Lecture 20. Review Session for the final exam.

Problem 1-5 60 pts. Slightly harder than Midterm 1 & 2.

Problem 6-8. 40pts. Generally easier than HW.

Problem 1a. Determine the interval of existence of an IVP with a linear ODE.

<u>Thm</u>: If $\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0. \end{cases}$ (linear ODE in the std. form)

satisfies (1) $p(t), g(t)$ are cont. on (a,b).

(2) $a < t_0 < b$,

then the IVP has a unique solution on (a,b)

To decide the interval of existence: (1) Find singularities.

(2) Plot them on the number line.

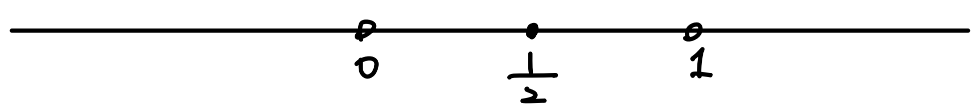
(3) Pick the interval containing t_0.

<u>Ex.</u>: $t(t-1)y' + \sin t \, y = e^{-t}$. $y(\frac{1}{2}) = 2$.

Std. form: $y' + \dfrac{\sin t}{t(t-1)} y = \dfrac{e^{-t}}{t(t-1)}$

$p(t) = \dfrac{\sin t}{t(t-1)}$ cont. everywhere except at $t=0$ or $t=1$.

$g(t) = \dfrac{e^{-t}}{t(t-1)}$ cont. everywhere except at $t=0$ or $t=1$



Interval of existence $= (0, 1)$.

Problem 1b. Solve an IVP (first-order linear ODE).

(1) Get the std. form: $y' + p(t)y = g(t)$.

(2) Find the integrating factor: $\mu(t) = e^{\int p(t)\,dt}$. (ignore $+c$ or $|\cdot|$)

(3) General solution: $y(t) = \dfrac{\int \mu(t) g(t)\,dt + C}{\mu(t)}$

Midterm 1: $t y' + (1-t)y = e^t \cos(3t)$.

(1) Std. form: $y' + \frac{1-t}{t} y = \frac{e^t \cos 3t}{t}$

(2) Int. factor: $\mu(t) = e^{\int \frac{1-t}{t}\,dt} = e^{\int (\frac{1}{t} - 1)\,dt} = e^{\ln t - t} = e^{\ln t} \cdot e^{-t} = te^{-t}$.

(3) Gen. sol'n: $y = \dfrac{\int \cancel{te^{-t}} \cdot \frac{\cancel{e^t} \cos 3t}{\cancel{t}}\,dt}{te^{-t}} = \dfrac{\int \cos 3t\,dt}{te^{-t}} = \dfrac{\frac{1}{3}\sin 3t + C}{te^{-t}}$

Initial condition: $y(\frac{\pi}{3}) = 0. \Rightarrow \dfrac{\frac{1}{3}\sin(3\cdot\frac{\pi}{3}) + C}{\frac{\pi}{3} e^{-\frac{\pi}{3}}} = 0.$ Recall $\sin(\pi) = 0$.

$\Rightarrow \dfrac{0 + C}{\text{non zero}} = 0 \Rightarrow C = 0.$

$y = \dfrac{\frac{1}{3}\sin 3t}{te^{-t}} = \dfrac{e^t \sin 3t}{3t}$

Problem 2a. Determine the region of (x_0, y_0) for which an IVP has a unique local sol'n. (equivalently, an IVP is reasonably formulated).

Generally, $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0. \end{cases}$

(1) Find out where f is continuous.

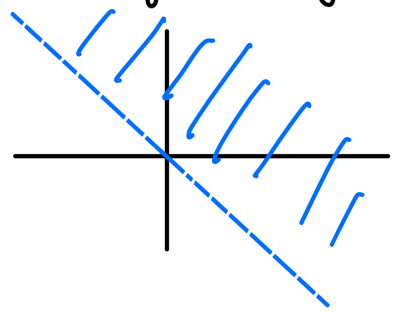
(2) Find out where $\frac{\partial f}{\partial y}$ is continuous.

(3) Interior of the region where f & $\frac{\partial f}{\partial y}$ are cont.

