First-order deformations of freely generated vertex algebras

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Outline

First-order deformations: classical version

Let A be a commutative associative algebra over $\mathbb{C}$ with a unit element $\mathbf{1}$. Let $m : A \otimes_{\mathbb{C}} A \to A$ be its multiplication.

Definition

A first-order deformation of A is defined by a linear map $m_1 : A \otimes_{\mathbb{C}} A \to A$, such that the vector space $A^t = \mathbb{C}[t]/(t^2) \otimes_{\mathbb{C}} A$ with the multiplication

$$m^t = m + tm_1$$

forms a commutative associative algebra with the unit element $\mathbf{1}$ over the base ring $\mathbb{C}[t]/(t^2)$.

Remark

Commutativity implies $m_1(a \otimes b) = m_1(b \otimes a)$. Associativity implies $m_1(a \otimes m(b \otimes c)) + m(a \otimes m_1(b \otimes c)) = m_1(m(a \otimes b) \otimes c) + m(m_1(a \otimes b) \otimes c)$.

Equivalent first-order deformations

Definition

Two first-order deformations $(A^t, m_{(1)}^t, \mathbf{1})$ and $(A^t, m_{(2)}^t, \mathbf{1})$ are equivalent, if there exists a $\mathbb{C}[t]/(t^2)$-linear isomorphism $f^t : A^t \to A^t$ whose restriction on A is of the form

$$f^t|_A = 1_A + tf_1,$$

where $f_1 : A \to A$ is a $\mathbb{C}$-linear map.

Remark

If $m_1 : A \otimes A \to A$ is given by the formula

$$m_1(a \otimes b) = m(a \otimes f_1(b)) - f_1(m(a \otimes b)) + m(f_1(a) \otimes b),$$

then the first-order deformation given by m_1 is trivial, i.e, equivalent to that given by the zero map.

Second Harrison cohomology

Definition

- A 2-cocycle of A is a linear map $\Phi : A \otimes A \to A$ such that for $a, b, c \in A$,

$$\Phi(a \otimes b) = \Phi(b \otimes a),$$
$$a \cdot \Phi(b \otimes c) - \Phi(ab \otimes c) + \Phi(a \otimes bc) - \Phi(a \otimes b) \cdot c = 0;$$

- A 2-coboundary of A is a linear map $\Phi : A \otimes A \to A$ given by

$$\Phi(a \otimes b) = a \cdot f(b) - f(a \cdot b) + f(a) \cdot b,$$

  where $f : A \to A$ is a $\mathbb{C}$-linear map.
- The second Harrison cohomology $H^2(A, A)$ is the quotient of the space of 2-cocycles modulo the subspace of 2-coboundaries.

Theorem

$H^2(A, A)$ corresponds bijectively to the set of equivalence classes of first-order deformations of A (as commutative associative algebras).

First-order deformation: vertex version

Let $(V, Y, \mathbf{1})$ be a grading-restricted vertex algebra.

Definition

A first-order deformation of V is defined by a linear map $Y_1 : V \otimes_{\mathbb{C}} V \to V((x))$, such that the vector space $V^t = \mathbb{C}[t]/(t^2) \otimes_{\mathbb{C}} V$ with the vertex operator

$$Y^t = Y + tY_1$$

and the vacuum $\mathbf{1}$ forms a grading-restricted vertex algebra over the base ring $\mathbb{C}[t]/(t^2)$.

For $u, v \in V$, write

$$Y_1(u, x)v = \sum_{n \in \mathbb{Z}} u_n^{def} v x^{-n-1}.$$

Then Y_1 should satisfy a series of conditions.

