Let us sumarize what we obtained in the previous lecture:

Let $\pi: X \to Y$ be an abelian cover with group G. Then G acts on $\pi_* \mathcal{O}_X$ and it splits $\pi_* \mathcal{O}_X$ as direct sum of isotypic components V_χ of character χ:

$$\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} V_\chi$$

V_χ is an invertible sheaf:

given $q \in Y$ and a fundamental neigh. V of q for π, then

$$\tau_V^\chi := \begin{cases} \sum_{g \in G} \overline{\chi(g)} \cdot g^* \mathbb{1}_U & \text{if } q \notin \operatorname{supp}(D) \\ \sum_{g \in G/\langle h \rangle} \overline{\chi(g)} \cdot g^*(t^{r_\chi^h} \mathbb{1}_U) & \text{if } q \in \operatorname{supp}(\Delta_h) \setminus \operatorname{Sing}(D) \end{cases}$$

where t is the local parameter of a irreducible component $T \subseteq X$, $T \cap U = (t=0)$, lying on $\Delta_h \cap V$, and $0 \le r_\chi^h \le |h|-1$ is the unique integer s.t. $\chi(h) = e^{\frac{2\pi i}{|h|} \cdot r_\chi^h}$.

Then
$$\begin{aligned} \mathcal{O}_{Y|V} &\xrightarrow{\sim} V_{\chi|V} \\ \alpha &\longmapsto \alpha \cdot \tau_V^\chi \end{aligned}$$
is an iso of sheaves.

Remark 1 By def. of τ_V^χ, then

$$\tau^\chi_{V|U} = \begin{cases} \mathbb{1}_U & \text{if } q \notin \operatorname{supp}(D) \\ t^{r_\chi^h} \cdot \mathbb{1}_U & \text{if } q \in \operatorname{supp}(\Delta_h) \setminus \operatorname{Sing}(D) \end{cases}$$

Similarly, given $g \in G/\langle h \rangle$, then

$$\tau^{\chi}_{V|_{g\cdot U}} = \begin{cases} \overline{\chi(g)} \cdot 1\!\!1_{g\cdot U} & \text{if } q \notin supp(D) \\ \overline{\chi(g)} \cdot (g^* t)^{r^h_\chi} \cdot 1\!\!1_{gU} & \text{if } q \in supp(\Delta_h) \setminus Sing(D) \end{cases}$$

Def $\mathcal{L}_\chi := V_\chi^{-1}$, so we have $\pi_* \mathcal{O}_X = \bigoplus_{\chi \in G^*} \mathcal{L}_\chi^{-1}$.

$\{D_g\}_{g\in G}$ and $\{\mathcal{L}_\chi\}_{\chi \in G^*}$ are called Building Data of $\pi : X \to Y$.

Remark 2 X connected $\Rightarrow$ $h^0(\mathcal{L}_\chi^{-1}) = 0$ $\forall \chi \neq 1_G$.

Indeed,

$$1 = h^0(X, \mathcal{O}_X) = h^0(Y, \pi_* \mathcal{O}_X) = \sum_{\chi \in G^*} h^0(Y, \mathcal{L}_\chi^{-1}) = \underbrace{h^0(Y, \mathcal{O}_Y)}_{1} + \sum h^0(\mathcal{L}_\chi^{-1})$$

$\Rightarrow$ $h^0(\mathcal{L}_\chi^{-1}) = 0$ $\forall \chi \neq 1_G$.

Remark 3 The cocycles of $\mathcal{L}_\chi = V_\chi^{-1}$ are

$$f_{V_2 V_1} = \frac{\tau^\chi_{V_2}}{\tau^\chi_{V_1}},$$

so a global (holom.) section of $\pi^* \mathcal{L}_\chi$ is

$$s_\chi = \{ (\pi^{-1}(V), \tau^\chi_V) \}_{V \subseteq Y \text{ fund. neigh.}}$$

