First-order deformations of freely generated vertex algebras

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Oct. 19, 2024

arXiv: 2408.16309.
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Outline

First-order deformations: classical version

Let A be a commutative associative algebra over $\mathbb{C}$ with a unit element $\mathbf{1}$. Let $m : A \otimes_{\mathbb{C}} A \to A$ be its multiplication.

Definition

A first-order deformation of A is defined by a linear map $m_1 : A \otimes_{\mathbb{C}} A \to A$, such that the vector space $A^t = \mathbb{C}[t]/(t^2) \otimes_{\mathbb{C}} A$ with the multiplication

$$m^t = m + tm_1$$

forms a commutative associative algebra with the unit element $\mathbf{1}$ over the base ring $\mathbb{C}[t]/(t^2)$.

Remark

Commutativity implies $m_1(a \otimes b) = m_1(b \otimes a)$. Associativity implies $m_1(a \otimes m(b \otimes c)) + m(a \otimes m_1(b \otimes c)) = m_1(m(a \otimes b) \otimes c) + m(m_1(a \otimes b) \otimes c)$.

Equivalent first-order deformations

Definition

Two first-order deformations $(A^t, m^t_{(1)}, \mathbf{1})$ and $(A^t, m^t_{(2)}, \mathbf{1})$ are equivalent, if there exists a $\mathbb{C}[t]/(t^2)$-linear isomorphism $f^t : A^t \to A^t$ whose restriction on A is of the form

$$f^t|_A = 1_A + tf_1,$$

where $f_1 : A \to A$ is a $\mathbb{C}$-linear map.

Remark

If $m_1 : A \otimes A \to A$ is given by the formula

$$m_1(a \otimes b) = m(a \otimes f_1(b)) - f_1(m(a \otimes b)) + m(f_1(a) \otimes b),$$

then the first-order deformation given by m_1 is trivial, i.e, equivalent to that given by the zero map.

Second Harrison cohomology

Definition

- A 2-cocycle of A is a linear map $\Phi : A \otimes A \to A$ such that for $a, b, c \in A$,

$$\Phi(a \otimes b) = \Phi(b \otimes a),$$
$$a \cdot \Phi(b \otimes c) - \Phi(ab \otimes c) + \Phi(a \otimes bc) - \Phi(a \otimes b) \cdot c = 0;$$

- A 2-coboundary of A is a linear map $\Phi : A \otimes A \to A$ given by

$$\Phi(a \otimes b) = a \cdot f(b) - f(a \cdot b) + f(a) \cdot b,$$

  where $f : A \to A$ is a $\mathbb{C}$-linear map.
- The second Harrison cohomology $H^2(A, A)$ is the quotient of the space of 2-cocycles modulo the subspace of 2-coboundaries.

Theorem

$H^2(A, A)$ corresponds bijectively to the set of equivalence classes of first-order deformations of A (as commutative associative algebras).

First-order deformation: vertex version

Let $(V, Y, \mathbf{1})$ be a grading-restricted vertex algebra.

Definition

A first-order deformation of V is defined by a linear map $Y_1 : V \otimes_{\mathbb{C}} V \to V((x))$, such that the vector space $V^t = \mathbb{C}[t]/(t^2) \otimes_{\mathbb{C}} V$ with the vertex operator

$$Y^t = Y + tY_1$$

and the vacuum $\mathbf{1}$ forms a grading-restricted vertex algebra over the base ring $\mathbb{C}[t]/(t^2)$.

For $u, v \in V$, write

$$Y_1(u, x)v = \sum_{n \in \mathbb{Z}} u_n^{def} v x^{-n-1}.$$

Then Y_1 should satisfy a series of conditions.