<u>Example.</u> $\begin{cases} y' = \sqrt{x + y} \\ y(x_0) = y_0 \end{cases}$

(1) $f(x,y) = \sqrt{x+y}$ cont. where $x+y \geq 0. \Rightarrow y \geq -x.$

(2) $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{1}{2\sqrt{x+y}} \cdot (0+1) = \dfrac{1}{2\sqrt{x+y}}$ Cont. where $x+y>0 \Rightarrow y > -x.$



(3) Region we want $= \{ (x_0, y_0) \mid y_0 > -x_0 \}.$

$$\frac{\partial}{\partial y}\left(\sqrt{x+y}\right) = \frac{\partial}{\partial y}\left(\sqrt{u}\right) \qquad (u = x+y)$$

$$= \frac{\partial}{\partial u}\sqrt{u} \cdot \frac{\partial u}{\partial y}. \qquad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{u}} \cdot \frac{\partial}{\partial y}(x+y) = \frac{1}{2\sqrt{x+y}} \cdot \left(\frac{\partial}{\partial y}x + \frac{\partial}{\partial y}y\right) = \frac{1}{2\sqrt{x+y}}(0+1).$$

Problem 2b. Find the explicit sol'n of an IVP. Determine the interval of existence.

Midterm 1. $(2y-2)\dfrac{dy}{dx} = -4x+3 \quad y(0) = 4.$

$(2y-2)\,dy = (-4x+3)\,dx.$

$y^2 - 2y = -2x^2 + 3x + C.$

$y(0) = 4 \Rightarrow C = 8.$

$y^2 - 2y \underset{+1}{} = -2x^2 + 3x + 8 \underset{+1}{}.$ Implicit solution.

$(y-1)^2 = -2x^2 + 3x + 9$

$y - 1 = \pm\sqrt{-2x^2+3x+9} \Rightarrow y = 1 + \sqrt{-2x^2+3x+9}.$ Explicit solution.

Pick + branch

Interval of existence. $-2x^2+3x+9 \geq 0 \Rightarrow 2x^2 - 3x - 9 \leq 0$

$(2x+3)(x-3) \leq 0.$

	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 3$	$x > 3$
	+	−	+
$2x+3$	−	+	+
$x-3$	−	−	+

$\Rightarrow -\frac{3}{2} \leq x \leq 3.$

Exercise 2.2.32. $(2y-2)\dfrac{dy}{dx} = 3x^2 + 4x + 2. \quad y(1) = -2.$

$(2y-2)\,dy = (3x^2+4x+2)\,dx \Rightarrow y^2 - 2y = x^3 + 2x^2 + 2x + C.$

$$y(1) = -2 \Rightarrow \quad 4 + 2\cdot 2 = 1 + 2 + 2 + C \Rightarrow C = 3.$$

$$\underset{+1}{y^2 - 2y} = \underset{+1}{x^3 + 2x^2 + 2x + 3}. \qquad \text{Implicit sol'n.}$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4. \Rightarrow y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4}.$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4} \qquad y(1) = -2 \Rightarrow \text{Pick the } - \text{ branch.}$$

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}. \qquad \text{Explicit solution.}$$

IoE: $x^3 + 2x^2 + 2x + 4 \geq 0.$

$x^2(x+2) + 2(x+2)$

$(x^2+2)(x+2) \geq 0.$

$\Rightarrow \; x \geq -2.$

	$x<-2$	-2	$x>-2$
	$-$		$+$
x^2+2	$+$		$+$
$x+2$	$-$		$+$

IoE: $(-2, +\infty)$

Problem 3: Given a sol'n of a homog. linear ODE, find the general sol'n.

If y_1 is a sol'n of the 2^{nd}-order linear homog. ODE

$$y'' + p(t)y' + q(t)y = 0 \qquad \text{Std. form.}$$

Set $y_2 = u\cdot y_1 \Rightarrow y_1 u'' + (2y_1' + p\cdot y_1)u' = 0.$ Separable ODE in u'.

Solve it $\Rightarrow u'$. Integrate $\Rightarrow u \Rightarrow y_2 \Rightarrow y = C_1 y_1 + C_2 y_2.$

Ignore abs. values or $+C$. as long as $u' \neq 0$.

Midterm 2. $y_1 = \frac{1}{t^3}$. $\quad t^2 y'' + 7t y' + 9y = 0$

(1) Std. form: $y'' + \frac{7}{t}y' + \frac{9}{t^2}y = 0. \qquad p(t) = \frac{7}{t}.$

(2) Formulate the ODE. $\frac{1}{t^3}\cdot u'' + \left(2\left(-\frac{3}{t^4}\right) + \frac{7}{t}\cdot\frac{1}{t^3}\right)u' = 0$

$$u'' + \left(-\frac{6}{t} + \frac{7}{t}\right)u' = 0.$$

$$u'' + \frac{1}{t}u' = 0 \Rightarrow \frac{du'}{dt} = -\frac{1}{t}u' \Rightarrow \frac{du'}{u'} = -\frac{1}{t}dt.$$

$$\Rightarrow \ln|u'| = -\ln|t| \Rightarrow u' = t^{-1} = \frac{1}{t} \Rightarrow u = \ln|t|.$$

$$\Rightarrow y_2 = \ln|t|\cdot\frac{1}{t^3} \Rightarrow y = C_1\frac{1}{t^3} + C_2\frac{\ln|t|}{t^3}.$$

HW6. 1b. $y_1 = e^t$, $(t-1)y'' - ty' + y = 0$.