Requirements on Y_1

1. The **d**-commutator formula implies that for $u, v \in V, n \in \mathbb{Z}$,

$$\mathrm{wt}\ u_n^{def} v = \mathrm{wt}\ u - n - 1 + \mathrm{wt}\ v.$$

2. The identity and creation properties implies that for $v \in V$,

$$Y_1(\mathbf{1}, x)v = 0, Y_1(v, x)\mathbf{1} = 0.$$

3. The D-derivative-commutator formula implies that for $v \in V$,

$$[D, Y_1(v, x)] = Y_1(Dv, x) = \frac{d}{dx} Y_1(v, x).$$

4. Skew-symmetry implies that for $u, v \in V$,

$$Y_1(u, x)v = e^{xD} Y_1(v, -x)u.$$

Requirements on Y_1 continued

5. The convergence of products requires that for $v' \in V', u_1, u_2, v \in V$,

$$\langle v', Y_1(u_1, z_1)Y(u_2, z_2)v\rangle + \langle v', Y(u_1, z_1)Y_1(u_2, z_2)v\rangle$$

converges absolutely in the region $|z_1| > |z_2| > 0$.

6. The convergence of iterates requires that for $v' \in V', u_1, u_2, v \in V$,

$$\langle v', Y_1(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle + \langle v', Y(Y_1(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

converges absolutely in the region $|z_2| > |z_1 - z_2| > 0$.

7. Associativity and rationality requires that the series in (5) and (6) converge to the same rational function with the only possible poles at $z_1 = 0, z_2 = 0$, and $z_1 = z_2$.

Remarks on Y_1 - weaker convergence requirement

Remark

For $v' \in V$, $u_1, u_2, v \in V$, The convergence is required on the sums

$$\langle v', Y_1(u_1, z_1)Y(u_2, z_2)v\rangle + \langle v', Y(u_1, z_1)Y_1(u_2, z_2)v\rangle$$

and

$$\langle v', Y_1(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle + \langle v', Y(Y_1(u_1, z_1 - z_2)u_2, z_2)v\rangle.$$

The individual summands

$$\langle v', Y_1(u_1, z_1)Y(u_2, z_2)v\rangle, \langle v', Y(u_1, z_1)Y_1(u_2, z_2)v\rangle,$$

$$\langle v', Y_1(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle, \langle v', Y(Y_1(u_1, z_1 - z_2)u_2, z_2)v\rangle.$$

generally do not have to converge.

Remarks on Y_1 - Jacobi-like identity

Remark

For $u, v \in V$, Y_1 also satisfies the following variant of Jacobi identity

$$
\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)\Big(Y_1(u, x_1)Y(v, x_2) + Y(u, x_1)Y_1(v, x_2)\Big) \\
& \quad + x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)\Big(Y_1(v, x_2)Y(u, x_1) + Y(v, x_2)Y_1(u, x_1)\Big) \\
& = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\Big(Y_1(Y(u, x_0)v, x_2) + Y(Y_1(u, x_0)v, x_2)\Big).
\end{aligned}
$$

If we write $Y_1(u, x) = \sum_{n\in\mathbb{Z}} u_n^{def} x^{-n-1}$, then for every $u, v \in V$, $m, n \in \mathbb{Z}$, the following commutator condition holds:

$$
[u_m^{def}, v_n] + [u_m, v_n^{def}] = \sum_{\alpha=0} \binom{m}{\alpha}\left(\left(u_\alpha^{def} v\right)_{m+n-\alpha} + (u_\alpha v)_{m+n-\alpha}^{def}\right).
$$

Equivalent first-order deformations

Definition

Two first-order deformations $(V^t, Y^t_{(1)}, \mathbf{1})$ and $(V^t, Y^t_{(2)}, \mathbf{1})$ are equivalent, if there exists a $\mathbb{C}[t]/(t^2)$-linear isomorphism $f^t : V^t \to V^t$ whose restriction on A is of the form

$$f^t|_V = 1_V + t\phi,$$

where $\phi : V \to V$ is a grading-preserving $\mathbb{C}$-linear map.

Remark

If $Y_1 : V \otimes V \to V((x))$ is given by the formula

$$Y_1(u,x)v = Y(\phi(u),x)v - \phi(Y(u,x)v) + Y(u,x)\phi(v),$$

then the first-order deformation given by Y_1 is trivial, i.e, equivalent to that given by the zero map.