Requirements on Y_1

1. The **d**-commutator formula implies that for $u, v \in V, n \in \mathbb{Z}$,

$$\text{wt } u_n^{def} v = \text{wt } u - n - 1 + \text{wt } v.$$

2. The identity and creation properties implies that for $v \in V$,

$$Y_1(\mathbf{1}, x)v = 0, Y_1(v, x)\mathbf{1} = 0.$$

3. The D-derivative-commutator formula implies that for $v \in V$,

$$[D, Y_1(v, x)] = Y_1(Dv, x) = \frac{d}{dx} Y_1(v, x).$$

4. Skew-symmetry implies that for $u, v \in V$,

$$Y_1(u, x)v = e^{xD} Y_1(v, -x)u.$$

Requirements on Y_1 continued

5. The convergence of products requires that for $v' \in V', u_1, u_2, v \in V$,
$$\langle v', Y_1(u_1, z_1)Y(u_2, z_2)v\rangle + \langle v', Y(u_1, z_1)Y_1(u_2, z_2)v\rangle$$
converges absolutely in the region $|z_1| > |z_2| > 0$.
6. The convergence of iterates requires that for $v' \in V', u_1, u_2, v \in V$,
$$\langle v', Y_1(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle + \langle v', Y(Y_1(u_1, z_1 - z_2)u_2, z_2)v\rangle$$
converges absolutely in the region $|z_2| > |z_1 - z_2| > 0$.
7. Associativity and rationality requires that the series in (5) and (6) converge to the same rational function with the only possible poles at $z_1 = 0, z_2 = 0$, and $z_1 = z_2$.

Remarks on Y_1 - weaker convergence requirement

Remark

For $v' \in V$, $u_1, u_2, v \in V$, The convergence is required on the sums

$$\langle v', Y_1(u_1, z_1)Y(u_2, z_2)v\rangle + \langle v', Y(u_1, z_1)Y_1(u_2, z_2)v\rangle$$

and

$$\langle v', Y_1(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle + \langle v', Y(Y_1(u_1, z_1 - z_2)u_2, z_2)v\rangle.$$

The individual summands

$$\langle v', Y_1(u_1, z_1)Y(u_2, z_2)v\rangle, \langle v', Y(u_1, z_1)Y_1(u_2, z_2)v\rangle,$$

$$\langle v', Y_1(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle, \langle v', Y(Y_1(u_1, z_1 - z_2)u_2, z_2)v\rangle.$$

generally do not have to converge.

Equivalent first-order deformations

Definition

Two first-order deformations $(V^t, Y^t_{(1)}, \mathbf{1})$ and $(V^t, Y^t_{(2)}, \mathbf{1})$ are equivalent, if there exists a $\mathbb{C}[t]/(t^2)$-linear isomorphism $f^t : V^t \to V^t$ whose restriction on A is of the form

$$f^t|_V = 1_V + t\phi,$$

where $\phi : V \to V$ is a grading-preserving $\mathbb{C}$-linear map.

Remark

If $Y_1 : V \otimes V \to V((x))$ is given by the formula

$$Y_1(u, x)v = Y(\phi(u), x)v - \phi(Y(u, x)v) + Y(u, x)\phi(v),$$

then the first-order deformation given by Y_1 is trivial, i.e, equivalent to that given by the zero map.

Second Harrison-Huang cohomology

Using analytic continuation and a technical convergence condition (called 1/2-composable condition), Yi-Zhi Huang defined the vertex analogue of Harrison cohomology, denoted by $H^2_{1/2}(V, V)$, and proved the following theorem.

Theorem (Huang, 2012)

$H^2_{1/2}(V, V)$ corresponds bijectively to the set of equivalence classes of first-order deformations of V (as vertex algebras).

Remark

There is a cohomology $H^2_\infty(V, V)$ defined by a much stronger convergence condition (called ∞-composable condition). This work proves that when V is a freely generated vertex algebra, then $H^2_\infty(V, V) = H^2_{1/2}(V, V)$.

Freely generated vertex algebras

Definition

Let $S = \{a^{(1)}, ..., a^{(r)}\}$ be a finite subset of V of homogeneous elements of positive weights. We say that V is freely generated by S, if V admits a PBW basis of the form

$$a^{(1)}_{-m^{(1)}_1} \cdots a^{(1)}_{-m^{(1)}_{k_1}} \cdots a^{(r)}_{-m^{(r)}_1} \cdots a^{(r)}_{-m^{(r)}_{k_r}} \mathbf{1}$$

where $k_1, ..., k_r \in \mathbb{N}$, $m^{(j)}_1 \geq \cdots \geq m^{(j)}_{k_j} \geq 1$ for each $j = 1, ..., r$.

