville Theorem 1.0.3. Moreover, if Σ is not a ruled surface, then* $\deg \Sigma \geq 2p_g - 4$*;*

(B) $\deg \Sigma \geq 3p_g + q(\Sigma) - 7$*, if S falls in the case (B) of the Beauville Theorem 1.0.3.*

Proof. Let us discuss the case (A). We observe that Σ is irreducible because S is irreducible and Φ_M is a morphism. In the case in which Φ_M is not a morphism, then we use the map $\Phi_M \circ \epsilon$, and the fact that $\widehat{S}$ is irreducible, if S is irreducible. (remember that $\widehat{S}$ is a blow-up of S).
Furthermore, Σ is not contained in a hyperplane, since Φ_M is non-degenerate. Then Theorem A.1.9 applies, and the thesis follows.

Let us consider now the case (B). In this case, Σ is a canonical surface, so a resolution $\eta \colon X \to \Sigma \subseteq \mathbb{P}^{p_g-1}$ of Σ is the canonical map of X. Note that Σ is the the canonical image also for the minimal model X_{min} of X. Jongmans Theorem A.1.10 applies to the minimal surface X_{min}, so that

$$\deg(\Sigma) = K^2_{X_{min}} \geq 3p_g(X_{min}) + q(X_{min}) - 7 = 3p_g + q(\Sigma) - 7,$$

and the thesis follows. □

Lemma 1.1.3. *Assume S is minimal. Then $K_S^2 \geq M^2$.*

Proof. By assumption, K_S is nef. Furthermore, M is nef, by its definition. Therefore, we have

$$\begin{aligned} K_S^2 &= K_S M + K_S F \\ &\geq K_S M = M^2 + MF \\ &\geq M^2. \end{aligned}$$

□

Theorem 1.1.4 (Bogomolov-Miyaoka-Yau Inequality, [Miy77], [Yau77], [Yau78])**.** *Any surface S of general type satisfies the inequality $K_S^2 \leq 9\chi(\mathcal{O}_S)$. Moreover, the equality holds if and only if S is the quotient of a ball in $\mathbb{C}^2$ by an infinite discrete group.*

Finally, we can establish the possibilities of Σ and d.

Theorem 1.1.5. *Let S be a surface with $p_g := p_g(S) \geq 3$, and such that the image $\Sigma \subseteq \mathbb{P}^{p_g-1}$ of the canonical map of S is a surface. Let d be the degree of the canonical map of S. Then*

(A) If S falls in the case (A) of the Beauville Theorem 1.0.3, then

(i) if Σ is ruled,

$$d \leq 9 + \frac{27 - 9q(S)}{p_g - 2}; \tag{1.2}$$

(ii) if Σ is not ruled,

$$d \leq \frac{1}{2}\left(9 + \frac{27 - 9q(S)}{p_g - 2}\right);$$

(B) Otherwise, if S falls in the case (B) of the Beauville Theorem 1.0.3, then

$$d \leq 3 + \frac{30 - 3q(\Sigma) - 9q(S)}{3p_g + q(\Sigma) - 7}.$$

Proof. Assume S is minimal. We apply in sequence the Lemmas 1.1.1, and 1.1.3 together with the Theorem 1.1.4 to get

$$\begin{aligned} d \deg \Sigma &\leq M^2 \\ &\leq K_S^2 \\ &\leq 9\chi(\mathcal{O}_S) = 9(p_g - q(S) + 1). \end{aligned}$$

Now we apply the Lemma 1.1.2, and we get

$$d\,(p_g - 2) \leq d \deg \Sigma \leq 9(p_g - q(S) + 1),$$

if S falls in the case (A) of the Beauville Theorem 1.0.3, and Σ is ruled. Otherwise, if Σ is not ruled, then

$$d\,(2p_g-4)\leq d\deg\Sigma\leq 9(p_g-q(S)+1).$$

Instead, if S falls in the case (B) of the Beauville Theorem, then

$$d\,(3p_g+q(\Sigma)-7)\leq d\deg\Sigma\leq 9(p_g-q(S)+1).$$

In the case in which S is not minimal, one can consider its minimal model and obtains the above inequalities. However, the canonical map of the minimal model of S factors through that of S, so that these maps have the same degree d, and image $\Sigma\subseteq\mathbb{P}^{p_g-1}$. Finally, it is sufficient to remember that the geometric genus and the irregularity are birational invariants. □

Many consequences of these inequalities can be drawn. Some of them are the following:

Remark 1.1.6.
1. $d\leq 9$, if $p_g\geq 30$, or $q(S)\geq 3$; Moreover, if Σ is non ruled, then $d\leq 4$, if $p_g\geq 30$, or $q(S)\geq 3$.

2. as noted first by Persson in [Per78, Prop. 5.7], the maximum possible degree is 36, and can be reached only if $p_g=3$, and $q(S)=0$. In this case, $\Sigma=\mathbb{P}^2$, a surface of degree 1, and we would have

$$36=d\deg\Sigma\leq K_S^2\leq 9(3-0+1)=36\implies K^2=36=9\chi(\mathcal{O}_S).$$

   Thus S is a quotient ball (from B-M-Y Theorem 1.1.4), with $K_S^2=36$, $p_g=3$, $q(S)=0$, and $|K_S|$ is base-point free;

3. $d\geq 23$ implies $q\leq 1$ and $p_g=3$, whilst $d\geq 28$ implies $q=0$ and $p_g=3$;

4. if $q(S)>0$, then the maximum possible degree is 27, when $p_g=3$, and $q(S)=1$. In this case, we would have $\Sigma=\mathbb{P}^2$, and

$$27=d\deg\Sigma\leq K_S^2\leq 9(3-1+1)=27\implies K_S^2=27=9\chi(\mathcal{O}_S).$$

   Thus S is a quotient ball (Thm. 1.1.4), with $K_S^2=27$, $p_g=3$, $q(S)=1$, and $|K_S|$ is base point free;

5. in case (B), the maximum possible value is 9, when $p_g=4$, and $q(S)=q(\Sigma)=0$. In this case, we would have

$$45=9(3\cdot4+0-7)\leq 9\deg\Sigma\leq K_S^2\leq 9(4-0+1)=45\implies\begin{cases}\deg(\Sigma)=5,\\ K_S^2=45=9\chi(\mathcal{O}_S).\end{cases}$$

   This means that S is a quotient ball (Thm. 1.1.4), with $K_S^2=45$, $p_g=4$, $q(S)=0$, and $|K_S|$ is base-point free. Moreover, Σ has to be a quintic surface in $\mathbb{P}^3$, with $q(\Sigma)=0$.

6. in case (B), if $d = 3$, then $q(S) \leq 3$.

The first natural question is: are the obtained inequalities in cases (A) and (B) sharp?
C. Rito uses in [Rit22] the Borisov-Keum equations of a fake projective plane and the Borisov-Keum equations of the Cartwright-Steger surface to show the existence of a surface with $p_g = 3$, $q = 0$, and canonical map of maximum degree 36, and of a surface with $p_g = 3$, $q = 1$, and canonical map of degree 27 (see the points 2. and 4. above). The first surface is an étale $\mathbb{Z}_2^2$-cover of the fake projective plane, and the second an étale $\mathbb{Z}_3$-cover of the Cartwright-Steger surface.
In the case (B), is not still clear if an example of a surface with canonical map of degree 9 exists (see the point 5. above). The best record in the literature is obtained independently by R. Pardini in [Par91b, Example 2.2] and by S. Tan in [Tan92]. They found the same surface, which is a $\mathbb{Z}_5^2$-cover of $\mathbb{P}^2$ ramified on five lines in general position. Here the surface has invariants $K_S^2 = 25$, $p_g = 4$, $q(S) = 0$, and Σ is a quintic surface in $\mathbb{P}^3$ with 10 rational double points as singularities. The degree of the canonical map is 5.

As we said at the beginning of the Chapter, we are majorly interested to investigate

[MLP23, **Question 5.2**]: For every $2 \leq d \leq 36$, does there exists any surface S with $p_g = 3$, and canonical map of degree d?

We point out that this question is well-posed thanks to points 2. and 3. remarked above. In particular, if we are interested to get examples with $d \geq 28$, then we have also to require that S is regular, and $p_g = 3$.
Therefore, the idea of the thesis is to study systematically a class of surfaces with such invariants. As we will see, the regular *product-quotient* surfaces of general type are good candidates, since they can be easily described and their canonical map is accessible to be studied.

1.2 Examples

Before going on to the Chapter 4 on product-quotient surfaces, we would like to mention which have been the first examples presented in the literature, with $p_g = 3$, and which degree of the canonical map is attained by these. We present in details those that can be easily described. Instead, we remand the remain to the respective papers.

The technique of their construction can be expressed by using the theory of *abelian covers* (for more details, please see [Par91a]).

Example 1.2.1. ($d = 2$) This example, in [Bea79], was known to M. Noether, as mentioned in [Enr49]. Let us take a surface Y with $p_g(Y) = 0$. Choose a line bundle on Y such that $|2L|$ is base-point free and $K_Y + L$ is a very ample divisor. Let $\pi\colon S \to Y$ be the double cover given by choosing a general element $D \in |2L|$, hence satisfying $2L \equiv D$. Then S is smooth, $K_S \equiv \pi^* (K_Y + L)$. Thus K_S is ample, so S is minimal of general type. Moreover, since $p_g(Y) = 0$, we have $|K_S| = \pi^*|K_Y + L|$. This means the canonical map of S is the composition with π and the embedding defined by the divisor $K_Y + L$.
We have constructed a family of surfaces S with the canonical map of degree 2, and unbounded p_g.
In particular, to get an example with $p_g = 3$, one can just take $Y = \mathbb{P}^2$ and look for the first line bundle L for which $K_Y + L$ is very ample, namely $L = 4h$, where h is the class of a line. In this case, the canonical map of S is the double cover π branched over a smooth optic, and its invariants are $K_S^2 = 2$, $q = 0$.
These are in fact the surfaces in Theorem 1.0.2 (iii), b.

Example 1.2.2. [$d = 3 \dots 9$] This example can be found in [MLP23, Example 4.5]. We take $G = \mathbb{Z}_2^2$, and Y a del Pezzo surface of degree $d \geq 3$. For every $0 \neq v \in G$ choose a curve $D_v \in |-K_Y|$ such that $\sum_v D_v$ is smooth normal crossing. The relations to ensure the existence of a G-cover are $2L_{10} \equiv 2L_{11} \equiv 2L_{01} \equiv -2K_Y$, whose only solution (since $\mathrm{Pic}(Y)$ is without torsion) is $L_{10} \equiv L_{11} \equiv L_{01} \equiv -K_Y$. The corresponding bi-double cover $\pi\colon S \to Y$ is smooth, we have $2K_S \equiv \pi^*(-K_Y)$, which is ample, because $d \geq 3$, and $-K_Y$ is ample. This means S is minimal of general type. We have $K_S^2 = d$, and $p_g = 3$, $q = 0$.
Finally, $|K_S|$ is spanned by the three curves $R_v := \pi^{-1}(D_v) = \frac{1}{2}\pi^*(D_v)$, so that $|K_S|$ is base-point free and so the canonical map of S is mapped d-to-1 to $\mathbb{P}^2$.

We would like to point out that for $d = 9$, then $Y = \mathbb{P}^2$, and this construction realizes one of the 11 families of smooth k-double covers of $\mathbb{P}^2$ with $p_g = 3$ founded in the classification cit.. And the end of that Chapter, the reader can find also a description of S by equations in the weighted projective space $\mathbb{P}(1^3, 3^3)$.

Example 1.2.3 ($d = 12$, Tan Example)**.** See [Tan03, Theorem 2. (6)].

Example 1.2.4. [$d = 16$, Persson Example] This is due to Persson in [Per78]. Let $Y = \mathbb{P}^2$, and let h be the class of a line. We take the group $G = \mathbb{Z}_2^4$, and consider the character $\chi_0 \in G^*$ sending each vector of $v \in G$ to the sum of its coordinates. Let us consider divisors D_v, which are a line, for $v \notin \ker \chi_0$, and $D_v = 0$, otherwise (hence they are in total 8 lines on $\mathbb{P}^2$). The relations to ensure the existence of a 4-double cover $\pi\colon S \to Y$

branched over those lines are $2L_\chi \equiv \sum_{v \notin \ker \chi} D_v$. We have

$$\begin{cases} 2L_{\chi_0} \equiv \sum_{v \notin \ker \chi_0} D_v = 8h \\ 2L_\chi \equiv \sum_{v \notin \ker \chi} D_v = \sum_{v \notin \ker \chi_0 \cup \ker \chi} D_v = 4h, \qquad \text{if} \qquad \chi \neq \chi_0, \end{cases}$$

whose only solution is $L_{\chi_0} \equiv 4h$, and $L_\chi \equiv 2h$, for $\chi \neq \chi_0$. We also assume that the lines D_v are in general position, so that S is smooth. Moreover, we have $K_S \equiv \pi^*(h)$, and so S is minimal of general type with $K_S^2 = 16$, $p_g = 3$ and $q = 0$. Finally, the linear system $|K_S| = \pi^*|K_Y + L_{\chi_0}| = \pi^*|h|$, so the canonical map of S is the covering π.

Example 1.2.5 ($d = 20$, Bin Example)**.** We refer the reader to [Bin21a] for a complete description. The author takes the group $G := \mathbb{Z}_2^4$, and a del Pezzo surface Y of degree 5. He constructs two 4-double covers $\pi_i \colon S_i \to Y$, $i = 1, 2$, with $p_g = 3$, $q = 0$, and $K^2 = 20$, for the first case, and 24, for the second one. Moreover, he shows that $|K_{S_1}|$ is base point free, so that the canonical map has degree $d = K_{S_1}^2 = 20$, whilst $|K_{S_2}|$ has fixed part, so that the degree of the canonical map decreases to 20.

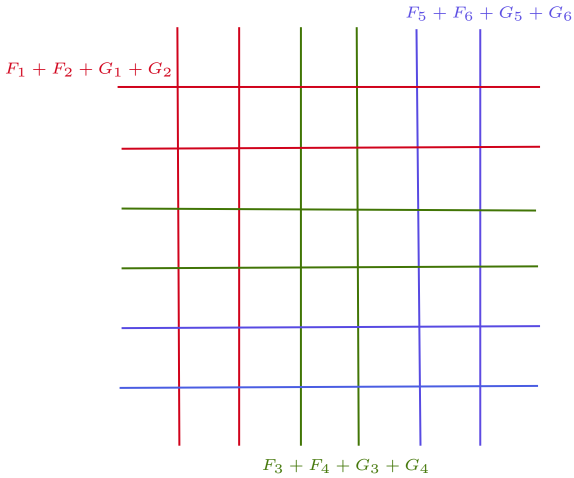
Example 1.2.6 ($d = 24$, Rito Example)**.** These are 4-double covers, Galois covers with Galois group G isomorphic to $(\mathbb{Z}_2)^4$, of a rational surface with $p_g = 3$, $K_S^2 = 24$, $q = 0$.

Example 1.2.7 ($d = 32$, Gleissner, Pignatelli, Rito Example)**.** We describe only one of the two examples in [CGP23] because they differ just by a different choice of the building data of the covering. Let us take $G := \mathbb{Z}_2^4$ and $Y := \mathbb{P}^1 \times \mathbb{P}^1$. Let $F_1, \ldots, F_6$ be six distinct vertical lines of Y, and $G_1, \ldots, G_6$ other six distinct horizontal lines. We denote by $e_1, \cdots, e_4$ a basis of G. By $e_{i_1 \cdots i_r}$ we mean $e_{i_1} + \cdots + e_{i_r}$. Let us consider the $\mathbb{Z}_2^4$-covering $\pi \colon S \to Y$ given by

$$D_{e_1} := F_1, \quad D_{e_2} := F_2, \quad D_{e_3} := F_3, \quad D_{e_4} := F_4, \quad D_{e_{13}} := F_5, \quad D_{e_{24}} := F_6,$$

$$D_{e_{234}} := G_1, \; D_{e_{134}} := G_2, \; D_{e_{124}} := G_3, \; D_{e_{123}} := G_4, \; D_{e_{14}} := G_5, \; D_{e_{23}} := G_6.$$

The branch locus of π is therefore something like



.

The covering π is well defined because there exists divisors L_χ, $\chi \in G^*$, satisfying Pardini's linear equations $2L_\chi \equiv \sum_{g \notin \ker(\chi)} D_g$. Since there is no 2-torsion in the Picard group of Y, then π is uniquely determined. In particular, we have $L_\chi \equiv 2F + 2G$, if $\chi \in \ker(e_{12}) \cap \ker(e_{34})$, $\chi \neq 0$, and $L_\chi \equiv F + 2G$, or $L_\chi \equiv 2F + G$, otherwise. S is smooth, with $p_g = 3$, and $q = 0$. The canonical system $|K_S|$ is generated by

$$\widehat{F_1} + \widehat{F_2} + \widehat{G_1} + \widehat{G_2}, \quad \widehat{F_3} + \widehat{F_4} + \widehat{G_3} + \widehat{G_4}, \quad \widehat{F_5} + \widehat{F_6} + \widehat{G_5} + \widehat{G_6},$$

where $\widehat{F_i} := \frac{1}{2}\pi^*(F_i)$, and $\widehat{G_j} := \frac{1}{2}\pi^*(G_j)$. Observe that $\widehat{F_i}\widehat{F_j} = \widehat{G_i}\widehat{G_j} = 0$, while $\widehat{F_i}\widehat{G_j} = 4$. This implies $K_S^2 = 32$. Finally, by looking to their image on Y, one verifies that the above three divisors have no common intersection (see also the above figure). Therefore $|K_S|$ is base-point free, and the degree of the canonical map amounts to $d = K_S^2 = 32$.

Example 1.2.8 ($d = 27, 36$, Rito's Examples)**.** See [Rit22].

Chapter 2

Galois coverings of Riemann surfaces

This chapter aims to discuss Galois covers of a Riemann surface.

Definition 2.0.1. Let C be a Riemann surface, and G be a finite group acting holomorphically on C. Then $\lambda\colon C \to C/G$ is called *Galois covering*. It can also be called *G-covering* if one needs to specify which is the group G acting on C.
Given a Riemann surface C', we say that (C, λ) is a *G-covering* of C' if $\lambda\colon C \to C/G$ is a G-covering, and $C/G \cong C'$.

We are going to investigate C/G, where G is a finite group acting holomorphically and faithfully on C.

Remark 2.0.2. The action of G on C can always be assumed to be faithful. If K is the normal subgroup of G of elements acting trivially on C, then $G' := G/K$ acts on C, and obviously $C/G = C/G'$.

Proposition 2.0.3. [Mir95, Prop. III.3.1] *Let C be a Riemann surface, $G \leq \mathrm{Aut}(C)$, and let $p \in C$. Suppose that the stabilizer subgroup* $\mathrm{Stab}(p)$ *is finite. Then* $\mathrm{Stab}(p)$ *is cyclic.*

Proposition 2.0.4. [Mir95, Prop. III.3.2] *Let C be a Riemann surface, let G be a finite group acting faithfully and holomorphically. Then the set of points of C with non trivial stabilizer is discrete.*

Remark 2.0.5. In the same assumption of the previous proposition, if C is compact, then only finitely many points have non trivial stabilizer.

The next goal is to define a complex structure on C/G, the only one for which the quotient map $\lambda\colon C \to C/G$ is holomorphic.

Theorem 2.0.6. [Mir95, Prop. III.3.3] *Let C be a Riemann surface and let $G \leq \mathrm{Aut}(G)$ finite. Fix a point $p \in C$. Then there is an open neighbourhood U of p such that:*

- *U is invariant under the action of* $\mathrm{Stab}(p)$*: $g(u) \in U$ for each $g \in$* $\mathrm{Stab}(p)$*;*
- *$U \cap g(U) = \emptyset$ for every $g \notin$* $\mathrm{Stab}(p)$*;*
- *the natural map $\alpha \colon U/\mathrm{Stab}(p) \to C/G$, induced by sending a point in U to its orbit, is a homeomorphism onto an open subset of C/G;*
- *no point of U except p is fixed by any element of* $\mathrm{Stab}(p)$*.*

Using the previous statement, we get the following structure theorem.

Theorem 2.0.7. [Mir95, Thm. III.3.4] *Let C be a Riemann surface and let $G \leq \mathrm{Aut}(C)$ finite. Then C/G has a unique structure of Riemann surface such that $\lambda \colon C \to C/G$ is holomorphic of degree $|G|$, and the ramification index $r_p(\lambda)$ of a point p equals $r_p(\lambda) = |\mathrm{Stab}(p)|$.*

Note that the previous theorem has many others implications.

Remark 2.0.8. The ramification locus of λ is the set of points of C having non trivial stabilizer.

Remark 2.0.9. Let $q \in C/G$ be a branch point of λ, and $p \in C$ be a point over q. The stabilizer of a point $g \cdot p$ over q is conjugated to $\mathrm{Stab}(p)$: $\mathrm{Stab}(g \cdot p) = g \cdot \mathrm{Stab}(p) \cdot g^{-1}$. In particular, they have the same order, so that

$$r_{g \cdot p}(\lambda) = r_p(\lambda) = |\mathrm{Stab}(p)|.$$

This justify the following

Definition 2.0.10. We say that $m \geq 1$ is the *ramification index* of a point $q \in C/G$ if it is the ramification index of a point p over q.

Note that $q \in C/G$ is a branch point if and only if its ramification index is at least two.

Remark 2.0.11. The number of points over $q \in C/G$ equals $|G|/m$, where m is the ramification index of q.

Remark 2.0.12. (Hurwitz formula for Galois coverings) The well-known Hurwitz formula assumes a nice homogeneous form if applied to Galois coverings. Let $q_1, \ldots, q_r$ be the set of branch points of λ of ramification indices $m_i \geq 2$ respectively. The ramification divisor of λ can be rearranged as

$$\begin{aligned}\sum_{p \in C} (r_p(\lambda) - 1)p &= \sum_{i=1}^{r} \left(\sum_{p \in \lambda^{-1}(q_i)} (r_p(\lambda) - 1)p \right) \\ &= \sum_{i=1}^{r} \left(\sum_{p \in \lambda^{-1}(q_i)} (m_i - 1)p \right) \\ &= \sum_{i=1}^{r} (m_i - 1)\lambda^{-1}(q_i),\end{aligned}$$

where $\lambda^{-1}(q_i)$ is the divisor of the fibre of q_i taken with the reduced structure. Let ω be a non-zero meromorphic 1-form of C/G. Then

$$div(\lambda^*\omega) = \lambda^* div(\omega) + \sum_{i=1}^{r} (m_i - 1)\lambda^{-1}(q_i). \tag{2.1}$$

By applying the degree operator to both divisors of the equation, we get the Hurwitz formula

$$\begin{aligned} 2g(C) - 2 &= \deg(\lambda)(2g(C/G) - 2) + \sum_{i=1}^{r} (m_i - 1)|\lambda^{-1}(q_i)| \\ &= |G|(2g(C/G) - 2) + \sum_{i=1}^{r} (m_i - 1)\frac{|G|}{m_i} \\ &= |G| \left(2g(C/G) - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) \right). \end{aligned} \tag{2.2}$$

We are interested to understand which is the action of $\mathrm{Stab}(p)$ locally around p.

Lemma 2.0.13. *(Linearization of the action,* [Mir95, Cor. III.3.5]*) Let C be a Riemann surface and let $G \leq \mathrm{Aut}(C)$ finite. Fix a point $p \in C$, and let m be the order of its stabilizer. Let $h \in \mathrm{Stab}(p)$ be a generator of the stabilizer subgroup. Then there is a local coordinate z on C centred at p such that h maps the point of coordinate z to the point of coordinate λz, where $\lambda = e^{\frac{2\pi i}{m}k}$ is one of the primitive m-roots of the unity.*

Definition 2.0.14. The previous lemma gives a bijection among the primitive m-roots of the unity and the generators of $\mathrm{Stab}(p)$. We denote by *local monodromy* of p the unique generator of $\mathrm{Stab}(p)$ acting by $z \mapsto e^{\frac{2\pi i}{m}} z$.

Remark 2.0.15. The local monodromy of $g \cdot p$ is the conjugate ghg^{-1} of h. In other word, the local monodromy of points of the same G-orbit are conjugated between them.

2.0.1 The Riemann Existence Theorem

Let C be a Riemann surface and let $G \leq \mathrm{Aut}(G)$ finite. By Theorem 2.0.7 we can define a structure of Riemann surface on $C' = C/G$ such that the quotient map $\lambda\colon C \to C'$ is holomorphic. Let $X := \{q_1, \dots, q_r\}$ be the set of branch points of λ. Then

$$\lambda\colon C \setminus \lambda^{-1}(X) \to C' \setminus X$$

is an étale covering.
We pose the following question: given a set of r points $X := \{q_1, \dots, q_r\}$ on

a Riemann surface C' and an étale Galois covering of $C' \setminus X$ is it possible to extend that covering to a Galois covering of the entire C'? Is the Riemann surface C unique up to isomorphism?

Theorem 2.0.16. *(Riemann Existence Theorem) Let $\overline{C}$ and C' be Riemann surfaces and let $X \subset C'$ be a finite set. Let*

$$\lambda_0 \colon \overline{C} \to C' \setminus X$$

be a proper étale covering.
Then λ can be extended to a branched covering of C', that is there exists a Riemann surface C, a proper biholomorphic map $\lambda\colon C \to C'$ and a biholomorphic map $\varphi\colon C \setminus \lambda^{-1} \to \overline{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
\overline{C} \arrow[d, "\lambda_0"'] & C \setminus \lambda^{-1}(X) \arrow[l, "\varphi"'] \arrow[r, hook] & C \arrow[d, "\lambda"] \\
C' \setminus X \arrow[rr, hook] & & C'
\end{tikzcd}$$

Moreover C is unique up to isomorphisms.

The Riemann Existence Theorem is extremely useful in many situations. We are going to use it in the next section together with the following classic result in algebraic topology:

Theorem 2.0.17. *(Existence Theorem of covering spaces) Let X be a topological space which is path-connected, locally path-connected, and semilocally simply connected. Then, for every subgroup $K \leq \pi_1(X, x_0)$, there exists a covering $\lambda\colon X_K \to X$ such that $\lambda_*(\pi_1(X_k, \widetilde{x})) = K$ for a suitable choice of the base point $\widetilde{x} \in \lambda^{-1}(x_0)$. The covering λ is unique up to equivalence of coverings.*

2.1 Description of Galois covers of the line via theoretical group data

A Galois covering may be often difficult to describe, especially with equations. Given a finite group G, we are particularly interested to determine any G-covering (C, λ) of $\mathbb{P}^1$ branched over r points $q_1, \dots, q_r$, up to topological equivalence. A surprisingly fact is that local monodromies $g_i \in G$ of points over q_i determine C, the action on C, and so the covering λ. We collect these local monodromies in a sequence $[g_1, \dots, g_r]$, which takes the name of *spherical system of generators.*

The chapter discusses the main results on the deep correspondence among G-coverings of the line (up to topological equivalence) and (classes) of spherical systems of generators.

Definition 2.1.1. Given a Riemann surface C' and a finite group G, we say that a couple (C, λ) is a *G-covering of C'* if G acts on C, and $\lambda\colon C \to C'$ is the quotient map, so it exhibits C' as the quotient of C via G.

Definition 2.1.2. We say that (C_1, λ_1) and (C_2, λ_2) are *topological equivalent* if there exists a orientation preserving homeomorphism $f\colon C_1 \to C_2$, and an automorphism $\varphi \in \mathrm{Aut}(G)$ such that $f(g \cdot p) = \varphi(g) \cdot f(p)$, for any $g \in G$, and $p \in C_1$. We say that (C_1, λ_1) and (C_2, λ_2) are *isomorphic* if moreover f is a biholomorphism.

Consider the set of G-coverings modulo isomorphism. The topological equivalence partitions it in equivalence classes, let $\mathcal{C}$ be one of them. González-Díez and Harvey showed in [GDH92] that $\mathcal{C}$ has a natural structure of connected complex manifold such that the natural map of $\mathcal{C}$ on the moduli space of curves mapping (C, λ) to C is analytic. More precisely, the manifold $\mathcal{C}$ is the normalization of its image $\widetilde{\mathcal{C}}$. In particular, $\widetilde{\mathcal{C}}$ is always an irreducible subvariety of the moduli space of curves.

Remark 2.1.3. Let C' be a Riemann surface. We remark that

- the genus g,
- the number r of points of the branch locus,
- the ramification indices $m_1, \dots, m_r \geq 2$,

are invariants up to topological equivalence of G-coverings (C, λ) of C'. The Hurwitz formula (2.2) establishes the relationship among the genus g of C, the genus g' of C', and the ramification indices m_i:

$$2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{2.3}$$

Definition 2.1.4. Up to re-label m_i, we can assume $2 \leq m_1 \leq \cdots \leq m_r$ The sequence $[m_1, \dots, m_r]$ is called *signature* of (C, λ).

Let us suppose now $C' = \mathbb{P}^1$, so we are working with G-coverings of the line.

Definition 2.1.5. We set $\mathcal{T}^r(G)$ the collection of all classes of G-coverings of $\mathbb{P}^1$ ramified over r points modulo topological equivalence.

Definition 2.1.6. A *spherical system of generators* (of length r) of G is a sequence $[g_1, \dots, g_r] \in G^r$ of elements of G such that $g_i \neq 1$ for all i, and

- $G = \langle g_1, \dots, g_r \rangle$;
- $g_1 \cdots g_r = 1$.

The sequence $[o(g_1), \dots, o(g_r)]$ is called *signature* of $[g_1, \dots, g_r]$.

Remark 2.1.7. For each signature $[m_1, \dots, m_r]$ consider the *orbifold group*

$$\mathbb{T}(m_1, \dots, m_r) := \langle \gamma_1, \dots, \gamma_r | \gamma_i^{m_i} = 1 = \gamma_1 \dots \gamma_r \rangle.$$

There is a natural bijection among the set of surjective homomorphisms $\mathbb{T}(m_1, \dots, m_r) \to G$ and the set of the spherical systems of generators $[g_1, \dots, g_r]$ of fixed signature $[m_1, \dots, m_r]$.

The bijection associates to any homomorphism φ the spherical system of generators $[\varphi(\gamma_1), \dots, \varphi(\gamma_r)]$.

Definition 2.1.8. We set $\mathcal{D}^r(G) \subset G^r$ the collection of all spherical systems of generators of G of length r.

Consider a finite group G, and a non-negative integer $r > 0$. Take the group $\widetilde{\mathcal{B}_r}$, whose presentation with generators $\sigma_1, \dots, \sigma_{r-1}$ is

$$\widetilde{\mathcal{B}_r} = \left\langle \sigma_1, \dots, \sigma_{r-1} \colon \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j| = 1 \\ (\sigma_{r-1} \cdots \sigma_1)^r = 1 & \end{array} \right\rangle.$$

We call such generators *Hurwitz moves*. We consider the following action of $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$ on $\mathcal{D}^r(G)$:

$$\Psi \cdot [g_1, \dots, g_r] := [\Psi(g_1), \dots, \Psi(g_r)], \qquad \Psi \in \mathrm{Aut}(G).$$

An Hurwitz move $\sigma_i \in \widetilde{\mathcal{B}_r}$ acts as

$$\sigma_i \cdot [g_1, \dots, g_r] := [g_1, \dots, g_{i-1}, g_i \cdot g_{i+1} \cdot g_i^{-1}, g_i, g_{i+2}, \dots g_r].$$

The action of the generators σ_i extends to an action of the entire $\widetilde{\mathcal{B}_r}$.
We finally have the following classical result

Theorem 2.1.9. *The collection of all classes of G-coverings of $\mathbb{P}^1$ ramified over r points modulo topological equivalence is in bijection with $\mathcal{D}^r(G)/\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$:*

$$\mathcal{T}^r(G) \cong \mathcal{D}^r(G)/\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}. \tag{2.4}$$

For a proof, we refer to the recent paper [GT22, Cor. 5.7].

We describe the bijection in Theorem 2.1.9. Take an element in the quotient $\mathcal{D}^r(G)/\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$, and choose a representative $[g_1, \dots, g_r]$ of it. From Remark 2.1.7 we obtain a surjective morphism $\mathbb{T}(m_1, \dots, m_r) \to G$, with $m_i = o(g_i)$.
We choose a finite set $X := \{q_1, \dots, q_r\}$ on $\mathbb{P}^1$, a point $q_0 \in \mathbb{P}^1 \setminus X$, and a *geometric basis* of the fundamental group of $\mathbb{P}^1 \setminus X$ with base point q_0:

Definition 2.1.10. Consider a smooth regular arc $\tilde{\eta}_i$ joining q_0 to q_{μ_i} (for some permutation μ). Assume that $\tilde{\eta}_i$ intersects only at q_0 and that the tangent vectors at q_0 are pairwise distinct and follow each other in the counterclockwise order (we orient S^2 by the outer normal). We define the loop η_i as follows:
Fix a small disk D around q_{μ_i}. The loop η_i starts at the point q_0, travels along $\tilde{\eta}_i$ till reaches the boundary ∂D, then makes a complete tour of ∂D counterclockwise and finally goes back to q_0 again along $\tilde{\eta}_i$.
Call always by η_i the homotopy class of the loop η_i. We say that $\eta_1, \ldots, \eta_r$ is a *geometric basis* (referred to μ) of $\pi_1(\mathbb{P}^1 \setminus X, q_0)$, $X = \{q_1, \ldots q_r\}$.

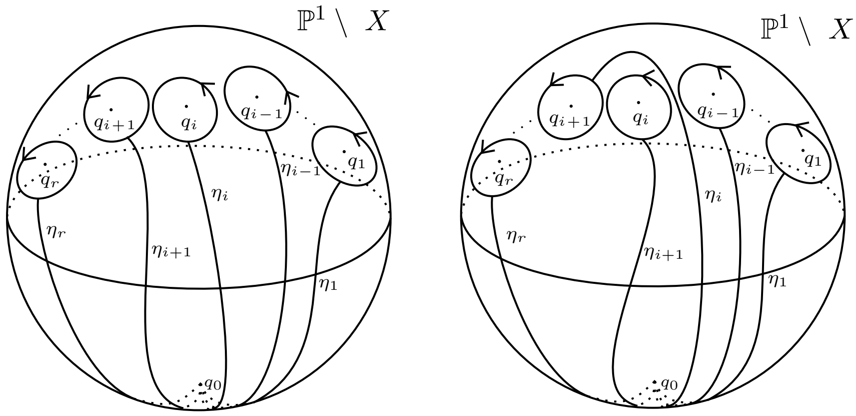


Figure 2.1: Two examples of geometric bases, the first with $\mu = \mathrm{Id}$, and the second with $\mu = (i \quad i+1)$.

Remark 2.1.11. The class of the loop η_i does not depend on the choice of the disk D. Moreover, if we replace $\widetilde{\eta_i}$ with different paths $\tilde{\delta}_i$, and called δ_i the resulted closed loops, then η_i and δ_i would be conjugated in $\pi_1(\mathbb{P}^1 \setminus X, q_0)$. In fact, define the loop ν that starts at q_0, travels along $\tilde{\eta}_i$ till ∂D, follows a piece of ∂D and finally goes back along $\tilde{\delta}_i$. Then $\delta_i = \nu \eta_i \nu^{-1}$.

Let us fix a geometric basis $\eta_1, \ldots, \eta_r$ referred to the permutation $\mu = \mathrm{Id}$. Observe that the product $\eta_1 \ldots \eta_r$ can be contracted to the point q_0 (see the figure (2.2) below). This is the only relation between the class of these loops. In other words, the choice of a geometric basis gives a presentation of $\pi_1(\mathbb{P}^1 \setminus X, q_0)$:

$$\Gamma_r := \langle \gamma_1, \ldots, \gamma_n | \; \gamma_1 \ldots \gamma_n \rangle, \qquad \Gamma_r \xrightarrow{\sim} \pi_1(\mathbb{P}^1 \setminus X, q_0), \quad \gamma_i \mapsto \eta_i. \tag{2.5}$$

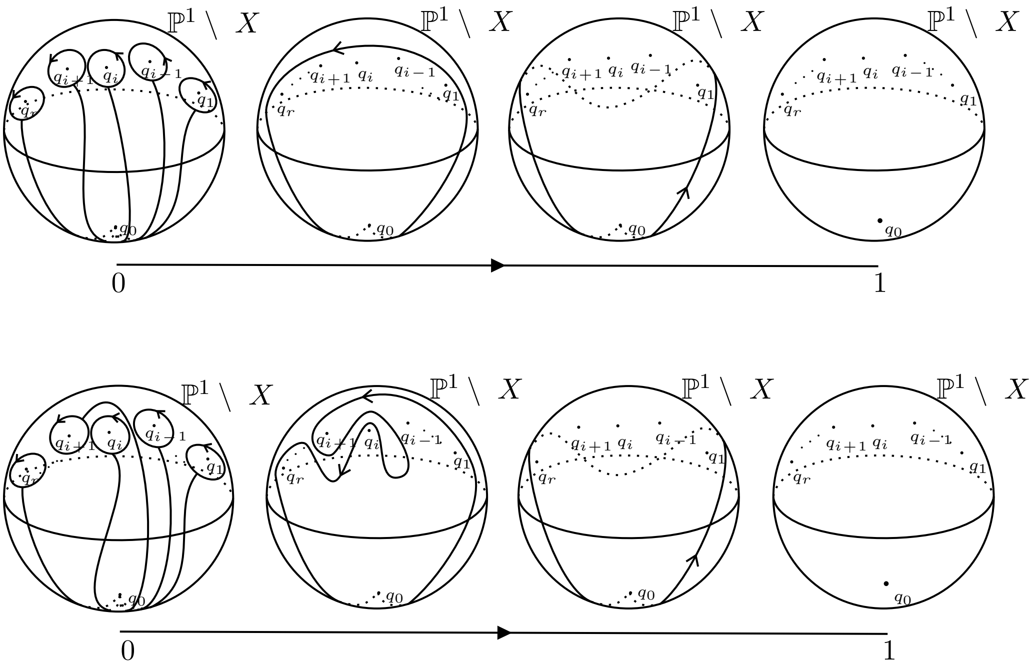


Figure 2.2: The contraction of the product loop $\eta_1 \cdots \eta_r$ to the point q_0, where the basis $\eta_1, \ldots, \eta_r$ is referred to $\mu = \mathrm{Id}$, in the first case, and to $\mu = (i \;\; i+1)$, in the second case.

Notice that $\mathbb{T}(m_1, \ldots, m_r)$ is a quotient by Γ_r. Therefore, the choice of a geometric basis gives a group homomorphism $\pi_1(\mathbb{P}^1 \backslash X, q_0) \to \mathbb{T}(m_1, \ldots, m_r)$ mapping, for all $i = 1, \ldots, r$, the geometric loop η_i around q_{μ_i} to γ_i.
The kernel of the composition $\pi_1(\mathbb{P}^1 \setminus X, q_0) \to \mathbb{T}(m_1, \ldots, m_r) \to G$ defines a unique étale G-covering of $\mathbb{P}^1 \setminus X$ from Existence Theorem of covering spaces 2.0.17.
By Riemann Existence Theorem 2.0.16, this completes to a G-covering (C, λ) of $\mathbb{P}^1$.
The bijection of Theorem 2.1.9 maps the class of $[g_1, \ldots, g_r]$ modulo $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$ to the class of (C, λ) modulo topological equivalence.

In particular Theorem 2.1.9 says that

1. if in the above construction we change

    - the set of spherical generators $[g_1, \ldots, g_r]$ by a set in the same orbit for the action of $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$, or
    - the points $q_0, q_1, \ldots, q_r$ with other $r+1$ points of $\mathbb{P}^1$, or
    - the choice of the geometric basis $\eta_1, \ldots, \eta_r$

    then the new obtained G-covering is topologically equivalent to (C, λ);

2. if (C_1, λ_1) and (C_2, λ_2) are obtained by spherical systems of generators that are not in the same $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$-orbit then (C_1, λ_1) and (C_2, λ_2) are not topologically equivalent.

3. every G-covering (C, λ) up to topological equivalence is obtained in this way by a spherical system of generators of G.

We discuss in details only the third point above. Hence we show how to get a spherical system of generators from a G-covering of the line.

Consider a G-covering $\lambda\colon C \to \mathbb{P}^1$ whose branch locus consists of r points. Take an orientation preserving homeomorphism, $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$, mapping this branch locus to $X = \{q_1, \dots, q_r\}$. Denote by C_2 the curve C with the complex structure making $\psi \circ \lambda\colon C_2 \to \mathbb{P}^1$ holomorphic. Then (C_2, λ_2) is topologically equivalent to (C, λ). Therefore, without lost of generality, we can assume that the branch locus of (C, λ) is exactly X.

Then $\lambda\colon C \setminus \lambda^{-1}(X) \to \mathbb{P}^1 \setminus X$ is an unramified covering.

We choose a point $p_0 \in \lambda^{-1}(q_0)$. Consider the normal subgroup

$$\lambda_* \pi_1(C \setminus \lambda^{-1}(X), p_0) \trianglelefteq \pi_1(\mathbb{P}^1 \setminus X, q_0), \tag{2.6}$$

and the monodromy map based at p_0

$$\pi_1(\mathbb{P}^1 \setminus X, q_0) \xrightarrow{\mathcal{L}_{p_0}} \lambda^{-1}(q_0), \qquad \eta \mapsto \mathcal{L}_{p_0}(\eta) := \tilde{\eta}_{p_0}(1). \tag{2.7}$$

This sends any class loop η of $\mathbb{P}^1 \setminus X$ based at q_0 to the end point of its (unique) class lifting $\tilde{\eta}_{p_0}$ on $C \setminus \lambda^{-1}(X)$, starting at the point p_0. Observe that (2.7) is surjective since $C \setminus \lambda^{-1}(X)$ is path-connected.

The point q_0 is not a branch point of the covering, so that the orbit $\lambda^{-1}(q_0)$ consists of $|G|$-points, and we can identify $\lambda^{-1}(q_0) \cong G$: $g \cdot p_0 \mapsto g$. Once such identification is fixed, then the monodromy map (2.7) is a group homomorphism: suppose $\mathcal{L}_{p_0}(\eta) = g \cdot p_0$, and $\mathcal{L}_{p_0}(\gamma) = h \cdot p_0$. Observe $\widetilde{\eta\gamma}_{p_0} = \tilde{\eta}_{p_0} \tilde{\gamma}_{g \cdot p_0}$, and the paths $\tilde{\gamma}_{g \cdot p_0}$ and $g \cdot \tilde{\gamma}_{p_0}$ are both lifting of γ at the initial point $g \cdot p_0$. Thus they are homotopic paths and have the same ending point. We can conclude

$$\mathcal{L}_{p_0}(\eta\gamma) = \widetilde{\eta\gamma}_{p_0}(1) = \tilde{\gamma}_{g \cdot p_0}(1) = g \cdot \tilde{\gamma}_{p_0}(1) = g \cdot (h \cdot p_0) = (gh) \cdot p_0.$$

Moreover, the kernel of (2.7) is exactly the normal subgroup (2.6). Notice that only the kernel of this map is uniquely determined by the covering.
Let us choose now a geometric basis $\eta_1, \dots, \eta_r$ we get the isomorphism (2.5). The composition $\Gamma_r \to \pi_1(\mathbb{P}^1 \setminus X, q_0) \xrightarrow{\mathcal{L}_{p_0}} G$ is an epimorphism which sends γ_i to some $g_i \in G$ (remember that (2.7) is surjective). We have therefore got a sequence $[g_1, \dots, g_r] \in G^r$ of elements of G, which are generating G, and whose product is 1. This is a *spherical system of generators* (of length r) of G as in the Definition 2.1.6.

In the subsection 2.1.1, we prove that g_i is the local monodromy (see the Def. 2.0.14) of a point over q_i, and the ramification index of q_i is exactly $m_i := o(g_i) \geq 2$. In particular, the signature of (C, λ) is the signature of the spherical system $[g_1, \dots, g_r]$, and the genus g of C can be computed by the signature using the Hurwitz formula (2.3).

Let us give some examples of how to use the Theorem 2.1.9:

Example 2.1.12. Let $p \geq 2$ be a prime number, and consider the group $G = \mathbb{Z}_p^2$. We are going to compute $\mathcal{T}^3(\mathbb{Z}_p^2)$, the collection of the $\mathbb{Z}_p^2$-coverings of the line (up to topological equivalence) ramified over 3 points. Consider a spherical system $[v_1, v_2, v_3]$. Note that v_1 and v_2 have to be linearly independent, since otherwise $v_3 = (p-1)v_1 + (p-1)v_2$, and so they would not generate the entire $\mathbb{Z}_p^2$. Now take the base change matrix $M \in \mathrm{Aut}(\mathbb{Z}_p^2)$ from the standard basis to $\{v_1, v_2\}$. Then $[v_1, v_2, v_3] = M \cdot [(1,0),(0,1),(p-1,p-1)]$. We can conclude that

$$\mathcal{T}^3(\mathbb{Z}_p^2) = \{[(F,\lambda)]\} \cong \mathcal{D}^3(\mathbb{Z}_p^2)/\mathrm{Aut}(\mathbb{Z}_p^2) \times \widetilde{\mathcal{B}_3} = \{[(1,0),(0,1),(p-1,p-1)]\}.$$

By the Hurwitz formula (2.3), the genus of the $\mathbb{Z}_p^2$-covering C of the line associated to such spherical system is:

$$2g(C) - 2 = p^2\left(-2 + 3 - \frac{3}{p}\right) \implies g(C) = \frac{(p-1)(p-2)}{2}.$$

The covering (F, λ) may be described as follows: take the Fermat curve of degree p in $\mathbb{P}^2$

$$F := \{x_0^p + x_1^p + x_2^p = 0\} \subset \mathbb{P}^2,$$

and define the action

$$\phi\colon \mathbb{Z}_p^2 \to \mathrm{Aut}(F), \quad (a,b) \mapsto [(x_0 : x_1 : x_2) \mapsto (x_0 : \zeta_p^a x_1 : \zeta_p^b x_2)], \quad \zeta_p := e^{\frac{2\pi i}{p}}.$$

This action has $3p$-points with non-trivial stabilizer. They form three orbits of length p. A representative of each orbit and a generator of the stabilizer is given by

point	$(-1:0:\zeta_p)$	$(-1:\zeta_p:0)$	$(0:-1:\zeta_p)$
generator	$(1,0)$	$(0,1)$	$(p-1,p-1)$

The quotient map is

$$\lambda\colon F \subset \mathbb{P}^2 \to \mathbb{P}^1, \qquad (x_0, x_1, x_2) \mapsto (x_1^p, x_2^p).$$

It is branched along $(0,1)$, $(1,0)$, and $(1,1)$.

Example 2.1.13. Let $G = S_3 = \langle \tau, \sigma \mid \tau^2 = \sigma^3 = 1, \sigma\tau = \tau\sigma^{-1} \rangle$ be the symmetric group of 3 elements. Consider $r = 3$, so we are going to determine $\mathcal{T}^3(S_3)$, the collection of S_3-coverings of the line (up to equivalence) ramified over $r = 3$ points. Let $[g_1, g_2, g_3]$ be a spherical system of S_3, and set $m_i := o(g_i) \in \{2, 3\}$. Without lost of generality, we can suppose $m_1 \leq m_2 \leq m_3$ (otherwise we re-order the g_i's by moving the system on its $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$-orbit through suitable Hurwitz moves). By the Hurwitz formula (2.3), the genus of the G-covering C associated to $[g_1, g_2, g_3]$ is

$$\begin{aligned} 2g(C) - 2 = 6\left(-2 + 3 - \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right)\right) \\ \implies g(C) = 4 - 3\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right). \end{aligned} \tag{2.8}$$

In particular, $3(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3})$ has to be an integer, which holds only for $m_1 = m_2 = m_3 = 3$, or $m_1 = m_2 = 2$ and $m_3 = 3$. The first case is not admissible, since there are not g_1, g_2, g_3 of order 3 generating S_3. It remains the second case. Observe that (2.8) gives $g(C) = 0$, hence $C = \mathbb{P}^1$.
The elements of order 2 of S_3 are τ, $\tau\sigma$, and $\tau\sigma^2$.

Since $[g_1, g_2, g_3]$ is a spherical system, then $g_3 = g_2^{-1} g_1^{-1}$, and $g_1 \neq g_2$ otherwise we would get $g_3 = 1$, because they have order 2. Thus the list of spherical systems $\mathcal{D}^3(S_3)$ with ordered signature $[2, 2, 3]$ is obtained just by choosing distinct $g_1, g_2 \in \{\tau, \tau\sigma, \tau\sigma^2\}$. Such list is

$$\begin{aligned} &[\tau, \tau\sigma, \sigma^2] \\ &[\tau, \tau\sigma^2, \sigma] = \begin{pmatrix} \tau \mapsto \tau \\ \sigma \mapsto \sigma^2 \end{pmatrix} \cdot [\tau, \tau\sigma, \sigma^2] = \sigma_1 \sigma_2^2 \sigma_1 \cdot [\tau, \tau\sigma, \sigma^2], \\ &[\tau\sigma, \tau, \sigma] = \begin{pmatrix} \tau \mapsto \tau\sigma \\ \sigma \mapsto \sigma^2 \end{pmatrix} \cdot [\tau, \tau\sigma, \sigma^2] = \sigma_1^2 \sigma_2^2 \sigma_1 \cdot [\tau, \tau\sigma, \sigma^2], \\ &[\tau\sigma^2, \tau\sigma, \sigma] = \begin{pmatrix} \tau \mapsto \tau\sigma^2 \\ \sigma \mapsto \sigma^2 \end{pmatrix} \cdot [\tau, \tau\sigma, \sigma^2] = \sigma_1^3 \sigma_2^2 \sigma_1 \cdot [\tau, \tau\sigma, \sigma^2], \\ &[\tau\sigma^2, \tau, \sigma^2] = \begin{pmatrix} \tau \mapsto \tau\sigma^2 \\ \sigma \mapsto \sigma \end{pmatrix} \cdot [\tau, \tau\sigma, \sigma^2] = \sigma_1 \cdot [\tau, \tau\sigma, \sigma^2], \\ &[\tau\sigma, \tau\sigma^2, \sigma^2] = \begin{pmatrix} \tau \mapsto \tau\sigma \\ \sigma \mapsto \sigma \end{pmatrix} \cdot [\tau, \tau\sigma, \sigma^2] = \sigma_1^2 \cdot [\tau, \tau\sigma, \sigma^2]. \end{aligned} \tag{2.9}$$

Observe that any automorphism $\Psi \in \mathrm{Aut}(S_3)$ acts on $[\tau, \tau\sigma, \sigma^2]$ as some element $\sigma \in \widetilde{\mathcal{B}_r}$.
Since any spherical system shares the same $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$-orbit of $[\tau, \tau\sigma, \sigma^2]$, then we can conclude

$$\mathcal{T}^3(S_3) = \{[(\mathbb{P}^1, \lambda)]\} \cong \mathcal{D}^3(S_3)/\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r} = \{[\tau, \tau\sigma, \sigma^2]\}.$$

An action of S_3 on $C = \mathbb{P}^1$ is

$$\tau \cdot (x_0, x_1) := (x_1, x_0), \qquad \sigma \cdot (x_0, x_1) := (\zeta_3 x_0, x_1), \qquad \zeta_3 := e^{\frac{2\pi i}{3}}.$$

The quotient map is

$$\lambda\colon \mathbb{P}^1 \to \mathbb{P}^1, \qquad (x_0, x_1) \mapsto (x_0^3 x_1^3, (x_0^6 + x_1^6)/2).$$

Example 2.1.14. Consider the group $G = S_3 \times \mathbb{Z}_p^2$, $p \geq 2$ prime number. We compute $\mathcal{T}^3(S_3 \times \mathbb{Z}_p^2)$. Up to apply suitable Hurwitz moves, we can assume that a spherical system $[(g_1, v_1), (g_2, v_2), (g_3, v_3)]$ has $o(g_1) \leq o(g_2) \leq o(g_3)$. Observe $g_i \neq 1$, otherwise S_3 would be generated by only one element, and this is not possible since it is not cyclic. The same argument holds for $\mathbb{Z}_p^2$, so that $v_i \neq 0$. This implies $[g_1, g_2, g_3] \in \mathcal{D}^3(S_3)$ is a spherical system of S_3, and $[v_1, v_2, v_3] \in \mathcal{D}^3(\mathbb{Z}_p^2)$ is a spherical system of $\mathbb{Z}_p^2$. However, we have proved that any ordered system of S_3 of the list (2.9) shares the same orbit of $[\tau, \tau\sigma, \sigma^2]$ by a suitable automorphism Ψ of S_3, and any system of $\mathbb{Z}_p^2$ shares the same orbit of $[(1,0),(0,1),(p-1,p-1)]$ by a base change matrix M. Thus (Ψ, M) sends $[(\tau,(1,0)),(\tau\sigma,(0,1)),(\sigma^2,(p-1,p-1))]$ to $[(g_1,v_1),(g_2,v_2),(g_3,v_3)]$. We have proved

$$\begin{aligned}\mathcal{T}^3(S_3 \times \mathbb{Z}_p^2) \cong& \frac{\mathcal{D}^3(S_3 \times \mathbb{Z}_p^2)}{\mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_p) \times \widetilde{\mathcal{B}_3}} \\ =&\{[(\tau,(1,0)),(\tau\sigma,(0,1)),(\sigma^2,(p-1,p-1))]\}.\end{aligned}$$

By the Hurwitz formula (2.3), the genus of the associated G-covering C is

$$2g(C) - 2 = \begin{cases} 24\left(-2+3-1-\frac{1}{6}\right) & \text{if } \ p=2 \\ 54\left(-2+3-\frac{2}{3}\right) & \text{if } \ p=3 \\ 6p^2\left(-2+3-\left(\frac{1}{2p}+\frac{1}{2p}+\frac{1}{3p}\right)\right) & \text{otherwise} \end{cases} \implies$$

$$g(C) = \begin{cases} 0 & \text{if } \ p=2 \\ 10 & \text{if } \ p=3 \\ (3p-1)(p-1) & \text{otherwise.} \end{cases}$$

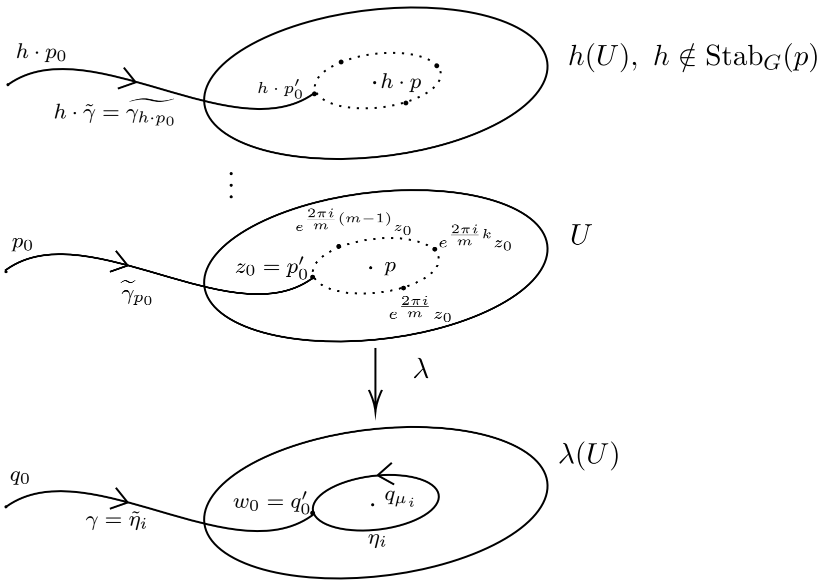
As we could expect, it becomes soon computationally difficult getting the $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$-orbits of $\mathcal{D}^r(G)$, when r or G increase. For this reason, several authors put an increased effort into the development of an efficient algorithm to compute such orbits, usually with the helping also of a computational algebra system (e.g. MAGMA, [BCP97]). A big step forward in this direction is given for instance in [CGP23], where the authors collect in a database a representative for any orbit of spherical systems of fixed genus g. At the moment, we have a complete list for any genus $g \leq 40$, with a few exceptions. We also mention their useful website https://mate.unipv.it/ghigi/tipitopo/. Here, chosen a finite group G and a signature m, one can ask to the database to pick-up one representative for any orbit of spherical systems of the group G having that pre-fixed signature m.

2.1.1 Monodromy and spherical systems of generators

Consider a G-covering of the line (C, λ) branched over $X = \{q_1, \dots, q_r\}$, a point $q_0 \notin X$, and a geometric basis $\eta_1, \dots, \eta_r$ referred to a permutation μ. Hence we have a presentation $\Gamma_r \to \pi_1(\mathbb{P}^1 \setminus X, q_0)$, which composed with the monodromy map $\mathcal{L}_{p_0}$ based at a point p_0 over q_0 gives an epimorphism $\Gamma_r \to G$ sending $\gamma_i \mapsto g_i$. We have got the spherical system $[g_1, \dots, g_r]$.

We are going to prove that any g_i is the local monodromy (see the Definition 2.0.14) of a point over q_{μ_i}, and its order is the ramification index $m := m_i$ of q_{μ_i}. To show this, let us consider a point p over q_{μ_i}.
Consider an invariant open neighbourhood U of p as in the Theorem 2.0.6. Hence for any $h \notin \mathrm{Stab}_G(p)$, then $h(U) \cap U = \emptyset$. Up to choose U small enough, we linearize the action of $\mathrm{Stab}_G(p)$ locally around p, by the Lemma 2.0.13. Hence it there exists a local coordinate z around p such that a generator g of $\mathrm{Stab}_G(p)$ acts as a multiplication of a primitive m-root of the unity, $g\colon z \mapsto e^{\frac{2\pi i}{m}k} z$, and the quotient map λ assumes the local normal form $z \mapsto w = z^m$.
Given $h \notin \mathrm{Stab}_G(p)$, note that $h(U)$ is an open neighbourhood of $h \cdot p$ with the same properties as U: there exists a local coordinate z_h around $h \cdot p$ such that the generator hgh^{-1} of $\mathrm{Stab}_G(h \cdot p)$ acts as $z_h \mapsto e^{\frac{2\pi i}{m}k} z_h$, and λ assumes locally around $h \cdot p$ the normal form $z_h \mapsto w = z_h^m$.

Consider the path $\gamma := \tilde{\eta}_i$, where $\tilde{\eta}_i$ is as the Definition 2.1.10; let us call by q_0' the intersection of γ and ∂D. Up to choose the disk D small enough, we can assume q_0' falls into the local chart $\lambda(U)$ of q_{μ_i}. Let us say $w_0 = w(q_0')$ is the local coordinate of q_0' on $\lambda(U)$. Call by p_0' the point over q_0' of the lifting of γ starting at p_0; it falls into only one of the open sets $h(U)$, $h \notin \mathrm{Stab}_G(p)$. Without lost of generalities, up to rename $h \cdot p$ to p, we can suppose that p_0' falls into U. Let $z_0 = z(p_0')$ be the local coordinate of p_0' on U. By construction, z_0 is one of the m-roots of w_0. The situation is the following



Let us change now the base point q_0 to q_0' via γ. The spherical system referred to the new geometric basis $\gamma^{-1}\eta_1\gamma, \ldots, \gamma^{-1}\eta_r\gamma$ is $[g_1, \ldots, g_r]$ again. In particular, by construction of γ, the loop $\gamma^{-1}\eta_i\gamma$ is the circle $t \mapsto e^{2\pi i t} w_0$ of radius $|w_0|$. Its unique lifting via $\lambda\colon z \mapsto z^m$ starting at z_0 is then $t \mapsto e^{\frac{2\pi i t}{m}} z_0$, whose ending point is $e^{\frac{2\pi i}{m}} z_0$. From the other side $\mathcal{L}_{p_0'}(\gamma^{-1}\eta_i\gamma) = g_i \cdot p_0'$. We have therefore proved $g_i \cdot z_0 = e^{\frac{2\pi i}{m}} z_0$.

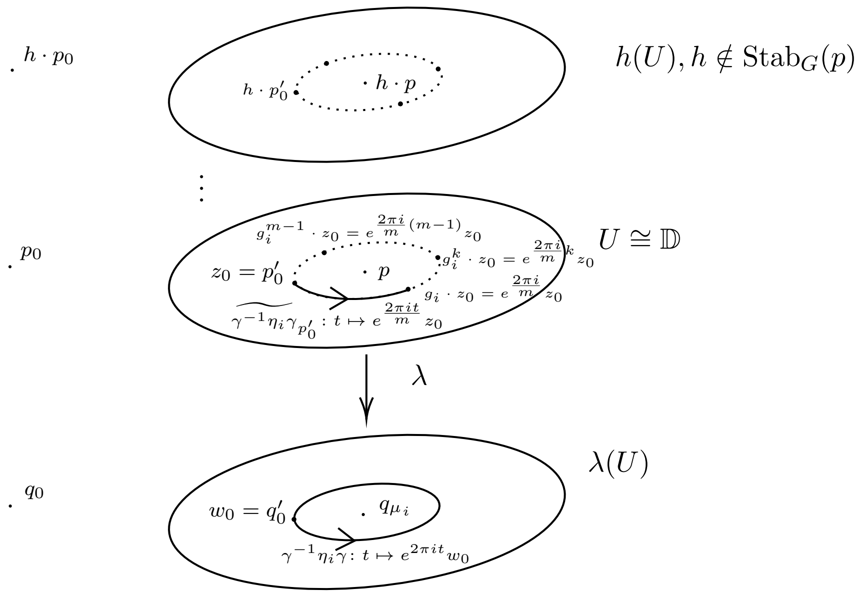


Figure 2.3: The lifting of the loop $\gamma^{-1}\eta_i\gamma\colon t \mapsto e^{2\pi it} w_0$ starting at the point $z_0 = p_0'$. We get $g_i \cdot z_0 = e^{\frac{2\pi i}{m}} z_0$.

Moreover, this shows $g_i \cdot z_0 \in g_i(U) \cap U \neq \emptyset$, so that $g_i \in \mathrm{Stab}_G(p)$. Since any element of $\mathrm{Stab}_G(p)$ is acting on U by the multiplication of a m-root of the unity and $g_i \cdot z_0 = e^{\frac{2\pi i}{m}} z_0$, then g_i has to act as $z \mapsto e^{\frac{2\pi i}{m}} z$, which proves g_i is the local monodromy of the point p.

Remark 2.1.15. We have immediately got also that hg_ih^{-1} is the local monodromy of the point $h \cdot p$, with $h \notin \mathrm{Stab}_G(p)$. Hence the local monodromies of points over q_{μ_i} are conjugated to each other.

