Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules

Yi-Zhi Huang, James Lepowsky and Lin Zhang

Abstract

This is the first part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. This theory generalizes the tensor category theory for modules for a vertex operator algebra previously developed in a series of papers by the first two authors to suitable module categories for a "conformal vertex algebra" or even more generally, for a "Möbius vertex algebra." We do not require the module categories to be semisimple, and we accommodate modules with generalized weight spaces. As in the earlier series of papers, our tensor product functors depend on a complex variable, but in the present generality, the logarithm of the complex variable is required; the general representation theory of vertex operator algebras requires logarithmic structure. This work includes the complete proofs in the present generality and can be read independently of the earlier series of papers. Since this is a new theory, we present it in detail, including the necessary new foundational material. In addition, with a view toward anticipated applications, we develop and present the various stages of the theory in the natural, general settings in which the proofs hold, settings that are sometimes more general than what we need for the main conclusions. In this paper (Part I), we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature.

Contents

In this paper, Part I of a series of eight papers, we give a detailed overview of logarithmic tensor category theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: The present paper, Part I, contains Sections 1 and 2. In Part II [HLZ2], which contains Section 3, we develop logarithmic formal calculus and study logarithmic intertwining operators. In Part III [HLZ3], which contains Section 4, we introduce and study intertwining maps and tensor product bifunctors. In Part IV [HLZ4], which contains Sections 5 and 6, we give constructions of the $P(z)$- and $Q(z)$-tensor product bifunctors using what we call "compatibility conditions" and certain other conditions. In Part V [HLZ5], which contains Sections 7 and 8, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. In Part VI [HLZ6], which contains Sections 9 and 10, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. In Part VII [HLZ7], which contains Section 11, we give sufficient conditions for the existence of the associativity isomorphisms. In Part VIII [HLZ8], which contains Section 12, we construct braided tensor category structure.

1 Introduction

A brief description of the present work

In the representation theory of many important algebraic structures, such as Lie algebras, groups (or group algebras), commutative associative algebras, Hopf algebras or quantum groups, tensor product operations among modules play a central role. They not only give new modules from known ones, but they of course also provide a powerful tool for studying modules. More significantly, suitable categories of modules for such algebras, equipped with tensor product operations and appropriate natural isomorphisms, and so on, become symmetric or braided tensor categories, and this tensor category structure is always used, even when it is not explicitly discussed.

Vertex operator algebras, and more generally, vertex algebras, are a fundamental class of algebraic structures whose extensive theory has been developed and used in recent years to provide the means to illuminate and to solve many problems in a wide variety of areas of mathematics and theoretical physics. In particular, the representation theory of vertex (operator) algebras plays deep roles in the construction and study of infinite-dimensional Lie algebra representations, of structures linking sporadic finite simple groups to string theory and to the theory of modular functions, and of knot invariants and 3-manifold invariants, in

mathematics; and of conformal field theory and string theory, in physics.

The present work is devoted to introducing and developing a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. This tensor category theory, and consequently, the representation theory, of vertex (operator) algebras, is much, much more elaborate and more difficult than that of Lie algebras, commutative associative algebras, Hopf algebras or quantum groups. In fact, the vertex-operator-algebraic analogues of even the most elementary parts of the tensor product theory of an algebra such as one of those are highly nontrivial, and the theory needs to be developed with completely new ideas and strategies (and with great care!). The present theory was what we needed to carry out in order to obtain the appropriate vertex-operator-algebraic analogue of the following routine triviality in the representation theory of (for example) Lie algebras: "Given a Lie algebra $\mathfrak{g}$, consider the symmetric tensor category of $\mathfrak{g}$-modules." A vertex operator algebra "wants to be" the space of primitive elements of a Hopf algebra (as is a Lie algebra, for example; this immediately yields the tensor category of modules), but a vertex operator algebra is *not* the space of primitive elements of any Hopf algebra, and this is the beginning of why the problem of constructing a tensor product theory and a tensor category theory of modules for a vertex operator algebra was (and is) hard. Yet it is at least as important to have a theory of tensor products and tensor categories of modules for a vertex operator algebra as it is in classical theories such as Lie algebra theory (where such tensor products and tensor categories of modules exist "automatically").

In Lie algebra theory (among other theories), many important module categories are semisimple, that is, every module is completely reducible, while on the other hand, many important module categories are not. Earlier, the first two authors developed a theory of braided tensor categories for the module category of a what we call a "finitely reductive" vertex operator algebra satisfying certain additional conditions; finite reductivity means that the module category is semisimple and that certain finiteness conditions hold. But it is just as natural and important to develop a theory for non-semisimple module categories in vertex operator algebra theory as it is in Lie algebra theory. Also, in any one of the classical theories such as Lie algebra theory, observing that there is a tensor category of modules is just as easy for not-necessarily-semisimple modules as it is for semisimple modules. For these and many other reasons, we considered it a natural problem to generalize the tensor category theory for vertex operator algebras from the finitely reductive case to the general case.

The present work accomplishes this goal, culminating in the construction of a braided tensor category structure on a suitable module category, not assumed semisimple, for a vertex (operator) algebra. It turns out the non-semisimplicity of modules is intimately linked to the presence of logarithms in the basic ingredients of the theory, beginning with intertwining operators among modules, and this is why we call the present theory "logarithmic tensor category theory." We must in fact consider "generalized modules"— structures for which a certain basic operator has generalized eigenvectors in addition to ordinary eigenvectors. This basic operator is contained in a natural copy of the three-dimensional simple Lie algebra, which plays the role of the Lie algebra of the group of Möbius symmetries; this Lie algebra is in turn a subalgebra of a natural copy of the Virasoro algebra, a central extension of a

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Lie algebra of conformal symmetries. In this work, we carry out our theory for suitable categories of generalized modules for a "conformal vertex algebra," which includes a copy of the Virasoro algebra, and even more generally, for a "Möbius vertex algebra," which has the Möbius symmetries but not all of the conformal symmetries. The present theory explicitly includes the earlier finitely reductive theory as a special case; however, the present theory is (necessarily) much more elaborate and subtle than the finitely reductive theory.