2-cochains: basic formulation

Let $\widetilde{V}_{z_1z_2}$ be the $\overline{V}$-valued rational function, i.e., the space of functions $f : \{(z_1, z_2) \in \mathbb{C}^2 : z_1 \neq z_2\} \to \overline{V}$ such that for every $v' \in V'$, $\langle v', f(z_1, z_2)\rangle$ is a rational function with the only possible pole at $z_1 = z_2$. Let $\Psi : V \otimes V \to \widetilde{V}_{z_1z_2}$ be a linear map. We use the notation

$$\Psi(v_1 \otimes v_2; z_1, z_2)$$

for the image of Ψ evaluated at z_1, z_2 in $\overline{V}$.
We consider only the linear maps Ψ compatible with the $\mathbf{d}$- and D-operators, i.e., for $v_1, v_2 \in V, a \in \mathbb{C}^\times$,

1. $a^{\mathbf{d}}\Psi(v_1 \otimes v_2; z_1, z_2) = \Psi(a^{\mathbf{d}}v_1 \otimes a^{\mathbf{d}}v_2; a^{-1}z_1, a^{-1}z_2)$
2. $\Psi(Dv_1 \otimes v_2; z_1, z_2) = \frac{\partial}{\partial z_1}\Psi(v_1 \otimes v_2; z_1, z_2)$,
   $\Psi(v_1 \otimes Dv_2; z_1, z_2) = \frac{\partial}{\partial z_2}\Psi(v_1 \otimes v_2; z_1, z_2)$,
   $D\Psi(v_1 \otimes v_2; z_1, z_2) = \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right)\Psi(v_1 \otimes v_2; z_1, z_2)$,

2-cochains: 1/2-composable condition

Definition

Ψ is 1/2-composable if for every $v' \in V', v_1, v_2, v_3 \in V$,

$$\langle v', Y(v_1, z_1)\Psi(v_2 \otimes v_3; z_2, z_3)\rangle + \langle v', \Psi(v_1 \otimes Y(v_2, z_2 - \zeta)Y(v_3, z_3 - \zeta)\mathbf{1}; z_1, \zeta)\rangle$$

and

$$\langle v', \Psi(Y(v_1, z_1 - \zeta)Y(v_2, z_2 - \zeta)\mathbf{1} \otimes v_3; \zeta, z_3)\rangle + \langle v', Y(v_3, z_3)\Psi(v_1 \otimes v_2; z_1, z_2)\rangle$$

converge absolutely in appropriate regions to a rational function with the only possible poles at $z_1 = z_2, z_1 = z_3$, and $z_2 = z_3$.

Remark

Again the individual summands are not required to be convergent.

Second Harrison-Huang cohomology

Definition

- A 2-cocycle of V is a linear map $\Psi : V \otimes V \to \widetilde{V}_{z_1 z_2}$, such that Ψ is compatible with $\mathbf{d}$- and D-operators, $1/2$-composable with the two series in the $1/2$-composable condition converging absolutely to the same rational function, and has the symmetry
$$\Psi(v_1 \otimes v_2; z_1, z_2) = \Psi(v_2 \otimes v_1; z_2, z_1).$$
- A 2-coboundary of V is a linear map $\Psi : V \otimes V \to \widetilde{V}_{z_1 z_2}$ given by
$$\begin{aligned}\Psi(a \otimes b; z_1, z_2) = \ & Y(a, z_1) Y(\phi(b), z_2)\mathbf{1} \\ & - \phi(Y(a, z_1) Y(b, z_2)\mathbf{1}) \\ & + Y(\phi(a), z_1) Y(b, z_2)\mathbf{1},\end{aligned}$$
where $\phi : V \to V$ is a grading-preserving $\mathbb{C}$-linear map.
- The second Harrison-Huang cohomology $H^2_{1/2}(V, V)$ is the quotient of the space of 2-cocycles modulo the subspace of 2-coboundaries.