We can state and prove

§9. Pardini Existence Theorem

Let Y be a smooth algebraic variety and let $\pi : X \to Y$ be an abelian cover of Y with Galois group G, X normal and building data $\{L_\chi\}_{\chi\in G^*}$, $\{D_g\}_g$.
Then, for any pairs of characters $\chi, \eta \in G^*$

$$(*) \qquad L_\chi + L_\eta = L_{\chi\eta} + \sum_{g\in G} \left\lfloor \frac{r_\chi^g + r_\eta^g}{|g|} \right\rfloor D_g$$

Conversely, given

- a collection of line bundles $\{L_\chi\}_{\chi\in G^*}$ of Y labeled by the characters of G;
- a collection of effective DIVISORS $\{D_g\}_{g\in G}$ indexed by the elements of G;

with the property that the linear equations $(*)$ hold for any pair $\chi, \eta \in G^*$, then it there exists an abelian cover $\pi : X \to Y$ with Galois group G.

If the cover so constructed is normal, then it has building data $\{L_\chi\}_{\chi\in G^*}$ and $\{D_g\}_{g\in G}$.

If Y is complete, then the building data determine the cover up to isomorph. of Galois covers.

Proof Let us consider the sections s_χ, $\chi \in G^*$ of $\pi^* \mathcal{L}_\chi$ defined above. We observe that a global section of $\pi^*(L_\chi \otimes L_\eta \otimes L_{\chi\eta}^{-1})$ on X is $\frac{s_\chi s_\eta}{s_{\chi\eta}}$, which is a G-invariant section by construction. Thus $\frac{s_\chi s_\eta}{s_{\chi\eta}}$ lives on Y, and it is a section of $L_\chi \otimes L_\eta \otimes L_{\chi\eta}^{-1}$, still by construction.

However, on some open set V (intersecting a irred. compon. Δ_h), given the fundamental neigh. $\mathcal{U} \subseteq X$ over V, then

$$\tau_V^\chi := \sum_{g \in G/\langle h \rangle} \overline{\chi(g)}\, g^*(t^{r_\chi^h} \mathbb{1}_\mathcal{U}) \quad \text{on } \pi^{-1}(V), \quad \text{and}$$

$$\tau^\chi_{V|\mathcal{U}} = t^{r_\chi^h} \cdot \mathbb{1}_\mathcal{U} \quad \text{on } \mathcal{U}$$

Thus, $$\frac{s_\chi s_\eta}{s_{\chi\eta}} = \frac{\tau_V^\chi \tau_V^\eta}{\tau_V^{\chi+\eta}} = \frac{t^{r_\chi^h} \cdot t^{r_\eta^h}}{t^{r_{\chi+\eta}^h}} \mathbb{1}_\mathcal{U} = t^{r_\chi^h + r_\eta^h - r_{\chi+\eta}^h} \cdot \mathbb{1}_\mathcal{U}$$

By definition of r_χ^h, then

$$r_\chi^h + r_\eta^h - r_{\chi\eta}^h = \begin{cases} |h| & \text{if } r_\chi^h + r_\eta^h \geq |h| \\ 0 & \text{otherwise} \end{cases}$$

Assume we are in the first case, so $\frac{s_\chi s_\eta}{s_{\chi\eta}} = t^{|h|} \mathbb{1}_\mathcal{U}$ on $\mathcal{U}$.

Similarly on gU, $g \in G/\langle h \rangle$, we have

$$\tau^{\chi}_{V|gU} = \overline{\chi}(g)\cdot (g^*t)^{r^h_\chi}\cdot \mathbb{1}_{gU}$$

$$\Rightarrow \frac{s_\chi s_\eta}{s_{\chi\eta}} = (g^*t)^{r^h_\chi + r^h_\eta - r^h_{\chi+\eta}}\,\mathbb{1}_{gU} = (g^*t)^{|h|}\,\mathbb{1}_{gU}$$

Thus, on $\pi^{-1}(V)$, we have

$$\frac{s_\chi s_\eta}{s_{\chi\eta}} = \sum_{g \in G/\langle h \rangle}' (g^*t)^{|h|}\cdot \mathbb{1}_{gU}$$

↖ G-invariant function living on V. Furthermore, it is exactly the function defining locally $\Delta_h \cap V$.

$$\Rightarrow \operatorname{div}\left(\frac{s_\chi s_\eta}{s_{\chi\eta}}\right)\Big|_V = \Delta_h$$

This holds for any irred. component Δ_h of D_h

Thus, all the divisors D_h, with the property that $r^h_\chi + r^h_\eta - r^h_{\chi+\eta} = |h|$ $\left(\text{namely } \left\lfloor \frac{r^h_\chi + r^h_\eta}{|h|} \right\rfloor = 1\right)$ occurs on $\operatorname{div}\left(\frac{s_\chi s_\eta}{s_{\chi\eta}}\right)$, and no others divisors occur on it. This proves $(*)$.

