

Reference:
J-P Serre
book

§5 Linear Representations of finite groups

Def Let V be a vect. space over \mathbb{C} of finite dimension n .

Given a finite group G , a linear representation of G in V is a homomorphism

$$G \xrightarrow{\rho} GL(V) \quad (\text{so } \rho_{st} = \rho_s \circ \rho_t \quad \forall s, t \in G),$$

where $GL(V)$ is the group of linear isomorphisms from V to itself. The degree of ρ is $\deg(\rho) := \dim_{\mathbb{C}}(V)$.

If we fix a basis of V , then any ρ_s can be represented by a invertible matrix R_s and

$$R_{st} = R_s \cdot R_t$$

In this case we say that ρ is represented in "matrix form".

Def We say that $\rho: G \rightarrow GL(V)$ and $\rho': G \rightarrow GL(V')$ are isomorphic repr. if it there exists a linear isomorphism

$$\tau: V \rightarrow V'$$

compatible with ρ and ρ' , in the sense that

$$\tau \circ \rho_s = \rho'_s \circ \tau \quad \forall s \in G$$

$$\begin{array}{ccc} V & \xrightarrow{\rho_s} & V \\ \downarrow \tau & \xrightarrow{\rho'_s} & \downarrow \tau \end{array}$$

In matrix form is equivalent to say that it there exists an invertible matrix T s.t.

$$T \cdot R_s = R_s' \cdot T \quad \forall s \in G.$$

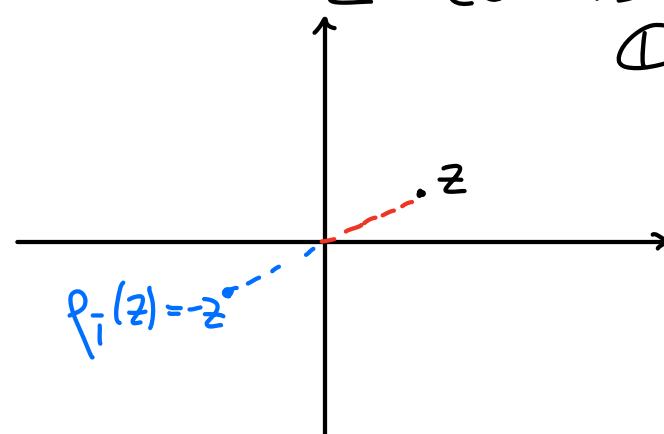
Rem Two isomorphic repr. have the same degree.

Rem 2 Fixed a basis of V , then the coordinate isomorphism $V \rightarrow \mathbb{C}^n$ is an isomorphism of ρ and the matrix form of ρ .

Examples 1) $\rho_{\text{triv}}: G \rightarrow \mathbb{C}^*$ is the trivial representation of G ;

2) $G = \mathbb{Z}_2, \rho: \mathbb{Z}_2 \rightarrow \mathbb{C}^*$ is a representation of \mathbb{Z}_2 .

Thus \mathbb{Z}_2 can be "represented" sending each complex number z to its opposite $-z$:



3) Given a representation of degree 1, $\rho: G \rightarrow \mathbb{C}^*$, since G is a finite group and so each element has finite order, then $\rho_g(z)$ is a root of the unity and $|\rho_g(z)| = \sqrt[\text{ord}(z)]{|z|}$

$$\text{then } |\rho_g(z)| = \sqrt[{\text{ord}(z)}]{|z|} \quad \forall z \in \mathbb{C}$$

4) Let $G = S_3 = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = 1, \tau\sigma = \sigma^2\tau \rangle$ the group of permutations of 3 elements. Then we define

$$\begin{aligned} \text{sgn} : S_3 &\longrightarrow \mathbb{C}^* \\ \tau &\longmapsto \text{sgn}(\tau) = -1 \\ \sigma &\longmapsto \text{sgn}(\sigma) = 1 \end{aligned}$$

← the sign of a permutation.

$$\begin{aligned} \rho : S_3 &\longrightarrow GL(\mathbb{C}^2) \\ \tau &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma &\longmapsto \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^2 \end{pmatrix} \end{aligned}$$

← Exercise:
Verify that it is well defined

5) (Regular Representation)

We consider the vector space V with a basis $(e_g)_{g \in G}$ indexed by the elements of G . We define the representation

$$\begin{aligned} \rho_{\text{reg}} : G &\longrightarrow GL(V) \\ g &\longmapsto \begin{pmatrix} V \rightarrow V \\ e_t \mapsto e_{gt} \end{pmatrix} \end{aligned}$$

ρ_{reg} is called regular representation

6) (Permutation Representation)

Assume G is acting on a finite set X , and let V be the vect. space with a basis $(e_x)_{x \in X}$

Then $\rho_{\text{perm}}: G \rightarrow \text{GL}(V)$

$$\begin{aligned} \mapsto & (V \rightarrow V) \\ & (e_x \mapsto e_{g \cdot x}) \end{aligned}$$

ρ_{perm} is called permutation representation

The regular representation is the permutation representation with $X = G$.