Remark 2.1.16. The order of $g_i \in G$ has to be $o(g_i) = m = m_i$. This follows because the action of g_i^k is $g_i^k \cdot z = e^{\frac{2\pi ik}{m}} z$, $0 \leq k \leq m-1$.
To see this, you can either simply compose k-times the local action of g_i, or to re-iterate the previous geometrical approach as follows:
Consider η_i^k instead of η_i. So $\gamma^{-1}\eta_i^k\gamma$ is the loop $t \mapsto e^{2\pi ikt} w_0$ starting at $w_0 = q_0'$ and going over the circle of radius $|w_0|$ k-times. It is sent to g_i^k via the monodromy map based at p_0', so that $\mathcal{L}_{p_0'}(\gamma^{-1}\eta_i^k\gamma) = g_i^k \cdot p_0'$. The lifting of $\gamma^{-1}\eta_i^k\gamma$ starting at $z_0 = p_0'$ via $\lambda\colon z \mapsto w = z^m$ is $t \mapsto e^{\frac{2\pi ik}{m}t} z_0$, whose ending point is then $e^{\frac{2\pi ik}{m}} z_0$. We have proved $g_i^k \cdot z_0 = e^{\frac{2\pi ik}{m}} z_0$, which forces g_i^k to act as required.

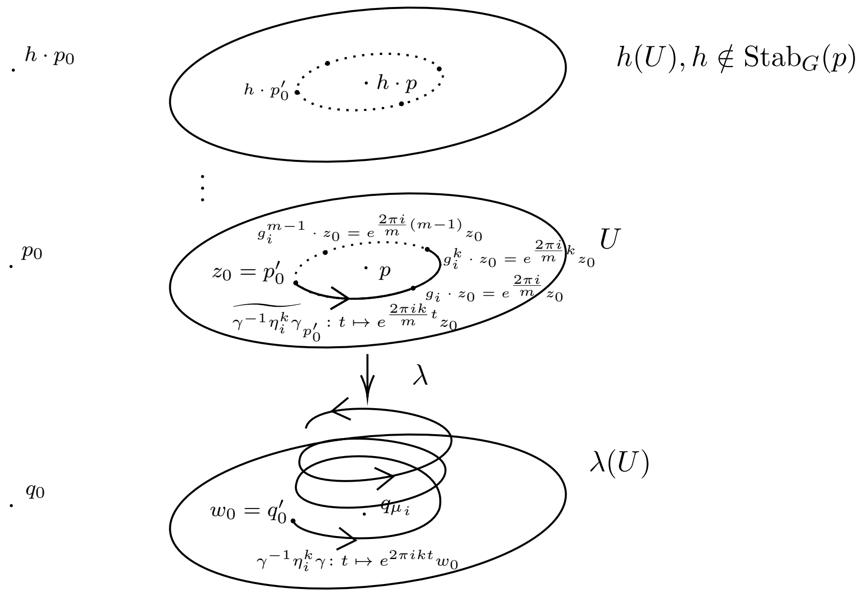


Figure 2.4: The lifting of the loop $\gamma^{-1}\eta_i^k\gamma\colon t \mapsto e^{2\pi ikt}w_0$ starting at the point $z_0 = p_0'$. We get $g_i \cdot z_0 = e^{\frac{2\pi ik}{m}} z_0$.

Chapter 3

Cyclic quotient singularities

Here we discuss cyclic quotient singularities of normal complex surfaces and their minimal resolution. Such singularities are crucial to be studied for sections 4.3 and 4.4 of the Chapter 4 since the quotient model of an unmixed type of a product-quotient surface has only finitely many cyclic singularities.

Definition 3.0.1. Let X be a normal complex variety. We say that a point $p \in X$ is a *quotient singularity* if there exists a neighbourhood U of p such that $U \cong \mathbb{C}^m/H$, where H is a finite subgroup of $\mathrm{Aut}(\mathbb{C}^m, 0)$, the group of automorphism of $\mathbb{C}^m$ fixing the origin 0.

Remark 3.0.2. Consider a subgroup $H \leq \mathrm{Aut}(\mathbb{C}^m, 0)$ and a change of coordinates $\phi\colon \mathbb{C}^m \to \mathbb{C}^m$ around 0, namely an automorphism of $\mathbb{C}^m$ fixing 0. Then this change determines a new group $H' := \{\phi \circ h \circ \phi^{-1} \colon h \in H\} \leq \mathrm{Aut}(\mathbb{C}^m, 0)$ conjugated to H, $H' = \phi H \phi^{-1} \cong H$, and an isomorphism $\mathbb{C}^m/H \to \mathbb{C}^m/H'$ sending $Hx \mapsto H'(\phi(x))$.
Another equivalent interpretation is that we have changed the action of H on $\mathbb{C}^m$ via the map ϕ; the new action may be different from the previous one, but the quotient $\mathbb{C}^m/H$ doesn't change, up to isomorphism.

Therefore, one can try to change coordinates on $\mathbb{C}^m$ in order to represent the singularity through a suitable subgroup H, which may be considered in some sense "canonical". Let us see how.

Lemma 3.0.3. *(Cartan, cf.* [Bri67, Lemma 2.2]*) If H is a finite subgroup of $\mathrm{Aut}(\mathbb{C}^m, 0)$, then there exists a system of coordinates such that the action of H is linear, namely $H \leq \mathrm{GL}(m, \mathbb{C})$.*

Proof. Let us define the map $\phi\colon \mathbb{C}^m \to \mathbb{C}^m$ sending $x \mapsto \sum_{h \in H} (J_0 h)^{-1} h \cdot x$, where $J_0 h$ denotes the Jacobian matrix of h at the point $x = 0$. We observe that ϕ is a change of coordinates around 0:

$$J_0\phi = \sum_{h \in H} (J_0 h)^{-1} J_0 h = |H| I_m \implies \det(J_0\phi) = |H| \neq 0.$$

Called by $z = (z_1, \dots, z_m)$ the coordinates of the target of the map ϕ, then the new action of H via ϕ is

$$\begin{aligned} g \cdot z = \left(\phi \circ g \circ \phi^{-1}\right) z &= \sum_{h \in H} (J_0 h)^{-1} h \cdot g \cdot (\phi^{-1} z) \\ &= J_0 g \sum_{h \in H} (J_0 hg)^{-1} (hg) \cdot (\phi^{-1} z) \\ &= (J_0 g) z. \end{aligned}$$

□

Remark 3.0.4. By the just proved Cartan's lemma 3.0.3, every quotient singularity $\mathbb{C}^m/H$ can be realized through a finite linear subgroup $H \leq \mathrm{GL}(m, \mathbb{C})$.

Lemma 3.0.5. *If H is a finite abelian subgroup of $\mathrm{GL}(m, \mathbb{C})$, and $h_1, \dots h_k$ are generators of order $r_1, \dots, r_k$ respectively, then there exists a system of coordinates of $\mathbb{C}^m$ such that H has the form*

$$H = \langle \begin{pmatrix} e^{\frac{2\pi i p_{11}}{r_1}} & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & e^{\frac{2\pi i p_{m1}}{r_1}} \end{pmatrix}, \dots, \begin{pmatrix} e^{\frac{2\pi i p_{1k}}{r_k}} & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & e^{\frac{2\pi i p_{mk}}{r_k}} \end{pmatrix} \rangle,$$

for some $p_{1i}, \dots, p_{mi}, r_i \in \mathbb{N}$, $i = 1, \dots k$.

Proof. Since H is finite, then each element of it has finite order and so it is a diagonalizable matrix. Moreover, H abelian implies that its elements are matrices commuting to each other, so that there exists a basis b of eigenvectors diagonalizing simultaneously all of them. In other words, called by $M = M_{b,e}(\mathrm{Id}_{\mathbb{C}^m})$ the base change matrix from the standard basis e to b, then MAM^{-1} is diagonal, for each $A \in H$. Thus, choose the change of coordinates $x \mapsto Mx$ to get an isomorphism $\mathbb{C}^m/H \to \mathbb{C}^m/H'$, where $H' = MHM^{-1}$ consists only of diagonal matrices. Now, the generators h_i of H corresponds to diagonal matrices D_i of H', and $D_i^{r_i} = I$ since h_i has order r_i. Therefore the eigenvalues of D_i are r_i-roots of the unity, and the thesis follows. □

The previous lemma justifies the following

Definition 3.0.6. A normal complex variety X has a *cyclic quotient singularity* in $p \in X$ if there exists a neighbourhood U of p such that $U \cong \mathbb{C}^m/H$, where H is a cyclic subgroup of $\mathrm{GL}(m, \mathbb{C})$ of the form

$$H = \left\langle \begin{pmatrix} e^{\frac{2\pi i p_1}{r}} & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & e^{\frac{2\pi i p_m}{r}} \end{pmatrix} \right\rangle,$$

for some $p_1, \dots p_m, r \in \mathbb{N}$. We say that $\frac{1}{r}(p_1, \dots, p_m)$ is the *type* of singularity of p.

Since we are interested on singularities on a surface, we consider from now a subgroup

$$H = \left\langle \begin{pmatrix} e^{\frac{2\pi i p}{r}} & 0 \\ 0 & e^{\frac{2\pi i q}{r}} \end{pmatrix} \right\rangle \leq \mathrm{GL}(2, \mathbb{C}),$$

in which case we will say that $\mathbb{C}^2/H$ is a singularity of type $\frac{1}{r}(p, q)$.

Remark 3.0.7. In dimension two the singular points which are quotient singularities through the action of a finite abelian group are cyclic quotient singularities. This follows directly from the classification of finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ (see [Mat02, Theorem 4.6.20]).

Remark 3.0.8. As we can expect, there are different way to describe a singularity. For example, a singularity of type $\frac{1}{5}(2, 3)$ is biholomorphic to one of type $\frac{1}{5}(1, 4)$, simply because $H = \langle g \rangle = \langle g^3 \rangle$, and so $\mathbb{C}^2/\langle g \rangle \cong \mathbb{C}^2/\langle g^3 \rangle$, where $g := \begin{pmatrix} e^{\frac{2\pi i 2}{5}} & 0 \\ 0 & e^{\frac{2\pi i 3}{5}} \end{pmatrix}$.

Definition 3.0.9. Take the set of formal symbols $\frac{1}{r}(p, q)$. We define on this set the following equivalence relation: we say that $\frac{1}{r_1}(p_1, q_1)$ is *equivalent* to $\frac{1}{r_2}(p_2, q_2)$ if $\mathbb{C}^m/H_1 \cong \mathbb{C}^m/H_2$, where $H_i := \left\langle \begin{pmatrix} e^{\frac{2\pi i p_i}{r_i}} & 0 \\ 0 & e^{\frac{2\pi i q_i}{r_i}} \end{pmatrix} \right\rangle$, $i = 1, 2$. A cyclic quotient singularity corresponds to an equivalence class.

The next lemma says that we can always pick up a "canonical" representative for any equivalence class. Before proving this, we give some easy remarks.

Remark 3.0.10. Each $\frac{1}{r}(p, q)$ is equivalent to $\frac{1}{r}([p]_r, [q]_r)$, where $[p]_r$ denotes the representative $\leq r$ of the class of p modulo r. Therefore, one can suppose without lost of generality that $0 \leq p, q < r$.

Remark 3.0.11. $\frac{1}{r}(p, q)$ is equivalent to $\frac{1}{r}(q, p)$. Remembering the Remark 3.0.2, it is sufficient to take the change of coordinates $(x, y) \mapsto (y, z)$, which gives a new cyclic group generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i p}{r}} & 0 \\ 0 & e^{\frac{2\pi i q}{r}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i q}{r}} & 0 \\ 0 & e^{\frac{2\pi i p}{r}} \end{pmatrix}.$$

Remark 3.0.12. Let (x, y) be the coordinates of $\mathbb{C}^2$. A singularity of type $\frac{1}{r}(0, q)$ is a smooth point (and it is always equivalent to some $\frac{1}{r'}(0, 1)$). In fact, the ring of the invariants of $\mathbb{C}^2/H$ is $\mathbb{C}[\mathbb{C}^2/H] = \mathbb{C}[x, y^r] \cong \mathbb{C}[X, Y]$, and so $\mathbb{C}^2/H \cong \mathbb{C}^2$ (see the Proposition 3.0.14).

One can ask himself if there could be others $H \leq \mathrm{GL}(2,\mathbb{C})$ giving a smooth quotient $\mathbb{C}^2/H \cong \mathbb{C}^2$.
A famous theorem by Chevalley and Sheppard-Todd says that given a finite group $H \leq \mathrm{GL}(k,\mathbb{C})$, then the quotient $\mathbb{C}^k/H$ is non-singular if and only if H is generated by *quasi-reflections* (matrices that diagonalise to $\mathrm{diag}(1,\ldots,1,e^{\frac{2\pi i}{p}})$, $p \in \mathbb{N}$).
Thus, in the case $k=2$, the only cyclic quotient singularities giving a smooth quotient are equivalent to $\frac{1}{r}(0,1)$.

Lemma 3.0.13. *Each cyclic quotient singularity of type $\frac{1}{r}(p,q)$, with $p,q \neq 0$, is equivalent to a cyclic quotient singularity of type $\frac{1}{n}(1,a)$, with $1 \leq a < n$, and $\gcd(a,n)=1$.*

Proof. **Claim:** take $d := \gcd(p,r)$. Therefore $p = p'd$, $r = r'd$, and $\gcd(p',r')=1$. Then $\frac{1}{r}(p,q)$ is equivalent to $\frac{1}{r'}(p',q)$. To prove this, call for brevity $g := \begin{pmatrix} e^{\frac{2\pi i p}{r}} & 0 \\ 0 & e^{\frac{2\pi i q}{r}} \end{pmatrix}$. We observe that

$$\mathbb{C}^2/\langle g\rangle \cong \left(\mathbb{C}^2/\langle g^{r'}\rangle\right) / \left(\langle g\rangle/\langle g^{r'}\rangle\right). \tag{3.1}$$

However, the action of $\langle g^{r'}\rangle$ on $\mathbb{C}^2$ gives a cyclic singularity of type $\frac{1}{r}(pr',qr') = \frac{1}{d}(0,q)$, which is smooth, by the Remark 3.0.12. Thus $\mathbb{C}^2/\langle g^{r'}\rangle \cong \mathbb{C}^2$, via the isomorphism $\phi\colon \langle g^{r'}\rangle(x,y) \mapsto (x,y^d)$. Let us call the coordinates of the new $\mathbb{C}^2$ by (X,Y).
We observe also that $\langle g\rangle/\langle g^{r'}\rangle \cong \langle g^d\rangle$, hence the isomorphism (3.1) becomes

$$\mathbb{C}^2/\langle g\rangle \cong \mathbb{C}^2/\langle g^d\rangle,$$

and the action of $\langle g^d\rangle$ on the new $\mathbb{C}^2$ is then

$$\begin{aligned} g^d \cdot (X,Y) = \phi(g^d \cdot (\phi^{-1}(X,Y))) &= \phi(g^d \cdot (\langle g^{r'}\rangle(x,y))) \\ &= \phi(\langle g^{r'}\rangle(g\cdot(x,y))) \\ &= \phi(\langle g^{r'}\rangle(e^{\frac{2\pi i p}{r}}x, e^{\frac{2\pi i q}{r}}y)) \\ &= (e^{\frac{2\pi i p}{r}}x, e^{\frac{2\pi i q d}{r}}y^d) \\ &= \begin{pmatrix} e^{\frac{2\pi i p'}{r'}} & 0 \\ 0 & e^{\frac{2\pi i q}{r'}} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \end{aligned}$$

Thus we can start the proof of the lemma by supposing $\frac{1}{r}(p,q)$ has $\gcd(p,r)=1$. However, this implies g^t generates $\langle g\rangle$, where $t = p^{-1}$ is the inverse of p modulo r, and so $\frac{1}{r}(p,q)$ is equivalent to $\frac{1}{r}(1,qt)$. By the Remark 3.0.11, $\frac{1}{r}(1,qt)$ is equivalent to $\frac{1}{r}(qt,1)$, for which the claim above applies. Therefore, $\frac{1}{r}(qt,1)$ is equivalent to some $\frac{1}{n}(a,1)$, which is equivalent to $\frac{1}{n}(1,a)$, with $1 \leq a < n$, and $\gcd(a,n)=1$. $\square$

It is useful the following standard result

Proposition 3.0.14. [Mum08, Sec. 7] [Rei03, Exs 4.3-4.4 for details] *Let G be a finite group acting on $\mathbb{C}^k$ by algebraic automorphisms. Write by $x_1, \ldots, x_k$ the coordinates of $\mathbb{C}^k$. Then the quotient $X := \mathbb{C}^k/G$ is an affine algebraic variety whose points correspond one-to-one with the orbits of the group action, and such that the ring $\mathbb{C}[X]$ of the functions defined on X are precisely the invariant polynomials on $\mathbb{C}^k$, namely $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots x_k]^G$. The ring $\mathbb{C}[X]$ is finitely generated. Moreover, if $f_1, \ldots f_N$ are invariant polynomials generating $\mathbb{C}[X]$, then X is defined as follows: take the ring homomorphism $\phi\colon \mathbb{C}[u_1, \ldots, u_N] \to \mathbb{C}[X]^G, \quad u_i \mapsto f_i$. Therefore the kernel of ϕ is the ideal of the relations between the generators f_i. We have*

$$X = V(\ker \phi) = \left\{ u \in \mathbb{C}^N \colon F(u) = 0, F \in \ker \phi \right\},$$

and the quotient map is

$$\mathbb{C}^k \to X \subseteq \mathbb{C}^N, \qquad \bar{x} := (x_1, \ldots, x_k) \mapsto (f_1(\bar{x}), \ldots, f_N(\bar{x})).$$

Example 3.0.15. The Remark 3.0.12 is the baby trivial example. The second non-trivial one is the singularity $\frac{1}{2}(1,1)$. Here the ring of the invariants of $X := \mathbb{C}^2/H$ is $\mathbb{C}[x^2, y^2, xy]$, therefore $X := V(u_3^2 - u_1u_2) \subseteq \mathbb{C}^3$ is the cone over a quadric, and the vertex of the cone is the singularity.

We are going to compute the ring of the invariants of a cyclic quotient singularity $X := \mathbb{C}^2/H$ of type $\frac{1}{n}(1,a)$, with $\gcd(a,n) = 1$. Let (x,y) be coordinates on $\mathbb{C}^2$. Since the action on $\mathbb{C}^2$ is linear and diagonal on the components x and y, then the ring of the invariants has to be generated by some monomials $x^\alpha y^\beta$, so that $\mathbb{C}[X]$ is identified just by the monoid of the positive quadrant of a lattice of $\mathbb{R}^2$, the lattice M of points (α, β) satisfying $\alpha + a\beta \equiv 0 \mod n$. Equivalently, $\beta \equiv (n-a')\alpha \mod n$, where $a' = a^{-1}$ is the inverse of a modulo n. Note that any (α,β) has to be an integer combination of $(0,0) = 1$, $(n,0) = x^n$, $(0,n) = y^n$, and $(1, n-a') = xy^{n-a'}$, which are trivially invariant polynomials. in other words, we have

$$\mathbb{C}[X] = \mathbb{C}[x^\alpha y^\beta : (\alpha,\beta) \in M \cap \mathbb{R}^2_{\geq 0}],$$

where M is the lattice

$$M := \{(\alpha,\beta)\colon \alpha + a\beta \equiv 0 \mod n\} = \mathbb{Z}(n,0) \oplus \mathbb{Z}(0,n) + \mathbb{Z}(1, n-a'). \tag{3.2}$$

We would like to find a (minimal) set of generators of $\mathbb{C}[X]$. Let us see how.

Definition 3.0.16. Let $1 \leq a < n$ be coprime integers. Then the *Hirzebruch-Jung continued fraction* of n/a is the expression

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ldots}} = [b_1, \ldots, b_l]. \tag{3.3}$$

For example,

$$\frac{7}{3} = [3,2,2] = 3 - \frac{1}{2 - \frac{1}{2}}.$$

Proposition 3.0.17. [Rei03, Prop. 2.2] *Let $1 \leq a < n$ be coprime integers and consider the lattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1, a)$. Then L contains the lattice $\mathbb{Z}^2$ as a sublattice of index n, and its other cosets are represented by the $n-1$ lattice points $\frac{1}{n}(j, [aj]_n)$ contained in the unit square of $\mathbb{R}^2$ (see the Figure 3.1). Define the Newton polygon as the convex hull Newton(L) in $\mathbb{R}^2$ of all non-zero lattice points in the positive quadrant*

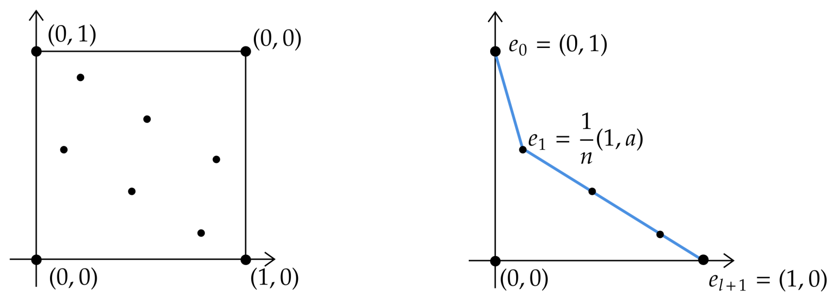


Figure 3.1: The lattice L and its Newton polygon

Write

$$e_0 = (0,1), \quad e_1 = \frac{1}{n}(1,a), \quad e_2, \cdots, e_l, \quad e_{l+1} = (1,0)$$

for the lattice points on the boundary of Newton(L). Then

(I) Any two consecutive lattice points e_i, e_{i+1} for $i = 0, \ldots, l$ form an oriented basis of L.

(II) Any three consecutive lattice points e_{i-1}, e_i, e_{i+1} for $i = 1, \ldots, l$ satisfy the relation

$$e_{i+1} + e_{i-1} = b_i e_i, \quad b_i \geq 2. \tag{3.4}$$

(III) The integers $b_1, \ldots, b_l$ in (II) are the entries of the continued fraction:

$$\frac{n}{a} = [b_1, \ldots b_l].$$

The relation in (II) can be viewed as a change of coordinates from the basis e_{i-1}, e_i to the next basis e_i, e_{i+1} expressed by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & b_i \end{pmatrix}$, that is,

$$e_i = 0 \cdot e_{i-1} + 1 \cdot e_i, \qquad e_{i+1} = -1 \cdot e_{i-1} + b_i \cdot e_i.$$

(IV) *The boundary points* $e_0, \ldots, e_{l+1}$ *are a minimal set of generators of the monoid* $L \cap \mathbb{R}^2_{\geq 0}$.

Proof. (I) Let us consider the parallelogram $\Pi := \langle 0, e_i, e_{i+1}, e_i + e_{i+1}\rangle$. By construction, the lower triangle $\Delta^- := \langle 0, e_i, e_{i+1}\rangle$ of Π does not contain lattice points other than its vertices. The same holds for the upper triangle $\Delta^+ := \langle e_i, e_{i+1}, e_1 + e_{i+1}\rangle$ since any vector v belonging on Δ^+ is such that $e_i + e_{i+1} - v \in \Delta^-$, and so the only lattice points of Δ^+ are its vertices. This means Π is a fundamental domain for L, in the sense that any point of $\mathbb{R}^2$ is obtained from a vector of Π by a translation of a vector of L. In fact, e_i, e_{i+1} is a basis of (the vector space) $\mathbb{R}^2$, and so any vector v can be written as $v = \alpha e_i + \beta e_{i+1}$. Then $v = v' + v''$, where

$$v' = (\alpha - \lfloor \alpha \rfloor)\, e_i + (\beta - \lfloor \beta \rfloor)\, e_{i+1} \in \Pi, \quad \text{and} \quad v'' = \lfloor \alpha \rfloor e_i + \lfloor \beta \rfloor e_{i+1} \in L$$

Thus, if $v \in L$, then $v' = v - v'' \in \Pi \cap L$, and so it has to being one of the vertices of Π, which proves e_i, e_{i+1} is a $\mathbb{Z}$-basis of L.

Remark 3.0.18. For plane lattices, the convexity condition is very strong, and implies that we get a $\mathbb{Z}$-basis of a lattice. This part fails in dimension ≥ 3.

(II) Since e_{i-1}, e_i is a $\mathbb{Z}$-basis, then $e_{i+1} = \alpha e_{i-1} + \beta e_i$, with $\alpha, \beta \in \mathbb{Z}$. However, e_i, e_{i+1} is a $\mathbb{Z}-$basis too, forcing α to be 1 or -1. However, from the figure, e_i is a positive combination of e_{i-1} and e_{i+1}, so that $\alpha = -1$ and $e_{i+1} + e_{i-1} = \beta e_i$, with $\beta > 0$. If $\beta = 1$, then e_i would be inside the Newton polygon of L, which contradicts the fact e_i is on the boundary of Newton(L). Therefore $\beta \geq 2$.

(III) We have proved that $e_2 + e_0 = b_1 e_1$, with $b_1 \geq 2$. Thus $e_2 = \frac{1}{n}(b_1, b_1 a - n)$ is in the unit square, which means $b_1 a - n \geq 0$. Actually we can not have $b_1 a - n \geq a$ otherwise e_2 would be above e_1 and this is not possible since e_2 belongs to the boundary of the Newton polygon of L. Then $b_1 \geq \lceil \frac{n}{a} \rceil$. By contradiction, if $b_1 > \lceil \frac{n}{a} \rceil$, then the point $v := e_0 + \big(b_1 - \lceil \frac{n}{a} \rceil\big)\, e_1 \in L$ would lie on the positive quadrant, and so e_2 would be write as $e_2 = v + \lceil \frac{n}{a} \rceil$, which contradicts once more e_2 is on the boundary. This proves $b_1 = \lceil \frac{n}{a} \rceil$.

The statement for $b_2, \ldots, b_l$ works recursively: write

$$L = (\mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_{l+1}) + \mathbb{Z} \cdot e_2.$$

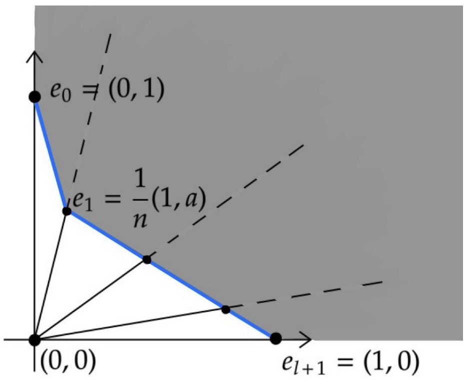
Change the standard basis to $\{e_{l+1}, e_1\}$ to get $L \cong \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n_1}(1, a_1)$, where n_1, a_1 are obtained writing e_2 in such basis:

$$e_2 = (b_1 - \frac{n}{a}) \cdot e_1 + \frac{1}{a} \cdot e_{l+1} = (\frac{1}{a}, b_1 - \frac{n}{a}) = \frac{1}{a}(1, b_1 a - n).$$

so that $n_1 := a$, and $a_1 := b_1 a - n$. Observe that $\gcd(a_1, n_1) = 1$. We can use what we have said above to get $e'_0 + e'_2 = \lceil \frac{n_1}{a_1} \rceil e'_1$, in the new basis. Here, observe that b_2 is exactly $\lceil \frac{n_1}{a_1} \rceil$, while e'_0 corresponds to e_1, e'_1 corresponds to e_2, and e'_2 is e_3, since the smaller cone of Newton(L) spanned by e_1, e_{l+1} is the same like before. We have got $e_1 + e_3 = b_2 e_2$. The result follows by recursion.

(IV) The points $e_0, \ldots, e_{l+1}$ generate $L \cap \mathbb{R}^2_{\geq 0}$ since any non-zero point of $L \cap \mathbb{R}^2_{\geq 0}$ has to fall in one of the sectors identified by a couple e_i, e_{i+1} (see the figure below). In particular, it has to live in the upper-sector $\{\alpha e_i + \beta e_{i+1} \colon \alpha \geq t, \beta \geq 1 - t, t \in [0, 1]\}$ of e_i, e_{i+1}, otherwise it would be a new lattice point of the boundary of Newton(L), which is not possible. This proves that it can be written as a non-negative integer combination of e_i, e_{i+1} (and so of $e_0, \ldots, e_{l+1}$).

Finally, they are a minimal set of generators because if $e_j = \sum_{i \neq j} \alpha_i e_i$, for some α_i non-negative integers, then e_j would not be on the boundary of Newton(L).



□

Corollary 3.0.19. *Let $X := \mathbb{C}^2/H$ be a cyclic quotient singularity of type $\frac{1}{n}(1, a)$. Let M be the lattice (3.2) of the invariant monomials of X. Then the Proposition 3.0.17 applies to $\frac{1}{n}M$. Let*

$$e_0 = (0, 1), \qquad e_1 = \frac{1}{n}(1, n - a'), \qquad \ldots, \qquad e_{k+1} = (1, 0)$$

be the lattice points of the boundary of the Newton polygon of $\frac{1}{n}M$.

Then a minimal set of generators of the ring $\mathbb{C}[X]$ is done by the corresponding invariant monomials of $ne_0, ne_1, \ldots, ne_{k+1} \in M$, that are

$$u_0 = y^n, \quad u_1 = xy^{n-a'}, \quad u_2, \ldots u_k, \quad u_{k+1} = x^n.$$

These monomials satisfy

$$u_{i-1}u_{i+1} = u_i^{a_i}, \qquad \textit{for} \quad i = 1, \ldots, k, \tag{3.5}$$

where a_i *are the entries of the continued fraction* $\frac{n}{n-a'} = [a_1, \ldots, a_k]$. *Note that the relations* (3.5) *are enough to specify all the* u_j *as rational expression of* u_0, u_1, *or any two consecutive monomials* u_i, u_{i+1}. *Thus the morphism* $\mathbb{C}^2 \to X \subset \mathbb{C}^{k+2}$, $(x, y) \mapsto (u_0, \ldots, u_{k+1})$, *is the quotient map. However, the above relations* (3.5) *are not enough to determine the image* X*; for a full set of generators of the ideal* $I(X)$ *we also need relations for* $u_i u_j$ *with* $|i - j| \geq 2$.

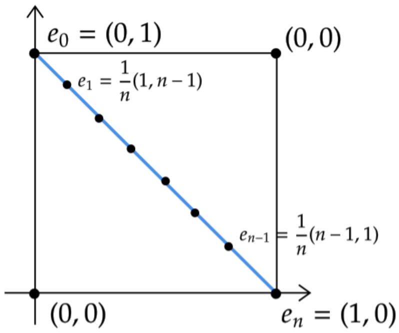
Proof. The point (IV) of the Proposition 3.0.17 permits to say $ne_0, \ldots, ne_{k+1}$ form a minimal set of generators of the monoid $M \cap \mathbb{R}^2_{\geq 0}$, so that their corresponding monomials form a minimal set of generators of $\mathbb{C}[X]$. Finally, the relations (3.5) comes out simply by transposing the relations of (II) between ne_{i-1}, ne_i, and ne_{i+1} to their corresponding invariant monomials. □

Example 3.0.20. Let (x, y) be the coordinates on $\mathbb{C}^2$. The computation of the ring of invariants comes out in a nice uniform way if we analyse the extreme cases $\frac{1}{n}(1, 1)$ and $\frac{1}{n}(1, n-1)$. We are going to use the Corollary 3.0.19.

1. $\frac{1}{n}(1, 1)$. Here $\frac{1}{n}M = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1, n-1)$, and $\frac{n}{n-1} = [2, \ldots, 2]$, whose length is $n - 1$. The points e_i of the boundary of the Newton polygon of $\frac{1}{n}M$ can be recursively computed trough the relations (3.4):

$$\begin{aligned} e_0 &= (0, 1), \qquad e_1 = \frac{1}{n}(1, n-1), \\ e_2 &= -e_0 + 2e_1 = \frac{1}{n}(2, n-2), \\ e_3 &= -e_1 + 2e_2 = \frac{1}{n}(3, n-3), \\ &\vdots \\ e_{n-1} &= -e_{n-3} + 2e_{n-2} = \frac{1}{n}(n-1, 1), \\ e_n &= (1, 0). \end{aligned}$$

Thus the Newton polygon of $\frac{1}{n}M$ and its boundary looks like the following figure



Then a minimal set of generators of the ring of invariant polynomials of $X = \mathbb{C}^2/H$ is given by

$$u_0 = y^n, \quad u_1 = xy^{n-1}, \quad \dots, \quad u_{n-1} = x^{n-1}y, \quad u_n = x^n.$$

The quotient map is

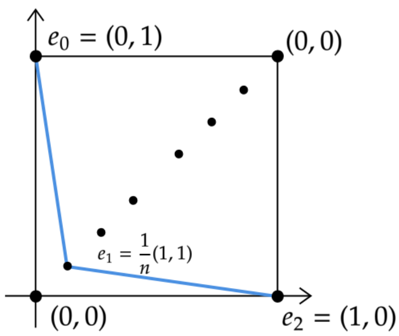
$$\mathbb{C}^2 \to X \subset \mathbb{C}^{n+1}, \quad (x, y) \mapsto (y^n, xy^{n-1}, \dots, x^{n-1}y, x^n),$$

and $X = \left\{ rk \begin{pmatrix} u_0 & u_1 & \dots & u_{n-1} \\ u_1 & u_2 & \dots & u_n \end{pmatrix} \leq 1 \right\} \subset \mathbb{C}^{n+1}$ is the cone over the rational normal curve of degree n in $\mathbb{P}^n$. Here, the singularity of type $\frac{1}{n}(1,1)$ is the vertex of the cone.

2. $\frac{1}{n}(1, n-1)$. We have $\frac{1}{n}M = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1,1)$, and $\frac{n}{1} = [n]$. The points of the boundary of the Newton polygon of $\frac{1}{n}M$ are then

$$e_0 = (0,1), \quad e_1 = \frac{1}{n}(1,1), \quad e_2 = (1,0)$$

The Newton polygon and its boundary looks like



A minimal set of generators of the ring of the invariants of $X = \mathbb{C}^2/H$ is then

$$u_0 = y^n, \qquad u_1 = xy, \qquad \text{and} \qquad u_2 = x^n.$$

The quotient map is

$$\mathbb{C}^2 \to X \subset \mathbb{C}^3, \qquad (x, y) \mapsto (y^n, xy, x^n),$$

and $X = \{Y^n - XZ = 0\} \subset \mathbb{C}^3$. Thus the singular point of X is a Rational Double Point of type A_{n-1} (see the Theorem 3.1.6 and the Remark 3.1.8).

The next step is to discuss the minimal resolution of a cyclic quotient singularity, see [BHPVdV04, Section III.5] or [Rei03] for major details.

Notation: Given x, y be two complex variables, and $v := \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \in \mathbb{Z}^2$, then $(xy)^v$ denotes the monomial $x^\alpha y^\beta$. We observe $\mathbb{Z}^2 \to \mathbb{C}[x, y], v \mapsto (xy)^v$ is a morphism of $\mathbb{Z}$-moduli.

Theorem 3.0.21. [Rei03, Thm. 3.2]*(Resolution of a cyclic quotient singularity) As usual, consider a singularity $X = \mathbb{C}^2/H$ of type $\frac{1}{n}(1, a)$. Let $a' := a^{-1}$ be the inverse of a modulo n. Write L for the overlattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{n}(1, a)$ of $\mathbb{Z}^2$, and by*

$$M = \{(\alpha, \beta) \colon \alpha + a\beta \equiv 0 \mod n\} \subseteq \mathbb{Z}^2$$

the dual lattice of L of invariant monomials defined at (3.2).

Let $f_0, \ldots, f_{k+1}$ be a set of minimal generators of the monoid $M \cap \mathbb{R}^2_{\geq 0}$ (obtained for instance via the Corollary 3.0.19; this means k is the length of the continued fraction of $\frac{n}{n-a'}$), and $\mathbb{C}^2 \to X \subset \mathbb{C}^{k+2}, (x, y) \mapsto ((xy)^{f_0}, \ldots, (xy)^{f_{k+1}})$ be the quotient map.
Write $\frac{n}{a} = [b_1, \ldots, b_l]$, and let $e_0, \ldots e_{l+1}$ be the lattice points of the boundary of the Newton polygon of L defined in the Proposition 3.0.17. For each $i = 0, \ldots, l$ let ξ_i, η_i be monomials forming the dual basis of M to e_i, e_{i+1}; that is, such that

$$e_i(\xi_i) = 1, e_i(\eta_i) = 0, \qquad e_{i+1}(\xi_i) = 0, e_{i+1}(\eta_i) = 1.$$

Denote by A_i the 2×2 matrix whose columns are the vectors e_i, e_{i+1}. Then X has a resolution of singularities $b \colon Y \to X \subset \mathbb{C}^{k+2}$ constructed as follows:

$$Y = Y_0 \cup_{\phi_0} Y_1 \cup \cdots \cup_{\phi_{l-1}} Y_l, \tag{3.6}$$

where each $Y_i \cong \mathbb{C}^2$ with coordinates ξ_i, η_i. For any $i = 0, \ldots, l-1$, the glueing $Y_i \cup Y_{i+1}$ consists of

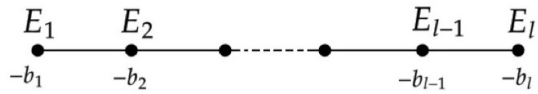
$$\phi_i \colon Y_i \setminus \{\xi_i = 0\} \to Y_{i+1} \setminus \{\eta_{i+1} = 0\}, \qquad (\xi_i, \eta_i) \mapsto \left(\xi_i^{b_{i+1}}\eta_i, \frac{1}{\xi_i}\right).$$

The resolution map is

$$b\colon Y \to X, \qquad (\xi_i,\eta_i) \mapsto \left((\xi_i\eta_i)^{A_i^t\cdot f_0},\dots,(\xi_i\eta_i)^{A_i^t\cdot f_{k+1}}\right), \quad i=0,\dots,l.$$

The exceptional curves of the resolution (namely, those which are contracted to the singular point of X) are exactly l, and have local equation $E_{i+1} = \{\eta_i = 0\} \cup_{\phi_i} \{\xi_{i+1} = 0\} \cong \mathbb{P}^1$, $i = 0,\dots,l-1$. Here $E_i^2 = -b_i$, $E_i \cdot E_{i+1} = 1$ for $i = 1,\dots,l-1$, and the intersection point is the origin $(\xi_i,\eta_i) = (0,0)$ of the piece Y_i. Instead, $E_i \cdot E_j = 0$ if $|i-j| \geq 2$.

The exceptional divisor $E = \bigcup_{i=1}^{l} E_i$ is called Hirzebruch-Jung string, and its dual configuration is



Proof. When you fix the dual basis e_i^*, e_{i+1}^* of M to e_i, e_{i+1} then you are constructing a new $Y_i = \mathbb{C}^2$ with coordinates ξ_i, η_i, and a map $Y_i = \mathbb{C}^2 \to X \subseteq \mathbb{C}^{k+2}$ which is an isomorphism outside the singular locus of X. This is defined as follows: any point of X can be written as $((xy)^{f_0},\dots,(xy)^{f_{k+1}})$ for suitable x and y. The vectors $f_0,\dots f_{k+1}$ are written in the dual standard basis of M, so let us write them in the new basis e_i^*, e_{i+1}^*: $A_i^t f_j = M_{\{e_i^*,e_{i+1}^*\},std}(\mathrm{Id}_{\mathbb{R}^2}) f_j$ is the vector of the coordinates of f_j in that new basis. Then $(xy)^{f_j} = \left((xy)^{e_i^*}(xy)^{e_{i+1}^*}\right)^{A_i^t f_j}$, and this suggests to define

$$\xi_i := (xy)^{e_i^*} = (xy)^{(A_i^{-1})^t\cdot\begin{pmatrix}1\\0\end{pmatrix}} \qquad \text{and} \qquad \eta_i := (xy)^{e_{i+1}^*} = (xy)^{(A_i^{-1})^t\cdot\begin{pmatrix}0\\1\end{pmatrix}}.$$

The map we are looking for is then $b_i\colon Y_i = \mathbb{C}^2 \to X \subset \mathbb{C}^{k+2}, (\xi_i,\eta_i) \mapsto ((\xi_i\eta_i)^{A_i^t f_0},\dots,(\xi_i\eta_i)^{A_i^t f_{k+1}})$. Actually, a point of the image of b_i falls on X because the relations of X translates as linear relations between the vectors $f_0,\dots,f_{k+1}$, and the same relations are satisfied by the vectors $A_i^t f_0,\dots,A_i^t f_{k+1}$. Hence the map is well defined. The map is algebraic since any $A_i^t f_j$ has integer non-negative entries, which are exactly $f_j(e_i), f_j(e_{i+1}) \in \mathbb{Z} \cap \mathbb{R}_{\geq 0}$.

Let us write the vectors e_i^*, e_{i+1}^* in function of the basis $\{f_0, f_1\}$. The coordinates of these vectors in that basis are the columns B_i of the change base matrix $B := M_{\{f_0,f_1\},\{e_i^*,e_{i+1}^*\}}(\mathrm{Id}_{\mathbb{R}^2})$. Then the inverse map $X \setminus \{0\} \to Y \setminus b_i^{-1}(0)$ is defined by sending $(u_0,\dots,u_{k+1}) \mapsto ((u_0u_1)^{B_1},(u_0u_1)^{B_2})$. Note that the inverse map of b_i is just rational (B is unimodular), and its definition does not to depend from the chosen basis $\{f_0, f_1\}$; if you would define it through another basis $\{f_j, f_{j+1}\}$, then the map would be just the same, in virtue of the linear relations of X between the vectors $f_0, \dots, f_{k+1}$.

We glue together the pieces $Y_0, \dots, Y_l$ by taking care that if $p \sim q$, with $p \in Y_i$ and $q \in Y_{i+1}$, then $b_i(p) = b_{i+1}(q)$. Let $i = 0, \dots, l-1$. The glueing ϕ_i is defined through the change base matrix $M_{\{e_i^*, e_{i+1}^*\}, \{e_{i+1}^*, e_{i+2}^*\}}(\mathrm{Id}_{\mathbb{R}^2})$, which is the transpose of the inverse matrix $\begin{pmatrix} 0 & -1 \\ 1 & b_{i+1} \end{pmatrix}$ defined at the point (III) of the Proposition 3.0.17:

$$(\xi_i, \eta_i) \to ((\xi_i \eta_i)^{\begin{pmatrix} b_{i+1} \\ 1 \end{pmatrix}}, (\xi_i \eta_i)^{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}) = (\xi_i^{b_{i+1}} \eta_i, \frac{1}{\xi_i}).$$

The curves that are contracted to the singular point of X are exactly l, say $E_1, \dots, E_l$. Fix $i = 1, \dots, l-1$. By a direct computation, E_i, E_{i+1} intersect transversally each other at the point $(\xi_i, \eta_i) = (0, 0)$ of the piece Y_i. Instead, their self-intersection $E_i^2 = -b_i$, $i = 1, \dots, l$, can be computed via elementary intersection theory. □

Let us give an example to understand better which is the approach.

Example 3.0.22. 1. $\frac{1}{n}(1,1)$. A minimal set of generators of $M \cap \mathbb{R}^2_{\geq 0}$ is computed at the point 1. of the Example 3.0.20:

$$f_0 = (0, n), \qquad f_1 = (1, n-1), \dots, \qquad f_n = (n, 0),$$

and the quotient map $\mathbb{C}^2 \to X = \left\{ rk \begin{pmatrix} u_0 & u_1 & \dots & u_{n-1} \\ u_1 & u_2 & \dots & u_n \end{pmatrix} \leq 1 \right\} \subset \mathbb{C}^{n+1}$ is given by $(x, y) \mapsto (y^n, xy^{n-1}, \dots, x^n)$. Instead, the points of the boundary of the Newton polygon of the lattice L are

$$e_0 = (0, 1), \qquad e_1 = \frac{1}{n}(1, 1), \qquad e_2 = (1, 0).$$

Thus we have got only two copies Y_0, Y_1 of $\mathbb{C}^2$, with coordinates ξ_0, η_0 and ξ_1, η_1 respectively. Consider the first copy Y_0. The monomials ξ_0, η_0 form the dual basis of M to e_0, e_1, and in function of x and y are

$$\xi_0 = (xy)^{(A_0^{-1})^t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = x^{-1} y, \qquad \text{and} \qquad \eta_0 = (xy)^{(A_0^{-1})^t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = x^n,$$

since $A_0 = M_{std, \{e_0, e_1\}}(\mathrm{Id}_{\mathbb{R}^2})$, and so the $(A_0^{-1})^t = M_{std, \{e_0^*, e_1^*\}}(\mathrm{Id}_{\mathbb{R}^2})$. To compute the resolution $b\colon Y \to X$ on Y_0 we need to write any invariant monomial $(xy)^{f_i}$ in function of ξ_0, η_0. This can be done easily writing any f_i from the standard basis to the new basis e_0^*, e_1^*. In other words, $(xy)^{f_i} = (\xi_0\eta_0)^{A_0^t \cdot f_i}$, and the resolution b is

$$(\xi_0, \eta_0) \mapsto \left((\xi_0\eta_0)^{A_0^t \cdot f_0}, \dots, (\xi_0\eta_0)^{A_0^t \cdot f_n}\right) = \left(\xi_0^n\eta_0, \xi_0^{n-1}\eta_0, \xi_0^{n-2}\eta_0, \dots, \eta_0\right).$$

Consider now the other copy Y_1, whose variables ξ_1, η_1 form the dual basis of M to e_1, e_2. They can be written in function of x and y as

$$\xi_1 = (xy)^{(A_1^{-1})^t \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = y^n, \qquad \text{and} \qquad \eta_1 = (xy)^{(A_1^{-1})^t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = xy^{-1},$$

where $A_1 = M_{std,\{e_1,e_2\}}(\mathrm{Id}_{\mathbb{R}^2})$. The resolution b on Y_1 is then

$$(\xi_1, \eta_1) \mapsto \left((\xi_1\eta_1)^{A_1^t \cdot f_0}, \dots, (\xi_1\eta_1)^{A_1^t \cdot f_n}\right) = \left(\xi_1, \xi_1\eta_1, \xi_1\eta_1^2, \dots, \xi_1\eta_1^n\right).$$

To compute how the copies Y_0 and Y_1 glue to each other observe that a couple of identified points has to be sent to the same point via b. This suggests to use the change base matrix $M := M_{\{e_0^*,e_1^*\},\{e_1^*,e_2^*\}}(\mathrm{Id}_{\mathbb{R}^2})$, which is the transpose of the inverse of the change base matrix from the base e_0, e_1 to the next basis e_1, e_2 defined at (III) of the Proposition 3.0.17. Therefore, the map $\phi_0 \colon Y_0 \setminus \{\xi_0 = 0\} \to Y_1 \setminus \{\eta_1 = 0\}$ is defined as

$$(\xi_0, \eta_0) \mapsto \left((\xi_0\eta_0)^{M \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}, (\xi_0\eta_0)^{M \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}\right) = \left(\xi_0^n\eta_0, \frac{1}{\xi_0}\right).$$

To summarize, $Y = Y_0 \cup_{\phi_0} Y_1$ where ϕ_0 is defined as above, and the resolution map of X is

$$\begin{aligned} b\colon Y \to X &= \left\{ rk \begin{pmatrix} u_0 & u_1 & \dots & u_{n-1} \\ u_1 & u_2 & \dots & u_n \end{pmatrix} \leq 1 \right\} \subset \mathbb{C}^{n+1} \\ (\xi_0, \eta_0) &\mapsto \left(\xi_0^n\eta_0, \xi_0^{n-1}\eta_0, \xi_0^{n-2}\eta_0, \dots, \eta_0\right) \\ (\xi_1, \eta_1) &\mapsto \left(\xi_1, \xi_1\eta_1, \xi_1\eta_1^2, \dots, \xi_1\eta_1^n\right). \end{aligned}$$

The exceptional locus of Y consists of only one rational curve $E_1 = \{\eta_0 = 0\} \cup_{\phi_0} \{\xi_1 = 0\}$ of self-intersection $-n$.

2. $\frac{1}{n}(1, n-1)$. A set of minimal generators of $M \cap \mathbb{R}^2_{\geq 0}$ is

$$f_0 = (0, n), \qquad f_1 = (1, 1) \qquad \text{and} \qquad f_2 = (n, 0),$$

and the quotient map $\mathbb{C}^2 \to X = \{Y^n - XZ = 0\} \subseteq \mathbb{C}^3$ is given by $(x, y) \mapsto (x^n, xy, y^n)$.

Instead, the points of the boundary of the Newton polygon of the lattice L are

$$e_0 = (0,1), \quad e_1 = \frac{1}{n}(1, n-1), \quad e_2 = \frac{1}{n}(2, n-2), \quad \dots, \quad e_n = (1,0).$$

Let ξ_i, η_i be monomials forming the dual basis of M to e_i, e_{i+1}, for $i = 0, \dots, n-1$. They can be written in function of x and y as

$$\xi_i = (xy)^{(A_i^{-1})^t \cdot \begin{pmatrix}1\\0\end{pmatrix}} = \frac{1}{x^{n-(i+1)}} y^{i+1} \quad \text{and} \quad \eta_i = (xy)^{(A_i^{-1})^t \cdot \begin{pmatrix}0\\1\end{pmatrix}} = x^{n-i}\frac{1}{y^i},$$

where $A_i = M_{std,\{e_i,e_{i+1}\}}(\mathrm{Id}_{\mathbb{R}^2}) = \frac{1}{n}\begin{pmatrix} i & i+1 \\ n-i & n-(i+1)\end{pmatrix}$. The resolution $b\colon Y \to X \subset \mathbb{C}^3$ of X is constructed as follows

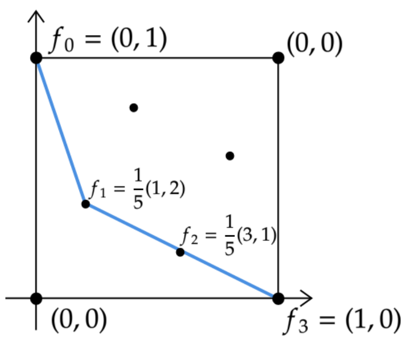
$$Y = Y_0 \cup_{\phi_0} \cdots \cup_{\phi_{n-2}} Y_{n-1},$$

where each $Y_i \cong \mathbb{C}^2$ with coordinates ξ_i, η_i. For any $i = 0, \dots, n-2$, the glueing $Y_i \cup Y_{i+1}$ is given by $\phi_i \colon Y_i \setminus \{\xi_i = 0\} \to Y_{i+1} \setminus \{\eta_{i+1} = 0\}$, $\quad (\xi_i, \eta_i) \mapsto (\xi_i^2\eta_i, \frac{1}{\xi_i})$, and the resolution map is

$$b\colon Y \to X = \{Y^n - XZ = 0\} \subset \mathbb{C}^3, \quad (\xi_i, \eta_i) \mapsto \left(\xi_i^{n-i}\eta_i^{n-(i+1)}, \xi_i\eta_i, \xi_i^i\eta_i^{i+1}\right).$$

The exceptional locus of Y consists of $n-1$ rational curves $E_{i+1} = \{\eta_i = 0\} \cup_{\phi_i} \{\xi_{i+1} = 0\}$, any of them with self intersection -2, for $i = 0, \dots n-1$. Here $E_i \cdot E_{i+1} = 1$, whilst $E_i \cdot E_j = 0$ for $|i-j| \geq 2$.

3. $\frac{1}{5}(1,2)$. We determine a minimal set of generators of $M \cap \mathbb{R}^2_{\geq 0}$ through the Corollary 3.0.19: here $\frac{5}{5-3} = [3,2]$, and the Newton polygon of $\frac{1}{5}M$ and its boundary are



Therefore the quotient map is $\mathbb{C}^2 \to X \subset \mathbb{C}^4$, $(x,y) \mapsto (y^5, xy^2, x^3y, x^5)$. Instead, the Newton boundary of L is

$$e_0 = (0,1), \qquad e_1 = \frac{1}{5}(1,2), \qquad e_2 = \frac{1}{5}(3,1), \qquad \text{and} \qquad e_3 = (1,0).$$

Taking dual basis gives

$$\xi_0 = x^{-2}y, \eta_0 = x^5, \qquad \xi_1 = x^{-1}y^3, \eta_1 = x^2y^{-1}, \qquad ,\xi_2 = y^5, \eta_2 = xy^{-3}.$$

Thus $Y = Y_0 \cup_{\phi_0} Y_1 \cup_{\phi_1} Y_2$, with 3 copies of $\mathbb{C}^2$ glued by

$$\phi_0 \colon \xi_1 = \xi_0^3\eta_0, \eta_1 = \frac{1}{\xi_0}, \qquad \text{and} \qquad \phi_1 \colon \xi_2 = \xi_1^2\eta_1, \eta_2 = \frac{1}{\xi_1}.$$

The resolution map is

$$\begin{aligned} b \colon Y &\to X \subseteq \mathbb{C}^4, \\ (\xi_0, \eta_0) &\mapsto (\xi_0^5\eta_0^2, \xi_0^2\eta_0, \xi_0\eta_0, \eta_0), \\ (\xi_1, \eta_1) &\mapsto (\xi_1^2\eta_1, \xi_1\eta_1, \xi_1\eta_1^2, \xi_1\eta_1^3), \\ (\xi_2, \eta_2) &\mapsto (\xi_2, \xi_2\eta_2, \xi_2^2\eta_2^3, \xi_2^3\eta_2^5). \end{aligned}$$

3.1 Normal surfaces with at most cyclic quotient singularities

Let us consider a normal surface X having at most a finite number of cyclic quotient singularities, and call by $\rho \colon S \to X$ its minimal resolution (obtained by applying the Theorem 3.0.21 to any singular point of X).

We denote by K_X the canonical (Weil) divisor on X corresponding to $i_*(\Omega^2_{X^0})$, where $i \colon X^0 \to X$ is the inclusion of the smooth locus of X. According to Mumford, we have an intersection product with values in $\mathbb{Q}$ for Weil divisors on a normal surface. We are going to consider in particular the self-intersection of the canonical divisor, $K_X^2 \in \mathbb{Q}$.

Let x be a singular point of X of type $\frac{1}{n}(1,a)$, with $1 \leq a < n$ coprime integers, and let

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}} = [b_1, \dots, b_l].$$

From the Theorem 3.0.21, the exceptional divisor of a minimal resolution of x is then an *Hirzebruch-Jung string* (for short HJ-string), namely $E = \bigcup_{i=1}^l E_i$ where all E_i are smooth rational curves, $E_i^2 = -b_i$, $E_i \cdot E_{i+1} = 1$ for $i = 0, \dots, l-1$, and $E_i \cdot E_j = 0$ otherwise. In a neighbourhood of x

$$K_S = \rho^* K_X + \sum_{i=1}^l r_i E_i, \tag{3.7}$$

where the rational numbers r_i are determined by the conditions

$$(K_S + E_j) \cdot E_j = 2g(E_j) - 2 = -2, \quad \text{and} \quad (K_S - \sum_{i=1}^l r_i E_i) \cdot E_j = 0, \; j = 1, \dots, l. \tag{3.8}$$

Lemma 3.1.1. *Let n and a be coprime integers with $1 \leq a < n$, and let $\frac{n}{a} = [b_1, \ldots, b_l]$. Let A be the intersection matrix determined by the Hirzebruch-Jung string of a singularity of type $\frac{1}{n}(1, a)$, i.e.*

$$A := \begin{pmatrix} -b_1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -b_2 & 1 & 0 & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & -b_{l-1} & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -b_l \end{pmatrix} \in M_l(\mathbb{Z}).$$

Then $\det(A) = (-1)^l n$.

Proof. By induction on l. If $l = 1$, then $a = 1$ and $b_1 = n$. We have $\det((-b_1)) = -b_1 = (-1)^1 n$. Suppose the formula is true for fractions whose HJ-string has length equal at most $l - 1$. We are going to prove the formula holds for l. Develops $\det(A)$ with respect the first column

$$\det(A) = -b_1 \det(A_{11}) - \det(A_{12}). \tag{3.9}$$

A_{11} is the intersection matrix of the HJ string $[b_2, \ldots, b_l]$, which can be computed as follows

$$\frac{n}{a} = b_1 - \frac{1}{[b_2, \ldots, b_l]} \implies [b_2, \ldots, b_l] = \frac{a}{ab_1 - n}$$

Therefore, by induction hypothesis, $\det(A_{11}) = (-1)^{l-1} a$. Instead, developing $\det(A_{12})$ with respect to the the first row, one sees that it is the determinant of the intersection matrix of the HJ string $[b_3, \ldots, b_l]$. Once more, we can compute it as

$$\frac{a}{ab_1 - n} = b_2 - \frac{1}{[b_3, \ldots, b_l]} \implies [b_3, \ldots, b_l] = \frac{ab_1 - n}{a(b_2 b_1 - 1) - n}.$$

By induction hypothesis, we have $\det(A_{12}) = (-1)^{l-2}(ab_1 - n)$. The equation (3.9) becomes

$$\det(A) = (-1)^l ab_1 - (-1)^l (ab_1 - n) = (-1)^l n.$$

□

Lemma 3.1.2. *([Bar99],[Hir53]) Let x be a singularity of type $\frac{1}{n}(1, a)$, with* $\gcd(n, a) = 1$. *Consider $\frac{n}{a} = [b_1, \ldots, b_l]$. In a neighbourhood of x we have*

$$K_S = \rho^* K_X + \sum_{i=1}^{l} r_i E_i.$$

Given $\lambda_0 = 0, \lambda_1 = 1$, and $\mu_0 = n, \mu_1 = a$, define

$$\lambda_{i+1} = -\lambda_{i-1} + b_i\lambda_i \qquad \text{and} \qquad \mu_{i+1} = -\mu_{i-1} + b_i\mu_i, \tag{3.10}$$

for $i = 1, \dots, l-1$. Then $nr_i = \lambda_i + \mu_i - n$, for $i = 1, \dots, l$.

Proof. The conditions (3.8) implies that

$$\begin{aligned} 0 = \left(K_S - \sum_{i=1}^{l} r_iE_i\right) \cdot E_j = b_j - 2 - \left(\sum_{i=1}^{l} r_iE_i\right) \cdot E_j \\ \Longrightarrow \left(\sum_{i=1}^{l} r_iE_i\right) \cdot E_j = b_j - 2. \end{aligned} \tag{3.11}$$

For short of notation, consider the $(n \times 1)$-vectors $\mathbf{r}$, $\mathbf{b}$, and $\mathbf{1}$ whose i-th entry of them is the number r_i, b_i, and 1 respectively, for $i = 1, \dots, l$. The latter equation of (3.11) is equivalent to ${}^te_jA\mathbf{r} = b_j - 2$, $j = 1, \dots, l$, where $e_1, \dots, e_l$ is the canonical basis, and A is the intersection matrix of the HJ-string of x, which is invertible by the Lemma 3.1.1. In other words, $\mathbf{r}$ is the only solution of the linear system $A\mathbf{x} = \mathbf{b} - 2 \cdot \mathbf{1}$.
Observe then $\mathbf{r} + \mathbf{1}$ solves the sublinear system $A'\mathbf{x} = 0$, where $A' := A(2, \dots, l-1|1, \dots, l) \in M_{l-2,l}(\mathbb{Z})$ is the matrix obtained by removing the first and last rows from A. A solution of the system A' can be constructed as follows: let us fix two numbers $t_0, t_1 \in \mathbb{R}$, and define the vector $\mathbf{v}(t_0, t_1) := \big(t_1, \dots, t_l\big)^t$, where t_i satisfies the recursive formula $t_{i+1} = -t_{i-1} + b_it_i$, $i = 1, \dots l-1$. By construction, $\mathbf{v}(t_0, t_1) \in \ker(A')$. Observe that the recursive formula between t_i, $i = 1, \dots, l-1$, can be re-written in two equivalent ways:

$$\begin{pmatrix} t_i \\ t_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & b_i \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & b_1 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix}, \tag{3.12}$$

$$\begin{pmatrix} t_{i+1} \\ t_i \end{pmatrix} = \begin{pmatrix} b_i & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_0 \end{pmatrix}. \tag{3.13}$$

By a simple check, the vectors $\mathbf{v}(0, 1)$ and $\mathbf{v}(n, a)$ are linearly independent, and so they form a basis of $\ker(A')$, which has dimension 2, since the rank of A' is $l-2$ (otherwise there could be a linear relation between the rows of A', which are those of A, and so A would not have maximal rank). Note that by construction

$$\mathbf{v}(0, 1) = (\lambda_1, \dots, \lambda_l)^t \qquad \text{and} \qquad \mathbf{v}(n, a) = (\mu_1, \dots, \mu_l)^t, \tag{3.14}$$

where λ_i and μ_i are those numbers defined at (3.10).
Remembering that $\mathbf{r} + \mathbf{1} \in \ker(A')$, then there exist suitable α, β such that

$$\mathbf{r} = \alpha\mathbf{v}(0, 1) + \beta\mathbf{v}(n, a) - \mathbf{1}. \tag{3.15}$$

The variables α, β can be found through the first and last equation of the linear system $A\mathbf{x} = \mathbf{b} - 2 \cdot \mathbf{1}$, whose we remember $\mathbf{r}$ is the only solution:

$$\begin{cases} -b_1 r_1 + r_2 = b_1 - 2 \\ r_{l-1} - b_l r_l = b_l - 2. \end{cases} \tag{3.16}$$

By (3.11) and (3.14), we need to know $\lambda_1, \lambda_2, \mu_1, \mu_2$ and $\lambda_{l-1}, \lambda_l, \mu_{l-1}, \mu_l$ to compute r_1, r_2, and r_{l-1}, r_l.
Let $1 \leq a' < n$ such that $a' = a^{-1}$ is the inverse of a modulo n. Then $\frac{n}{a'} = [b_l, \ldots, b_1]$, and we get

$$\begin{pmatrix} b_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & b_l \end{pmatrix} \begin{pmatrix} n \\ a' \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 \\ -1 & b_l \end{pmatrix} \begin{pmatrix} n \\ a' \end{pmatrix} = \begin{pmatrix} b_{l-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_l \\ \lambda_{l-1} \end{pmatrix}. \tag{3.17}$$

The last identity follows from (3.13). From the other side, we have

$$\begin{pmatrix} b_l \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & b_{l-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & b_1 \end{pmatrix} \begin{pmatrix} n \\ a \end{pmatrix} = \begin{pmatrix} \mu_{l-1} \\ \mu_l \end{pmatrix}. \tag{3.18}$$

The last identity follows from (3.12). Putting together the equations (3.17) and (3.18), we get

$$\begin{aligned} &\lambda_1 = 1, \quad \lambda_2 = b_1, \quad \lambda_{l-1} = -n + b_l a', \quad \lambda_l = a', \\ &\mu_1 = a, \quad \mu_2 = -n + b_1 a, \quad \mu_{l-1} = b_l, \quad \mu_l = 1. \end{aligned} \tag{3.19}$$

Thus $r_1 = \alpha + \beta a - 1$, $r_2 = \alpha b_1 + \beta(-n + b_1 a) - 1$, $r_{l-1} = \alpha(-n + b_l a') + \beta b_l - 1$, and $r_l = \alpha a' + \beta - 1$. The system (3.16) becomes

$$\begin{cases} -b_1(\alpha + \beta a - 1) + \alpha b_1 + \beta(-n + b_1 a) - 1 = b_1 - 2 \\ \alpha(-n + b_l a') + \beta b_l - 1 - b_l(\alpha a' + \beta - 1) = b_l - 2. \end{cases} \Longrightarrow$$

$$\begin{cases} -\beta n + b_1 - 1 = b_1 - 2 \\ -\alpha n + b_l - 1 = b_l - 2. \end{cases}$$

Thus $\alpha = \beta = \frac{1}{n}$, and replacing them to the Equation (3.15), we get $n\mathbf{r} = \mathbf{v}(0, 1) + \mathbf{v}(n, a) - n\mathbf{1}$. □

Lemma 3.1.3. *For a singular point x of X of type $\frac{1}{n}(1, a)$, with $1 \leq a < n$ coprime integers, we have in a neighbourhood of x*

$$K_S^2 = K_X^2 - \left(-2 + \frac{2 + a + a'}{n} + \sum_{i=1}^{l} (b_i - 2) \right).$$

Here $a' = a^{-1}$ is the inverse of a modulo n, $1 \leq a' < n$, and b_i are given by $\frac{n}{a} = [b_1, \ldots, b_l]$.