Let us assume now to have a set of line bundles $\{\mathcal{L}_\chi\}_{\chi \in G^*}$ and divisors $\{D_g\}_{g \in G}$ of Y for which equations $(*)$ hold.

We consider the vector bundle $V\left(\bigoplus_{\chi \neq 1_G} \mathcal{L}_\chi\right) \xrightarrow{\pi} Y$.

It is always possible to choose a open cover of Y, $\{V\}_{V \in Y}$ such that any V trivializes

simultaneously L_χ, $V(L_\chi)(\pi^{-1}(V)) \to Y \times \mathbb{C}$
$p \mapsto (\pi(p), u_\chi(p))$

Thus we have local coordinates $(u_\chi : \chi \in G^* \setminus \{1_G\})$ trivializing $V(\bigoplus_{\chi \neq 1_G} L_\chi)$ on $\pi^{-1}(V)$.

For any $g \in G$, we define the action on $V(\bigoplus_{\chi \neq 1_G} L_\chi)$:

$$(u_\chi : \chi \in G^* \setminus \{1_G\}) \xmapsto{g} (\chi(g) u_\chi : \chi \in G^* \setminus \{1_G\})$$

The local action is compatible with the change of the chart, so it extends to the entire $V(\bigoplus_{\chi \neq 1_G} L_\chi)$. For any $g \in G$, let us choose $s_g \in H^0(Y, \mathcal{O}_Y(D_g))$ with $\mathrm{div}(s_g) = D_g$.

Finally, we define X on the local chart $\pi^{-1}(V)$:

$$X \cap \pi^{-1}(V) := \left\{ u_\chi \cdot u_\eta = \left(\prod_{r_\chi^h + r_\eta^h \geq |h|} s_h \right) \cdot u_{\chi\eta} \right\}$$

We notice that from $(*)$, then we can glue these sets and obtain $X := \bigcup_{V \subseteq Y} X \cap \pi^{-1}(V)$.

By construction of X, then the action of G on $V(\bigoplus_{\chi \neq 1_G} L_\chi)$ extends to an action of X.

Thus, we have a Galois covering $\pi|_X : X \to Y$, $(q, u_\chi : \chi \in G \setminus 1_G) \mapsto q$ of Y with group G.

Assume X is normal, so the theory of norm. ab. covers holds for $\pi_{|X}: X \to Y$, which has then some building data. The ram. locus of $\pi_{|X}$ consists of those points with no triv. stab. It is easy to see from equations that $p \in X$ has no trivial stab $\Leftrightarrow$ $p \in \mathrm{supp}(\mathrm{div}(\sigma_h))$ for some σ_h. Thus, $D = \sum'_{h \in G} \mathrm{supp}(\mathrm{div}(\sigma_h)) = \sum'_{h \in G} \mathrm{red}(D_h)$ where $\mathrm{red}(D_h)$ is the reduced divisor of D_h. However, X normal forces D_h to be already reduced (we will see this in the next lectures when we will study the normality of a standard abelian cover). This means $D = \sum'_{h \in G} D_h$. To the other side,

$$\pi_* \mathcal{O}_{X|V} \cong \bigoplus_{\chi \in \mathrm{Irr}(V)} \mathcal{O}_{Y|V} \cdot u_\chi^V$$

so the cocycles of the line bundles V_χ^{-1} are given by $g_{21} = \frac{u_\chi^{V_2}}{u_\chi^{V_1}}$ on $V_1 \cap V_2$, which are by construction of $V(\bigoplus_{\chi \in \mathrm{Irr}(G)} L_\chi)$ the cocycles of L_χ.

