§5 Linear Representations of finite groups (Reference: J-P Serre book)

Def Let V be a vect. space $/\mathbb{C}$ of finite dimension n. Given a finite group G, a linear representation of G in V is a homomorphism

$$G \xrightarrow{\rho} GL(V) \quad (\text{so } \rho_{st} = \rho_s \circ \rho_t \quad \forall s,t \in G),$$

where $GL(V)$ is the group of linear isomorphisms from V to itself. The degree of ρ is $\deg(\rho) := \dim_{\mathbb{C}}(V)$.

If we fix a basis of V, then any ρ_s can be represented by a invertible matrix R_s and

$$R_{st} = R_s \cdot R_t$$

In this case we say that ρ is represented in "matrix form".

Def We say that $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ are isomorphic repr. if it there exists a linear isomorphism

$$\tau : V \to V'$$

compatible with ρ and ρ', in the sense that

$$\tau \circ \rho_s = \rho'_s \circ \tau \qquad \forall s \in G$$

$$\begin{tikzcd} V \arrow[r, "\rho_s"] \arrow[d, "\tau"'] & V \arrow[d, "\tau"] \\ V' \arrow[r, "\rho'_s"] & V' \end{tikzcd}$$

In matrix form is equivalent to say that it there exists an invertible matrix T s.t.

$$T \cdot R_s = R'_s \cdot T \quad \forall s \in G.$$

Rem Two isomorphic repr. have the same degree.

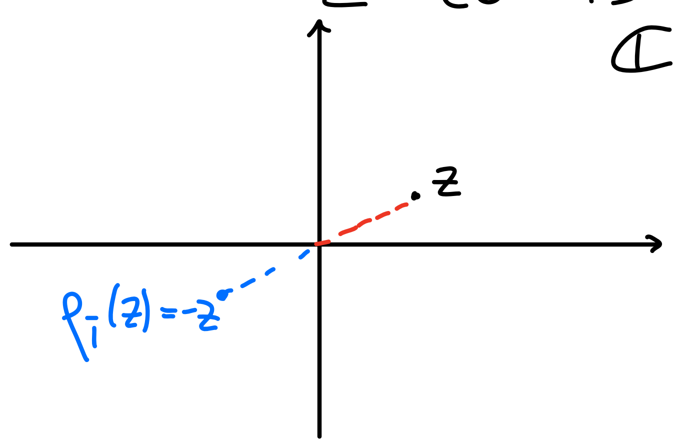
Rem 2 Fixed a basis of V, then the coordinate isomorphism $V \to \mathbb{C}^n$ is an isomorphism of ρ and the matrix form of ρ.

Examples 1) $\rho_{triv}: G \to \mathbb{C}^*$, $s \mapsto 1$ is the trivial representation of G;

2) $G = \mathbb{Z}_2$, $\rho: \mathbb{Z}_2 \to \mathbb{C}^*$, $\bar{1} \mapsto -Id_{\mathbb{C}}$ is a representation of $\mathbb{Z}_2$.

Thus $\mathbb{Z}_2$ can be "represented" sending each complex number z to its opposite $-z$:

$\mathbb{C}$



3) Given a representation of degree 1, $\rho : G \to \mathbb{C}^*$, since G is a finite group and so each element has finite order, then $\rho_s(z)$ is a rooth of the unity and $\sqrt[\operatorname{ord}(s)]{|z|}$

then $|\rho_s(z)| = \sqrt[\operatorname{ord}(s)]{|z|} \quad \forall z \in \mathbb{C}$

4) Let $G = S_3 = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = 1, \ \tau\sigma = \sigma^2\tau \rangle$ the group of permutation of 3 elements. Then we define

$$\begin{aligned} \operatorname{sgn} : S_3 &\longrightarrow \mathbb{C}^* \\ \tau &\longmapsto \operatorname{sgn}(\tau) = -1 \\ \sigma &\longmapsto \operatorname{sgn}(\sigma) = 1 \end{aligned}$$

← the sign of a permutation.

$$\begin{aligned} \rho : S_3 &\longrightarrow GL(\mathbb{C}^2) \\ \tau &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma &\longmapsto \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^2 \end{pmatrix} \end{aligned}$$

← Exercise: Verify that it is well defined

5) (Regular Representation)