Def Given a representation $\rho: G \rightarrow \text{GL}(V)$ and a vector subspace W of V , we say that W is invariant if for each $s \in G$

$$\rho_s(v) \in W \quad \forall v \in W$$

In this case it is well defined the repr. $\rho_s^W: G \rightarrow \text{GL}(W)$

$$\mapsto (\rho_s|_W: W \rightarrow W)$$

ρ_s^W is called subrepresentation of ρ .

Example Given the regular representation ρ_G , then $W := \langle \sum_{s \in G} e_s \rangle$ is invariant, so

P_{reg}^W is a subrepresentation (of degree 1) of freq . In this case $P_{\text{reg}}^W = P_{\text{friv}}$.

We will compute all subrepresentations of P_{reg}

Remark: We can always find an invariant subspace of V :

$$V^G := \{v \in V \mid p_s(v) = v \quad \forall s \in G\}$$

Clearly, V^G can eventually be 0 or V .

Moreover, $p_s^{V^G} = \text{Id}_{V^G} \quad \forall s \in G$.

It is always possible to project any vector of V to V^G . (We remind that a projection over a subspace W is a linear map $\pi: V \rightarrow V$ s.t. $\pi(V) = W$ and $\pi \circ \pi = \pi$)

Thm (Reynolds Operator, important in fluidodynamics invariant theory)
There is a natural projector onto V^G :

$$\begin{aligned} \pi: V &\longrightarrow V \\ v &\longmapsto \pi(v) := \frac{1}{|G|} \sum_{s \in G} p_s(v) \end{aligned}$$

Furthermore, it holds the following equality

$$\dim(V^G) = \frac{1}{|G|} \cdot \sum_{s \in G} \text{Tr}(p_s).$$

Proof $\text{Tr}(\pi(v)) = \frac{1}{|G|} \sum_{s \in G} \text{Tr}(p_s(v)) = \frac{1}{|G|} \sum_{s \in G} p_s(v) = \pi(v)$
 $\Rightarrow \pi(v) \in V^G \Rightarrow \pi(V) \subseteq V^G.$

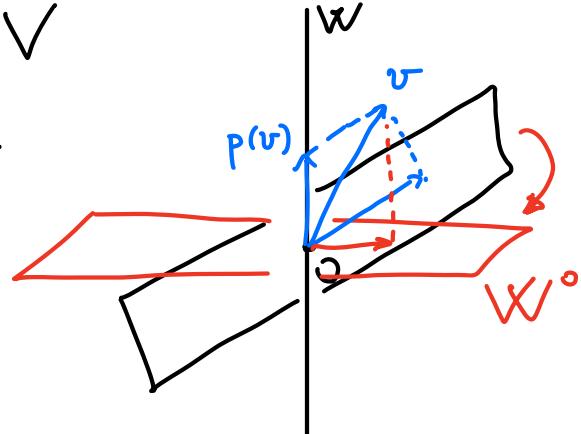
Instead, given $v \in V^G$, then $\pi(v) = v \in \pi(V) \Rightarrow \pi(V) = V^G$. Finally, we can write $V = \text{ker}(\pi) \oplus V^G$, so after completing a basis of V^G to a basis of V , we have in that basis that the associated matrix to π is $\begin{pmatrix} 0_{\dim(\text{ker})} \\ \text{Id}_{\dim V^G} \end{pmatrix} \Rightarrow \dim(V^G) = \text{Tr}(\pi) \blacksquare$

Thus Given an invariant subspace W of P , then there exists a complement W° of W which is invariant.

proof Consider a complement subspace of W and let $p: V \rightarrow V$ be the projection on W .

Then we can define a new linear map p^0

$$p^0 := \frac{1}{|G|} \cdot \sum_{S \in G} p_S \circ p \circ p_S^{-1}$$



We notice that p^0 fixes W :

$$p^0(w) = \frac{1}{|G|} \sum_{S \in G} p_S(p(p_S^{-1}(w))) = \underbrace{\frac{1}{|G|} |G|}_w w = w \quad \forall w \in W.$$

and it has image in W as $p(w) = w$ and

W is invariant. Thus the image is W and the kernel of the map is a complement of W° of W : $W \oplus W^\circ = V$.

Finally, $p_S \circ p^\circ = p^\circ \circ p_S$ and so

given $v \in W^\circ$, then

$$p^\circ(p_S(v)) = p_S(p^\circ(v)) = p_S(0) = 0$$

$\Rightarrow W^\circ$ is invariant.