Second Harrison-Huang cohomology

Theorem (Huang, 2012)

$H^2_{1/2}(V, V)$ corresponds bijectively to the set of equivalence classes of first-order deformations of V (as vertex algebras).

Remark

In contrast, there is a cohomology $H^2_\infty(V, V)$ defined by a ∞-composable condition, which requires a much stronger convergence, namely, for $l, m, n \in \mathbb{N}$, $v_1, ..., v_{l+m+n} \in V, v' \in V'$

$$\begin{aligned}&\langle v', Y(v_1, z_1) \cdots Y(v_l, z_l)\\ &\quad \cdot \Psi(Y(v_1, z_{l+1} - \zeta) \cdots Y(v_{l+m}, z_{l+m-\zeta})\mathbf{1}\\ &\qquad \otimes Y(v_{l+m+1}, z_{l+m+1} - \eta) \cdots Y(v_{l+m+n}, z_{l+m+n-\eta})\mathbf{1}; \zeta, \eta)\rangle\end{aligned}$$

converge absolutely in an appropriate region. The corresponding cohomology is denoted by $H^2_\infty(V, V)$.

Freely generated vertex algebras

Definition

Let $S = \{a^{(1)}, ..., a^{(r)}\}$ be a finite subset of V of homogeneous elements of positive weights. We say that V is freely generated by S, if V admits a PBW basis of the form

$$a^{(1)}_{-m_1^{(1)}} \cdots a^{(1)}_{-m_{k_1}^{(1)}} \cdots a^{(r)}_{-m_1^{(r)}} \cdots a^{(r)}_{-m_{k_r}^{(r)}} \mathbf{1}$$

where $k_1, ..., k_r \in \mathbb{N}$, $m_1^{(j)} \geq \cdots \geq m_{k_j}^{(j)} \geq 1$ for each $j = 1, ..., r$.

To classify the first-order deformations of such vertex algebras. It suffices to determine the singular parts of the $Y_1(a^{(i)}, x)a^{(j)}$, namely, $(a^{(i)})_m^{def} a^{(j)}$ for $i, j = 1, ..., r, m \geq 0$. It suffices to check that the commutator condition on these singular parts. Also, in this case, we have

$$H^2_{1/2}(V, V) = H^2_\infty(V, V).$$

Supporting Theorem 1

Let V be a vertex algebra freely generated by $\{a^{(1)}, ..., a^{(r)}\}$.
Let Y_1 be the operator associated with a first-order deformation.

Proposition

For fixed $i, j = 1, ..., r$, if $Y_1(a^{(i)}, x)a^{(j)}$ is regular, then there exists a grading-preserving $\mathbb{C}$-linear map $\phi : V \to V$ such that

$$Y_1(a^{(i)}, x)a^{(j)} = Y_\phi(a^{(i)}, x)a^{(j)}.$$

Theorem

Upon a trivial deformation Y_ϕ, one may assume that for $i, j = 1, ..., r$,

$$Y_1(a^{(i)}, x)a^{(j)} = Y_1^-(a^{(i)}, x)a^{(j)} + \frac{1}{2}\left(e^{xD}Y_1^-(a^{(j)}, -x)a^{(i)} - Y_1^-(a^{(i)}, x)a^{(j)}\right).$$

Supporting Theorem 1 - Remarks

Theorem

Upon a trivial deformation Y_ϕ, one may assume that for $i, j = 1, ..., r$,

$$Y_1(a^{(i)}, x)a^{(j)} = Y_1^-(a^{(i)}, x)a^{(j)} + \frac{1}{2}\left(e^{xD}Y_1^-(a^{(j)}, -x)a^{(i)} - Y_1^-(a^{(i)}, x)a^{(j)}\right).$$

Remark

- The fact Y_1 being uniquely determined by its singular part is crucial in proving that $H^2_{1/2}(V, V) = H^2_\infty(V, V)$.
- When there exists relations among the PBW basis, the previous proposition and the supporting theorem fail. The failure of the proof clearly indicates the the direction for the generalization towards strongly generated vertex algebras.

