P7.5

(a) $\mu_x = E[x] = \sum_x x\, p(x) = \sum_x x\left[p(x|y=1)p(y=1) + p(x|y=2)p(y=2)\right]$

$$= \frac{1}{2}\left[\sum_x x\, N(x|\mu_1, \Sigma_1) + \sum_x x\, N(x|\mu_2, \Sigma_2)\right]$$

$$= \frac{1}{2}(\mu_1 + \mu_2)$$

$$\Sigma_x = E[(x-\mu_x)(x-\mu_x)^T] = E\left[\left(\tfrac{1}{2}x - \tfrac{1}{2}\mu_1 + \tfrac{1}{2}x - \tfrac{1}{2}\mu_2\right)\left(\tfrac{1}{2}x - \tfrac{1}{2}\mu_1 + \tfrac{1}{2}x - \tfrac{1}{2}\mu_2\right)^T\right]$$

$$= \frac{1}{4}E\left[(x-\mu_1)(x-\mu_1)^T + (x-\mu_1)(x-\mu_2)^T + (x-\mu_2)(x-\mu_1)^T + (x-\mu_2)(x-\mu_2)^T\right]$$

$$= \frac{1}{4}E\left[\underline{(x-\mu_1)(x-\mu_1)^T} + \underline{(x-\mu_1)(x-\mu_1)^T} + \underset{\Delta}{(x-\mu_1)(\mu_1-\mu_2)^T} + \underline{(x-\mu_2)(x-\mu_2)^T} + \underset{\Delta}{(x-\mu_2)(\mu_2-\mu_1)^T} + \underline{(x-\mu_2)(x-\mu_2)^T}\right]$$

$$= \frac{1}{2}E[(x-\mu_1)(x-\mu_1)^T] + \frac{1}{2}E[(x-\mu_2)(x-\mu_2)^T] + \frac{1}{4}E[(x-\mu_1-x+\mu_2)(\mu_1-\mu_2)^T]$$

$$= \frac{1}{2}(\Sigma_1 + \Sigma_2) \underset{\Delta}{-} \frac{1}{4}(\mu_1-\mu_2)(\mu_1-\mu_2)^T$$

I believe I'm correct.

(b) Using the result of (a), we have $\mu_x = \frac{1}{2}(\mu_1+\mu_2) = 0$

and $\Sigma_x = \frac{1}{2}(\Sigma_1+\Sigma_2) - \frac{1}{4}(\mu_1-\mu_2)(\mu_1-\mu_2)^T = \begin{bmatrix} 1-\alpha^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$

so, if $1-\alpha^2 > \sigma^2$, choose eigenvector $e_1 = [1, 0]^T$, $z = x_1$

$\alpha < \sqrt{1-\sigma^2}$

$$p(z|y=1) = p(x_1 = z|y=1) = N(z|\alpha, 1)$$
$$p(z|y=2) = p(x_1 = z|y=2) = N(z|-\alpha, 1)$$

otherwise, if $\alpha > \sqrt{1-\sigma^2}$, we have $z = x_2$, and

$$p(z|y=1) = p(x_2 = z|y=1) = N(z|0, \sigma^2)$$
$$p(z|y=2) = p(x_2 = z|y=2) = N(z|0, \sigma^2)$$