Proof. In a neighbourhood of x we have $K_S = \rho^* K_X + \sum_{i=1}^{l} r_i E_i$, where r_i are the rational numbers computed in the Lemma 3.1.2. Since $\rho\colon S \to X$ is birational and contracts the rational curves E_i, then

$$K_S^2 = K_X^2 + \left(\sum_{i=1}^{l} r_i E_i\right)^2. \tag{3.20}$$

However

$$\begin{aligned}\left(\sum_{i=1}^{l} r_i E_i\right)^2 &= \left(\sum_{i=1}^{l} r_i E_i\right) \cdot (K_S - \rho^* K_X) = \sum_{i=1}^{l} r_i E_i \cdot K_S \\ &= \sum_{i=1}^{l} r_i(-2 - E_i^2) = \frac{1}{n}\sum_{i=1}^{l} n r_i (b_i - 2) \\ &= \frac{1}{n}\sum_{i=1}^{l} (\lambda_i + \mu_i - n)(b_i - 2).\end{aligned}$$

The last equality follows from the Lemma 3.1.2. We have got

$$\begin{aligned}\left(\sum_{i=1}^{l} r_i E_i\right)^2 &= \frac{1}{n}\sum_{i=1}^{l} (\lambda_i + \mu_i - n)(b_i - 2) \\ &= \frac{1}{n}\sum_{i=1}^{l} (\lambda_i + \mu_i)(b_i - 2) - \sum_{i=1}^{l}(b_i - 2) \\ &= \frac{1}{n}\sum_{i=1}^{l-1} (\lambda_i + \mu_i)(b_i - 2) + \frac{1}{n}(\lambda_l + \mu_l)(b_l - 2) - \sum_{i=1}^{l}(b_i - 2).\end{aligned} \tag{3.21}$$

Extending the first sum

$$\begin{aligned}\sum_{i=1}^{l-1} (\lambda_i + \mu_i)(b_i - 2) &= \sum_{i=1}^{l-1}(\lambda_i b_i - 2\lambda_i) + \sum_{i=1}^{l-1}(\mu_i b_i - 2\mu_i) \\ &= \sum_{i=1}^{l-1}(\lambda_{i-1} + \lambda_{i+1} - 2\lambda_i) + \sum_{i=1}^{l-1}(\mu_{i-1} + \mu_{i+1} - 2\mu_i).\end{aligned}$$

These two sums are telescopic sums, and so

$$\sum_{i=1}^{l-1} (\lambda_i + \mu_i)(b_i - 2) = -(\lambda_{l-1} - \lambda_0) + \lambda_l - \lambda_1 - (\mu_{l-1} - \mu_0) + \mu_l - \mu_1. \tag{3.22}$$

Thanks to the computation (3.19) done during the proof of the Lemma 3.1.2 and to the Equation (3.22), then the Equation (3.21) becomes

$$\begin{aligned}\left(\sum_{i=1}^{l} r_i E_i\right)^2 &= \frac{1}{n}(-\lambda_{l-1}+\lambda_0-\lambda_1-\mu_{l-1}+\mu_0-\mu_1+(\lambda_l+\mu_l)(b_l-1))-\sum_{i=1}^{l}(b_i-2)\\ &= \frac{1}{n}\left(n-b_l a'-1-b_l+n-a+(1+a')(b_l-1)\right)-\sum_{i=1}^{l}(b_i-2)=\\ &= \frac{1}{n}\left(2n-(2+a+a')\right)-\sum_{i=1}^{l}(b_i-2)\\ &= -\left(-2+\frac{2+a+a'}{n}+\sum_{i=1}^{l}(b_i-2)\right).\end{aligned} \tag{3.23}$$

The thesis follows by replacing the value obtained in the Equation (3.23) to the Equation (3.20). $\square$

Definition 3.1.4. A singular point x of a normal surface X is a *Rational Double Point* (for short RDP) or *Du Val singularity* if the exceptional divisor $E = \cup E_i$ of the minimal resolution $\rho\colon S \to X$ of the singularities of X consists of a three of smooth rational curves E_i, and $K_S \cdot E_i = 0$, or equivalently $E_i^2 = -2$.

Definition 3.1.5. ([Rei87, Definition 1.1]) A normal variety X of dimension n has *canonical singularities* if

1. for some $n \geq 1$, then the (Weil) divisor nK_X is Cartier;
2. if $\rho\colon Y \to X$ is a resolution of the singularities of X and $E = \cup E_i$ is the exceptional divisor of ρ, then

$$nK_Y = \rho^* nK_X + \sum a_i E_i, \qquad a_i \geq 0.$$

In dimension 2, canonical singularities are the same as Rational Double Points, as the following theorem states:

Theorem 3.1.6. *(cf.* [Mat02, Theorem 4.6.7]*) Let x be a singular point on a normal surface X. Then x is a canonical singularity if and only if it is a Rational Double Point. Moreover, x is locally analytically determined by the dual graph of the exceptional divisor of the minimal resolution of its* *singularity, which is one of the following* 5 *types:*

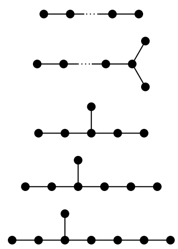
A_n

D_n

E_6

E_7

E_8



More precisely, an analytic neighbourhood of x is biholomorphic to a neighbourhood of the origin of one of the following hypersurfaces of $\mathbb{C}^3$:

$$\begin{aligned} x^2+y^2+z^{n+1} &= 0 \quad \textit{if the graph is} \quad A_n; \\ x^2+y^2z+z^{n-1} &= 0 \quad \textit{if the graph is} \quad D_n; \\ x^2+y^3+z^4 &= 0 \quad \textit{if the graph is} \quad E_6; \\ x^2+y^3+yz^3 &= 0 \quad \textit{if the graph is} \quad E_7; \\ x^2+y^3+z^5 &= 0 \quad \textit{if the graph is} \quad E_8; \end{aligned}$$

Remark 3.1.7. Let X be a surface with at most canonical singularities and $\rho\colon S \to X$ be a minimal resolution of the singularities of X. Then (see [Mat02, Theorem 4.6.2])

$$K_S = \rho^* K_X.$$

Remark 3.1.8. A cyclic quotient singularity is a canonical singularity, or equivalently a Rational Double Point, if and only if is of type $\frac{1}{n}(1, n-1)$. Here, $H \subset \mathrm{SL}(2, \mathbb{C})$, and the exceptional divisor of its minimal resolution consists of $n-1$ rational smooth curves of self-intersection -2, whose dual graph is A_{n-1}.

To prove this, take a cyclic quotient singularity of type $\frac{1}{n}(1, a)$, whose continued fraction $\frac{n}{a} = [b_1, \ldots, b_l]$ has length l. Assume it is a canonical singularity, or equivalently a RDP. From the Theorem 3.0.21, then the exceptional divisor of the minimal resolution of $\frac{1}{n}(1, a)$ consists of l rational smooth curves E_i, whose dual graph is of type A_l, and with self-intersection $E_i^2 = -b_i$. However, we are assuming that $\frac{1}{n}(1, a)$ is a RDP so that $E_i^2 = -b_i = -2$. This means

$$\frac{n}{a} = [2, \ldots, 2] = \frac{l+1}{l} \implies n = l+1, \quad a = l = n-1.$$

Let us compute now the Euler characteristic $e(S)$ of a minimal resolution of the singularities of X.

Lemma 3.1.9. *Let X be a normal variety having at most a finite number of cyclic quotient singularities, and X° its smooth locus. Consider $\rho\colon S \to X$*

the minimal resolution of the singularities of X. For any singular point x of type $\frac{1}{n}(1,a)$, write by l_x the length of continued fraction $\frac{n}{a} = [b_1, \dots, b_{l_x}]$. Then

$$e(S) = e(X^\circ) + \sum_x (l_x + 1).$$

Proof. Fix a singular point x and set $l := l_x$. Consider the three $\rho^{-1}(x) = \bigcup_{i=1}^{l} E_i$ of the exceptional curves of x. From the Theorem 3.0.21 we know that $E_i \cdot E_{i+1} = 1$, and $E_i \cdot E_j = 0$ otherwise. Thus we can write

$$\begin{aligned} e(\rho^{-1}(x)) &= e(\mathbb{P}^1 \setminus \{\text{point}\}) + e(\bigcup_{i=2}^{l} E_i) \\ &= 1 + e(\bigcup_{i=2}^{l} E_i) = \dots \\ &= l - 1 + e(E_l) = l + 1. \end{aligned}$$

The thesis follows by applying this result recursively for any singular point of X

$$e(S) = e(S \setminus \bigcup_x \rho^{-1}(x)) + \sum_x e(\rho^{-1}(x)) = e(X^\circ) + \sum_x (l_x + 1).$$

□

Definition 3.1.10. Let X be a normal complex surface having at most a finite number of cyclic quotient singularities. A *representation of the basket of singularities of* X is a multiset

$$\mathcal{B}(X) := \left\{ \lambda \times \left(\frac{1}{n}(1,a) \right) : X \text{ has exactly } \lambda \text{ singularities of type } \frac{1}{n}(1,a) \right\}.$$

For instance, $\mathcal{B}(X) = \left\{ 2 \times \frac{1}{5}(1,1), 1 \times \frac{1}{5}(1,4) \right\}$ means that X has 2 singularities of type $\frac{1}{5}(1,1)$ and one singularity of type $\frac{1}{5}(1,4)$.

Remark 3.1.11. We observe that a normal surface X with only cyclic quotient singularities has different representation of its basket. For instance,

$$\left\{ 2 \times \frac{1}{7}(1,3) \right\}, \left\{ 1 \times \frac{1}{7}(1,3), 1 \times \frac{1}{7}(1,5) \right\}, \left\{ 2 \times \frac{1}{7}(1,5) \right\}$$

represent the same basket of singularities of X.

This justifies the next definition

Definition 3.1.12. Consider the set of multisets of the form

$$\mathcal{B}(X) := \left\{ \lambda \times \left(\frac{1}{n}(1,a) \right) : a, n, \lambda \in \mathbb{N}, a < n, \gcd(a,n) = 1 \right\},$$

together with the equivalence relation given by "$\frac{1}{n}(1,a)$ *is equivalent to* $\frac{1}{n}(1,a')$", where $a' = a^{-1}$ in $(\mathbb{Z}/\mathbb{Z}n)^*$. A basket of singularities is then an equivalence class.

Chapter 4

Product-Quotient Surfaces

In the 30's L. Campedelli and L. Godeaux have constructed the first examples of surfaces of general type with $p_g = 0$. Such surfaces have been known later on as numerical Campedelli surfaces, and numerical Godeaux surfaces. They have been ones of the possible exceptions encountered by E. Bombieri in the 70's in its famous Theorem 1.0.2 on the structure of the multicanonical map Φ_{nK_S} of surfaces of general type. This is one of the reasons why several authors started to study them, and tried to give more examples.

The idea of Godeaux was to consider the quotient of simpler surfaces by the free action of a finite group. In this spirit, Beauville (see [Bea96, pg. 118]) proposed a simpler construction of surfaces of general type, by considering the quotient of a product of two curves C_1 and C_2 by the free action of a finite group. Moreover, he gave an explicit example by taking the quotient of two Fermat curves of degree 5 in $\mathbb{P}^2$.

After [Cat00] many authors started studying the surfaces that appear as quotient of a product of curves.

They are revealed to be a very useful tool for building new examples of algebraic surfaces and studying their geometry in an accessible way. Apart from other works, that mainly deal with irregular surfaces, we want to mention the complete classification of surfaces isogenous to a product with $p_g = q = 0$ [BCG08] and the classification for $p_g = 1$ and $q = 0$ under the assumption that the action is diagonal [Gle15], the rigid but not infinitesimally rigid manifolds [BP21] of Bauer and Pignatelli that gave a negative answer to a question of Kodaira and Morrow [MK71, p.45] and also the infinite series of n-dimensional infinitesimally rigid manifolds of general type with non-contractible universal cover for each $n \geq 3$, provided by Frapporti and Gleissner[FG23].

The chapter is organized as follows: in the first two sections we study the action of a finite group on a product of curves, and we give the formal definition of *product-quotient* surfaces of unmixed and mixed type. In Section 4.3 and 3.1 we study the type of singularities of their quotient model,

which are necessary to determine their invariants. This is the main content of Section 4.4.
The next Section 4.5 aims to describe regular product-quotient surfaces of general type through a set of group data. This is one of the most crucial steps for trying to get a classification of them via an efficient algorithm, which has to be implemented in a computational algebra system (e.g. MAGMA [BCP97]). One of the most difficult implementation problems is to get an efficient database. Indeed usually a few millions of group data determine the same product-quotient surface up to isomorphism. We present a new approach to the problem and we give new results which permits to avoid those repetitions.
The next section proves that the classification problem with fixed self-intersection K^2 and characteristic χ is theoretically manageable. These results are raised in the Section 4.8, where we present a classification algorithm.

4.1 Automorphism group of a product of curves

Let C_1 and C_2 be two Riemann surfaces of respective genus $g_i \geq 2$. In general, the automorphism group of $C_1 \times C_2$ contains $\mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$. It could happens that they are equal, although this is not always true.
An easy counterexample is obtained by taking two copies of the same curve C. In fact, the involution $\iota\colon C \times C \to C \times C$, $(x, y) \mapsto (y, x)$, is an automorphism of $C \times C$ not belonging to the product $\mathrm{Aut}(C)^2$.
Moreover, this suggests also that $\mathrm{Aut}(C)^2 \rtimes_\varphi \mathbb{Z}_2$ is contained in $\mathrm{Aut}(C \times C)$ via the injective homomorphism Φ defined through ι

$$\Phi\colon \mathrm{Aut}(C)^2 \rtimes_\varphi \mathbb{Z}_2 \hookrightarrow \mathrm{Aut}(C \times C), \qquad (f, g, a) \mapsto (f, g) \circ \iota^a.$$

Here, the semidirect product $\mathrm{Aut}(C)^2 \rtimes_\varphi \mathbb{Z}_2$ is defined by the homomorphism $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{Aut}(C)^2)$ sending $\overline{1} \mapsto \begin{pmatrix} \mathrm{Aut}(C)^2 \to \mathrm{Aut}(C)^2 \\ (f,g) \mapsto (g,f) \end{pmatrix}$.
A surprisingly fact is that

- $\mathrm{Aut}(C_1 \times C_2) = \mathrm{Aut}(C_1) \times \mathrm{Aut}(C_2)$, if C_1 and C_2 are not isomorphic to each other;
- $\mathrm{Aut}(C \times C) = \mathrm{Aut}(C)^2 \rtimes_\varphi \mathbb{Z}_2$.

This result follows directly once we prove

Lemma 4.1.1. *(Rigidity Lemma* [Cat00, Lemma 3.8]*) Let $f\colon C_1 \times C_2 \to B_1 \times B_2$ be a surjective morphism between products of curves. Assume that both B_1 and B_2 have genus ≥ 2. Then, after possible exchanging B_1 with B_2, there are holomorphic maps $f_i\colon C_i \to B_i$ such that $f(x, y) = (f_1(x), f_2(y))$.*

4.2 Group action on a product of curves

Consider two Riemann surfaces C_1 and C_2 of genus $g_i \geq 2$. Let G be a finite group acting faithfully on the product $C_1 \times C_2$. By what we have said in the previous Section 4.1, the action of G on $C_1 \times C_2$ can be only of two types:

- *Unmixed:* G acts independently (non necessarily faithfully) on each factor $G \hookrightarrow \mathrm{Aut}(C_i)$, and the action of G on $C_1 \times C_2$ is diagonal:

  $$g(x, y) = (g(x), g(y)).$$

  This happens when C_1 and C_2 are not isomorphic to each other;

- *Mixed:* here the two curves C_i are the same curve C. In this case, we have $G \hookrightarrow \mathrm{Aut}(C)^2 \rtimes_\varphi \mathbb{Z}_2$ and so there could be elements of G that exchange the two factors via the involution ι. Define $G^0 := G \cap \mathrm{Aut}(C)^2$, so that $G = G^0 \rtimes_\varphi (G \cap \mathbb{Z}_2)$.
  Observe that the action of G^0 is unmixed, so it acts on each factor and the action on the product is diagonal.

Definition 4.2.1. Let C_1 and C_2 be two curves of genus $g_i \geq 2$. Let G be a finite group acting on each of them and consider its diagonal action on $C_1 \times C_2$. Then $X := (C_1 \times C_2)/G$ is called *quotient model of unmixed type*, and the minimal resolution S of the singularities of X is called *product-quotient* surface (of unmixed type).

Let us define also

Definition 4.2.2. Let C be a curve of genus $g \geq 2$. A (faithful) action of G on $C \times C$ is said to be *mixed* if G is not contained in $\mathrm{Aut}(C)^2$. In this case, we say that $X := (C \times C)/G$ is a *quotient model* (of *mixed type*) and the minimal resolution S of the singularities of X is called *product-quotient* surface (of *mixed type*).

From now on, we are going to analyse product-quotient surfaces of unmixed type.
The situation can be described through the following commutative hexago-

nal diagram:

$$\begin{tikzcd}
 & & C_1 \times C_2 \arrow[dll, "p_1"'] \arrow[drr, "p_2"] \arrow[dd, "\lambda_{12}"] & & \\
C_1 \arrow[dd, "\lambda_1"] & & & & C_2 \arrow[dd, "\lambda_2"] \\
 & & (C_1 \times C_2)/G \arrow[dll, "f_1"'] \arrow[drr, "f_2"] \arrow[dd, "\lambda"] & & \\
C_1/G & & & & C_2/G \\
 & & C_1/G \times C_2/G. \arrow[ull] \arrow[urr] & &
\end{tikzcd} \tag{4.1}$$

We will refer to the notation of the maps fixed in above picture (4.1) for the rest of the chapter.

Observe that the reason why we have considered $g_i \geq 2$ is given by the following

Lemma 4.2.3. *Let G be a finite group acting on a product $C_1 \times C_2$, where C_i are two Riemann surfaces of respective genus g_i. Let $S \to X = (C_1 \times C_2)/G$ be the minimal resolution of the singularities of $X = (C_1 \times C_2)/G$. If S is of general type, then $g_i \geq 2$.*
Conversely, if the quotients C_i/G have genera ≥ 2 (and therefore also $g_i \geq 2$), then S is of general type.

Proof. Suppose S is of general type. Then the Corollary A.2.10 applies to the quotient map $\lambda_{12} \colon C_1 \times C_2 \to (C_1 \times C_2)/G$, and we have $\kappa(C_1 \times C_2) \geq \kappa(S) = 2$. Thus $C_1 \times C_2$ is of general type too. However, by the Theorem A.2.8, we have $\kappa(C_1) + \kappa(C_2) = \kappa(C_1 \times C_2) = 2$. Therefore $\kappa(C_i) = 1$ for each i, and this is equivalent to say that $g_i \geq 2$, from the Remark A.2.6 .
Conversely, suppose that both C_i/G have genus ≥ 2. Then apply the Corollary A.2.10 to the map $\lambda\colon (C_1 \times C_2)/G \to C_1/G \times C_2/G$ to get $\kappa(S) \geq \kappa(C_1/G \times C_2/G) = \kappa(C_1/G) + \kappa(C_2/G) \geq 1 + 1 = 2$. □

Remark 4.2.4. In general is not true that if $g_i \geq 2$, then S is of general type. A list of examples can be found in [BP16, Table 1, pg. 341].

Since we want to construct surfaces S of general type as the minimal resolution of the singularities of $(C_1 \times C_2)/G$, we shall consider C_i of genus $g_i \geq 2$.

Remark 4.2.5. Consider a product-quotient surface S of general type of quotient model $(C_1 \times C_2)/G$ (of unmixed type). Then we can suppose without lost of generalities that G acts faithfully on both factors C_1 and

C_2.
To prove this, let us consider the actions $\phi_i \colon G \to \mathrm{Aut}(C_i)$ with kernels K_i. If both of K_i are trivial, then we are nothing to do, so let us suppose for instance that $K_1 \neq \{1\}$. Since K_i are normal subgroups of G, then G acts also on C_1/K_2 and C_2/K_1. Moreover, K_1 is contained in the kernels of the actions $G \to \mathrm{Aut}(C_i/K_j)$, $i \neq j$, and so it makes sense to take the actions $G/K_1 \to \mathrm{Aut}(C_i/K_j)$ on C_i/K_j.
By construction, we have a natural isomorphism

$$(C_1 \times C_2)/G \to (C_1/K_2 \times C_2/K_1)\,/\,(G/K_1)\,, \quad G(p_1,p_2) \mapsto (G/K_1)(K_2p_1, K_1p_2).$$

Furthermore, we get the quotient $(C_1/K_2 \times C_2/K_1)\,/\,(G/K_1)$ is of general type, and so C_i/K_j have always genus ≥ 2, by the Lemma 4.2.3.

To summarize, we have constructed two new curves C_i/K_j of genus ≥ 2 such that the group G/K_1 acts on them, and the obtained quotient model is still isomorphic to $(C_1 \times C_2)/G$. Since we have assumed that K_1 is non-trivial, then $|G/K_1| < |G|$. Thus we can repeat the same procedure in a finite number of steps until the actions on the two curves is faithful.

4.3 Singularities of a quotient model of unmixed type

In this section we investigate the types of singularities of the quotient model $X := (C_1 \times C_2)/G$ of unmixed type. Many of these results are taken from [BP12].

Proposition 4.3.1. *Let $X := (C_1 \times C_2)/G$ be a quotient model of unmixed type. Then*

1. *(analytic neighbourhood of a point $G(p_1,p_2) \in X$)*
   Consider a point (p_1,p_2) of C, and for any $i = 1,2$ take a $\mathrm{Stab}(p_i)$-invariant neighbourhood U_i of p_i as in the Theorem 2.0.6.
   Linearize the action of $\mathrm{Stab}(p_i)$ locally around p_i as the Lemma 2.0.13.
   Let x be the local coordinate of $U_1 \cong V_1 \subseteq \mathbb{C}$ around p_1, and y that of $U_2 \cong V_2 \subseteq \mathbb{C}$ around p_2.
   Then a generator of $\mathrm{Stab}(p_1,p_2) = \mathrm{Stab}(p_1) \cap \mathrm{Stab}(p_2)$ acts naturally on $U_1 \times U_2 \cong V_1 \times V_2 \subseteq \mathbb{C}^2$ simultaneously on both factors by multiplication of some n-roots of the unity, $n := |\mathrm{Stab}(p_1,p_2)|$:

   $$(x,y) \mapsto (e^{\frac{2\pi i}{n}a}x, e^{\frac{2\pi i}{n}b}y), \qquad 1 \leq a,b \leq n.$$

   Moreover

   $$\alpha\colon\ (V_1 \times V_2)\,/\begin{pmatrix} e^{\frac{2\pi i}{n}a} & 0 \\ 0 & e^{\frac{2\pi i}{n}b} \end{pmatrix} \cong (U_1 \times U_2)\,/\mathrm{Stab}(p_1,p_2) \xrightarrow{\sim} X$$

is an homeomorphism onto an open neighbourhood of $G(p_1, p_2)$ *of* X*. In other words,* $X \cong (V_1 \times V_2) / \begin{pmatrix} e^{\frac{2\pi i}{n}a} & 0 \\ 0 & e^{\frac{2\pi i}{n}b} \end{pmatrix}$ *locally around* $G(p_1, p_2)$*;*

2. *the singular points of* X *are only cyclic quotient singularities;*

3. *Let* $\mathrm{Sing}(X)$ *be the set of singular points of* X*. Then* $\mathrm{Sing}(X)$ *is the branch locus of the map* λ_{12}*, and*

$$\mathrm{Sing}(X) \subset \bigcup_{q_1 \in Crit(\lambda_1), q_2 \in Crit(\lambda_2)} \lambda^{-1}(q_1, q_2),$$

where $Crit(\lambda_i)$ *is the set of branch points of* λ_i*. In particular,* $\mathrm{Sing}(X)$ *is finite.*

Proof. 1. Since the action on $C_1 \times C_2$ is diagonal, then $\mathrm{Stab}(p_1, p_2) = \mathrm{Stab}(p_1) \cap \mathrm{Stab}(p_2)$. By Proposition 2.0.3, then $\mathrm{Stab}(p_i)$ is a cyclic subgroup of G, so that $\mathrm{Stab}(p_1, p_2)$ is cyclic too; let us say g is a generator. If x is the local coordinate of $U_1 \cong V_1$ around p_1, and y that of $U_2 \cong V_2$ around p_2, then g acts locally as $x \mapsto e^{\frac{2\pi i}{n}a}x$ around p_1, and as $y \mapsto e^{\frac{2\pi i}{n}b}y$ locally around p_2, for suitable $1 \leq a, b \leq n$. Therefore, g acts locally around (p_1, p_2) as $(x, y) \mapsto \left(e^{\frac{2\pi i}{n}a}x, e^{\frac{2\pi i}{n}b}y\right)$.
Observe that the map α is injective, by construction of U_i: if $\mathrm{Stab}(p_1, p_2)(q_1, q_2)$ and $\mathrm{Stab}(p_1, p_2)(q_1', q_2')$ are sent to the same class $G(q_1, q_2) = G(q_1', q_2')$, then there exists $h \in G$ such that $q_1' = h \cdot q_1 \in U_1$, and $q_2' = h \cdot q_2 \in U_2$. However, $U_i \cap (h \cdot U_i) = \emptyset$ if $h \notin \mathrm{Stab}(p_i)$. This forces h to belong to $\mathrm{Stab}(p_i)$, and so to $\mathrm{Stab}(p_1, p_2)$.
The map α is also continuous and open since, composed with the projection $U_1 \times U_2 \to (U_1 \times U_2) / \mathrm{Stab}(p_1, p_2)$, it gives the quotient application $(\lambda_{12})_{|U_1 \times U_2}$:

$$\begin{tikzcd} U_1 \times U_2 \arrow[r] \arrow[dr, "i_{U_1 \times U_2}"'] & (U_1 \times U_2) / Stab(p_1, p_2) \arrow[r, "\alpha"] & X = (C_1 \times C_2)/G \\ & C_1 \times C_2 \arrow[ur, "\lambda_{12}"'] & \end{tikzcd}$$

2. is straightforward. Point 1. proves X is locally isomorphic to $X \cong V_1 \times V_2 \subseteq \mathbb{C}^2$, if $\mathrm{Stab}(p_1, p_2) = \{1\}$, and $(V_1 \times V_2) / \langle \begin{pmatrix} e^{\frac{2\pi i}{n}a} & 0 \\ 0 & e^{\frac{2\pi i}{n}b} \end{pmatrix} \rangle$ otherwise. In particular, the point $G(p_1, p_2)$ is either smooth, or it is a cyclic quotient singularity of type $\frac{1}{n}(a, b)$.

3. Since $G(p_1, p_2)$ is not singular when $\mathrm{Stab}(p_1, p_2) = \{1\}$, then any singular point of X is contained in the branch locus of λ_{12}. Conversely, take a branch point $G(p_1, p_2)$ of λ_{12}, so that $\mathrm{Stab}(p_1, p_2)$ is not trivial of

order $n \geq 1$. Let g be a generator. The local action of g around (p_1, p_2) is $(x, y) \to (e^{\frac{2\pi i}{n}a}x, e^{\frac{2\pi i}{n}b}y)$, for suitable $1 \leq a, b \leq n$. Therefore $G(p_1, p_2)$ is a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ which is not smooth, otherwise $\frac{1}{n}(a, b)$ would be equivalent to $\frac{1}{n}(1, 0)$, from the Remark 3.0.12. This would mean either a or b is equal to n, and g would act locally around p_i as the identity. Since the action of G on C_i is supposed to be faithful, then $g = 1$, which contradicts $n > 1$. We have proved $G(p_1, p_2)$ is singular.

Now take a branch point $G(p_1, p_2)$ of λ_{12}. Then the stabilizer of (p_1, p_2) is not trivial, which forces p_1 and p_2 to being ramification points of λ_1, and λ_2 respectively. Hence their images $q_i = \lambda_i(p_i)$ are branch points. We have proved $G(p_1, p_2) \in \lambda^{-1}(q_1, q_2)$, with q_i branch points of λ_i.
From Proposition 2.0.4, the branch locus of λ_i is finite, so that there are a finite number of couples (q_1, q_2) for which q_i is a branch point of λ_i. Since λ is by construction a finite map (not necessarily Galois, but of order $|G|$) then $\lambda^{-1}(q_1, q_2)$ is a finite set too. We have proved $\lambda^{-1}(Crit(\lambda_1) \times Crit(\lambda_2))$ is finite, and so the same holds for $\operatorname{Sing}(X)$. □

By the point 3. of the Proposition 4.3.1, we have seen that a point $G(p_1, p_2)$ of X may be singular only if it belongs to a fibre $\lambda^{-1}(q_1, q_2)$, with $q_i := \lambda_i(p_i)$ branch point of λ_i. Let us count the points of the fibre $\lambda^{-1}(q_1, q_2)$.

Proposition 4.3.2. *([BP12, Prop. 1.16]) Consider a point $(q_1, q_2) \in C_1/G \times C_2/G$ and, fixed a point $p_i \in C_i$ over q_i, denote by H_i the stabilizer of that point. Consider the right action of H_i on G, and take the quotients G/H_i. Then*

(1) there is a G-equivariant bijective map $(\lambda \circ \lambda_{12})^{-1}(q_1, q_2) \to G/H_1 \times G/H_2$, where the G-action on the target is given by left multiplication (simultaneously on both factors);

(2) Each point of $\lambda^{-1}(q_1, q_2)$ is in one-to-one correspondence with an orbit of the H_1-(left) action on G/H_2. In other words, there is a bijection map $\lambda^{-1}(q_1, q_2) \to (G/H_2)/H_1$.

Proof. (1) Observe that $(\lambda \circ \lambda_{12})^{-1}(q_1, q_2) := \{(g_1 \cdot p_1, g_2 \cdot p_2) : g_1, g_2 \in G\}$. Therefore define simply the map sending each point $(g_1 \cdot p_1, g_2 \cdot p_2) \mapsto (g_1 H_1, g_2 H_2)$.

(2) Quotient the previous map of the point (1) with respect to the action of G. This map is a correspondence since the map of (1) is G-equivariant. Moreover, observe that the quotient $(G/H_1 \times G/H_2)/G$ is in natural bijection with $(G/H_2)/H_1$ via the map $G(g_1H_1, g_2H_2) \mapsto H_1(g_1^{-1}g_2H_2)$. To summarize, we have the following correspondences

$$\underbrace{\left((\lambda \circ \lambda_{12})^{-1}(q_1, q_2)\right)/G}_{=\lambda_{12}((\lambda\circ\lambda_{12})^{-1}(q_1,q_2))=\lambda^{-1}(q_1,q_2)} \xrightarrow{\sim} (G/H_1 \times G/H_2)/G \xrightarrow{\sim} (G/H_2)/H_1 .$$

□

It remains to study the types of cyclic quotient singularities of the points lying on the fibre $\lambda^{-1}(q_1, q_2)$. In order to do this, we state and prove the following

Proposition 4.3.3. *([BP12, Prop. 1.18]) The notation is the same like the Proposition 4.3.2. Let $g \in H_1$ be a generator of H_1, which is also the local monodromy (see Definition 2.0.14) of the point p_1 over q_1. Similarly, take $h \in H_2$ the local monodromy of the point p_2 over q_2.*
An element $[t] \in (G/H_2)/H_1$ corresponds to a cyclic quotient singularity of type $\frac{1}{n}(1, a)$, where $n := |H_1 \cap tH_2t^{-1}|$, and a is given as follows: let δ be the minimal positive number such that there exists $1 \leq \gamma \leq o(h)$ with $g^\delta = th^\gamma t^{-1}$. Then $a := \frac{n\gamma}{o(h)}$.

Proof. An element $[t]$ corresponds to the point $G(p_1, t \cdot p_2) \in \lambda^{-1}(q_1, q_2)$, so n is the cardinality of a stabilizer of a point (on $C_1 \times C_2$) over it. In particular, the stabilizer of the point $(p_1, t \cdot p_2)$ is exactly $H_1 \cap tH_2t^{-1}$.

By definition, the local monodromy g of p_1 acts in local analytic coordinates on C_1 as $x \mapsto e^{\frac{2\pi i}{o(g)}}x$. Instead, since h is the local monodromy of p_2, then $t \cdot h \cdot t^{-1}$ is the local monodromy of $t \cdot p_2$. Therefore $t \cdot h \cdot t^{-1}$ acts in local analytic coordinates around $t \cdot p_2$ on C_2 as $y \mapsto e^{\frac{2\pi i}{o(h)}}y$.
By construction of δ, we claim that $\langle g^\delta \rangle = \mathrm{Stab}(p_1, t \cdot p_2)$, and $o(g) = n\delta$: let $g^\alpha = th^\beta t^{-1} \in \mathrm{Stab}(p_1, t \cdot p_2) = H_1 \cap tH_2t^{-1}$, and divide α by δ; $\alpha = a\delta + b$, with $0 \leq b < \delta$. Then $g^b = (th^\beta t^{-1})(th^\gamma t^{-1})^{-a} = th^{\beta - a\gamma}t^{-1}$. From the minimality of δ, then b is forced to be 0, which shows $g^\alpha = (g^\delta)^a \in \langle g^\delta \rangle$. In particular, $o(g^\delta) = n$.
It remains to show $o(g) = n\delta$. Divide $o(g)$ by δ; $o(g) = a\delta + b$, with $0 \leq b < \delta$. Then $g^b = (th^\gamma t^{-1})^{-a} = th^{-\gamma a}t^{-1}$. From the minimality of δ, then b is forced to be 0. Thus $\gcd(o(g), \delta) = \delta$, and we obtain

$$n = o(g^\delta) = \frac{o(g)}{\gcd(o(g), \delta)} = \frac{o(g)}{\delta} \implies o(g) = n\delta.$$

Then g^δ acts (diagonally) on (x, y) as

$$(x, y) \mapsto (e^{\frac{2\pi i}{o(g)}\delta}x, e^{\frac{2\pi i}{o(h)}\gamma}y) = \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{\frac{2\pi i}{n}\frac{n\gamma}{o(h)}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This shows that $a = \frac{n\gamma}{o(h)}$. □

Remark 4.3.4. Observe that the number of singularities and their type over the fibre $\lambda^{-1}(q_1, q_2)$ does not depend from the choice of the local monodromies g and h on their conjugacy classes. If you replace p_1 by $\nu_1 \cdot p_1$, and p_2 by $\nu_2 \cdot p_2$, then the local monodromies g and h are respectively replaced by their conjugated g^{ν_1}, and h^{ν_2}. Applying the Proposition 4.3.3, we get a different singularity for each $[t]$, but the whole set of singularities above (q_1, q_2) does not change.

To better understand the Proposition 4.3.3 let us give an example.

Example 4.3.5. Consider the special linear group $G := \mathrm{SL}(2,5)$ and take the local monodromies g and h of two points p_1 and p_2 over a fibre $\lambda^{-1}(q_1, q_2)$. We compute the types of the singularities of the points of this fibre through g and h under the assumption that $g = h =: s$ is of order 5.

Thus $H_1 = H_2 = \langle s \rangle$ is a subgroup of order 5, and G/H_2 consists of 24 elements. The (left)-action of H_1 on G/H_2 gives 4 orbits of length 5 and 4 orbits of length 1. The latter ones arising from the points kH_2 fixed with respect the action of H_1. These points have to satisfy

$$s \cdot (kH_2) = kH_2 \iff k^{-1} \cdot s \cdot k = s^j, \quad \text{for some} \quad j.$$

We note that s is conjugate only to itself and s^4, and not to s^2 and s^3. In other words, if $s = k^{-1} \cdot s \cdot k$ and $s^4 = r^{-1} \cdot s \cdot r$, then the fixed points of the action are kH_2 and rH_2.
In particular, it turns out that two of the four fixed points k_1H_2, k_2H_2 satisfy the condition $s = k_i^{-1} \cdot s \cdot k_i$, while the other two fixed points r_1H_2, r_2H_2 satisfy $s^4 = r_i^{-1} \cdot s \cdot r_i$.

By the Proposition 4.3.3, the 4 orbits of cardinality 5 give smooth points of X; instead $\{k_1H_2\}, \{k_2H_2\}$ give two points with $n = 5$, $a = 1$, and $\{r_1H_2\}$, $\{r_2H_2\}$ give other two points with $n = 5$, $a = 4$.
To summarize, the fibre $\lambda^{-1}(q_1, q_2)$ consists of 8 points; 4 of them are smooth points of X, 2 of them are singular of type $\frac{1}{5}(1,1)$, and the remain two are singular of type $\frac{1}{5}(1,4)$.

In the Section 4.7, we are going to show how much is useful the Proposition 4.3.3, and how we can use it once one describes product-quotient surfaces via theoretical group data.

4.4 On the invariants of a product-quotient surface of unmixed type

In this Section, we follow the flow of [BP12], and compute the invariants of an unmixed product-quotient surface S of quotient model $X := (C_1 \times C_2)/G$.

Definition 4.4.1. ([BP12, Definition 1.5]) Let x be a singularity of type $\frac{1}{n}(1,a)$ with $\gcd(n,a) = 1$, and let $1 \leq a' < n$ be the inverse of a modulo n, $a' = a^{-1}$. Write $\frac{n}{a}$ as a continued fraction

$$\frac{n}{a} = b_1 - \cfrac{1}{b_2 - \frac{1}{b_3 - \dots}} = [b_1, \dots, b_l]$$

as in the Definition 3.0.16. We define the following correction terms

- $k_x := k(\frac{1}{n}(1,a)) = -2 + \frac{2+a+a'}{n} + \sum_{i=1}^{l}(b_i - 2) \geq 0$;
- $e_x := e(\frac{1}{n}(1,a)) = l + 1 - \frac{1}{n} \geq 0$;
- $B_x := 2e_x + k_x$.

Let $\mathcal{B}$ be the basket of singularities of X (recall X is normal and has at most a finite number of cyclic quotient singularities, by the Proposition 4.3.1). Then we denote by

$$k(\mathcal{B}) := \sum_x k_x, \qquad e(\mathcal{B}) := \sum_x e_x, \qquad B(\mathcal{B}) := \sum_x B_x.$$

Theorem 4.4.2. *Let $\rho\colon S \to X$ be the minimal resolution of singularities of $X = (C_1 \times C_2)/G$. Then the self-intersection of the canonical divisor of S and its Euler characteristic are equal to*

$$K_S^2 = \frac{8(g_1-1)(g_2-1)}{|G|} - k(\mathcal{B}),$$
$$e(S) = \frac{4(g_1-1)(g_2-1)}{|G|} + e(\mathcal{B}).$$

Proof. Apply the Lemma 3.1.3 recursively for any singular point to X:

$$K_S^2 = K_X^2 - \sum_x k_x = K_X^2 - k(\mathcal{B}).$$

However, $\lambda_{12}\colon C_1 \times C_2 \to X$ has finite branch locus, and so $K_{C_1\times C_2} = \lambda_{12}^* K_X$, which implies by projection formula

$$|G|K_X^2 = (p_1^*K_{C_1} + p_2^*K_{C_2})^2 = 2\deg(K_{C_1})\deg(K_{C_2}) = 8(g_1-1)(g_2-1).$$

Let us discuss how to compute $e(S)$. By the Lemma 3.1.9 we get

$$e(S) = e(X^\circ) + \sum_x (l_x + 1),$$

where X° is the smooth locus of X. Here x is a singularity of type $\frac{1}{n_x}(1, a_x)$, and l_x is the length of continued fraction of n_x/a_x.

From the point 3. of Proposition 4.3.1, the singular locus of X is exactly the branch locus of $\lambda_{12}\colon C_1 \times C_2 \to X$, so that $\lambda_{12}\colon Z^\circ \to X^\circ$ is étale, where $Z^\circ := (C_1 \times C_2) \setminus \lambda_{12}^{-1}(\mathrm{Sing}(X))$. Therefore, we have

$$\begin{aligned} e(X^\circ) = \frac{e(Z^\circ)}{|G|} &= \frac{e(C_1)e(C_2)}{|G|} - \frac{e(\lambda_{12}^{-1}(\mathrm{Sing}(X)))}{|G|} \\ &= \frac{4(g_1-1)(g_2-1)}{|G|} - \frac{1}{|G|}\sum_x |\lambda_{12}^{-1}\{x\}| \\ &= \frac{4(g_1-1)(g_2-1)}{|G|} - \sum_x \frac{1}{n_x}. \end{aligned}$$

The latter equality holds since $|\lambda_{12}^{-1}\{x\}| = |G|/|\mathrm{Stab}(p_1, p_2)|$, where (p_1, p_2) is a point over x, and we have $|\mathrm{Stab}(p_1, p_2)| = n_x$, because X is locally isomorphic in a neighbourhood of x to the quotient $V_1 \times V_2/\mathrm{Stab}(p_1, p_2)$ (see the point 1. of Proposition 4.3.1).

It follows that

$$\begin{aligned} e(S) &= e(X^\circ) + \sum_x (l_x + 1) \\ &= \frac{4(g_1 - 1)(g_2 - 1)}{|G|} - \sum_x \frac{1}{n_x} + \sum_x (l_x + 1) \\ &= \frac{4(g_1 - 1)(g_2 - 1)}{|G|} + e(\mathcal{B}). \end{aligned}$$

□

Corollary 4.4.3. *Let $\rho\colon S \to X = (C_1 \times C_2)/G$ be the minimal resolution of singularities of X. Then*

$$K_S^2 = 8\chi(S) - \frac{1}{3}B(\mathcal{B}).$$

Proof. By the Theorem 4.4.2, we have

$$2e(S) = K_S^2 + k(\mathcal{B}) + 2e(\mathcal{B}) \implies e(S) = \frac{K_S^2 + B(\mathcal{B})}{2}.$$

By Noether's formula we obtain

$$12\chi(S) = K_S^2 + e(S) = \frac{3K_S^2 + B(\mathcal{B})}{2} \implies K_S^2 = 8\chi(S) - \frac{B(\mathcal{B})}{3}.$$

□

Corollary 4.4.4.

$$K_S^2 \leq 8\chi(S),$$

and the equality holds if and only if $X = (C_1 \times C_2)/G$ is smooth, or, equivalently, isogenous to a product of curves.

Proof. Observe $B(\mathcal{B}) \geq 0$. Moreover, if the equality holds, then $B(\mathcal{B}) = 0$, and so no singular points occur to X. □

It remains to compute the irregularity q, and the geometric genus p_g of S. The canonical linear system of S is amply studied in the Chapter 5, so we refer that chapter for more details. Briefly speaking, given $\rho\colon X \to S$ be the minimal resolution of the singularities of $X = (C_1 \times C_2)/G$, then $(\rho^{-1})^* : H^{i,0}(S) \to H^{i,0}(X^\circ)$ is a natural monomorphism, X° smooth locus of X. Since X has only cyclic quotient singularities (see Prop. 4.3.1), it is also an epimorphism, by Freitag's theorem [Fre71, Satz 1]. The composition of such map together with the pullback $\lambda_{12}^*\colon H^{i,0}(X^\circ) \to H^{i,0}(C_1 \times C_2)$ shows that $H^{i,0}(S)$ is sent isomorphically to the invariant subspace $H^{i,0}(C_1 \times C_2)^G$. We have obtained

$$H^{i,0}(S) \cong H^{i,0}(C_1 \times C_2)^G. \tag{4.2}$$

Theorem 4.4.5. *Let $\rho\colon S \to X$ be the minimal resolution of singularities of $X = (C_1 \times C_2)/G$. Then the irregularity of S and its geometric genus are equal to*

$$\begin{aligned} q &= g(C_1/G) + g(C_2/G), \\ p_g &= \chi(S) + q - 1 \\ &= \frac{(g_1-1)(g_2-1)}{|G|} + \frac{1}{12}\left(e(\mathcal{B}) - k(\mathcal{B})\right) + g(C_1/G) + g(C_2/G) - 1. \end{aligned}$$

Proof. Apply the above formula (4.2) for $i = 1$, and use the Künneth formula:

$$H^{1,0}(S) \cong H^{1,0}(C_1 \times C_2)^G \cong \left(H^{1,0}(C_1)^G \otimes H^{0,0}(C_2)\right) \oplus \left(H^{0,0}(C_1) \otimes H^{1,0}(C_2)^G\right).$$

This gives $q = h^0(\Omega^1_S) = g(C_1/G) + g(C_2/G)$. Now write $p_g = \chi(S) + q - 1$; the thesis follows by using Noether's formula $12\chi(S) = K_S^2 + e(S)$, and replacing the values of K_S^2 and $e(S)$ as stated by the Theorem 4.4.2. □

Corollary 4.4.6. *A product-quotient surface S is regular, namely $q = 0$, if and only if $C_i/G \cong \mathbb{P}^1$. In other words, the curves C_i are G-coverings of $\mathbb{P}^1$.*

In the Example 4.7.3 of the Section 4.7, we compute the basket of singularities of the quotient model X of certain product-quotient surfaces S, and then we apply the formulas above to perform their invariants.

We shall now list some properties of the basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ of a product-quotient surface.

Lemma 4.4.7. *Let $X = (C_1 \times C_2)/G$ be as above. There exists a representation of the basket*

$$\mathcal{B}(X) = \left\{\lambda_1 \times \frac{1}{n_1}(1, a_1), \ldots, \lambda_R \times \frac{1}{n_R}(1, n_R)\right\}$$

such that

$$\sum \lambda_i \frac{a_i}{n_i} \in \mathbb{Z}.$$

Proof. Consider the fibration $f_1\colon X \to C_1/G$, and let $F_1, \ldots, F_r$ be the singular fibres taken with the reduced structure. Let $\widetilde{F_i}$ be the strict transform of F_i on S. Then, by [Pol10, Proposition 2.8], for a suitable representation of the basket

$$\sum \lambda_i \frac{a_i}{n_i} = -\sum \widetilde{F_i}^2 \in \mathbb{Z}.$$

□

Corollary 4.4.8. *Suppose X admits at most nodes, i.e. every singularity is of type $\frac{1}{2}(1,1)$: Then $2 \leq K_S^2 \leq 8\chi(S)$ is an even number.*

Proof. Let λ be the number of singularities of X. Since any of them is of type $\frac{1}{2}(1,1)$, then Lemma 4.4.7 implies

$$\lambda \cdot \frac{1}{2} \in \mathbb{Z} \implies \lambda \ \text{ even.}$$

Then $e(\mathcal{B}) \in \mathbb{Z}$. This together with $k(\mathcal{B}) = 0$ implies $B(\mathcal{B}) = 2e(\mathcal{B})$ is even, and so $K_S^2 = 8\chi(S) - \frac{1}{3}B(\mathcal{B})$ is even too. $\square$

Definition 4.4.9. A multiset

$$\mathcal{B} := \left\{ \lambda_1 \times \frac{1}{n_1}(1, a_1), \dots, \lambda_R \times \frac{1}{n_R}(1, n_R) \right\}$$

is called *a possible basket of singularities* for (K^2, χ) is and only if satisfies the following conditions:

- there is a representation of $\mathcal{B}$, say

  $$\mathcal{B} := \left\{ \lambda'_1 \times \frac{1}{n'_1}(1, a'_1), \dots, \lambda'_R \times \frac{1}{n'_R}(1, n'_R) \right\}$$

  such that $\sum \lambda'_i \frac{a'_i}{n'_i} \in \mathbb{Z}$,

- $B(\mathcal{B}) = 3(8\chi(S) - K^2)$.

It is clear now that the basket of the quotient model X of a product-quotient S is a possible basket of singularities for the pair $(K_S^2, \chi(S))$.

4.5 Counting product-quotient surfaces arising from a pair of topological types of G-coverings

Consider a regular product-quotient surface S of quotient model $X := (C_1 \times C_2)/G$. By the Corollary 4.4.6, then $C_i/G \cong \mathbb{P}^1$. The situation is the following:

$$\begin{tikzcd}
 & & C_1 \times C_2 \arrow[dll, "p_1"'] \arrow[d, "\lambda_{12}"] \arrow[drr, "p_2"] & & \\
C_1 \arrow[d, "\lambda_1"] & & (C_1 \times C_2)/G \arrow[dll, "f_1"'] \arrow[d, "\lambda"] \arrow[drr, "f_2"] & & C_2 \arrow[d, "\lambda_2"] \\
\mathbb{P}^1 \cong C_1/G & & & & C_2/G \cong \mathbb{P}^1 \\
 & & C_1/G \times C_2/G \cong \mathbb{P}^1 \times \mathbb{P}^1, \arrow[ull] \arrow[urr] & &
\end{tikzcd} \tag{4.3}$$

In the Chapter 2.1 we have seen that $\lambda_i \colon C_i \to \mathbb{P}^1$ are both described, up to deformation, by the $\operatorname{Aut}(G) \times \widetilde{\mathcal{B}_r}$ (resp. $\operatorname{Aut}(G) \times \widetilde{\mathcal{B}_s}$)-orbit of a spherical system of generators of G.
Conversely, a pair of spherical systems of generators $[g_1, \dots, g_r]$, and $[h_1, \dots, h_s]$ gives G-coverings (C_1, λ_1), and (C_2, λ_2) of $\mathbb{P}^1$, and so a family of regular product-quotient surfaces S of quotient model $X := (C_1 \times C_2)/G$. However, if we replace one $[h_1, \dots, h_s]$ with another spherical system of generators in the same $\operatorname{Aut}(G) \times \widetilde{\mathcal{B}_s}$-orbit, then we get a family of product-quotient surfaces that may be different.

In this section we are going to determine exactly how many families of product-quotient surfaces one obtains by a pair of topological types of G-coverings of $\mathbb{P}^1$.

Definition 4.5.1. Let us call by $\mathcal{T}^{r,s}(G)$, the collection of all families of regular product-quotient surfaces, whose natural fibrations λ_i are G-coverings C_i of $\mathbb{P}^1$ branched over r and s points respectively.

Remark 4.5.2. In the above definition the order of (C_1, λ_1) and (C_2, λ_2) is relevant. Thus exchanging them gives a natural bijection from $\mathcal{T}^{r,s}(G) \to \mathcal{T}^{s,r}(G)$ which sends families to isomorphic families of surfaces.

We give a generalization of Theorem 2.1.9 for product-quotient surfaces (see [BP12] and [BCGP12]).

Proposition 4.5.3. *There is a natural bijection among* $\mathcal{T}^{r,s}(G)$ *and*

$$\frac{\mathcal{D}^r(G) \times \mathcal{D}^s(G)}{\operatorname{Aut}(G) \times \widetilde{\mathcal{B}_r} \times \widetilde{\mathcal{B}_s}},$$

where $(\Psi, \sigma_1, \sigma_2) \in \operatorname{Aut}(G) \times \widetilde{\mathcal{B}_r} \times \widetilde{\mathcal{B}_s}$ *acts on* $([g_1, \dots, g_r], [h_1, \dots, h_s])$ *via*

$$\begin{aligned}(\Psi, \sigma_1, \sigma_2) \cdot ([g_1, \dots, g_r], [h_1, \dots, h_s]) &= \\ &= (\sigma_1 \cdot [\Psi(g_1), \dots, \Psi(g_r)], \sigma_2 \cdot [\Psi(h_1), \dots, \Psi(h_s)]),\end{aligned}$$

in the notation of Section 2.1.

The bijection in Proposition 4.5.3 is given by a map $\mathcal{D}^r(G) \times \mathcal{D}^s(G) \to T^{r,s}(G)$ as follows.

Consider a pair of spherical systems of generators $[g_1, \ldots, g_r]$ and $[h_1, \ldots, h_s]$. We fix points $q_0, q_1, \ldots, q_r \in \mathbb{P}^1$ and a geometric basis $\eta_1, \ldots, \eta_r$ as in the Definition 2.1.10, where η_i is a class loop based at q_0 around the point q_i. In this way, following the description of Section 2.1 we get from the first spherical system $[g_1, \ldots, g_r]$ a G-covering of the line (C_1, λ_1) whose branch locus is $\{q_1, \ldots, q_r\}$. In fact, we obtain an isomorphism among G and the automorphism group of the covering (C_1, λ_1), and g_i is the local monodromy of a point over q_i.

Similarly, we fix others $q'_0, q'_1, \ldots, q'_s \in \mathbb{P}^1$, and a geometric basis $\eta'_1, \ldots, \eta'_s$, where η'_i is a class loop based at q'_0 around q'_i. The spherical system $[h_1, \ldots, h_s]$ gives then another G-covering of the line (C_2, λ_2), and an isomorphism among G and the automorphism group of (C_2, λ_2).

Then the diagonal action of G on $C_1 \times C_2$ gives a product-quotient surface S whose quotient model is $(C_1 \times C_2)/G$.

The map $\mathcal{D}^r(G) \times \mathcal{D}^s(G) \to T^{r,s}(G)$ sends the pair of spherical systems $([g_1, \ldots, g_r], [h_1, \ldots, h_s])$ to the family of S.

Let us discuss how $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r} \times \widetilde{\mathcal{B}_s}$ acts on this construction.

We show that acting with $\Psi \in \mathrm{Aut}(G)$ on $[g_1, \ldots, g_r]$ and $[h_1, \ldots, h_s]$ the isomorphic class of S does not change. Acting with Ψ, we obtain the same G-coverings (C_1, λ_1) and (C_2, λ_2), but the isomorphisms among G and the automorphism groups of (C_1, λ_1) and (C_2, λ_2) are both modified by composition with Ψ. Then we obtain the same product $C_1 \times C_2$ and the action of $G \times G$ on it has been modified by composition with $\Psi \times \Psi$. Since $\Psi \times \Psi$ sends the diagonal to itself, then we obtain a surface isomorphic to S.

The group $\widetilde{\mathcal{B}_r}$ acts only on the first spherical system of generators $[g_1, \ldots, g_r]$ replacing (C_1, λ_1) with a topological equivalent G-covering (C'_1, λ'_1) as described in Section 2.1. By the result of González-Díez and Harvey in [GDH92] mentioned there, then (C_1, λ_1) and (C'_1, λ'_1) are in the same irreducible connected family of G-coverings. In particular, the action of $\widetilde{\mathcal{B}_r}$ on the given construction connects surfaces of the same family.

An analogous statement holds for the action of $\widetilde{\mathcal{B}_s}$ on a spherical system of generators $[h_1, \ldots, h_s]$.

As discussed at the beginning of this section, to each family of product-quotient surfaces we have a naturally associated pair of topological types of G-coverings, thus giving a surjective map $\mathcal{T}^{r,s}(G) \twoheadrightarrow \mathcal{T}^r(G) \times \mathcal{T}^s(G)$. By Proposition 4.5.3 and Theorem 2.1.9 we obtain the following commutative diagram

$$\begin{tikzcd} \mathcal{T}^{r,s}(G) \arrow[r, leftrightarrow] \arrow[d, two heads] & \frac{\mathcal{D}^r(G)\times\mathcal{D}^s(G)}{\mathrm{Aut}(G)\times\widetilde{\mathcal{B}_r}\times\widetilde{\mathcal{B}_s}} \arrow[d, dashed, two heads, "\pi"] \\ \mathcal{T}^r(G)\times\mathcal{T}^s(G) \arrow[r, leftrightarrow] & \frac{\mathcal{D}^r(G)}{\mathrm{Aut}(G)\times\widetilde{\mathcal{B}_r}}\times\frac{\mathcal{D}^s(G)}{\mathrm{Aut}(G)\times\widetilde{\mathcal{B}_s}} \end{tikzcd} \tag{4.4}$$

Here the map π is defined as the only map making the diagram commutative. Such π sends the class of a pair of spherical systems of generators $[V_1, V_2]$ to the pair of classes $([V_1], [V_2])$.

We are going to find the inverse image of each point $([V_1], [V_2])$ by π, which translates in determining each family of product-quotient surfaces afforded by the pair of topological types of G-coverings, the first given by $[V_1]$, and the second by $[V_2]$.

Definition 4.5.4. Let $V := [g_1, \dots, g_r]$ be a spherical system of generators. The group of automorphisms of *braid type* on V is the following subgroup of $\mathrm{Aut}(G)$

$$\mathcal{B}\mathrm{Aut}(G, V) := \{\varphi \in \mathrm{Aut}(G) \colon \exists\, \sigma \in \widetilde{\mathcal{B}_r} \text{ such that } \varphi \cdot V = \sigma \cdot V\}.$$

Observe that $\mathcal{B}\mathrm{Aut}(G, V)$ is a subgroup of $\mathrm{Aut}(G)$: let $\varphi_1, \varphi_2 \in \mathcal{B}\mathrm{Aut}(G, V)$, then

$$\left(\varphi_1 \circ \varphi_2^{-1}\right) \cdot V = \varphi_1(\sigma_2^{-1} \cdot V) = \sigma_2^{-1} \cdot (\varphi_1 \cdot V) = (\sigma_2^{-1}\sigma_1) \cdot V$$

for some $\sigma_1, \sigma_2 \in \widetilde{\mathcal{B}_r}$. Thus $\varphi_1 \circ \varphi_2^{-1} \in \mathcal{B}\mathrm{Aut}(G, V)$.

Remark 4.5.5. If you replace V by V' on its $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$-orbit, let us say $V' := (\Psi, \sigma)\cdot V$, then the subgroup $\mathcal{B}\mathrm{Aut}(G, V')$ is conjugate to $\mathcal{B}\mathrm{Aut}(G, V)$:

$$\mathcal{B}\mathrm{Aut}(G, V') = \Psi \circ \mathcal{B}\mathrm{Aut}(G, V) \circ \Psi^{-1}.$$

Note that $\Psi \in \mathcal{B}\mathrm{Aut}(G, V)$ implies $\mathcal{B}\mathrm{Aut}(G, V') = \mathcal{B}\mathrm{Aut}(G, V)$.

Definition 4.5.6. Let V_1 and V_2 be a pair of spherical systems of generators of G. We will say that two automorphisms $\Phi, \Psi \in \mathrm{Aut}(G)$ are $(V_1, V_2)-$related, and we will write

$$\Phi \sim_{V_1,V_2} \Psi$$

if the following holds: there exist $\varphi_1 \in \mathcal{B}\mathrm{Aut}(G, V_1), \varphi_2 \in \mathcal{B}\mathrm{Aut}(G, V_2)$ such that

$$\Psi = \varphi_1 \circ \Phi \circ \varphi_2.$$

The relation $\sim_{V_1,V_2}$ is clearly an equivalence relation on $\mathrm{Aut}(G)$. We denote by $Q\mathrm{Aut}(G)_{V_1,V_2}$ the quotient of $\mathrm{Aut}(G)$ by $\sim_{V_1,V_2}$.

In other words $Q\mathrm{Aut}(G)_{V_1,V_2}$ is the set of double cosets

$$Q\mathrm{Aut}(G)_{V_1,V_2} = \mathcal{B}\mathrm{Aut}(G,V_1)\backslash\mathrm{Aut}(G)/\mathcal{B}\mathrm{Aut}(G,V_2).$$

Remark 4.5.7. Replacing V_1 and V_2 by two spherical systems of generators in the same orbits, $V_1' = (\Psi_1,\sigma_1)\cdot V_1$ and $V_2' = (\Psi_2,\sigma_2)\cdot V_2$, then by the Remark 4.5.5 we have

$$\Phi \sim_{V_1,V_2} \Psi \iff \Psi_1\circ\Phi\circ\Psi_2^{-1} \sim_{V_1',V_2'} \Psi_1\circ\Psi\circ\Psi_2^{-1}.$$

Moreover, the bijection $\Phi\mapsto\Psi_1\circ\Phi\circ\Psi_2^{-1}$ induces a bijection among the quotients

$$Q\mathrm{Aut}(G)_{V_1,V_2} \leftrightarrow Q\mathrm{Aut}(G)_{V_1',V_2'}, \qquad [\Phi]\mapsto[\Psi_1\circ\Phi\circ\Psi_2^{-1}]. \tag{4.5}$$

that only depends on $V_1, V_2, V_1'.V_2'$ and not on the choice of Ψ_1,Ψ_2.

Theorem 4.5.8. *We consider the map π defined at (4.4). Let us fix a point $x\in\frac{\mathcal{D}^r(G)}{Aut(G)\times\widetilde{\mathcal{B}_r}}\times\frac{\mathcal{D}^s(G)}{Aut(G)\times\widetilde{\mathcal{B}_s}}$, and let us choose a pair of spherical systems of generators V_1 and V_2 such that $x=([V_1],[V_2])$. The following hold:*

1. *given $\Phi\in\mathrm{Aut}(G)$, then*
$$[V_1,\Phi\cdot V_2]\in\frac{\mathcal{D}^r(G)\times\mathcal{D}^s(G)}{\mathrm{Aut}(G)\times\widetilde{\mathcal{B}_r}\times\widetilde{\mathcal{B}_s}}$$
depends only by class of Φ in $Q\mathrm{Aut}(G)_{V_1,V_2}$.

2. *The map*
$$\begin{aligned} Q\mathrm{Aut}(G)_{V_1,V_2} &\longrightarrow \pi^{-1}(x)\\ [\Phi] &\mapsto [V_1,\Phi\cdot V_2]. \end{aligned} \tag{4.6}$$
is bijective. In particular, $|\pi^{-1}(x)| = |Q\mathrm{Aut}(G)_{V_1,V_2}|$.

3. *If we replace V_1 by V_1' in the same $\mathrm{Aut}(G)\times\widetilde{\mathcal{B}_r}$-orbit, and V_2 by V_2' in the same $\mathrm{Aut}(G)\times\widetilde{\mathcal{B}_s}$-orbit, then the bijective maps in (4.5) and (4.6) form a commutative triangle*

$$\begin{tikzcd} Q\mathrm{Aut}(G)_{V_1',V_2'} \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow] & \\ & \pi^{-1}(x) \\ Q\mathrm{Aut}(G)_{V_1,V_2} \arrow[ur, leftrightarrow] & \end{tikzcd}$$

Proof. 1. Let us consider an automorphism $\Phi' = \varphi_1 \circ \Phi \circ \varphi_2$ in the same class of Φ in $Q\mathrm{Aut}(G)_{V_1,V_2}$, where $\varphi_1 \in \mathcal{B}\mathrm{Aut}(G, V_1)$ and $\varphi_2 \in \mathcal{B}\mathrm{Aut}(G, V_2)$. Then

$$\begin{aligned}[V_1, \Phi' \cdot V_2] &= [V_1, (\varphi_1 \circ \Phi \circ \varphi_2)\, V_2] \\ &= [\varphi_1^{-1} \cdot V_1, (\Phi \circ \varphi_2) \cdot V_2] \\ &= [\sigma_1^{-1} \cdot V_1, \Phi \cdot (\sigma_2 \cdot V_2)] \\ &= [\sigma_1^{-1} \cdot V_1, \sigma_2 \cdot (\Phi \cdot V_2)] = [V_1, \Phi \cdot V_2].\end{aligned}$$

2. Point 1. proves that the map 4.6 is well-defined. Let us consider an element $[V_1', V_2'] \in \pi^{-1}(x)$, hence V_1' is in the same orbit of V_1 and V_2' is in the same orbit of V_2. We write

$$V_1' = (\Psi_1, \sigma_1) \cdot V_1 \qquad \text{and} \qquad V_2' = (\Psi_2, \sigma_2) \cdot V_2,$$

where $(\Psi_1, \sigma_1) \in \mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r}$, and $(\Psi_2, \sigma_2) \in \mathrm{Aut}(G) \times \widetilde{\mathcal{B}_s}$. Then

$$[V_1', V_2'] = [\Psi_1 \cdot V_1, \Psi_2 \cdot V_2] = [V_1, \left(\Psi_1^{-1} \cdot \Psi_2\right) \cdot V_2].$$

This proves (4.6) is surjective.