Generating function approach

Instead of focusing on normal-ordered products of the generators, we start by focusing on the series

$$Y_1(s_1, z_1)Y(s_2, z_2)s_3$$

for $s_1, s_2, s_3 \in S$ of minimal weights. Recall from the product side that

$$Y_1(s_1, z_1)Y(s_2, z_2)s_3 + Y(s_1, z_1)Y_1(s_2, z_2)s_3$$

converges absolutely to a $\overline{V}$-valued rational function.
Observe: $Y_1(s_i, z_2)s_j$ is uniquely determined by its singular parts.
Thus, $Y(s_1, z_1)Y_1(s_2, z_2)s_3$ converges.
Conclusion: $Y_1(s_1, z_1)Y(s_2, z_2)s_3$ also converges!
Generally, $Y_1(s_1, z_1)Y(s_2, z_2)\cdots Y(s_n, z_n)s_{n+1}$ converges for all $s_1, ..., s_n \in S$.

Generating function approach

Write

$$Y_1(s_1, z_1)Y(s_2, z_2)s_3 = Y_1(s_1, z_1)Y^+(s_2, z_2)s_3 + Y_1(s_1, z_1)Y^-(s_2, z_2)s_3$$

Observe: $Y^-(s_2, z_2)s_3$ involves finitely many terms.
Thus, $Y_1(s_1, z_1)Y^-(s_2, z_2)s_3$ also converges.
Conclusion: $Y_1(s_1, z_1)Y^+(s_2, z_2)s_3$ also converges!
We use the E-notation for the analytic continuation of the series.
With the convergence, we may set

$$E\left(Y_1(s_1, z_1)Y^+(s_2, z_2)s_3\right) = \sum_i \frac{p_i^{(123)}(z_1, z_2)}{z_1^{q_1}(z_1 - z_2)^{q_{12}}} b_i.$$

where $\{b_i\}$ is a basis of V, $p_i(z_1, z_2)$ are polynomial, $q_1, q_2, q_{12} \in \mathbb{N}$.

Iterate side

Recall that the assocativity requires the iterate side

$$Y_1(Y(s_1, z_1 - z_2)s_2, z_2)s_3 + Y(Y_1(s_1, z_1 - z_2)s_2, z_2)s_3$$

to converge absolutely to the same $\overline{V}$-valued rational function as that from the product side.
Apply skew-symmetry of Y_1 and Y,

$$e^{z_2 D}Y_1(s_3, -z_2)Y(s_1, z_1 - z_2)s_2 + e^{z_2 D}Y(s_3, -z_2)Y_1(s_1, z_1 - z_2)s_2$$

Since $e^{z_2 D}$ does not interfere with the convergence, we know that

$e^{z_2 D}Y_1(s_3, -z_2)Y(s_1, z_1 - z_2)s_2$ and thus $e^{z_2 D}Y_1(s_3, -z_2)Y^+(s_1, z_1 - z_2)s_2$

converge. We may set

$$E\left(e^{z_2 D}Y_1(s_3, -z_2)Y^+(s_1, z_1 - z_2)s_2\right) = \sum_i \frac{p^{(312)}(z_1, z_2)}{(-z_2)^{q_2}(-z_1)^{q_{12}}} e^{z_2 D} b_i$$

Cocycle equation

We then obtain the following cocycle equation concerning Y_1

$$\begin{aligned}
&E\Big(Y_1(s_1,z_1)Y^+(s_2,z_2)s_3\Big)\\
&\quad + E\Big(Y_1(s_1,z_1)Y^+(s_2,z_2)s_3\Big) + \Big(Y(s_1,z_1)Y_1(s_2,z_2)s_3\Big)\\
&= E\Big(e^{z_2D}Y_1(s_3,-z_2)Y^+(s_1,z_1-z_2)s_2\Big)\\
&\quad + E\Big(e^{z_2D}Y_1(s_3,-z_2)Y^-(s_1,z_1-z_2)s_2\Big)\\
&\quad + E\Big(e^{z_2D}Y(s_3,-z_2)Y_1(s_1,z_1-z_2)s_2\Big),
\end{aligned}$$