We consider the vector space V with a basis $(e_s)_{s \in G}$ indixed by the elements of G. We define the representation

$$\begin{aligned} \rho_{reg} : G &\longrightarrow GL(V) \\ s &\longmapsto \left(\begin{aligned} V &\to V \\ e_t &\mapsto e_{st} \end{aligned} \right) \end{aligned}$$

ρ_{reg} is called regular representation

6) (Permutation Representation)

Assume G is acting on a finite set X, and let V be the vect. space with a basis $(e_x)_{x\in X}$

Then
$$\rho_{perm} : G \to GL(V)$$
$$s \longmapsto \begin{pmatrix} V \longrightarrow V \\ e_x \mapsto e_{s\cdot x} \end{pmatrix}$$

ρ_{perm} is called permutation representation

The regular representation is the permutation representation with $X = G$.

Def Given a representation $\rho : G \to GL(V)$ and a vector subspace W of V, we say that W is invariant if for each $s\in G$

$$\rho_s(v) \in W \quad \forall v \in W$$

In this case it is well defined the repr.
$$\rho_s^W : G \to GL(W)$$
$$s \longmapsto (\rho_s|_W : W \to W)$$

ρ_s^W is called subrepresentation of ρ.

Example Given the regular representation ρ_{reg}, then $W := \langle \sum_{s\in G} e_s \rangle$ is invariant, so

ρ_{reg}^W is a subrepresentation (of degree 1) of ρ_{reg}. In this case $\rho_{reg}^W = \rho_{triv}$.

We will compute all subrepresentations of ρ_{reg}

Remark: We can always find an invariant subspace of V:

$$V^G := \{ v \in V \mid \rho_s(v) = v \quad \forall s \in G \}$$

Clearly, V^G can eventually be 0 or V.

Moreover, $\rho_s^{V^G} = Id_{V^G} \quad \forall s \in G$.

It is always possible to project any vector of V to V^G. (We remind that a projection over a subspace W is a linear map $\pi : V \to V$ s.t. $\pi(V) = W$ and $\pi \circ \pi = \pi$)

Thm (Reynolds Operator, important in fluidodynamics + invariant theory)

There is a natural projector onto V^G:

$$\pi : V \longrightarrow V$$
$$v \longmapsto \pi(v) := \frac{1}{|G|} \sum_{s \in G} \rho_s(v)$$

Furthermore, it holds the following equality

$$\dim(V^G) = \frac{1}{|G|} \cdot \sum_{s \in G} Tr(\rho_s).$$

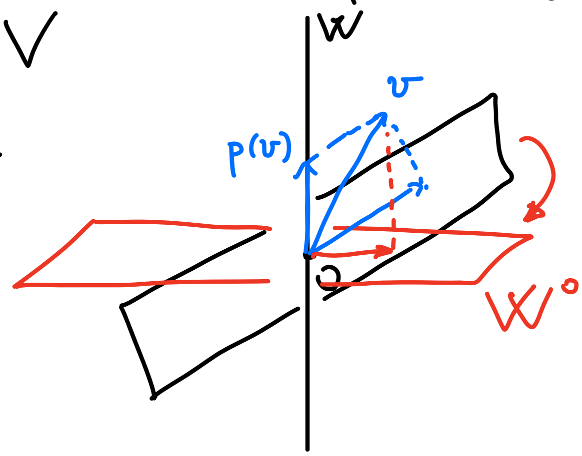
proof $\rho_t(\pi(v)) = \frac{1}{|G|} \sum_{s \in G} \rho_{ts}(v) = \frac{1}{|G|} \sum_{s \in G} \rho_s(v) = \pi(v)$

$\Rightarrow \pi(v) \in V^G \Rightarrow \pi(V) \subseteq V^G$.

Instead, given $v \in V^G$, then $\pi(v) = v \in \pi(V) \Rightarrow \pi(V) = V^G$. Finally, we can write $V = \mathrm{Ker}(\pi) \oplus V^G$, so after completing a basis of V^G to a basis of V, we have in that basis that the associat. matrix to π is $\begin{pmatrix} 0_{\dim(\ker)} & \\ & Id_{\dim V^G} \end{pmatrix} \Rightarrow \dim(V_G) = Tr(\pi)$ ▨

Thm Given an invariant subspace W of ρ, then there exists a complement W^0 of W which is invariant.

proof Consider a complement subspace of W and let $p: V \to V$ be the projection on W.