In both the finitely reductive and the logarithmic generality, even the construction of the tensor product (generalized) modules is nontrivial; the correct tensor product module of two modules (when it exists) is not at all based on the tensor product vector space of the two underlying vector spaces. Moreover, the construction of the necessary natural associativity isomorphisms among triples of modules is highly nontrivial. While in classical tensor product theories the natural associativity isomorphisms among triples of modules are given by the usual trivial maps, in the tensor product theory of modules for a vertex (operator) algebra, the corresponding statement is not at all true, and indeed, there are not even any candidates for easy associativity isomorphisms. These and many related issues require the present tensor product and tensor category theory to be elaborate.

A crucial discovery in the work of the first two authors in the finitely reductive case was the existence of natural tensor products of two or more elements in the algebraic completions of tensor product modules. All of the categorical structures and properties are formulated, constructed and/or proved using tensor products of elements. In the finitely reductive case, tensor products of elements were defined using intertwining operators (without logarithms). In order to develop the tensor category theory in the general setting of the present work, it is again crucial to establish the existence of tensor products of elements and to prove the fundamental properties of these tensor product elements, and to do this, we are inevitably led to the development of the theory of logarithmic intertwining operators.

The structures of tensor product module, natural associativity isomorphisms, and resulting braided tensor category structure incorporating these, constructed in the present work, are assumed to exist in a number of research works in mathematics and physics. The results in the present work allow one to remove assumptions of this type. We provide a mathematical foundation for such results and for ongoing and future research involving the representation theory of vertex (operator) algebras.

In fact, what we actually construct in this work is structure much stronger than braided tensor category structure: The natural associativity isomorphisms are constructed by means of a "logarithmic operator product expansion" theorem for logarithmic intertwining operators. This logarithmic operator product expansion is in fact the starting point of "logarithmic conformal field theory," which has been studied extensively by physicists as well as mathematicians. Here, this logarithmic operator product expansion is established as a mathematical theorem.

Moreover, our constructions and proofs in this work actually give what the first two authors have called "vertex-tensor-categorical structure," in which the tensor product bifunctors depend crucially on complex variables. This structure is necessary for producing the desired braided tensor category struture, through the use of the tensor product elements and

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logarithmic operator product expansion mentioned above, and our construction of braided tensor category structure involves a "limiting process" in which the complex-analytic information is "forgotten" and only the "topological" information associated with braided tensor category structure is retained. When we perform this specialization to the "limiting case" of braided tensor category structure, tensor products of three or more elements are no longer defined.

The word "algebra" appears in the phrases "vertex operator algebra" and "vertex algebra," but beginning at the stage of the theory where one must compose intertwining operators, or rather, intertwining maps, among (generalized) modules, one must use analysis as well as algebra, starting even from the definition of composition of intertwining maps. The kind of algebra on which the theory is largely based, and which is needed throughout, is called "formal calculus," which we must in fact extensively develop in the course of the work. We must also enhance formal calculus with a great deal of analytic reasoning, and the synthesized theory is no longer "pure algebra."

This work includes the complete proofs in the present generality and can be read independently of the first two authors' earlier series of papers carrying out the finitely reductive theory. Since this is a new theory, we present it in detail, including the necessary new foundational material. In addition, we develop and present the various stages of the theory in the natural, general settings in which the proofs hold, settings that are sometimes more general than what we need for the main conclusions. This will allow for the future use of the intermediate results in a variety of directions.

Later in the Introduction, we mention some of the recent applications of the present theory, and we include a discussion of some recent literature. We state the main results of the present work at the end of the Introduction.

The main results presented here have been announced in [HLZ1].

Introduction

In a series of papers ([HL1], [HL4], [HL5], [HL6], [HL7], [H2]), the first two authors have developed a tensor product and tensor category theory for modules for a vertex operator algebra under suitable conditions. A structure called "vertex tensor category structure" (see [HL4]), which is much richer than tensor category structure, has thereby been established for many important categories of modules for classes of vertex operator algebras, since the conditions needed for invoking the general theory have been verified for these categories. The most important such families of examples of this theory are listed in Section 1.1 below. In the present work, which has been announced in [HLZ1], we generalize this tensor category theory to a larger family of module categories, for a "conformal vertex algebra," or even more generally, for a "Möbius vertex algebra," under suitably relaxed conditions. A conformal vertex algebra is just a vertex algebra in the sense of Borcherds [B] equipped with a conformal vector satisfying the usual axioms; a Möbius vertex algebra is a variant of a "quasi-vertex operator algebra" as in [FHL]. Central features of the present work are that we do not require the modules in our categories to be completely reducible and that we accommodate modules with generalized weight spaces.

As in the earlier series of papers, our tensor product functors depend on a complex variable, but in the present generality, the logarithm of the complex variable is required. The first part of this work is devoted to the study of logarithmic intertwining operators and their role in the construction of the tensor product functors. The remainder of this work is devoted to the construction of the appropriate natural associativity isomorphisms between triple tensor product functors, to the proof of their fundamental properties, and to the construction of the resulting braided tensor category structure. This leads to vertex tensor category structure for further important families of examples, or, in the Möbius case, to "Möbius vertex tensor category" structure.