Let us consider $[\Phi_1]$ and $[\Phi_2]$ in $Q\mathrm{Aut}(G)_{V_1,V_2}$ such that

$$[V_1, \Phi_2 \cdot V_2] = [V_1, \Phi_1 \cdot V_2].$$

We are going to show that $[\Phi_2] = [\Phi_1]$. Since $(V_1, \Phi_2 \cdot V_2)$ and $(V_1, \Phi_1 \cdot V_2)$ share the same orbit, then there exists $(\Psi, \sigma_1, \sigma_2) \in \mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r} \times \widetilde{\mathcal{B}_s}$ such that

$$(V_1, \Phi_2 \cdot V_2) = (\Psi, \sigma_1, \sigma_2) \cdot (V_1, \Phi_1 \cdot V_2) \implies \begin{cases} \Psi \cdot V_1 = \sigma_1^{-1} \cdot V_1 \\ \left(\Phi_1^{-1} \circ \Psi^{-1} \circ \Phi_2\right) \cdot V_2 = \sigma_2 \cdot V_2. \end{cases}$$

Therefore, $\varphi_1 := \Psi \in \mathcal{B}\mathrm{Aut}(G, V_1)$ and $\varphi_2 := \Phi_1^{-1} \circ \Psi^{-1} \circ \Phi_2 \in \mathcal{B}\mathrm{Aut}(G, V_2)$. Finally, we have

$$\Phi_2 = \varphi_1 \circ \Phi_1 \circ \varphi_2,$$

which proves $[\Phi_2] = [\Phi_1]$, and so that (4.6) is injective.

3. It is an immediate consequence from the definition of the map (4.5). □

Theorem 4.5.8 gives a perfect enumeration of the families of regular product-quotient surfaces corresponding to an **ordered** pair of topological types of G-coverings of the projective line. In fact, in the Remark 4.5.2 we have observed that exchanging (C_1, λ_1) and (C_2, λ_2) defines an involution on $\bigcup \mathcal{T}^{r,s}(G)$ connecting isomorphic families.

If we are interested in counting the families given by two different topological types of G-coverings, then it is sufficient to choose an order of them and then apply Theorem 4.5.8.
However, to enumerate the families of product-quotient surfaces associated to twice the same topological type we need to study how the exchange of the factors acts on $Q\mathrm{Aut}(G)_{V,V}$.

Proposition 4.5.9. *The exchange of the factors acts on $Q\mathrm{Aut}(G)_{V,V}$ as the involution*

$$Q\mathrm{Aut}(G)_{V,V} \to Q\mathrm{Aut}(G)_{V,V}, \qquad [\Phi] \mapsto [\Phi^{-1}].$$

Proof. The exchange of the factors is a map from $\pi^{-1}([V],[V])$ to itself sending each $[V, \Phi \cdot V]$ to $[\Phi \cdot V, V] = [V, \Phi^{-1} \cdot V]$. □

Corollary 4.5.10. *Let (C_1, λ_1) and (C_2, λ_2) be two G-coverings of $\mathbb{P}^1$ and let V_1 and V_2 be their spherical systems of generators respectively. Then the cardinality of the set of families of product-quotient surfaces given by (C_1, λ_1) and (C_2, λ_2) is equal to*

1. *the cardinality of $Q\mathrm{Aut}(G)_{V_1,V_2}$, if (C_1, λ_1) and (C_2, λ_2) are not topological equivalent;*

2. *the cardinality of $Q\mathrm{Aut}(G)_{V_1,V_1}/\left(\Phi \mapsto \Phi^{-1}\right)$, if (C_1, λ_1) and (C_2, λ_2) are topological equivalent.*

Let us give an example:

Example 4.5.11. Let $G = S_3 \times \mathbb{Z}_p^2$, $p \geq 3$ prime number.

We are going to compute all regular product-quotient surfaces with quotient model $(C_1 \times C_2)/G$ where the natural fibrations $\lambda_1 \colon C_1 \to \mathbb{P}^1$ and $\lambda_2 \colon C_2 \to \mathbb{P}^1$ are both ramifying over three points.

From Example 2.1.14 we can say that

$$\frac{\mathcal{D}^3(S_3 \times \mathbb{Z}_p^2)}{\mathrm{Aut}(S_3 \times \mathbb{Z}_p^2) \times \widetilde{\mathcal{B}_3}} = \{V\},$$

with

$$V := [(\tau, (1,0)), (\tau\sigma, (0,1)), (\sigma^2, (p-1, p-1))].$$

We need to compute the subgroup $\mathcal{B}\mathrm{Aut}(G, V) \leq \mathrm{Aut}(S_3 \times \mathbb{Z}_p^2)$.

Firstly we note that, since we have assumed $p \geq 3$, then

$$\mathrm{Aut}(S_3 \times \mathbb{Z}_p^2) \cong \mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_p).$$

In fact, any automorphism of $\mathrm{Aut}(S_3 \times \mathbb{Z}_p^2)$ preserves the factors. This is obvious for the second factor since it is the centre of the group. For the first factor, we note that it is generated by the only elements of order two of the group, and then it is a characteristic subgroup.

Hence every element of $\mathcal{B}\mathrm{Aut}(G, V)$ can be written as a pair (Ψ, M), where $\Psi \in \mathrm{Aut}(S_3)$, and $M \in \mathrm{GL}_2(\mathbb{Z}_p)$.

The action of $\widetilde{\mathcal{B}_3}$ on $[(1,0),(0,1),(p-1,p-1)]$ permutes its entries, since $\mathbb{Z}_p^2$ is abelian. Therefore, the automorphisms $M \in \mathrm{GL}_2(\mathbb{Z}_p)$ of braid type on it are those permuting its entries. Such automorphisms belong to the subgroup $\langle M_1, M_2 \rangle \cong S_3$ generated by

$$M_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad M_2 := \begin{pmatrix} p-1 & 0 \\ p-1 & 1 \end{pmatrix}.$$

Let (Ψ, M) be of braid type on V, and let η be a braid in $\widetilde{\mathcal{B}_3}$ such that $(\Psi, M) \cdot V = \eta \cdot V$. We observe that the signature of V is $[2p, 2p, 3p]$: since the third number is different from the other two, and the automorphisms send elements in elements of the same order, then the permutation image of η in S_3 fix the number three. This implies that M fixes $(p-1, p-1)$, so $M \in \langle M_1 \rangle \cong \mathbb{Z}_2$. Therefore,

$$\mathcal{B}\mathrm{Aut}(G, V) \leq \mathrm{Aut}(S_3) \times \langle M_1 \rangle \cong S_3 \times \mathbb{Z}_2.$$

Let us choose two generators of $\mathrm{Aut}(S_3)$: let Ψ_1 be the inner automorphism given by τ and let Ψ_2 be the inner automorphism of σ^2. They act as

$$\Psi_1 = \begin{pmatrix} \tau \mapsto \tau \\ \sigma \mapsto \sigma^2 \end{pmatrix}, \quad \Psi_2 = \begin{pmatrix} \tau \mapsto \tau\sigma^2 \\ \sigma \mapsto \sigma^2 \end{pmatrix}$$

We observe that (Ψ_1, Id) and $(\Psi_2 \circ \Psi_1, M_1)$ are of braid type on V, since they act on V respectively as the braids $\sigma_1\sigma_2^2\sigma_1$ and σ_1. Since they generate the whole $\mathrm{Aut}(S_3) \times \langle M_1 \rangle$ then

$$\mathcal{B}\mathrm{Aut}(G, V) = \mathrm{Aut}(S_3) \times \langle M_1 \rangle \cong S_3 \times \mathbb{Z}_2.$$

Now we can compute

$$Q\mathrm{Aut}(S_3 \times \mathbb{Z}_p^2)_{V,V}.$$

Firstly, we observe that by definition of $\sim_{V,V}$ we have the following natural identification

$$Q\mathrm{Aut}(S_3 \times \mathbb{Z}_p^2)_{V,V} = {}_{\mathcal{B}\mathrm{Aut}(G,V)} \backslash^{(\mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_p))} /_{\mathcal{B}\mathrm{Aut}(G,V)}.$$

Since $\mathcal{B}\mathrm{Aut}(G, V) = \mathrm{Aut}(S_3) \times \langle M_1 \rangle$ contains the subgroup $\mathrm{Aut}(S_3) \times \{1\}$, which is normal in $\mathrm{Aut}(S_3) \times \mathrm{GL}_2(\mathbb{Z}_p)$, then we have the following natural identification

$$Q\mathrm{Aut}(S_3 \times \mathbb{Z}_p^2)_{V,V} \cong {}_{\langle M_1 \rangle} \backslash^{\mathrm{GL}_2(\mathbb{Z}_p)} /_{\langle M_1 \rangle}. \tag{4.7}$$

More precisely, the correspondence sends $[(\mathrm{Id}_{S_3}, A)] \leftrightarrow [A]$.

From (4.4) and Theorem 4.5.8 we can conclude that

$$\mathcal{T}^{3,3}(S_3 \times \mathbb{Z}_p^2) \cong Q\mathrm{Aut}(G)_{V,V} \cong {}_{\langle M_1 \rangle}\backslash^{\mathrm{GL}_2(\mathbb{Z}_p)}/_{\langle M_1 \rangle}.$$

However, we are majorly interested to find the set of families of product-quotient surfaces given by V, V. As proved in the Corollary 4.5.10, it is sufficient to determine

$$Q\mathrm{Aut}(G)_{V_1,V_1} / \left(\Phi \mapsto \Phi^{-1}\right).$$

This is the quotient of $\mathrm{GL}_2(\mathbb{Z}_p)$ by the simultaneous action of the three involutions $A \mapsto M_1 A$, $A \mapsto AM_1$ and $A \mapsto A^{-1}$. These involutions generate a group of order 8 isomorphic to a dihedral group. Hence

$$Q\mathrm{Aut}(G)_{V_1,V_1} / \left(\Phi \mapsto \Phi^{-1}\right) \cong \mathrm{GL}_2(\mathbb{Z}_p)/D_4. \tag{4.8}$$

We have proved that regular product-quotient surfaces with quotient model $(C_1 \times C_2)/G$ where the natural fibrations $\lambda_1 \colon C_1 \to \mathbb{P}^1$ and $\lambda_2 \colon C_2 \to \mathbb{P}^1$ are both ramifying over three points are in bijection with $\mathrm{GL}_2(\mathbb{Z}_p)/D_4$.

This bijection can be described as follows. Consider the Riemann surface C_1 with an action $\phi\colon G \to \mathrm{Aut}(C_1)$ such that the quotient $C_1/G \cong \mathbb{P}^1$ and the associated spherical system is V. Considering a matrix $A \in \mathrm{GL}_2(\mathbb{Z}_p)$ and let C_2 be a copy of C_1 with $G = S_3 \times \mathbb{Z}_p^2$ acting by $\phi_A := \phi \circ (\mathrm{Id}, A)^{-1}$. Then the product-quotient surface of quotient model $C_1 \times C_2/G$ corresponds to $[A]$.

We count these product-quotient surfaces, computing the cardinality of the right term of (4.8).

Notice that there two involutions $A \mapsto M_1 A$ and $A \mapsto AM_1$ of D_4 that obviously don't fix any matrix. These are suitable reflections of the group D_4. Hence the non-trivial inertia subgroups of D_4 are subgroups not containing these reflections. These are five subgroups:

1. $\langle A \mapsto A^{-1} \rangle \cong \mathbb{Z}_2$;
2. $\langle A \mapsto M_1 A^{-1} M_1 \rangle \cong \mathbb{Z}_2$;
3. the centre $\langle A \mapsto M_1 A M_1 \rangle \cong \mathbb{Z}_2$;
4. $\langle A \mapsto A^{-1}, A \mapsto M_1 A M_1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$;
5. $\langle A \mapsto A^{-1} M_1 \rangle \cong \mathbb{Z}_4$.

First we note that the last three groups are those containing the centre. Then the matrices with inertia group equal to one of the last three cases are those stabilized by the centre.

Hence we first compute the elements stabilized by the centre. They are the doubly symmetric matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \qquad a \neq \pm b.$$

They are in total $(p-1)^2$.

Recall that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The elements stabilized by $A \mapsto A^{-1}$ are of two types, those of determinant 1, and those of determinant -1. Those of determinant 1 are two, $\pm I$. They are double symmetric, hence their inertia group is the one isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Conversely, if a matrix A with determinant -1 has inertia group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $M_1 A$ has determinant 1, and the same inertia group. This implies $\{\pm I, \pm M_1\}$ are the only matrices with inertia group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Instead, the matrices having inertia group $\langle A \mapsto A^{-1} \rangle$ are

$$\left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = 1 \right\} \setminus \{\pm M_1\}.$$

They are in total $p(p+1) - 2 = (p-1)(p+2)$.

Furthermore, the set of matrices stabilized by $\langle A \mapsto M_1 A^{-1} M_1 \rangle$ are in bijection with the set of matrices stabilized by $\langle A \mapsto A^{-1} \rangle$ via multiplication of M_1. Therefore they are in total $(p-1)(p+2)$ again.

Recall that $\mathrm{GL}_2(\mathbb{Z}_p)$ has $(p^2-1)(p^2-p)$ elements. The number of matrices with trivial stabilizer is

$$(p^2-1)(p^2-p) - (p-1)^2 - 2(p-1)(p+2) = (p+1)(p-1)(p^2-p-3).$$

It remains to determine the matrices that have inertia group isomorphic to $\mathbb{Z}_4$. We will see that they exist if and only if

$$p \equiv 1 \mod 4 \qquad \text{and} \qquad (-4)^{\frac{p-1}{4}} \equiv 1 \mod p. \tag{4.9}$$

As remarked above, they are also stabilized by the centre, hence are double symmetric. A double symmetric matrix $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is equal to $A^{-1} M_1 = \frac{1}{a^2-b^2}\begin{pmatrix} -b & a \\ a & -b \end{pmatrix}$ if and only if

$$\begin{cases} a = \frac{-b}{a^2-b^2} \\ b = \frac{a}{a^2-b^2} \end{cases}$$

However, the pair (a, b) can never be the zero vector, and so we get $(a^2 - b^2)^2 = -1$. Setting $\delta = a^2 - b^2$, then we obtain $\delta^2 = -1$, which implies

$p \equiv 1 \mod 4$, and $a = \delta b$. Note that $\delta = \det A = a^2 - b^2 = -2b^2$, hence $\delta^2 = -1 = 4b^4$. This means b is a root of the equation $4x^4 = -1$ that has solutions if and only if $(-4)^{\frac{p-1}{4}} \equiv 1 \mod p$ (from primitive element theorem).

Finally, replacing a by $\delta b = -2b^3$ in the matrix A, we get

$$A = b \begin{pmatrix} -2b^2 & 1 \\ 1 & -2b^2 \end{pmatrix}.$$

Assume (4.9) does not hold. In this case, the cardinality of $\mathrm{GL}_2(\mathbb{Z}_p)/D_4$ is equal to

$$\frac{(p+1)(p-1)(p^2-p-3)}{8} + 2\frac{(p-1)(p+2)}{4} + \frac{(p-1)^2-4}{4} + 2 = \\ = \frac{(p-1)(p+1)(p^2-p+3)}{8} + 1.$$

Instead, if p verifies (4.9), then thew cardinality of $\mathrm{GL}_2(\mathbb{Z}_p)/D_4$ is equal to

$$\frac{(p+1)(p-1)(p^2-p-3)}{8} + 2\frac{(p-1)(p+2)}{4} + \frac{(p-1)^2-4-4}{4} + 2 + \frac{4}{2} = \\ = \frac{(p-1)(p+1)(p^2-p+3)}{8} + 2.$$

4.6 Finiteness of the classification problem

This section is a continuation to the Section 4.4, and generalize the results of [BP12, Sec 1.1] by removing the assumption $\chi = 1$ there and following step-by-step the same arguments. Fixed a pair $(K^2, \chi) \in \mathbb{Z} \times \mathbb{Z}$, we shall classify regular product-quotient surfaces S of general type having such prescribed self-intersection $K_S^2 = K^2$, and Euler characteristic $\chi(S) = \chi$.
With the spirit to obtain good candidates to investigate the main Question 5.2 of Chapter 1, we are going to consider the case $\chi = 4$ later on. In other words, in virtue of the Corollary 4.4.4, we are looking for families of smooth regular product-quotient surfaces of general type with geometric genus p_g equal to three.

The first Lemma shows that, for every pair $(K^2, \chi) \in \mathbb{Z} \times \mathbb{Z}$, there are only finitely many possible baskets of singularities.

Lemma 4.6.1. *Let $C \in \mathbb{Q}$ be fixed. Then there are finitely many baskets $\mathcal{B}$ such that $B(\mathcal{B}) = C$.*
More precisely, we have

i. $|\mathcal{B}| \leq \frac{C}{3}$;

ii. *if* $\lambda \times \frac{1}{n}(1, a) \in \mathcal{B}$, *and* $\frac{n}{a} = [b_1, \ldots, b_l]$, *then* $\lambda \sum b_i \leq C$.

Proof. Observe first that $B(\frac{1}{n}(1,a)) = \frac{a+a'}{n} + \sum b_i \geq 3$. In particular,

$$C = B(\mathcal{B}) \geq 3|\mathcal{B}|,$$

which shows i. The point ii. is obvious. $\square$

Now we consider regular product-quotient surfaces S of general type with fixed $\chi(S) = \chi \in \mathbb{Z}$. Let $\lambda_i \colon C_i \to \mathbb{P}^1$, $i = 1, 2$, be two G-covers associated to S, with genus $g(C_i) \geq 2$ respectively. These covers induce appropriate spherical systems of G

$$[g_1, \dots, g_r], \qquad [h_1, \dots, h_s],$$

with signature $[m_1, \dots, m_r]$, and $[n_1, \dots, n_s]$ respectively, such that the Riemann-Hurwitz relation (2.3) holds.

We need the following

Definition 4.6.2. Fix an r-tuple of natural numbers $t := [m_1, \dots, m_r]$, and a basket of singularities $\mathcal{B}$. Then we associate to these the following numbers:

$$\Theta(t) := -2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right);$$

$$\alpha(t, \mathcal{B}) := \frac{12\chi + k(\mathcal{B}) - e(\mathcal{B})}{6\Theta(t)}.$$

We recall

Definition 4.6.3. The minimal positive integer I_x such that $I_x K_X$ is Cartier in x is called the *index* of the singularity x.

The index of X is the minimal positive integer I such that I is Cartier. In particular, $I = \mathrm{lcm}_{x \in \mathrm{Sing} X} I_x$.

It is well known (see for instance [Mat02, Thm. 4.6.20]) that the index of a cyclic quotient singularity $\frac{1}{n}(1,a)$ is

$$I_x = \frac{n}{\gcd(n, a+1)}.$$

We shall bound now, for fixed K^2, χ, and $\mathcal{B}$, the possibilities for

- $|G|$;
- $t_1 := [m_1, \dots, m_r]$,
- $t_2 := [n_1, \dots, n_s]$,

of a product-quotient surface S with $K_S^2 = K^2$, $\chi(S) = \chi$, and basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ equal to $\mathcal{B}$.

Proposition 4.6.4. [BP12, compare Prop. 1.14] *Fix $(K^2, \chi) \in \mathbb{Z} \times \mathbb{Z}$, and fix a possible basket of singularities $\mathcal{B}$ for (K^2, χ). Let S be a product-quotient surface S of general type such that*

i. $K_S^2 = K^2$;

ii. $\chi(S) = \chi$;

iii. the basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ equals $\mathcal{B}$.

Then

a) $g(C_1) = \alpha(t_2, \mathcal{B}) + 1$, $g(C_2) = \alpha(t_1, \mathcal{B}) + 1$;

b) $|G| = \frac{8\alpha(t_1,\mathcal{B})\alpha(t_2,\mathcal{B})}{K^2+k(\mathcal{B})}$;

c) $r, s \leq \frac{K^2+k(\mathcal{B})}{2} + 4$;

d) m_i *divides* $2\alpha(t_1, \mathcal{B})I$, n_j *divides* $2\alpha(t_2, \mathcal{B})I$;

e) there are at most $|\mathcal{B}|/2$ indices i such that m_i does not divide $\alpha(t_1, \mathcal{B})$, and similarly for the n_j;

f) $m_i \leq \frac{1+I\frac{K^2+k(\mathcal{B})}{2}}{f(t_1)}$, $n_i \leq \frac{1+I\frac{K^2+k(\mathcal{B})}{2}}{f(t_2)}$, *where I is the index of X, and* $f(t_1) := \max(\frac{1}{6}, \frac{r-3}{2})$, $f(t_2) := \max(\frac{1}{6}, \frac{s-3}{2})$;

g) except for at most $|\mathcal{B}|/2$ indices i, the sharper inequality $m_i \leq \frac{1+\frac{K^2+k(\mathcal{B})}{4}}{f(t_1)}$ holds, and similarly for the n_j.

Remark 4.6.5. Note that *b*) shows t_1 and t_2 determines the order of G. *c*) and *f*) implies there are only finitely many possibilities for the types t_1, t_2. Instead, *d*), *e*), and *g*) are strictly necessary to obtain an efficient algorithm.

Proof. *a*) Observe that from the Corollary 4.4.3, then

$$\begin{aligned}\Theta(t_1)\alpha(t_1, \mathcal{B}) =& \frac{1}{2}\frac{24\chi + 2k(\mathcal{B}) - 2e(\mathcal{B})}{6} = \frac{24\chi - B(\mathcal{B}) + 3k(\mathcal{B})}{6} \\ &= 3\frac{8\chi - \frac{B(\mathcal{B})}{3} + k(\mathcal{B})}{12} = \frac{K^2 + k(\mathcal{B})}{4},\end{aligned}$$

and then by the Theorem 4.4.2 and Hurwitz' formula, we have

$$\alpha(t_1, \mathcal{B}) = \frac{K^2 + k(\mathcal{B})}{4\Theta(t_1)} = \frac{8(g(C_1)-1)(g(C_2)-1)}{4|G|\left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right)\right)} = \frac{8(g(C_1)-1)(g(C_2)-1)}{4(2g(C_1)-2)}.$$

b)
$$|G| = \frac{8(g(C_1)-1)(g(C_2)-1)}{K^2+k(\mathcal{B})} = \frac{8\alpha(t_1,\mathcal{B})\alpha(t_2,\mathcal{B})}{K^2+k(\mathcal{B})}.$$

c) Note that $r \le 2\Theta(t_1)+4$. On the other hand, since $g(C_j) \ge 2$, then $1 \le \alpha(t_j,\mathcal{B}) = \frac{K^2+k(\mathcal{B})}{4\Theta(t_1)}$. This implies $(0 \le)\ \Theta(t_i) \le \frac{K^2+k(\mathcal{B})}{4}$.

d) Each m_i is the branching index of a branch point p_i of $\lambda_1\colon C_1 \to C_1/G \cong \mathbb{P}^1$. Let F_i be the fibre of the map $f_1\colon X \to C_1/G$. Then $F_i = m_iW_i$ for some irreducible Weil divisor W_i. We have

$$2\alpha(t_1,\mathcal{B}) = 2g(C_2)-2 = K_XF_i = m_iK_XW_i.$$

Therefore
$$\frac{2\alpha(t_1,\mathcal{B})I}{m_i} = (IK_X)W_i \in \mathbb{Z}.$$

e) By [Ser96], if F_i contains a singular point of X, then it contains at least 2 singular points. Therefore, there are at most $|\mathcal{B}|/2$ indices i, $1 \le i \le r$, such that $F_i \cap \mathrm{Sing}X \neq \emptyset$.
For all other indices j we have $F_j \cap \mathrm{Sing}X = \emptyset$. Then W_j is Cartier and K_X is Cartier in a neighbourhood of W_j. In particular, $\frac{\alpha(t_1,\mathcal{B})}{m_i} = \frac{K_XW_j}{2} \in \mathbb{Z}$.

f) Note that $\Theta(t_1)+\frac{1}{m_i} \ge \frac{r-3}{2}$. Moreover, $\Theta(t_i) > 0$ implies that $r \ge 3$. Obviously, if $r=3$, since $\Theta(2,2,m) = -\frac{1}{m} < 0$, then $\Theta(t_1)+\frac{1}{m_i} \ge \frac{1}{6}$. Therefore, $\Theta(t_1)+\frac{1}{m_i} \ge f(t_1)$, whence $m_i \le \frac{1+\Theta(t_1)m_i}{f(t_1)}$.

By *d)* $m_i \le 2\alpha(t_1,\mathcal{B})I = \frac{K^2+k(\mathcal{B})}{2\Theta(t_1)}I$. This implies

$$m_i \le \frac{1+\Theta(t_1)m_i}{f(t_1)} \le \frac{1+\Theta(t_1)\frac{K^2+k(\mathcal{B})}{2\Theta(t_1)}I}{f(t_1)} \le \frac{1+\frac{K^2+k(\mathcal{B})}{2}I}{f(t_1)}.$$

g) This is proved by the same argument as in *f)*, using *e)* instead of *d)*. $\square$

4.7 How to read the basket $\mathcal{B}$ from the local monodromies

This section is directly related to the Section 4.3. Our next goal is to describe explicitly how the couple of spherical systems

$$[g_1,\dots,g_r] \qquad \text{and} \qquad [h_1,\dots,h_s]$$

associated respectively to $G-$coverings (C_1,λ_1) and (C_2,λ_2) determines the singularities of the quotient model $X = (C_1\times C_2)/G$.
Let us say λ_1 has branch locus consisting of r-points $q_1,\dots,q_r$. In the Subsection 2.1.1 we have proved that g_i is the local monodromy of a point p_i over q_i; let us call the stabilizer of p_i by $H_i := \langle g_i\rangle$. Similarly, we say that

$q'_1, \ldots, q'_s$ is the branch locus of λ_2, and then h_j is the local monodromy of a point p'_j over q'_j; denote by $H'_j := \langle h_j \rangle$ be the stabilizer of p'_j.
From the Proposition 4.3.1, then the only points of X that may be singular are those belonging to a fibre $\lambda^{-1}(q_i, q'_j)$, with $1 \leq i \leq r, 1 \leq j \leq s$. The Proposition 4.3.2 shows that

$$\left(G/H'_j\right)/H_i \to \lambda^{-1}(q_i, q'_j), \qquad [t] \mapsto G(p_i, tp'_j)$$

is a bijection, and the Proposition 4.3.3 establishes the type of singularity of the point $G(p_i, tp'_j)$ for any fixed $[t]$: let δ be the minimal positive number such that there exists $1 \leq \gamma \leq o(h_j)$ with $g_i^\delta = th_j^\gamma t^{-1}$. Then $G(p_i, tp'_j)$ is a cyclic quotient singularity of type $\frac{1}{n}(1, a)$, with $n = n(i, j, t) := |H_i \cap tH'_j t^{-1}|$, and $a = a(i, j, t) := \frac{n\gamma}{o(h_j)}$.
To summarize, for any

- $1 \leq i \leq r$,
- $1 \leq j \leq s$,
- $[t] \in \left(G/H'_j\right)/H_i$,

we apply recursively the Proposition 4.3.3 to establish the type of singularity of the point $G(p_i, tp'_j)$.
The set of singular points of X is therefore equal to

$$\mathrm{Sing}(X) = \{G(p_i, tp'_j) : n(i, j, t) \neq 1, 1 \leq i \leq r, 1 \leq j \leq s, [t] \in (G/H'_j)/H_i\},$$

and a representation of the basket of singularities of X is then

$$\mathcal{B}(X) = \{\frac{1}{n(i,j,t)}(1, a(i,j,t)) \colon n(i,j,t) \neq 1, \\ 1 \leq i \leq r, 1 \leq j \leq s, [t] \in (G/H'_j)/H_i\}.$$

Remark 4.7.1. Note that $a = a(i, j, t)$ and $n = n(i, j, t)$ may be not coprime. Define by $d = d(i, j, t) := \gcd(n, a)$. Then $\frac{n}{d}$ and $\frac{a}{d}$ are coprime, and by the claim of the Lemma 3.0.13, then $\frac{1}{n}(1, a)$ is equivalent to $\frac{1}{\frac{n}{d}}(1, \frac{a}{d})$.

Remark 4.7.2. By the Remark 4.3.4, then we conclude that the basket $\mathcal{B}(X)$ of a quotient model X described by a couple of spherical systems $[g_1, \ldots, g_r]$, and $[h_1, \ldots, h_s]$ depends only by the conjugacy classes of g_i and h_j.

To better understand how to compute the basket of singularities of X via the Proposition 4.3.3 let us give an example.

Example 4.7.3. We are going to study the basket of singularities of some of the product-quotient surfaces obtained in the Example 4.5.11. Here we assume for simplicity $p \geq 5$. Let us consider the same spherical system V of 4.5.11:

$$V = [(\tau, e_1), (\tau\sigma, e_2), (\sigma^2, (p-1)(e_1 + e_2))] =: [g_1, g_2, g_3],$$

where e_1, e_2 is the standard basis of $\mathbb{Z}_p^2$ for simplicity of notation. Let $A \in \mathrm{GL}_2(\mathbb{Z}_p)$ be an automorphism such that

$$Ae_1,\ Ae_2,\ A(e_1 + e_2) \notin \langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_1 + e_2 \rangle.$$

We twist V by A, and so consider the new spherical system $A \cdot V =: [h_1, h_2, h_3]$.

The couple of spherical systems V and $A \cdot V$ describes a product-quotient surface S of quotient model $X := (C_1 \times C_2)/G$.

Let us start to consider the singular points raised from the couple $g_1 = (\tau, e_1)$, and $h_1 = (\tau, Ae_1)$. The quotient $G/\langle h_1 \rangle$ consists of $3p$ points. Now consider the left action of $\langle g_1 \rangle$ on $G/\langle h_1 \rangle$. The orbits $[g\langle h_1 \rangle]$ may have only length $2p$, p, or 2. A point with trivial stabilizer is $(\sigma, 0)\langle h_1 \rangle$. In fact, $g_1^k \cdot (\sigma, 0)\langle h_1 \rangle = (\sigma, 0)\langle h_1 \rangle$ implies

$$\sigma^2\tau^k\sigma \in \langle \tau \rangle, \qquad \text{and} \qquad ke_1 \in \langle Ae_1 \rangle.$$

From our assumption on A, then this is possible only for $k = 0$, or p. If $k = p$, then $\sigma^2\tau^k\sigma = \sigma^2\tau\sigma = \tau\sigma^2 \in \langle \tau \rangle$, a contradiction. Hence $k = 0$, so $(\sigma, 0)\langle h_1 \rangle$ has trivial stabilizer.

The points $g\langle h_1 \rangle$ whose stabilizer is of order 2 are fixed necessarily by g_1^p, and they have to satisfy

$$g_1^p \cdot g\langle h_1 \rangle = g\langle h_1 \rangle \implies g_1^p = gh_1^p g^{-1} \implies g\langle h_1 \rangle = (\tau, ke_1)\langle h_1 \rangle, \quad k = 0, \ldots, p-1.$$

A straightforward computation shows that these points share the same $\langle g_1 \rangle$-orbit $[\langle h_1 \rangle]$:

$$(\tau, e_1)^{k_1 - k_2} \cdot (\tau, k_2 e_1)\langle h_1 \rangle = (\tau, e_1)^{k_1 - k_2} \cdot \left((\tau, k_2 e_1)(\tau, Ae_1)^{p[k_1 - k_2]_2} \right) \langle h_1 \rangle = (\tau, k_1 e_1)\langle h_1 \rangle.$$

This shows $(G/\langle h_1 \rangle)/\langle g_1 \rangle = \{[(\sigma, 0)\langle h_1 \rangle], [\langle h_1 \rangle]\}$. The first orbit gives a smooth point of X, and the second gives a singular point of type $\frac{1}{2}(1,1)$.

Let us going on by considering the couple g_1, and h_2. The quotient by $\langle h_2 \rangle$ consists of $3p$ points. The left action of $\langle g_1 \rangle$ on the quotient has one orbit $[(\sigma, 0)\langle h_2 \rangle]$ of cardinality $2p$, and $[(\tau\sigma^2, 0)\langle h_2 \rangle]$ of cardinality p. The first gives a smooth point, while the second a singular point of type $\frac{1}{2}(1,1)$.

Using a similar argument, we obtain a singular point of type $\frac{1}{2}(1,1)$ for each of the two pairs g_2, h_1, and g_2, h_2.

Now consider g_1, and h_3. Since $p \geq 5$, then the quotient by $\langle h_3 \rangle$ has cardinality $2p$, and the quotient by $\langle g_1 \rangle$ consists only of one orbit $[(\sigma, 0)\langle h_2 \rangle]$. This gives a smooth point.

The same argument works for the pairs g_2, h_3 and g_3, h_1, and g_3, h_2 which gives then each a smooth point respectively.

It remains the couple g_3 and h_3. The quotient by $\langle h_3 \rangle$ has cardinality $2p$. The left action by $\langle g_3 \rangle$ on the quotient has one orbit $[(\sigma, 0)\langle h_3 \rangle]$ of length p and $[(\tau, 0)\langle h_3 \rangle]$ of length p. Let us analyse the first point coming from $[(\sigma, 0)\langle h_3 \rangle]$. the smallest positive δ such that $g_3^\delta = (\sigma, 0)h_3^\gamma(\sigma^2, 0)$ is $\delta = p$, and $\gamma = p$, so that the singular point is of type $\frac{1}{3}(1, 1)$. Instead, the smallest positive δ such that $g_3^\delta = (\tau, 0)h_3^\gamma(\tau, 0)$ is $\delta = p$, and $\gamma = 2p$. Therefore the last singular point is of type $\frac{1}{3}(1, 2)$.

The basket of singularities of X given by V and $A \cdot V$ is therefore

$$\mathcal{B}(X) = \left\{ 4 \times \frac{1}{2}(1,1),\ \frac{1}{3}(1,1),\ \frac{1}{3}(1,2) \right\}.$$

Once that we have computed the basket of singularities of X, we can determine its invariants K_S^2, and $\chi(S)$, by applying the formulas of the Theorem 4.4.2, and the Corollary 4.4.3:

$$\begin{aligned}
k(\mathcal{B}) &= -2 + \frac{2+1+1}{3} + (3-2) = \frac{1}{3}, \\
e(\mathcal{B}) &= 4\left(1 + 1 - \frac{1}{2}\right) + \left(1 + 1 - \frac{1}{3}\right) + \left(2 + 1 - \frac{1}{3}\right) = 11 - \frac{2}{3}, \\
B(\mathcal{B}) &= 22 - \frac{4}{3} + \frac{1}{3} = 21.
\end{aligned}$$

Thus, replacing the genera $g(C_i)$ by the values computed in the Example 2.1.14, we obtain

$$\begin{aligned}
K_S^2 &= \frac{8(3p^2 - 4p)^2}{6p^2} - \frac{1}{3} = \frac{(6p-8-1)(6p-8+1)}{3} = (2p-3)(6p-7), \\
\chi(S) &= \frac{1}{8}\left(K_S^2 + \frac{1}{3}B(\mathcal{B})\right) = \frac{(2p-3)(6p-7)}{8} + \frac{7}{8} = \frac{(p-1)(3p-5)}{2} + 1.
\end{aligned}$$

4.8 Description and implementation of the classification algorithm

Fixed a pair $(K^2, \chi) \in \mathbb{N}\times\mathbb{N}$, the next goal is to use the results of the previous sections to write a MAGMA script to find all minimal regular surfaces S of general type with $K_S^2 = K^2$, and $\chi(S) = \chi$, which are product-quotient surfaces. After that, we specialize our classification to those surfaces with $(K^2, \chi) = (32, 4)$.

A commented version of the MAGMA code is available here:

https://fefe9696.github.io/FedericoFallucca/pubbl.html#PhDTh

We describe here the strategy, and explain how the most important scripts work. The strategy is the same of [BP12]. Most of the scripts are modification of those in [BP12]. Since those scripts were written under the assumption $\chi = 1$, we generalize all of them to allow any value of χ. At the end of the section we will indicate which are the other main improvements we did.

First of all, we fix the couple (K^2, χ). Note that by minimality of S, and by the Corollary 4.4.4, then $K^2 \in \{1, \dots, 8\chi\}$, and the case $K^2 = 8\chi$ corresponds to those surfaces whose quotient model X is smooth.

Step 1: The script **Baskets** lists all the *possible basket of singularities* for (K^2, χ) as in the Definition 4.4.9. Indeed, there are only finitely many of them by the Lemma 4.6.1. As in the Lemma 4.6.1, the input is $3(8\chi - K^2)$, so to get for instance all baskets for $(K^2, \chi) = (28, 4)$, we need *Basket*(12).

Step 2: From the Proposition 4.6.4 once we know the basket of singularities of X, then there are finitely many possible signatures. **ListOfTypes** computes them using the inequalities we have proved in the Proposition 4.6.4. Here the input is K^2, and χ, so *ListOfTypes* first computes *Baskets(*$3(8\chi - K^2)$*)*, and then computes for each basket all numerically compatible signatures. The output is a list of pairs, the first element of each pair being a basket, and the second element being the list of all signatures compatible with that basket.

Step 3: Every surface produces two signatures, one for each curve C_i, both compatible with the basket of singularities of X; if we know the signatures and the basket, then Proposition 4.6.4 *b*) tells us the order of G. **ListGroups**, whose input is K^2, and χ, first computes *ListOfTypes*(K^2, χ). Then for each pair of signatures in the output, it calculates the order of the group. Next it searches for the groups of given order which admit appropriate spherical systems of generators corresponding to both signatures. Here we use the database in [CGP23] if we are in one of the cases classified there, otherwise we use the function *FindGenerators* developed in the work [CGP23]. For each affirmative answer, it stores the triple (basket, pair of signatures, group) in a list, which is the main output.

The script has some shortcuts:

- If the pair of signatures $[m_1, \ldots, m_r]$ and $[n_1, \ldots, n_s]$ admits orbifold groups (see the Remark 2.1.7) $\mathbb{T}(m_1, \ldots, m_r)$ and $\mathbb{T}(n_1, \ldots, n_s)$ such that the orders of their abelianization are coprime numbers, then G is forced to be a perfect group:

$$\gcd\left(|\mathbb{T}(m_1, \ldots, m_r)^{ab}|, |\mathbb{T}(n_1, \ldots, n_s)^{ab}|\right) = 1 \implies G^{ab} = \{1\}.$$

  In fact, the orbifold (surjective) homomorphisms $\mathbb{T}(m_1, \ldots, m_r) \to G$, and $\mathbb{T}(n_1, \ldots, n_s) \to G$, extend to surjective homomorphisms

$$\begin{aligned} \mathbb{T}(m_1, \ldots, m_r)^{ab} &\to G^{ab}, \\ \mathbb{T}(n_1, \ldots, n_s)^{ab} &\to G^{ab}. \end{aligned}$$

  This means $|G^{ab}|$ has to divide at the same time $|\mathbb{T}(m_1, \ldots, m_r)^{ab}|$, and $|\mathbb{T}(n_1, \ldots, n_s)^{ab}|$.

  MAGMA knows all perfect groups of order ≤ 50000, and then *ListGroups* checks first if there are perfect groups of the right order: if not, this case can not occur.

- If:

  - either the expected order of the group is 1024 or bigger than 2000, since MAGMA does not have a list of the finite groups of this order;
  - or the order is a number as *e.g.*, 1728, where there are too many isomorphism classes of groups;

  then *ListGroups* just stores these cases in a list, secondary output of the script. These "exceptional" cases have to be considered separately.

Step 4: The basket of singularities of a surface described by a couple of spherical systems $[g_1, \ldots, g_r]$ and $[h_1, \ldots, h_s]$ depends only by the conjugacy classes of g_i and h_j, from the Remark 4.7.2. **ExistingSurfaces** runs on the output of *ListGroups*(K^2, χ), and throws away all triples giving rise only to surfaces whose singularities do not correspond to the basket.

Step 5: Each triple (basket, pair of signatures, group) in the output *ExistingSurfaces*(K^2, χ) gives many different pairs of appropriate spherical systems of generators. On them there is the action of $\mathrm{Aut}(G) \times \widetilde{\mathcal{B}_r} \times \widetilde{\mathcal{B}_s}$ described in the Section 4.5. Therefore, the script **FindSurfaces** uses Theorem 4.5.8 and Corollary 4.5.10 to pick up only one pair of spherical systems of generators for any family of product-quotient surfaces compatible with the triple (basket, pair of signatures, group). Thus, the output is a list of (basket, sph1, sph2, group), where sph1 and sph2 are spherical systems of group compatible with pair of signatures and basket.

Remark 4.8.1. The main novelties respect to the program in [BP12] are the following:

- The program works for every $\chi \in \mathbb{N}$ whereas the original scripts in [BP12] were assuming $\chi = 1$;
- The shortcut using perfect groups is a generalization of the similar shortcut in [BP12], where it was applied only to the case when one of the signatures is $[2, 3, 7]$;
- The main improvement is **Step 5** which is essentially new and much more performing than the analogous procedure in [BP12]. Here we use our main Theorem 4.5.8 of this chapter in combination with the database and the script *FindGenerators* developed in [CGP23]. Moreover, we use these last tools from [CGP23] to speed up **Step 3** as well.

4.9 Classification of regular product-quotient surfaces isogenous to a product of curves with geometric genus of three

Regular product-quotient surfaces isogenous to a product of curves with $p_g = 3$ are those with $K_S^2 = 32$ and $\chi = 4$.

We have run the function *FindSurfaces* described in the previous section on each triple of the output of *ListGroups*(K^2, χ), where $K^2 = 32$, and $\chi = 4$. This has given the following

Theorem 4.9.1. *Let S be a product-quotient surface of general type with $p_g = 3$, $q = 0$, and $K_S^2 = 32$ of quotient model $(C_1 \times C_2)/G$. Assume that both the topological types of the G-action on C_1 and C_2 are in the database* [CGP23, 11^{th} of June 2023]. *Then S realizes one of the* 213 *families of surfaces of general type described in tables 4.1 and 4.2.*

Note that they are minimal surfaces, since the quotient model X is smooth, and the canonical divisor of $C_1 \times C_2$ (which is nef, because $g(C_i) \geq 2$, see Lemma 4.2.3) is the pullback of that of X.

Remark 4.9.2. The database [CGP23, 11^{th} of June 2023] currently covers, g being the genus of the curve, d being the order of the group, and r being the length of the signature:

- all topological types with $g \leq 40$ except

  $g = 28$, group $G(18, 4)$, signature 2^{10}

  $g = 34$, group $G(18, 4)$, signature $2^{10}, 3$

  $g = 37$, group $G(18, 4)$, signature 2^{12}

$g = 40$, group $G(18, 4)$, signature $2^{10}, 3^2$

$g = 40$, group $G(12, 3)$, signature $2^{13}, 3^3$

- all cases with $r = 3$, $g \leq 64$, $d \leq 2000$
- all cases with $r = 4$, $g \leq 100$ except

  $g = 76$, group $G(125, 5)$, signature 5^4

  $g = 88$, group $G(696, 35)$, signature $2^3, 4$

- all cases with $r = 5$ and $g \leq 100$ except:

  $g = 75$, group $G(148, 3)$, signature $2^3, 4^2$

  $g = 100$, group $G(81, 2)$, signature 9^5

The tables 4.1 and 4.2 contain the following informations

- t_1 and t_2 are the signatures of the pair of spherical systems of generators defining a family of product-quotient surfaces;
- N is the number of irreducible families; our tables have only 74 lines, but we collect in the same line N families, which share all the other data. The number of lines counted with multiplicity N is 213 (which is the number of families of Theorem 4.9.1.)

For the groups occurring in tables 4.1 and 4.2, we use the following notation: we denote by $\mathbb{Z}_d$ the cyclic group of order d, S_n is the symmetric group in n letters, $\mathcal{A}_n$ is the alternating group.
$PSL(2, 7)$ is the group of 2×2 matrices over $\mathbb{F}_7$ with determinant 1 modulo the subgroup generated by $-\mathrm{Id}$.
$SO(3, 7)$ is the group of 3×3 orthogonal matrices over $\mathbb{F}_7$ having determinant 1.
$D_{p,q,r} := \langle x, y | x^p, y^q, xyx^{-1}y^{-r} \rangle$, and $D_n = D_{2,n,-1}$ is the usual dihedral group of order $2n$.
Finally, $G(n, k)$ is the $k-$th group of order n in the MAGMA database of small groups.

To prove the main Theorem 4.9.1, it remains to show that the cases skipped by *ListGroups* can not occur. It turns out that no other surfaces occur from the skipped cases. The techniques that we have used to exclude these cases are those developed in [BCP06], [BP12, Chp. 3] and [Fra12, Sec 6.3]. For this reason, we decided not to provide further details on how we excluded them.

Table 4.1

t_1	t_2	G	N
2^5	2^{12}	$\mathbb{Z}_2^3$	3
2^6	2^8	$\mathbb{Z}_2^3$	3
3^4	3^7	$\mathbb{Z}_3^2$	2
3^5	3^5	$\mathbb{Z}_3^2$	1
$2^3, 4$	2^{12}	$\mathbb{Z}_2 \times D_4$	6
2^5	$2^5, 4^2$	$\mathbb{Z}_2 \times D_4$	4
2^5	$2^2, 4^4$	$\mathbb{Z}_2 \times D_4$	1
$2^3, 4^2$	2^6	$\mathbb{Z}_2 \times D_4$	2
2^6	2^6	$\mathbb{Z}_2 \times D_4$	1
$2^2, 4^2$	$2^2, 4^4$	$G(16, 3)$	2
$2^2, 4^2$	$2^5, 4^2$	$G(16, 3)$	6
$2^3, 4^2$	$2^3, 4^2$	$G(16, 3)$	2
2^5	2^8	$\mathbb{Z}_2^4$	13
2^6	2^6	$\mathbb{Z}_2^4$	6
$2^2, 3^2$	$2^2, 4^4$	S_4	1
$2^4, 3$	4^4	S_4	1
$2, 3, 4^2$	2^6	S_4	1
2^5	$2^5, 6$	$\mathbb{Z}_2^2 \times S_3$	1
$2^3, 4$	$2^2, 4^4$	$G(32, 27)$	2
$2^3, 4$	$2^5, 4^2$	$G(32, 27)$	30
$2^2, 4^2$	$2^3, 4^2$	$G(32, 27)$	1
2^5	$2^3, 4^2$	$G(32, 27)$	4
$2^2, 4^2$	2^6	$G(32, 27)$	4
$2^3, 4$	$2^2, 4^4$	$G(32, 28)$	1
2^5	2^6	$\mathbb{Z}_2^2 \times D_4$	4
2^5	$2^3, 4^2$	$\mathbb{Z}_2^2 \times D_4$	2
$2^2, 4^2$	$2^3, 4^2$	$G(32, 22)$	7
$2^2, 4^2$	4^4	$G(32, 6)$	1
2^5	$2^3, 4^2$	$G(32, 49)$	1
$2^3, 3$	$2^2, 4^4$	$\mathbb{Z}_2 \times S_4$	4
$4^2, 6$	2^6	$\mathbb{Z}_2 \times S_4$	3
$2^3, 6$	4^4	$\mathbb{Z}_2 \times S_4$	1
$2^3, 6$	$2^3, 4^2$	$\mathbb{Z}_2 \times S_4$	1
$2^2, 4^2$	$2^4, 3$	$\mathbb{Z}_2 \times S_4$	2
$2^2, 4^2$	$2^2, 6^2$	$\mathbb{Z}_2 \times S_4$	1
2^5	$2, 3, 4^2$	$\mathbb{Z}_2 \times S_4$	1
$3, 4^2$	$2^2, 4^4$	$G(48, 30)$	3
$2^3, 4$	$2^5, 6$	$S_3 \times D_4$	1
$2^2, 4^2$	$2^2, 4, 12$	$G(48, 14)$	1
7^3	7^3	$\mathbb{Z}_7^2$	7
$3, 5^2$	2^6	$\mathcal{A}_5$	2
5^3	$2^4, 3$	$\mathcal{A}_5$	1
5^3	3^4	$\mathcal{A}_5$	1

Table 4.2

t_1	t_2	G	N
$2^3, 4$	2^6	$G(64, 73)$	1
$2^3, 4$	$2^3, 4^2$	$G(64, 73)$	4
$2^3, 4$	4^4	$G(64, 138)$	1
$2^3, 4$	$2^3, 4^2$	$G(64, 138)$	6
2^5	2^5	$G(64, 211)$	1
2^5	2^5	$\mathbb{Z}_2^2 \times D_8$	1
$2^2, 4^2$	$2^2, 4^2$	$G(64, 60)$	3
$2^2, 4^2$	$2^2, 4^2$	$G(64, 71)$	1
$2^2, 4^2$	$2^2, 4^2$	$G(64, 75)$	1
$2, 4, 6$	$2^2, 4^4$	$\mathrm{GL}_2(\mathbb{Z}_4)$	10
$4^2, 6$	$2^2, 4^2$	$\mathrm{GL}_2(\mathbb{Z}_4)$	1
$3, 4^2$	$2^3, 4^2$	$G(96, 227)$	1
$2^3, 3$	4^4	$G(96, 227)$	3
$3, 4^2$	2^6	$G(96, 227)$	3
$2^3, 4$	$2^3, 4, 12$	$G(96, 89)$	1
$2^3, 6$	$2^2, 4^2$	$\mathbb{Z}_2^2 \times S_4$	1
$2, 5, 6$	4^4	S_5	2
$2, 5, 6$	$2^3, 4^2$	S_5	1
$3, 4^2$	$2^4, 5$	S_5	1
$3^2, 7$	$2^2, 4^2$	$PSL(2, 7)$	1
4^3	$3, 7^2$	$PSL(2, 7)$	4
$2, 4, 6$	$2^3, 4^2$	$G(192, 955)$	7
$2, 4, 6$	4^4	$G(192, 955)$	2
$2^3, 4$	$4^2, 6$	$G(192, 955)$	1
$2, 4, 6$	$2^2, 10^2$	$\mathbb{Z}_2 \times S_5$	1
4^3	4^3	$G(256, 295)$	3
4^3	4^3	$G(256, 298)$	2
4^3	4^3	$G(256, 306)$	2
$2, 3, 14$	$2^2, 4^2$	$\mathbb{Z}_2 \times PSL(2, 7)$	1
$2, 6, 7$	4^3	$\mathbb{Z}_2 \times PSL(2, 7)$	2
$2, 6, 7$	$2, 8^2$	$SO(3, 7)$	2

Recalling the Remark 1.1.6, part 3., the surfaces of general type with a high degree of the canonical map have p_g equal to 3. It is natural to search among the 213 families of Theorem 4.9.1 for surfaces with canonical map of high degree.

Note that the canonical map of these surfaces, if not composed with a pencil, has degree at most 32.

The only surfaces in the literature with canonical map equal to 32 are the two families in [GPR22] (see the Example 1.2.7). They are two of the six families of Table 4.1 with group $G = \mathbb{Z}_2^4$ and signatures 2^6 and 2^6.

Furthermore, the authors proved in [GPR22, Prop. 5.3] that these two examples are the only product-quotient surfaces with G abelian and $p_g = 3$ having degree of the canonical map equal to 32.
However, there are other families from Theorem 4.9.1 with canonical map of degree equal to 32. For instance, using the tools developed in the following Chapters, we proved that the surfaces in the family with group $G = \mathbb{Z}_2 \times D_4$ and both signatures 2^6 of Theorem 4.9.1 have also canonical map of degree 32. We decide not to include the discussion of this specific family in this thesis.

We will discuss the canonical map of the seven product-quotient surfaces with group $\mathbb{Z}_7^2$ of Theorem 4.9.1 in Section 6.1 and the canonical map of one of the four families with group $G := \mathbb{Z}_2^2 \times D_4$ and signatures 2^5 and 2^6 in Section 6.3.

Chapter 5

On the canonical map of a product-quotient surface

In this chapter, we investigate the canonical map of a product-quotient surface. Their canonical map was studied, in the special case of the surfaces isogenous to a product, in [Cat18].

In the first section we prove the main Theorem 5.1.8, called from us *Base locus formula*. This is a formula for the base locus of the subsystem of the canonical system of a Riemann surface C given by an isotypic component of the action of a finite group G on C. Furthermore, in Corollary 5.1.11 assuming $C/G \cong \mathbb{P}^1$ we give an expression of this formula in terms of spherical systems of generators under the assumption that the associated irreducible character is of degree one.

In the second section we study the structure of the canonical map of a product-quotient surface S. We give a decomposition of $H^{2,0}(S)$ in terms of irreducible characters of G. In the Theorem 5.2.8, we determine the base locus of the subsystems of $|K_S|$ corresponding to characters of degree one.

Finally, Section 5.3 is devoted to the study of the degree of the canonical map of a product-quotient surface, whenever this is not composed with a pencil.

To compute such degree, we need to know the degree of the image of the canonical map, that is one if p_g is equal to three, and the self-intersection of the mobile part of a subsystem of the canonical system of a suitable blow-up of $C_1 \times C_2$.

Finally, Theorem 5.4.3 gives a *Correction term formula* computing this self-intersection when p_g is equal to three and all characters involved in the decomposition of $H^{2,0}(S)$ are of degree one.

5.1 Isotypic components of canonical representations of actions on curves and base loci

Let C be a curve, $G < \mathrm{Aut}(C)$ be a finite group, $C' := C/G$ its quotient, and let $\lambda\colon C \to C'$ be the quotient map. Let g and g' be the genera of C and C' respectively.
G acts on $H^{1,0}(C)$ via the cotangent representation:

$$(g \cdot \omega)_p := (dg^{-1})_p \omega_{g^{-1}\cdot p},$$

which is called *canonical representation*. Let us denote by χ_{can} the character afforded by the canonical representation, which takes the name of *canonical character*. By classical representation theory the character of a representation determines the representation up to isomorphism. Furthermore, any representation can be splitted as a direct sum of irreducible representations. In our case, we write

$$H^{1,0}(C) = \bigoplus_{\chi \in Irr(G)} H^{1,0}(C)^{\chi}.$$

Here $H^{1,0}(C)^{\chi}$ is the *isotypic component* of $H^{1,0}(C)$ of character χ, namely that G-invariant subspace such that the restriction of the canonical representation is isomorphic to $\langle \chi_{can}, \chi\rangle$-times the irreducible representation afforded by the character χ.
In terms of characters, the above splitting translates as

$$\chi_{can} = \sum_{\chi \in Irr(G)} \langle \chi_{can}, \chi\rangle \cdot \chi.$$

Thus, the canonical representation of G can be determined just by knowing the scalar products $\langle \chi_{can}, \chi\rangle$. They can be computed through the well-known *Chevalley-Weil formula*, which uses the local monodromies (see the Definition 2.0.14) of points of the branch locus of λ.
We recall the beautiful thesis [Gle16] for a complete description how to use the Chevalley-Weil formula. What is important to remark here is simply that we use the algorithm developed in [Gle16] and implemented in the computational algebra system MAGMA to compute the canonical character χ_{can} of any Galois branched covering.

The aim of this section is to investigate the base locus of the associated subsystem $|K_C|^{\chi}$ given by the isotypic component $H^{1,0}(C)^{\chi}$. Let us give first some preliminary results.

Notation: Given a point $q \in C'$, the divisor $\lambda^{-1}(q)$ is considered with the reduced structure.

Lemma 5.1.1. *Consider a G-invariant subspace $W \subseteq H^{1,0}(C)$. For any $p \in \lambda^{-1}(q)$, let t_p be the minimal order of vanishing of a 1-form in $|W|$ at*

p. Then all t_p are equal to the same number, denoted by t_q. Therefore the base locus of $|W|$ is a union of orbits

$$Bs(|W|) = \sum_q t_q \lambda^{-1}(q).$$

Furthermore, there exists a general form $\omega \in W$ vanishes of order exactly t_q at each $p \in \lambda^{-1}(q)$.

Proof. For every point $p \in \lambda^{-1}(q)$, it there exists a 1-form ω_p in W vanishing at p with order t_p, by the definition of t_p. Given $g \in G$, then $g \cdot \omega_p$ belongs to the invariant subspace W too, and it vanishes at $g \cdot p$ with multiplicity t_p, so that $t_{g \cdot p} \leq t_p$. Hence all t_p are equal to the same number, denoted as t_q.

We observe that a generic linear combination ω of the obtained $|\lambda^{-1}(q)|$ 1-forms ω_p vanishes with order t_q at each point of $\lambda^{-1}(q)$. □

Remark 5.1.2. Let $\omega \in W$ be the 1-form of the Lemma 5.1.1, with vanishing order t_q at each point $p \in \lambda^{-1}(q)$. Given $g \in G$, then $g \cdot \omega \in W$ is a 1-form with vanishing order t_q at each point $p \in \lambda^{-1}(q)$.

Let $H^{1,0}(C)^\chi$ be the isotypic component of $H^{1,0}(C)$ of irreducible character χ.

Lemma 5.1.3. *Let $f \in \mathcal{M}(C/G) = \mathcal{M}(C)^G$ be a non-zero invariant meromorphic function. Denote by $H^{1,0}(C)^\chi_f$ the subspace of $H^{1,0}(C)^\chi$ consisting of forms ω such that $f\omega$ is a holomorphic form. Then*

$$f \colon H^{1,0}(C)^\chi_f \to f \cdot H^{1,0}(C)^\chi_f \subseteq H^{1,0}(C), \qquad \omega \mapsto f\omega \tag{5.1}$$

is a G-equivariant isomorphism. In particular, $f \cdot H^{1,0}(C)^\chi_f$ is a G-invariant subspace of $H^{1,0}(C)^\chi$.

Proof. $H^{1,0}(C)^\chi_f$ is G-invariant: given $g \in G$ and $\omega \in H^{1,0}(C)^\chi_f$, then $f(g \cdot \omega) = g \cdot (f\omega)$ is holomorphic since f is G-invariant, and $f\omega$ is holomorphic. This shows immediately also that the map of (5.1) is G-equivariant. From Schur Lemma, then the image of (5.1) is contained in $H^{1,0}(C)^\chi$. However, f is not the zero function, so (5.1) is injective. □

Definition 5.1.4. Let X be a Riemann surface and $q \in X$. Let us define

$$k_q := \min \left\{ m \in \mathbb{N} : h^0(X, mq) \geq 2 \right\}$$

the *minimal non-gap of q*. k_q is therefore the smallest number such that X admits a non-constant meromorphic function f with only one pole at q, of order $-k_q$.
Moreover, any non-constant f has order at q exactly $-k_q$. Indeed, by definition of k_q, then $H^0(X, (k_q - 1)q) \cong \mathbb{C}$ consists only of constant functions, so that

$$-(k_q - 1) > \mathrm{ord}_q(f) \geq -k_q \implies \mathrm{ord}_q(f) = -k_q.$$

Remark 5.1.5. From Riemann-Roch theorem we have

$$h^0(X,(g(X)+1)q) = h^0(X, K-(g(X)+1)q) + 2 \geq 2.$$

Therefore

$$k_q \leq g(X)+1.$$

In other words, k_q is the minimum of the complement of the set of the Weierstrass gaps for q. In particular, $k_q = g(X)+1$, if q is not a Weierstrass point, or $k_q < g(X)+1$, otherwise.

Let $q \in C'$ be a branch point of λ. The stabilizers of the points lying on q are cyclic subgroups of G and they are conjugated to each other. Thus the order of the stabilizers depends only on q, denoted as m_q.
We remind the Definition 2.0.14 of local monodromy:

Definition. Let us fix a point $p \in \lambda^{-1}(q)$. Given a generator h of $Stab(p)$, there exists a coordinate z in C such that the action of h in a neighborhood of p corresponds to $z \to \lambda z$, where λ is one of the m_q-roots of the unity. This gives a bijection among the primitive m_q-roots of the unity and the generators of $Stab(p)$. We denote by *local monodromy* of p the unique generator of $Stab(p)$ acting by $z \to e^{\frac{2\pi i}{m_q}} z$.

Remark 5.1.6. The *local monodromy* of another point $g \cdot p$ over q is the conjugate ghg^{-1} of h. In other words, the *local monodromy* of points lying over q are conjugated to each other.

Let $\chi \in Irr(G)$ be a character afforded by an irreducible representation ρ_χ, and denote by $|K_C|^\chi$ the associated subsystem of the canonical linear system of C given by the isotypic component $H^{1,0}(C)^\chi$.
Lemma 5.1.1 applies to $H^{1,0}(C)^\chi$, so the base locus of $|K_C|^\chi$ is

$$Bs(|K_C|^\chi) = \sum_q t_q^\chi \lambda^{-1}(q).$$

We have the following

Lemma 5.1.7. *Let us fix a point $q \in C/G$. Let h be the local monodromy of a point $p \in \lambda^{-1}(q)$. There exist*

$$a_q^\chi \in \{j \in [0,\dots,m_q-1] \colon e^{\frac{2\pi i}{m_q}j} \in \mathrm{Spec}(\rho_\chi(h))\}$$

and a non-negative integer $0 \leq k_q^\chi < k_q \leq g(C/G)+1$ such that

$$t_q^\chi = m_q - a_q^\chi - 1 + k_q^\chi m_q.$$

where k_q is the minimal non-gap of q in the Definition 5.1.4.
The values a_q^χ and k_q^χ depends only from q and χ and not by the choice of $p \in \lambda^{-1}(q)$.

Proof. The order of h in G is m_q, by definition of h.
We observe that the action on $H^{1,0}(C)^\chi$ of h is diagonalizable, and its spectrum is contained in the set of the m_q-roots of the unity. Hence the action of h decomposes $H^{1,0}(C)^\chi$ as

$$H^{1,0}(C)^\chi = \bigoplus_{j=0}^{m_q-1} V_j,$$

where V_j is the eigenspace of eigenvalue ξ^j, and ξ is the first m_q-root of the unity (V_j may be zero, whenever ξ^j is not an eigenvalue of h).