Proposition

For fixed $i, j = 1, ..., r$, if $Y_1(a^{(i)}, x)a^{(j)}$ is regular, then there exists a grading-preserving $\mathbb{C}$-linear map $\phi : V \to V$ such that

$$Y_1(a^{(i)}, x)a^{(j)} = Y_\phi(a^{(i)}, x)a^{(j)}.$$

Supporting Theorem 1 - Remarks

Theorem

Upon a trivial deformation Y_ϕ, one may assume that for $i, j = 1, ..., r$,

$$Y_1(a^{(i)}, x)a^{(j)} = Y_1^-(a^{(i)}, x)a^{(j)} + \frac{1}{2}\left(e^{xD}Y_1^-(a^{(j)}, -x)a^{(i)} - Y_1^-(a^{(i)}, x)a^{(j)}\right).$$

Remark

- When there exists relations among the PBW basis, the previous proposition and the supporting theorem fail. The failure of the proof clearly indicates the the direction for the generalization towards strongly generated vertex algebras.
- The fact Y_1 being uniquely determined by its singular part is crucial in proving that $H^2_{1/2}(V, V) = H^2_\infty(V, V)$. In particular, with the convergence of $Y_1(s_1, z_1)Y(s_2, z_2)s_3 + Y(s_1, z_1)Y_1(s_2, z_2)s_3$, we have

$$Y_1(s_1, z_1)Y(s_2, z_2)s_3 \text{ converges.}$$

Generating function approach

Write

$$Y_1(s_1, z_1)Y(s_2, z_2)s_3 = Y_1(s_1, z_1)Y^+(s_2, z_2)s_3 + Y_1(s_1, z_1)Y^-(s_2, z_2)s_3$$

Observe: $Y^-(s_2, z_2)s_3$ involves finitely many terms.
Thus, $Y_1(s_1, z_1)Y^-(s_2, z_2)s_3$ also converges.
Conclusion: $Y_1(s_1, z_1)Y^+(s_2, z_2)s_3$ also converges!
We use the E-notation for the analytic continuation of the series.
With the convergence, we may set

$$E\left(Y_1(s_1, z_1)Y^+(s_2, z_2)s_3\right) = \sum_i \frac{p_i^{(123)}(z_1, z_2)}{z_1^{q_1}(z_1 - z_2)^{q_{12}}} b_i.$$

where $\{b_i\}$ is a basis of V, $p_i^{(123)}(z_1, z_2)$'s are polynomials, $q_1, q_{12} \in \mathbb{N}$.

Cocycle equation

If we use skew-symmetry to reorganize the iterate side of the associativity requirement, we obtain the following cocycle equation concerning Y_1

$$E\Big(Y_1(s_1,z_1)Y^+(s_2,z_2)s_3\Big)+E\Big(Y_1(s_1,z_1)Y^-(s_2,z_2)s_3\Big)+E\Big(Y(s_1,z_1)Y_1(s_2,z_2)s_3\Big)$$

$$=E\Big(e^{z_2D}Y_1(s_3,-z_2)Y^+(s_1,z_1-z_2)s_2\Big)+E\Big(e^{z_2D}Y_1(s_3,-z_2)Y^-(s_1,z_1-z_2)s_2\Big)$$

$$+E\Big(e^{z_2D}Y(s_3,-z_2)Y_1(s_1,z_1-z_2)s_2\Big),$$

together with the skew-symmetry

$$Y_1(s_i,z)s_j=e^{zD}Y(s_j,-z)s_i.$$

where

$$E\Big(Y_1(s_1,z_1)Y^+(s_2,z_2)s_3\Big)=\sum_i\frac{p_i^{(123)}(z_1,z_2)}{z_1^{q_1}(z_1-z_2)^{q_{12}}}b_i,$$

$$E\Big(e^{zD}Y_1(s_3,-z_2)Y^+(s_1,z_1-z_2)s_3\Big)=\sum_i\frac{p_i^{(312)}(-z_2,z_1-z_2)}{(-z_2)^{q_2}(-z_1)^{q_{12}}}e^{zD}b_i.$$

with $p_i^{(123)}$'s and $p_i^{(312)}$'s are polynomial functions, $q_1,q_2,q_{12}\in\mathbb{N}$.