We emphasize that we develop our representation theory (tensor category theory) in a very general setting; the vertex (operator) algebras that we consider are very general, and the "modules" that we consider are very general. We call them "generalized modules"; they are not assumed completely reducible. Many extremely important (and well-understood) vertex operator algebras have semisimple module categories, but in fact, now that the theory of vertex operator algebras and of their representations is as highly developed as it has come to be, it is in fact possible, and very fruitful, to work in the greater generality. Focusing mainly on the representation theory of those vertex operator algebras for which every module is completely reducible would be just as restrictive as focusing, classically, on the representation theory of semisimple Lie algebras as opposed to the representation theory of Lie algebras in general. In addition, once we consider suitably general vertex (operator) algebras, it is unnatural to focus on only those modules that are completely reducible. As we explain below, such a general representation theory of vertex (operator) algebras requires logarithmic structure.

A general representation theory of vertex operator algebras is crucial for a range of applications, and we expect that it will be a foundation for future developments. One example is that the original formulation of the uniqueness conjecture [FLM2] for the moonshine module

vertex operator algebra $V^\natural$ (again see [FLM2]) requires (general) vertex operator algebras whose modules might not be completely reducible. Another example is that this general theory is playing a deep role in the (mathematical) construction of conformal field theories (cf. [H10], [H11], [H12], [H13], [L]), which in turn correspond to the perturbative part of string theory. Just as the classical (general) representation theory of groups, or of Lie groups, or of Lie algebras, is not about any particular group or Lie group or Lie algebra (although one of its central goals is certainly to understand the representations of particular structures), the general representation theory of suitably general vertex operator algebras is "background independent," in the terminology of string theory. In addition, the general representation theory of vertex (operator) algebras can be thought of as a "symmetry" theory, where vertex (operator) algebras play a role analogous to that of groups or of Lie algebras in classical theories; deep and well-known analogies between the notion of vertex operator algebra and the classical notion of, for example, Lie algebra are discussed in several places, including [FLM2], [FHL] and [LL].

As we mentioned above, the present work includes the complete proofs in the present generality and can be read independently of the earlier series of papers of the first two authors constructing tensor categories. Our treatment is based on the theory of vertex operator algebras and their modules as developed in [B], [FLM2], [FHL], [DL] and [LL]. Throughout the work, we must, and do, develop new algebraic and analytic methods, including a synthesis of the "formal calculus" of vertex operator algebra theory with analysis.

1.1 Tensor category theory for finitely reductive vertex operator algebras

The main families for which the conditions needed for invoking the first two authors' general tensor category theory have been verified, thus yielding vertex tensor category structure [HL4] on these module categories, include the module categories for the following classes of vertex operator algebras (or, in the last case, vertex operator superalgebras):

1. The vertex operator algebras V_L associated with positive definite even lattices L; see [B], [FLM2] for these vertex operator algebras and see [D1], [DL] for the conditions needed for invoking the general tensor category theory.

2. The vertex operator algebras $L(k, 0)$ associated with affine Lie algebras and positive integers k; see [FZ] for these vertex operator algebras and [FZ], [HL8] for the conditions.

3. The "minimal series" of vertex operator algebras associated with the Virasoro algebra; see [FZ] for these vertex operator algebras and [Wa], [H3] for the conditions.

4. Frenkel, Lepowsky and Meurman's moonshine module $V^\natural$; see [FLM1], [B], [FLM2] for this vertex operator algebra and [D2] for the conditions.

5. The fixed point vertex operator subalgebra of $V^\natural$ under the standard involution; see [FLM1], [FLM2] for this vertex operator algebra and [D2], [H4] for the conditions.

6. The "minimal series" of vertex operator superalgebras (suitably generalized vertex operator algebras) associated with the Neveu-Schwarz superalgebra and also the "unitary series" of vertex operator superalgebras associated with the $N = 2$ superconformal algebra; see [KW] and [Ad2] for the corresponding $N = 1$ and $N = 2$ vertex operator superalgebras, respectively, and [Ad1], [Ad3], [HM1], [HM2] for the conditions.

In addition, vertex tensor category structure has also been established for the module categories for certain vertex operator algebras built from the vertex operator algebras just mentioned, such as tensor products of such algebras; this is carried out in certain of the papers listed above.

For all of the six classes of vertex operator algebras (or superalgebras) listed above, each of the algebras is "rational" in the specific sense of Huang-Lepowsky's work on tensor category theory. This particular "rationality" property is easily proved to be a sufficient condition for insuring that the tensor product modules exist; see for instance [HL5]. Unfortunately, the phrase "rational vertex operator algebra" also has several other distinct meanings in the literature. Thus we find it convenient at this time to assign a new term, "finite reductivity," to our particular notion of "rationality": We say that a vertex operator algebra (or superalgebra) V is *finitely reductive* if:

1. Every V-module is completely reducible.

2. There are only finitely many irreducible V-modules (up to equivalence).

3. All the fusion rules (the dimensions of the spaces of intertwining operators among triples of modules) for V are finite.

We choose the term "finitely reductive" because we think of the term "reductive" as describing the complete reducibility—the first of the conditions (that is, the algebra "(completely) reduces" every module); the other two conditions are finiteness conditions.