Then we can define a new linear map

$$p^0 := \frac{1}{|G|} \cdot \sum_{s \in G} \rho_s \circ p \circ \rho_s^{-1}$$

We notice that p^0 fixes W:

$$p^0(w) = \frac{1}{|G|} \sum \rho_s \, p(\underbrace{\rho_s^{-1}(w)}_{\substack{\cap \\ W}}) = \frac{1}{|G|} |G| \cdot w = w \quad \forall w \in W.$$

and it has image in W as $p(W) = W$ and

W is invariant. Thus the image is W and the kernel of the map is a complement of W^0 of W: $W \oplus W^0 = V$.

Finally, $\rho_s \circ p^0 = p^0 \circ \rho_s$ and so given $v \in W^0$, then

$$p^0(\rho_s(v)) = \rho_s(p^0(v)) = \rho_s(0) = 0$$

$\Rightarrow W^0$ is invariant. $\blacksquare$

Def The direct sum of two representations $\rho: G \to GL(V)$, $\eta: G \to GL(W)$ is

$$\rho \oplus \eta : G \longrightarrow GL(V \oplus W)$$
$$s \longmapsto \begin{array}{l} V \oplus W \longrightarrow V \oplus W \\ (v, w) \mapsto (\rho_s(v), \eta_s(w)) \end{array}$$

Remark From the previous theorem, we have that ρ is isomorphic to

$$\rho \cong \rho^W \oplus \rho^{W^0}$$

as $V \cong W \oplus W^0$ and W and W^0 are invariant.

Def An irreducible representation is a repr. $\rho: G \to GL(V)$ whose invariant subspaces are only 0 and V.

Thm Every representation is the finite sum of irreducible representations.

proof By induction on the dimension of V.
If $\dim V = 0$, then it is trivial.
Assume then $\dim V > 0$. If $\rho: G \to GL(V)$ is irreducible, then we are done.
Otherwise, it there exists a proper invariant subspace $0 \subsetneq W \subsetneq V$, and an invariant complement W^0, $V = W \oplus W^0$.
Then $\rho = \rho^W \oplus \rho^{W^0}$ and the inductive hypothesis applies as $\dim W < \dim V$
$\dim W^0 < \dim V$ ∎

Rem Thus V can be decomposed as a direct sum of irreducible representations $W_1, \dots W_k$: $V = W_1 \oplus \dots \oplus W_k$
It is natural to ask if the decomposition is unique. The answer is clearly NO. For instance
$\rho: \mathbb{Z}_2 \to GL(\mathbb{C}^2)$ has inv. subsp. $\langle e_1 \rangle$ and $\langle e_2 \rangle$
$\bar{1} \mapsto -Id$

but also $\langle e_1 + e_2 \rangle$ and $\langle e_2 \rangle$: $\mathbb{C}^2 = \langle e_1 + e_2 \rangle \oplus \langle e_2 \rangle = \langle e_1 \rangle \oplus \langle e_2 \rangle$
What will not change will be the NUMBER of irreduc. represent. isomorphic to a given W.

Def The *dual representation* of $\rho: G \to GL(V)$ is $\rho^*: G \to GL(V^*)$ where $\rho^*_g(f) := f \circ \rho_{g^{-1}}$.

Def Given $\rho_1: G \to GL(V)$, $\rho_2: G \to GL(W)$ we can define

$$\rho_1 \otimes \rho_2 : G \to GL(V \otimes W)$$
$$g \longmapsto \begin{pmatrix} V\otimes W \to V \otimes W \\ e_{i_1} \otimes e_{i_2} \mapsto \rho_g(e_{i_1}) \otimes \rho_g(e_{i_2}) \end{pmatrix}$$

which is called *tensor product representat.* of ρ_1 and ρ_2.

We remind that $V \otimes V = Alt^2(V) \oplus Sym^2(V)$ where $Alt^2(V)$ is given by a basis

$$e_i \otimes e_j - e_j \otimes e_i \qquad i \neq j$$

and $Sym^2(V)$ is given by a basis

$$e_i \otimes e_j + e_j \otimes e_i$$

$\left(\dim(Alt^2(V)) = \frac{n(n-1)}{2}\right.$ and $\left.\dim(Sym^2(V)) = \frac{n(n+1)}{2}\right.$.

We observe that $Alt^2(V)$ and $Sym^2(V)$ are invariant with respect to $\rho \otimes \rho : G \to GL(V \otimes V)$, so $\rho \otimes \rho$ is never irreducible and can be written as a direct sum of two repres., called the *Alternating square* and *Symmetric Square*.