(1) Std. form: $y'' - \frac{t}{t-1}y' + \frac{1}{t-1}y = 0$. $p(t) = -\frac{t}{t-1}$

(2) Formulate the ODE for u'. $e^t u'' + \left(2e^t - \frac{t}{t-1}e^t\right)u' = 0$.

$$u'' + \left(2 - \frac{t}{t-1}\right)u' = 0 \Rightarrow u'' + \frac{t-2}{t-1}u' = 0 \Rightarrow \frac{du'}{dt} = -\frac{t-2}{t-1}u' \Rightarrow \frac{du'}{u'} = -\frac{t-2}{t-1}dt$$

$$\frac{2(t-1)}{t-1} - \frac{t}{t-1} = \frac{2t-2-t}{t-1} = \frac{t-2}{t-1}$$

$$\ln|u'| = -\int \frac{t-2}{t-1}dt = -\int \frac{t-1-1}{t-1}dt = -\int \left(1 - \frac{1}{t-1}\right)dt$$

$$= -\left(t - \ln|t-1|\right) = -t + \ln|t-1|.$$

Exp. both sides $\Rightarrow u' = e^{-t+\ln|t-1|} = e^{-t} \cdot e^{\ln|t-1|} = (t-1)e^{-t}$.

$$u = \int \underbrace{(t-1)}_{D}\underbrace{e^{-t}}_{I}dt = (t-1)(-e^{-t}) - \int 1 \cdot (-e^{-t})dt = -te^{-t} \cancel{+e^{-t}} \cancel{-e^{-t}} = -te^{-t}.$$

$$\Rightarrow y_2 = u \cdot y_1 = -te^{-t} \cdot e^t = -t. \Rightarrow y = C_1 e^t + C_2 t.$$

Problem 4a. Gen. sol'n of higher-order lin. homog. ODE.

(1) Aux. eqn. (2) Factorization by grouping

or Taking complex roots.

Example: $2y''' + 3y'' + 3y' + 2 = 0$. Aux. eqn: $2r^3 + 3r^2 + 3r + 2 = 0$.

$2(r^3+1) + 3r(r+1) = 0$. Recall: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
$a^3 + b^3 = (a+b)(a^2-ab+b^2)$.

$$2(r+1)(r^2-r+1) + 3r(r+1) = 0$$

$$(r+1)(2r^2-2r+2+3r) = (r+1)(2r^2+r+2) = 0. \Rightarrow r+1=0 \text{ or } 2r^2+r+2=0$$

$$\Rightarrow r=-1 \text{ or } r = \frac{-1 \pm \sqrt{1^2 - 4\times2\times2}}{4} = -\frac{1}{4} \pm \frac{\sqrt{-15}}{4} = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$$

$$y = C_1 e^{-t} + C_2 e^{-\frac{1}{4}t}\cos\left(\frac{\sqrt{15}}{4}t\right) + C_3 e^{-\frac{1}{4}t}\sin\left(\frac{\sqrt{15}}{4}t\right).$$

Example: $y^{(4)} + 16y = 0$ Aux. eqn. $r^4 + 16 = 0. \Rightarrow r^4 = -16 = 16 \cdot e^{i(\pi + 2k\pi)}$.

$-1 = e^{i\pi}$ $\quad e^{i\theta} = e^{i(\theta + 2k\pi)}$

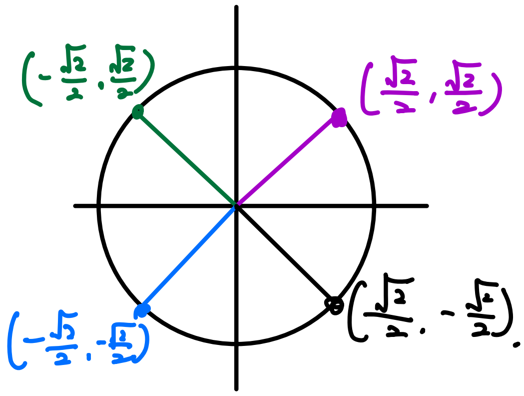
$$r^4 = 16 e^{i(\pi+2k\pi)} \Rightarrow r = \left(16 e^{i(\pi+2k\pi)}\right)^{\frac{1}{4}}$$

$(ab)^c = a^c b^c$.

$$= 16^{\frac{1}{4}} \cdot e^{i(\pi+2k\pi)\cdot\frac{1}{4}}$$

$(a^b)^c = a^{bc}$

$$r = 2 e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)} = \begin{cases} 2 e^{i\left(\frac{\pi}{4}\right)} & k=0 \\ 2 e^{i\left(\frac{3\pi}{4}\right)} & k=1 \\ 2 e^{i\left(\frac{5\pi}{4}\right)} & k=2 \\ 2 e^{i\left(\frac{7\pi}{4}\right)} & k=3 \end{cases}$$