The vertex-algebraic study of tensor category structure on module categories for certain vertex algebras was stimulated by the work of Moore and Seiberg [MS1] [MS2], in which, in the study of what they termed "rational" conformal field theory, they obtained a set of polynomial equations based on the assumption of the existence of a suitable operator product expansion for "chiral vertex operators" (which correspond to intertwining operators in vertex algebra theory) and observed an analogy between the theory of this set of polynomial equations and the theory of tensor categories. Earlier, in [BPZ], Belavin, Polyakov, and Zamolodchikov had already formalized the relation between the (nonmeromorphic) operator product expansion, chiral correlation functions and representation theory, for the Virasoro algebra in particular, and Knizhnik and Zamolodchikov [KZ] had established fundamental relations between conformal field theory and the representation theory of affine Lie algebras. As we have discussed in the introductory material in [HL4], [HL5] and [HL8], such study of conformal field theory is deeply connected with the vertex-algebraic construction and study of tensor categories, and also with other mathematical approaches to the construction of tensor categories in the spirit of conformal field theory. Concerning the latter

approaches, we would like to mention that the method used by Kazhdan and Lusztig, especially in their construction of the associativity isomorphisms, in their breakthrough work in [KL1]–[KL5], is related to the algebro-geometric formulation and study of conformal-field-theoretic structures in the influential works of Tsuchiya-Ueno-Yamada [TUY], Drinfeld [Dr] and Beilinson-Feigin-Mazur [BFM]. See also the important work of Deligne [De], Finkelberg ([Fi1] [Fi2]), Bakalov-Kirillov [BK] and Nagatomo-Tsuchiya [NT1] on the construction of tensor categories in the spirit of this approach to conformal field theory, and also the discussions in Remark 1.8 and in Section 1.5 below.

1.2   Logarithmic tensor category theory

The semisimplicity of the module categories mentioned in the examples above is related to another property of these modules, namely, that each module is a direct sum of its "weight spaces," which are the eigenspaces of a special operator $L(0)$ coming from the Virasoro algebra action on the module. But there are important situations in which module categories are not semisimple and in which modules are not direct sums of their weight spaces. Notably, for the vertex operator algebras $L(k, 0)$ associated with affine Lie algebras, when the sum of k and the dual Coxeter number of the corresponding Lie algebra is not a nonnegative rational number, the vertex operator algebra $L(k, 0)$ is not finitely reductive, and, working with Lie algebra theory rather than with vertex operator algebra theory, Kazhdan and Lusztig constructed a natural braided tensor category structure on a certain category of modules of level k for the affine Lie algebra ([KL1], [KL2], [KL3], [KL4], [KL5]). This work of Kazhdan-Lusztig in fact motivated the first two authors to develop an analogous theory for vertex operator algebras rather than for affine Lie algebras, as was explained in detail in the introductory material in [HL1], [HL4], [HL5], [HL6], and [HL8]. However, this general theory, in its original form, did not apply to Kazhdan-Lusztig's context, because the vertex-operator-algebra modules considered in [HL1], [HL4], [HL5], [HL6], [HL7], [H2] are assumed to be the direct sums of their weight spaces (with respect to $L(0)$), and the non-semisimple modules considered by Kazhdan-Lusztig fail in general to be the direct sums of their weight spaces. Although their setup, based on Lie theory, and ours, based on vertex operator algebra theory, are very different (as was discussed in the introductory material in our earlier papers), we expected to be able to recover (and further extend) their results through our vertex operator algebraic approach, which is very general, as we discussed above. This motivated us, in the present work, to generalize the work of the first two authors by considering modules with generalized weight spaces, and especially, intertwining operators associated with such generalized kinds of modules. As we discuss below, this required us to use logarithmic intertwining operators and logarithmic formal calculus, and we have been able to construct braided tensor category structure, and even vertex tensor category structure, on important module categories that are not semisimple. Using the present theory, the third author ([Zha1], [Zha2]) has indeed recovered the braided tensor category structure of Kazhdan-Lusztig, and has also extended it to vertex tensor category structure. While in our theory, logarithmic structure plays a fundamental role, in this Kazhdan-Lusztig work, logarithmic structure does not show up explicitly.

From the viewpoint of the general representation theory of vertex operator algebras, it would be unnatural to study only semisimple modules or only $L(0)$-semisimple modules; focusing only on such modules would be analogous to focusing only on semisimple modules for general (nonsemisimple) finite-dimensional Lie algebras. And as we have pointed out, working in this generality leads to logarithmic structure; the general representation theory of vertex operator algebras requires logarithmic structure.

Logarithmic structure in conformal field theory was in fact first introduced by physicists to describe disorder phenomena [Gu]. A lot of progress has been made on this subject. We refer the interested reader to the review articles [Ga], [Fl2], [RaT] and [Fu], and references therein. One particularly interesting class of logarithmic conformal field theories is the class associated to the triplet $\mathcal{W}$-algebras of central charge $1 - 6\frac{(p-1)^2}{p}$. We will discuss these algebras, and generalizations of them, including references, in Section 1.5 below. The paper [FHST] initiated a study of a possible generalization of the Verlinde conjecture for rational conformal field theories to these theories; see also [FG]. The paper [Fu] assumed the existence of braided tensor category structures on the categories of modules for the vertex operator algebras considered; together with [H14], the present work gives a construction of these structures. The paper [CF] used the results in the present work as announced in [HLZ1].

Here is how such logarithmic structure also arises naturally in the representation theory of vertex operator algebras: In the construction of intertwining operator algebras, the first author proved (see [H8]) that if modules for the vertex operator algebra satisfy a certain cofiniteness condition, then products of the usual intertwining operators satisfy certain systems of differential equations with regular singular points. In addition, it was proved in [H8] that if the vertex operator algebra satisfies certain finite reductivity conditions, then the analytic extensions of products of the usual intertwining operators have no logarithmic terms. In the case when the vertex operator algebra satisfies only the cofiniteness condition but not the finite reductivity conditions, the products of intertwining operators still satisfy systems of differential equations with regular singular points. But in this case, the analytic extensions of such products of intertwining operators might have logarithmic terms. This means that if we want to generalize the results in [HL1], [HL4]–[HL7], [H2] and [H8] to the case in which the finite reductivity properties are not always satisfied, we have to consider intertwining operators involving logarithmic terms.