Let $\omega_j \in V_j$ be an eigenvector. We determine the vanishing order of ω_j at the point p. By definition of local monodromy, it there exists a local coordinate z such that the action of h in a neighborhood of p is $z \mapsto \xi z$. We write $\omega_j = f(z)dz$ locally around this neighborhood of p. We get

$$\begin{aligned} \xi^j f(z)dz &= h \cdot (f(z)dz) \\ &= (h^{-1})^*(f(z)dz) \\ &= f(\xi^{m_q-1}z)\xi^{m_q-1}dz. \end{aligned}$$

Hence f satisfies $f(\xi^{m_q-1}z) = \xi^{j+1}f(z)$, forcing it to be $f = z^{m_q-j-1}g(z^{m_q})$, for some holomorphic function g. Hence $\mathrm{ord}_p(\omega_j)$ is congruent to $m_q - j - 1$ modulo m_q.

Applying Lemma 5.1.1 to $W = H^{1,0}(C)^\chi$ we find a form $\omega \in H^{1,0}(C)^\chi$ with vanishing order t_q^χ at each point of $\lambda^{-1}(q)$. Writing ω as a $\omega = \sum_{j=0}^{m_q-1} \omega_j$, with $\omega_j \in V_j$. Since ω_j has different order at p, then

$$t_q^\chi = \mathrm{ord}_p(\omega) = \min_{\omega_j \neq 0}\{\mathrm{ord}_p(\omega_j)\}.$$

In other words, there exists $j_0 \in [0, \dots, m_q - 1]$ such that $t_q^\chi = \mathrm{ord}_p(\omega_{j_0})$.

Since ω_{j_0} is an eigenvector of eigenvalue ξ^{j_0}, then $t_q^\chi = \mathrm{ord}_p(\omega_{j_0})$ is congruent to $m_q - j_0 - 1$ modulo m_q; let us say $t_q^\chi = m_q - j_0 - 1 + k_{j_0}m_q$, for some non-negative integer k_{j_0}.

We claim that $k_{j_0} < k_q$. By contradiction, if $k_{j_0} \geq k_q$, then we use the definition of k_q to pick up a meromorphic function $f \in \mathcal{M}(C/G) = \mathcal{M}(C)^G$ with only one pole at q of order $\mathrm{ord}_q(f) = -k_q$. In this case, then $f\omega$ is a holomorphic form. Indeed, by definition of f, the only poles of $f\omega$ that may occur lie on $\lambda^{-1}(q)$, but the order of $f\omega$ at each $g \cdot p \in \lambda^{-1}(q)$ is

$$\begin{aligned} \mathrm{ord}_{g\cdot p}(f\omega) &= \mathrm{ord}_{g\cdot p}(\omega) + \mathrm{ord}_{g\cdot p}(f) \\ &= t_q^\chi - k_q m_q \\ &= m_q - j_0 - 1 + (k_{j_0} - k_q)m_q \geq 0. \end{aligned}$$

Furthermore, by the Lemma 5.1.3, then $f\omega \in H^{1,0}(C)^\chi$. However, this would contradict the definition of t_q^χ, since $\mathrm{ord}_p(f\omega) = t_q^\chi - k_q m_q < t_q^\chi$.

To summarize, we have proved

$$t_q^\chi = m_q - j_0 - 1 + k_{j_0} m_q,$$

where j_0 is one of the integers such that $\xi^{j_0} \in \operatorname{Spec}(\rho_\chi(h))$, and $k_{j_0} < k_q$.

It remains to prove that these integers does not depend from the choice of $p \in \lambda^{-1}(q)$. Set $a_q^\chi(p) := j_0$, and $k_q^\chi(p) := k_{j_0}$. If we repeat the proof using another point $g \cdot p \in \lambda^{-1}(q)$, then we get integers $a_q^\chi(g \cdot p) \in [0, \dots, m_q - 1]$ and $k_q^\chi(g \cdot p) < k_q$ satisfying

$$t_q^\chi = m_q - a_q^\chi(g \cdot p) - 1 + k_q^\chi(g \cdot p) m_q.$$

Hence it holds at the same time

$$m_q - a_q^\chi(g \cdot p) - 1 + k_q^\chi(g \cdot p) m_q = t_q^\chi = m_q - a_q^\chi(p) - 1 + k_q^\chi(p) m_q$$

so

$$a_q^\chi(p) - a_q^\chi(g \cdot p) = \left(k_q^\chi(p) - k_q^\chi(g \cdot p)\right) m_q \implies \begin{cases} k_q^\chi(g \cdot p) = k_q^\chi(p) \\ a_q^\chi(g \cdot p) = a_q^\chi(p). \end{cases}$$

□

Theorem 5.1.8. *(Base locus formula) The base locus of $|K_C|^\chi$ is*

$$Bs(|K_C|^\chi) = \sum_q \left(m_q - a_q^\chi - 1 + k_q^\chi m_q\right) \lambda^{-1}(q),$$

where the non-negative integers a_q^χ and k_q^χ are those defined in the Lemma 5.1.7.

Proof. It is sufficient to apply Lemma 5.1.7 to every point $q \in C/G$. □

Remark 5.1.9. Under suitable assumptions it is possible to determine exactly a_q^χ and k_q^χ.

For instance, if $C/G \cong \mathbb{P}^1$, then $k_q = g(C/G) + 1 = 1$, for any $q \in \mathbb{P}^1$. Hence $k_q^\chi = 0$, and we get

$$t_q^\chi = m_q - a_q^\chi - 1.$$

Moreover, if one of the following holds

- χ is an irreducible character of degree 1, or
- the local monodromy h is in the centre of G,

then $\rho_\chi(h) = \frac{\chi(h)}{\chi(1)} \cdot \mathrm{Id}$ is a multiple of the identity. In particular, $a_q^\chi \in [0, \dots, m_q - 1]$ is the only integer such that $\chi(h) = e^{\frac{2\pi i}{m_q} a_q^\chi} \chi(1)$.

This is obvious when the character has degree one. When the local monodromy is central, it is the following lemma that we take from [Cat18].

Lemma 5.1.10. *Let χ be a character afforded by a irreducible representation $\rho_\chi\colon G \to \mathrm{Aut}(V)$. If $h \in G$ is in the centre of G, then $\rho_\chi(h)$ is a multiple of the identity. Precisely, $\rho_\chi(h) = \frac{\chi(h)}{\chi(1)} \cdot \mathrm{Id}_V$.*

Proof. The action of h on V diagonalizes V as a direct sum of eigenspaces

$$V = \bigoplus_{j=0}^{o(h)-1} V_j$$

where V_j is an eigenspace (possibly zero) of eigenvalue ξ^j, and $\xi = e^{\frac{2\pi i}{o(h)}}$ is the first $o(h)$-root of the unity.

Let us fix j, and consider an eigenvector $v \in V_j$. For any $g \in G$, then $\rho_\chi(g)v$ is an eigenvector of eigenvalue ξ^j for the operator $\rho_\chi(ghg^{-1})$, which is equal to $\rho_\chi(h)$, since h is in the centre of G. In other words, $\rho_\chi(g)v \in V_j$. This proves V_j is G-invariant. However, ρ_χ is irreducible, and so $V = V_j$. This translates as $\rho_\chi(h) = \xi^j \cdot \mathrm{Id}_V$. Finally, we compute

$$\chi(h) = \mathrm{Tr}(\rho_\chi(h)) = \xi^j \dim(V) = \xi^j \chi(1) \implies \xi^j = \frac{\chi(h)}{\chi(1)}.$$

$\square$

We deduce then the following immediate consequence from Theorem 5.1.8 and Remark 5.1.9:

Corollary 5.1.11. *Assume $C/G \cong \mathbb{P}^1$, and χ is an irreducible character of degree 1. Then*

$$Bs(|K_C|^\chi) = \sum_q \left(m_q - a_q^\chi - 1\right) \lambda^{-1}(q),$$

where $a_q^\chi \in [0, \dots m_q - 1]$ is the only non-negative integer such that $\chi(h) = e^{\frac{2\pi i}{m_q} a_q^\chi}$, with h local monodromy of a point p over q.

5.2 The canonical system of a product-quotient surface

Let G be a finite group acting on a Riemann surface C_i, $i = 1, 2$. According to the previous section, then G induces the canonical representation on $H^{1,0}(C_i)$, afforded by the canonical character χ^i_{can}.
Let S be the product-quotient surface of quotient model $X := (C_1 \times C_2)/G$. This section studies the canonical system of S.

Theorem 5.2.1. *Every G-invariant global holomorphic 2-form of $C_1 \times C_2$ extends uniquely to a global holomorphic 2-form on the minimal resolution of the singularities $\rho\colon S \to X$ of X. It holds*

$$H^{2,0}(S) = H^{2,0}(C_1 \times C_2)^G = \bigoplus_{\chi \in Irr(G)} \left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G. \qquad (5.2)$$

Furthermore,

$$p_g(S) = \sum_{\chi \in Irr(G)} \langle \chi^1_{can}, \chi\rangle \cdot \langle \chi^2_{can}, \overline{\chi}\rangle.$$

Proof. Denote by X° the smooth locus of X, i.e. the locus of the image of that points of $C_1 \times C_2$ with trivial stabilizer. Each global holomorphic 2-form of X° extends uniquely to a global holomorphic 2-form of $C_1 \times C_2$, via the pullback map $\lambda^*_{12}\colon H^{2,0}(X^\circ) \to H^{2,0}(C_1 \times C_2)$, resulting a monomorphism onto the invariant subspace $H^{2,0}(C_1 \times C_2)^G$. On the other side, the minimal resolution of the singularities $\rho\colon S \to X$ is an isomorphism on X°, hence $\left(\rho^{-1}\right)^*: H^{2,0}(S) \to H^{2,0}(X^\circ)$ is a monomorphism. Furthermore, each global holomorphic 2-form on the smooth locus X° of X extends uniquely to a global holomorphic 2-form on S, by Freitag's theorem [Fre71, Satz 1], so $\left(\rho^{-1}\right)^*$ is an epimorphism too.
Thus $H^{2,0}(S)$ is sent isomorphically via $\lambda^*_{12} \circ (\rho^{-1})^*$ onto the invariant subspace $H^{2,0}(C_1 \times C_2)^G \subsetneq H^{2,0}(C_1 \times C_2)$. Finally, by applying Künneth formula and writing $H^{1,0}(C_i)$ as the direct sum of isotypic components, we get

$$H^{2,0}(C_1 \times C_2)^G = \bigoplus_{\chi,\eta \in Irr(G)} \left(H^{1,0}(C_1)^\chi \otimes H^{1,0}(C_2)^{\eta}\right)^G.$$

Formula (5.2) follows just by applying Schur lemma: the dimension of any piece of the sum is $\langle \chi^1_{can}, \chi\rangle \cdot \langle \chi^2_{can}, \eta\rangle \cdot \langle \chi\eta, 1\rangle$. However $\langle \chi\eta, 1\rangle = \langle \chi, \overline{\eta}\rangle$, which is equal to 1 only for $\eta = \overline{\chi}$, and 0 otherwise. □

Remark 5.2.2. As already discussed at (4.2), then one can say in general that

$$H^{i,0}(S) = H^{i,0}(C_1 \times C_2)^G$$

by Freitag's theorem [Fre71, Satz 1]. Hence, another immediate consequence firstly observed by Serrano in [Ser96, Prop. 2.2] is a formula for the irregularity of S:

$$q(S) = g(C_1/G) + g(C_2/G).$$

In particular, S is regular if and only if $C_i/G \cong \mathbb{P}^1$.

Let us remind the following classical lemma of representation theory:

Lemma 5.2.3. *Let us consider an irreducible representation $\phi_\chi\colon G \to GL(V)$ afforded by a character χ, of degree $n := \chi(1)$. Consider a basis $v_1, \dots, v_n$ of V and its dual basis $e_1, \dots, e_n$ of V^*. Then*

$$(V \otimes V^*)^G = \langle v_1 \otimes e_1 + \dots + v_n \otimes e_n\rangle.$$

Proof. The dimension of $(V \otimes V^*)^G$ is

$$\dim((V \otimes V^*)^G) = \langle \chi\overline{\chi}, 1\rangle = \langle \chi, \chi\rangle = 1.$$

Thus it is sufficient to prove $v_1 \otimes e_1 + \cdots + v_n \otimes e_n$ is an invariant vector. Write $g \in G$ as a matrix in the fixed basis on V, $g = (g_{ij})$, and in the dual basis on V^*, $g = (h_{ij})$. However, by definition of the action of G induced on V^*, $g = (h_{ij}) = \left((g_{ij})^{-1}\right)^t$. Then the statement follows by a direct computation

$$\begin{aligned} g \cdot \left(\sum_k v_k \otimes e_k\right) &= \sum_k \left(\sum_i g_{ik} v_i\right) \otimes \left(\sum_j h_{jk} e_j\right) \\ &= \sum_{i,j} \left(\sum_k g_{ik} h_{jk}\right) \cdot v_i \otimes e_j. \end{aligned}$$

The sum $\sum_k g_{ik} h_{jk}$ is the (i,j)-entry of the product matrix between (g_{ij}) and $(h_{ij})^t = (g_{ij})^{-1}$, which is equal to 1, if $i = j$, and 0 otherwise. □

We use the previous lemma to say that:

Remark 5.2.4. It is possible to describe a basis of the invariant subspace $(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$.
Let us consider the irreducible representation $\phi_\chi \colon G \to GL(V)$ of character χ. Let $n := \chi(1)$ be the degree of ϕ_χ. Then $H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}$ is the direct sum of certain number of copies of $V \otimes V^*$ (the exact number is $\langle \chi^1_{can}, \chi\rangle \cdot \langle \chi^2_{can}, \overline{\chi}\rangle$). Consequently its invariant subspace $(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}})^G$ is a direct sum of the same number of copies of the invariant subspace $(V \otimes V^*)^G$, which is always one dimensional:

$$\dim\left((V \otimes V^*)^G\right) = \langle \chi\overline{\chi}, 1\rangle = \langle \chi, \chi\rangle = 1.$$

Let us fix a basis $\{\omega_1, \ldots, \omega_n\}$ of V and the (dual) basis $\{\eta_1, \ldots, \eta_n\}$ on V^*. Hence, denote by $\{\omega_1^k, \ldots, \omega_n^k\}$ the corresponding basis of the k-th copy of V on $H^{1,0}(C_1)^{\chi}$, $k = 1, \ldots, \langle \chi^1_{can}, \chi\rangle$ [resp. by $\{\eta_1^l, \ldots, \eta_n^l\}$ the corresponding basis of the l-th copy of V^* on $H^{1,0}(C_2)^{\overline{\chi}}$, $l = 1, \ldots, \langle \chi^2_{can}, \overline{\chi}\rangle$]. Lemma 5.2.3 applies for any copy of $(V \otimes V^*)^G$, so that

$$(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}})^G = \bigoplus_{k,l} \langle \omega_1^k \otimes \eta_1^l + \cdots + \omega_n^k \otimes \eta_n^l\rangle. \tag{5.3}$$

Definition 5.2.5. We denote by $|K_{C_1 \times C_2}|^G$ the linear subsystem of $|K_{C_1 \times C_2}|$ spanned by p_g invariant 2-forms of $C_1 \times C_2$ defining Φ_{K_S}.

We give a theoretical description of the canonical map Φ_{K_S} of S. From Theorem 5.2.1, the (rational) map $\Phi_{K_S} \circ \lambda_{12}$ is induced by the linear subsystem $|K_{C_1\times C_2}|^G$. Therefore such map is not defined on the base locus of $|K_{C_1\times C_2}|^G$. The situation is the following:

$$\begin{tikzcd}
 & X & & \\
C_1 \times C_2 \arrow[ur, "\lambda_{12}"] \arrow[rr, dashed, "\lambda_{12}"] \arrow[d] \arrow[dr, "\Phi_{K_{C_1\times C_2}}"'] \arrow[rrr, dashed, bend right, "\Phi_{|K_{C_1\times C_2}|^G}"] & & S \arrow[ul, "\rho"'] \arrow[r, dashed, "\Phi_{K_S}"] & \mathbb{P}^{p_g-1} \\
\mathbb{P}^{g_1-1} \times \mathbb{P}^{g_2-1} \arrow[r, "Segre"] & \mathbb{P}^{g_1g_2-1}. \arrow[urr, dashed, "proj"'] & &
\end{tikzcd}$$

Let us fix a basis of $H^{1,0}(C_1)$ and $H^{1,0}(C_2)$. Then $\Phi_{K_S} \circ \lambda_{12}$ is the composition of the product of the canonical maps of C_1 and C_2 with the Segre embedding in $\mathbb{P}^{g_1g_2-1}$, together with the projection map $proj$. This latter map sends a basis of 2-forms of $C_1 \times C_2$ to a basis of invariant 2-forms defining Φ_{K_S}.
We can use Remark 5.2.4 to give an explicit description of $proj$, which is defined in coordinates as follows:
Let us fix coordinates ${}^{\chi}x_{ij}^{kl}$ on $\mathbb{P}^{g_1g_2-1}$, with $1 \le i,j \le \chi(1)$, and $1 \le k \le \langle \chi^1_{can}, \chi\rangle$, $1 \le l \le \langle \chi^2_{can}, \overline{\chi}\rangle$. Then

$$proj\left(\left({}^{\chi}x_{ij}^{kl} : \chi, i, j, k, l\right)\right) = \left({}^{\chi}x_{11}^{kl} + \cdots + {}^{\chi}x_{nn}^{kl} : \quad \chi \in Irr(G), n = \chi(1), k, l\right).$$

5.2.1 Base locus of the canonical system of a product-quotient surface

Given an irreducible character $\chi \in Irr(G)$, we have the following series of inclusions

$$\left(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G \subseteq H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}} \subseteq H^{2,0}(C_1 \times C_2).$$

Let us define the associated subsystems of $|K_{C_1\times C_2}|$ given by these subspaces.

Definition 5.2.6. We denote by $|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}$ and by $\left(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}\right)^G$ the associated subsystems of $|K_{C_1\times C_2}|$ given by $H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}$ and $\left(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}\right)^G$, respectively.

Theorem 5.2.1 permits us to describe the base locus of $|K_{C_1\times C_2}|^G$ in terms of the base locus of its pieces $\left(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}\right)^G$, $\chi \in Irr(G)$. Precisely, we have

$$Bs(|K_{C_1\times C_2}|^G) = \bigcap_{\langle \chi^1_{can}, \chi\rangle \neq 0,\ \langle \chi^2_{can}, \overline{\chi}\rangle \neq 0} Bs\left(\left(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}\right)^G\right). \tag{5.4}$$

Notation: Let us denote by

$$B_q^{vert} := \{q\} \times C_2/G, \qquad \text{and} \qquad B_l^{hor} := C_1/G \times \{l\},$$

where $q \in C_1/G$ and $l \in C_2/G$. Instead, R_q^{vert} and R_l^{hor} denote the reduced inverse images on $C_1 \times C_2$ of B_q^{vert} and B_l^{hor}:

$$R_q^{vert} := \frac{1}{m_q}(\lambda \circ \lambda_{12})^* \left(\{q\} \times C_2/G\right), \qquad \text{and} \qquad R_l^{hor} := \frac{1}{m_l}(\lambda \circ \lambda_{12})^* \left(C_1/G \times \{l\}\right).$$

Remark 5.2.7. With this notation, then the branch locus of $\lambda \circ \lambda_{12} \colon C_1 \times C_2 \to C_1/G \times C_2/G$ is the grid

$$B_q^{vert} := \{q\} \times C_2/G, \qquad \text{and} \qquad B_l^{hor} := C_1/G \times \{l\}$$

with $q \in Crit(\lambda_1)$ and $l \in Crit(\lambda_2)$.

Base Locus formula theorem 5.1.8 gives a formula for the base locus of $|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}$.

Theorem 5.2.8. *The (schematic) base locus of the linear subsystem $|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}$ of $|K_{C_1 \times C_2}|$ is pure in codimension 1 and is amount to*

$$Bs(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}) = \sum_{q \in Crit(\lambda_1)} t_q^{\chi} R_q^{vert} + \sum_{l \in Crit(\lambda_2)} t_l^{\overline{\chi}} R_l^{hor} \tag{5.5}$$

where t_q^{χ} and $t_l^{\overline{\chi}}$ are the non-negative integers computed in Lemma 5.1.7.

Corollary 5.2.9. *Let χ be a character of degree 1. Then*

$$(H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}})^G = H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}$$

and the base locus of its associated linear subsystem $\left(|K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}\right)^G = |K_{C_1}|^{\chi} \otimes |K_{C_2}|^{\overline{\chi}}$ is given by the formula (5.5) of Theorem 5.2.8.

Assume furthermore that $C_i/G \cong \mathbb{P}^1$, for $i = 1, 2$. Then t_q^{χ} and $t_l^{\overline{\chi}}$ of (5.5) are the unique non-negative integers with $0 \leq t_q^{\chi} \leq m_q - 1$ and $0 \leq t_l^{\overline{\chi}} \leq m_l - 1$ satisfying

$$\chi(h) = e^{\frac{2\pi i}{m_q}(m_q - t_q^{\chi} - 1)} \qquad \textit{and} \qquad \chi(g) = e^{\frac{2\pi i}{m_l}(t_l^{\overline{\chi}} + 1)},$$

where h is the local monodromy of a point over q, and g is the local monodromy of a point over l.

Proof. The first sentence is straightforward, since every $v \otimes w \in H^{1,0}(C_1)^{\chi} \otimes H^{1,0}(C_2)^{\overline{\chi}}$ is G-invariant

$$g \cdot (v \otimes w) = (\chi(g)v) \otimes (\overline{\chi}(g)w) = |\chi(g)| v \otimes w = v \otimes w.$$

The rest of the thesis follows from Remark 5.1.9. □

5.3 The degree of the canonical map

We are going to establish when Φ_{K_S} is composed with a pencil and, if the answer is negative, to compute its degree.

Instead to work on S it is convenient to work on $C_1 \times C_2$. Therefore, let us consider

$$\begin{tikzcd} C_1 \times C_2 \arrow[r, dashed, "\lambda_{12}"] \arrow[dr, "\Phi_{K_{C_1\times C_2}}"'] & S \arrow[r, dashed, "\Phi_{K_S}"] & \Sigma \subseteq \mathbb{P}^{p_g-1} \\ & \mathbb{P}^{g_1g_2-1}. \arrow[ur, dashed, "proj"'] & \end{tikzcd}$$

The (a priori rational) map $\Phi_{K_S} \circ \lambda_{12}$ is induced by the linear subsystem $|K_{C_1\times C_2}|^G$ of $|K_{C_1\times C_2}|$ generated by p_g invariant 2-forms defining Φ_{K_S}.

We *resolve the indeterminacy* of $\Phi_{K^G_{C_1\times C_2}} = \Phi_{K_S} \circ \lambda_{12}$ by a sequence of blowups, as explained in the textbook [Bea96, Theorem II.7]:

$$\begin{tikzcd} \widehat{C_1 \times C_2} \arrow[r] \arrow[dr, "\Phi_{\widehat{M}}"'] & C_1 \times C_2 \arrow[d, dashed, "\Phi_{K^G_{C_1\times C_2}}"] \\ & \mathbb{P}^{p_g-1}. \end{tikzcd}$$

Here the morphism $\Phi_{\widehat{M}}$ is induced by the base-point free linear system $|\widehat{M}|$ obtained as follow: let $|M|$ be the mobile part of $|K_{C_1\times C_2}|^G$.
We blow up the base-points of $|M|$, take the pullback of $|M|$ and remove the fixed part of this new linear system. We repeat the procedure, until we obtain a base-point free linear system $|\widehat{M}|$.

Lemma 5.3.1. *The map Φ_{K_S} is not composed with a pencil if and only if $\widehat{M}^2$ is positive.*

Proof. The map Φ_{K_S} is composed with a pencil if and only if $\Phi_{\widehat{M}}$ is composed with a pencil. Such map is composed with a pencil, so its image Σ is a curve, if and only if we are able to pick-up two general hyperplanes H_1 and H_2 of $\mathbb{P}^{p_g-1}$ such that $H^2_{|C} = H_1 \cdot H_2 \cdot \Sigma = 0$. However, $\widehat{M} = \Phi^*_{\widehat{M}}(H)$, hence $H^2_{|C}$ is zero if and only if $\widehat{M}^2$ is equal to zero. □

Let us suppose $\widehat{M}^2 > 0$, so that Φ_{K_S} is not composed with a pencil, and its image has dimension 2. In this case, then $\Phi_{\widehat{M}}$ is a finite morphism, and

$$\widehat{M}^2 = \deg(\Phi_{\widehat{M}})\deg(\Sigma) = \deg(\Phi_{K_S})\deg(\Sigma)|G|.$$

Thus

$$\deg(\Phi_{K_S}) = \frac{1}{|G| \cdot \deg(\Sigma)}\widehat{M}^2. \tag{5.6}$$

The above formula may be hard to handle, since both $\deg(\Sigma)$ and $\widehat{M}^2$ are not easy to determine. However, we are interested to study the case p_g equal to three. In this case, then $\Phi_{\widehat{M}}$ is onto, so the image is $\mathbb{P}^2$, a surface of degree 1. Hence $\deg(\Sigma) = 1$ and we obtain

$$\deg(\Phi_{K_S}) = \frac{1}{|G|}\widehat{M}^2. \tag{5.7}$$

Let us write $\widehat{M}^2$ as $\widehat{M}^2 = M^2 - (M^2 - \widehat{M}^2)$. M^2 is the self-intersection of the mobile part of $|K_{C_1\times C_2}|^G$. Instead, $M^2 - \widehat{M}^2$ is the sum of the correction terms arising from each isolated base-point of $|M|$.

M^2 is computable once we know the fixed part of the linear subsystem $|K_{C_1\times C_2}|^G$ (even without the assumption p_g equal to three).

Lemma 5.3.2. *Suppose that any irreducible character χ such that $\langle \chi^1_{can}, \chi\rangle \neq 0$ and $\langle \chi^2_{can}, \overline{\chi}\rangle \neq 0$ has degree 1. Then the fixed part of the linear system $|K_{C_1\times C_2}|^G$ is*

$$Fix\left(|K_{C_1\times C_2}|^G\right) = \sum_{q\in Crit(\lambda_1)} \left(\min_{\chi:\ \langle \chi^1_{can},\chi\rangle\neq 0, \langle \chi^2_{can},\overline{\chi}\rangle\neq 0} t_q^{\chi}\right) R_q^{vert} + \sum_{l\in Crit(\lambda_2)} \left(\min_{\chi:\ \langle \chi^1_{can},\chi\rangle\neq 0, \langle \chi^2_{can},\overline{\chi}\rangle\neq 0} t_l^{\overline{\chi}}\right) R_l^{hor}. \tag{5.8}$$

Proof. The fixed part of $|K_{C_1\times C_2}|^G$ is the common divisor of the fixed parts of that pieces $\left(|K_{C_1}|^{\chi}\otimes|K_{C_2}|^{\overline{\chi}}\right)^G$ that are non-empty, for χ irreducible character. They are non-empty whenever $\langle \chi^1_{can}, \chi\rangle \neq 0$ and $\langle \chi^2_{can}, \overline{\chi}\rangle \neq 0$. By assumption, then χ is of degree 1. Thus Corollary 5.2.9 applies, so that the the fixed part of $\left(|K_{C_1}|^{\chi}\otimes|K_{C_2}|^{\overline{\chi}}\right)^G$ is amount to

$$\sum_{q\in Crit(\lambda_1)} t_q^{\chi} R_q^{vert} + \sum_{l\in Crit(\lambda_2)} t_l^{\overline{\chi}} R_l^{hor}.$$

The common divisor of these fixed parts is the right member of 5.8. □

By definition of M, then

$$M \equiv K_{C_1\times C_2} - Fix(|K_{C_1\times C_2}|^G).$$

Suppose to stay under the hypothesis of Lemma 5.3.2. Thus $Fix(|K_{C_1\times C_2}|^G)$ is a union of fibres. To compute M^2 is then sufficient to know the intersections

$$K_{C_1\times C_2}\cdot R_q^{vert}, \qquad K_{C_1\times C_2}\cdot R_l^{hor}, \qquad \left(R_q^{vert}\right)^2, \qquad \left(R_l^{hor}\right)^2 \quad \text{and} \quad R_q^{vert}\cdot R_l^{hor}.$$

We compute them.
R_q^{vert} can be written as sum of $|G|/m_q$ components $\{g\cdot p\}\times C_2$, with p point over q, and $g\in G$. $\{g\cdot p\}\times C_2$ has self-intersection zero (since two points are always homologous on a connected variety, and then the fibres of $f_1\circ\lambda_{12}$ are always numerically equivalent). Thus we can use the *genus formula* to get

$$K_{C_1\times C_2}\cdot(\{g\cdot p\}\times C_2)=2g(C_2)-2-(\{g\cdot p\}\times C_2)^2=2g(C_2)-2.$$

The same reasoning works for an horizontal divisor R_l^{hor}. Thus, we have got

$$K_{C_1\times C_2}\cdot R_q^{vert}=\frac{|G|}{m_q}\left(2g(C_2)-2\right),\qquad K_{C_1\times C_2}\cdot R_l^{hor}=\frac{|G|}{m_l}\left(2g(C_1)-2\right).$$

Analogously,

$$\left(R_q^{vert}\right)^2=\left(R_l^{hor}\right)^2=0,\qquad\text{and}\qquad R_q^{vert}\cdot R_l^{hor}=\frac{|G|^2}{m_q m_l}.$$

It remains to determine $M^2-\widehat{M}^2$.

5.4 The correction term to the self-intersection of a 2-dimensional linear system with only isolated base points

As remarked in the previous Section 5.3, $M^2-\widehat{M}^2$ is the sum of the correction terms arising from each isolated base-point of $|M|$, the mobile part of the linear subsystem $|K_{C_1\times C_2}|^G$.

The contribution to the correction term of any isolated base-point may be easily computed under the assumption that any irreducible character χ such that $\langle\chi^1_{can},\chi\rangle\neq 0$ and $\langle\chi^2_{can},\overline{\chi}\rangle\neq 0$ has degree 1.

Let us fix a base-point $(p_1,p_2)\in C_1\times C_2$ of the mobile part $|M|$. The point p_1 is over $q\in C_1/G$ and p_2 is over $l\in C_2/G$. Let us fix an irreducible character χ. We can always choose a general basis of $H^{1,0}(C_1)^\chi$ such that each one-form of the basis has the minimum vanishing order t_q^χ at p_1, which is the natural number computed in Lemma 5.1.7.
Similarly, we can choose a general basis of $H^{1,0}(C_2)^{\overline{\chi}}$ such that each one-form of the basis has minimum vanishing order $t_l^{\overline{\chi}}$ at p_2. The choice of this pair of bases gives via tensor product a natural basis of $H^{1,0}(C_1)^\chi\otimes H^{1,0}(C_2)^{\overline{\chi}}$, which is a G-invariant subspace from the assumption χ is of degree one. This permits us to conclude that the divisors spanning the linear subsystem $|K_{C_1\times C_2}|^G$ can be written in a neighbourhood of (p_1,p_2) as

$$t_q^\chi R_q^{vert}+t_l^{\overline{\chi}}R_l^{hor},\qquad \chi\ \text{ such that }\quad \langle\chi^1_{can},\chi\rangle\neq 0,\langle\chi^2_{can},\overline{\chi}\rangle\neq 0.$$

Finally, it is sufficient to remove the fixed part of $|K_{C_1\times C_2}|^G$ computed in Lemma 5.3.2 to get how the divisors spanning $|M|$ are written in a neighbourhood of (p_1, p_2). So, the linear system $|M|$ is generated by p_g divisors locally near (p_1, p_2) of the form

$$a_1 R_q^{vert} + b_1 R_l^{hor}, \qquad \dots \qquad a_{p_g} R_q^{vert} + b_{p_g} R_l^{hor}.$$

Since we assumed that (p_1, p_2) is a base-point and $|M|$ has not fixed components, then without loss of generality $a_1 = b_2 = 0$.
Note that R_q^{vert} and R_l^{hor} are smooth and intersect transversally at (p_1, p_2). In Theorem 5.4.3 we give a general formula to compute directly the contribution of (p_1, p_2) to the correction term $M^2 - \widehat{M}^2$ whenever p_g is equal to three.

The slightly more general setting is the following:
Let $|M|$ be a (not necessarily complete) two-dimensional linear system on a surface S spanned by D_1, D_2, and D_3. Assume that $|M|$ has only isolated base-points, smooth for S, and that in a neighborhood of a basepoint p we can write the divisors D_i as

$$D_1 = aH, \quad D_2 = bK \quad \text{and} \quad D_3 = cH + dK.$$

Here H and K are reduced, smooth, and intersect transversally at p and a, b, c, d are non-negative integers, $b \leq a$.

Let $|\widehat{M}|$ be the linear system obtained as follows: we blow-up the base-point p, take the pullback of the mobile part of $|M|$ and remove the fixed part of this new linear system. If an infinitely near point of p is a base-point for this linear system, then repeat the procedure, until we obtain a (not necessarily complete) linear system $|\widehat{M}|$ such that no infinitely near point of p is a base point of $|\widehat{M}|$. The linear system $|\widehat{M}|$ is called *strict transform* of $|M|$ at p.

Lemma 5.4.1. *Assume that $bc + ad \geq ab$. Then $\widehat{M}^2 = M^2 - ab$.*

Proof. We prove the lemma by induction on (a, b), with $b \leq a$. Here we are considering the lexicographic order $\leq$ defined on the lower half plane $\Delta^{\geq} := \{(a, b) \colon a \geq b\} \subseteq \mathbb{N} \times \mathbb{N}$ as follows:

$$(a', b') \leq (a, b) \text{ if and only if } a' < a \text{ or } a' = a \text{ and } b' \leq b.$$

In this case, $\Delta^{\geq}$ admits the *well-ordering principle* and so it holds the *mathematical induction.*
Suppose that $(a, b) = 0$. Then $|M|$ is base-point free and so $\widehat{M} = M^2 = M^2 - ab$. Now suppose that the statement is true for $(a', b') < (a, b)$. We aim to prove it for (a, b). We blow up the base-point p, take the pullback of the divisors D_i, and remove the fixed part, which is the exceptional divisor

bE of the blowup. In fact the pullback of D_3 contains $c+d$ times E and $c+d \geq b$, thanks to $b \leq a$ and to the assumption $bc+ad \geq ab$:

$$a(c+d) \geq bc+ad \geq ab, \qquad \text{so} \qquad c+d \geq b.$$

Restricted to the preimage of our neighborhood of p, these divisors are:

$$a\widehat{H}+(a-b)E, \qquad b\widehat{K} \qquad \text{and} \qquad c\widehat{H}+d\widehat{K}+(c+d-b)E.$$

Here, $\widehat{H}$ and $\widehat{K}$ are the strict transforms of H and K. Let $|\widehat{M}|$ be the linear system generated by these three divisors, then $\widehat{M}^2 = M^2 - b^2$. If $a = b$ or $b = 0$, then $|\widehat{M}|$ is base-point free and we are done. Otherwise, on the preimage, the linear system $|\widehat{M}|$ has precisely one new base-point: the intersection point of $\widehat{K}$ and E. Locally near this point the three divisors spanning $|\widehat{M}|$ are:

$$(a-b)E, \qquad b\hat{K} \qquad \text{and} \qquad d\hat{K}+(c+d-b)E.$$

We need to distinguish two cases, when $(a-b) < b$ or when $(a-b) \geq b$. In the first case $(a-b) < b$ we get $(b, a-b) < (a, b)$. We define new coefficients $a' := b$, $b' := a-b$, $c' := d$ and $d' := c+d-b$. Otherwise if $(a-b) \geq b$, then $(a-b, b) < (a, b)$, and we define $a' := a-b$, $b' := b$, $c' := c+d-b$, and $d' := d$. For both cases, the new coefficients fulfill the inductive hypothesis, because:
Thanks to $bc+ad \geq ab$, we have

$$\begin{aligned} b'c'+a'd' &= (a-b)d+b(c+d-b) \\ &= ad+bc-b^2 \\ &\geq ab-b^2 = (a-b)b \\ &= a'b'. \end{aligned}$$

By induction, the self-intersection of the new linear system $\widehat{M}$ is equal to

$$\widehat{M}^2 = (M^2-b^2)-b(a-b) = M^2-ab.$$

□

Lemma 5.4.2. *Assume that $bc+ad \leq ab$. Then $\widehat{M}^2 = M^2-(ad+bc)$.*

Proof. We prove the lemma by induction, once more on (a, b), with $b \leq a$. Thus we consider the lexicographic order $\leq$ on $\Delta^{\geq}$, as we have done in the proof of the Lemma 5.4.1.

Suppose that $(a, b) = 0$. Then $|M|$ is base-point free and so $\widehat{M} = M^2 = M^2-(0d+0c)$. Now suppose that the statement is true for $(a', b') < (a, b)$. Our aim is to prove it for (a, b). We blow up the base-point p, take the pullback of the divisors D_i, and remove the fixed part, which is the exceptional divisor $(c+d)E$ of the blowup, if $c+d \leq b$, or the divisor bE, otherwise. Hence we need to distinguish two cases.

Let us suppose first that $c+d \leq b$ $(\leq a)$. Restricted to the preimage of our neighborhood of p, the divisors are

$$a\widehat{H} + (a-(c+d))E, \qquad b\widehat{K} + (b-(c+d))E \qquad \text{and} \qquad c\widehat{H} + d\widehat{K}.$$

Here, $\widehat{H}$ and $\widehat{K}$ are the strict transforms of H and K. Let $|\widehat{M}|$ be the linear system generated by these three divisors, then $\widehat{M}^2 = M^2 - (c+d)^2$. On the preimage, the linear system $|\widehat{M}|$ has precisely two new base-points: the intersection points of $\widehat{H}$ and $\widehat{K}$ with E. Locally near these points the three divisors spanning $|\widehat{M}|$ are respectively

$$a\widehat{H} + (a-(c+d))E, \qquad (b-(c+d))E \qquad \text{and} \qquad c\widehat{H},$$

and

$$(a-(c+d))E, \qquad b\widehat{K} + (b-(c+d))E \qquad \text{and} \qquad d\widehat{K}.$$

We claim that for both points the coefficients of the three divisors satisfy the assumption of the Lemma 5.4.1.
Let us verify it for the first point $\widehat{H} \cap E$: if $c \geq (b-(c+d))$, then define $a' := c$, $b' := b-(c+d)$, $c' := a$, and $d' := a-(c+d)$, otherwise define $a' := b-(c+d)$, $b' := c$, $c' := a-(c+d)$, and $d' := a$. For both the cases $d' \geq b'$ so that $b'c' + a'd' \geq a'd' \geq a'b'$.
Regarding the second point $\widehat{K} \cap E$, we have: if $d \geq (a-(c+d))$, then define $a' := d$, $b' := a-(c+d)$, $c' := b$, and $d' := b-(c+d)$, otherwise define $a' := a-(c+d)$, $b' := d$, $c' := b-(c+d)$, $d' := b$. In the first case $c' \geq a'$, while in the second case $d' \geq b'$. Therefore we get $b'c' + a'd' \geq a'b'$ for both cases.
Thus the Lemma 5.4.1 applies for both points and the self-intersection of the new linear system $\widehat{M}$ at the final step is amount to

$$\widehat{M}^2 = (M^2 - (c+d)^2) - (b-(c+d))c - (a-(c+d))d = M^2 - (ad+bc).$$

It remains to discuss the case $c+d \geq b$.
As we have already done before, we blow up the base-point p, take the pullback of the divisors D_i, and remove the fixed part, which this time is the exceptional divisor bE of the blowup. Restricted to the preimage of our neighborhood of p, these divisors are:

$$a\widehat{H} + (a-b)E, \qquad b\widehat{K} \qquad \text{and} \qquad c\widehat{H} + d\widehat{K} + (c+d-b)E.$$

Here $\widehat{M}^2 = M^2 - b^2$. If $b=0$ or $a=b$, then $|\widehat{M}|$ is base-point free. In the first case $b=0$, we get $ad = bc + ad \leq ab = 0$, so $\widehat{M}^2 = M^2 - b^2 = M^2 = M^2 - (ad+bc)$, and we are done. In the second case $a=b$, we get, thanks to the assumptions $ad + bc \leq ab$ and $b \leq c + d$, that

$$\begin{aligned} a(c+d) &= ad + bc \\ &\leq ab \qquad , \qquad \text{so} \qquad c + d = b = a. \\ &\leq a(c+d) \end{aligned}$$

Also in this case we are done, because $\widehat{M}^2 = M^2 - b^2 = M^2 - (ad + bc)$. It remains to consider when $a - b = 0$ or $b = 0$ does not hold. In this case, on the preimage, the linear system $|\widehat{M}|$ would have precisely one new base-point: the intersection point of $\widehat{K}$ and E. Locally near this point the three divisors spanning $|\widehat{M}|$ are:

$$(a-b)E, \qquad b\hat{K} \qquad \text{and} \qquad d\hat{K} + (c + d - b)E.$$

We need to distinguish two cases, when $(a - b) < b$ or when $(a - b) \geq b$. In the first case $(a - b) < b$ we get $(b, a - b) < (a, b)$. We define new coefficients $a' := b$, $b' := a - b$, $c' := d$ and $d' := c + d - b$. Otherwise if $(a - b) \geq b$, then $(a - b, b) < (a, b)$, and we define $a' := a - b$, $b' := b$, $c' := c + d - b$, and $d' := d$. For both cases, the new coefficients fulfill the inductive hypothesis, because:
Thanks to $bc + ad \leq ab$, we have

$$\begin{aligned} b'c' + a'd' &= (a - b)d + b(c + d - b) \\ &= ad + bc - b^2 \\ &\leq ab - b^2 = (a - b)b \\ &= a'b'. \end{aligned}$$

By induction, the self-intersection of the new linear system $\widehat{M}$ is equal to

$$\begin{aligned} \widehat{M}^2 &= (M^2 - b^2) - (a'd' + b'c') \\ &= M^2 - b^2 - (ad + bc - b^2) \\ &= M^2 - (ad + bc). \end{aligned}$$

□

By applying Lemma 5.4.1 and Lemma 5.4.2 it follows directly

Theorem 5.4.3 (Correction Term Formula)**.** *Let $|M|$ be a two-dimensional linear system on a surface S spanned by D_1, D_2, and D_3. Assume that $|M|$ has only isolated base-points, smooth for S, and that in a neighborhood of a basepoint p we can write the divisors D_i as*

$$D_1 = aH, \quad D_2 = bK \quad \text{and} \quad D_3 = cH + dK.$$

Here H and K are reduced, smooth, and intersect transversally at p and a, b, c, d are non-negative integers, $b \leq a$. Let $|\widehat{M}|$ be the strict transform of $|M|$ along p. Then

$$\widehat{M}^2 = M^2 - \min\{ab,\ ad + bc\}.$$

Chapter 6

Examples with a high degree of the canonical map

In this Chapter, we are going to show some examples of families of product-quotient surfaces with a high degree of the canonical map. Some of them are picked up from the classification of product-quotient surfaces with $p_g = 3$, $q = 0$ and $K^2 = 32$ attained in Chapter 4 (see Theorem 4.9.1).
Some of the presented examples realize a degree of the canonical map not yet discovered in the literature. These degrees are $d = 10, 11, 13, 14, 15$, and 18.
Any example is described without using the language of product-quotient surfaces. In this way, the Chapter can be read separately from the rest of the thesis and the reader does not necessarily need to know the theory of product-quotient surfaces. Therefore, we simply decided to present every family from the equations defining the G-coverings (C_1, λ_1) and (C_2, λ_2), and fixing a suitable action of the group G on both of them.
However, we also describe them as product-quotient surfaces presenting a corresponding pair of spherical generators. This allows to compute the degree of the canonical map by using the results in Chapter 5.

6.1 Examples with degree $d = 5, 7, 10, 11$, and 14

This is a joint work [FG23] together with Dr. C. Gleissner, whom I had the pleasure to know during my visit to the Universität Bayreuth, Germany, as a guest Ph.D. student.

Let F be the Fermat septic curve

$$F = \{x_0^7 + x_1^7 + x_2^7 = 0\} \subset \mathbb{P}^2.$$

In this section we construct a series of surfaces S, as quotients of a product of two copies of F, modulo a suitable diagonal action of the group $\mathbb{Z}_7^2$. For any surface S, we determine the canonical map Φ_{K_S} and compute its degree.

They are the seven product-quotient surfaces with group $G = \mathbb{Z}_7^2$ of Theorem 4.9.1.
On the first copy of F we define the action of $\mathbb{Z}_7^2$ as

$$\phi\colon \mathbb{Z}_7^2 \to \mathrm{Aut}(F), \quad (a,b) \mapsto [(x_0 : x_1 : x_2) \mapsto (x_0 : \zeta_7^a x_1 : \zeta_7^b x_2)], \quad \zeta_7 := e^{\frac{2\pi i}{7}}.$$

This action has 21 points with non trivial stabilizer. They form three orbits of length 7. A representative of each orbit and a generator of the stabilizer is given by:

point	$(-1:0:\zeta_7)$	$(-1:\zeta_7:0)$	$(0:-1:\zeta_7)$
generator	$(1,0)$	$(0,1)$	$(6,6)$

Note that the automorphisms $\phi(a,b)$ are precisely the deck transformations of the cover

$$\lambda\colon F \to \mathbb{P}^1, \qquad (x_0 : x_1 : x_2) \mapsto (x_1^7 : x_2^7).$$

The cover has degree 49 and is branched along $(0:1)$, $(1:0)$ and $(-1:1)$. In particular $F/\mathbb{Z}_7^2 \simeq \mathbb{P}^1$ and π is the quotient map.

Note that in the Example 2.1.12 we have shown (F, λ) is the unique $\mathbb{Z}_7^2$-covering of $\mathbb{P}^1$ up to topological equivalence.
On the second copy of F, for which we use the homogenous variables $y = (y_0 : y_1 : y_2)$, the group acts by $\phi \circ A$, where $A \in \mathrm{Aut}(\mathbb{Z}_7^2)$ is an automorphism depending on the specific example. The explicit choices for A are stated in the tables below. To write the canonical systems of the corresponding unmixed quotients

$$S := (F \times F)/\mathbb{Z}_7^2 \qquad \text{modulo the diagonal actions} \qquad \phi \times (\phi \circ A),$$

we need to fix a suitable basis of the space $H^0(F, \Omega_F^1)$ of global holomorphic 1-forms on F. In affine coordinates such a basis is given by

$$\{\omega_{jk} := u^j v^{k-6} du \mid j + k \leq 4\}, \qquad \text{where} \qquad u := \frac{x_1}{x_0} \quad \text{and} \quad v := \frac{x_2}{x_0}.$$

Note that:

I) The action of $\mathbb{Z}_7^2$ on $H^0(F, \Omega_F^1)$ under pullback with ϕ is

$$\phi(a,b)^*(\omega_{jk}) = \zeta_7^{a(j+1)+b(k-6)} \omega_{jk}.$$

Note that this is not the canonical representation defined in Section 5.2 but its composition with the map $g \mapsto g^{-1}$, which is an automorphism of G since G is abelian.

II) By Theorem 5.2.1, the space of canonical sections $H^0(K_S)$ is isomorphic to the invariant subspace

$$H^0(K_S) \cong \big(H^0(\Omega_F^1) \otimes H^0(\Omega_F^1)\big)^{\mathbb{Z}_7^2},$$

where the action on the tensor product is diagonal, i.e. $(a, b) \in \mathbb{Z}_7^2$ acts via

$$\phi(a, b)^* \otimes \phi\big(A(a, b)\big)^*.$$

The observations I) and II) imply:

Lemma 6.1.1. *A basis of $H^0(K_S)$ is given by the $\mathbb{Z}_7^2$-invariant tensors $\omega_{jklm} := \omega_{jk} \otimes \omega_{lm}$. A tensor ω_{jklm} is invariant if and only if for all $(a, b) \in \mathbb{Z}_7^2$ it holds:*

$$a(j+1)+b(k-6)+a'(l+1)+b'(m-6) \equiv 0 \mod 7, \quad \textit{where} \quad \begin{pmatrix} a' \\ b' \end{pmatrix} := A \begin{pmatrix} a \\ b \end{pmatrix}.$$

We can now state and prove our main result:

Theorem 6.1.2. *For all $A \in \mathrm{Aut}(\mathbb{Z}_7^2)$ in the Table 6.1, the diagonal action $\phi \times (\phi \circ A)$ of $\mathbb{Z}_7^2$ on the product of two Fermat septics is free. The quotient is a regular smooth projective surface S of general type with $p_g = 3$. A basis of $H^0(K_S)$, the canonical map Φ_{K_S} in projective coordinates and its degree are stated in the Table 6.1.*
The image of the canonical map of the last surface is the conic $\{z_1^2 = z_0 z_2\} \subset \mathbb{P}^2$. The surfaces no. 3, 4, 5, *and* 6 *of the table are the first known examples of surfaces with* $\deg(\Phi_{K_S}) = 10$, 11, *and* 14.

The degrees 5 and 7 of the first and second surfaces have also been realized by a different construction [MLP23, Example 4.5].

Remark 6.1.3. The surfaces S in the Table 6.1 are examples of unmixed surfaces isogenous to a product of curves. More precisely, they are *Beauville surfaces*, which are by definition the *rigid surfaces* isogenous to a product. Rigidity means that they do not admit nontrivial deformations. This is equivalent to the fact that the quotient curves C_i/G are isomorphic to $\mathbb{P}^1$ and the quotient maps $C_i \to C_i/G \simeq \mathbb{P}^1$ are branched in three points.

Using Theorem 4.5.8 and Corollary 4.5.10 we find that there are exactly seven families of product-quotient surfaces isogenous to a product of the form $(F \times F)/\mathbb{Z}_7^2$ given by the seven classes of the matrices in table 6.1.

Using the MAGMA [BCP97] algorithm from the paper [GPR22] one can classify all regular unmixed surfaces isogenous to a product of curves with $p_g = 3$ and abelian group G. Among them are the unmixed Beauville surfaces with $p_g = 3$ and abelian group. The latter form seven biholomorphism classes, which are exactly the surfaces in the table of our theorem.

Table 6.1

No	A	Basis of $H^0(K_S)$	$\Phi_{K_S}(x,y)$	$\deg(\Phi_{K_S})$
1.	$\begin{pmatrix} 3 & 3 \\ 6 & 2 \end{pmatrix}$	$\{\omega_{0203}, \omega_{1004}, \omega_{3112}\}$	$(x_0^2x_2^2y_0y_2^3 : x_0^3x_1y_2^4 : x_1^3x_2y_0y_1y_2^2)$	5
2.	$\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$	$\{\omega_{1022}, \omega_{2131}, \omega_{4010}\}$	$(x_0^3x_1y_1^2y_2^2 : x_0x_1^2x_2y_1^3y_2 : x_1^4y_0^3y_1)$	7
3.	$\begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}$	$\{\omega_{1304}, \omega_{2210}, \omega_{3012}\}$	$(x_1x_2^3y_2^4 : x_1^2x_2^2y_0^3y_1 : x_0x_1^3y_0y_1y_2^2)$	10
4.	$\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$	$\{\omega_{0011}, \omega_{1202}, \omega_{2040}\}$	$(x_0^4y_0^2y_1y_2 : x_0x_1x_2^2y_0^2y_2^2 : x_0^2x_1^2y_1^4)$	11
5.	$\begin{pmatrix} 3 & 3 \\ 6 & 4 \end{pmatrix}$	$\{\omega_{0103}, \omega_{1310}, \omega_{3031}\}$	$(x_0^3x_2y_0y_2^3 : x_1x_2^3y_0^3y_1 : x_0x_1^3y_1^3y_2)$	14
6.	$\begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$	$\{\omega_{0101}, \omega_{1313}, \omega_{3030}\}$	$(x_0^3x_2y_0^3y_2 : x_1x_2^3y_1y_2^3 : x_0x_1^3y_0y_1^3)$	14
7.	$\begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix}$	$\{\omega_{0202}, \omega_{2121}, \omega_{4040}\}$	$(x_0^2x_2^2y_0^2y_2^2 : x_0x_1^2x_2y_0y_1^2y_2 : x_1^4y_1^4)$	$\mathrm{im}(\Phi_{K_{S_7}}) = \{z_1^2 = z_0z_2\} \subset \mathbb{P}^2$

proof of Theorem 6.1.2. First we show that the seven diagonal actions $\phi \times (\phi \circ A)$ on $F \times F$ are free. Indeed, as remarked above, the non-trivial stabilizers of the points on the first copy of F are generated by $(1,0)$, $(0,1)$ and $(6,6)$. However, none of these elements have a fixed point on the second copy of F under the twisted actions $\phi \circ A$. Thus, the actions are free and the quotient surfaces S are smooth, projective and of general type. The latter holds because the genus of the Fermat septic is $g(F) = 15 \geq 2$. Moreover, they are regular surfaces by the Corollary 4.4.6, since $F/\mathbb{Z}_7^2$ is biholomorphic to $\mathbb{P}^1$. The geometric genus of each S is therefore equal to

$$p_g = \chi(\mathcal{O}_S) - 1 = \frac{(g(F)-1)^2}{|\mathbb{Z}_7^2|} - 1 = \frac{14^2}{49} - 1 = 3.$$

Using Lemma 6.1.1, we compute a basis of $H^0(K_S)$ for each surface S. Replacing the affine variables by $\frac{x_i}{x_0}$ and $\frac{y_j}{y_0}$ and multiplying by $x_0^4 y_0^4$ we obtain the bi-quartics that define the canonical map.
It remains to determine the degree of Φ_{K_S} for each surface S. Following the strategy explained in Section 5.3, we resolve the indeterminacy of $\Phi_{|K_{F\times F}|^{\mathbb{Z}_7^2}} = \Phi_{K_S} \circ \lambda_{12}$ by a sequence of blowups, where $\lambda_{12} \colon F \times F \to S$ is the quotient map, and $|K_{F\times F}|^{\mathbb{Z}_7^2}$ is the subsystem of $|K_{F\times F}|$ generated by three invariant 2-forms defining Φ_{K_S}. Hence we have

$$\begin{tikzcd}
\widehat{F \times F} \arrow[r] \arrow[dr, "\Phi_{\widehat{M}}"'] & F \times F \arrow[d, dashed, "\Phi_{K_{F\times F}^{\mathbb{Z}_7^2}}"] \\
 & \mathbb{P}^2.
\end{tikzcd}$$

Here the morphism $\Phi_{\widehat{M}}$ is induced by the base-point free linear system $|\widehat{M}|$ obtained as follow: let $|M|$ be the mobile part of $|K_{F\times F}|^{\mathbb{Z}_7^2}$.
We blow up the base-points of $|M|$, take the pullback of $|M|$ and remove the fixed part of this new linear system. We repeat the procedure, until we obtain a base-point free linear system $|\widehat{M}|$.
The self-intersection $\widehat{M}^2$ is positive if and only if $\Phi_{\widehat{M}}$ is not composed with a pencil, by Lemma 5.3.1. In this case $\Phi_{\widehat{M}}$ is onto and it holds:

$$\deg(\Phi_{K_S})|\mathbb{Z}_7^2| = \deg(\Phi_{\widehat{M}}) = \widehat{M}^2 \implies \deg(\Phi_{K_S}) = \frac{1}{49}\widehat{M}^2.$$

For the computation of the resolution, it is convenient to write the divisors of the bi-quartics defining $\Phi_{K_{F\times F}^{\mathbb{Z}_7^2}}$ as linear combinations of the reduced curves $F_j := \{x_j = 0\}$ and $G_k := \{y_k = 0\}$ on $F \times F$. Note that F_j and G_k intersect transversally in $|\mathbb{Z}_7^2| = 49$ points and $(F_j, F_k) = (G_j, G_k) = 0$, for all j, k. Thus, these curves can be illustrated as a grid of 21 vertical and 21 horizontal lines.
Consider the third surface in the table. Here, the divisors of the three bi-quartics spanning the canonical system $|K_{F\times F}|^{\mathbb{Z}_7^2}$ are:

$$F_1 + 3F_2 + 4G_2, \qquad 2F_1 + 2F_2 + 3G_0 + G_1 \qquad \text{and} \qquad F_0 + 3F_1 + G_0 + G_1 + 2G_2. \tag{6.1}$$

The fixed part of $|K_{F\times F}|^{\mathbb{Z}_7^2}$ is F_1 and the mobile part $|M|$ has precisely $4 \times 49 = 196$ base-points given by the intersection among:

$$F_1 \cap G_2, \quad F_2 \cap G_0, \quad F_2 \cap G_1, \quad \text{and} \quad F_2 \cap G_2.$$

In this case observe that $M^2 = (3F_2 + 4G_2)^2 = 24 \times 49$. We can perform the computation of the difference $M^2 - \widehat{M}^2$ by applying the *Correction Term Formula* 5.4.3 recursively to each base-point of $|M|$.
For simplicity, let us call by p_{ij} a point belonging to the intersection $F_i \cap G_j$. Then
-Around p_{12}, the divisors are given by $4G_2$, F_1, and $2F_1 + 2G_2$. In the notation of the Theorem 5.4.3, $a = 4$, $b = 1$ and $c = d = 2$. This implies $ad + bc = 10 \geq 4 = ab$. The correction term is $ab = 4$;
-Around p_{20}, the divisors are $3F_2$, $2F_2 + 3G_0$ and G_0. In this case $a = 3, b = 1, c = 2, d = 3$ and the correction term is $ab = 3$;
-Around p_{21}, we have $3F_2$, $2F_2 + G_1$ and G_1, which yields 3 as correction term;
-Around p_{22}, we have $3F_2 + 4G_2$, $2F_2$ and $2G_2$, thus the correction term is 4.
The degree of the canonical map is therefore given by

$$\begin{aligned}\deg(\Phi_{K_S}) &= \frac{1}{49}\left(M^2 - (M^2 - \widehat{M}^2)\right) \\ &= \frac{1}{49}\left((3F_2 + 4G_2)^2 - 4 \times 49 - 3 \times 49 - 3 \times 49 - 4 \times 49\right) = 10.\end{aligned}$$

The degree of the canonical map of all other surfaces can be computed in the same way. □

Remark 6.1.4. The degrees of the canonical maps of the surfaces of Theorem 6.1.2 may be computed from the associated pair of spherical systems of generators. The strategy is that developed in Chapter 5.

Consider the third surface S in Table 6.1. It is described by the following pair of spherical systems of generators:

$$[(1,0),(0,1),(6,6)] \quad \text{and} \quad [(5,6),(3,6),(6,2)].$$

Let $\epsilon_1 \colon \mathbb{Z}_7^2 \to \mathbb{C}^*$ be the character $\mathbb{Z}_7^2$ defined by sending $(1,0)$ to the first 7-root of the unity ζ_7, and $(0,1)$ to 1. Similarly, ϵ_2 sends $(1,0)$ to 1 and $(0,1)$ to ζ_7. The group of characters of $\mathbb{Z}_7^2$ is generated by ϵ_1 and ϵ_2.

By applying Chevalley-Weil formula [Gle16, Thm. 1.3.3], then

$$\langle \chi^1_{can}, \epsilon_1^{\alpha}\epsilon_2^{\beta}\rangle = -1 + \frac{\alpha+\beta+[6(\alpha+\beta)]_7}{7}$$
$$\langle \chi^2_{can}, \epsilon_1^{\alpha}\epsilon_2^{\beta}\rangle = -1 + \frac{[5\alpha+6\beta]_7+[3\alpha+6\beta]_7+[6\alpha+2\beta]_7}{7}.$$

where χ^1_{can} is the canonical character of the first copy of F, and χ^2_{can} that of the second copy.

The pairs (α,β) for which $\epsilon_1^{\alpha}\epsilon_2^{\beta}$ occurs on χ^1_{can} and $\overline{\epsilon_1^{\alpha}\epsilon_2^{\beta}} = \epsilon_1^{7-\alpha}\epsilon_2^{7-\beta}$ occurs on χ^2_{can} are

$$(\alpha,\beta) \in \{(3,6),(4,4),(5,3)\}.$$

From Theorem 5.2.1 we have $H^{2,0}(S) = \left(H^{1,0}(F)\otimes H^{1,0}(F)\right)^{\mathbb{Z}_7^2}$ decomposes into three pieces of dimension one:

$$H^{1,0}(F)^{\epsilon_1^3\epsilon_2^6}\otimes H^{1,0}(F)^{\epsilon_1^4\epsilon_2^1}, \quad H^{1,0}(F)^{\epsilon_1^4\epsilon_2^4}\otimes H^{1,0}(F)^{\epsilon_1^3\epsilon_2^3}, \quad H^{1,0}(F)^{\epsilon_1^5\epsilon_2^3}\otimes H^{1,0}(F)^{\epsilon_1^2\epsilon_2^4}.$$

Theorem 5.2.8 determines which is respectively a generator of the associated sublinear system given by each of these pieces:

$$3R^{vert}_{(0,1)} + R^{vert}_{(-1,1)} + R^{hor}_{(0,1)} + 2R^{hor}_{(1,0)} + R^{hor}_{(-1,1)},$$
$$2R^{vert}_{(0,1)} + 2R^{vert}_{(1,0)} + R^{hor}_{(0,1)} + 3R^{hor}_{(-1,1)},$$
$$R^{vert}_{(0,1)} + 3R^{vert}_{(1,0)} + 4R^{hor}_{(1,0)}.$$

Thus, the above three divisors are spanning the linear system $|K_{F\times F}|^{\mathbb{Z}_7^2}$. Notice that what we have obtained agrees with (6.1) in the proof of the Theorem 6.1.2. Indeed, we observe that

$$F_0 = R^{vert}_{(-1,1)}, \qquad F_1 = R^{vert}_{(0,1)}, \qquad F_2 = R^{vert}_{(1,0)},$$
$$G_0 = R^{hor}_{(-1,1)}, \qquad G_1 = R^{hor}_{(0,1)}, \qquad G_2 = R^{hor}_{(1,0)}.$$

Finally, it is sufficient to follow the rest of the proof of Theorem 6.1.2 starting from (6.1) to perform the degree of the canonical map.

6.2 Examples with degree $12, 13, 15, 16,$ and 18

This work can be found in [Fal22]. In this section we construct a series of surfaces S, as quotients of a product of the two curves C_1 and C_2, modulo a suitable diagonal action of the group $S_3 \times \mathbb{Z}_3^2$. For any surface S, we determine the canonical map Φ_{K_S} and compute its degree.

These surfaces have $p_g = 3$ and $K_S^2 = 24$, so they are not in the list of Theorem 4.9.1. We have found them however by using the script of Section 4.8 for $(K^2, \chi) = (24, 4)$.

Notation: Let us denote by σ and τ a rotation (3-cycle) and a reflection (transposition) of the group S_3 respectively. Consider also the three irreducible characters of S_3, so the trivial character 1, the character sgn computing the sign of a permutation, and the only 2-dimensional irreducible character $\mu := \frac{1}{2}(\chi_{reg} - sgn - 1)$, where χ_{reg} is the character of the regular representation of S_3.
Let us fix a basis e_1, e_2 of $\mathbb{Z}_3^2$ and consider the dual characters ϵ_1, ϵ_2 of e_1 and e_2, i.e. the characters defined by

$$\epsilon_i(ae_1 + be_2) := \zeta_3^{a\delta_{1i}+b\delta_{2i}}, \qquad \zeta_3 := e^{\frac{2\pi i}{3}},$$

where δ_{ij} is the Kronecker delta.