Homomorphism Representation

Given $\rho : G \to GL(V)$ and $\eta : G \to GL(W)$, then we have a natural representation on $Hom(V, W)$:

$$\begin{aligned} Hom(\rho,\eta) : G &\to Hom(V,W) \\ s &\mapsto \left(\begin{aligned} Hom(V,W) &\to Hom(V,W) \\ F &\mapsto \eta(s) \circ F \circ \rho(s^{-1}) \end{aligned} \right) \end{aligned}$$

Remark There is always an invariant subspace

$$\boxed{Hom^G(V,W) := \{ F : V \to W \mid \eta(s) \circ F \circ \rho(s^{-1}) = F \ \forall s \in G \}}$$

We remind the natural isomorphism in linear algebra

$$\begin{aligned} \Theta : V^* \otimes W &\longrightarrow Hom(V,W) \\ f \otimes w &\longmapsto \left(\begin{aligned} V &\to W \\ v &\mapsto f(v) \cdot w \end{aligned} \right) \end{aligned}$$

(whose inverse is not natural and it is defined by the choice of a bases of V $(e_1, \dots e_n)$:

$$\begin{aligned} Hom(V,W) &\longrightarrow V^* \otimes W \\ f &\longmapsto \sum_{i=1}^{n} e_i^* \otimes f(e_i) \end{aligned}$$
)

As you can expect, Θ is an isomorphism of repr. among $Hom(\rho,\eta)$ and $\rho^* \otimes \eta$:

$\forall s \in G,$

$$\begin{tikzcd} V^* \otimes W \arrow[r, "\rho^* \otimes \eta_s"] \arrow[d, "\Theta"'] & V^* \otimes W \arrow[d, "\Theta"] \\ Hom(V,W) \arrow[r] & Hom(V,W) \end{tikzcd}$$

Thus $Hom(\rho,\eta) \cong \rho^* \otimes \eta$.

SCHUR LEMMA

Let $\rho: G \to GL(V)$, $\eta: G \to GL(W)$ be two irreducible representations of G, and let f be a linear map from V to W s.t.

$$\eta_s \circ f = f \circ \rho_s \quad \forall s \in G.$$

Then

(1) if ρ and η are <u>NOT</u> isomorphic, $f \equiv 0$;

(2) if $V = W$ and $\rho = \eta$, then $f = \lambda \cdot Id$,

where $\lambda = \frac{Tr(f)}{n}$, $n = \dim(V)$

proof (1) If $f = 0$ is trivial, assume $f \neq 0$.

We claim that $Ker(f)$ and $Im(f)$ are invariant subspaces of V and W respect.

Given $v \in Ker(f)$, then $f(\rho_s(v)) = \rho_s(f(v)) = 0$

$\Rightarrow \rho_s(v) = 0$;

Given $f(v) \in Im(f)$, then $\rho_s(f(v)) = f(\rho_s(v)) \in Im(f)$.

However, ρ and η are irreducible, so the possibilities are $Ker(f) = \{0\}$ and $Im(f) = W$, which means f is an isomorphism, so ρ and η are iso, or $Ker(f) = V$, $Im(f) = 0$, which means $f = 0$.

2) Let v be an eigenvector of f with eigenv. λ. Then $Ker(f - \lambda I) \neq \{0\}$ and $f - \lambda I$ satisfies

$$\rho_s \circ (f - \lambda I) = (f - \lambda I) \circ \rho_s \quad \forall s \in G$$

$\Rightarrow$ from (1) we have $f - \lambda I = 0 \Rightarrow f = \lambda I$ $\blacksquare$

§5.1 Character of a Representation

Let $\rho: G \to GL(V)$ be a representation of G. There is also another object that does not change when ρ is replaced by a isomorphic representation; this object is the trace $Tr(\rho_s)$.

Def The character χ_ρ of the representation ρ is the function
$$\chi_\rho : G \longrightarrow \mathbb{C}$$
$$s \longmapsto Tr(\rho_s)$$

As we will see, the character of ρ completely determine ρ.