Logarithmic structure also appears naturally in modular invariance results for vertex operator algebras and in the genus-one parts of conformal field theories. For a vertex operator algebra V satisfying certain finiteness and reductivity conditions, Zhu proved in [Zhu2] a modular invariance result for q-traces of products of vertex operators associated to V-modules. Zhu's result was generalized to the case involving twisted vertex operators by Dong, Li and Mason in [DLM] and to the case of q-traces of products of one intertwining operator and arbitrarily many vertex operators by Miyamoto in [Miy1]. In [Miy2], Miyamoto generalized Zhu's modular invariance result to a modular invariance result involving the logarithm of q for vertex operator algebras not necessarily satisfying the reductivity condition. In [H9], for vertex operator algebras satisfying certain finiteness and reductivity conditions, by overcoming the difficulties one encounters if one tries to generalize Zhu's method, the first

author was able to prove the modular invariance for q-traces of products and iterates of more than one intertwining operator, using certain differential equations and duality properties for intertwining operators. If the vertex operator algebra satisfies only Zhu's cofiniteness condition but not the reductivity condition, the q-traces of products and iterates of intertwining operators still satisfy the same differential equations, but now they involve logarithms of all the variables. To generalize the general Verlinde conjecture proved in [H12] and the modular tensor category structure on the category of V-modules obtained in [H13], one will need such general logarithmic modular invariance. See [FHST] for research in this direction.

In [Mil1], Milas introduced and studied what he called "logarithmic modules" and "logarithmic intertwining operators." See also [Mil2]. Roughly speaking, logarithmic modules are weak modules for a vertex operator algebra that are direct sums of generalized eigenspaces for the operator $L(0)$. We will call such weak modules "generalized modules" in this work. Logarithmic intertwining operators are operators that depend not only on powers of a (formal or complex) variable x, but also on its logarithm $\log x$.

The special features of the logarithm function make the logarithmic theory very subtle and interesting. In order to develop our logarithmic tensor category theory, we were required to considerably develop:

1. Formal calculus, beyond what had been developed in [FLM2], [FHL], [HL5]–[HL7], [H2] and [LL], in particular. (Formal calculus has been developed in a great many works.)

2. What we may call "logarithmic formal calculus," which involves arbitrary powers of formal variables and of their formal logarithms. This logarithmic formal calculus has been extended and exploited by Robinson [Ro1], [Ro2], [Ro3].

3. Complex analysis involving series containing *arbitrary real* powers of the variables.

4. Complex analysis involving series containing nonnegative integral powers of the logarithms of the variables, in the presence of arbitrary real powers of the variables.

5. A blending of these themes in order to formulate and to prove many interchange-of-limit results necessary for the construction of the ingredients of the logarithmic tensor category theory and for the proofs of the fundamental properties.

Our methods intricately combine both algebra and analysis, and must do so, since the statements of the results themselves are both algebraic and analytic. See Remark 1.7 below for a discussion of these methods and their roles in this work.

As we mentioned above, one important application of our generalization is to the category $\mathcal{O}_\kappa$ of certain modules for an affine Lie algebra studied by Kazhdan and Lusztig in their series of papers [KL1]–[KL5]. It has been shown in [Zha1] and [Zha2] by the third author that, among other things, all the conditions needed to apply our theory to this module category are satisfied. As a result, it is proved in [Zha1] and [Zha2] that there is a natural vertex tensor category structure on this module category, giving in particular a new construction, in the context of general vertex operator algebra theory, of the braided tensor category structure on

$\mathcal{O}_\kappa$. This construction does not use the Knizhnik-Zamolodchikov equations. The methods used in [KL1]–[KL5] were very different; the Knizhnik-Zamolodchikov equations play an essential role in their construction, while the present theory is very general.

The triplet $\mathcal{W}$-algebras belong to a different class of vertex operator algebras, satisfying certain finiteness, boundedness and reality conditions. In this case, it has been shown in [H14] by the first author that all the conditions needed to apply the theory carried out in the present work to the category of grading-restricted modules for the vertex operator algebra are also satisfied. Thus, by the results obtained in this work, there is a natural vertex tensor category structure on this category.

In addition to these logarithmic issues, another way in which the present work generalizes the earlier tensor category theory for module categories for a vertex operator algebra is that we now allow the algebras to be somewhat more general than vertex operator algebras, in order, for example, to accommodate module categories for the vertex algebras V_L where L is a nondegenerate even lattice that is not necessarily positive definite (cf. [B], [DL]); see [Zha1].

What we accomplish in this work, then, is the following: We generalize essentially all the results in [HL5], [HL6], [HL7] and [H2] from the category of modules for a vertex operator algebra to categories of suitably generalized modules for a conformal vertex algebra or a Möbius vertex algebra equipped with an additional suitable grading by an abelian group. The algebras that we consider include not only vertex operator algebras but also such vertex algebras as V_L where L is a nondegenerate even lattice, and the modules that we consider are not required to be the direct sums of their weight spaces but instead are required only to be the (direct) sums of their "generalized weight spaces," in a suitable sense. In particular, in this work we carry out, in the present greater generality, the construction theory for the "$P(z)$-tensor product" functor originally done in [HL5], [HL6] and [HL7] and the associativity theory for this functor—the construction of the natural associativity isomorphisms between suitable "triple tensor products" and the proof of their important properties, including the isomorphism property—originally done in [H2]. This leads, as in [HL4], [HL9], to the proof of the coherence properties for vertex tensor categories, and in the Möbius case, the coherence properties for Möbius vertex tensor categories.

For simplicity, we present our theory only for a conformal vertex algebra or a Möbius vertex algebra and not for their superalgebraic analogues, but in fact our theory generalizes routinely to a conformal vertex superalgebra or a Möbius vertex superalgebra equipped with an additional suitable grading by an abelian group; here we are referring only to the usual sign changes associated with the "odd" subspace of a vertex superalgebra, and not to any superconformal structure.