We consider the projective space $\mathbb{P}^3$ with homogeneous coordinates $x_0, \ldots, x_3$ and the weighted projective space $\mathbb{P}^3(1,1,1,2)$ with homogeneous coordinates $y_0, \ldots, y_3$. Here y_3 is the variable of weight 2. We take the curves $C_1 \subseteq \mathbb{P}^3$ and $C_2 \subseteq \mathbb{P}^3(1,1,1,2)$ as follows

$$C_1\colon \begin{cases} x_2^3 = x_0^3 - x_1^3 \\ x_3^3 = x_0^3 + x_1^3 \end{cases}, \qquad C_2\colon \begin{cases} y_2^3 = y_0^3 + y_1^3 \\ y_3^3 = y_0^6 + y_1^6 - 2\lambda y_0^3 y_1^3 \end{cases}, \lambda \neq -1, 1$$

Both curves are smooth, in fact this is the reason why we assume $\lambda \neq -1, 1$ in the definition of C_2.

On the first curve C_1 we consider the action of $S_3 \times \mathbb{Z}_3^2$ given by

$$\begin{aligned} &\phi_1\colon S_3 \times \mathbb{Z}_3^2 \to \mathrm{Aut}(C_1), \\ &\quad \big(\sigma^i\tau^j, (a,b)\big) \mapsto [(x_0 : x_1 : x_2 : x_3) \mapsto (\zeta_3^i x_{[j]} : x_{[j+1]} : (-1)^j \zeta_3^{2a+2i} x_2 : \zeta_3^{2b+2i} x_3)]. \end{aligned}$$

We leave to the reader to checking that this defines an action.

Note that the automorphisms $\phi_1(\sigma^i\tau^j, (a,b))$ are precisely the deck transformations of the cover

$$\lambda_1\colon C_1 \xrightarrow{9:1} \mathbb{P}^1 \xrightarrow{6:1} \mathbb{P}^1, \quad (x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_1) \mapsto \big(x_0^3 x_1^3 : (x_0^6 + x_1^6)/2\big). \tag{6.2}$$

In particular $C_1/\big(S_3 \times \mathbb{Z}_3^2\big) \simeq \mathbb{P}^1$ and λ_1 is the quotient map.

We point out that the intermediate quotient $\mathbb{P}^1 \xrightarrow{6:1} \mathbb{P}^1$ is the unique S_3-covering of $\mathbb{P}^1$ ramified over three points described in the Example 2.1.13. Instead, the curve C_1 is the unique $S_3 \times \mathbb{Z}_3^2$-covering of $\mathbb{P}^1$ described in the Example 2.1.14, for $p = 3$.

The cover λ_1 is branched along $p_1 := (1 : 1)$, $p_2 := (0 : 1)$ and $p_3 := (-1 : 1)$, corresponding to the three orbits of the points with non trivial stabilizer, of respective length $9, 18$ and 9. A representative of each orbit and a generator of the stabilizer is given by:

	p_1	p_2	p_3
representative	$(1 : 1 : 0 : \sqrt[3]{2})$	$(1 : 0 : 1 : 1)$	$(1 : -\zeta_3 : \sqrt[3]{2} : 0)$
generator	$g_1 := (\tau, (1, 0))$	$g_2 := (\sigma^2, (2, 2))$	$g_3 := (\sigma\tau, (0, 1))$

On the second curve C_2 the action ϕ_2 is defined as

$$\begin{aligned}\phi_2\colon S_3 \times \mathbb{Z}_3^2 &\to \mathrm{Aut}(C_2),\\ \big(\sigma^i\tau^j, (a, b)\big) &\mapsto [(y_0 : y_1 : y_2 : y_3) \mapsto (\zeta_3^i y_{[j]} : y_{[j+1]} : \zeta_3^{a+2b+2i} y_2 : \zeta_3^{2a+2b+i} y_3)].\end{aligned}$$

As in the previous case, we leave to the reader to checking that this defines a group action and note that the automorphisms $\phi_2(\sigma^i\tau^j, (a, b))$ are precisely the deck transformations of the cover

$$\lambda_2\colon C_2 \xrightarrow{9:1} \mathbb{P}^1 \xrightarrow{6:1} \mathbb{P}^1, \qquad (y_0 : y_1 : y_2 : y_3) \mapsto (y_0 : y_1) \mapsto \big(y_0^3 y_1^3 : (y_0^6 + y_1^6)/2\big)\,.$$

Hence $C_2/\big(S_3 \times \mathbb{Z}_3^2\big) \simeq \mathbb{P}^1$ and λ_2 is the quotient map. The cover is branched along $q_1 := (1 : 1)$, $q_2 := (0 : 1)$, $q_3 := (1 : \lambda)$ and $q_4 := (-1 : 1)$, corresponding to the four orbits of the points with non trivial stabilizer, of respective length $27, 18, 18$ and 9. Note that the points q_j are pairwise distinct under the assumption $\lambda \neq -1, 1$.

A representative of each orbit and a generator of the stabilizer is given by:

	q_1	q_2	q_3
rep	$(1 : \zeta_3 : \sqrt[3]{2} : \sqrt[3]{2 - 2\lambda})$	$(0 : 1 : 1 : 1)$	$(1 : \sqrt[3]{\lambda - \sqrt{\lambda^2 - 1}} : \sqrt[3]{1 + \lambda - \sqrt{\lambda^2 - 1}} : 0)$,
gen	$h_1 := (\sigma\tau, 0)$	$h_2 := (\sigma, (1, 0))$	$h_3 := (\mathrm{Id}, (1, 1))$

	q_4
rep	$(1 : -1 : 0 : \sqrt[3]{2 + 2\lambda})$.
gen	$h_4 := (\tau, (1, 2))$

We compute the action of $S_3 \times \mathbb{Z}_3^2$ on $H^0(C_i, \Omega^1_{C_i})$.

By standard adjunction theory $H^0(C_1, \Omega^1_{C_1})$ is isomorphic to $H^0(C_1, \mathcal{O}_{C_1}(2))$, isomorphism mapping a monomial $x_0^{2-\alpha-\beta-\gamma} x_1^\alpha x_2^\beta x_3^\gamma$ to the 1-form $\omega_{\alpha\beta\gamma}$ that in affine coordinates is

$$\omega_{\alpha\beta\gamma} := u^\alpha v^{\beta-2} t^{\gamma-2} du, \qquad \text{where} \qquad u := \frac{x_1}{x_0} \quad v := \frac{x_2}{x_0} \quad \text{and} \qquad t := \frac{x_3}{x_0}.$$

The character of the *canonical* representation of C_1, the action of $S_3 \times \mathbb{Z}_3^2$ on $H^0(C_1, \Omega^1_{C_1})$, can be computed by the standard Chevalley-Weil formula and is amount to

$$\chi^1_{can} = \epsilon_1^2 \cdot \epsilon_2^2 + sgn \cdot \epsilon_1 \cdot \epsilon_2 + sgn \cdot \epsilon_2 + sgn \cdot \epsilon_1 + \mu \cdot \epsilon_1 \cdot \epsilon_2 + \mu \cdot \epsilon_1^2 \cdot \epsilon_2 + \mu \cdot \epsilon_1 \cdot \epsilon_2^2. \quad (6.3)$$

We give an explicit decomposition in irreducible subspaces. Using the expression in affine coordinates we obtain

$$\begin{aligned}(\sigma^i\tau^j, &(a,b)) \cdot \omega_{\alpha\beta\gamma} = \phi_1(\big(\sigma^i\tau^j,(a,b)\big)^{-1})^*(\omega_{\alpha\beta\gamma}) \\ &= (-1)^{j(\beta-1)}\zeta_3^{a(\beta-2)+b(\gamma-2)+(\alpha-(2\alpha+\beta+\gamma-2)[j]+2\beta+2\gamma-7)i}\omega_{(\alpha-(2\alpha+\beta+\gamma-2)[j])\beta\gamma}.\end{aligned}$$

A tedious but straightforward computation gives the following decomposition:

$$\begin{aligned}H^0(C_1, \Omega^1_{C_1}) =& \langle\omega_{011}\rangle_{\epsilon_1^2\cdot\epsilon_2^2} \oplus \langle\omega_{100}\rangle_{sgn\cdot\epsilon_1\cdot\epsilon_2} \oplus \langle\omega_{020}\rangle_{sgn\cdot\epsilon_2} \oplus \langle\omega_{002}\rangle_{sgn\cdot\epsilon_1} \oplus \\ & \langle\omega_{000},\omega_{200}\rangle_{\mu\cdot\epsilon_1\cdot\epsilon_2} \oplus \langle\omega_{010},\omega_{110}\rangle_{\mu\cdot\epsilon_1^2\cdot\epsilon_2} \oplus \langle\omega_{001},\omega_{101}\rangle_{\mu\cdot\epsilon_1\cdot\epsilon_2^2}.\end{aligned}$$

Similarly, adjunction theory gives an isomorphism among $H^0(C_2, \Omega^1_{C_2})$ and $H^0(C_2, \mathcal{O}_{C_2}(4))$ mapping a monomial $y_0^{4-\alpha-\beta-2\gamma} y_1^\alpha y_2^\beta y_3^\gamma$ to the 1-form $\omega'_{\alpha\beta\gamma}$ that in affine coordinates is

$$\omega'_{\alpha\beta\gamma} := (u')^\alpha (v')^{\beta-2}(t')^{\gamma-2} du', \qquad \text{where} \qquad u' := \frac{y_1}{y_0} \quad v' := \frac{y_2}{y_0} \quad \text{and} \qquad t' := \frac{y_3}{y_0^2}.$$

We obtain a basis of 19 dimension space $H^0(C_2, \mathcal{O}_{C_2}(4))$ by taking the 22 monomials of degree 4 in the variables y_j and removing $y_0y_2^3$, $y_1y_2^3$ and y_2^4, that can be expressed in terms of the other monomials using the cubic equation defining C_2. Accordingly we get a basis of $H^0(C_2, \Omega^1_{C_2})$ by removing from that set $\omega'_{\alpha\beta\gamma}$ the 1-forms $\omega'_{040}, \omega'_{030}$ and ω'_{130}. The *canonical* character of C_2 is given by Chevalley-Weil as

$$\begin{aligned}\chi^2_{can} =& sgn \cdot \epsilon_1^2 \cdot \epsilon_2 + sgn \cdot \epsilon_1^2 \cdot \epsilon_2^2 + sgn \cdot \epsilon_1 \cdot \epsilon_2 + sgn \cdot \epsilon_1 + sgn \cdot \epsilon_2^2 + \mu \cdot \epsilon_1 \\ & + \mu \cdot \epsilon_2 + 2\mu \cdot \epsilon_2^2 + sgn \cdot \epsilon_1^2 + \epsilon_1^2 + \mu \cdot \epsilon_1^2 + \mu \cdot \epsilon_1 \cdot \epsilon_2,\end{aligned} \quad (6.4)$$

and the action on $H^0(C_2, \Omega^1_{C_2})$ computed in affine coordinates as above is

$$\begin{aligned}(\sigma^i\tau^j, &(a,b)) \cdot \omega'_{\alpha\beta\gamma} = \phi_2(\big(\sigma^i\tau^j,(a,b)\big)^{-1})^*(\omega'_{\alpha\beta\gamma}) \\ &= (-1)^j\zeta_3^{a(2\beta+\gamma)+b(\beta+\gamma-4)+(\alpha-(2\alpha+\beta+2\gamma-4)[j]+2\beta+\gamma+1)i}\omega'_{(\alpha-(2\alpha+\beta+2\gamma-4)[j])\beta\gamma}.\end{aligned}$$

Another tedious computation gives the decomposition

$$\begin{aligned}H^0(C_2, \Omega^1_{C_2}) =& \langle\omega'_{002}\rangle_{sgn\cdot\epsilon_1^2\cdot\epsilon_2} \oplus \langle\omega'_{021}\rangle_{sgn\cdot\epsilon_1^2\cdot\epsilon_2^2} \oplus \langle\omega'_{120}\rangle_{sgn\cdot\epsilon_1\cdot\epsilon_2} \\ & \oplus \langle\omega'_{101}\rangle_{sgn\cdot\epsilon_1} \oplus \langle\omega'_{200}\rangle_{sgn\cdot\epsilon_2^2} \oplus \langle\omega'_{001},\omega'_{201}\rangle_{\mu\cdot\epsilon_1} \oplus \langle\omega'_{011},\omega'_{111}\rangle_{\mu\cdot\epsilon_2} \\ & \oplus \big(\langle\omega'_{000},\omega'_{400}\rangle \oplus \langle\omega'_{100},\omega'_{300}\rangle\big)_{\mu\cdot\epsilon_2^2} \oplus \langle\omega'_{010}+\omega'_{310}\rangle_{sgn\cdot\epsilon_1^2} \\ & \oplus \langle\omega'_{010}-\omega'_{310}\rangle_{\epsilon_1^2} \oplus \langle\omega'_{110},\omega'_{210}\rangle_{\mu\cdot\epsilon_1^2} \oplus \langle\omega'_{220},\omega'_{020}\rangle_{\mu\cdot\epsilon_1\cdot\epsilon_2}.\end{aligned}$$

We consider unmixed quotients $S := (C_1 \times C_2)/\left(S_3 \times \mathbb{Z}_3^2\right)$ modulo a diagonal action $\phi_1 \times (\phi_2 \circ \Psi)$, where Ψ is one of the automorphisms of $S_3 \times \mathbb{Z}_3^2$. Firstly we study the singularities of S. We observe that C_1 and C_2 have stabilizers of order $6, 3$ and 6 and $2, 3, 3$ and 6 respectively. Hence 18 points of C_1 and 36 points of C_2 have stabilizer of even order. However $S_3 \times \mathbb{Z}_3^2$ has only three elements of order 2 and they are in the same conjugacy class. This means that each of these three elements fix exactly $6 \cdot 12 = 72$ points of $C_1 \times C_2$. Thus S can never be smooth and if it admits only nodes, then they are in total $3 \cdot 72/27 = 8$.
Now let us consider the following automorphisms of $S_3 \times \mathbb{Z}_3^2$

$$\begin{aligned}
&\Psi_1 = Id, && \Psi_2 = \left(\begin{cases} \sigma \mapsto \sigma \\ \tau \mapsto \tau\sigma \end{cases}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\right), \\
&\Psi_3 = \left(\begin{cases} \sigma \mapsto \sigma^2 \\ \tau \mapsto \tau \end{cases}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\right), && \Psi_4 = \left(\begin{cases} \sigma \mapsto \sigma^2 \\ \tau \mapsto \tau \end{cases}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\right).
\end{aligned} \tag{6.5}$$

A direct computation shows us that for these four choices of Ψ the surface S has exactly 8 nodes and no other singularities.

Remark 6.2.1. We use the Theorem 4.5.8 proving that the families of product-quotient surfaces given by these two topological types of group actions on curves having only eight nodes as singularities are those presented in this section.

The vector space $H^0(K_S)$ is isomorphic to the invariant subspace $\left(H^0(\Omega^1_{C_1}) \otimes H^0(\Omega^1_{C_2})\right)^{S_3 \times \mathbb{Z}_3^2}$, where the action on the tensor product is diagonal, i.e. $\left(\sigma^i\tau^j, (a,b)\right) \in S_3 \times \mathbb{Z}_3^2$ acts via

$$\phi_1(\left(\sigma^i\tau^j, (a,b)\right)^{-1})^* \otimes \phi_2(\Psi(\left(\sigma^i\tau^j, (a,b)\right)^{-1}))^*. \tag{6.6}$$

For each character η of $S_3 \times \mathbb{Z}_3^2$ define its twist by Ψ as

$$\eta_\Psi := \eta \circ \Psi^{-1}.$$

Pulling back $H^0(K_S)$ to $C_1 \times C_2$ we obtain from Theorem 5.2.1 that

Lemma 6.2.2. *A basis of $H^0(K_S)$ is given by the $\left(S_3 \times \mathbb{Z}_3^2\right)$-invariant 2-forms of $H^0(\Omega^1_{C_1}) \otimes H^0(\Omega^1_{C_2})$ with respect to the action (6.6). Hence*

$$\left(H^0(\Omega^1_{C_1}) \otimes H^0(\Omega^1_{C_2})\right)^{S_3 \times \mathbb{Z}_3^2} = \bigoplus_{\eta \neq 0} \left(H^0(\Omega^1_{C_1})_\eta \otimes H^0(\Omega^1_{C_2})_{\overline{\eta_\Psi}}\right)^{S_3 \times \mathbb{Z}_3^2},$$

where $H^0(\Omega^1_{C_i})_\eta$ is the isotypic component of $H^0(\Omega^1_{C_i})$ of character η. Moreover

$$p_g = \langle \chi^1_{can} \cdot \chi^2_{can}, 1 \rangle = \sum_{\eta \neq 0} \langle \chi^1_{can}, \eta \rangle \cdot \langle \chi^2_{can}, \overline{\eta_\Psi} \rangle.$$

Denote by $\omega_{jklmrs} := \omega_{jkl} \otimes \omega'_{mrs}$. We can now state and prove our main result:

Theorem 6.2.3. *For all $\Psi \in \mathrm{Aut}(S_3 \times \mathbb{Z}_3^2)$ in (6.5), the diagonal action $\phi_1 \times (\phi_2 \circ \Psi)$ of $S_3 \times \mathbb{Z}_3^2$ on the product of the two curves C_1 and C_2 is not free. The quotient is a canonical model of a regular surface S of general type with $K_S^2 = 24$, $p_g = 3$ and with 8 rational double points as singularities of type $\frac{1}{2}(1,1)$. A basis of $H^0(K_S)$, the canonical map Φ_{K_S} in projective coordinates and its degree are stated in the Table 6.2.*

Proof. We have already mentioned that for all Ψ in (6.5) the action is not free and the quotient S has 8 singularities of type $\frac{1}{2}(1,1)$ and no other singularities. The genus of the two curves is $g(C_i) \geq 2$, hence $C_1 \times C_2$ has ample canonical divisor and so S has ample canonical divisor too. It follows S is a canonical model.

The self-intersection of the canonical divisor of each S is amount to

$$K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|S_3 \times \mathbb{Z}_3^2|} = 24.$$

They are regular surfaces, because they do not possess any non-zero holomorphic one-forms, since $C_i/\left(S_3 \times \mathbb{Z}_3^2\right)$ is biholomorphic to $\mathbb{P}^1$. The geometric genus of each S is therefore equal to (see Theorem 4.4.5)

$$p_g = \chi(\mathcal{O}_S) - 1 = \frac{(g(C_1)-1)(g(C_2)-1)}{|S_3 \times \mathbb{Z}_3^2|} + \frac{1}{12}\left(8 \cdot \frac{3}{2}\right) - 1 = 3.$$

Using Lemma 6.2.2 we have computed a basis of $H^0(K_S)$. In fact since we have proved that $p_g = 3$ it is enough to verify that the given elements of the table are invariant for the corresponding action. Applying the explicit isomorphisms from $H^0(C_1, \Omega^1_{C_1})$ to $H^0(C_1, \mathcal{O}_{C_1}(2))$ and from $H^0(C_2, \Omega^1_{C_2})$ to $H^0(C_2, \mathcal{O}_{C_2}(4))$ we obtain the product of quadrics and quartics defining the canonical map in the table.

It remains to determine the degree of Φ_{K_S} for each surface S. Instead to work on S it is convenient to work on $C_1 \times C_2$, following the strategy of Section 5.3:

$$\begin{tikzcd}
C_1 \times C_2 \arrow[r, "\lambda_{12}"] \arrow[dr, "\Phi_{K_{C_1 \times C_2}}"'] & S \arrow[r, dashed, "\Phi_{K_S}"] & \mathbb{P}^2 \\
& \mathbb{P}^{10 \cdot 19 - 1} \arrow[ur, dashed] &
\end{tikzcd}.$$

That the map $\Phi_{K_S} \circ \lambda_{12}$ is induced by the sublinear system $|K_{C_1 \times C_2}|^{S_3 \times \mathbb{Z}_3^2}$ of $|K_{C_1 \times C_2}|$ generated by the three invariant 2-forms defining Φ_{K_S}.

We resolve the indeterminacy of $\Phi_{K_{C_1 \times C_2}^{S_3 \times \mathbb{Z}_3^2}} = \Phi_{K_S} \circ \lambda_{12}$ by a sequence of blowups

$$\begin{tikzcd}
\widehat{C_1 \times C_2} \arrow[r] \arrow[dr, "\Phi_{\widehat{M}}"'] & C_1 \times C_2 \arrow[d, dashed, "\Phi_{K^{S_3\times\mathbb{Z}_3^2}_{C_1\times C_2}}"] \\
& \mathbb{P}^2.
\end{tikzcd}$$

Here the morphism $\Phi_{\widehat{M}}$ is induced by the base-point free linear system $|\widehat{M}|$ obtained as follow: let $|M|$ be the mobile part of $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$.
We blow up the base-points of $|M|$, take the pullback of the mobile part $|M|$ and remove the fixed part of this new linear system. We repeat the procedure until we obtain a base-point free linear system $|\widehat{M}|$.

From Lemma 5.3.1, then the self-intersection $\widehat{M}^2$ is positive if and only if $\Phi_{\widehat{M}}$ is not composed with a pencil. In this case $\Phi_{\widehat{M}}$ is onto and it holds:

$$\deg(\Phi_{K_S}) = \frac{1}{|S_3 \times \mathbb{Z}_3^2|}\deg(\Phi_{\widehat{M}}) = \frac{1}{54}\widehat{M}^2.$$

For the computation of the resolution, it is convenient to write the divisors of the product of quadrics and quartics defining Φ_{K_S} (and hence $\Phi_{K^{S_3\times\mathbb{Z}_3^2}_{C_1\times C_2}}$) as linear combinations of the curves $F_j := \{x_j = 0\}$ and $G_k := \{y_k = 0\}$ on $C_1 \times C_2$. We point out that these curves are reduced and intersect pairwise transversally thanks to the assumption $\lambda \neq -1, 1$. In particular $(F_j, F_k) = (G_j, G_k) = 0$ and $(F_j, G_k) = 81$, for $k \neq 3$, while $(F_j, G_3) = 162$. Consider the first surface in the table. Here, the divisors of the three products of quadrics and quartics spanning the subsystem $|\Phi_{K^{S_3\times\mathbb{Z}_3^2}_{C_1\times C_2}}|$ are:

$$F_0 + F_1 + 2G_2 + G_3, \qquad 2F_2 + 2G_0 + 2G_1 \qquad \text{and} \qquad 2F_3 + 4G_2. \tag{6.7}$$

Then $|\Phi_{K^{S_3\times\mathbb{Z}_3^2}_{C_1\times C_2}}|$ has not fixed part so that

$$M^2 = (\lambda_{12}^* K_S)^2 = |S_3 \times \mathbb{Z}_3^2| \cdot K_S^2 = 54 \cdot 24.$$

Furthermore, $|\Phi_{K^{S_3\times\mathbb{Z}_3^2}_{C_1\times C_2}}|$ has precisely 81 (non reduced) isolated base-points $F_2 \cap G_2$. We can perform the computation of the difference $M^2 - \widehat{M}^2$ by applying the *Correction term formula* 5.4.3, recursively for each base-point of $|\Phi_{K^{S_3\times\mathbb{Z}_3^2}_{C_1\times C_2}}|$:
In a neighbourhood of each of these base-points the three divisors are respectively

$$2G_2, \qquad 2F_2 \qquad \text{and} \qquad 4G_2.$$

Since F_2 and G_2 are transversal we are in the situation of the Theorem 5.4.3, with $H = G_2$ and $K = F_2$, $a = 4, b = c = 2$, and $d = 0$. This implies

$ad + bc = 4 \leq ab = 8$. The correction term is $ab + cd = 4$ for each of the 81 base-points. Thus

$$M^2 - \widehat{M}^2 = 4 \cdot 81.$$

The degree of the canonical map is therefore given by

$$\deg(\Phi_{K_S}) = \frac{1}{54}\widehat{M}^2 = \frac{1}{54}\left(M^2 - (M^2 - \widehat{M}^2)\right) = \frac{1}{54}(54 \cdot 24 - 4 \cdot 81) = 18.$$

Now we take in exam the second surface in our table. Here the subsystem $|\Phi_{K^{S_3 \times \mathbb{Z}_3^2}_{C_1 \times C_2}}|$ has not fixed part and it is spanned by:

$$D_1 := 2F_2 + G_0 + G_1 + G_3, \quad D_2 := 2F_3 + 2G_0 + 2G_1 \quad \text{and} \quad D_3 := F_2 + 2G_2 + \Delta,$$

where $\Delta = (\zeta_3 x_0 y_0^2 - x_1 y_1^2)$. The (set-theoretical) base locus is

$$F_2 \cap G_0, F_2 \cap G_1, \quad \Delta \cap G_0, \Delta \cap G_1, \quad \text{and} \quad \Delta \cap F_3 \cap G_3.$$

We remark that the other pieces of the base locus are empty. In fact that points would belong in some $F_i \cap F_j$ or $G_i \cap G_j$ and we have already mentioned that they are pairwise disjoint.

We determine the correction term to the self intersection number for each of these base-points of $|\Phi_{K^{S_3 \times \mathbb{Z}_3^2}_{C_1 \times C_2}}|$.

We consider first the 81 points $F_2 \cap G_i$, for $i = 0, 1$. Here F_2 and G_i intersect transversally on each of them. Around one of these points, the divisors D_k are given by $G_i + 2F_2$, $2G_i$ and F_2. We are in the situation of the Theorem 5.4.3, with $H = G_i$ and $K = F_2$, $a = d = 2$ and $b = c = 1$. Hence $ad + bc = 5 \geq 2 = ab$, which yields $ab = 2$ as correction term.

We let go on to the 81 base-points $\Delta \cap G_i$. The local coordinates around one of these points are $X := x_j/x_i$ and $Y := y_i/y_j$, where $j = 0, 1, j \neq i$. So the divisors D_k are respectively given by

$$\{Y = 0\}, \qquad 2\{Y = 0\} \qquad \text{and} \qquad \{\zeta_3^{1+i} Y^2 - X = 0\}.$$

Thus D_1 and D_3 intersect transversally in $(0, 0)$ and we fall down once more in the situation of the Theorem 5.4.3. Here $H = D_3$ and $K = D_1$, $a = b = 1$, $c = 0$ and $d = 2$. Since $ad + bc = 2 \geq 1 = ab$, then the correction term is amount to $ab = 1$.

We consider finally the points $\Delta \cap F_3 \cap G_3$. These points satisfy the equations

$$\begin{cases} y_3^3 = y_0^6 + y_1^6 - 2\lambda y_0^3 y_1^3 & = 0 \\ x_3^2 = x_0^3 + x_1^3 & = 0\ . \\ \zeta_3 x_0 y_0^2 - x_1 y_1^2 & = 0 \end{cases} \tag{6.8}$$

The last two equations imply that $x_1^3 = -x_0^3$ and

$$x_0^3 y_0^6 = (\zeta_3 x_0 y_0^2)^3 = (x_1 y_1^2)^3 = x_1^3 y_1^6 = -x_0^3 y_1^6.$$

Thus $y_0^6 + y_1^6 = 0$ and comparing it with the first equation of 6.8 we get $\lambda y_0^3 y_1^3 = 0$. Therefore $\Delta \cap F_3 \cap G_3$ is non empty only if $\lambda = 0$.
Let us suppose $\lambda \neq 0$. Then

$$M^2 - \widehat{M}^2 = 2 \cdot 2 \cdot 81 + 2 \cdot 81 = 6 \cdot 81,$$

and the degree of the canonical map is amount to

$$\deg(\Phi_{K_S}) = \frac{1}{54}\left(M^2 - (M^2 - \widehat{M}^2)\right) = \frac{1}{54}(54 \cdot 24 - 6 \cdot 81) = 15.$$

It remains to consider the case when $\lambda = 0$. The base-points $\Delta \cap F_3 \cap G_3$ are the following 54 ones:

$$t_k := \left(\left(1 : -\zeta_3^{k_1} : \sqrt[3]{2}\zeta_3^{k_2} : 0\right), \left(1 : e^{\frac{\pi i}{6}}\zeta_6^{k_3} : \sqrt[6]{2}e^{\frac{\pi i}{12}(1-2[k_3]_2)}\zeta_3^{k_4} : 0\right)\right),$$
$$\text{with} \qquad k_1 + k_3 \equiv 2 \mod 3,$$

where $k_i = 0, 1, 2$, for $i \neq 3$, and $k_3 = 0, \ldots, 5$. Fix coordinates $X := x_1/x_0 + \zeta_3^2$ and $Y := y_1/y_0 - e^{\frac{\pi i}{6}}$ around one of these points, for example that one for $k = (2, 0, 0, 0)$. The divisors D_k are locally given by

$$\{Y = 0\}, \qquad 2\{X = 0\} \qquad \text{and} \qquad \{Y(2e^{\frac{\pi i 5}{6}} + Y - 2e^{\frac{\pi i 5}{6}}X - XY) = 0\} = \{Y = 0\}.$$

In this case $H = \{X = 0\}$ and $K = \{Y = 0\}$ and $a = 2$ and $b = d = 1$, $c = 0$. The correction term is $ab = 2$.
Hence

$$M^2 - \widehat{M}^2 = 2 \cdot 2 \cdot 81 + 2 \cdot 81 + 2 \cdot 54 = 6 \cdot 81 + 2 \cdot 54.$$

The degree of the canonical map is therefore given by

$$\deg(\Phi_{K_S}) = \frac{1}{54}\left(M^2 - (M^2 - \widehat{M}^2)\right) = \frac{1}{54}(54 \cdot 24 - 6 \cdot 81 - 2 \cdot 54) = 13.$$

The degree of the canonical map of all other surfaces of Table 6.2 can be computed in a similar way. $\square$

Table 6.2

No	Ψ	Basis of $H^0(K_S)$	$\Phi_{K_S}(x,y)$	$\deg(\Phi_{K_S})$
1.	Id	$\{\omega_{100021}, \omega_{020200}, \omega_{002040}\}$	$(x_0x_1y_2^2y_3 : x_2^2y_0^2y_1^2 : x_3^2y_2^4)$	18
2.	Ψ_2	$\{\omega_{020101}, \omega_{002200}, \zeta_3\omega_{010020} - \omega_{110220}\}$	$(x_2^2y_0y_1y_3 : x_3^2y_0^2y_1^2 : x_2y_2^2(\zeta_3x_0y_0^2 - x_1y_1^2))$	$\begin{cases} 15 & \text{if} \quad \lambda \neq 0 \\ 13 & \text{if} \quad \lambda = 0 \end{cases}$
3.	Ψ_3	$\{\omega_{100002}, \omega_{020040}, \omega_{001220} + \omega_{101020}\}$	$(x_0x_1y_3^2 : x_2^2y_2^4 : x_3y_2^2(x_0y_1^2 + x_1y_0^2))$	$\begin{cases} 18 & \text{if} \quad \lambda \neq 0 \\ 16 & \text{if} \quad \lambda = 0 \end{cases}$
4.	Ψ_4	$\{\omega_{100120}, \omega_{020101}, \omega_{000020} + \omega_{200220}\}$	$(x_0x_1y_0y_1y_2^2 : x_2^2y_0y_1y_3 : y_2^2(x_0^2y_0^2 + x_1^2y_1^2))$	12

Remark 6.2.4. In this case we can also compute the degree of the canonical map with the strategy developed in Chapter 5 only for the family no. 1 in Table 6.2 since in the other cases there is one character χ of degree two such that χ occurs on χ^1_{can} and $\overline{\chi}$ occurs on χ^2_{can}.

We compute the degree of the canonical map of the surface no. 1. in Table 6.2 with the strategy developed in Chapter 5.

Consider the first surface S in Table 6.1. It is described by the following pair of spherical systems of generators:

$$[(\tau,(1,0)),(\sigma^2,(2,2)),(\sigma\tau,(0,1))] \quad \text{and} \quad [(\sigma\tau,0),(\sigma,(1,0)),(\mathrm{Id},(1,1)),(\tau,(1,2))].$$

We apply Chevalley-Weil formula [Gle16, Thm. 1.3.3] to both the curves C_1 and C_2 to perform the canonical characters χ^1_{can} and χ^2_{can}. They are (6.3) and (6.4) respectively.

We notice that the irreducible characters χ such that χ occurs on χ^1_{can} and $\overline{\chi}$ occurs on χ^2_{can} have degree one and are precisely:

$$sgn\cdot\epsilon_1\cdot\epsilon_2, \qquad sgn\cdot\epsilon_2, \qquad \text{and} \qquad sgn\cdot\epsilon_1.$$

From Theorem 5.2.1 we have $H^{2,0}(S) = \left(H^{1,0}(C_1)\otimes H^{1,0}(C_2)\right)^{S_3\times\mathbb{Z}_3^2}$ decomposes into three pieces of dimension one:

$$H^{1,0}(C_1)^{sgn\cdot\epsilon_1\cdot\epsilon_2}\otimes H^{1,0}(C_2)^{sgn\cdot\epsilon_1^2\cdot\epsilon_2^2}, \quad H^{1,0}(C_1)^{sgn\cdot\epsilon_2}\otimes H^{1,0}(C_2)^{sgn\cdot\epsilon_2^2},$$

$$H^{1,0}(C_1)^{sgn\cdot\epsilon_1}\otimes H^{1,0}(C_2)^{sgn\cdot\epsilon_1^2}.$$

Theorem 5.2.8 determines which is respectively a generator of the associated linear subsystem given by each of these pieces:

$$\begin{aligned}
&R^{vert}_{(0,1)} + R^{hor}_{(1,\lambda)} + 2R^{hor}_{(-1,1)},\\
&2R^{vert}_{(1,1)} + 2R^{hor}_{(0,1)},\\
&2R^{vert}_{(-1,1)} + 4R^{hor}_{(-1,1)}.
\end{aligned}$$

Thus, the above three divisors are spanning the linear system $|K_{C_1\times C_2}|^{S_3\times\mathbb{Z}_3^2}$. Notice that what we have obtained agrees with (6.7) in the proof of the Theorem 6.2.3. Indeed, we observe that

$$\begin{aligned}
F_2 &= R^{vert}_{(1,1)}, \qquad & F_0+F_1 &= R^{vert}_{(0,1)}, \qquad & F_3 &= R^{vert}_{(-1,1)},\\
G_0+G_1 &= R^{hor}_{(0,1)}, \qquad & G_3 &= R^{hor}_{(1,\lambda)}, \qquad & G_2 &= R^{hor}_{(-1,1)}.
\end{aligned}$$

Finally, it is sufficient to follow the rest of the proof of Theorem 6.2.3 starting from (6.1) to compute the degree of the canonical map.

6.3 Example with degree 24

This is a joint work with Dr. Davide Frapporti. The surfaces S constructed here are a quotient of a product of the two curves C_1 and C_2 modulo a suitable diagonal action of the group $D_4 \times \mathbb{Z}_2^2$.

Notation: Let $D_4 = \langle r, t | r^4, t^2, trt^{-1} = r^3 \rangle$ be the dihedral group of order 8, where r and t denote a rotation (of order 4) and a reflection (of order 2) of the square respectively. Consider also the five irreducible characters of D_4, so the trivial character 1 and the characters

$$\mu_1 \colon D_4 \to \mathbb{Z}_2^2 \to \mathbb{C}^*, \quad t^j r^i \mapsto (-1)^i, \qquad \mu_2 \colon D_4 \to \mathbb{Z}_2^2 \to \mathbb{C}^*, \quad t^j r^i \mapsto (-1)^j,$$
$$\mu_1 \cdot \mu_2 \colon D_4 \to \mathbb{Z}_2^2 \to \mathbb{C}^*, \quad t^j r^i \mapsto (-1)^{i+j}, \qquad \chi := \frac{1}{2}\left(\chi_{reg} - (1 + \mu_1 + \mu_2 + \mu_1 \cdot \mu_2)\right).$$

where χ_{reg} is the character of the regular representation of D_4.
Let us fix a basis e_1, e_2 of $\mathbb{Z}_2^2$ and consider the dual characters ϵ_1, ϵ_2 of e_1 and e_2, i.e. the characters defined by

$$\epsilon_i(ae_1 + be_2) := (-1)^{a\delta_{1i} + b\delta_{2i}},$$

where δ_{ij} is the Kronecker delta.

We consider the projective space $\mathbb{P}^4(1,1,1,1,2)$, with homogeneous coordinates $x_0, \cdots, x_4$. Here x_4 is the variable of weight 2. Let us consider also the projective space $\mathbb{P}^5$ with homogeneous coordinates $y_0, \ldots, y_5$.

We take the curves $C_1 \subseteq \mathbb{P}^4(1,1,1,1,2)$, and $C_2 \subseteq \mathbb{P}^5$ as follows

$$C_1 \colon \begin{cases} x_1^2 = \frac{\lambda_1 - 1}{2} x_3^2 + \frac{\lambda_1 + 1}{2} x_0^2 \\ x_2^2 = \frac{\lambda_1 + 1}{2} x_3^2 + \frac{\lambda_1 - 1}{2} x_0^2 \\ 4x_4^2 = \lambda_2^2 (x_3^2 + x_0^2)^2 - (x_3^2 - x_0^2)^2 \end{cases}, \qquad \lambda_i \notin \{-1, 0, 1\}, \quad \text{and} \quad \lambda_2 \neq \pm\lambda_1,$$

$$C_2 \colon \begin{cases} y_2^2 = y_0^2 - y_1^2 \\ y_3^2 = \mu_3^2 y_0^2 - y_1^2 \\ y_4^2 = (\mu_1 y_0 - y_1)(\mu_2 y_0 - y_1) \\ y_5^2 = (\mu_1 y_0 + y_1)(\mu_2 y_0 + y_1) \end{cases}, \qquad \mu_i \notin \{-1, 0, 1\}, \quad \text{and} \quad \mu_i \neq \pm\mu_j, i \neq j.$$

Both curves are smooth, in fact this is the reason why we assume those restrictions on the coefficients λ_i and μ_j in the definition of C_1 and C_2.

On the first curve C_1, we consider the action $\phi_1 \colon D_4 \times \mathbb{Z}_2^2 \to \mathrm{Aut}(C_1)$ given by

$$(1, (a, b)) \mapsto [\mathbf{x} \mapsto (x_0 : (-1)^{a+b} x_1 : (-1)^a x_2 : (-1)^b x_3 : (-1)^{a+b} x_4)],$$
$$\left(t^j r^i, (0,0)\right) \mapsto [\mathbf{x} \mapsto (x_{3[i]} : (-1)^{j + \frac{i(i+1)}{2}} x_{1+[i]} : (-1)^{\frac{i(i-1)}{2}} x_{2-[i]} : x_{3-3[i]} : (-1)^{i+j} x_4)].$$

We leave to the reader to checking that this action is well-defined.

Note that the automorphisms $\phi_1(t^j r^i,(a,b))$ are precisely the deck transformations of the cover

$$\pi_1\colon C_1 \xrightarrow{16:1} \mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^1, \qquad \mathbf{x} \mapsto (x_3^2+x_0^2 : x_3^2-x_0^2) \mapsto \left((x_3^2+x_0^2)^2 : (x_3^2-x_0^2)^2\right).$$

In particular $C_1/\left(D_2\times\mathbb{Z}_2^2\right)\simeq\mathbb{P}^1$ and π_1 is the quotient map. The cover is branched over $p_1:=(1:0)$, $p_2:=(1:1)$ and $p_3:=(1:\lambda_1^2)$, $p_4:=(1:\lambda_2^2)$ and $p_5:=(0:1)$, corresponding to the five orbits of the points with non trivial stabilizer, each one of length 16. Note that p_j are pairwise distinct thanks to the assumptions $\lambda_i\notin\{-1,0,1\}$, and $\lambda_2\neq\pm\lambda_1$. A representative of each orbit and a generator of the stabilizer is given by:

	p_1	p_2
repres.	$(1:\sqrt{\lambda_1}:\sqrt{\lambda_1}:1:\lambda_2)$	$(0:\sqrt{\lambda_1-1}:\sqrt{\lambda_1+1}:\sqrt{2}:\sqrt{\lambda_2^2-1})$
gen.	$g_1:=(tr,(0,0))$	$g_2:=(tr^2,(0,1))$

	p_3
repres.	$(\sqrt{1-\lambda_1}:0:\sqrt{2\lambda_1}:\sqrt{1+\lambda_1}:\sqrt{\lambda_2^2-\lambda_1^2})$
gen.	$g_3:=(tr^2,(1,0))$

	p_4	p_5
repres.	$(\sqrt{1-\lambda_2}:\sqrt{\lambda_1-\lambda_2}:\sqrt{\lambda_1+\lambda_2}:\sqrt{1+\lambda_2}:0)$	$(i:i:1:1:i)$
gen.	$g_4:=\left(r^2,(1,0)\right)$	$g_5:=(tr^3,(0,1))$

.

On the second curve C_2 the action $\phi_2\colon D_2\times\mathbb{Z}_2^2\to\mathrm{Aut}(C_2)$ is defined as

$$(1,(a,b))\mapsto[\mathbf{y}\mapsto(y_0:y_1:(-1)^{a+b}y_2:(-1)^b y_3:y_4:y_5)],$$
$$\left(t^j r^i,(0,0)\right)\mapsto[\mathbf{y}\mapsto(y_0:(-1)^j y_1:y_2:(-1)^{i+j}y_3:(-1)^{\frac{i(3i+1)}{2}+ij}y_{4+[j]}:(-1)^{\frac{i(3i-1)}{2}+ij}y_{5-[j]})].$$

As in the previous case, we leave it to the reader to check that this action is well-defined and note that the automorphisms $\phi_2(t^j r^i,(a,b))$ are precisely the deck transformations of the cover

$$\pi_2\colon C_2 \xrightarrow{16:1} \mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^1, \qquad \mathbf{y}\mapsto(y_0:y_1)\mapsto\left(y_0^2:y_1^2\right).$$

Hence $C_2/\left(D_2\times\mathbb{Z}_2^2\right)\simeq\mathbb{P}^1$ and π_2 is the quotient map. The cover is branched over $q_1:=(0:1)$, $q_2:=(1:0)$, $q_3:=(1:1)$, $q_4:=(1:\mu_1^2)$, $q_5:=(1:\mu_2^2)$ and $q_6:=(1:\mu_3^2)$, corresponding to the six orbits of the points with non-trivial stabilizer, each one of length 16. Note that the points q_j are pairwise distinct under the assumptions $\mu_i\notin\{-1,0,1\}$, and $\mu_i\neq\pm\mu_j, i\neq j$. A representative of each orbit and a generator of the stabilizer is given by:

	q_1	q_2
representative	$(0:1:i:i:1:1)$	$(1:0:1:\mu_3:\sqrt{\mu_1\mu_2},\sqrt{\mu_1\mu_2})$
generator	$h_1:=\left(tr^3,(1,1)\right)$	$h_2:=(tr,(1,0))$

	q_3
representative	$(1:1:0:\sqrt{\mu_3^2-1},\sqrt{(\mu_1-1)(\mu_2-1)}:\sqrt{(\mu_1+1)(\mu_2+1)})$
generator	$h_3 := (1,(1,0))$

	q_4
representative	$(1:\mu_1:\sqrt{1-\mu_1^2}:\sqrt{\mu_3^2-\mu_1^2}:0:\sqrt{2\mu_1(\mu_2+\mu_1)})$
generator	$h_4 := (t,(0,0))$

	q_5
representative	$(1:\mu_2:\sqrt{1-\mu_2^2}:\sqrt{\mu_3^2-\mu_2^2}:0:\sqrt{2\mu_2(\mu_2+\mu_1)})$
generator	$h_5 := (t,(0,0))$

	q_6
representative	$(1:\mu_3:\sqrt{1-\mu_3^2}:0:\sqrt{(\mu_1-\mu_3)(\mu_2-\mu_3)}:\sqrt{(\mu_1+\mu_3)(\mu_2+\mu_3)})$.
generator	$h_6 := \left(r^2,(1,1)\right)$

We compute the action of $D_4\times\mathbb{Z}_2^2$ on $H^0(C_i,\Omega^1_{C_i})$. By the adjunction formula $H^0(C_1,\Omega^1_{C_1})$ is isomorphic to $H^0(C_1,\mathcal{O}_{C_1}(2))$, isomorphism mapping a monomial $x_0^{2-\alpha-\beta-\gamma-2\delta}x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta$ to the 1-form $\omega_{\alpha\beta\gamma\delta}$ that in local affine coordinates is

$$\omega_{\alpha\beta\gamma\delta} := u^{\alpha-1}v^{\beta-1}t^\gamma s^{\delta-1}dt, \quad \text{where} \quad u:=\frac{x_1}{x_0} \quad v:=\frac{x_2}{x_0}, \quad t:=\frac{x_3}{x_0} \quad \text{and} \quad s:=\frac{x_4}{x_0^2}.$$

We obtain a basis of 9 dimension space $H^0(C_1,\mathcal{O}_{C_1}(2))$ by taking the 11 monomials of degree 2 in the variables $x_0,\dots x_3,x_4$, and removing x_1^2, x_2^2, that can be expressed in terms of the other monomials using the quadratic equations defining C_1. Accordingly we get a basis of $H^0(C_1,\Omega^1_{C_1})$ by removing from that set $\omega_{\alpha\beta\gamma\delta}$ the 1-forms $\omega_{2000},\omega_{0200}$.

The *canonical* character of C_1 is given by the Chevalley-Weil formula as

$$\chi^1_{can} = \mu_2\cdot\epsilon_1+\mu_2+\mu_1\cdot\epsilon_1+\chi+\chi\cdot\epsilon_2+\mu_1\cdot\epsilon_1\cdot\epsilon_2+\epsilon_1\cdot\epsilon_2.$$

We give an explicit decomposition in irreducible subspaces. Using the expression in affine coordinates we obtain

$$\begin{aligned}(t^jr^i,(a,b))\cdot\omega_{\alpha\beta\gamma\delta} &= \phi_1(\left(t^jr^i,(a,b)\right)^{-1})^*(\omega_{\alpha\beta\gamma\delta}) = \\ &= (-1)^K\,\omega_{(\alpha-(\alpha-\beta)[i])(\beta+(\alpha-\beta)[i])(\gamma-\gamma[i]+(2-(\alpha+\beta+\gamma+2\delta))[i])\delta}\end{aligned}$$

where

$$\begin{aligned}K := a(\alpha+\beta+\delta-1)+b(\alpha+\gamma+\delta-1)+j(\alpha+\delta)+\frac{3i(3i+1)}{2}(\alpha+\beta)\\ +i(\beta+\delta-1+j(\alpha+\beta)).\end{aligned}$$

A tedious but straightforward computation gives the following decomposition:

$$\begin{aligned}H^0(C_1,\Omega^1_{C_1}) =&\langle\omega_{1100}\rangle_{\mu_2\cdot\epsilon_1}\oplus\langle\omega_{0001}\rangle_{\mu_2}\oplus\langle\omega_{0010}\rangle_{\mu_1\cdot\epsilon_1}\oplus\langle\omega_{1000},\omega_{0110}\rangle_\chi\oplus\\ &\langle\omega_{1010},\omega_{0100}\rangle_{\chi\cdot\epsilon_2}\oplus\langle\omega_{0000}+\omega_{0020}\rangle_{\mu_1\cdot\epsilon_1\cdot\epsilon_2}\oplus\langle\omega_{0000}-\omega_{0020}\rangle_{\epsilon_1\cdot\epsilon_2}.\end{aligned}$$

Similarly, adjunction theory gives an isomorphism among $H^0(C_2, \Omega^1_{C_2})$ and $H^0(C_2, \mathcal{O}_{C_2}(2))$ mapping a monomial $y_0^{2-\alpha-\beta-\gamma-\delta-\theta} y_1^\alpha y_2^\beta y_3^\gamma y_4^\delta y_5^\theta$ to the 1-form $\omega'_{\alpha\beta\gamma\delta\theta}$ that in affine coordinates is

$$\begin{aligned} \omega'_{\alpha\beta\gamma\delta\theta} &:= (u')^\alpha (v')^{\beta-1} (t')^{\gamma-1} (s')^{\delta-1} (r')^{\theta-1} du', \qquad \text{where} \\ u' := \frac{y_1}{y_0} \quad v' &:= \frac{y_2}{y_0} \quad t' := \frac{y_3}{y_0} \quad s' := \frac{y_4}{y_0} \quad \text{and} \qquad r' := \frac{y_5}{y_0}. \end{aligned}$$

We obtain a basis of 17 dimension space $H^0(C_2, \mathcal{O}_{C_2}(2))$ by taking the 21 monomials of degree 2 in the variables y_j and removing y_2^2, y_3^2, y_4^2, and y_5^4, that can be expressed in terms of the other monomials using the quadratic equations defining C_2. Accordingly we get a basis of $H^0(C_2, \Omega^1_{C_2})$ by removing from that set $\omega'_{\alpha\beta\gamma\delta\theta}$ the 1-forms $\omega'_{02000}, \omega'_{00200}$, ω'_{00020}, and ω'_{00002}. The *canonical* character of C_2 is given by Chevalley-Weil as

$$\begin{aligned} \chi^2_{can} = 2\mu_1{\cdot}\mu_2{\cdot}\epsilon_1 + \mu_2{\cdot}\epsilon_1 + 2\chi{\cdot}\epsilon_1{\cdot}\epsilon_2 + \mu_2{\cdot}\epsilon_1{\cdot}\epsilon_2 + \mu_1{\cdot}\mu_2{\cdot}\epsilon_2 + \mu_2{\cdot}\epsilon_2 + \mu_1{\cdot}\mu_2{\cdot}\epsilon_1{\cdot}\epsilon_2 + \\ + \chi + \chi \cdot \epsilon_1 + \epsilon_1 + \mu_2, \end{aligned}$$

and the action on $H^0(C_2, \Omega^1_{C_2})$ computed in affine coordinates as above is

$$\begin{aligned} (t^j r^i, (a,b)) \cdot \omega'_{\alpha\beta\gamma\delta\theta} &= \phi_2(\left(t^j r^i, (a,b)\right)^{-1})^*(\omega'_{\alpha\beta\gamma\delta\theta}) = \\ &= (-1)^K \, \omega'_{\alpha\beta\gamma((1-[i])\delta+[i]\theta)([i]\delta+(1-[i])\theta)} \end{aligned}$$

where

$$\begin{aligned} K := a(\beta-1) + b(\beta+\gamma+\delta+\theta) + j + i(\alpha+\beta-1) + ij(\delta+\theta) + \\ + \left(\frac{3i(3i+1)}{2} + \frac{[j]_2([j]_2+1)}{2}\right)\delta + \frac{3i(3i-1)}{2}\theta. \end{aligned}$$

Another tedious computation gives the decomposition

$$\begin{aligned} H^0(C_2, \Omega^1_{C_2}) = &\left(\langle \omega'_{00000}\rangle \oplus \langle \omega'_{20000}\rangle\right)_{\mu_1\cdot\mu_2\cdot\epsilon_1} \oplus \langle \omega'_{10000}\rangle_{\mu_2\cdot\epsilon_1} \oplus \left(\langle \omega'_{10001}, \omega'_{10010}\rangle \oplus \langle \omega'_{00010}, \omega'_{00001}\rangle\right)_{\chi\cdot\epsilon_1\cdot\epsilon_2} \\ &\oplus \langle \omega'_{10100}\rangle_{\mu_2\cdot\epsilon_1\cdot\epsilon_2} \oplus \langle \omega'_{11000}\rangle_{\mu_1\cdot\mu_2\cdot\epsilon_2} \oplus \langle \omega'_{01000}\rangle_{\mu_2\cdot\epsilon_2} \oplus \langle \omega'_{00100}\rangle_{\mu_1\cdot\mu_2\cdot\epsilon_1\cdot\epsilon_2} \\ &\oplus \langle \omega'_{01001}, \omega'_{01010}\rangle_\chi \oplus \langle \omega'_{00101}, \omega'_{00110}\rangle_{\chi\cdot\epsilon_1} \oplus \langle \omega'_{00011}\rangle_{\epsilon_1} \oplus \langle \omega'_{01100}\rangle_{\mu_2}. \end{aligned}$$

We consider the unmixed quotient $S := (C_1 \times C_2)/\left(D_4 \times \mathbb{Z}_2^2\right)$ modulo the diagonal action $\phi_1 \times \phi_2$. We observe that the non-trivial stabilizers of the points on the first curve C_1 are generated by $(tr, (0,0))$, $(tr^2, (0,1))$, $(tr2, (1,0))$, $(r^2, (1,0))$, and $(tr^3, (0,1))$. However, none of these elements have a fixed point on the second curve C_2 under the action ϕ_2. Thus, the action $\phi_1 \times \phi_2$ is free and the quotient surface S is smooth, projective, and of general type. The latter holds because the genus of the two curves is

$g(C_i) \geq 2$, hence $C_1 \times C_2$ has ample canonical divisor, and so K_S is ample too.

By Theorem 5.2.1, the vector space $H^0(K_S)$ is isomorphic to the invariant subspace $\left(H^0(\Omega^1_{C_1}) \otimes H^0(\Omega^1_{C_2})\right)^{D_4\times\mathbb{Z}_2^2}$, where the action on the tensor product is diagonal, i.e. $\left(\sigma^i\tau^j,(a,b)\right) \in D_4 \times \mathbb{Z}_2^2$ acts via

$$\phi_1(\left(\sigma^i\tau^j,(a,b)\right)^{-1})^* \otimes \phi_2(\left(\sigma^i\tau^j,(a,b)\right)^{-1})^*. \tag{6.9}$$

Let denote by $\omega_{jklmrstvu} := \omega_{jklm} \otimes \omega'_{rstvu}$. We can now state and prove our main result:

Theorem 6.3.1. *The diagonal action $\phi_1\times\phi_2$ of $D_4\times\mathbb{Z}_2^2$ on the product of the two curves C_1 and C_2 is free. The quotient is a smooth projective surface S of general type with $K_S^2 = 32$, and $p_g = 3$. In particular, S is quasi-abelian, a basis of $H^0(K_S)$ is $\{\omega_{110010000}, \omega_{000101100}, \omega_{100001001} + \omega_{011001010}\}$, the canonical map Φ_{K_S} in projective coordinates is*

$$\Phi_{K_S}(\mathbf{x},\mathbf{y}) = (x_1x_2y_0y_1 : x_4y_2y_3 : y_2(x_0x_1y_5 + x_2x_3y_4)),$$

and its degree d is equal to 24.

Proof. We have already mentioned that the action is free, and the quotient S is smooth, projective, and of general type.

Hence the self-intersection of the canonical divisor of each S amounts to

$$K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|D_4\times\mathbb{Z}_2^2|} = \frac{8\cdot 8\cdot 16}{32} = 32.$$

They are regular surfaces, because they do not possess any non-zero holomorphic one-forms, since $C_i/\left(D_4\times\mathbb{Z}_2^2\right)$ is biholomorphic to $\mathbb{P}^1$. The geometric genus of each S is therefore equal to (see Theorem 4.4.5)

$$p_g = \chi(\mathcal{O}_S) - 1 = \frac{(g(C_1)-1)(g(C_2)-1)}{|D_4\times\mathbb{Z}_2^2|} - 1 = \frac{8\cdot 16}{32} - 1 = 3.$$

Since we have proved that $p_g = 3$, we leave it to the reader to verify that the given 2-forms are invariant for the action (6.9). Applying the explicit isomorphisms from $H^0(C_1,\Omega^1_{C_1})$ to $H^0(C_1,\mathcal{O}_{C_1}(2))$ and from $H^0(C_2,\Omega^1_{C_2})$ to $H^0(C_2,\mathcal{O}_{C_2}(2))$ we obtain the bi-quadrics defining the canonical map.

It remains to determine the degree of Φ_{K_S}. Instead of working on S it is convenient to work on $C_1\times C_2$ by following the strategy of Section 5.3:

$$\begin{tikzcd}
C_1\times C_2 \arrow[r, "\lambda_{12}"] \arrow[dr, "\Phi_{K_{C_1\times C_2}}"'] & S \arrow[r, dashed, "\Phi_{K_S}"] & \mathbb{P}^2 \\
 & \mathbb{P}^{9\cdot 17-1} \arrow[ur, dashed] &
\end{tikzcd}.$$

The map $\Phi_{K_S} \circ \lambda_{12}$ is induced by the sublinear system $|K_{C_1 \times C_2}|^{D_4 \times \mathbb{Z}_2^2}$ of $|K_{C_1 \times C_2}|$ generated by the three invariant 2-forms defining Φ_{K_S}.

We resolve the indeterminacy of $\Phi_{K_{C_1\times C_2}^{D_4 \times \mathbb{Z}_2^2}} = \Phi_{K_S} \circ \lambda_{12}$ by a sequence of blowups,

$$\begin{tikzcd} \widehat{C_1 \times C_2} \arrow[r] \arrow[dr, "\Phi_{\widehat{M}}"'] & C_1 \times C_2 \arrow[d, dashed, "\Phi_{K_{C_1\times C_2}^{D_4\times\mathbb{Z}_2^2}}"] \\ & \mathbb{P}^2. \end{tikzcd}$$

Here the morphism $\Phi_{\widehat{M}}$ is induced by the base-point free linear system $|\widehat{M}|$ obtained as follow: let us consider the mobile part $|M|$ of $|K_{C_1 \times C_2}|^{D_4 \times \mathbb{Z}_2^2}$. We blow up the base-points of $|M|$, take the pullback of $|M|$ and remove the fixed part of this new linear system. We repeat the procedure until we obtain a base-point free linear system $|\widehat{M}|$.

From Lemma 5.3.1 the self-intersection $\widehat{M}^2$ is positive if and only if $\Phi_{\widehat{M}}$ is not composed with a pencil. In this case $\Phi_{\widehat{M}}$ is onto and it holds:

$$\deg(\Phi_{K_S}) = \frac{1}{|D_4 \times \mathbb{Z}_2^2|} \deg(\Phi_{\widehat{M}}) = \frac{1}{32} \widehat{M}^2.$$

For the computation of the resolution, it is convenient to write the divisors of the bi-quadrics defining Φ_{K_S} (and hence $\Phi_{K_{C_1\times C_2}^{D_4 \times \mathbb{Z}_2^2}}$) as linear combinations of the curves $F_j := \{x_j = 0\}$ and $G_k := \{y_k = 0\}$ on $C_1 \times C_2$. We point out that these curves are reduced. The F_j's are pairwise disjoint, and the G_k's as well, by the assumption on the coefficients λ_i and μ_j. Moreover F_j and G_k meet transversally in each one of their 128 (if $j \neq 4$), resp. 256 (if $j = 4$), intersection points. Summarizing, $(F_j, F_k) = (G_j, G_k) = 0$ and $(F_j, G_k) = 128$, for $j \neq 4$, while $(F_4, G_k) = 256$.
The divisors spanning the subsystem $|K_{C_1 \times C_2}|^{D_4 \times \mathbb{Z}_2^2}$ are:

$$D_1 := F_1 + F_2 + G_0 + G_1, \qquad D_2 := F_4 + G_2 + G_3 \qquad \text{and} \qquad D_3 := G_2 + \Delta,$$

where $\Delta = (x_0 x_1 y_5 + x_2 x_3 y_4)$. Hence $|K_{C_1 \times C_2}|^{D_4 \times \mathbb{Z}_2^2}$ has not fixed part and

$$M^2 = (\lambda_{12}^* K_S)^2 = |D_4 \times \mathbb{Z}_2^2| \cdot K_S^2 = 32^2.$$

The (set-theoretical) base locus of $|M| = |K_{C_1 \times C_2}|^{D_4 \times \mathbb{Z}_2^2}$ is contained in

$$F_1 \cap G_2, \quad F_2 \cap G_2, \quad G_0 \cap F_4 \cap \Delta, \quad \text{and} \quad G_1 \cap F_4 \cap \Delta$$

We remark that the other pieces of the base locus are empty. In fact, those points would belong to some $F_i \cap F_j$, or $G_i \cap G_j$.

We determine the correction term to the self intersection number for each of these base-points of $|M|$.

We consider first the 128 points $F_i \cap G_2$, for $i = 1, 2$. Here F_i and G_2 intersect transversally. Around one of these points, the divisors D_k are given by F_i, G_2, and G_2. We are in the situation of the *Correction Term formula* 5.4.3 with $H = F_i$ and $K = G_2$, $a = b = d = 1$ and $c = 0$. Hence the correction term is $ab = ad + bc = 1$.

We go on showing that no other base-points arise from the loci $G_0 \cap F_4 \cap \Delta$, and $G_1 \cap F_4 \cap \Delta$. The points $G_0 \cap F_4 \cap \Delta$ satisfy the equations

$$\begin{cases} 4x_4^2 = \lambda_2^2(x_3^2 + x_0^2)^2 - (x_3^2 - x_0^2)^2 & = 0 \\ y_4^2 = (\mu_1 y_0 - y_1)(\mu_2 y_0 - y_1) & = y_1^2 \\ y_5^2 = (\mu_1 y_0 + y_1)(\mu_2 y_0 + y_1) & = y_1^2 \\ x_0 x_1 y_5 + x_2 x_3 y_4 & = 0 \end{cases}. \tag{6.10}$$

The last equation implies that $x_0^2 x_1^2 y_5^2 = x_2^2 x_3^2 y_4^2$, so, through the second and the third equations, we get

$$x_0^2 x_1^2 = x_2^2 x_3^2. \tag{6.11}$$

Now we use the equations defining C_1 to substitute in the equation (6.11) the values x_1^2 and x_2^2 in function of x_0^2 and x_3^2:

$$x_0^2 \left(\frac{\lambda_1 - 1}{2} x_3^2 + \frac{\lambda_1 + 1}{2} x_0^2 \right) = \left(\frac{\lambda_1 + 1}{2} x_3^2 + \frac{\lambda_1 - 1}{2} x_0^2 \right) x_3^2 \implies \frac{\lambda_1 + 1}{2} (x_3^4 - x_0^4) = 0.$$

We remember that we have assumed $\lambda_1 \neq -1$, so $x_3^4 = x_0^4$. Finally, substituting x_3^2 with $\pm x_0^2$ in the first equation of the system (6.10), we get

$$\begin{aligned} 0 &= \lambda_2^2(x_3^2 + x_0^2)^2 - (x_3^2 - x_0^2)^2 = 4\lambda_2^2 x_0^4, \quad \text{if} \quad x_3^2 = x_0^2 \\ 0 &= \lambda_2^2(x_3^2 + x_0^2)^2 - (x_3^2 - x_0^2)^2 = -4x_0^4, \quad \text{if} \quad x_3^2 = -x_0^2. \end{aligned}$$

The condition $\lambda_2 \neq 0$ that we have assumed for C_1 and $x_0 \neq 0$ permits us to conclude that both cases are not allowed. Hence $G_0 \cap F_4 \cap \Delta$ is empty.