Prop The following holds:

(1) $\chi_\rho(1) = n$, $n = \dim(V)$;

(2) $\chi_\rho(s^{-1}) = \overline{\chi_\rho(s)}$ $\forall s \in G$;

(3) $\chi_\rho(t s t^{-1}) = \chi_\rho(t)$ $\forall t, s \in G$ (So the values of χ depends only on the conjugacy classes of G)

proof (1) and (3) are trivial as the trace of $\rho_t \circ \rho_s \circ \rho_{t^{-1}}$ is the same as ρ_s (invariance of the trace up to similar matrices)

For (2), we remind that any matrix of finite order is diagonalizable, so let $\lambda_1, \dots \lambda_n$ be the eigenvalues of ρ_s. Then $\frac{1}{\lambda_1}, \dots \frac{1}{\lambda_n}$ are the eigenvalues of $\rho_{s^{-1}}$. However
$$\rho_s(v_i) = \lambda_i v_i \Rightarrow \rho_{s^{ord(s)}}(v_i) = v_i = \lambda_i^{ord(s)} v_i$$

and so $\lambda_i^{\mathrm{ord}(s)} = 1 \Rightarrow |\lambda_i| = 1 \Rightarrow \lambda_i \overline{\lambda_i} = 1$

This means

$$\mathrm{Tr}(\rho_{s^{-1}}) = \frac{1}{\lambda_1} + \dots + \frac{1}{\lambda_n} = \overline{\lambda_1} + \dots + \overline{\lambda_n} = \overline{\mathrm{Tr}(\rho_s)}.$$

$\blacksquare$

Prop 2 Given $\rho: G \to GL(V)$ with charc. χ and $\eta: G \to GL(W)$ with characters χ_ρ and χ_η, then

(Dual rep.) $\chi_{\rho^*} = \overline{\chi_\rho}$;

(Direct Sum rep.) $\chi_{\rho \oplus \eta} = \chi_\rho + \chi_\eta$;

(Tensor rep.) $\chi_{\rho \otimes \eta} = \chi_\rho \cdot \chi_\eta$;

(Alt. square rep.) $\chi_{Alt^2 \rho}(s) = \frac{1}{2}\left(\chi_\rho^2(s) - \chi_\rho(s^2)\right)$

(Sym. square rep.) $\chi_{Sym^2 \rho}(s) = \frac{1}{2}\left(\chi_\rho^2(s) + \chi_\rho(s^2)\right)$

Examples

1) The character of the trivial representation is

$$\chi_{triv} = 1 \quad \forall s \in G;$$

2) $\rho_{reg}(s)$ sends $e_t \to e_{st}$, so the associated matrix has only zeros on the diagonal unless $s = 1_G$, in which case all the elements on the diagonal are 1.

Thus $\chi_{\rho_{reg}}(s) = \begin{cases} |G| & \text{if } s = 1_G \\ 0 & \text{otherwise} \end{cases}$.

3) $\rho_{perm}(s)$ sends $e_x \to e_{s\cdot x}$ which is the same e_x iff $s \in \mathrm{Stab}(x)$. Thus, let $\mathrm{Fix}(s) := \{x \in X \mid s\cdot x = x\} \leq G$. We have that $\chi_{\rho_{perm}}(s) = |\mathrm{Fix}(s)|$.

Def When we have two complex valued functions

$$f: G \to \mathbb{C}, \quad g: G \to \mathbb{C}$$

of G we can always define the scalar product

$$(f|g) := \frac{1}{|G|}\sum_{s\in G} f(s)\cdot\overline{g(s)}$$

Thus, given two characters χ_ρ and χ_η, we can always compute the (a priori complex) number $(\chi_\rho|\chi_\eta)$.

Rem:

Using Reynolds operator, we proved

$$\dim V_G = \frac{1}{|G|}\sum_{s\in G} \mathrm{Tr}(\rho_s)$$

that now can be rewritten as $\boxed{\dim V^G = (\chi_\rho|\chi_{triv})}$.

Thm Given $\rho: G \to GL(V)$, $\eta: G \to GL(W)$ with characters χ_ρ and χ_η, then the number $(\chi_\rho|\chi_\eta)$ is always an integer equal to

$$\boxed{(\chi_\rho|\chi_\eta) = \dim_{\mathbb{C}}(\mathrm{Hom}^G(V,W))}$$

proof We have seen that $\mathrm{Hom}(\rho,\eta) \cong \rho^* \otimes \eta$, so its character is $\overline{\chi_\rho}\cdot\chi_\eta$.