Def The direct sum of two representations $\rho: G \rightarrow GL(V)$, $\eta: G \rightarrow GL(W)$ is

$$\rho \oplus \eta: G \rightarrow GL(V \oplus W)$$

$$\downarrow \mapsto V \oplus W \rightarrow V \oplus W$$

$$(v, w) \mapsto (\rho_S(v), \eta_S(w))$$

Remark From the previous theorem, we have that ρ is isomorphic to

$$\rho \cong \rho^W \oplus \rho^{W^\circ}$$

as $V \cong W \oplus W^\circ$ and W and W° are invariant.

Def An irreducible representation is a repr. $\rho: G \rightarrow GL(V)$ whose invariant subspaces are only 0 and V .

Thm Every representation is the finite sum of irreducible representations.

Proof By induction on the dimension of V .
If $\dim V=0$, then it is trivial.

Assume then $\dim V>0$. If $\rho: G \rightarrow GL(V)$ is irreducible, then we are done.

Otherwise, if there exists a proper invariant subspace $0 \subsetneq W \subsetneq V$, and an invariant complement W° : $V = W \oplus W^\circ$.

Then $\rho = \rho^W \oplus \rho^{W^\circ}$ and the inductive hypothesis applies as $\dim W < \dim V$
 $\dim W^\circ < \dim V$ □

Rem Thus V can be decomposed as a direct sum of irreducible representations W_1, \dots, W_k : $V = W_1 \oplus \dots \oplus W_k$

It is natural to ask if the decomposition is unique. The answer is clearly no. For instance $\rho: \mathbb{Z}_2 \rightarrow GL(\mathbb{C}^2)$ has inv. subsp. $\langle e_1 \rangle$ and $\langle e_2 \rangle$

but also $\langle e_1 + e_2 \rangle$ and $\langle e_2 \rangle$: $\mathbb{C}^2 = \langle e_1 + e_2 \rangle \oplus \langle e_2 \rangle = \langle e_1 \rangle \oplus \langle e_2 \rangle$

What will not change will be the NUMBER of irreduc. represent. isomorphic to a given W .

Def The dual representation of $\rho: G \rightarrow GL(V)$ is $\rho^*: G \rightarrow GL(V^*)$ where $\rho_s^*(f) := f \circ \rho_{s^{-1}}$.

Def Given $\rho_1: G \rightarrow GL(V)$, $\rho_2: G \rightarrow GL(W)$

we can define $\rho_1 \otimes \rho_2: G \rightarrow GL(V \otimes W)$

$$\begin{aligned} & \downarrow \\ & V \otimes W \rightarrow V \otimes W \\ & e_i \otimes e_j \mapsto \rho_1(e_i) \otimes \rho_2(e_j) \end{aligned}$$

which is called tensor product represent. of ρ_1 and ρ_2 .

We remind that $V \otimes V = Alt^2(V) \oplus Sym^2(V)$ where $Alt^2(V)$ is given by a basis

$$e_i \otimes e_j - e_j \otimes e_i \quad i \neq j$$

and $Sym^2(V)$ is given by a basis

$$e_i \otimes e_j + e_j \otimes e_i$$

$$(\dim(Alt^2(V))) = \frac{n(n-1)}{2} \text{ and } \dim(Sym^2(V)) = \frac{n(n+1)}{2}.$$

We observe that $Alt^2(V)$ and $Sym^2(V)$ are invariant with respect to $\rho \otimes \rho: G \rightarrow GL(V \otimes V)$, so $\rho \otimes \rho$ is never irreducible and can be written as a direct sum of two repres., called the Alternating square and Symmetric Square.

Moromorphism Representation

Given $\rho: G \rightarrow GL(V)$ and $\eta: G \rightarrow GL(W)$, then we have a natural representation on $\text{Hom}(V, W)$:

$$\begin{aligned} \text{Hom}(\rho, \eta): G &\rightarrow \text{Hom}(V, W) \\ s &\mapsto \left(\begin{array}{l} \text{Hom}(V, W) \xrightarrow{\quad} \text{Hom}(V, W) \\ F \mapsto \eta(s) \circ F \circ \rho(s^{-1}) \end{array} \right) \end{aligned}$$

Remark There is always an invariant subspace

$$\text{Hom}^G(V, W) := \{F: V \rightarrow W \mid \eta(s) \circ F \circ \rho(s^{-1}) = F \ \forall s \in G\}$$

We remind the natural isomorphism in linear algebra

$$\begin{aligned} \Theta: V^* \otimes W &\longrightarrow \text{Hom}(V, W) \\ f \otimes w &\longmapsto \left(\begin{array}{l} V \longrightarrow W \\ v \mapsto f(v) \cdot w \end{array} \right) \end{aligned}$$

(whose inverse is not natural and it is defined by the choice of a basis of V (e_1, \dots, e_n))

$$\begin{aligned} \text{Hom}(V, W) &\longrightarrow V^* \otimes W \\ f &\longmapsto \sum_{i=1}^n e_i^* \otimes f(e_i) \end{aligned}$$

As you can expect, Θ is an isomorphism of repr. among $\text{Hom}(\rho, \eta)$ and $\rho^* \otimes \eta$:

$$\begin{array}{ccc} V^* \otimes W & \xrightarrow{\rho^* \otimes \eta} & V^* \otimes W \\ \Theta \downarrow & & \downarrow \Theta \\ \text{Hom}(V, W) & \longrightarrow & \text{Hom}(V, W) \end{array}$$

Thus $\text{Hom}(\rho, \eta) \cong \rho^* \otimes \eta$.