Support Theorem 2

By permuting the indices $(1 \to 2 \to 3 \to 1)$, the cocycle equation and the skew-symmetry form a linear nonhomogeneous system of equations about the coefficients of the polynomial functions $p_i^{(123)}, p_i^{(231)}$, and $p_i^{(312)}$. The corresponding homogeneous system is

$$E\left(Y_1(s_i, z_1)Y^+(s_j, z_2)s_k\right) = E\left(e^{z_2 D}Y_1(s_k, -z_2)Y^+(s_i, z_1 - z_2)s_i\right)$$
$$Y_1(s_i, z)s_j = e^{zD}Y(s_j, -z)s_i$$

with $(i, j, k) = (1, 2, 3), (2, 3, 1)$, and $(3, 1, 2)$.

Theorem

The complementary solution of the cocycle equation (i.e., the general solution of the homogeneous system) gives a trivial deformation Y_ϕ for some grading-preserving $\mathbb{C}$-linear map $\phi : V \to V$.

Support Theorem 2 - Remarks

Theorem

The complementary solution of the cocycle equation (i.e., the general solution of the homogeneous system) gives a trivial deformation Y_ϕ for some grading-preserving $\mathbb{C}$-linear map $\phi : V \to V$.

Remark

- The theorem extends to any number of variables and generators, through a technical induction process.
- Consequently, to solve the cocycle equation, it suffices to find a particular solution.
- The theorem does not hold when relations exist among the PBW basis vectors. Likewise, the failure of the proof clearly indicates the direction for generalizing towards strongly generated vertex algebras.

Modes of Y_1 on the PBW basis

Let $S = \{a^{(1)}, ..., a^{(r)}\}$. Write

$$Y_1(a^{(i)}, x)a^{(j)} = B(a^{(i)}, a^{(j)})\mathbf{1}x^{-\mathrm{wt}(a^{(i)})-\mathrm{wt}(a^{(j)})} + \sum_{m\in\mathbb{Z}} M_m(a^{(i)}, a^{(j)})x^{-m-1}$$

Also write

$$Y_1(a^{(i)}, x) = \sum_{n\in\mathbb{Z}} (a^{(i)})_n^{def} x^{-n-1} \in \mathrm{End}(V)[[x, x^{-1}]].$$

We define the Y_1-modes for $a^{(i)}$ as follows:

1. $(a^{(i)})_m^{def}\mathbf{1} = 0$ for every $m \in \mathbb{Z}$.
2. If $i \leq j$, then

$$(a^{(i)})_m^{def} a^{(j)} = \begin{cases} 0 & \text{if } m \geq \mathrm{wt}(a^{(i)}) + \mathrm{wt}(a^{(j)}) \\ B(a^{(i)}, a^{(j)})\mathbf{1} & \text{if } m = \mathrm{wt}(a^{(i)}) + \mathrm{wt}(a^{(j)}) - 1 \\ M_m(a^{(i)}, a^{(j)}) & \text{if } m < \mathrm{wt}(a^{(i)}) + \mathrm{wt}(a^{(j)}) - 1, \end{cases}$$