However, from the previous remark applied to the vector space $\mathrm{Hom}(V, W)$ we have

$$\dim \mathrm{Hom}^G(V, W) = (\bar{\chi}_\rho \cdot \chi_\eta \mid \chi_{triv}) = (\chi_\rho \mid \chi_\eta) \quad \blacksquare$$

Corollary (IMPORTANT)

(1) if ρ and η are irreducible represent, then

$$(\chi_\rho \mid \chi_\eta) = \begin{cases} 1 & \text{if } \rho \text{ and } \eta \text{ are isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

(thus $\{\chi_\rho \mid \rho \text{ irreduc. rep}\}$ form an orthonormal system!

(2) Given a representation $\rho: G \to GL(V)$, suppose it decomposes in irred. rep. $V = W_1 \oplus \cdots \oplus W_k$.
Let $\eta: G \to GL(W)$ be a irreducible representation,
Then the number of $W_i \subseteq V$ isomorphic to W equals the number $(\chi_\eta \mid \chi_\rho)$.
This number does not depend on the decomposition

(3) Two representations with the same character are isomorphic;

(4) A representation is irreducible if and only if $(\chi_\rho, \chi_\rho) = 1$.

proof (1) $(\chi_\rho \mid \chi_\eta) = \dim_{\mathbb{C}}(\mathrm{Hom}^G(V,W))$

However ρ and η are irreducible, so by Schur Lemma homomorphism represt. in $\mathrm{Hom}^G(V,W)$ is an isomorphism rep. of ρ and η.

If η and ρ are NOT iso, then $\mathrm{Hom}^G(V,W)=0$.

Instead, if it there exists an isomorphism $F: V \to W$, then

$$\mathrm{Hom}^G(V,W) \xrightarrow{\sim} \mathrm{Hom}^G(V,V) \text{ is } \underline{\text{iso}}$$
$$g \longmapsto F^{-1} \circ g$$

However, from Schur Lemma (2),

$$\mathrm{Hom}^G(V,V) = \langle \mathrm{Id}_V \rangle$$

and so $\mathrm{Hom}^G(V,W)$ is one dimensional.

(2) We have $V = W_1 \oplus \cdots \oplus W_k$, let $\rho_1, \dots \rho_k$ be their irreduc. representations. Then $\chi_\rho = \chi_{\rho_1} + \dots + \chi_{\rho_k}$ and so by the previous point

$$(\chi_\rho \mid \chi_\eta) = \#\{J \mid \rho_J \text{ is iso with } \eta\}$$

(3) If ρ and η have the same character χ, then they contain the same irreducible representations the same number of times.

Thus ρ and η are iso;

(4) ($\Rightarrow$) is proved in (1)

($\Leftarrow$) Assume that $V = m_1 W_1 \oplus \dots \oplus m_k W_k$ where m_i is the number of times the representation ρ_i is occurring on V.

Then $\chi_\rho = m_1 \chi_{\rho_1} + \dots + m_k \chi_{\rho_k}$

and so
$$1 = (\chi_\rho \mid \chi_\rho) = m_1^2 + \dots + m_k^2$$
$\Rightarrow \exists ! J$ s.t. $m_J = 1$ and the others are zero $\Rightarrow$ $\chi_\rho = \chi_{\rho_J}$ $\Rightarrow$ ρ and ρ_J are iso $\Rightarrow$ ρ is irreducible. ▨

Remark We can now find the irreducible rep. contained in ρ_{reg}. We observe that
$$(\chi_{reg} \mid \chi) = \frac{1}{|G|} \cdot |G| \cdot \chi(1_G) = \chi(1_G)$$
so χ irreducible occurs on χ_{reg} with multiplicity $\chi(1_G)$.

This means that there are only finitely many irreducible characters $\chi_1, \dots \chi_k$ and are all of them contained in χ_{reg}. In particular, it holds
$$\boxed{\begin{aligned} \chi_{reg} &= \sum_{i=1}^{k} \chi_i(1_G) \cdot \chi_i \\ |G| &= \sum_{i=1}^{k} \chi_i^2(1_G) \end{aligned}}$$

Def A *class function* is a function $f: G \to \mathbb{C}$ satisfying $f(tst^{-1}) = f(t)$ $\forall t, s \in G$.
The space of class functions of G is denoted by $CF(G)$.

Notice that this space contains every character of G.