It remains to study the points $G_1 \cap F_4 \cap \Delta$, which satisfy the equations

$$\begin{cases} 4x_4^2 = \lambda_2^2(x_3^2 + x_0^2)^2 - (x_3^2 - x_0^2)^2 & = 0 \\ y_4^2 = (\mu_1 y_0 - y_1)(\mu_2 y_0 - y_1) & = \mu_1 \mu_2 y_0^2 \\ y_5^2 = (\mu_1 y_0 + y_1)(\mu_2 y_0 + y_1) & = \mu_1 \mu_2 y_0^2 \\ x_0 x_1 y_5 + x_2 x_3 y_4 & = 0 \end{cases}. \tag{6.12}$$

The last equation implies that $x_0^2 x_1^2 y_5^2 = x_2^2 x_3^2 y_4^2$, so, through the second and the third equations, we get

$$\left(x_0^2 x_1^2 - x_2^2 x_3^2 \right) \mu_1 \mu_2 = 0.$$

However, by the assumptions on C_2, we have $\mu_1, \mu_2 \neq 0$, so $x_0^2 x_1^2 = x_2^2 x_3^2$, that is the equation (6.11). Thus, following exactly the flow of the computations done after (6.11), we obtain a contradiction also in this case. Therefore $G_1 \cap F_4 \cap \Delta$ is empty.

Then

$$M^2 - \widehat{M}^2 = 1 \cdot 128 + 1 \cdot 128 = 256.$$

The degree of the canonical map amounts to

$$\deg(\Phi_{K_S}) = \frac{1}{32}\left(M^2 - (M^2 - \widehat{M}^2)\right) = \frac{1}{32}\left(32^2 - 256\right) = 24.$$

$\square$

Chapter 7

Unbounded families with canonical map of degree 4

This is a joint work [FP21] together with Prof. Roberto Pignatelli.

An *unbounded family* of surfaces is a sequence of surfaces S_n with an arbitrarily large Euler characteristic $\chi(\mathcal{O}_{S_n})$, namely $\lim_{n\to\infty}\chi(\mathcal{O}_{S_n}) = +\infty$.

In Chapter 1 we have seen from a pioneering work of Beauville [Bea79] and a Theorem of X. Gang [Xia86] that the degree δ of the canonical map of a surface S, if we assume a large enough Euler characteristic, is bounded from above by 8. Precisely, this follows by point (i) of Theorem 1.1.5:

$$\delta \leq 9 + \frac{27 - 9q(S)}{p_g - 2} \leq 9 \qquad \text{for} \qquad \chi(\mathcal{O}_S) \geq 31$$

and by [Xia86, Theorem 3], which excludes the case $\delta = 9$ for $\chi(\mathcal{O}_S) \geq 134$. Recall that the degree of the canonical map is a birational invariant, hence we can assume S is minimal.

In the beautiful survey by M. Mendes Lopes and R. Pardini [MLP23] can be found, among other things, examples of unbounded surfaces with canonical map of degree $\delta \in \{2, 4, 6, 8\}$.

The *slope* μ of a minimal surface S is defined as $\mu(S) := \frac{K_S^2}{\chi(\mathcal{O}_S)}$. By Bogomolov-Miyaoka-Yau inequality 1.1.4 $\mu(S) \leq 9$. Furthermore, for any unbounded family S_n of minimal surfaces whose canonical map has a degree of δ we have

$$\delta(\chi(\mathcal{O}_S) - 3) \leq K_S^2 \leq 9\chi(\mathcal{O}_S) \implies \liminf \mu(S_n) \in [\delta, 9]. \tag{7.1}$$

The above sentence 7.1 follows easily by applying in sequence lemmas 1.1.3, 1.1.1, and 1.1.2 proved in Chapter 1. This raises the following question (compare [MLP23, Question 5.6]):

Question: For all δ, which are the accumulation points of the slopes in the range $[\delta, 9]$ of unbounded families of minimal surfaces whose canonical map has a degree of δ?

We know only three constructions of unbounded families of minimal surfaces whose canonical map has a degree of 4.

The first, mentioned in [MLP23], is obtained by taking the product of two hyperelliptic curves. All these surfaces have slope 8.

The second, see [Bin21b, Remark 3], is a construction as Galois cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$; they also have $\lim \mu(S_n)$ equal to 8.

The last, constructed by F. J. Gallego and G. P. Purnaprajna, give unbounded families with $\lim \mu(S_n)$ equal either to 8 or to 4, see the last column of [GP08, Table at page 5491]

In this chapter, we show that $\lim \mu(S_n)$, when S_n is an unbounded family of minimal surfaces whose canonical map has a degree of 4, may assume infinitely many different values. More precisely, we prove the main Theorem 7.3.1.

All these surfaces are product-quotient surfaces, those studied in Chapter 4. Their canonical map is studied in Chapter 5.

7.1 Generalized Wiman Curves

By a classical result of Harvey and Wiman ([Har66, Wim95]) an automorphism of a curve of genus g at least 2 has order at most $4g+2$. Moreover, there is exactly one curve of genus g with an automorphism of order $4g+2$ for each integer $g \geq 2$, usually referred in literature as the *Wiman curve of genus g*.

Definition 7.1.1 (Generalized Wiman curves). Consider two positive integers $n, d \geq 1$.

A generalized Wiman curve of type n, d is a curve in the weighted projective space $\mathbb{P}\left(1, 1, \left\lceil \frac{nd}{2} \right\rceil\right)$ defined by an equation of the form

$$y^2 = x_0^{[nd]_2} f(x_0^n, x_1^n)$$

where f is a homogenous polynomial of degree d in the two variables x_0, x_1 without multiple roots such that neither x_0 nor x_1 divide f.

Remark 7.1.2. The assumptions on the polynomial f ensure that any generalized Wiman curve is smooth.

By adjunction a generalized Wiman curve C of type n, d has genus $g = \left\lceil \frac{nd}{2} \right\rceil - 1$. In fact a basis of $H^0(C, K_C)$ is given by the monomials

$$x_0^{\left\lceil \frac{nd}{2} \right\rceil - 2}, x_0^{\left\lceil \frac{nd}{2} \right\rceil - 3} x_1, \ldots, x_0 x_1^{\left\lceil \frac{nd}{2} \right\rceil - 3}, x_1^{\left\lceil \frac{nd}{2} \right\rceil - 2} \tag{7.2}$$

A generalized Wiman curve of type n, d has the following two natural commuting automorphisms

$$\iota\colon (x_0, x_1, y) \mapsto (x_0, x_1, -y) \qquad \rho\colon (x_0, x_1, y) \mapsto (x_0, e^{\frac{2\pi i}{n}} x_1, y)$$

of respective order 2 and n. This shows

1. all generalized Wiman curves are hyperelliptic, ι being their hyperelliptic involution;

2. a generalized Wiman curve of type $2g+1, 1$ is the Wiman curve of genus g.

Since ι is the hyperelliptic involution, ι acts on $H^0(C, K_C)$ as the multiplication by -1. The points fixed by ι are the $2g+2$ points of the divisor $y = 0$.

Definition 7.1.3. We will say that ρ is the *rotation* of C.

We conclude this section by studying the action of the rotation.

Proposition 7.1.4. *The action of ρ on the locus $x_0x_1 \neq 0$ has all orbits of order n.*

The divisor $x_1 = 0$ is given by two points, both fixed by ρ.

If both n and d are odd, then the divisor $x_0 = 0$ is given by one single point, fixed by ρ. Else the divisor $x_0 = 0$ is given by two distinct points, fixed by ρ if d is even and exchanged by ρ if d is odd.

The monomials in (7.2) are eigenvalues for the induced action of ρ on $H^0(C, K_C)$. More precisely ρ acts on them as

$$x_0^{\lceil \frac{nd}{2} \rceil - 2 - a} x_1^a \mapsto e^{(a+1)\frac{2\pi i}{n}} x_0^{\lceil \frac{nd}{2} \rceil - 2 - a} x_1^a \tag{7.3}$$

Proof. The rotation lifts the automorphism of $\mathbb{P}^1 = C/\iota$ acting as $(x_0, x_1) \mapsto (x_0, e^{\frac{2\pi i}{n}} x_1)$, which fixes only the two points $x_0x_1 = 0$, so the analogous statement holds for ρ.

By the definition of f the point $(x_0, x_1) = (0, 1)$ is a branching point of the hyperelliptic $2:1$ map $C \to \mathbb{P}^1$ if and only if both n and d are odd, in which case the divisor $x_0 = 0$ in C is a single (double) point, that is therefore fixed by ρ. Else, if nd is even, $x_0 = 0$ is formed by two distinct points with homogeneous coordinates $(x_0, x_1, y) = (0, 1, \pm \bar{u}_0)$ for some $\bar{u}_0 \neq 0$. These two points are either fixed or exchanged by ρ. By the properties of the weighted projective space they are fixed by ρ if and only if $\left(e^{-\frac{2\pi i}{n}}\right)^{\frac{nd}{2}} = 1$. We conclude the analysis of the divisor $x_0 = 0$ by observing that the last equation is verified if and only if d is even.

Since the point $(x_0, x_1) = (1, 0)$ is not a branching point of the hyperelliptic map, the divisor $x_1 = 0$ is made by two distinct points with coordinates $(x_0, x_1, y) = (1, 0, \pm \bar{u}_1)$ for some $\bar{u}_1 \neq 0$, both obviously fixed by ρ.

The function $z := x_1/x_0$ is a local coordinate in both of then, on which ρ acts as $z \mapsto e^{\frac{2\pi i}{n}} z$. The adjunction formula maps a monomial $x_0^{\lceil \frac{nd}{2} \rceil - 2 - a} x_1^a$ to the form that locally restricts to $z^a dz$ and therefore ρ acts on it as the multiplication by $e^{(a+1)\frac{2\pi i}{n}}$. □

7.2 Wiman product-quotient surfaces

Definition 7.2.1. For all integers n, d_1, d_2 and for all $1 \leq k \leq n-1$ with $\gcd(k,n) = 1$ we define a *Wiman product-quotient surface of type* n, d_1, d_2 with *shift* k to be the minimal resolution S of the singularities of its *quotient model* $X := (C_1 \times C_2)/\,H$ where

- C_j, $j = 1, 2$ is a generalized Wiman curve of type n, d_j;
- $H \subset \mathrm{Aut}(C_1 \times C_2)$ is the cyclic subgroup of order n generated by the automorphism
$$(x, y) \mapsto \left(\rho_1 x, \rho_2^k y\right).$$

where ρ_j is the rotation of C_j.

Denote the hyperelliptic involution of C_j by ι_j. Then $\mathrm{Aut}(C_1 \times C_2)$ contains a subgroup of order 4 generated by $(\iota_1, 1)$ and $(1, \iota_2)$. The corresponding quotient of $C_1 \times C_2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Since this group commutes with H and it intersects H trivially, it defines a subgroup $K \cong (\mathbb{Z}/2\mathbb{Z})^2$ of $\mathrm{Aut}(X)$. Note that X/K is dominated by $\mathbb{P}^1 \times \mathbb{P}^1$ and therefore it is rational.

Lemma 7.2.2. *The canonical map of S factors through the rational surface X/K.*

Proof. By the Kuenneth formula
$$H^0(C_1 \times C_2, K_{C_1 \times C_2}) \cong H^0(C_1, K_{C_1}) \otimes H^0(C_2, K_{C_2})$$
and then both involutions $(\iota_1, 1)$ and $(1, \iota_2)$ act on $H^0(C_1 \times C_2, K_{C_1 \times C_2})$ as the multiplication by -1. From Theorem 5.2.1, the pull-back map sends $H^0(S, K_S) = H^0(X, K_X)$ isomorphically onto the invariant subspace $H^0(C_1 \times C_2, K_{C_1 \times C_2})^H$. It follows that all elements of K act on $H^0(S, K_S) = H^0(X, K_X)$ as a multiple of the identity.

This implies that $H^0(S, K_S)$ cannot separate two points in the same orbit by the action of K. □

In the "degenerate" case $n = 1$, $S = X$ is the product of the two hyperelliptic curves C_1 and C_2. Assuming $d_1, d_2 \geq 5$ (to have genera at least 2) we find an unbounded family of surfaces with canonical map of degree 4 as those mentioned in [MLP23].

The degree of the canonical map remains in fact 4 also for bigger n.

Theorem 7.2.3. *Let S be a Wiman product-quotient surface of type n, d_1, d_2 and assume $n \geq 2$.*

1. *If $d_1, d_2 \geq 3$, then K_S is nef.*
2. *If $d_1 \geq 4$, $d_2 \geq 5$ then the canonical map of S has degree 4.*

Proof. We denote by x_0, x_1, y the coordinates of the weighted projective space containing C_1 as in Definition 7.1.1, and by $\bar{x}_0, \bar{x}_1, \bar{y}$ the analogous coordinates for C_2. By the Kuenneth formula the monomials

$$m_{a,b} := x_0^{\left\lceil \frac{nd_1}{2} \right\rceil -2-a} \bar{x}_0^{\left\lceil \frac{nd_2}{2} \right\rceil -2-b} x_1^a \bar{x}_1^b$$

form a basis of eigenvectors for the action of the $\left(\rho_1, \rho_2^k\right)$ of H on $H^0(C_1 \times C_2, K_{C_1 \times C_2})$ with respective eigenvalues $e^{(a+1+k(b+1))\frac{2\pi i}{n}}$. So a basis of $H^0(S, K_S)$ is given by the monomials

$$\{m_{a,b} | n \text{ divides } a+1+k(b+1)\} \tag{7.4}$$

1. Pulling back $H^0(S, K_S)$ to $C_1 \times C_2$ we obtain a linear system $|K_{C_1 \times C_2}|^H$ defined by the vector subspace $V \subset H^0(C_1 \times C_2, K_{C_1 \times C_2})$ generated by the monomials $m_{a,b}$ in (7.4).

   We claim that if both d_j are at least 3, then the base locus of $|K_{C_1 \times C_2}|^H$ is finite.

   We first note that the divisor defined by each $m_{a,b}$ on $C_1 \times C_2$ is a linear combination of the 4 divisors $x_0 = 0$, $\bar{x}_0 = 0$, $x_1 = 0$, $\bar{x}_1 = 0$. Then the base locus of $|K_{C_1 \times C_2}|^H$ is contained in the union of these 4 divisors.

   We show that the intersection of the base locus of $|K_{C_1 \times C_2}|^H$ with $x_1 = 0$ is finite. It suffices to prove that there is a monomial in V of the form $m_{0,b}$. In other words, that there is an integer $0 \leq b \leq \left\lceil \frac{nd_2}{2} \right\rceil - 2$ so that n divides $1 + k(b+1)$, which is equivalent to ask that the remainder class of b module n is the unique class solving the corresponding congruence. Since $d_2 \geq 3$, $\left\lceil \frac{nd_2}{2} \right\rceil - 2 \geq n-1$ and therefore we can find a b in our range for any such a class, giving a monomial $m_{0,b}$ in V.

   A similar argument show that the intersection of the base locus of $|K_{C_1 \times C_2}|^H$ with each of the other three divisors $x_0 = 0$, $\bar{x}_0 = 0$, $\bar{x}_1 = 0$ is finite, by showing the existence of a monomial in V of respective type $m_{\left\lceil \frac{nd_1}{2} \right\rceil -2, b}$, $m_{a, \left\lceil \frac{nd_2}{2} \right\rceil -2}$ and $m_{a,0}$. This concludes the proof of the claim.

   Since the base locus of $|K_{C_1 \times C_2}|^H$ is finite, the base locus of $|K_X|$ is finite too whereas the base locus of $|K_S|$ may contain some irreducible curves, all exceptional for the map $S \to X$, the minimal resolution of the singularities of X. In particular there is no (-1)-curve in the base locus of $|K_S|$. But a (-1)-curve on a surface S is always in the base locus of $|K_S|$! So S is a minimal surface, in the sense that it does not contain (-1)-curve. Since the canonical system is not empty, then S minimal implies that K_S is nef.

2. If $d_1 \geq 4$, $d_2 \geq 5$, arguing as above, we can find a monomial of the form $m_{0,b}$ in V such that also $m_{0,b+n}$, $m_{n,b}$, $m_{n,b+n}$ belong to V. These 4 monomials map $C_1 \times C_2$ as $x_0^n\bar{x}_0^n, x_0^n\bar{x}_1^n, x_1^n\bar{x}_0^n, x_1^n\bar{x}_1^n$ onto a smooth quadric $Q \subset \mathbb{P}^3$. Then the canonical image of S, dominating Q, is a surface as well.

   Choose a general point $q \in Q$. Its preimage in $C_1 \times C_2$ has cardinality $(2n)^2$, giving $4n$ points of S. The group K acts freely on them, giving n smooth points $q_1, \dots, q_n$ of X/K. We know by Lemma 7.2.2 that the canonical map of X factors through X/K; we finish the proof by showing that it separates the q_j.

   The automorphism $(\rho_1, 1)$ of $C_1 \times C_2$ commutes with H, so it defines an automorphism ρ_X of X. This automorphism commutes with K, so inducing a further automorphism ρ_K of order n of X/K. A straightforward direct computation shows that ρ_K permutes the q_j cyclically.

   Now choose a monomial in V of the form $m_{1,c}$. Then the action of $(\rho_1, 1)$ on the vector subspace of V generated by $m_{1,c}, m_{0,b}, m_{0,b+n}, m_{n,b}$, $m_{n,b+n}$ has exactly two distinct eigenvalues, which differ by a primitive $n-$th root of the unity. This implies that the canonical map of X separates the q_j.

$\square$

Remark 7.2.4. The statement of Theorem 7.2.3 is not meant to be sharp. For example, essentially the same proof shows that part (2) extends to the case $d_1 = 3$ with the possible exception $n = 2$.

Remark 7.2.5. The proof of Theorem 7.2.3, part (1) shows that the canonical system of these surfaces has no fixed components.

In fact, it contains all the elements necessary to explicitly compute the base locus of the canonical system, by describing its pull-back on $C_1 \times C_2$, the base locus of the linear system $|K_{C_1 \times C_2}|^H$.

Consider for example the first case $n = 2$, $d_1 = d_2 = 3$. In this case $k = 1$. Then the given basis of $H^0(S, K_S)$ is $\{x_0\bar{x}_0, x_1\bar{x}_1\}$. This implies that the base locus of $|K_{C_1 \times C_2}|^H$ is formed by 8 simple points, four defined by $x_0 = \bar{x}_1 = 0$ and four defined by $\bar{x}_0 = x_1 = 0$. The involution defining S as quotient of $C_1 \times C_2$ acts on these eight points freely, so $H^0(S, K_S)$ has exactly four simple base points, their images.

By Proposition 7.1.4 this involution fixes exactly 4 points, those at $x_1 = \bar{x}_1 = 0$, inducing 4 singular points of type A_1 on S. The standard formulas from Theorem 4.4.2 and Theorem 4.4.5 give $K_S^2 = 4$ and $p_g(S) = q(S) = 2$, confirming that the canonical system is a pencil with 4 base points.

7.3 Unbounded sequences of Wiman Product-Quotient surfaces

In this section we only consider Wiman product-quotient surfaces of type n, d_1, d_2 with both d_1, d_2 even.

Identifying a point of X with an orbit of the action of H on $C_1 \times C_2$, the singular points of X correspond to the orbits of cardinality smaller than n.

By Proposition 7.1.4 the orbits of the rotation of a generalized Wiman curve of type n, d with d even are all of order n with 4 exceptions, 4 fixed points. So X has 16 singular points. A straightforward application of Proposition 4.3.3 shows that 8 are of type $\frac{1}{n}(1,k)$ and 8 of type $\frac{1}{n}(1,n-k)$.

We consider the invariant γ of the basket introduced in [BP16, Section 4]: it vanishes by [BP16, Proposition 4.4] since the basket contains as many points of type $\frac{1}{n}(1,k)$ as of type $\frac{1}{n}(1,n-k)$. By [BP16, Proposition 4.1] $K_S^2 = 8\chi(\mathcal{O}_S) - 2\gamma - l = 8\chi(\mathcal{O}_S) - l$ where l is the number of exceptional curves of $S \to X$.

Therefore

$$8 - \mu(S) = \frac{8\chi(\mathcal{O}_S) - K_S^2}{\chi(\mathcal{O}_S)} = \frac{l}{\chi(\mathcal{O}_S)} = \frac{l}{\frac{\left(n\frac{d_1}{2}-2\right)\left(n\frac{d_2}{2}-2\right)}{n} + 4\left(1-\frac{1}{n}\right)}$$

Writing the continued function of $\frac{n}{k}$

$$\frac{n}{k} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \dots}}$$

then ([Rie74, Section 3]) the number of irreducible components of the resolution above two singular points of respective type $\frac{1}{n}(1,k)$ and $\frac{1}{n}(1,n-k)$ equals $1 + \sum(b_j - 1)$, so

$$8 - \mu(S) = \frac{8\left(1 + \sum(b_j - 1)\right)}{\frac{\left(n\frac{d_1}{2}-2\right)\left(n\frac{d_2}{2}-2\right)}{n} + 4\left(1-\frac{1}{n}\right)} \approx_{n\to\infty} \frac{32}{d_1 d_2}\frac{1 + \sum(b_j - 1)}{n} \tag{7.5}$$

In the simplest case $k = 1$ we obtain $\frac{1+\sum(b_j-1)}{n} = \frac{1+n-1}{n} = 1$ and then

Theorem 7.3.1. *There is an unbounded sequence S_n of surfaces that have canonical map of degree 4 such that*

$$\lim_{n\to\infty} \mu(S_n) = 8\left(1 - \frac{1}{m}\right)$$

for all positive integers $m \geq 6$ that are not prime numbers.

Proof. Write $m = ab$ with $a \geq 2$, $b \geq 3$ and pick a sequence of Wiman product-quotient surfaces S_n of type $n, 2a, 2b$ and shift 1.

We are in the assumptions of Theorem 7.2.3, part (2) ($d_1 = 2a \geq 4$, $d_2 = 2b \geq 6 > 5$) so the canonical map of S_n has degree 4.

Finally, by (7.5)

$$\lim_{n\to\infty} \mu(S_n) = 8 - \frac{32}{d_1 d_2}\frac{1+\sum(b_j-1)}{n} = 8 - \frac{8}{ab}\frac{1+n-1}{n} = 8\left(1-\frac{1}{ab}\right).$$

□

7.4 Further questions and possible generalizations

We have studied some natural generalizations of this construction giving surfaces with canonical map of degree 4. Unfortunately they do not lead to a substantial improvement of our main result, so we have decided not to include them in this work. However, we mention them here for completeness.

We obtain in fact similar results for Wiman product quotient surfaces where the d_j are not both even. One can also consider hyperelliptic curves of equation $y^2 = x_0 x_1 f(x_0^n, x_1^n)$. All these generalizations lead to surfaces with canonical map of degree 4 and slope in the same range $\left[6+\frac{2}{3}, 8\right]$.

The other possible generalization is by considering shifts other that 1. More precisely, consider a sequence of positive integers k_n, with $1 \leq k_n \leq n-1$, $\gcd(k_n, n) = 1$. Then a sequence S_n of Wiman product-quotient surfaces of type $n, 2a, 2b$ and shift k_n has

$$\lim_{n\to\infty} \mu(S_n) = 8 - 8\frac{1}{m}\lim_{n\to\infty} \sigma\left(\frac{k_n}{n}\right).$$

where

$$\sigma\left(\frac{k}{n}\right) := \frac{1+\sum(b_j-1)}{n}.$$

Obviously $\sigma\left(\frac{k}{n}\right) > 0$, $\sigma\left(\frac{1}{n}\right) = 1$. It is known [TU22, Lemma 3.3] that $\sigma \leq 1$. An independent proof has been sent us by J. Stevens.

Question: *What are the possible limits of* $\left\{\sigma\left(\frac{k}{n}\right)\right\} \subset [0,1]$ *for sequences of rational numbers* $\frac{k}{n}$ *with unbounded denominators?*

Note $\lim_{n\to\infty} \sigma\left(\frac{m}{mn+1}\right) = \frac{1}{m}$. We could not obtain any sequence with limit neither zero nor of the form $\frac{1}{m}$. If there were more possible limits, this construction would improve our main result.

Chapter 8

Smooth k-double covers of the plane of geometric genus 3

This is a joint work [FP23] together with Prof. Roberto Pignatelli.

In this chapter we classify all smooth surfaces with geometric genus equal to three and an action of a group G isomorphic to $(\mathbb{Z}/2)^k$ such that the quotient is a plane. We find 11 families, listed in the main Theorem 8.4.1. We compute the canonical map of all of them, finding in particular a family of surfaces with canonical map of degree 16 that we could not find in the literature. We discuss the quotients by all subgroups of G finding several K3 surfaces with symplectic involutions. In particular we show that six families are families of triple K3 burgers in the sense of Laterveer. These families are those listed in Corollary 8.5.3.

The surfaces of general type with geometric genus 3 are interesting to be studied from several points of view. A first interest, which is the central one of this thesis, comes from the study of the degree of the canonical map. We recall the Remark 1.1.6, point 3: a surface of general type with a high degree of the canonical map have p_g equal to 3. We remind to the beautiful survey [MLP23] (see also Section 1.2) to the known examples with a high degree of the canonical map. We just mention here that most of these examples are obtained as Galois covers of rational surfaces with Galois group isomorphic to $(\mathbb{Z}/2)^k$: see for instance the examples with canonical map of degree 32 in [GPR22] and those of degree 20 in [Bin21a].

On the other hand, a classical conjecture of Claire Voisin, describing how $0-$cycles on a surface S should behave when pulled-back to a self-product of enough copies of S, led Laterveer to the definition of triple K3 burgers. These are surfaces with $p_g = 3$ provided with three pairwise commuting involutions such that the quotients are K3 surfaces. Studying them, Laterveer proved in [Lat21a] Voisin's conjecture for some family of surfaces, including a family of surfaces with $p_g = 3$ (that he calls "Garbagnati surfaces of type G3") with an action of $(\mathbb{Z}/2)^2$ whose quotient is $\mathbb{P}^2$. This leads us to the problem, interesting by itself, of studying and classifying all surfaces with $p_g = 3$ with an action of a group isomorphic to $G = (\mathbb{Z}/2)^k$ such that the quotient S/G is isomorphic to $\mathbb{P}^2$ and then studying their geometry, by investigating their canonical map and if they are triple $K3$ burgers. We call surfaces like these $k-$double planes for short.

The main argument to attain our classification is that by using the standard formulas for abelian covers, if the Galois group is of the form $(\mathbb{Z}/2\mathbb{Z})^k$, the numerical class of all divisors D_g is determined by the characteristic line bundles L_χ. We give the explicit formula in Theorem 8.1.11. So we first compute the possible L_χ, that is easy, and then deduce from it the class of each divisor.

This is (unfortunately) not true for general abelian group, since different numerical class of divisors may give the same characteristic sheaves L_χ, see Remark 8.1.7. So a similar analysis for different groups may be harder. However, there are several interesting examples of Galois covers of rational surfaces with Galois group of the form $(\mathbb{Z}/p\mathbb{Z})^k$, see for example Section 6.1 and [BGvBP22], so also such a classification would be desirable.

The chapter is organized as follows.

In section 1 we recall the general theory of abelian covers and prove the just mentioned Theorem 8.1.11 when the group is of the form $(\mathbb{Z}/2\mathbb{Z})^k$. In section 2 we recall the known results on the canonical systems of abelian covers. Note that in these two sections we use the multiplicative notation for G^* since it is more efficient for writing the general theory, whereas in the other sections we switch to the additive notation which is more convenient for the computations.

In section 3 we study and classify all the smooth $k-$double planes, obtaining the 11 mentioned families in terms of the branch divisors D_g and of the characteristic sheaves L_χ.

In section 4 we prove Theorem 8.4.1, and then we study each family separately. For each family we write explicit equations in a weighted projective space, and describe the quotients by all subgroups of G, determining all the $K3$ surfaces obtained in this way and the symplectic involutions on them.

Finally, in the last section, we determine which families are families of triple K3 burgers.

Notation: A Galois cover is a finite morphism $\pi\colon X \to Y$ among algebraic varieties with the property that there is a subgroup G of $\mathrm{Aut}(X)$ such that π factors as the composition of the quotient map $X \to X/G$ with an isomorphism $X/G \cong Y$. We will always assume Y to be irreducible, whereas we find it convenient for the general theory of Galois covers not to do any analogous assumption for X. The finite group G is the Galois group of π.

An abelian cover is a Galois cover whose Galois group is an abelian group. A $k-$double cover is an abelian cover whose Galois group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$. A $k-$double plane is a $k-$double cover of $\mathbb{P}^2$.

8.1 Abelian covers

In this section we collect some preliminary results on abelian covers, mostly well known.

Let π be an abelian cover with Y smooth and X normal. Following [Par91a], we decompose the direct image of the structure sheaf of X as a sum of line bundles corresponding to the characters of G

$$\pi_*\mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}.$$

By the Zariski-Nagata purity theorem, the branch locus of π is a divisor. We call this divisor D when it is taken with the reduced structure. The ramification divisor R of π is the preimage $\pi^{-1}(D)$, also taken with the reduced structure.

Let T be an irreducible component of R. By [Par91a, Lemma 1.1] the elements of G fixing all points of T form a cyclic subgroup H of G, the inertia group of T. By [Par91a, Lemma 1.2], there is a unique character $\psi\colon H \to \mathbb{C}^*$, a generator for the group of characters H^*, and a uniformizing parameter t for $\mathcal{O}_{X,T}$ such that, for all $h \in H$, h acts as

$$t \mapsto \psi(h)t.$$

This gives a natural decomposition

$$R = \sum_{\substack{H<G \text{ cyclic} \\ \psi \text{ generating } H^*}} R_{H,\psi}$$

of the ramification divisor as follows: if T is an irreducible component of R, then T is a summand of $R_{H,\psi}$ if and only if its inertia group is H and the corresponding character is χ.

As in [FP97] we observe that there is a natural bijection among the pairs (H,ψ) as above and the group G, associating to each element $g \in G$ the subgroup $H = \langle g \rangle$ generated by it and the unique character $\psi \in H^*$ with the property that $\psi(g) = e^{\frac{2\pi i}{|H|}}$. So we can set $R_g := R_{H,\psi}$ and write $R = \sum_{g\in G} R_g$.

Since G is abelian, if T_1 and T_2 are two irreducible components of R in the same $G-$orbit, they share the same inertia group H and the same character ψ, so T_1 and T_2 belong to the same summand R_g. Therefore there are reduced divisors D_g (denoted $D_{H,\psi}$ in [Par91a]) such that $R_g = \pi^{-1}(D_g)$. These give a decomposition of the branch divisor

$$D = \sum_{g \in G} D_g.$$

Definition 8.1.1. [Par91a, Definition 2.1] The building data of an abelian cover $\pi\colon X \to Y$ are the line bundles $\mathcal{L}_\chi$ and the reduced effective divisors D_g introduced above.

Note that, if 0 is the identity of G, $D_0 = 0$. Analogously, if 1 is the trivial character of G, $\mathcal{L}_1 \cong \mathcal{O}_Y$.

Remark 8.1.2. X is connected (equivalently: irreducible) if and only if, for all $\chi \neq 1$, $H^0(\mathcal{L}_\chi^{-1}) = 0$.

The building data determine the cover in the following sense.

Definition 8.1.3. Let $\pi\colon X \to Y$ be an abelian cover with Galois group G, Y smooth and X normal. Fix an element $g \in G$ and a character $\chi \in G^*$. Let $o(g)$ be the order of g. Then there exists a unique integer $0 \leq r_g^\chi \leq o(g) - 1$ such that

$$\chi(g) = e^{r_g^\chi \cdot \frac{2\pi i}{o(g)}}.$$

Given a further character $\chi' \in G^*$ we set moreover

$$\varepsilon^g_{\chi,\chi'} = \begin{cases} 1 & \text{if } r_g^\chi + r_g^{\chi'} \geq o(g) \\ 0 & \text{else} \end{cases}.$$

Theorem 8.1.4. [Par91a, Theorem 2.1 and Corollary 3.1] *Let $\pi\colon X \to Y$ be an abelian cover with Galois group G, Y smooth and X normal.*

Then for all $\chi, \chi' \in G^*$

$$\mathcal{L}_\chi \otimes \mathcal{L}_{\chi'} \cong \mathcal{L}_{\chi \cdot \chi'} \otimes \mathcal{O}_X\left(\sum_{g \in G} \varepsilon^g_{\chi,\chi'} \cdot D_g\right). \tag{8.1}$$

Conversely, given an abelian group G and a smooth irreducible variety Y assume that we have

a line bundle $\mathcal{L}_\chi$ on Y for each character $\chi \in G^*$ and

an effective divisor D_g for all $g \in G$

satisfying (8.1), and with the property that the divisor $D = \sum D_g$ is reduced.

Then there is a unique Galois cover $\pi\colon X \to Y$ whose Galois group is G, and whose building data are the L_χ and the D_g, such that X is normal.

Equation (8.1) shows that the divisors D_g determine the line bundles $\mathcal{L}_\chi$ up to torsion as follows.

Definition 8.1.5. For all χ set $L_\chi \in \operatorname{Pic}(Y) = \operatorname{Div}(Y)/\sim$ for the divisor class of the invertible sheaf $\mathcal{L}_\chi$. We use the additive notation for the torsion product in $\operatorname{Pic}(Y)$.

Corollary 8.1.6. [Par91a, see Proposition 2.1]

$$o(\chi)L_\chi \equiv \sum_{g\in G} \frac{o(\chi)r_g^\chi}{o(g)} D_g.$$

In particular

$$L_\chi \equiv_{num} \sum_{g\in G} \frac{r_g^\chi}{o(g)} D_g.$$

Proof. Note first that by definition of r_g^χ, for all $k \in \mathbb{N}$, $r_g^{\chi^k}$ is the remainder of the Euclidean division of kr_g^χ by $o(g)$. Then

$$\mathcal{L}_\chi^k \cong \mathcal{L}_{\chi^k}\left(\sum_{g\in G} \left\lfloor \frac{kr_g^\chi}{o(g)} \right\rfloor D_g\right)$$

follows by induction on k applying (8.1) to the products $\mathcal{L}_\chi \otimes \mathcal{L}_{\chi^{k-1}}$.

For $k = o(\chi)$ we obtain the stated formula since $\mathcal{L}_1 \cong \mathcal{O}_X$ and $\frac{o(\chi)r_g^\chi}{o(g)}$ is integral. $\square$

In particular, if $\operatorname{Pic}(Y)$ is torsion free (for example if Y is rational) then the divisors do determine the line bundles.

In the next sections we are going to walk in the opposite direction: first we look for the "good" possible $\mathcal{L}_\chi$ and then we find suitable divisors D_g realizing them.

Of course the divisors will be free to move in their linear equivalence class. We find it important to notice that for general G the line bundles $\mathcal{L}_\chi$ do not determine even the linear equivalence class of the divisors D_g. In fact this fails already for cyclic groups of order 5 of more. We just write one example of this phenomenon.

Example 8.1.7. Set $G = \mathbb{Z}/5\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$

Then the following choices

$$\begin{array}{lllll} \deg D_{\overline{0}} = 0 & \deg D_{\overline{1}} = 2 & \deg D_{\overline{2}} = 0 & \deg D_{\overline{3}} = 0 & \deg D_{\overline{4}} = 2 \\ \deg D_{\overline{0}} = 0 & \deg D_{\overline{1}} = 1 & \deg D_{\overline{2}} = 1 & \deg D_{\overline{3}} = 1 & \deg D_{\overline{4}} = 1 \end{array}$$

give both Galois covers with Galois group G and $\mathcal{L}_\chi \cong \mathcal{O}_{\mathbb{P}^1}(2)$ for all $\chi \neq 1$.

In contrast, we show in the forthcoming Theorem 8.1.11 that when $G \cong (\mathbb{Z}/2\mathbb{Z})^k$ then L_χ determine the linear equivalence class of the divisors D_g up to torsion.

We first need a Lemma on the sums of the r_g^χ for general abelian covers.

Definition 8.1.8. The natural isomorphism $G \to G^{**}$ allows each g in G to be considered as a character of G^*, which we will also denote by g, by setting

$$g(\chi) = \chi(g).$$

Then $\ker g$ is the subgroup of G^* of the characters χ such that $\chi(g) = 1$. In other words

$$\chi \in \ker g \Leftrightarrow g \in \ker \chi.$$

Let $\mathcal{H}$ be a subgroup of G^*, possibly of the form $\ker g$. For all $g \in G$ we will denote by $g_{|\mathcal{H}}$ the element of $\mathcal{H}^*$ obtained restricting g to $\mathcal{H}$.

Lemma 8.1.9. *For all $g \in G$, for each subgroup $\mathcal{H}$ of G^*,*

$$\sum_{\chi\in\mathcal{H}} r_g^\chi = \frac{|\mathcal{H}|}{2} o(g) \left(1 - \frac{1}{o(g_{|\mathcal{H}})}\right) \tag{8.2}$$

In particular

$$\sum_{\chi\in G^*} r_g^\chi = \frac{|G|}{2} \left(o(g) - 1\right). \tag{8.3}$$

Proof. Since $r_g^\chi = 0$ if and only if $\chi \in \ker g$, then the number of addenda of $\sum_{\chi\in\mathcal{H}} r_g^\chi$ that are equal to zero is exactly $\left|\ker g_{|\mathcal{H}}\right| = \frac{|\mathcal{H}|}{o(g_{|\mathcal{H}})}$.

The remaining $|\mathcal{H}|\left(1 - \frac{1}{o(g_{|\mathcal{H}})}\right)$ addenda are integers between 1 and $o(g)-1$. Since $r_g^\chi \neq 0$ implies $r_g^\chi + r_g^{\chi^{-1}} = o(g)$ it follows that their average equals $\frac{o(g)}{2}$, thus giving the result. □

It follows that

Proposition 8.1.10.

$$\sum_{\chi\in G^*} L_\chi \equiv_{num} \frac{|G|}{2} \sum_{g\in G} \left(1 - \frac{1}{o(g)}\right) D_g. \tag{8.4}$$

Moreover, for every $g \in G$,

$$\sum_{\chi\in\ker g} L_\chi \equiv_{num} \frac{|G|}{2o(g)} \sum_{h\in G} \left(1 - \frac{1}{o(h_{|\ker g})}\right) D_h. \tag{8.5}$$

Proof. By Corollary 8.1.6 $L_\chi \equiv_{num} \sum_{g\in G} \frac{r_g^\chi}{o(g)} D_g$.

Summing over all χ and using (8.3) we obtain (8.4).

Setting $\mathcal{H} = \ker g$ and summing only on the characters in $\mathcal{H}$, using (8.2) and $|\mathcal{H}| = \frac{|G|}{o(g)}$ we obtain (8.5). □

Now we can give the announced formula for the linear systems of the divisors D_g in terms of the L_χ when the group is $(\mathbb{Z}/2\mathbb{Z})^k$.

Theorem 8.1.11. *Let $\pi\colon X \to Y$ be a $k-$double cover, Y smooth and X normal, with set of data $\mathcal{L}_\chi$, D_g. Then for all $g \in G$*

$$D_g \equiv_{num} \frac{1}{2^{k-2}} \left(\sum_{\chi \notin \ker g} L_\chi - \sum_{\chi \in \ker g} L_\chi \right).$$

Proof. Let us fix an element $g \in G = (\mathbb{Z}/2\mathbb{Z})^k$, $g \neq 0$.

We note that for all h in $(\mathbb{Z}/2\mathbb{Z})^k$, $o(h_{|\ker g})$ equals 1 if $h \in \langle g \rangle = \{0, g\}$ and 2 otherwise. Then by (8.5)

$$\sum_{\chi \in \ker g} L_\chi \equiv_{num} 2^{k-2} \sum_{h \in G} \left(1 - \frac{1}{o(h_{|\ker g})} \right) D_h = 2^{k-3} \sum_{g \notin \langle h \rangle} D_h.$$

By (8.4), recalling that $D_0 = 0$, we obtain $\sum_{\chi \in G^*} L_\chi \equiv_{num} 2^{k-2} \sum_{h \in G} D_h$ and then

$$D_g = D_g + D_0 = \sum_{h \in G} D_h - \sum_{h \notin \langle g \rangle} D_h = \frac{1}{2^{k-2}} \left(\sum_{\chi \in G^*} L_\chi - 2 \sum_{\chi \in \ker g} L_\chi \right) =$$
$$= \frac{1}{2^{k-2}} \left(\sum_{\chi \notin \ker g} L_\chi - \sum_{\chi \in \ker g} L_\chi \right).$$

□

8.2 The canonical system of an abelian cover

A canonical divisor K_X on a normal variety X is a Weil divisor, the closure of a canonical divisor of the smooth part of X (see [Rei87, (1.5)]).

If $\pi\colon X \to Y$ is a $G-$cover, then G acts on $\pi_*(\mathcal{O}_X(K_X))$ inducing a decomposition on it in eigenspaces

$$\pi_*(\mathcal{O}_X(K_X)) = \bigoplus_{\chi \in G^*} \pi_*(\mathcal{O}_X(K_X))^{(\chi)}$$

Theorem 8.2.1. [BP21, Proposition 2.4], [Par91a, Proposition 4.1, c) for the case when X is smooth] *Let $\pi\colon X \to Y$ be an abelian cover, with X normal and Y smooth, whose building data are $\mathcal{L}_\chi$ and D_g. Then*

$$(\pi_*\mathcal{O}_X(K_X))^{(\chi)} \cong \mathcal{O}_Y(K_Y) \otimes \mathcal{L}_{\chi^{-1}}. \tag{8.6}$$

Consider a global section $\sigma \in H^0(\mathcal{O}_Y(K_Y) \otimes \mathcal{L}_{\chi^{-1}})$, and let $(\sigma) \in \mathrm{Div}(Y)$ be the induced effective divisor. By (8.6) σ determines an element of $H^0(\pi_*\mathcal{O}_X(K_X)) \cong H^0(\mathcal{O}_X(K_X))$, whose divisor is, by the proof of [BP12, Proposition 2.4] (compare also [Lie03, Section 3.4]),

$$\pi^*(\sigma) + \sum_g (o(g) - r_g^{\chi^{-1}} - 1) R_g. \tag{8.7}$$

It follows

Proposition 8.2.2. *Assume that all not empty linear systems $|K_Y + L_\chi|$ are base-point-free.*

Then the base locus of $|K_X|$ equals

$$\bigcap_{\substack{\chi \in G^*:\\ |K_Y+L_\chi| \neq \emptyset}} \quad \bigcup_{\substack{g \in G:\\ r_g^\chi \neq o(g)-1}} R_g$$

Proof. Since $|K_Y + L_\chi|$ is base-point-free, if the linear subsystem of $|K_X|$ corresponding to $H^0(\mathcal{O}_X(K_X))^{(\chi^{-1})}$ is not empty, by (8.7) its base locus equals

$$\bigcup_{\substack{g \in G:\\ r_g^\chi \neq o(g)-1}} R_g.$$

Since these linear subsystems generate $|K_X|$, its base locus equals their intersection. □

We recall that all complete linear systems on $\mathbb{P}^n$ are base-point-free, so Proposition 8.2.2 gives a complete description of the base locus of the canonical system of any abelian cover of a projective space. For $k-$double covers of $\mathbb{P}^n$ we obtain as in [GPR22, Section 2] (see also [Cat99, Section 2]),

Corollary 8.2.3. *Let $\pi\colon X \to \mathbb{P}^n$ be a $k-$double cover with building data L_χ, D_g. Then $|K_X|$ is base-point-free if and only if*

$$\bigcap_{\chi : \deg L_\chi \geq n+1} \quad \bigcup_{g \in \ker \chi} D_g = \emptyset.$$

8.3 Smooth k-double planes with p_g=3

Definition 8.3.1. A smooth $k-$double plane is a $k-$double cover $\pi\colon X \to \mathbb{P}^2$ such that all D_g are smooth, each two of them intersect transversally, and no point in $\mathbb{P}^2$ belongs to three of them.

In particular the branch divisor $D = \sum D_g$ is a smooth normal crossing divisor.

The assumption ensures the smoothness of X.

Proposition 8.3.2. *Let $\pi\colon X \to \mathbb{P}^2$ be a smooth $k-$double plane. Then X is smooth.*

Proof. This is a special case of [Par91a, Proposition 3.1] (see also [Man01, Proposition 3.14]). □

Notation

It is convenient to consider G and G^* as vector spaces over the field with 2 elements as in [Man01, Setup 3.2]. We are thus going to switch to the additive notation, so for example the sheaf $\mathcal{L}_1$ will be $\mathcal{L}_0$ from now on, and for each character χ we will write $-\chi$ for the character that was called χ^{-1} in the previous section.

Denote by $e_1, \cdots, e_k$ the standard basis of $G = (\mathbb{Z}/2)^k$ and by $\epsilon_1, \ldots, \epsilon_k$ the dual basis of G^*.

To every smooth $k-$double plane $\pi\colon X \to \mathbb{P}^2$ we consider its building data L_χ, D_g and the numbers

$$d_g := \deg D_g, \qquad\qquad l_\chi := \deg L_\chi.$$

Note that $d_0 = l_0 = 0$.

Note moreover that since $G = (\mathbb{Z}/2\mathbb{Z})^k$, for each $\chi \in G^*$, $\chi = -\chi$. We will use this often in the following computations.

Definition 8.3.3. We will say that a smooth $k-$double plane with $p_g = 3$ is

of type A if $l_{\epsilon_1} = 4$, $l_\chi \in \{1,2\}$ for all $\chi \notin \langle \epsilon_1 \rangle$

of type B if $l_{\epsilon_1} = l_{\epsilon_2} = l_{\epsilon_1+\epsilon_2} = 3$, $l_\chi \in \{1,2\}$ for all $\chi \notin \langle \epsilon_1, \epsilon_2 \rangle$

of type C if $l_{\epsilon_1} = l_{\epsilon_2} = l_{\epsilon_3} = 3$, $l_\chi \in \{1,2\}$ for all $\chi \notin \{0, \epsilon_1, \epsilon_2, \epsilon_3\}$

By (8.6) for a smooth $k-$double plane $\pi\colon X \to \mathbb{P}^2$

$$p_g(X) = h^0(\mathcal{O}_X(K_X)) = h^0(\pi_*(\mathcal{O}_X(K_X))) = \sum_{\chi \in G^*} h^0(\mathcal{O}_{\mathbb{P}^2}(l_\chi - 3)), \quad (8.8)$$

so in all cases of Definition 8.3.3 we obtain $p_g(X) = 3$. Conversely

Proposition 8.3.4. *Up to automorphisms of G every smooth $k-$double plane with $p_g(X) = 3$ falls in one of the three cases in Definition 8.3.3.*

Proof. Since X is connected, for all $\chi \neq 0$, $H^0(\mathcal{L}_\chi^{-1}) = 0$ and thus $l_\chi > 0$.

By (8.8) $l_\chi \leq 4$ and either there is only one χ with $l_\chi \geq 3$, in which case $l_\chi = 4$, or there are three χ with $l_\chi \geq 3$, all with $l_\chi = 3$.

Using an automorphism of G, we can reduce the former case to "type A", and the latter case either to "type B" or "type C", depending if the three special characters are linearly dependent or not. □

We now look at when a $k-$double plane with $p_g = 3$ has canonical system base-point-free.

Lemma 8.3.5. *Let $\pi\colon X \to \mathbb{P}^2$ be a smooth $k-$double plane with $p_g = 3$ of type t. Then $|K_X|$ is base-point-free if and only if*

$d_g = 0$ *for all* $g \in \ker \epsilon_1$ *when* $t = A$*;*

$d_g = 0$ *for all* $g \in \ker \epsilon_1 \cap \ker \epsilon_2$ *when* $t = B$*;*

$d_g = 0$ *for all* $g \in \bigcup_{1 \leq i < j \leq 3} (\ker \epsilon_i \cap \ker \epsilon_j)$ *when* $t = C$*.*

Proof. By Corollary 8.2.3 $|K_X|$ is base-point-free if and only if

$$\bigcap_{\chi : l_\chi \geq 3} \bigcup_{g \in \ker \chi} D_g = \emptyset.$$

For type A we deduce $d_g = 0$ for all $g \in \ker \epsilon_1$.

In the remaining cases we have three characters χ with $l_\chi = 3$. We first show that for each χ with $l_\chi = 3$ there is at least one $g \in \ker \chi$ such that $d_g \neq 0$. In fact, in this case $K_{\mathbb{P}^2} + L_\chi = 0$ and thus by (8.7) $\sum_{g \in \ker \chi} R_g$ is a canonical divisor. If $\sum_{g \in \ker \chi} d_g$ vanished, then this canonical divisor would vanish, and thus $\mathcal{O}_X(K_X)$ would be isomorphic to $\mathcal{O}_X$, contradicting $p_g = 3$.

For type C we obtain that $|K_X|$ is base-point-free if and only the following intersection of three divisors

$$\left(\bigcup_{g \in \ker \epsilon_1} D_g \right) \cap \left(\bigcup_{g \in \ker \epsilon_2} D_g \right) \cap \left(\bigcup_{g \in \ker \epsilon_3} D_g \right) \tag{8.9}$$

vanishes, and by our last remark all three divisors are not empty. Then if there is a g such that $d_g \neq 0$, belonging to two different $\ker \epsilon_i$, then any intersection point among D_g and one of the $D_h \neq 0$ in the kernel of the third ϵ_j is in (8.9), and thus $|K_X|$ is not base-point-free.

Conversely, if $d_g = 0$ for all $g \in \bigcup_{1 \leq i < j \leq 3} \ker \epsilon_i \cap \ker \epsilon_j$ then the three divisors we are intersecting in (8.9) have no common components, and thus the intersection is empty since D is a smooth normal crossing divisor.

For type B the result follows similarly using that $\ker \epsilon_1 \cap \ker \epsilon_2 = \ker \epsilon_1 \cap \ker (\epsilon_1 + \epsilon_2) = \ker \epsilon_2 \cap \ker (\epsilon_1 + \epsilon_2)$. □

We can now classify the $k-$double planes with $p_g = 3$, by considering separately the three cases in Definition 8.3.3.

For type A we obtain a special case of the situation classified in [DG14, Theorem 1.1].

Proposition 8.3.6. *The smooth* $k-$*double planes with* $p_g = 3$ *of type A form four families, one for each value of* $k = 1, \ldots, 4$*.*

In all cases π *is the canonical map of* X*,* $|K_X| = |\pi^* \mathcal{O}_{\mathbb{P}^2}(1)|$ *is base-point-free and*

$l_0 = 0$	$l_{\epsilon_1} = 4$	$l_\chi = 2$ *for all remaining* χ
$d_g = 0$ *for all* $g \in \ker \epsilon_1$		$d_g = 2^{4-k}$ *for all* $g \notin \ker \epsilon_1$

Proof. By (8.1), for all $\chi \in G^*$, $l_\chi + l_{\chi+\epsilon_1} = l_{\epsilon_1} + \sum_{g\in G} \varepsilon^g_{\chi,\chi+\epsilon_1} d_g \geq l_{\epsilon_1} = 4$.

Since for χ not in $\langle \epsilon_1 \rangle$ we have $l_\chi \leq 2$, it follows $l_\chi = 2$.

It follows moreover $\sum_{g\in G} \varepsilon^g_{\chi,\chi+\epsilon_1} d_g = 0$ so $d_g = 0$ for all g that are neither in $\ker \chi$ nor in $\ker \chi + \epsilon_1$. Varying $\chi \in G^*$ this shows that $d_g = 0$ for all $g \in \ker \epsilon_1$.

Then by Lemma 8.3.5 $|K_X|$ is base-point-free. In fact $H^0(\mathcal{O}_X(K_X)) = H^0(\mathcal{O}_X(K_X))^{(\epsilon_1)}$: this implies that the canonical map is composed with π. In fact since $\pi_*(\mathcal{O}_X(K_X))^{(\epsilon_1)} \cong \mathcal{O}_{\mathbb{P}^2}(1)$, π is exactly the canonical map of X and $|K_X| = |\pi^*\mathcal{O}_{\mathbb{P}^2}(1)|$.

Finally by Theorem 8.1.11, for all $g \notin \ker \epsilon_1$,

$$d_g = \frac{\sum_{\chi\notin\ker g} l_\chi - \sum_{\chi\in\ker g} l_\chi}{2^{k-2}} = \frac{\big(4 + (2^{k-1}-1)\cdot 2\big) - \big(0 + (2^{k-1}-1)\cdot 2\big)}{2^{k-2}} = 2^{4-k}.$$

It follows $k \leq 4$.

We leave to the reader the easy check that all 4 cases do exist by checking that (8.1) holds for them. □

To study the next two cases, we preliminarily note that Corollary 8.1.6 may be rewritten as $l_\chi = \frac{1}{2}\sum_{g\notin\ker\chi} d_g$ or equivalently

$$\forall \chi \in G^* \qquad \sum_{g\in\ker\chi} d_g = d - 2l_\chi \tag{8.10}$$

where $d := \deg D = \sum_g d_g$.

For type B we obtain only one family.

Proposition 8.3.7. *The smooth $k-$double planes of type B with $p_g = 3$ form one family, with $k = 2$. These surfaces have a canonical system that is base-point-free and*

$$\begin{array}{ll} l_0 = 0 & \qquad l_\chi = 3 \textit{ for } \chi \neq 0 \\ d_0 = 0 & \qquad d_g = 3 \textit{ for } g \neq 0 \end{array}$$

Proof. We note that G is the union of the three subgroups $\ker \epsilon_1$, $\ker \epsilon_2$ and $\ker(\epsilon_1+\epsilon_2)$, which pairwise intersect in $\ker(\epsilon_1 \cap \epsilon_2)$. It follows that

$$\sum_{g\in\ker(\epsilon_1\cap\epsilon_2)} d_g = \frac{1}{2}\left(-d + \sum_{g\in\ker\epsilon_1} d_g + \sum_{g\in\ker\epsilon_2} d_g + \sum_{g\in\ker(\epsilon_1+\epsilon_2)} d_g \right) \overset{(8.10)}{=} \\ = d - (l_{\epsilon_1} + l_{\epsilon_2} + l_{\epsilon_1+\epsilon_2}) = d - 9,$$

so $d \geq 9$.

On the other hand, since $l_\chi \leq 2$ for all $\chi \notin \langle \epsilon_1, \epsilon_2 \rangle$

$$2^{k-2}d \stackrel{(8.4)}{=} \sum_{\chi \in G^*} l_\chi = 9 + \sum_{\chi \notin \langle \epsilon_1, \epsilon_2 \rangle} l_\chi \leq 9 + \left(2^k - 4\right) \cdot 2 = 2^{k+1} + 1.$$

so $d \leq 8 + \frac{1}{2^{k-2}}$.

Since by assumption $k \geq 2$, we conclude that $d = 9$ and $k = 2$.

The d_g follow by Theorem 8.1.11. Since $\epsilon_1 \cap \epsilon_2$ is trivial, Lemma 8.3.5 ensures that the canonical system is base-point-free.

We leave to the reader to check that equations (8.1) are verified. □

For type C we obtain six families. In order to write them clearly we introduce the following rather standard notation.

Notation

The weight $w(g)$ of an element $g = (g_1, \ldots, g_k) \in (\mathbb{Z}/2\mathbb{Z})^k$ is the number of g_i different from zero.

For every $h \leq k$ we denote by $w_h(g)$ the number of g_i different from zero with $i \leq h$.

In the following we apply this notation to both the elements of G and of G^*.

We note that by Lemma 8.3.5 the canonical system of a k-double plane with $p_g = 3$ of type C is base-point-free if and only if $\sum_{g|w_3(g) \leq 1} d_g = 0$.

Let us set $\epsilon := \sum_{i=1}^3 \epsilon_i$. We note that $g \in \ker \epsilon$ if and only if $w_3(g)$ is even. It follows that

$$2 \sum_{g|w_3(g) \leq 1} d_g = 3d - \sum_g w_3(g) d_g - \sum_{g|w_3(g)\ even} d_g = 3d - \sum_{i=1}^3 \left(\sum_{g \notin \ker \epsilon_i} d_g \right) - \sum_{g \in \ker \epsilon} d_g,$$

from which, by (8.10)

$$\sum_{g|w_3(g) \leq 1} d_g = \frac{1}{2}\left(3d - 2\sum l_{\epsilon_i} - d + 2l_\epsilon\right) = d + l_\epsilon - \sum_{i=1}^3 l_{\epsilon_i} = d + l_\epsilon - 9. \quad (8.11)$$

We consider first those surfaces whose canonical system is base-point-free.

Proposition 8.3.8. *The smooth $k-$double planes with $p_g = 3$ of type C with canonical system base-point-free form the following five families.*

(C3) $k = 3$, $l_0 = 0$ *and*

$$\begin{array}{lll} l_\chi = 3 \ \textit{if}\ w(\chi) = 1, & l_\epsilon = 1, & l_\chi = 2 \ \textit{otherwise}; \\ d_g = 0 \ \textit{if}\ w(g) \leq 1, & & d_g = 2 \ \textit{otherwise}. \end{array}$$

(C4) $k = 4$, $l_0 = 0$ *and*

$$\begin{array}{lll} l_\chi = 3 \text{ if } w(\chi) = 1, & l_\epsilon = 1, & l_\chi = 2 \text{ otherwise;} \\ d_g = 0 \text{ if } w_3(g) \leq 1, & & d_g = 1 \text{ otherwise.} \end{array}$$

(D3) $k = 3$, $l_0 = 0$ *and*

$$\begin{array}{lll} l_\chi = 3 \text{ if } w(\chi) = 1, & l_\epsilon = 2, & l_\chi = 1 \text{ otherwise;} \\ d_g = 0 \text{ if } w(g) \leq 1, & d_{e_1+e_2+e_3} = 4, & d_g = 1 \text{ otherwise.} \end{array}$$

(D4) $k = 4$, $l_0 = 0$ *and*

$$\begin{array}{lll} l_\chi = 3 \text{ if } w_3(\chi) = 1, & l_\chi = 1 \text{ if } w_3(\chi) = w(\chi) = 2 & l_\chi = 2 \text{ otherwise;} \\ & \quad \text{or } w_3(\chi) \in \{0, 3\}, \chi \notin \{0, \epsilon\} & \\ d_g = 2 \text{ if } w_3(g) = 3, & d_g = 1 \text{ if } w_3(g) = w(g) = 2 & d_g = 0 \text{ otherwise.} \end{array}$$

(D5) $k = 5$, $l_0 = 0$ *and*

$$\begin{array}{lll} l_\chi = 3 \text{ if } w_3(\chi) = 1, & l_\chi = 1 \text{ if } w_3(\chi) = w(\chi) = 2 & l_\chi = 2 \text{ otherwise;} \\ & \quad \text{or } w_3(\chi) \in \{0, 3\}, \chi \notin \{0, \epsilon\} & \\ & d_g = 1 \text{ if } w_3(g) = w(g) = 2 \text{ or } w_3(g) = 3, & d_g = 0 \text{ otherwise.} \end{array}$$

Proof. Since we are assuming that the canonical system is base-point-free, by Lemma 8.3.5 and (8.11)

$$d = 9 - l_\epsilon$$

and we have to distinguish two cases, depending if $l_\epsilon = 1$ or 2.

We start with the case $l_\epsilon = 1$. Then $d = 8$.

By (8.4) $\sum_{\chi \in G^*} l_\chi = 8 \cdot 2^{k-2} = 2^{k+1}$ so the average of the l_χ equals 2. We know the values of five l_χ: $l_0 = 0$, $l_\epsilon = 9 - 8 = 1$ and the three $l_{\epsilon_i} = 3$; their average equals 2 as well. Since for all remaining χ, $l_\chi \leq 2$ we conclude that they all equal 2. By Theorem 8.1.11 $d_{e_1+e_2} = 2^{4-k}$ so $k = 3$ or 4. In both cases we deduce all other d_g by 8.1.11 obtaining the cases (C3) and (C4) in the statement.

Otherwise $l_\epsilon = 2$ and $d = 7$. Then by (8.10) $\sum_{g \in \ker \epsilon_i} d_g = 1$, so for each $i = 1, 2, 3$ there exists a unique $g \in \ker \epsilon_i$ such that $d_g \neq 0$, that we denote by g_i, and $d_{g_i} = 1$.

We show that the g_i are linearly dependent by proving that the vector subspace

$$V = \bigcap_{i=1}^{3} \ker g_i \subset G^*$$

has at most codimension 2.

First we note that if χ is a character with $l_\chi = 1$ different from $\epsilon_1 + \epsilon_2$, $\epsilon_1 + \epsilon_3$ and $\epsilon_2 + \epsilon_3$, then it belongs to V. In fact then for all $i \in \{1, 2, 3\}$ it holds $l_{\chi+\epsilon_i} \leq 2$ and then by (8.1)

$$\sum_{\substack{g \notin \ker \chi \\ g \in \ker \epsilon_i}} d_g = \sum_{\substack{g \notin \ker \chi \\ g \notin \ker(\chi+\epsilon_i)}} d_g = l_\chi + l_{\chi+\epsilon_i} - l_{\epsilon_i} \leq 1 + 2 - 3 = 0.$$

Then we note that there are at least two χ in V with $l_\chi \neq 1$: 0 and ϵ. So, setting $A := \# \{\chi \in G^* | l_\chi = 1\}$, then $\#V \geq A - 3 + 2 = A - 1$. On the other hand $A = 2 \cdot 2^k + 1 - \sum_{\chi \in G^*} l_\chi \overset{(8.4)}{=} 2^{k+1} + 1 - 7 \cdot 2^{k-2} = 2^{k-2} + 1$. Therefore

$$\#V \geq A - 1 = 2^{k-2}. \tag{8.12}$$

proving the claim that the g_i are linearly dependent.

By Lemma 8.3.5 $g_i \neq g_j$ when $i \neq j$, so $g_3 = g_1 + g_2$, and V has exactly codimension 2, and (8.12) is an equality. We complete ϵ to a basis $\epsilon, \epsilon_4, \ldots, \epsilon_k$ of V. Then $\epsilon_1, \ldots, \epsilon_k$ is a basis of G^* respect to which $V = \{\chi | w_3(\chi) \in \{0, 3\}\}$. Since (8.12) is an equality we know exactly which l_χ are equal to 1: those in V different from 0 and ϵ, plus the three characters $\epsilon_1 + \epsilon_2$, $\epsilon_1 + \epsilon_3$ and $\epsilon_2 + \epsilon_3$.

Note that respect to the basis $e_1, \ldots, e_k$ of G dual to $\epsilon_1, \ldots, \epsilon_k$ we have

$$g_1 = e_2 + e_3, \qquad g_2 = e_1 + e_3, \qquad g_3 = e_1 + e_2.$$

Finally we compute all d_g from the l_χ using Theorem 8.1.11. For $g = e_1 + e_2 + e_3$ we obtain

$$d_{e_1+e_2+e_3} = \frac{1}{2^{k-2}} \left(\sum_{w_3(\chi) \text{ odd}} l_\chi - \sum_{w_3(\chi) \text{ even}} l_\chi \right)$$

We note that l_χ appears in this expression with the opposite sign of $l_{\chi+\epsilon}$.

