Problem 5.2

the estimated probability distribution is

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} \tilde{k}(x - x_i)$$

(a) $$\hat{\mu} = E_{\hat{p}}[x] = \int \hat{p}(x)\, x\, dx = \int \frac{1}{n}\sum_{i=1}^{n} \tilde{k}(x-x_i)\, x\, dx = \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(x-x_i)\, x\, dx$$

$$= \frac{1}{n}\sum_{i=1}^{n} \int \left[\tilde{k}(x-x_i)(x-x_i) + \tilde{k}(x-x_i)x_i\right] dx$$

$$= \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(x-x_i)(x-x_i)\,dx + \int \tilde{k}(x-x_i)\,x_i\, dx$$

$$= \frac{1}{n}\sum_{i=1}^{n} \underbrace{\int \tilde{k}(x-x_i)(x-x_i)\,d(x-x_i)}_{0} + x_i \underbrace{\int \tilde{k}(x-x_i)\, d(x-x_i)}_{1}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i \quad \checkmark$$

(b) $$\hat{\Sigma} = \mathrm{cov}_{\hat{p}}(x) = \int \hat{p}(x)(x-\hat{\mu})(x-\hat{\mu})^T dx = \int \frac{1}{n}\sum_{i=1}^{n} \tilde{k}(x-x_i)(x-\hat{\mu})(x-\hat{\mu})^T dx$$

$$= \frac{1}{n}\sum_{i=1}^{n} \int \tilde{k}(x-x_i)\left[(x-x_i)(x-x_i)^T + (x_i-\mu)(x_i-\mu)^T + 2x_i(x-x_i) - 2\mu(x-x_i)\right] dx$$

$$= \frac{1}{n}\sum_{i=1}^{n} \underbrace{\int \tilde{k}(x-x_i)(x-x_i)(x-x_i)^T dx}_{H} + \underbrace{\int \tilde{k}(x-x_i)(x_i-\mu)(x_i-\mu)^T dx}_{(x_i-\mu)(x_i-\mu)^T} + \underbrace{2\int \tilde{k}(x-x_i)(x-x_i)(x_i-\mu)\,dx}_{0}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[H + (x_i-\mu)(x_i-\mu)^T\right]$$

$$= H + \frac{1}{n}\sum_{i=1}^{n} (x_i-\mu)(x_i-\mu)^T$$

(c) the properties of the KDE is related to its kernel function and samples. the mean is unbiased estimation.