Thm Let f be a class function on G, $\rho: G \to GL(V)$ a repr. of G. We define the homomorphism

$$\rho_f := \sum_{s \in G} f(s) \cdot \rho_s$$

If V is irreducible of degree n, then ρ_f is an homothety of ratio $\lambda = \frac{|G|}{n} (f | \overline{\chi_\rho})$.

proof $\rho_t \rho_f = \Big(\sum_{s \in G} \underbrace{f(s)}_{f(tst^{-1})} \rho_{tst^{-1}}\Big) \rho_t = \rho_f \rho_t \Rightarrow$ by Schur Lemma

$$\rho_f = \lambda \, Id_V \quad \text{where} \quad \lambda = \frac{Tr(\rho_f)}{n} = \frac{|G| \cdot (f | \overline{\chi}_\rho)}{n}. \qquad \blacksquare$$

Thm (1) $Irr(G) := \{\text{irreducible characters of } G\}$ is an orthonormal basis of $CF(G)$;

(2) The number of irreducible characters is equal to the number of conjugacy classes of G.

proof (1) $\chi_1, \ldots \chi_k$ irreducible characters of G.
We can decompose $CF(G)$ as direct sum of $\langle \chi_1, \ldots \chi_k \rangle$ and its orthogonal complement.

Thus it is sufficient to prove that if $f \in CF(G)$ verifies $(\chi_i | f) = 0 \ \forall i = 1, \dots k \Rightarrow f = 0$.

Let us consider $\rho_f = \sum_{s \in G} f(s) \rho_s$ for any represent. ρ. The previous thm. shows that ρ_f is zero on any irreducible represent. of ρ as $(\chi_i | f) = 0$. Thus ρ_f is identically zero for any repr. ρ. Let us consider the regular representation ρ_{reg}. Then

$$0 = (\rho_{reg})_f = \sum_{s \in G} f(s) (\rho_{reg})_s$$

$$\Rightarrow \quad 0 = (\rho_{reg})_f (e_{1_G}) = \sum_{s \in G} f(s) \cdot e_s \quad \Rightarrow f(s) = 0 \ \forall s \in G$$

(2) Another basis of $CF(G)$ is given by

$\{ \mathbb{1}_{\mathrm{Conj}(x)} : x \in G \}$ where $\mathrm{Conj}(x) = \{ t x t^{-1} \mid t \in G \}$.

Thus $\#\text{conj classes} = \dim_{\mathbb{C}} CF(G) = \# \mathrm{Irr}(G)$ ▨

Corollary A Group is abelian if and only if all the irreducible representations of G are 1-dimensional

proof Using the regular representation, we have $|G| = \chi_1^2(1_G) + \dots \chi_k^2(1_G)$

wher $k = \#$ conj classes of G.

However G is abelian $\Leftrightarrow$ $k = |G|$ $\Leftrightarrow$

$$\chi_1(1_G) = \dots = \chi_k(1_G) = 1 \qquad \blacksquare$$

Def Given $\rho: G \to GL(V)$ repres. and an irreducible repr. $\eta: G \to GL(W)$, the isotypic component W^η of ρ of character η is the biggest invariant subspace of V isomorphic to some copies of the same representation η.

Thus, $\chi_{\rho_{W^\eta}} = \langle \chi_\rho | \chi_\eta \rangle \cdot \chi_\eta$.

Remark With this notation, we have a canonical unique decomposition of $\rho: G \to GL(V)$ as a direct sum of isotypic components:

$$V = W^{\eta_1} \oplus \dots \oplus W^{\eta_k}$$ ← We are just putting together the isomorph. represent.

We can use a generalization of Reynold operator to construct a projection of V to the isotypic component of char. η.

Thm (Reynold Operator of Character η)

Let $\rho: G \to GL(V)$ repr. and η be a irreducible repres. Let W^η be the isoty. component of character ρ. Then

$$\Pi_\eta := \frac{1}{|G|} \sum_{s \in G} \overline{\chi_\eta(s)} \cdot \rho_s$$

is a projection on W^η.