We have proved $\{L_\chi\}_\chi$ and $\{D_g\}_{g \in G}$ are the building data of $\pi_{|X}: X \to Y$.

Regardless the uniqueness, we discuss it below. $\blacksquare$

Def Given a variety Y, a finite group G, and a set of line bundles $\{L_\chi\}_\chi$ and divisors $\{D_g\}_{g\in G}$ on Y satisfying eq. (*) above, then we define the standard abelian cover $\pi: X \to Y$, the cover constructed in the proof of Pardini Ex. Thm.

In this case, we refer to $\{L_\chi\}_\chi, \{D_g\}$ as to the building data of the cover, and to $D = \sum_{g\in G} D_g$ as the branch locus also if X is not normal.

Lemma Any normal abelian cover is standard. Instead, not all standard abelian covers are normal.

proof

Assume to have a Galois covering $\pi: X \to Y$ with group G, X normal, and Building Data $\{L_\chi\}_\chi, \{D_g\}_g$. Let $X' \xrightarrow{\pi'} Y$ be the abelian covering with group G constructed as above from $\{L_\chi\}_\chi, \{D_g\}_g$, where we choose as sections σ_h of $\Delta_h \leq D_h$ exactly the invariant function $\sigma_{h|V} := \sum'_{g\in G/\langle h\rangle} (g^* t)^{(h)} \cdot \mathbb{1}_{gU}$ on Y.

We remind that $s_\chi := \{(\pi^{-1}(V), \tau_V^\chi)\}_{V \subseteq Y}$ are global sections of $\pi^* L_\chi$, $\chi \in G^*$. Then an isomorphism among $X \xrightarrow{\pi} Y$ and $X' \xrightarrow{\pi'} Y$ is given by

$$X \xrightarrow{\Psi} X'$$
$$p \longmapsto (\pi(p), s_\chi(p), \chi \in G^* \setminus 1_G)$$

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr] & X' \arrow[d, "\pi'"] \\ & Y \end{tikzcd}$$

$\blacksquare$

We will study the smoothness and normality of a standard abelian cover in terms of $\{L_\chi\}_\chi$ and $\{D_g\}_g$.

Prop (Uniqueness of Pardini Existence Thm.)
Let $\pi: X \to Y$ and $\pi': X' \to Y$ be two standard abelian covers with group G and same building data $\{L_\chi\}_\chi$, $\{D_g\}_g$. If Y is complete, then π and π' are isomorphic as Galois covers

proof We skip it, it is the final Part of Pardini, Theorem 2.1. $\blacksquare$

Remark The previous Prop gives the uniqueness part of the Pardini Ex. Thm.

Indeed, let $\pi : X \to Y$ and $\pi' : X' \to Y$ be two normal abelian covers with group G and same building data $\{L_\chi\}_{\chi \in G^*}$, $\{D_g\}_{g \in G}$.

Then $\pi : X \to Y$ is isom. with its standard ab. cover $\pi_{st} : X_{st} \to Y$, which has building data $\{L_\chi\}_\chi$, $\{D_g\}_g$ via the sections σ_g of D_g,

and the same holds for $\pi' : X' \to Y$ and $\pi'_{st} : X'_{st} \to Y$, whose isomorphism is given by sections σ'_g of D_g.

However, π_{st} and π'_{st} have the same $\{L_\chi\}_\chi$ and $\{D_g\}_g$, so by the prev. Prop. are isomorphic by an isomorphism $\phi_{st} : X_{st} \to X'_{st}$. Thus, $\phi := (\Psi')^{-1} \circ \phi_{st} \circ \Psi : X \to X'$ is an iso:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\Psi"', "\wr"] & X' \arrow[d, "\wr"', "\Psi'"] \\ X_{st} \arrow[r, "\sim", "\phi_{st}"'] & X'_{st} \end{tikzcd}$$

Remark The number r_χ^h is very easy to compute for an elementary abelian p-group $G \cong (\mathbb{Z}/p)^k$. Indeed, all the elements of G and G^* have the same order p, so $0 \le r_\chi^h \le p-1$ is the unique integer s.t. $\chi(h) = e^{\frac{2\pi i}{p} r_\chi^h}$.