$e^{i\theta} = \cos\theta + i\sin\theta$. $\quad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

$$= \begin{cases} 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \underline{\sqrt{2} + \sqrt{2} i} & k=0 \\ 2\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \underline{-\sqrt{2} + \sqrt{2} i} & k=1 \\ 2\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) = \underline{-\sqrt{2} - \sqrt{2} i} & k=2 \\ 2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right) = \underline{\sqrt{2} - \sqrt{2} i} & k=3. \end{cases}$$

$$y = C_1 e^{\sqrt{2}t}\cos\sqrt{2}t + C_2 e^{\sqrt{2}t}\sin\sqrt{2}t + C_3 e^{-\sqrt{2}t}\cos\sqrt{2}t + C_4 e^{-\sqrt{2}t}\sin\sqrt{2}t.$$

Problem 4b. Determine the final ansatz.

Midterm 2. $\quad y^{(4)} + 3y''' + 4y'' + 12y' = 2te^{-3t} + 6\cos 2t + e^{-t}\cos 3t.$

Aux. eqn. $r^4 + 3r^3 + 4r^2 + 12r = r^3(r+3) + 4r(r+3) = (r+3)(r^3+4r) = r(r+3)(r^2+4) = 0.$

$\Rightarrow r_1 = 0,\ r_2 = -3,\ r_3 = 2i,\ r_4 = -2i.$

$g_1(t) = 2te^{-3t}$. First ansatz: $Y_1 = e^{-3t}(At + B)$ exp. coeff. $= -3$. single root

Final ansatz: $Y_1 = e^{-3t}t(At+B) = e^{-3t}(At^2 + Bt)$.

$g_2(t) = 6\cos 2t$. First ansatz $Y_2 = C\cos 2t + D\sin 2t$. exp. coeff. $= 2i$. single root

Final ansatz $Y_2 = Ct\cos 2t + Dt\sin 2t$.

$g_3(t) = e^{-t}\cos 3t$. First ansatz $Y_3 = e^{-t}(E\cos 3t + F\sin 3t)$ exp. coeff. $= -1+3i$ not a root.

Final ansatz. $Y_3 = e^{-t}(E\cos 3t + F\sin 3t)$.

$$Y = e^{-2t}(At^2+Bt) + Ct\cos 2t + Dt\sin 2t + Ee^{-t}\cos 3t + Fe^{-t}\sin 3t.$$

Problem 5. Gen. sol'n of linear ODE w/ const. coeffs.

Midterm 2. $y''' + 2y'' + 2y' = \underset{0}{2t} + \underset{0}{6} + \underset{i}{3\cos t} - \underset{i}{\sin t}.$

Aux eqn. : $r^3 + 2r^2 + 2r = r(r^2 + 2r + 2) = 0. \Rightarrow r_1 = 0,\ r_2 = -1+i,\ r_3 = -1-i.$

Comp. sol'n: $y_c = C_1 + C_2 e^{-t}\cos t + C_3 e^{-t}\sin t.$

$g_1(t) = 2t+6.$ First ansatz $= At+B.$ exp. coeff. $=0$ single root.

Final ansatz $Y_1 = At^2 + Bt.$ $Y_1' = 2At+B,$ $Y_1'' = 2A,$ $Y_1''' = 0.$

$Y_1''' + 2Y_1'' + 2Y_1' = 0 + 2\cdot 2A + 2(2At+B) = 4At + 4A + 2B = 2t+6$

$4A = 2.$ $4A + 2B = 6 \Rightarrow$ $A = \frac{1}{2},$ $B = 2.$ $Y_1 = \frac{1}{2}t^2 + 2t.$

$g_2(t) = 3\cos t - \sin t.$ First ansatz $= C\cos t + D\sin t.$ exp. coeff. $= i,$ not a root.

$Y_2 = C\cos t + D\sin t,$ $Y_2' = -C\sin t + D\cos t,$

$Y_2'' = -C\cos t - D\sin t,$ $Y_2''' = C\sin t - D\cos t.$

$$Y_2''' + 2Y_2'' + 2Y_2' = \cos t(-D - 2C + 2D) + \sin t(C - 2D - 2C)$$

$$= (-2C + D)\cos t + (-C - 2D)\sin t = 3\cos t - \sin t.$$

$$\begin{cases} -2C + D = 3 \\ -C - 2D = -1 \end{cases} \Rightarrow \begin{cases} C = -1 \\ D = 1 \end{cases} \qquad Y_2 = -\cos t + \sin t.$$

$$y = C_1 + C_2 e^{-t}\cos t + C_3 e^{-t}\sin t + \frac{1}{2}t^2 + 2t + \sin t - \cos t.$$