Modes of Y_1 on the PBW basis

3. If $i > j$, set
$$(a^{(i)})^{def}_m a^{(j)} = \mathrm{Res}_x x^m e^{xD} Y_1(a^{(j)}, -x)a^{(i)}.$$
$(a^{(i)})^{def}_m \mathbf{1} = 0$ for every $m \in \mathbb{Z}$.
4. If $m < 0$, $1 \le i \le r$, $1 \le j_1 \le \cdots \le j_p \le r$, set $(a^{(i)})^{def}_m a^{(j_1)}_{-n_1} \cdots a^{(j_p)}_{-n_p}\mathbf{1}$
$$= \frac{1}{2}\sum_{k=1}^{p} a^{(j_1)}_{-n_1}\cdots a^{(j_{k-1})}_{-n_{k-1}} \cdot \sum_{\alpha\ge 0}\binom{m}{\alpha}\left(((a^{(i)})^{def}_{\alpha} a^{(j_k)})_{m-n_k-\alpha} + (a^{(i)}_{\alpha} a^{(j_k)})^{def}_{m-n_k-\alpha}\right)$$
$$\cdot\, a^{(j_{k-1})}_{-n_{k-1}}\cdots a^{(j_p)}_{-n_p}\mathbf{1}.$$
5. If $m \ge 0$ and $1 \le j_1 \le \cdots \le j_p \le r$, set $(a^{(i)})^{def}_m a^{(j_1)}_{-n_1} \cdots a^{(j_p)}_{-n_p}\mathbf{1}$ as
$$a^{(j_1)}_{-n_1}(a^{(i)})^{def}_m a^{(j_2)}_{-n_2}\cdots a^{(j_p)}_{-n_p}\mathbf{1}$$
$$+ (a^{(j_1)})^{def}_{-n_1} a^{(i)}_m a^{(j_2)}_{-n_2}\cdots a^{(j_p)}_{-n_p}\mathbf{1} - a^{(i)}_m (a^{(j_1)})^{def}_{-n_1}(a^{(j_2)})_{-n_2}\cdots a^{(j_p)}_{-n_p}\mathbf{1}$$
$$+ \sum_{\alpha=0}^{\infty}\binom{m}{\alpha}\left(((a^{(i)})^{def}_{\alpha} a^{(j_1)})_{m-n_1-\alpha} + (a^{(i)}_{\alpha} a^{(j_1)})^{def}_{m-n_1-\alpha}\right) a^{(j_2)}_{-n_2}\cdots a^{(j_p)}_{-n_p}\mathbf{1}.$$

Main theorem

Theorem

The so-defined Y_1 satisfies the cocycle equation if the following commutator condition holds: for every $1 \leq i, j, k \leq r, m, n \in \mathbb{N}$,

$$
\begin{aligned}
&[(a^{(i)})^{def}_m, a^{(j)}_n]a^{(k)} + [a^{(i)}_m, (a^{(j)})^{def}_n]a^{(k)} \\
&= \sum_{\alpha=0}^{\infty} \binom{m}{\alpha} \left(\left((a^{(i)})^{def}_{\alpha} a^{(j)} \right)_{m+n-\alpha} a^{(k)} + \left(a^{(i)}_{\alpha} a^{(j)} \right)^{def}_{m+n-\alpha} a^{(k)} \right),
\end{aligned}
$$

In this case, $Y_1 : S \to End(V)[[x, x^{-1}]]$ uniquely extends to $Y_1 : V \to End(V)[[x, x^{-1}]]$. Moreover, $H^2_{1/2}(V, V) = H^2_{\infty}(V, V)$.

Remark

Note that the modes involved are all nonnegative. So starting from an ansatz of $Y_1^-(a^{(i)}, x)a^{(j)}$ with finitely many structural constants, there are only finitely many relations to check.

Examples

1. In case V is the universal Virasoro VOA, or the universal affine VOA, or the universal Zamolodchikov VOA, we computed that $\dim_{\mathbb{C}} H^2_{1/2}(V, V) = 1$.
2. In case V is the Heisenberg VOA of rank r of level l,

$$\dim_{\mathbb{C}} H^2_{1/2}(V, V) = \begin{cases} \binom{r}{3} & \text{if } l \neq 0, \\ 3\binom{r+1}{3} & \text{if } l = 0. \end{cases}$$

Future work in consideration

1. Infinitely many generators (universal two-paramter W_∞-algebra).
2. Passing to the quotients. Rigidity conjecture.
3. Cup products and Gerstenhaber brackets.
   Batalin-Vilkovisky structure. Deligne conjecture.
4. Noncommutative deformations.

The end

Thank you for your attention!

Happy birthday to Jim!