SCHUR LEMMA

Let $\rho: G \rightarrow GL(V)$, $\eta: G \rightarrow GL(W)$ be two irreducible representations of G , and let f be a linear map from V to W s.t.

$$\eta_s \circ f = f \circ \rho_s \quad \forall s \in G.$$

Then

- (1) if ρ and η are NOT isomorphic, $f = 0$;
- (2) if $V = W$ and $\rho = \eta$, then $f = \lambda \cdot \text{Id}$, where $\lambda = \frac{\text{Tr}(f)}{n}$, $n = \dim(V)$

proof (1) If $f = 0$ is trivial, assume $f \neq 0$.

We claim that $\ker(f)$ and $\text{Im}(f)$ are invariant subspaces of V and W respect.

Given $v \in \ker(f)$, then $f(\rho_s(v)) = \rho_s(f(v)) = 0$
 $\Rightarrow \rho_s(v) = 0$;

Given $f(v) \in \text{Im}(f)$, then $\rho_s(f(v)) = f(\rho_s(v)) \in \text{Im}(f)$.

However, ρ and η are irreducible, so the possibilities are $\ker(f) = \{0\}$ and $\text{Im}(f) = W$, which means f is an isomorphism, so ρ and η are iso,

or $\ker(f) = V$, $\text{Im}(f) = 0$, which means $f = 0$.

2) Let v be an eigenvector of f with eigenvalue λ . Then $\ker(f - \lambda I) \neq \{0\}$ and $f - \lambda I$ satisfies

$$\rho_s \circ (f - \lambda I) = (f - \lambda I) \circ \rho_s \quad \forall s \in G$$

\Rightarrow from (1) we have $f - \lambda I = 0 \Rightarrow f = \lambda I$ □

§5.1 Character of a Representation

Let $\rho: G \rightarrow GL(V)$ be a representation of G .

There is also another object that does not change when ρ is replaced by an isomorphic representation; this object is the trace $\text{Tr}(\rho_s)$.

Def The character χ_ρ of the representation ρ is the function $\chi_\rho: G \longrightarrow \mathbb{C}$

$$s \longmapsto \text{Tr}(\rho_s)$$

As we will see, the character of ρ completely determines ρ .

Prop The following holds:

$$(1) \chi_\rho(1) = n, \quad n = \dim(V);$$

$$(2) \chi_\rho(s^{-1}) = \overline{\chi_\rho(s)} \quad \forall s \in G;$$

$$(3) \chi_\rho(tst^{-1}) = \chi_\rho(t) \quad \forall t, s \in G$$

(So the values of χ_ρ depends only on the conjugacy classes of G)

proof (1) and (3) are trivial as the trace of

$\rho_t \circ \rho_s \circ \rho_{t^{-1}}$ is the same as ρ_s (invariance of the trace up to similar matrices)

For (2), we remind that any matrix of finite order is diagonalizable, so let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of ρ_s . Then $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$ are the eigenvalues of $\rho_{s^{-1}}$. However

$$\rho_s(v_i) = \lambda_i v_i \Rightarrow \rho_{s^{\text{ord}(s)}}(v_i) = \sigma_i = \lambda_i^{(\text{ord}(s))} v_i$$

and so $\lambda_i^{\text{ord}(s)} = 1 \Rightarrow |\lambda_i| = 1 \Rightarrow \lambda_i \bar{\lambda}_i = 1$

This means

$$\text{Tr}(\rho_{S^{-1}}) = \frac{1}{\lambda_1} + \dots + \frac{1}{\lambda_n} = \overline{\lambda}_1 + \dots + \overline{\lambda}_n = \overline{\text{Tr}(\rho_S)}. \quad \square$$

Prop 2

Given $\rho: G \rightarrow GL(V)$ with charc. X and $\eta: G \rightarrow GL(W)$ with characters X_ρ and X_η , then

(Dual rep.) $X_{\rho^*} = \overline{X_\rho} ;$

(Direct Sum rep.) $X_{\rho \oplus \eta} = X_\rho + X_\eta ;$

(Tensor rep.) $X_{\rho \otimes \eta} = X_\rho \cdot X_\eta ;$

(Alt. square rep.) $X_{\text{Alt}^2 \rho}(s) = \frac{1}{2} (X_\rho(s^2) - X_\rho(s^2))$

(Sym. square rep.) $X_{\text{Sym}^2 \rho}(s) = \frac{1}{2} (X_\rho(s^2) + X_\rho(s^2))$

Examples

1) The character of the trivial representation is

$$X_{\text{triv}} = 1 \quad \forall s \in G;$$

2) $\text{Freq}(s)$ sends $e_t \rightarrow e_{st}$, so the associated matrix has only zeros on the diagonal unless $s = 1_G$, in which case all the elements on the diagonal are 1.