Since $w_3(\chi) = 3 - w_3(\chi + \epsilon)$, then $\chi \in V = \{\chi | w_3(\chi) \in \{0, 3\}\}$ if and only if $\chi + \epsilon \in V$. We have proved that, if χ does not belong to $\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$ then $l_\chi = 1$ if $\chi \in V$ and $l_\chi = 2$ otherwise. So the contributions of the l_χ not in $\langle \epsilon_1, \epsilon_2, \epsilon_3 \rangle$ cancel each other out and

$$d_{e_1+e_2+e_3} = \frac{1}{2^{k-2}} \left(l_{\epsilon_1} + l_{\epsilon_2} + l_{\epsilon_3} + l_\epsilon - l_{\epsilon_1+\epsilon_2} - l_{\epsilon_1+\epsilon_3} - l_{\epsilon_2+\epsilon_3} \right) = \frac{8}{2^{k-2}} = 2^{5-k}$$

so $k \leq 5$ and we obtain a family for each $k = 3, 4, 5$. We leave to the reader the computation of the remaining d_g, giving the families $(D3)$, $(D4)$ and $(D5)$. $\square$

Finally we consider those $k-$double planes with $p_g = 3$ whose canonical system is not base-point-free, and see that they provide exactly one more family.

Proposition 8.3.9. *The smooth $k-$double planes with $p_g = 3$ whose canonical system is not base-point-free are of type C and form one family, with $k = 3$, $l_0 = 0$ and*

$$l_{\epsilon_i} = 3 \quad l_{\epsilon_1+\epsilon_2} = 1 \qquad\qquad l_\chi = 2 \text{ otherwise}$$
$$d_{e_1+e_2+e_3} = 3 \quad d_{e_1+e_2} = 2 \quad d_{e_3} = d_{e_1+e_3} = d_{e_2+e_3} = 1 \quad d_0 = d_{e_1} = d_{e_2} = 0$$

Their canonical system has four simple base points, the preimages of the two points in the intersection of the line D_{e_3} and the conic $D_{e_1+e_2}$.

Proof. By propositions 8.3.6 and 8.3.7 these double planes are of type C. Thus, using (8.4)

$$d \cdot 2^{k-2} = \sum_{\chi \in G^*} l_\chi = 9 + \sum_{\chi \notin \{\epsilon_j\}} l_\chi \leq 2^{k+1} + 1 \tag{8.13}$$

from which we deduce, since $k \geq 3$, $d \leq 8 + \frac{1}{2^{k-2}} \leq 8 + \frac{1}{2}$. So $d \leq 8$.

We recall that the existence of base points for the canonical system is equivalent to $\sum_{g | w_3(g) \leq 1} d_g \neq 0$. On the other hand by (8.11)

$$\sum_{g | w_3(g) \leq 1} d_g = d + l_\epsilon - 9 \leq l_\epsilon - 1.$$

We conclude that

$$l_\epsilon = 2 \qquad\qquad d = 8 \qquad\qquad \sum_{g | w_3(g) \leq 1} d_g = 1$$

and thus there is an unique $h \in G$ with $d_h = 1$ and $w_3(h) = 1$. Note that exactly one of the three characters ϵ_j is not in $\ker h$.

The inequality in (8.13) fails to be an equality exactly by 1. This means that there is exactly one character η with $l_\eta = 1$. By the expression of d_h in term of the l_χ in Theorem 8.1.11 we deduce that $\eta \notin \ker h$ (or d_h would be negative) and $d_h = \frac{1}{2^{k-3}}$. So $k = 3$.

Using an automorphism of G we can now assume without loss of generality $\eta = \epsilon_1 + \epsilon_2$. We have now computed all l_ϵ: we leave to the reader to compute all d_g by applying Theorem 8.1.11.

By (8.7) the canonical system $|K_X|$ is generated by the following three divisors

$$R_{e_3} + R_{e_2+e_3} \qquad\qquad R_{e_3} + R_{e_1+e_3} \qquad\qquad R_{e_1+e_2}$$

and then by the smoothness assumption the base locus is the schematic intersection $R_{e_1+e_2} \cap R_{e_3}$.

The line D_{e_3} and the conic $D_{e_1+e_2}$ intersect transversally in two points. Above each of them there are two points of X, stabilized by the index two subgroup $\langle e_1 + e_2, e_3 \rangle$, the intersection points of $R_{e_1+e_2} \cap R_{e_3}$. A straightforward local computation shows that $R_{e_1+e_2}$ and R_{e_3} are transversal. $\square$

8.4 The eleven families

In the previous section we have proved that the smooth k-double planes with $p_g = 3$ form 11 families. In this section we will study these families.

Notation

We will denote each family by a letter and a number. The number is the exponent k of the Galois group, while the letter reminds the type. In particular the 4 families in Proposition 8.3.6 give surfaces of type A1, A2, A3 and A4, while the surfaces in Proposition 8.3.7 form the family B2. There are more families of surfaces of type C with the same Galois group, so for these we need to use more letters: we will use the letters C, D and E. Precisely the surfaces in Proposition 8.3.8 are named, as already specified in that statement, as C3, C4, D3, D4 and D5, while the surfaces in Proposition 8.3.9 form the family E3.

All these surfaces have ample canonical class, since it is numerically the pull-back of an ample class of $\mathbb{P}^2$ (see *e.g.* [Par91a, Proof of Proposition 4.2]). Their irregularity vanishes, for example since their geometric genus is 3 by construction and the Euler characteristic is 4 by [Par91a, (4.8)].

For each family we compute the degree of the canonical map.

Theorem 8.4.1. *All smooth k-double covers S of the plane with geometric genus 3 are regular surfaces with ample canonical class.*

The canonical map φ_{K_S} is a morphism of degree K_S^2 on $\mathbb{P}^2$ unless S of type E3, in which case the canonical map is a rational map of degree $K_S^2 - 4 = 4$ undefined at 4 points.

Each family is unirational. The modular dimension of each family, that is the dimension of its image in the Gieseker moduli space of the surfaces of general type, equals $4 + 2^{6-k}$ with one exception, the family B2, whose dimension is 19.

The values of K_S^2, of $\deg \varphi_{K_S}$ and of the modular dimension of each family are listed in the following table:

Family	A1	A2	A3	A4	B2	C3	C4	D3	D4	D5	E3
K_S^2	2	4	8	16	9	8	16	2	4	8	8
$\deg \varphi_{K_S}$	2	4	8	16	9	8	16	2	4	8	4
mod. dim.	36	20	12	8	19	12	8	12	8	6	12

Proof. Each surface S is a Galois cover $\pi\colon S \to \mathbb{P}^2$. By the Leray spectral sequence, $H^1(S, \mathcal{O}_S) \cong H^1(\mathbb{P}^2, \pi_*\mathcal{O}_S) \cong \bigoplus_\chi H^1\left(\mathbb{P}^2, \mathcal{L}_\chi^{-1}\right)$. Since every line bundle on $\mathbb{P}^2$ has trivial first cohomology group, it follows $h^1(S, \mathcal{O}_S) = 0$.

The value of the self-intersection of the canonical class follows by the formula (see [Par91a, (4.8)])

$$K^2 = 2^k \left(-3 + \frac{1}{2}\sum_{g\in G} d_g\right)^2$$

By Propositions 8.3.6, 8.3.7, 8.3.8, 8.3.9 the canonical system of S is base point free unless S is of type E3, in which case it has four simple base points. So (blowing up the base points in this last case) we get a surface with canonical system having movable part of self intersection as in the second line of the table above, so strictly positive. Then the canonical map is not composed with a pencil. Since $p_g = 3$ then the canonical map of this surface is a morphism on $\mathbb{P}^2$ of the given degree.

The families are parametrized by a Zariski open subset of a product of projective spaces, the complete linear systems to which the divisors $|D_g|$, quoted by the faithful action of PGL(3), a group of dimension 8. Since the surfaces are of general type, their automorphism group is finite and therefore it contains only finitely many subgroups of the form $(\mathbb{Z}/2\mathbb{Z})^k$, which implies that the map from this quotient to the Gieseker moduli space of the surfaces of general type is finite. So the modular dimension of each family equals

$$-8 + \sum \dim |D_g|$$

which gives the modular dimensions in the table above. As an example, the family $E3$ depends on the choice of three lines, a conic and a cubic so its modular dimension is

$$-8 + 3\cdot 2 + 5 + 9 = 12.$$

□

For each family we will first give explicit equations of the surfaces embedded in a suitable weighted projective space, computed by using the equations in [Cat08, Section 6] (see also [Man01, Section 3.3]) as follows.

We consider a weighted projective space whose first three variables x_0, x_1, x_2 of weight 1. The group acts trivially on them: in fact the k-double cover is the map on $\mathbb{P}^2$ given by them. Each branch divisor divisor D_g, $g = \sum_1^k i_j e_j$, is defined by a homogeneous polynomial in the x_j, the polynomial $f_{i_1\cdots i_k}(x_j) \in \mathbb{C}[x_0, x_1, x_2]$. If $D_g = 0$ then $f_{i_1\cdots i_k}(x_j) = 1$.

Then we add variables $y_{i_1\cdots i_k}$, $i_j \in \{0,1\}$, meaning that e_j acts on $y_{i_1\cdots i_k}$ via multiplication by $(-1)^{i_j}$. The equations

$$y_{r_1\cdots r_k} y_{s_1\cdots s_k} = y_{t_1\cdots t_k} \prod_{\substack{\sum i_j r_j, \sum i_j s_j \\ \text{both odd}}} f_{i_1\cdots i_k} \quad \text{when all } r_j + s_j + t_j \text{ are even} \tag{8.14}$$

define an embedding of these surfaces in the weighted projective with variables $x_j, y_{i_1\cdots i_k}$. The weight of the variable $y_{i_1\cdots i_k}$ is the positive integer $l_{\sum_j i_j \epsilon_j}$.

Sometimes these equations allow to eliminate some variables, embedding the surfaces in a weighted projective space of smaller dimension. For example for the family A2 we find the equation $y_{11}y_{01} = y_{10}$, using it to eliminate y_{10} gives an embedding in a smaller dimensional weighted projective space. In the following we will eliminate all the variables that we can eliminate, to give simpler equations.

Then we will discuss all "intermediate" quotients, the quotients of these surfaces by subgroups of the Galois group of the cover, with a focus on $K3$ surfaces and symplectic involutions.

8.4.1 Family A1

These surfaces have $K^2 = 2$.

They are the hypersurfaces of degree 8 in $\mathbb{P}(1^3, 4)$, with variables x_0, x_1, x_2, y_1,

$$y_1^2 = f_1(x_j),$$

with $\deg f_1 = 8$.

These are the Horikawa surfaces in [Hor76][Theorem 1.6.(i)]. This is the Example 1.2.1 due to Beauville.

8.4.2 Family A2

These surfaces have $K^2 = 4$.

These are the complete intersections of two quartics in $\mathbb{P}(1^3, 2^2)$, with variables $x_0, x_1, x_2, y_{11}, y_{01}$,

$$y_{11}^2 = f_{10}(x_j) \qquad\qquad y_{01}^2 = f_{11}(x_j)$$

with $\deg f_\bullet = 4$.

There are three intermediate quotients: the quotient by e_1 and e_1+e_2 are double planes branched on quartics, so del Pezzo surfaces of degree 2. The quotient by e_2 is a double plane branched on both quartics, so a degeneration of the family A1, a Horikawa surface with 16 nodes.

This family is in [DG14, Theorem 1.1.(5)]. These surfaces were also studied by Horikawa, see [Hor78, Theorem 2.1].

8.4.3 Family B2

These surfaces have $K^2 = 9$.

They are embedded in $\mathbb{P}(1^3, 3^3)$, with variables $x_0, x_1, x_2, y_{10}, y_{01}, y_{11}$, defined by the equations

$$\mathit{Rank} \begin{pmatrix} f_{10}(x_j) & y_{10} & y_{11} \\ y_{10} & f_{11}(x_j) & y_{01} \\ y_{11} & y_{01} & f_{01}(x_j) \end{pmatrix} = 1$$

with $\deg f_\bullet = 3$. This the Example 1.2.2, for $d = 9$.

The three intermediate quotients are double planes branched on the union of two cubics: three $K3$ surfaces with 9 nodes.

We met this family in [Cat99, Example 6] and [Gar19, Proposition 6.3]. They are also studied in [Lat21a] and [GP22].

8.4.4 Family A3

These surfaces have $K^2 = 8$.

They are embedded in $\mathbb{P}(1^3, 2^6)$, with variables $x_0, x_1, x_2, y_{010}, y_{001}, y_{110}, y_{101}, y_{011}, y_{111}$, defined by the equations

$$Rank \begin{pmatrix} f_{111}(x_j) & y_{010} & y_{001} & y_{111} \\ y_{010} & f_{110}(x_j) & y_{011} & y_{101} \\ y_{001} & y_{011} & f_{101}(x_j) & y_{110} \\ y_{111} & y_{101} & y_{110} & f_{100}(x_j) \end{pmatrix} = 1$$

with $\deg f_\bullet = 2$.

The quotients by $\ker \epsilon_1$ are double planes branched on the union of 4 conics, degenerations of the family A1 with 24 nodes. The quotients by each of the other 6 subgroups of index 2 are double planes branched on the union of 2 conics, del Pezzo surfaces of degree 2 with 4 nodes.

The quotients by a subgroup $\langle g \rangle$ of index 4 behave differently according to if g belongs to $\ker \epsilon_1$ or not. If $g \in \ker \epsilon_1$ the quotient is a degeneration of the family A2 with 16 nodes. Otherwise, for the remaining four g, the quotients are 2-double planes such that each of the three branching divisors is a conic. By, *e.g* [BP21, Propositions 2.4-2.5 and their proof] they have $p_g = 0$ and bicanonical sheaf trivial, so they are Enriques surfaces.

These surfaces are in [DG14, Theorem 1.1.(3)], where the authors give them through equations of a different (not normal) birational model.

8.4.5 Family C3

These surfaces have $K^2 = 8$.

They are embedded in $\mathbb{P}(1^4, 2^3)$, with variables $x_0, x_1, x_2, y_{111}, y_{110}, y_{101}, y_{011}$, defined by the equations

$$Rank \begin{pmatrix} f_{110}(x_j) & y_{011} & y_{101} \\ y_{011} & f_{101}(x_j) & y_{110} \\ y_{101} & y_{110} & f_{011}(x_j) \end{pmatrix} = 1 \qquad y_{111}^2 = f_{111}(x_j)$$

with $\deg f_\bullet = 2$.

The Galois group has seven subgroups of index 2, the three of the form $\ker \epsilon_i$, the three of the form $\ker \epsilon_i + \epsilon_j$, and $\ker \epsilon$.

The quotients by a subgroup of the form $\ker \epsilon_i$ are double planes branched on the union of 3 conics, so K3 surfaces with 12 nodes. The quotients by a subgroup of the form $\ker \epsilon_i + \epsilon_j$ are double planes branched on the union of 2 conics, so del Pezzo surfaces of degree 2 with 4 nodes. The quotients by $\ker \epsilon$ are double planes branched on one conic, so $\mathbb{P}^1 \times \mathbb{P}^1$.

The quotients by a subgroup $\langle g \rangle$ of index 4 are 2−double planes as follows. If $g = e_1 + e_2 + e_3$ then the three branching divisors are three smooth conics, so the quotients are smooth Enriques surfaces. If g is of the form $e_i + e_j$ then one of the branching divisors is empty, one is a smooth conic, and the last is union of two conics: the quotients are K3 surfaces with 8 nodes. If g is one of the e_i then two divisors are conics whereas the third is the union of two conics: they are surfaces with $K^2 = 4$, $p_g = 2$ and 8 nodes.

Then each surface in this family dominates six different K3 surfaces. Let us give names to them. Let $U_{i,j}$ be the K3 with 8 nodes obtained quoting by $\langle e_i + e_j \rangle$ and let V_k be the K3 with 12 nodes obtained quoting by $\ker \epsilon_k$. Then these K3 are naturally subdivided in three pairs by double covers $V_{i,j} \to U_k$ (here $k \notin \{i, j\}$) branched on 8 nodes and nowhere else, quotient of $V_{i,j}$ by the symplectic involution induced by e_i. The $V_{i,j}$ are special cases of the K3 surfaces considered in [vGS07, 3.5], where the plane quartic considered there splits as union of two conics.

8.4.6 Family D3

These surfaces have $K^2 = 2$.

They are embedded in $\mathbb{P}(1^6, 2)$ with variables $x_0, x_1, x_2, y_{110}, y_{101}, y_{011}, y_{111}$, defined by the equations

$$Rank \begin{pmatrix} f_{110}(x_j) & y_{011} & y_{101} \\ y_{011} & f_{101}(x_j) & y_{110} \\ y_{101} & y_{110} & f_{011}(x_j) \end{pmatrix} = 1 \qquad y_{111}^2 = f_{111}(x_j)$$

with $\deg f_{110} = \deg f_{101} = \deg f_{011} = 1$ and $\deg f_{111} = 4$. Note that these equations are identical to those of the family C3, the only difference being in the degrees.

The quotients by a subgroup of the form $\ker \epsilon_i$ are double planes branched on the union of 2 lines and one quartic, so K3 surfaces with 9 nodes. The quotients by a subgroup of the form $\ker \epsilon_i + \epsilon_j$ are double planes branched on the union of 2 lines, so del Pezzo surfaces of degree 8 with 1 node. The quotients by $\ker \epsilon$ are double planes branched on one quartic, so smooth del Pezzo surfaces of degree 2.

The quotients by a subgroup $\langle g \rangle$ of index 4 are $2-$double planes as follows. If $g = e_1 + e_2 + e_3$ then the three branching divisors are lines, so the quotients are projective planes $\mathbb{P}^2$. If g is of the form $e_i + e_j$ then one of the branching divisors is empty, one is a smooth quartic, and the last is union of two lines: the quotients are K3 surfaces with 2 nodes. If g is one of the e_i then two divisors are lines whereas the third is the union of a line and a quartic: they are surfaces with $K^2 = 1$, $p_g = 2$ and 8 nodes.

Then each surface in this family dominates six different K3 surfaces naturally subdivided in three pairs as in the previous case. More precisely, let $U_{i,j}$ be the K3 with 2 nodes obtained quoting by $\langle e_i + e_j \rangle$ and let V_k be the K3 with 9 nodes obtained quoting by $\ker \epsilon_k$. Then we have double covers $V_{i,j} \to U_k$, $k \notin \{i, j\}$, branched on 8 nodes and nowhere else, quotient of $V_{i,j}$ by the symplectic involution induced by e_i. These are again special cases of the K3 surfaces considered in [vGS07, 3.5], where the plane conic considered there splits as union of two lines.

We finally note that, since the quotient by $e_1 + e_2 + e_3$ represents these surfaces as double cover of the plane, these surfaces are a degeneration of the surfaces in the family A1, special Horikawa surfaces in the family of [Hor76, Theorem 1.6.(i)] with extra automorphisms.

8.4.7 Family E3

These surfaces have $K^2 = 8$.

They are embedded in $\mathbb{P}(1^4, 2^3, 3^2)$, with variables x_0, x_1, x_2, y_{110}, y_{101}, y_{011}, y_{111}, y_{100}, y_{010}, defined by the equations

$$Rank\begin{pmatrix} f_{110}f_{111} & y_{100} & y_{010} \\ y_{100} & f_{101} & y_{110} \\ y_{010} & y_{110} & f_{011} \end{pmatrix} = 1, \qquad Rank\begin{pmatrix} f_{110}f_{001} & y_{011} & y_{101} \\ y_{011} & f_{101} & y_{110} \\ y_{101} & y_{110} & f_{011} \end{pmatrix} = 1,$$

$$Rank\begin{pmatrix} f_{111} & y_{100} & y_{111} \\ y_{100} & f_{110}f_{101} & y_{011} \\ y_{111} & y_{011} & f_{001} \end{pmatrix} = 1, \qquad Rank\begin{pmatrix} f_{111} & y_{010} & y_{111} \\ y_{010} & f_{110}f_{011} & y_{101} \\ y_{111} & y_{101} & f_{001} \end{pmatrix} = 1.$$

with $\deg f_{101} = \deg f_{011} = \deg f_{001} = 1$, $\deg f_{110} = 2$ and $\deg f_{111} = 3$.

The quotients by the subgroup $\ker \epsilon_1$ or $\ker \epsilon_2$ are double planes branched on the union of one line, one conic and one cubic, so K3 surfaces with 11 nodes. The quotients by the subgroup $\ker \epsilon_3$ are branched on the union of three lines and one cubic, so K3 surfaces with 12 nodes. The quotients by $\ker \epsilon_1 + \epsilon_2$ are branched on the union of two lines, so del Pezzo surfaces of degree 8 with 1 node. The quotients by $\ker \epsilon_1 + \epsilon_3$ or $\ker \epsilon_2 + \epsilon_3$ are branched on the union of two lines and a conic, so del Pezzo surfaces of degree 2 with 5 nodes. The quotients by $\ker \epsilon$ are branched on the union of one line and one cubic, so del Pezzo surfaces of degree 2 with 3 nodes.

The quotients by a subgroup $\langle g \rangle$ of index 4 are 2−double planes as follows. If $g = e_1 + e_2 + e_3$ then two of the branching divisors are lines and the third is the union of a line and a conic, so the quotients are del Pezzo surfaces of degree 1 with 4 nodes. If $g = e_1 + e_2$ then one divisor is empty, the second is the union of two lines, the third is the union of a line and a cubic, and the quotients are K3 surfaces with 8 nodes. If g is $e_1 + e_3$ or $e_2 + e_3$ then one of the branching divisors is a line, one is a cubic, and the last is union of a line and a conic: the quotients have $K^2 = p_g = 1$ and 4 nodes. If $g = e_3$ then two divisors are lines and the third is union of a conic and a cubic: the quotients have $K^2 = 1$, $p_g = 2$ and 12 nodes. If g is e_1 or e_2 then one divisor is the union of two lines, one is a conic and the last is the union of a line and a cubic, giving surfaces with $K^2 = 4$, $p_g = 2$ and 8 nodes.

Then each surface in this family dominates four different K3 surfaces. We get only one symplectic involution by the construction, on the K3 surface with 8 nodes quotient by $e_1 + e_2$. The symplectic involution is induced by e_1, and the quotient is the K3 with 12 nodes obtained by $\ker \epsilon_3$. The two K3 surfaces with 11 nodes are both dominated by a surface of general type with $K^2 = p_g = 1$.

8.4.8 Family A4

These surfaces have $K^2 = 16$.

They are embedded in $\mathbb{P}(1^3, 2^{14})$ with variables x_0, x_1, x_2, y_{1111}, y_{0100}, y_{0010}, y_{0001}, y_{1011}, y_{1101}, y_{1110}, y_{0110}, y_{0101}, y_{0011}, y_{1010}, y_{1100}, y_{0111}, y_{1001}, defined by the equations

$$Rank \begin{pmatrix} f_{1000}f_{1011} & y_{1111} & y_{1011} & y_{1100} \\ y_{1111} & f_{1110}f_{1101} & y_{0100} & y_{0011} \\ y_{1011} & y_{0100} & f_{1100}f_{1111} & y_{0111} \\ y_{1100} & y_{0011} & y_{0111} & f_{1010}f_{1001} \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{1000}f_{1101} & y_{1111} & y_{1101} & y_{1010} \\ y_{1111} & f_{1110}f_{1011} & y_{0010} & y_{0101} \\ y_{1101} & y_{0010} & f_{1010}f_{1111} & y_{0111} \\ y_{1010} & y_{0101} & y_{0111} & f_{1100}f_{1001} \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{1000}f_{1110} & y_{1111} & y_{1110} & y_{1001} \\ y_{1111} & f_{1101}f_{1011} & y_{0001} & y_{0110} \\ y_{1110} & y_{0001} & f_{1001}f_{1111} & y_{0111} \\ y_{1001} & y_{0110} & y_{0111} & f_{1100}f_{1010} \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{1100}f_{1101} & y_{0100} & y_{0110} & y_{1010} \\ y_{0100} & f_{1110}f_{1111} & y_{0010} & y_{1110} \\ y_{0110} & y_{0010} & f_{1010}f_{1011} & y_{1100} \\ y_{1010} & y_{1110} & y_{1100} & f_{1000}f_{1001} \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{1100}f_{1110} & y_{0100} & y_{0101} & y_{1001} \\ y_{0100} & f_{1101}f_{1111} & y_{0001} & y_{1101} \\ y_{0101} & y_{0001} & f_{1001}f_{1011} & y_{1100} \\ y_{1001} & y_{1101} & y_{1100} & f_{1000}f_{1010} \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{1010}f_{1110} & y_{0010} & y_{0011} & y_{1001} \\ y_{0010} & f_{1011}f_{1111} & y_{0001} & y_{1011} \\ y_{0011} & y_{0001} & f_{1001}f_{1101} & y_{1010} \\ y_{1001} & y_{1011} & y_{1010} & f_{1000}f_{1100} \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{1000}f_{1111} & y_{1011} & y_{1101} & y_{1110} \\ y_{1011} & f_{1100}f_{1011} & y_{0110} & y_{0101} \\ y_{1101} & y_{0110} & f_{1010}f_{1101} & y_{0011} \\ y_{1110} & y_{0101} & y_{0011} & f_{1001}f_{1110} \end{pmatrix} = 1$$

with $\deg f_\bullet = 1$. This is Persson example 1.2.4.

The quotients by $\ker \epsilon_1$ are double planes branched on the union of 8 lines, degenerations of the family A1 with 28 nodes. The quotients by each of the other 6 subgroups of index 2 are double planes branched on the union of 4 lines. They are del Pezzo surfaces of degree 2 with 6 nodes.

The quotients by a subgroup H of index 4 behave differently according to if H is contained in $\ker \epsilon_1$ or not. If $H \subset \ker \epsilon_1$ the quotients are degenerations of the family A2 with 24 nodes. Otherwise, the quotients are Enriques surfaces with 6 nodes.

The quotients by a subgroup $\langle g \rangle$ of index 8 also behave differently according to if g belongs to $\ker \epsilon_1$ or not. If $g \in \ker \epsilon_1$ the quotients are degenerations of the family A3 with 32 nodes. Otherwise the quotients are numerical Campedelli surfaces, surfaces with $p_g = 0$, $K^2 = 2$ and ample canonical class.

Note that these surfaces are then double covers of numerical Campedelli surfaces: in fact they were first found by Persson in this way in [Per78, Ex. 5.8]. They are also in [DG14, Theorem 1.1.(1)], where the authors give them through equations of a different (not normal) birational model.

8.4.9 Family C4

These surfaces have $K^2 = 16$.

They are embedded in $\mathbb{P}(1^4, 2^{11})$ with variables x_0, x_1, x_2, y_{1110}, y_{1001}, y_{0011}, y_{0101}, y_{1111} y_{0111}, y_{1101}, y_{1011}, y_{0001}, y_{1100}, y_{1010}, y_{0110} defined by the equations

$$Rank \begin{pmatrix} f_{1110} & y_{1001} & y_{0101} & y_{0011} & y_{1111} & y_{1110} \\ y_{1001} & f_{1010}f_{1100}f_{0111} & y_{1100}f_{1100} & y_{1010}f_{1010} & y_{0110}f_{0111} & y_{0111} \\ y_{0101} & y_{1100}f_{1100} & f_{1100}f_{0110}f_{1011} & y_{0110}f_{0110} & y_{1010}f_{1011} & y_{1011} \\ y_{0011} & y_{1010}f_{1010} & y_{0110}f_{0110} & f_{1010}f_{0110}f_{1101} & y_{1100}f_{1101} & y_{1101} \\ y_{1111} & y_{0110}f_{0111} & y_{1010}f_{1011} & y_{1100}f_{1101} & f_{1101}f_{1011}f_{0111} & y_{0001} \\ y_{1110} & y_{0111} & y_{1011} & y_{1101} & y_{0001} & f_{1111} \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{1100}f_{1101} & y_{0110} & y_{1010} \\ y_{0110} & f_{1010}f_{1011} & y_{1100} \\ y_{1010} & y_{1100} & f_{0110}f_{0111} \end{pmatrix} = 1$$

with $\deg f_\bullet = 1$.

We describe only the intermediate quotients that are K3 surfaces.

We find three intermediate K3 surfaces with 15 nodes, the quotients by $\ker \epsilon_i$, $i = 1, 2, 3$, double planes branched on six lines. Each of them is double covered by a $K3$ with 14 nodes, the quotient by $(\ker \epsilon_i) \cap (\ker \epsilon_j + \epsilon_k)$, $\{i, j, k\} = \{1, 2, 3\}$ with a symplectic involution by e_j. Note that each of these last surfaces is double covered by two further intermediate quotients with $p_g = 1$, the quotients by $e_j + e_k$ and $e_j + e_k + e_4$, both giving surfaces with K ample, $K^2 = 2$ and 8 nodes. There are special case f the "special Horikawa surfaces" considered in [Lat21b]. These pairs of $K3$ are again a specialization of [vGS07, 3.5], where all plane curves splits as union of lines.

8.4.10 Family D4

These surfaces have $K^2 = 4$.

They are embedded in $\mathbb{P}(1^8) = \mathbb{P}^7$ with variables $x_0, x_1, x_2, y_{1100}, y_{1010}, y_{0110}, y_{0001}, y_{1111}$ we take the surfaces defined by the equations

$$Rank \begin{pmatrix} f_{1100}(x_j) & y_{0110} & y_{1010} \\ y_{0110} & f_{1010}(x_j) & y_{1100} \\ y_{1010} & y_{1100} & f_{0110}(x_j) \end{pmatrix} = 1 \quad y_{1111}^2 = f_{1110}(x_j) \quad y_{0001}^2 = f_{1111}(x_j)$$

with $f_\bullet$ general of respective degrees $\deg f_{1100} = \deg f_{1010} = \deg f_{0110} = 1$ and $\deg f_{1110} = \deg f_{1111} = 2$.

The intermediate quotients that are K3 surfaces form three towers of three K3s corresponding to the chain of subgroups, for $i, j \leq 3$, $i \neq j$

$$\langle e_i + e_j \rangle \subset \langle e_i + e_j, e_4 \rangle \subset \langle e_i, e_j, e_4 \rangle$$

giving three towers of double covers between K3 surfaces $U_{i,j} \to V_{i,j} \to W_{i,j}$ with respectively 4, 10 and 13 nodes.

8.4.11 Family D5

These surfaces have $K^2 = 8$.

They are embedded in $\mathbb{P}(1^{12}) = \mathbb{P}^{11}$ with variables x_0, x_1, x_2, y_{11000}, y_{10100}, y_{01100}, y_{00010}, y_{00001}, y_{11110}, y_{11101}, y_{00011}, y_{11111} defined by the equations

$$Rank \begin{pmatrix} f_{11111}(x_j) & y_{00010} & y_{00001} & y_{11111} \\ y_{00010} & f_{11110}(x_j) & y_{00011} & y_{11101} \\ y_{00001} & y_{00011} & f_{11101}(x_j) & y_{11110} \\ y_{11111} & y_{11101} & y_{11110} & f_{11100}(x_j) \end{pmatrix} = 1$$

$$Rank \begin{pmatrix} f_{11000}(x_j) & y_{01100} & y_{10100} \\ y_{01100} & f_{10100}(x_j) & y_{11000} \\ y_{10100} & y_{11000} & f_{01100}(x_j) \end{pmatrix} = 1$$

with $f_\bullet$ general of degree 1.

There are 48 intermediate quotients that are $K3$ surfaces, divided in three families, each of them giving several towers of three consecutive double covers between (four) K3 surfaces. One for each pair $i \neq j$, $i, j = 1, 2, 3$. Namely for each pair of subgroups $H_4 \subset H_8$ with $|H_d| = d$ and

$$\langle e_i + e_j \rangle \subset H_4 \subset H_8 \subset \langle e_i, e_j, e_4, e_5 \rangle$$

we obtain a tower of 4 K3 surfaces with respectively 8, 12, 14 and 15 nodes, with the surfaces with 8 and 15 nodes depending only on i and j.

8.5 Burgers

We recall Laterveer's definition [Lat19, Definition 3.1]

Definition 8.5.1. A surface S is called a triple K3 burger if the following conditions are satisfied:

(0) S is minimal, of general type, with $q = 0$ and $p_g = 3$;

(i) there exist involutions $\sigma_j \colon S \to S$ $(j = 0, 1, 2)$ that commute with one another, and such that the quotients $\overline{X}_j := S/\langle\sigma_j\rangle$ $(j = 0, 1, 2)$ are birational to a K3 surface X_j;

(ii) there is an isomorphism

$$((p_0)^*, (p_1)^*, (p_2)^*) : H^2(\overline{X}_0, \mathcal{O}) \oplus H^2(\overline{X}_1, \mathcal{O}) \oplus H^2(\overline{X}_2, \mathcal{O}) \to H^2(S, \mathcal{O}),$$

where $p_j \colon S \to \overline{X}_j$ denotes the quotient morphism.

Laterveer's original definition included also the third condition (iii) that the involutions respect the canonical divisor: $\sigma_j^*|K_S| = |K_S|$. We removed that because it is automatic since the pull-back of a canonical divisor by an automorphism is the divisor of the pull-back of the corresponding differential form.

Our surfaces not of type A are natural candidates to be triple K3 burger. In fact

Proposition 8.5.2. *Let S be a smooth $k-$double plane not of type A.*

If S is of type B2 set, in the notation of the previous section, $\sigma_0 = e_1 + e_2$, $\sigma_1 = e_1$ and $\sigma_2 = e_2$. Otherwise set $\sigma_0 = e_1 + e_2$, $\sigma_1 = e_2 + e_3$ and $\sigma_2 = e_1 + e_3$. Then there is an isomorphism

$$((p_0)^*, (p_1)^*, (p_2)^*) : H^2(\overline{X}_0, \mathcal{O}) \oplus H^2(\overline{X}_1, \mathcal{O}) \oplus H^2(\overline{X}_2, \mathcal{O}) \to H^2(S, \mathcal{O}),$$

where $p_j \colon S \to \overline{X}_j$ denotes the quotient morphism.

Proof. Let S be a smooth $k-$double plane of type C. So we are considering now the families C3, C4, D3, D4, D5 and E3, and not considering the family B2 yet.

We know that $H^0(S, K_S)^\chi = 0$ unless $\chi = \epsilon_1$, ϵ_2, or ϵ_3. More precisely

$$\mathbb{C}^3 \cong H^0(S, K_S) = p_0^* H^0(\overline{X}_0, K_{\overline{X}_0}) \oplus p_1^* H^0(\overline{X}_1, K_{\overline{X}_1}) \oplus p_2^* H^0(\overline{X}_2, K_{\overline{X}_2}) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

which implies the stated isomorphism by the standard Serre duality.

If S is of type B2 the proof follows by the same argument replacing ϵ_3 with $\epsilon_1 + \epsilon_2$. $\square$

The following consequence has already been proved by Laterveer for the surfaces of type B2 in [Lat19, Remark 3.4]

Corollary 8.5.3. *The families B2, C3, D3, D4, D5 and E3 are families of triple K3 burgers.*

Proof. All our surfaces have ample canonical class, so condition (0) is automatic.

In Proposition 8.5.2 we have chosen involutions σ_j in each case and proved condition 2 for them. About condition (i), we have shown that the surfaces $\overline{X}_j$ are nodal K3 surfaces in the previous section. $\square$

We note that the surfaces in the family C4 are not triple K3 burgers since the quotients $\overline{X}_j$ are three surfaces of general type, more precisely surfaces with K ample, $K^2 = 2$, $p_g = 1$, $q = 0$ and 8 nodes. However each of them is a double cover of a K3 surface with 14 nodes.

Appendix A

Appendix

A.1 Classical results on Surfaces

This section lists and sometimes proves the main results on surfaces that are freely given in the thesis. They may be presented with strong hypothesis, which are the same ones encountered in the course of the thesis and for which they therefore apply.

If you are interested in some more general result, we suggest to see [GH78], [Bea96]

Here S and X are smooth projective surfaces.

Theorem A.1.1. *Let A, B two effective divisors of S such that $A - B$ is also an effective divisor C, namely $A = B + C$. Then*

1. *$C + |B| \subseteq |A|$;*
2. *$h^0(S, O_S(B)) \leq h^0(S, O_S(A))$;*
3. *Let us denote by c a local defining function of C. The induced (rational) map Φ_B factorizes through Φ_A:*

$$\begin{tikzcd} S \arrow[r, dashed, "\Phi_A"] \arrow[dr, dashed, "\Phi_B"'] & \mathbb{P}(H^0(S, O_S(A))^*) \arrow[d, dashed, "\pi_B"] \\ & \mathbb{P}(H^0(S, O_S(B))^*), \end{tikzcd}$$

where the projection map π_B is the (rational) projective dual map induced by the injective linear map

$$H^0(S, O_S(B)) \xrightarrow{\otimes C} H^0(S, O_S(A)) \ , \qquad s \mapsto c \cdot s. \tag{A.1}$$

The (schematic) base-locus of Φ_B amounts to $Bs(\Phi_B) = Bs(\Phi_A) + C$.

Furthermore, the following are equivalent *(a).* $C + |B| = |A|$;

(b). $h^0(S, O_S(B)) = h^0(S, O_S(A))$;

(c). π_B *is an isomorphism (namely, in suitable projective coordinates corresponds to the identity). In other words* Φ_A *and* Φ_B *are the same map;*

Proof. The point 1 is straightforward. Let us prove the point 2. Consider the natural multiplication map (A.1) given by C. This map is well defined thanks to $C + B = A$: if s is a global section of $O_S(B)$, then $\mathrm{div}(s) \equiv B$, and so $\mathrm{div}(c \cdot s) = \mathrm{div}(c) + \mathrm{div}(s) \equiv C + B = A$, hence $c \cdot s$ is a global section of $O_S(A)$.

This map is naturally injective: if $c \cdot s = c \cdot s'$, then

$$\begin{aligned} \mathrm{div}(c) + \mathrm{div}(s) &= \mathrm{div}(c \cdot s) \\ &= \mathrm{div}(c \cdot s') \qquad \Longrightarrow \mathrm{div}(s) = \mathrm{div}(s'), \quad \text{and so} \quad s' = \lambda s. \\ &= \mathrm{div}(c) + \mathrm{div}(s'), \end{aligned} \tag{A.2}$$

If s is zero, then also s' is zero and we are done. However, if s is no-zero, one can use $c \cdot s = c \cdot s'$ and (A.2) to say $c \cdot s = \lambda\, (c \cdot s)$, that implies $\lambda = 1$, and $s' = \lambda s = s$. Hence $\otimes C$ is injective, and so the point 2 follows.
About the point 3, one can consider the projective dual map of $\otimes C$, that in general is not a morphism. This map is not defined on the functional maps of $H^0(S, O_S(A))$ that composed with $\otimes C$ are zero. This means also that $\pi_B \circ \Phi_A$ is not defined exactly on the points of C and on the base locus of Φ_A. It remains to prove $\Phi_B = \pi_B \circ \Phi_A$, that is a direct check

$$(\pi_B \circ \Phi_A)\,(p) = [ev_p^A \circ (\otimes C)] = [c(p) ev_p^B] = [ev_p^B] = \Phi_B(p).$$

Now we prove (a) implies (b). It is sufficient to prove that $\otimes C$ is surjective. Given $t \in H^0(S, O_S(A))$, from (a), there exists an effective divisor $D \equiv B$ such that $\mathrm{div}(t) = C + D$. By using the identification $|B| \cong \mathbb{P}(H^0(S, O_S(B)))$, one can say that D is the divisor of a global holomorphic section $s \in H^0(S, O_S(B))$. Therefore

$$\mathrm{div}(t) = C + D = C + \mathrm{div}(s) = \mathrm{div}(c \cdot s), \qquad \text{and so} \qquad t = \lambda(c \cdot s).$$

About (b) implies (c), it is sufficient to observe that $\otimes C$ becomes an isomorphism, and so its projective dual map is an isomorphism too. In this case, we observe that Φ_B and Φ_A would be the same map, up to the projective transformation π_B.
It remains to prove (a) when holds (c). Let $D \in |A|$; by the identification of $|A| \cong \mathbb{P}(H^0(S, O_S(A)))$ it there exists a global section s such that $D = \mathrm{div}(t)$. However, π_B is an isomorphism so $\otimes C$ results to being an

isomorphism too. In particular, it there exists $s \in H^0(S, O_S(B))$ such that $t = c \cdot s$, and so $D = \mathrm{div}(t) = \mathrm{div}(c) + \mathrm{div}(s) \equiv C + B \in C + |B|$. The theorem is proved. □

Definition A.1.2. Let $\pi\colon S \to X$ be a morphism.

1. Consider a vector bundle $E \xrightarrow{b} X$ of rank n. The pullback bundle π^*E on S is the fibred product $S \otimes_X E := \{(p, e) : \pi(p) = b(e)\}$ together with the projection map $S \otimes_X E \xrightarrow{\pi_1} S$ on the first factor. Given an open cover $\{V_\alpha\}_\alpha$ of X and the trivializations $\{\phi_\alpha\colon b^{-1}(V_\alpha) \to V_\alpha \times \mathbb{C}^n\}_\alpha$ of the vector bundle E, then
$$\phi'_\alpha\colon \pi_1^{-1}\left(\pi^{-1}(V_\alpha)\right) \to \pi^{-1}(V_\alpha) \times \mathbb{C}^n, \quad (p, e) \mapsto (p, \phi_\alpha(\pi(p)))$$
are the trivializations of the vector bundle π^*E. Therefore, π^*E is given by the co-cycles $\{g_{\beta\alpha} \circ \pi\}_{\beta\alpha}$, where $\{g_{\beta\alpha}\}_{\beta\alpha}$ are the co-cycles of E;[1]

2. Let us suppose that π is surjective, and consider a divisor D on X. We define the pullback π^*D as the divisor on S whose local defining function is $f_\alpha \circ \pi$, where f_α is a local defining function of D. [2]

Remark A.1.3. If $\pi\colon S \dashrightarrow X$ is a rational map, then it is possible to extend the definition of the pullback π^*E. Let us consider an open cover $\{V_\alpha\}_\alpha$ of X and the co-cycles $\{g_{\beta\alpha}\}_{\beta\alpha}$ of E. Let U be the maximal definition domain of π; hence $S \setminus U$ is a finite set of points, let us say $p_1, \ldots, p_k$. Take the collection of local holomorphic functions $(g_{\beta\alpha} \circ \pi) : \pi^{-1}(V_\alpha) \to GL(\mathbb{C}^n)$, that are defined on U. Then, by Hartogs theorem, $g_{\beta\alpha} \circ \pi$ can be uniquely extended to a map that is holomorphic also on the points $p_1, \ldots, p_k$. We denote such a map always by $g_{\beta\alpha} \circ \pi$. Furthermore, the properties
$$(g_{\alpha\alpha} \circ \pi) = 1, \quad (g_{\beta\alpha} \circ \pi) = (g_{\alpha\beta} \circ \pi)^{-1}, \quad \text{and} \quad (g_{\gamma\beta} \circ \pi)\,(g_{\beta\alpha} \circ \pi) = (g_{\gamma\alpha} \circ \pi)$$
hold not only outside of $p_1, \ldots, p_k$, thanks to the uniqueness of the extension. Therefore $\{g_{\beta\alpha} \circ \pi\}_{\beta\alpha}$ are co-cycles of a line bundle on S, that we will call pullback bundle, and will be denoted as π^*E.

Theorem A.1.4. *Let $\pi\colon S \dashrightarrow X$ be a dominant map. Given a divisor D of X, then*

1. *$\pi^*O_X(D) = O_S(\pi^*D)$;*

[1] One can observe that a problem occur, if π would not be surjective. In fact, it could happen that some V_α is not contained in the image of π, and so ϕ'_α would not makes sense. In this case, we can simply avoid considering that open set V_α, and the definition can be extended for π morphism.

[2] One can define the pullback of a divisor in general when $\pi(S) \not\subseteq D$. This requirement is always satisfied if π is surjective or a dominant map.

2. The pullback map

$$\pi^*\colon H^0(X, O_X(D)) \to H^0(S, O_S(\pi^* D)), \qquad s \mapsto s \circ \pi$$

is injective.

Proof. The point 1 is a direct consequence of the definitions. Consider a trivializating open cover $\{V_\alpha\}_\alpha$ of the line bundle $O_X(D)$, and denote by s_α the local function of a section $s \in H^0(X, O_X(D))$ in the open set V_α. The pullback map is well defined since for any $s \in H^0(X, O_X(D))$, then $s_\alpha \circ \pi$ is holomorphic on $\pi^{-1}(V_\alpha)$, except for at most a finite number of points where π is not defined. However, from Hartogs theorem, then $s_\alpha \circ \pi$ can be extended to a unique map holomorphic also on that points. We denote this map always by $s_\alpha \circ \pi$. Therefore $s \circ \pi := \{s_\alpha \circ \pi\}_\alpha$ is a global section of $O_S(\pi^* D)$, and π^* is well-defined.

Furthermore, the map is injective because if $\pi^* s = s \circ \pi$ and $\pi^* s' = s' \circ \pi$ are equal, then their local holomorphic functions $s_\alpha \circ \pi$, $s'_\alpha \circ \pi$ are equal at any open set $\pi^{-1}(V_\alpha)$ of S. However, s_α and s'_α are equal on $\pi(\pi^{-1}(V_\alpha))$, which is dense in V_α, since π is dominant. This means that s and s' are equal everywhere. □

Let us consider now the canonical bundles ω_S and ω_X. Any global holomorphic 2-forms of X can be (uniquely) lifted to a global holomorphic 2-form of S, in the following way: Let $\omega = g(z_1, z_2) dz_1 \wedge dz_2$ be a 2-form on X written in local coordinates $\mathbf{z} := (z_1, z_2)$ around a point $q \in X$. Given a point $p \in \pi^{-1}(q)$ and fixed local coordinates $\mathbf{x} := (x_1, x_2)$ around p, we can compose locally ω with π to get

$$g(\pi(x_1, x_2)) \det(J_{\mathbf{zx}}\pi) dx_1 \wedge dx_2,$$

where $J_{\mathbf{zx}}\pi$ is the Jacobian matrix of π in those local coordinates. We observe that if one changes coordinates $\mathbf{x}'$ around p and $\mathbf{z}'$ around q, then

$$\begin{aligned} g(\pi(x'_1, x'_2)) \det(J_{\mathbf{z}'\mathbf{x}'}\pi) &= g(\pi(x_1, x_2)) \det(J_{\mathbf{zz}'}) \left(\det(J_{\mathbf{z}'\mathbf{z}}) \det(J_{\mathbf{zx}}\pi)\right) \det(J_{\mathbf{x}'\mathbf{x}})) \\ &= g(\pi(x_1, x_2)) \det(J_{\mathbf{zx}}\pi) \det(J_{\mathbf{x}'\mathbf{x}}). \end{aligned}$$

This means that

$$\frac{g(\pi(x'_1, x'_2)) \det(J_{\mathbf{z}'\mathbf{x}'}\pi)}{g(\pi(x_1, x_2)) \det(J_{\mathbf{zx}}\pi)} = \det(J_{\mathbf{x}'\mathbf{x}}),$$

that is the co-cycle in the coordinates $\mathbf{x}', \mathbf{x}$ of the canonical bundle ω_S. This justify the next definition

Definition A.1.5. Given a global 2-holomorphic form $\omega = \{g(\mathbf{z})\}_\mathbf{z}$ of X, its pullback $\pi^*\omega$ is the collection of holomorphic functions $\pi^*\omega := \{(g \circ \pi) \det(J_{\mathbf{zx}}\pi)\}_\mathbf{x}$. By construction, $\pi^*\omega$ is a global 2-holomorphic form of S.

To one side, we have obtained the pullback $\pi^*\omega$ of a 2-form ω, and, from the other side, the pullback $\pi^*\mathrm{div}(\omega)$ of the divisor of ω. The natural question is: Is there a relation between $\mathrm{div}(\pi^*\omega)$ and $\pi^*\mathrm{div}(\omega)$?

Theorem A.1.6. *(Hurwitz formula) Let $\pi\colon S \to X$ be a surjective morphism. Then the following identity of divisors holds*

$$div(\pi^*\omega) = \pi^*\, div(\omega) + R,$$

where R is an effective divisor, the ramification locus of π.[3] *In particular, one can give a more precise description of the ramification divisor*

$$R = \sum_{j=1}^{q}(e_j - 1)C_j + \sum_{i=1}^{p} r_i E_i,$$

where C_j are irreducible curves whose image with respect to π is a curve Γ_j, while E_i are curves contracted by π. Here e_j is the ramification index of C_j, namely the coefficient of C_j in the divisor $\pi^*\Gamma_j$, and $r_i \geq 0$.

Proof. By the definition of $\pi^*\omega$, we have $\pi^*\omega = (g \circ \pi)\det(J_{\mathbf{zx}}\pi)dx_1 \wedge dx_2$ in local coordinates $\mathbf{x} = (x_1, x_2)$, where $\omega = g(z_1, z_2)dz_1 \wedge dz_2$ in local coordinates $\mathbf{z} = (z_1, z_2)$. Therefore

$$\mathrm{div}(\pi^*\omega) = \pi^*\mathrm{div}(\omega) + \mathrm{div}(\det(J_{\mathbf{zx}}\pi)) = \pi^*\mathrm{div}(\omega) + R.$$

Now we prove the remain part of the statement. Let us consider an irreducible curve $C := C_j$, with image the curve $\Gamma := \Gamma_j$, and denote $e := e_j$ its ramification index. Fixed a point p on C, it there exists local coordinates $\mathbf{x}$ around p and $\mathbf{z}$ around $\pi(p) \in \Gamma$ such that π in this local coordinates is $(x_1, x_2) \mapsto (x_1^e, x_2)$. We observe that $C = \{x_1 = 0\}$, and $\Gamma = \{z_1 = 0\}$, in those local coordinates. But now

$$\det(J_{\mathbf{zx}}\pi) = \det\begin{pmatrix} ex_1^{e-1} & 0 \\ 0 & 1 \end{pmatrix} = ex_1^{e-1},$$

which proves $e - 1$ is the coefficient of C in the divisor R. □

Corollary A.1.7. *Let $\pi\colon S \to X$ is a surjective morphism. Any holomorphic 2-form ω of X lefts uniquely to a global holomorphic 2-form $\pi^*\omega$ of S, and the following commutative diagram holds*

$$\begin{tikzcd}
H^0(X, O_X(K_X)) \arrow[r, hook, "\pi^*"] \arrow[dr, hook] & H^0(S, O_S(K_S)) \\
& H^0(S, O_S(\pi^*K_X)). \arrow[u, hook, "\otimes R"']
\end{tikzcd}$$

[3]The ramification locus of a morphism is always a pure codimension 1 subvariety, by the Zariski-Nagata purity theorem.

Here R is the ramification locus of π. Moreover, one has

$$\begin{tikzcd}
S \arrow[rr, "\pi"] \arrow[d, dashed, "\Phi_{K_S}"'] & & X \arrow[d, dashed, "\Phi_{K_X}"] \\
\mathbb{P}(H^0(S, O_S(K_S))^*) \arrow[rr, dashed, "\pi^*"'] & & \mathbb{P}(H^0(X, O_S(K_X))^*),
\end{tikzcd}$$

and the canonical map of S factors through π if and only if $p_g(S) = p_g(X)$.

Proof. By the Hurwitz formula Theorem A.1.6, we have that $\mathrm{div}(\pi^*\omega) = \pi^*\mathrm{div}(\omega) + R$, with $\pi^*\mathrm{div}(\omega)$ and R effective divisors. We have fallen in the hypothesis of the Theorem A.1.1, with $A := K_S$, $B := \pi^*K_X$, and $C := R$. Hence the Theorem A.1.1 applies, and $\otimes R$ is injective. The other map $H^0(X, O_X(K_X)) \to H^0(S, O_S(\pi^*K_S))$ is injective too, by the Theorem A.1.4. The commutativity of the first diagram is straightforward, and from this follows also that π^* is injective.
Let us prove now the commutativity of the second diagram. Let $p \in S$, then

$$(\pi^* \circ \Phi_{K_S})(p) = \pi^*[ev_p] = [\det(J\pi_p) ev_{\pi(p)}] = [ev_{\pi(p)}] = (\Phi_{K_X} \circ \pi)(p).$$

Finally, Φ_{K_S} factors through π if and only if π^* is an isomorphism, that happens if and only if $p_g(S) = p_g(X)$. □

Theorem A.1.8 (Clifford Theorem)**.** *Let C be an irreducible smooth curve, and D be a divisor of C such that $0 \le \deg(D) \le 2g(C) - 2$. Then*

$$h^0(C, O_C(D)) \le \frac{1}{2}\deg(D) + 1.$$

Proof. For the proof, we remind to [GH78, pg. 251] □

Theorem A.1.9. *Let $\Sigma \subseteq \mathbb{P}^n$ be an irreducible surface, not contained in an hyperplane. Then $\deg(\Sigma) \ge n - 1$; furthermore, if Σ is not ruled, then $\deg(\Sigma) \ge 2n - 2$.*

Proof. Let $\eta\colon S \to \Sigma \subseteq \mathbb{P}^n$ be a resolution of Σ. Denote by $|H|$ the inverse image of the linear system of hyperplanes of $\mathbb{P}^n$. Let us pick up a generic smooth curve $C \in |H|$ and consider $D := H_{|C}$. We have the following exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, bend right, "\otimes O_S(H)"] & O_S(-C) \arrow[r] & O_S \arrow[r] & O_C \arrow[r] & 0 \\
0 \arrow[r] & O_S \arrow[r] & O_S(H) \arrow[r] & O_C(D) \arrow[r] & 0.
\end{tikzcd}$$

The long exact sequence in cohomology

$$0 \longrightarrow \underbrace{H^0(S, O_S)}_{\cong\, \mathbb{C}} \longrightarrow H^0(S, O_S(H)) \xrightarrow{\beta} H^0(C, O_C(D)) \longrightarrow \cdots$$

implies that $h^0(S, O_S(H)) \leq h^0(C, O_C(D)) + 1$. We observe that $|H|$ is the system that induces the resolution η, so $h^0(S, O_S(H)) = n + 1$, since Σ is not contained in an hyperplane. Therefore $n \leq h^0(C, O_C(D))$.

We also observe that $C^2 = \deg(D) = \deg(\Sigma)$, by construction.

Now we need to distinguish two cases:

($HK \geq 0$) In this case, by the genus formula of smooth curves on surfaces, we would have $\deg(D) = C^2 \leq C^2 + HK = 2(g(C) - 1)$. We can therefore apply Clifford Theorem A.1.8, and get $h^0(C, O_C(D)) \leq \frac{1}{2}\deg(D) + 1$. Finally

$$n \leq h^0(C, O_C(D)) \leq \frac{1}{2}\deg(D) + 1 \implies \deg(\Sigma) = \deg(D) \geq 2n - 2.$$

($HK < 0$) We observe that this happens only if Σ is ruled. In fact, by the Enriques-Kodaira classification, if $k(S) = -\infty$, namely $h^0(S, O_S(dS)) = 0$ for every $d \geq 0$, then S is ruled. Hence, if S is non-ruled, then it must there exists $T \in |dK| \neq \emptyset$ for some d sufficiently large. However, our divisor H is nef, and T is effective. This means $dHK = HT \geq 0$, and so $HK \geq 0$.

Now we finish to prove the statement of the theorem. We observe that in this case $\deg(D) = C^2 \geq C^2 + HK = 2(g(C) - 2)$. In particular, this implies $h^1(C, O_C(D)) = h^0(C, O_C(K_C - D)) = 0$. We apply now the Riemann-Roch theorem for curves

$$n \leq h^0(C, O_C(D)) = \deg(D) - g(C) + 1 \implies \deg(\Sigma) = \deg(D) \geq n - 1.$$

□

Theorem A.1.10 (Jongmans Theorem, 1947)**.** *Any minimal surface S of general type with a birational canonical map satisfies the inequality*

$$K_S^2 \geq 3p_g(S) + q(S) - 7.$$

Proof. See [Deb82, Thm. 3.2]. □

A.2 The Kodaira dimension

Now let us define the Kodaira dimension of a compact complex manifold X. Roughly speaking, the Kodaira dimension is a useful tool to measure the

size of its *canonical model*.
In the case of surfaces, it divides them into four classes, according to the "ampleness" of their canonical divisor. Any of such classes have been intensively studied in the literature. The most important result in this direction is the the so-called Enriques-Kodaira classification [BHPVdV04, Theorem VI.1.1].

Definition A.2.1. The *canonical ring* of X is the ring

$$R(K_X) := \bigoplus_{d\geq 0} H^0\left(X, dK_X\right).$$

The *canonical model* of X is the projective scheme $\mathrm{Proj}\left(R(K_X)\right)$.
Here $P_d := \dim H^0(X, dK_X)$ is called *d-plurigenus* of X.
In particular, $p_g := P_1$ is called *geometric genus* of X.
The *Kodaira dimension* of X is denoted by $\kappa(X)$ and it is $-\infty$ if the *plurigenera* $P_d := \dim H^0(X, dK_X)$ are zero for any $d > 0$; otherwise is the minimum k such that P_d/d^k is bounded, namely

$$\limsup_{d\to+\infty} \frac{P_d}{d^k} < \infty.$$

Theorem A.2.2. *([BHPVdV04, Theorem I.7.2] or [Uen75, Theorem 8.1]) Let X be a compact complex manifold. Let us denote by $\deg tr(R(X))$ the degree of trascendency of the canonical ring $R(X)$ over $\mathbb{C}$. Then*

$$\kappa(X) = \begin{cases} -\infty & \textit{if} \quad R(X) \cong \mathbb{C} \\ \deg tr(R(X)) - 1 & \textit{otherwise} \end{cases}.$$

Remark A.2.3. ([Har77, page 421]) Let Φ_{mK_X} be the rational map to the projective space associated with the linear system $|mK_X|$. Then the Kodaira dimension of X is the maximal dimension of the images of Φ_{mK_X}, $m \geq 1$.[4] Moreover, $\kappa(X)$ is a birational invariant and it assumes values $-\infty, 0, \ldots, n$, with $n := \dim(X)$.

The above Remark A.2.3 justifies the following definitions

Definition A.2.4. The Kodaira dimension of an algebraic variety X is the Kodaira dimension of a resolution $\rho\colon \widehat{X} \to X$ of the singularities of X.

Definition A.2.5. We say that a variety X of dimension $n := \dim(X)$ is of *general type* if $\kappa(X) = n$, or equivalently if

$$\limsup_{d\to+\infty} \frac{P_d}{d^n} > 0.$$

[4] If $|nK_X| = \emptyset$, then $\Phi_{nK_S} = \emptyset$ and we say that $\dim(\emptyset) = -\infty$.

Remark A.2.6. ([Bea96, Example VII.2]) In the case of curves, it is easy to determine the Kodaira dimension in terms of its genus. Let C be a smooth curve of genus g. Then

- $\kappa(C) = -\infty \iff g = 0 \iff C \cong \mathbb{P}^1$;
- $\kappa(C) = 0 \iff g = 1$;
- $\kappa(C) = 1 \iff g \geq 2$.

In general, we can say something more about the Plurigenera of X with Kodaira dimension 0, without so much effort.

Remark A.2.7. X has Kodaira dimension 0 if and only if the Plurigenera $P_d = 0$ or 1, but not always 0.
In fact, if X has Kodaira dimension 0, then it would exists at least one d_0 such that $P_{d_0} \neq 0$. Now, by contradiction suppose $P_{d_0} \geq 2$. Then we would have at least two independent sections $s_0, s_1 \in H^0(X, d_0K_X)$. In this way, we would get a subsystem $\langle s_0^i s_1^{m-i} : i = 0, \ldots, m\rangle \subseteq H^0(X, (md_0)K_X)$ generated by $m+1$ sections. We claim that they are also linearly independent. To see this, consider a linear combination of them such that $\sum_{i=0}^m a_i s_0^j s_1^{m-i} = 0$. The (complex) polynomial $p(x_0, x_1) := \sum_{i=0} a_i x_0^j x_1^{m-j}$ can be decomposed in irreducible polynomials of degree 1; let us say $p(x_0, x_1) = \prod_{i=0}^m (\alpha_i x_0 + \beta_i x_1)$. Since $p(s_0, s_1) = 0$, then we would get $X = \cup_{i=0}^m \{\alpha_i s_0 + \beta_i s_1 = 0\}$. However, X is irreducible and so it must there exists j such that $X = \{\alpha_j s_0 + \beta_j s_1 = 0\}$. In other words, we would get $\alpha_j s_0 + \beta_j s_1 = 0$, which implies $\alpha_j = \beta_j = 0$, since s_0, s_1 are linearly independent. To conclude, we get $p(x_0, x_1) = 0$ and this implies $a_i = 0$ for each $i = 0, \ldots m$.
Since we have proved that $H^0(X, (md_0)K_X)$ contains a subspace of dimension $m+1$, then $P_{md_0} \geq m+1$, which contradicts the fact that P_d is bounded.

Theorem A.2.8. [Uen75, page 69]*) If X_1 and X_2 are connected compact complex manifolds, then $\kappa(X_1 \times X_2) = \kappa(X_1) + \kappa(X_2)$.*

For sake of simplicity, we state and prove the following only for the case of algebraic surfaces.

Theorem A.2.9. *Let $\pi \colon S \to X$ be a surjective morphism of smooth algebraic surfaces. Then $\kappa(S) \geq \kappa(X)$.*

Proof. By the Hurwitz formula A.1.6, we have $dK_S = \pi^*(dK_X) + D$ for any $d \geq 1$, where D is an effective divisor (in particular, D is d-times the ramification divisor of π).
Therefore, by Theorems A.1.1 and A.1.4 then the pullback $\pi^* \colon H^0(X, dK_X) \hookrightarrow H^0(S, \pi_1^*(dK_X)) \subseteq H^0(S, dK_S)$ is injective. In other words, $h^0(dK_S) \geq h^0(dK_X)$.

Define $k := \kappa(S)$; if $k \leq 0$, then the thesis follows immediately. Otherwise, we would get

$$\limsup_{d \to \infty} \frac{h^0(dK_S)}{d^k} \geq \limsup_{d \to \infty} \frac{h^0(dK_X)}{d^k},$$

hence $k(X) \leq k = k(S)$. $\square$

Corollary A.2.10. *Let $f\colon Y \to X$ be a surjective morphism of algebraic surfaces, with Y and X not necessarily smooth. Then $\kappa(Y) \geq \kappa(X)$.*

Proof. Let $\rho_Y\colon \widehat{Y} \to Y$ and $\rho_X\colon \widehat{X} \to X$ be two resolutions of the singularities of Y and X. We are going to show $\kappa(\widehat{Y}) \geq \kappa(\widehat{X})$. Consider the natural rational map $\widehat{Y} \dashrightarrow \widehat{X}$ and resolve its indeterminacy by a finite number of blow-ups $b\colon \widehat{Y}' \to \widehat{Y}$. We have therefore a morphism $\pi\colon \widehat{Y}' \to \widehat{X}$ fitting in the following commutative diagram

$$\begin{tikzcd}
\widehat{Y}' \arrow[r, "b"] \arrow[dr, "\pi"'] & \widehat{Y} \arrow[r, "\rho_Y"] \arrow[d, dashed] & Y \arrow[d, "f"] \\
& \widehat{X} \arrow[r] & X
\end{tikzcd}.$$

Since f is surjective, then π is a surjective morphism too. Apply now the Theorem A.2.9 to the map π to get $\kappa(\widehat{Y}') \geq \kappa(\widehat{X})$. However, the Kodaira dimension is a birational invariant by the Remark A.2.3, and so $\kappa(\widehat{Y}) = \kappa(\widehat{Y}') \geq \kappa(\widehat{X})$. $\square$

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