The general structure of much of this work essentially follows that of [HL5], [HL6], [HL7] and [H2]. However, the results here are much stronger and much more general than in these earlier works, and in addition, many of the results here have no counterparts in those works. Moreover, many ideas, formulations and proofs in this work are necessarily quite different from those in the earlier papers, and we have chosen to give some proofs that are new even in the finitely reductive case studied in the earlier papers.

Some of the new ingredients that we are introducing into the theory here are: an analysis of logarithmic intertwining operators, including "logarithmic formal calculus"; a notion of "$P(z_1, z_2)$-intertwining map" and a study of its properties; new "compatibility conditions"; considerable generalizations of virtually all of the technical results in [HL5], [HL6], [HL7] and [H2]; and perhaps most significantly, the analytic ideas and methods that are sketched in Remark 1.7 below.

The contents of the sections of this work are as follows: In the rest of this Introduction we compare classical tensor product and tensor category theory for Lie algebra modules with tensor product and tensor category theory for vertex operator algebra modules. One crucial difference between the two theories is that in the vertex operator algebra setting, the theory depends on an "extra parameter" z, which must be understood as a (nonzero) complex variable rather than as a formal variable (although one needs, and indeed we very heavily use, an extensive "formal calculus," or "calculus of formal variables," in order to develop the theory). We also discuss recent applications of the present theory and some related literature and state the main results of the present work. In Section 2 we recall and extend some basic concepts in the theory of vertex (operator) algebras. We use the treatments of [FLM2], [FHL], [DL] and [LL]; in particular, the formal-calculus approach developed in these works is needed for the present theory. Readers can consult these works for further detail. We also set up notation and terminology that will be used throughout the present work, and we describe the main categories of (generalized) modules that we will consider. In Section 3 we introduce the notion of logarithmic intertwining operator as in [Mil1] and present a detailed study of the basic properties of these operators. At the beginning of this section we introduce and prove results about logarithmic formal calculus, including a general "formal Taylor theorem." In Sections 4 and 5 we present the notions of $P(z)$- and $Q(z)$-intertwining maps, and based on this, the definitions and constructions of $P(z)$- and $Q(z)$-tensor products, generalizing considerations in [HL5], [HL6] and [HL7]. The constructions of the tensor product functors require certain "compatibility conditions" and "local grading restriction conditions." The proofs of some of the results in Section 5 are postponed to Section 6. In Section 7 the convergence condition for products and iterates of intertwining maps introduced in [H2] is generalized to the present context. More importantly, in this section we start to develop the complex analysis approach that we will heavily use in later sections. The new notion of $P(z_1, z_2)$-intertwining map, generalizing the corresponding concept in [H2], is introduced and developed in Section 8. This will play a crucial role in the construction of the natural associativity isomorphisms. In Section 9 we prove important conditions that are satisfied by vectors in the dual space of the vector-space tensor product of three modules that arise from products and from iterates of intertwining maps. This leads us to study elements in this dual space satisfying suitable compatibility and local grading restriction conditions. In this section we extensively use our complex analysis approach, including, in particular, for proving that the order of many iterated summations can be interchanged. By relating the subspaces considered in Section 9, we construct the associativity isomorphisms in Section 10. In Section 11, we generalize certain sufficient conditions for the existence of the associativity isomorphisms in [H2], and we prove the relevant conditions using differential equations. In

Section 12, we establish the coherence properties of our braided tensor category structure.

1.3   The Lie algebra case

In this section and the next, we compare classical tensor product and tensor category theory for Lie algebra modules with the present theory for vertex operator algebra modules, and in fact it is heuristically useful to start by considering tensor product theory for Lie algebra modules in a somewhat unusual way in order to motivate our approach for the case of vertex (operator) algebras.

In the theory of tensor products for modules for a Lie algebra, the tensor product of two modules is defined, or rather, constructed, as the vector-space tensor product of the two modules, equipped with a Lie algebra module action given by the familiar diagonal action of the Lie algebra. In the vertex algebra case, however, the vector-space tensor product of two modules for a vertex algebra is *not* the correct underlying vector space for the tensor product of the vertex-algebra modules. In this section we therefore consider another approach to the tensor category theory for modules for a Lie algebra—an approach, based on "intertwining maps," that will show how the theory proceeds in the vertex algebra case. Then, in the next section, we shall lay out the corresponding "road map" for the tensor category theory in the vertex algebra case, which we then carry out in the body of this work.

We first recall the following elementary but crucial background about tensor product vector spaces: Given vector spaces W_1 and W_2, the corresponding tensor product structure consists of a vector space $W_1 \otimes W_2$ equipped with a bilinear map

$$W_1 \times W_2 \longrightarrow W_1 \otimes W_2,$$

denoted

$$(w_{(1)}, w_{(2)}) \mapsto w_{(1)} \otimes w_{(2)}$$

for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, such that for any vector space W_3 and any bilinear map

$$B : W_1 \times W_2 \longrightarrow W_3,$$

there is a unique linear map

$$L : W_1 \otimes W_2 \longrightarrow W_3$$

such that

$$B(w_{(1)}, w_{(2)}) = L(w_{(1)} \otimes w_{(2)})$$

for $w_{(i)} \in W_i$, $i = 1, 2$. This universal property characterizes the tensor product structure $W_1 \otimes W_2$, equipped with its bilinear map $\cdot \otimes \cdot$, up to unique isomorphism. In addition, the tensor product structure in fact exists.