Thus $X_{\text{Freq}}(s) = \begin{cases} |G| & \text{if } s = 1_G \\ 0 & \text{otherwise} \end{cases}$.

3) $\rho_{\text{perm}}(s)$ sends $e_x \rightarrow e_{s \cdot x}$ which is the same e_x iff $s \in \text{Stab}(x)$. Thus, let $\text{Fix}(s) := \{x \in X \mid s \cdot x = x\} \leq G$. We have that $X_{\rho_{\text{perm}}}(s) = |\text{Fix}(s)|$.

Def When we have two complex valued functions $f: G \rightarrow \mathbb{C}$, $g: G \rightarrow \mathbb{C}$ of G we can always define the scalar product

$$(f|g) := \frac{1}{|G|} \sum_{s \in G} f(s) \cdot \overline{g(s)}$$

Thus, given two characters X_p and X_q , we can always compute the (a priori complex) number $(X_p | X_q)$.

Rem:

Using Reynolds operator, we proved

$$\dim V_G = \frac{1}{|G|} \sum_{s \in G} \text{Tr}(\rho_s)$$

that now can be rewritten as $\boxed{\dim V^G = (X_p | X_{\text{triv}})}$.

Thm Given $\rho: G \rightarrow GL(V)$, $\eta: G \rightarrow GL(W)$ with characters X_p and X_q , then the number $(X_p | X_q)$ is always an integer equal to

$$\boxed{(X_p | X_q) = \dim \mathbb{C}(H^G_{\rho, \eta}(V, W))}$$

proof We have seen that $H^G_{\rho, \eta}(\rho, \eta) \cong \rho^* \otimes \eta$, so its character is $\overline{X_p} \cdot X_q$.

However, from the previous remark applied to the vector space $\text{Hom}(V, W)$ we have

$$\dim \text{Hom}^G(V, W) = (\bar{\chi}_p \cdot \chi_\eta | \chi_{\text{triv}}) = (\chi_p | \chi_\eta)$$

■

Corollary (IMPORTANT)

(1) if p and η are irreducible represent, then

$$(\chi_p | \chi_\eta) = \begin{cases} 1 & \text{if } p \text{ and } \eta \text{ are isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

(thus $\{\chi_p \mid p \text{ irred. rep}\}$ form an orthonormal system !)

(2) Given a representation $p: G \rightarrow GL(V)$, suppose it decomposes in irred. rep. $V = W_1 \oplus \dots \oplus W_k$.

Let $\eta: G \rightarrow GL(W)$ be a irreducible representation, Then the number of $W_i \subseteq V$ isomorphic to W equals the number $(\chi_\eta | \chi_p)$.

This number does not depend on the decomposition

- (3) Two representations with the same character are isomorphic;
- (4) A representation is irreducible if and only if $(\chi_p, \chi_p) = 1$.

proof (1) $(\chi_p | \chi_\eta) = \dim_{\mathbb{C}} (\text{Hom}^G(V, W))$

However ρ and η are irreducible, so by Schur Lemma homomorphism repres. in $\text{Hom}^G(V, W)$ is an isomorphism rep. of ρ and η .

If η and ρ are NOT iso, then $\text{Hom}^G(V, W) = 0$. Instead, if it there exists an isomorphism $F: V \rightarrow W$, then

$$\begin{aligned} \text{Hom}^G(V, W) &\xrightarrow{\sim} \text{Hom}^G(V, V) \text{ is } \underline{\text{iso}} \\ g &\longmapsto F^{-1} \circ g \end{aligned}$$

However, from Schur Lemma (2),

$$\text{Hom}^G(V, V) = \langle \text{Id}_V \rangle$$

and so $\text{Hom}^G(V, W)$ is one dimensional.

(2) We have $V = W_1 \oplus \dots \oplus W_k$, let ρ_1, \dots, ρ_k be their irreduc. representations. Then $\chi_\rho = \chi_{\rho_1} + \dots + \chi_{\rho_k}$ and so by the previous point

$$(\chi_\rho | \chi_\eta) = \#\{j \mid \rho_j \text{ is iso with } \eta\}$$

(3) If ρ and η have the same character χ , then they contain the same irreducible representations the same number of times.

Thus ρ and η are iso;

(4) (\Rightarrow) is proved in (1)

(\Leftarrow) Assume that $V = w_1 W_1 \oplus \dots \oplus w_k W_k$ where w_i is the number of times the representation p_i is occurring on V .