For instance, for $G = \langle e_1, e_2 \rangle \cong \mathbb{Z}_5^2$, $G^* = \langle \varepsilon_1, \varepsilon_2 \rangle$, $r_{\varepsilon_1 + \varepsilon_2}^{2e_1 + e_2} = 3$.

Example 1. $Y = \mathbb{P}^2(y_0, y_1, y_2)$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e_1, e_2 \rangle$

$D_{e_1} := (y_0 = 0)$, $D_{e_2} = (y_1 = 0)$, $D_{e_1+e_2} = (y_2 = 0)$

Can we construct a $\mathbb{Z}_2 \times \mathbb{Z}_2$-cover of $\mathbb{P}^2$ with branch locus $D_{e_1} + D_{e_2} + D_{e_1+e_2}$?

We need to determine, if they exists, line bundles $\{\mathcal{L}_\chi\}_\chi$ such that Pardini equations hold. We can try to find them using Pardini Equations:

$$2\mathcal{L}_{\varepsilon_1} = \underset{\substack{\| \\ \mathcal{L}_0 = 0}}{\mathcal{L}_{2\varepsilon_1}} + \sum_{\substack{g:\ \gamma_{\varepsilon_1}^g = 1 \\ \Updownarrow \\ g \notin \ker(\varepsilon_1)}} D_g = D_{e_1} + D_{e_1+e_2} \equiv 2H$$

$\Rightarrow$ we need to choose $\mathcal{L}_{\varepsilon_1} \equiv \mathcal{O}_{\mathbb{P}^2}(H)$.

$\Downarrow$ Pic($\mathbb{P}^2$) has not torsion

Similarly, $2\mathcal{L}_{\varepsilon_2} \equiv 0 + \sum_{g \notin \ker(\varepsilon_2)} D_g = D_{e_2} + D_{e_1+e_2} = 2H$

$$\Rightarrow \mathcal{L}_{\varepsilon_2} \equiv \mathcal{O}_{\mathbb{P}^2}(H)$$

Instead, $2\mathcal{L}_{\varepsilon_1+\varepsilon_2} \equiv 0 + \sum_{g \notin \ker(\varepsilon_1+\varepsilon_2)} D_g = D_{e_1} + D_{e_2} = 2H$

$$\Rightarrow \mathcal{L}_{\varepsilon_1+\varepsilon_2} \equiv \mathcal{O}_{\mathbb{P}^2}(H)$$

$2H \equiv \mathcal{L}_{\varepsilon_1} + \mathcal{L}_{\varepsilon_2} = \mathcal{L}_{\varepsilon_1+\varepsilon_2} + \sum_{g \notin \ker(\varepsilon_1) \cap \ker(\varepsilon_2)} D_g = \mathcal{L}_{\varepsilon_1+\varepsilon_2} + D_{e_1+e_2} \equiv 2H$ ✓

$2H \equiv \mathcal{L}_{\varepsilon_1} + \mathcal{L}_{\varepsilon_1+\varepsilon_2} = \mathcal{L}_{\varepsilon_2} + D_{e_1} \equiv 2H$ ✓ $\qquad 2H \equiv \mathcal{L}_{\varepsilon_2} + \mathcal{L}_{\varepsilon_1+\varepsilon_2} = \mathcal{L}_{\varepsilon_1} + D_{e_2} = 2H$ ✓

Thus, $\mathcal{L}_{\varepsilon_1} = \mathcal{L}_{\varepsilon_2} = \mathcal{L}_{\varepsilon_1+\varepsilon_2} \equiv \mathcal{O}_Y(H)$ and $D_{e_1} \equiv D_{e_2} \equiv D_{e_3} \equiv H$ satisfy Pardini Equations $\Rightarrow \exists!$ Galois Covering of $\mathbb{P}^2$ with group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and b.data $\{\mathcal{L}_\chi\}_\chi$, $\{D_g\}_g$.

This abelian cover is that of Example 4 of the 1st Lecture