together with the skew-symmetry

$$Y_1(s_i,z)s_j = e^{zD}Y(s_j,-z)s_i.$$

Support Theorem 2

By permuting the indices $(1 \to 2 \to 3 \to 1)$, the cocycle equation and the skew-symmetry form a linear nonhomogeneous system of equations concerning the coefficients of the polynomial functions
$p_i^{(123)}(z_1, z_2), p_i^{(231)}(z_1, z_2), p_i^{(312)}(z_1, z_2)$.
The corresponding homogeneous system is

$$E\left(Y_1(s_i, z_1)Y^+(s_j, z_2)s_k\right) = E\left(e^{z_2 D}Y_1(s_k, -z_2)Y^+(s_i, z_1 - z_2)s_j\right)$$

$$Y_1(s_i, z)s_j = e^{zD}Y(s_j, -z)s_i$$

with $(i, j, k) = (1, 2, 3), (2, 3, 1)$, and $(3, 1, 2)$.

Theorem

The complementary solution of the cocycle equation (i.e., the general solution of the homogeneous system) give trivial deformations Y_ϕ for some grading-preserving $\mathbb{C}$-linear map $\phi : V \to V$.

Support Theorem 2 - Remarks

Theorem

The complementary solution of the cocycle equation (i.e., the general solution of the homogeneous system) give trivial deformations Y_ϕ for some grading-preserving $\mathbb{C}$-linear map $\phi : V \to V$.

Remark

- The theorem extends to any number of variables and generators, through a technical induction process.
- Consequently, to solve the cocycle equation, it suffices to find a particular solution.
- The theorem does not hold when relations exist among the PBW basis vectors. Likewise, the failure of the proof clearly indicates the direction for generalizing towards strongly generated vertex algebras.

Modes of Y_1 on the PBW basis

Let $S = \{a^{(1)}, ..., a^{(r)}\}$. Write

$$Y_1(a^{(i)}, x)a^{(j)} = B(a^{(i)}, a^{(j)})\mathbf{1}x^{-\mathrm{wt}(a^{(i)})-\mathrm{wt}(a^{(j)})} + \sum_{m\in\mathbb{Z}} M_m(a^{(i)}, a^{(j)})x^{-m-1}$$

Also write

$$Y_1(a^{(i)}, x) = \sum_{n\in\mathbb{Z}} (a^{(i)})_n^{def} x^{-n-1} \in \mathrm{End}(V)[[x, x^{-1}]].$$

We define the Y_1-modes for $a^{(i)}$ as follows:

1. $(a^{(i)})_m^{def}\mathbf{1} = 0$ for every $m \in \mathbb{Z}$.
2. If $i \leq j$, then

$$(a^{(i)})_m^{def} a^{(j)} = \begin{cases} 0 & \text{if } m \geq \mathrm{wt}(a^{(i)}) + \mathrm{wt}(a^{(j)}) \\ B(a^{(i)}, a^{(j)})\mathbf{1} & \text{if } m = \mathrm{wt}(a^{(i)}) + \mathrm{wt}(a^{(j)}) - 1 \\ M_m(a^{(i)}, a^{(j)}) & \text{if } m < \mathrm{wt}(a^{(i)}) + \mathrm{wt}(a^{(j)}) - 1, \end{cases}$$