Then $X_p = w_1 X_{p_1} + \dots + w_k X_{p_k}$

and so

$$1 = (X_p | X_p) = w_1^2 + \dots + w_k^2$$

$\Leftrightarrow \exists j$ s.t. $w_j = 1$ and the others are zero $\Rightarrow X_p = X_{p_j} \Rightarrow p$ and p_j are iso $\Rightarrow p$ is irreducible. \blacksquare

Remark We can now find the irreducible rep. contained in X_{reg} . We observe that

$$(X_{\text{reg}} | X) = \frac{1}{|G|} \cdot |G| \cdot \chi(1_G) = \chi(1_G)$$

so X irreducible occurs on X_{reg} with multiplicity $\chi(1_G)$.

This means that there are only finitely many irreducible characters χ_1, \dots, χ_n and are all of them contained in X_{reg} . In particular, it holds

$$\boxed{X_{\text{reg}} = \sum_{i=1}^k \chi_i(1_G) \cdot \chi_i}$$

$$|G| = \sum_{i=1}^k \chi_i^2(1_G)$$

Def A class function is a function $f: G \rightarrow \mathbb{C}$ satisfying $f(tst^{-1}) = f(t) \quad \forall t, s \in G$.
 The space of class functions of G is denoted by $CF(G)$.

Notice that this space contains every character of G .

Thm Let f be a class function on G , $\rho: G \rightarrow GL(V)$ a repr. of G . We define the homomorphism

$$\rho_f := \sum_{s \in G} f(s) \cdot \rho_s$$

If V is irreducible of degree n , then ρ_f is an homothety of ratio $\lambda = \frac{|G|}{n} (f | \bar{\chi}_\rho)$.

Proof $\rho_t \rho_f = \left(\sum_{s \in G} f(s) \underbrace{\rho_{tst^{-1}}}_{\rho(f(tst^{-1}))} \right) \rho_t = \rho_f \rho_t \Rightarrow$ by Schur Lemma $\rho_f = \lambda \text{Id}_V$ where $\lambda = \frac{\text{Tr}(\rho_f)}{n} = \frac{|G| \cdot (f | \bar{\chi}_\rho)}{n}$. \blacksquare

Thm (1) $\text{Irr}(G) := \{ \text{irreducible characters of } G \}$
 is an orthonormal basis of $CF(G)$;
 (2) The number of irreducible characters is equal to the number of conjugacy classes of G .

Proof (1) χ_1, \dots, χ_k irreducible characters of G .
 We can decompose $CF(G)$ as direct sum of $\langle \chi_1, \dots, \chi_k \rangle$ and its orthogonal complement.

Thus it is sufficient to prove that if $f \in CF(G)$ verifies $(\chi_i | f) = 0 \quad \forall i=1, \dots, k \Rightarrow f = 0$. Let us consider $p_f = \sum_{s \in G} f(s) p_s$ for any represent. p . The previous thm. shows that p_f is zero on any irreducible represent. of p as $(\chi_i | f) = 0$. Thus p_f is identically zero for any repr. p . Let us consider the regular representation p_{reg} . Then

$$0 = (p_{reg})_f = \sum_{s \in G} f(s) (p_{reg})_s \\ \Rightarrow 0 = (p_{reg})_f(e_G) = \sum_{s \in G} f(s) \cdot e_s \Rightarrow f(s) = 0 \quad \forall s \in G$$

(2) Another basis of $CF(G)$ is given by $\{1_{\text{cong}(x)} : x \in G\}$ where $\text{cong}(x) = \{t + xt^{-1} \mid t \in G\}$. Thus $\#\text{cong classes} = \dim_{\mathbb{C}} CF(G) = \#\text{Irr}(G)$ \square

Corollary A Group is abelian if and only if all the irreducible representations of G are 1-dimensional

Proof Using the regular representation, we have $|G| = \chi_1^2(1_G) + \dots + \chi_k^2(1_G)$ when $k = \#\text{cong classes}$ of G .

However G is abelian $\Leftrightarrow |k| = |G| \Leftrightarrow$
 $\chi_1(1_G) = \dots = \chi_k(1_G) = 1$ □

Def Given $\rho: G \rightarrow GL(V)$ repres. and an irreducible repr. $\eta: G \rightarrow GL(W)$, the isotypic component W^η of ρ of character η is the biggest invariant subspace of V isomorphic to some copies of the same representation η .

Thus, $X_{\rho|W^\eta} = \langle X_\rho | X_\eta \rangle \cdot X_\eta$.

Remark With this notation, we have a canonical unique decomposition of $\rho: G \rightarrow GL(V)$ as a direct sum of isotypic components:

$$V = W^{\eta_1} \oplus \dots \oplus W^{\eta_K}$$

We are just putting together
the isomorph. represent.

We can use a generalization of Reynolds operator to construct a projection of V to the isotypic component of char. η .

Thm (Reynold Operator of character η)

Let $\rho: G \rightarrow GL(V)$ repr. and η be a irreducible repres. Let W^η be the isotypic component of character η . Then

$$\pi_\eta := \frac{\eta(1_G)}{|G|} \sum_{g \in G} \overline{\chi_\eta(g)} \cdot P_g$$

is a projection on W^η .