Furthermore, given a basis $e_1, \dots, e_n$ of V, then if $V = W^{\eta_1} \oplus \dots \oplus W^{\eta_k}$, we have

$$\Pi_{\eta_1}(e_1), \dots \quad \Pi_{\eta_1}(e_n) \quad (\text{generates } W^{\eta_1})$$

$$\vdots$$

$$\Pi_{\eta_k}(e_1), \dots, \quad \Pi_{\eta_k}(e_n) \quad (\text{generates } W^{\eta_k})$$

generate the entire space V.

proof We apply the previous result and obtain that Π_η restricted to any irr. repr. W_J of character η_J is an homothopy of ratio

$$\lambda = \frac{(\chi_\eta | \chi_{\eta_J})}{n_J} = \begin{cases} 1 & \text{if } \chi_\eta = \chi_{\eta_J} \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow \Pi_\eta$ is the identity on W_J if it is isomorph. to η, and zero otherwise. Thus Π_η is the identity on the isotypic component of charact. η and zero otherwise.

We can write $V = W^{\eta_1} \oplus \cdots \oplus W^{\eta_k}$ and so $x \in V$ can be written as $x = x_1 + \cdots + x_k \Rightarrow$

$$\pi_\eta(x) = \pi_\eta(x_1) + \cdots + \pi_\eta(x_k) = x_j \quad (\eta = \eta_j)$$

$\Rightarrow \pi_\eta$ is the projection on W^η ▨

FINAL EXAMPLE

$S_3 = \langle \sigma, \tau \mid \tau^2 = \sigma^3 = 1, \ \tau\sigma = \sigma^2\tau \rangle$, $|S_3| = 6$

We want to find all the possible irr. repres.:

Conjugacy classes are $Conj(\sigma) = \{\sigma, \sigma^2\}$

$Conj(1) = \{1\}$

$Conj(\tau) = \{\tau, \tau\sigma^2, \tau\sigma\}$

$\Rightarrow \# Irr(S_3) = 3$.

However, we already constructed 2 natural characters of S_3:

$\chi_{triv} : S_3 \to \mathbb{C}^*$

$sgn : S_3 \to \mathbb{C}^*$

$+$

$\chi_{reg} : S_3 \to \mathbb{C}^*$

The last character χ is then computable using

$$\chi_{reg} = \chi_{triv} + sgn + \chi(1_G) \cdot \chi$$

$\Rightarrow$ at 1_G we have $|S_3| = 6 = 1 + 1 + \chi^2(1_G) \Rightarrow$

$\chi(1_G) = 2$, and $\boxed{\chi = \frac{1}{2}(\chi_{reg} - \chi_{triv} - sgn)}$

Actually we can also prove that the

irreducible representation of deg. 2 is

$$\rho : S_3 \longrightarrow GL(\mathbb{C}^2)$$
$$\tau \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma \longmapsto \begin{pmatrix} \xi_3 & 0 \\ 0 & \xi_3^2 \end{pmatrix}$$

To prove that ρ is irreducible it is sufficient to prove $(\chi_\rho | \chi_\rho) = 1$.

We can order the irr. characters in a table called CHARACTER TABLE:

	$Conj(1)$	$Conj(\sigma)$	$Conj(\tau)$
1	1	1	1
sgn	1	1	-1
χ	2	-1	0

Let us construct a basis of isotypic components of the regular representation χ_{reg}:

$$\rho_{reg} : S_3 \to GL(\mathbb{C}^6) \qquad e_1 \; e_\tau \; e_{\tau\sigma} \; e_{\tau\sigma^2} \; e_\sigma \; e_{\sigma^2}$$

$e_1 \leftarrow$ this generate $W^{\chi_{triv}}$

$$\pi_{sgn}(e_1) = \frac{1}{6}\left(e_1 - e_{\tau\sigma} - e_\tau - e_{\tau\sigma^2} + e_{\sigma^2} + e_\sigma \right)$$ $\leftarrow$ this generate W^{sgn}

$$\pi_\chi(e_1) = \frac{1}{6}\left(2e_1 - e_\sigma - e_{\sigma^2} \right)$$
$$\pi_\chi(e_\sigma) = \frac{1}{6}\left(2e_\sigma - e_{\sigma^2} - e_1 \right)$$
$$\pi_\chi(e_\tau) = \frac{1}{6}\left(2e_\tau - e_{\tau\sigma^2} - e_{\tau\sigma} \right)$$
$$\pi_\chi(e_{\tau\sigma}) = \frac{1}{6}\left(2e_{\tau\sigma} - e_\tau - e_{\tau\sigma^2} \right)$$

$\leftarrow$ they generate W^χ isotypic comp. of charact χ for ρ_{reg}, which contains 2-times the irr. represent. $\rho = \rho_\chi$ above.