Modes of Y_1 on the PBW basis

3. If $i > j$, set
$$(a^{(i)})^{def}_m a^{(j)} = \mathrm{Res}_x x^m e^{xD} Y_1(a^{(j)}, -x)a^{(i)}.$$
$(a^{(i)})^{def}_m \mathbf{1} = 0$ for every $m \in \mathbb{Z}$.
4. If $m < 0$, $1 \leq i \leq r$, $1 \leq j_1 \leq \cdots \leq j_p \leq r$, set $(a^{(i)})^{def}_m a^{(j_1)}_{-n_1} \cdots a^{(j_p)}_{-n_p}\mathbf{1}$
$$= \frac{1}{2}\sum_{k=1}^{p} a^{(j_1)}_{-n_1} \cdots a^{(j_{k-1})}_{-n_{k-1}} \cdot \sum_{\alpha \geq 0} \binom{m}{\alpha} \left(((a^{(i)})^{def}_\alpha a^{(j_k)})_{m-n_k-\alpha} + (a^{(i)}_\alpha a^{(j_k)})^{def}_{m-n_k-\alpha} \right)$$
$$\cdot\, a^{(j_{k-1})}_{-n_{k-1}} \cdots a^{(j_p)}_{-n_p}\mathbf{1}.$$
5. If $m \geq 0$ and $1 \leq j_1 \leq \cdots \leq j_p \leq r$, set $(a^{(i)})^{def}_m a^{(j_1)}_{-n_1} \cdots a^{(j_p)}_{-n_p}\mathbf{1}$ as
$$a^{(j_1)}_{-n_1}(a^{(i)})^{def}_m a^{(j_2)}_{-n_2} \cdots a^{(j_p)}_{-n_p}\mathbf{1}$$
$$+ (a^{(j_1)})^{def}_{-n_1} a^{(i)}_m a^{(j_2)}_{-n_2} \cdots a^{(j_p)}_{-n_p}\mathbf{1} - a^{(i)}_m (a^{(j_1)})^{def}_{-n_1} (a^{(j_2)})_{-n_2} \cdots a^{(j_p)}_{-n_p}\mathbf{1}$$
$$+ \sum_{\alpha=0}^{\infty} \binom{m}{\alpha} \left(((a^{(i)})^{def}_\alpha a^{(j_1)})_{m-n_1-\alpha} + (a^{(i)}_\alpha a^{(j_1)})^{def}_{m-n_1-\alpha} \right) a^{(j_2)}_{-n_2} \cdots a^{(j_p)}_{-n_p}\mathbf{1}.$$

Main theorem

Theorem

The so-defined Y_1 satisfies the cocycle equation if the following commutator condition holds: for every $1 \leq i, j, k \leq r, m, n \in \mathbb{N}$,

$$
\begin{aligned}
&[(a^{(i)})^{def}_m, a^{(j)}_n]a^{(k)} + [a^{(i)}_m, (a^{(j)})^{def}_n]a^{(k)} \\
= &\sum_{\alpha=0}^{\infty} \binom{m}{\alpha} \left(\left((a^{(i)})^{def}_\alpha a^{(j)} \right)_{m+n-\alpha} a^{(k)} + \left(a^{(i)}_\alpha a^{(j)} \right)^{def}_{m+n-\alpha} a^{(k)} \right),
\end{aligned}
$$

In this case, $Y_1 : S \to End(V)[[x, x^{-1}]]$ uniquely extends to $Y_1 : V \to End(V)[[x, x^{-1}]]$. Moreover, $H^2_{1/2}(V, V) = H^2_\infty(V, V)$.

Remark

Note that the modes involved are all nonnegative. So starting from an ansatz of $Y_1^-(a^{(i)}, x)a^{(j)}$ with finitely many structural constants, there are only finitely many relations to check.

Examples

1. In case V is the universal Virasoro VOA, or the universal affine VOA, or the universal Zamolodchikov VOA, we computed that $\dim_{\mathbb{C}} H^2_{1/2}(V,V) = 1$.
2. In case V is the Heisenberg VOA of rank r of level l,

$$\dim_{\mathbb{C}} H^2_{1/2}(V,V) = \begin{cases} \binom{r}{3} & \text{if } l \neq 0, \\ 3\binom{r+1}{3} & \text{if } l = 0. \end{cases}$$

Future work in consideration

1. Infinitely many generators (universal two-paramter W_∞-algebra).
2. Passing to the quotients. Rigidity conjecture.
3. Cup products and Gerstenhaber brackets.
   Batalin-Vilkovisky structure. Deligne conjecture.
4. Noncommutative deformations.

The end

Thank you for your attention!