Furthermore, given a basis e_1, \dots, e_n of V , then if $V = W^{\eta_1} \oplus \dots \oplus W^{\eta_k}$, we have

$$\pi_{\eta_1}(e_1), \dots, \pi_{\eta_1}(e_n) \quad (\text{generates } W^{\eta_1})$$

⋮

$$\pi_{\eta_k}(e_1), \dots, \pi_{\eta_k}(e_n) \quad (\text{generates } W^{\eta_k})$$

generate the entire space V .

Proof We apply the previous result and obtain that π_η restricted to any irred. repr. W_J of character η_J is an isomophy of ratio $\lambda = \frac{(\eta|_J) \cdot \chi_\eta |_{W_J}}{n_J} = \begin{cases} 1 & \text{if } \chi_\eta = \chi_{\eta_J} \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow \pi_\eta$ is the identity on W_J if it is isomorph. to η_J , and zero otherwise. Thus π_η is the identity on the isotypic component of charact. η and zero otherwise.

We can write $V = W^{n_1} \oplus \dots \oplus W^{n_k}$ and so $x \in V$ can be written as $x = x_1 + \dots + x_k \Rightarrow$

$$\pi_{\eta}(x) = \pi_1(x_1) + \dots + \pi_{\eta}(x_k) = x_j \quad (\eta = \eta_j)$$

$\Rightarrow \pi_{\eta}$ is the projection on W^{η} □

FINAL EXAMPLE

$$S_3 = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = 1 \rangle, \quad |S_3| = 6$$

We want to find all the possible irr. repres.

Conjugacy classes are $\text{Conj}(\sigma) = \{\sigma, \sigma^2\}$

$$\text{Conj}(1) = \{1\}$$

$$\text{Conj}(\tau) = \{\tau, \tau\sigma^2, \tau\sigma\}$$

$$\Rightarrow \# \text{Irr}(S_3) = 3.$$

However, we already constructed 2 natural characters of S_3 :

$$\chi_{\text{triv}} : S_3 \rightarrow \mathbb{C}^*$$

$$\text{sgn} : S_3 \rightarrow \mathbb{C}^*$$

$$\chi_{\text{reg}}^+ : S_3 \rightarrow \mathbb{C}^*$$

The last character χ is then computable using

$$\chi_{\text{reg}} = \chi_{\text{triv}} + \text{sgn} + \chi(1_G) \cdot \chi$$

$$\Rightarrow \text{at } 1_G \text{ we have } |S_3| = 6 = 1 + 1 + \chi(1_G) \Rightarrow$$

$$\chi(1_G) = 2, \text{ and}$$

$$\boxed{\chi = \frac{1}{2} (\chi_{\text{reg}} - \chi_{\text{triv}} - \text{sgn})}$$

Actually we can also prove that the

irreducible representation of step. 2 is

$$\rho: S_3 \rightarrow GL(\mathbb{C}^2)$$

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$G \mapsto \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^{-2} \end{pmatrix}$$

To prove that ρ is irreducible it is sufficient to prove $(X_\rho | X_\rho) = 1$.

We can order the ir. characters in a table

called CHARACTER TABLE :

	$\text{Conj}(1)$	$\text{Conj}(\sigma)$	$\text{Conj}(\tau)$
1	1	1	1
sgn	1	1	-1
X	2	-1	0

Let us construct a basis of isotypic components of the regular representation X_{reg} :

$$\text{Pre}_g: S_3 \rightarrow GL(\mathbb{C}^6)$$

$$\pi_{\text{sgn}}(e_1) = \frac{1}{6} (e_1 - e_{\tau\sigma} - e_\tau - e_{\tau\sigma^2} + e_{\sigma^2} + e_6)$$

$$\pi_X(e_1) = \frac{1}{6} (2e_1 - e_6 - e_{6^2})$$

$$\pi_X(e_6) = \frac{1}{6} (2e_6 - e_{6^2} - e_1)$$

$$\pi_X(e_\tau) = \frac{1}{6} (2e_\tau - e_{\tau\sigma^2} - e_{\tau\sigma})$$

$$\pi_X(e_{\tau\sigma}) = \frac{1}{6} (2e_{\tau\sigma} - e_\tau - e_{\tau\sigma^2})$$

$e_1 \leftarrow$ this generate $W^{X_{\text{reg}}}$

$e_6 \leftarrow$ this generate W^{sgn}

$\left. \begin{array}{l} \text{they generate } W^X \\ \text{isotypic comp. of} \\ \text{charact } X \text{ for Pre}_g, \text{ which} \\ \text{contains 2-times the irr.} \\ \text{represent. } \rho = \rho_X \text{ above.} \end{array} \right\}$

BONUS: A natural question that may arise is why do we chose the trace to uniquely determine a representation instead the determinant.