As was illustrated in [HL4], and as is well known, the notion of tensor product of modules for a Lie algebra can be formulated in terms of what can be called "intertwining maps": Let W_1, W_2, W_3 be modules for a fixed Lie algebra V. (We are calling our Lie algebra V because we shall be calling our vertex algebra V, and we would like to emphasize the

analogies between the two theories.) An *intertwining map of type* $\binom{W_3}{W_1 W_2}$ is a linear map $I : W_1 \otimes W_2 \longrightarrow W_3$ (or equivalently, from what we have just recalled, a bilinear map $W_1 \times W_2 \longrightarrow W_3$) such that

$$\pi_3(v)I(w_{(1)} \otimes w_{(2)}) = I(\pi_1(v)w_{(1)} \otimes w_{(2)}) + I(w_{(1)} \otimes \pi_2(v)w_{(2)}) \tag{1.1}$$

for $v \in V$ and $w_{(i)} \in W_i$, $i = 1, 2$, where π_1, π_2, π_3 are the module actions of V on W_1, W_2 and W_3, respectively. (Clearly, such an intertwining map is the same as a module map from $W_1 \otimes W_2$, equipped with the tensor product module structure, to W_3, but we are now temporarily "forgetting" what the tensor product module is.)

A *tensor product of the V-modules W_1 and W_2* is then a pair (W_0, I_0), where W_0 is a V-module and I_0 is an intertwining map of type $\binom{W_0}{W_1 W_2}$ (which, again, could be viewed as a suitable bilinear map $W_1 \times W_2 \longrightarrow W_0$), such that for any pair (W, I) with W a V-module and I an intertwining map of type $\binom{W}{W_1 W_2}$, there is a unique module homomorphism $\eta : W_0 \longrightarrow W$ such that $I = \eta \circ I_0$. This universal property of course characterizes (W_0, I_0) up to canonical isomorphism. Moreover, it is obvious that the tensor product in fact exists, and may be constructed as the vector-space tensor product $W_1 \otimes W_2$ equipped with the diagonal action of the Lie algebra, together with the identity map from $W_1 \otimes W_2$ to itself (or equivalently, the canonical bilinear map $W_1 \times W_2 \longrightarrow W_1 \otimes W_2$). We shall denote the tensor product (W_0, I_0) of W_1 and W_2 by $(W_1 \boxtimes W_2, \boxtimes)$, where it is understood that the image of $w_{(1)} \otimes w_{(2)}$ under our canonical intertwining map $\boxtimes$ is $w_{(1)} \boxtimes w_{(2)}$. Thus $W_1 \boxtimes W_2 = W_1 \otimes W_2$, and under our identifications, $\boxtimes = 1_{W_1 \otimes W_2}$.

Remark 1.1 This classical explicit construction of course shows that the tensor product functor exists for the category of modules for a Lie algebra. For vertex algebras, it will be relatively straightforward to *define* the appropriate tensor product functor(s) (see [HL4], [HL5], [HL6], [HL7]), but it will be a nontrivial matter to *construct* this functor (or more precisely, these functors) and thereby prove that the (appropriate) tensor product of modules for a (suitable) vertex algebra exists. The reason why we have formulated the notion of tensor product module for a Lie algebra in the way that we just did is that this formulation motivates the correct notion of tensor product functor(s) in the vertex algebra case.

Remark 1.2 Using this explicit construction of the tensor product functor and our notation $w_{(1)} \boxtimes w_{(2)}$ for the tensor product of elements, the standard natural associativity isomorphisms among tensor products of triples of Lie algebra modules are expressed as follows: Since $w_{(1)} \boxtimes w_{(2)} = w_{(1)} \otimes w_{(2)}$, we have

$$\begin{aligned}
(w_{(1)} \boxtimes w_{(2)}) \boxtimes w_{(3)} &= (w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}, \\
w_{(1)} \boxtimes (w_{(2)} \boxtimes w_{(3)}) &= w_{(1)} \otimes (w_{(2)} \otimes w_{(3)})
\end{aligned}$$

for $w_{(i)} \in W_i$, $i = 1, 2, 3$, and so the canonical identification between $w_{(1)} \otimes (w_{(2)} \otimes w_{(3)})$ and $(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}$ gives the standard natural isomorphism

$$\begin{aligned}
(W_1 \boxtimes W_2) \boxtimes W_3 &\rightarrow W_1 \boxtimes (W_2 \boxtimes W_3) \\
(w_{(1)} \boxtimes w_{(2)}) \boxtimes w_{(3)} &\mapsto w_{(1)} \boxtimes (w_{(2)} \boxtimes w_{(3)}). \tag{1.2}
\end{aligned}$$

This collection of natural associativity isomorphisms of course satisfies the classical coherence conditions for associativity isomorphisms among multiple nested tensor product modules—the conditions that say that in nested tensor products involving any number of tensor factors, the placement of parentheses (as in (1.2), the case of three tensor factors) is immaterial; we shall discuss coherence conditions in detail later. Now, as was discovered in [H2], it turns out that maps analogous to (1.2) can also be constructed in the vertex algebra case, giving natural associativity isomorphisms among triples of modules for a (suitable) vertex operator algebra. However, in the vertex algebra case, the elements "$w_{(1)} \boxtimes w_{(2)}$," which indeed exist (under suitable conditions) and are constructed in the theory, lie in a suitable "completion" of the tensor product module rather than in the module itself. Correspondingly, it is a nontrivial matter to construct the triple-tensor-product elements on the two sides of (1.2); in fact, one needs to prove certain convergence, under suitable additional conditions. Even after the triple-tensor-product elements are constructed (in suitable completions of the triple-tensor-product modules), it is a delicate matter to construct the appropriate natural associativity maps, analogous to (1.2), to prove that they are well defined, and to prove that they are isomorphisms. In the present work, we shall generalize these matters (in a self-contained way) from the context of [H2] to a more general one. In the rest of this section, for triples of modules for a Lie algebra, we shall now describe a construction of the natural associativity isomorphisms that will seem roundabout and indirect, but this is the method of construction of these isomorphisms that will give us the correct "road map" for the corresponding construction (and theorems) in the vertex algebra case, as in [HL5], [HL6], [HL7] and [H2].

