Problem 3.7

(a) $p(D|\pi) = \prod_{i=1}^{n} p(x_i|\pi) = \prod_{i=1}^{n} \left[\pi^{x_i}(1-\pi)^{1-x_i}\right] = \pi^{\sum_{i=1}^{n} x_i}(1-\pi)^{\sum_{i=1}^{n}(1-x_i)} = \pi^{s}(1-\pi)^{n-s}$

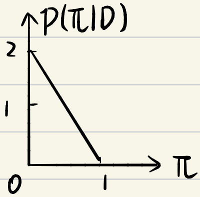
where $s = \sum_{i=1}^{n} x_i$

(b) $$p(\pi|D) = \frac{p(D|\pi)p(\pi)}{p(D)} = \frac{p(D|\pi)p(\pi)}{\int_0^1 p(D|\pi)p(\pi)d\pi} = \frac{\pi^s(1-\pi)^{n-s}}{\int_0^1 \pi^s(1-\pi)^{n-s}d\pi}$$

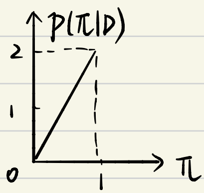
$$= \frac{\pi^s(1-\pi)^{n-s}}{\frac{s!(n-s)!}{(n+1)!}} = \frac{(n+1)!}{s!(n-s)!}\pi^s(1-\pi)^{n-s}$$

when $n=1$, $p(\pi|D) = \frac{2\pi^s(1-\pi)^{1-s}}{s!(1-s)!}$ $\begin{cases} \text{when } s=0, & p(\pi|D) = 2-2\pi \\ \text{when } s=1, & p(\pi|D) = 2\pi \end{cases}$

plot.



s=0



s=1

(c) $$p(x|D) = \int p(x|\pi)p(\pi|D)d\pi = \int \pi^x(1-\pi)^{1-x}\frac{(n+1)!}{s!(n-s)!}\pi^s(1-\pi)^{n-s}d\pi$$

$$= \frac{(n+1)!}{s!(n-s)!}\int \pi^{x+s}(1-\pi)^{n-s+1-x}d\pi = \frac{(n+1)!(x+s)!(n-s+1-x)!}{s!(n-s)!(n+2)!}$$

when $x=0$, $p(x|D) = \frac{n-s+1}{n+2} = 1-\frac{s+1}{n+2}$; when $x=1$, $p(x|D) = \frac{s+1}{n+2}$

so $p(x|D) = \left(\frac{s+1}{n+2}\right)^x\left(1-\frac{s+1}{n+2}\right)^{1-x}$

the effective Bayesian estimate of π is $\left[\pi = \frac{s+1}{n+2}\right]$

explanation: just add one sample of $x=1$ and one sample of $x=0$

(d) MLE $\pi = \frac{1}{n}\sum_{i=1}^{n} x_i$

if π obey uniform prior, the MAP estimate is the same with MLE. $\pi = \frac{1}{n}\sum_{i=1}^{n} x_i$

MAP is more convincing than MLE, but when relate to the uniform prior, MAP is the same with MLE

(e) 1° $\arg\max_{\pi} P(\pi|D) = \arg\max_{\pi} P(D|\lambda)P(\lambda) = \arg\max_{\pi} \pi^{s}(1-\pi)^{n-s}\,2\pi$

$$= \arg\max_{\pi} 2\pi^{s+1}(1-\pi)^{n-s}$$

$$m = \log[2\pi^{s+1}(1-\pi)^{n-s}] = \log 2 + (s+1)\log\pi + (n-s)\log(1-\pi)$$

$$\frac{\partial m}{\partial \pi} = \frac{s+1}{\pi} - \frac{n-s}{1-\pi} = 0 \Rightarrow \boxed{\pi = \frac{s+1}{n+1}}$$

Bayesian. $P(\theta|D) = \frac{2\pi^{s+1}(1-\pi)^{n-s}}{\int 2\pi^{s+1}(1-\pi)^{n-s}d\pi} = Be(s+2,\ n-s+1)$ $\quad \mu(Be) = \boxed{\frac{s+2}{n+3}}$

2° $\arg\max_{\pi} P(\pi|D) = \arg\max_{\pi} P(D|\lambda)P(\lambda) = \arg\max_{\pi} \pi^{s}(1-\pi)^{n-s}(2-2\pi)$

$$= \arg\max_{\pi} 2\pi^{s}(1-\pi)^{n+1-s}$$

$$m = \log[2\pi^{s}(1-\pi)^{n+1-s}] = \log 2 + s\log\pi + (n+1-s)\log(1-\pi)$$

$$\frac{\partial m}{\partial \pi} = \frac{s}{\pi} - \frac{n+1-s}{1-\pi} = 0 \Rightarrow \boxed{\pi = \frac{s}{n+1}}$$ $\quad \alpha = s+1 \quad \beta = n-s+2$

Bayesian. $P(\theta|D) = \frac{2\pi^{s}(1-\pi)^{n+1-s}}{\int 2\pi^{s}(1-\pi)^{n+1-s}d\pi} = Be(s+1,\ n-s+2)$ $\quad \mu(Be) = \boxed{\frac{s+1}{n+3}}$