Or, more in general, why do we not choose one of the other coefficients of the characteristic polynomial

$$P_{P_s}(z) := \det(P_s - z \text{Id}) ? \quad (\text{here we fix } s \in G)$$

Indeed, they are invariant by similarity, so are class functions.

First of all, we observe that all of them can be viewed as a character, so as a trace of a representation.

Indeed, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of P_s . Then

$$\det(P_s - z \text{Id}) = \prod_{i=1}^n (z - \lambda_i) = \sum_{j=0}^n (-1)^{n-j} \left(\sum_{i_1 < \dots < i_{n-j}} \lambda_{i_1} \cdots \lambda_{i_{n-j}} \right) z^j$$

It is not so difficult to prove that if we consider the representation $\underbrace{p \otimes \dots \otimes p}_{(n-s)-\text{times}}$ and the invariant subspace $\Lambda^{n-j} V$,

then the character of the subrepresentation of $(p \otimes \dots \otimes p)^{\Lambda^{n-j} V}$ is

exactly $\chi_{\Lambda^{n-j} p}(z) = \sum_{i_1 < \dots < i_{n-j}} \lambda_{i_1} \cdots \lambda_{i_{n-j}}$

Thus, we can write $\boxed{\det(P_s - z \text{Id}) = \sum_{j=0}^n (-1)^{n-j} \chi_{\Lambda^{n-j} p}(z) z^j}$

Since the trace completely determine p , then we could write the coefficient of the characteristic polynomial in function of χ_p .

Theorem

The j -th coefficient of the characteristic polynomial of P_S , or equivalently the character of the $(n-j)$ -th alternating repr. $\Lambda^{n-j} P_S$, can be written in function of X_P as follows:

$$X_{\Lambda^{n-j} P}(s) = \frac{1}{(n-j)!} \det$$

$$\begin{pmatrix} X_P(s) & 1 & 0 & \cdots & \cdots & 0 \\ X_P(s^2) & X_P(s) & 2 & 0 & \cdots & 0 \\ X_P(s^3) & X_P(s^2) & X_P(s) & 3 & \cdots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ X_P(s^{n-j}) & X_P(s^{n-j-1}) & \cdots & \cdots & \cdots & X_P(s) \end{pmatrix}$$

In particular,

$$(\det P)(s) = \frac{1}{n!} \det$$

$$\begin{pmatrix} X_P(s) & 1 & 0 & \cdots & \cdots & 0 \\ X_P(s^2) & X_P(s) & 2 & 0 & \cdots & 0 \\ X_P(s^3) & X_P(s^2) & X_P(s) & 3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ X_P(s^n) & X_P(s^{n-1}) & \cdots & \cdots & \cdots & X_P(s) \end{pmatrix}$$

proof We need to write $s_k = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$ in function of the sum of the powers $S_m = \sum_{i=1}^n x_i^m$. This formula is called WARING FORMULA:

$$s_k = \frac{1}{n!} \det \begin{pmatrix} S_1 & 1 & 0 & \cdots & ? \\ S_2 & S_1 & 2 & \cdots & ? \\ S_3 & S_2 & S_1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & n-1 \\ S_n & S_{n-1} & \cdots & \cdots & S_1 \end{pmatrix}$$

$$\text{However, } S_1 = \sum_{i=1}^n \lambda_i + \dots + \lambda_n = \chi_p(\beta)$$

$$S_2 = \sum_{i=1}^n \lambda_i^2 + \dots + \lambda_n^2 = \chi_p(\beta^2)$$

$$\vdots$$

$$S_n = \sum_{i=1}^n \lambda_i^n + \dots + \lambda_n^n = \chi_p(\beta^n)$$

so the thesis follows. □

Thus, if $\rho: G \rightarrow GL(V)$ has dimension 2, then

$$\det(\rho)(\beta) = \frac{1}{2!} \det \begin{pmatrix} \chi_p(\beta) & 1 \\ \chi_p(\beta^2) & \chi_p(\beta) \end{pmatrix} = \frac{1}{2} (\chi_p^2(\beta) - \chi_p(\beta^2))$$

For instance, if $G = S_3$ and ρ is the irreducible repres. of degree 2, $\chi_\rho = \frac{1}{2} (\chi_{\text{reg}} - 1 - \text{sgn})$

$$\Rightarrow \det(\rho)(\beta) = \begin{cases} 1 & \beta \in \text{Conj}(1) \\ \frac{1}{2}(2+1)=1 & \beta \in \text{Conj}(6) \\ \frac{1}{2}(0-2)=-1 & \beta \in \text{Conj}(1) \end{cases}$$

$$\Rightarrow \det \rho = \text{sgn}$$

However, ρ and sgn are NOT isomorphic but have the same determinant.

This shows that determinant does not determine the representation such as the trace.

Exercise: Prove that the other coefficients of the characteristic polynomial does not determine the representation.