A significant feature of the constructions in the earlier works (and in the present work) is that the tensor product of modules W_1 and W_2 for a vertex operator algebra V is the contragredient module of a certain V-module that is typically a *proper* subspace of $(W_1 \otimes W_2)^*$, the dual space of the vector-space tensor product of W_1 and W_2. In particular, our treatment, which follows, of the Lie algebra case will use contragredient modules, and we will therefore restrict our attention to *finite-dimensional* modules for our Lie algebra. It will be important that the double-contragredient module of a Lie algebra module is naturally isomorphic to the original module. We shall sometimes denote the contragredient module of a V-module W by W', so that $W'' = W$. (We recall that for a module W for a Lie algebra V, the corresponding contragredient module W' consists of the dual vector space W^* equipped with the action of V given by: $(v \cdot w^*)(w) = -w^*(v \cdot w)$ for $v \in V$, $w^* \in W^*$, $w \in W$.)

Let us, then, now restrict our attention to finite-dimensional modules for our Lie algebra V. The dual space $(W_1 \otimes W_2)^*$ carries the structure of the classical contragredient module of the tensor product module. Given any intertwining map of type $\binom{W_3}{W_1 W_2}$, using the natural linear isomorphism

$$\text{Hom}(W_1 \otimes W_2, W_3) \xrightarrow{\sim} \text{Hom}(W_3^*, (W_1 \otimes W_2)^*) \tag{1.3}$$

we have a corresponding linear map in $\text{Hom}(W_3^*, (W_1 \otimes W_2)^*)$, and this must be a map of V-modules. In the vertex algebra case, given V-modules W_1 and W_2, it turns out that with a suitable analogous setup, the union in the vector space $(W_1 \otimes W_2)^*$ of the ranges of

all such V-module maps, as W_3 and the intertwining map vary (and with W_3^* replaced by the contragredient module W_3'), is the correct candidate for the contragredient module of the tensor product module $W_1 \boxtimes W_2$. Of course, in the Lie algebra situation, this union is $(W_1 \otimes W_2)^*$ itself (since we are allowed to take $W_3 = W_1 \otimes W_2$ and the intertwining map to be the canonical map), but in the vertex algebra case, this union is typically much smaller than $(W_1 \otimes W_2)^*$. In the vertex algebra case, we will use the notation $W_1 \boxminus W_2$ to designate this union, and if the tensor product module $W_1 \boxtimes W_2$ in fact exists, then

$$
\begin{aligned}
W_1 \boxtimes W_2 &= (W_1 \boxminus W_2)', && (1.4) \\
W_1 \boxminus W_2 &= (W_1 \boxtimes W_2)'. && (1.5)
\end{aligned}
$$

Thus in the Lie algebra case we will write

$$
W_1 \boxminus W_2 = (W_1 \otimes W_2)^*, \qquad\qquad (1.6)
$$

and (1.4) and (1.5) hold. (In the Lie algebra case we prefer to write $(W_1 \otimes W_2)^*$ rather than $(W_1 \otimes W_2)'$, because in the vertex algebra case, $W_1 \otimes W_2$ is typically not a V-module, and so we will not be allowed to write $(W_1 \otimes W_2)'$ in the vertex algebra case.)

The subspace $W_1 \boxminus W_2$ of $(W_1 \otimes W_2)^*$ was in fact further described in the following terms in [HL5] and [HL7], in the vertex algebra case: For any map in $\mathrm{Hom}(W_3', (W_1 \otimes W_2)^*)$ corresponding to an intertwining map according to (1.3), the image of any $w_{(3)}' \in W_3'$ under this map satisfies certain subtle conditions, called the "compatibility condition" and the "local grading restriction condition"; these conditions are not "visible" in the Lie algebra case. These conditions in fact precisely describe the proper subspace $W_1 \boxminus W_2$ of $(W_1 \otimes W_2)^*$. We will discuss such conditions further in Section 1.4 and in the body of this work. As we shall explain, this idea of describing elements in certain dual spaces was also used in constructing the natural associativity isomorphisms between triples of modules for a vertex operator algebra in [H2].

In order to give the reader a guide to the vertex algebra case, we now describe the analogue for the Lie algebra case of this construction of the associativity isomorphisms. To construct the associativity isomorphism from $(W_1 \boxtimes W_2) \boxtimes W_3$ to $W_1 \boxtimes (W_2 \boxtimes W_3)$, it is equivalent (by duality) to give a suitable isomorphism from $W_1 \boxminus (W_2 \boxtimes W_3)$ to $(W_1 \boxtimes W_2) \boxminus W_3$ (recall (1.4), (1.5)).

Rather than directly constructing an isomorphism between these two V-modules, it turns out that we want to embed both of them, separately, into the single space $(W_1 \otimes W_2 \otimes W_3)^*$. Note that $(W_1 \otimes W_2 \otimes W_3)^*$ is naturally a V-module, via the contragredient of the diagonal action, that is,

$$
\begin{aligned}
(\pi(v)\lambda)(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) &= -\lambda(\pi_1(v)w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
&\quad - \lambda(w_{(1)} \otimes \pi_2(v)w_{(2)} \otimes w_{(3)}) \\
&\quad - \lambda(w_{(1)} \otimes w_{(2)} \otimes \pi_3(v)w_{(3)}), \qquad (1.7)
\end{aligned}
$$

for $v \in V$ and $w_{(i)} \in W_i$, $i = 1, 2, 3$, where π_1, π_2, π_3 are the module actions of V on W_1, W_2 and W_3, respectively. A concept related to this is the notion of *intertwining map from*

$W_1 \otimes W_2 \otimes W_3$ *to a module* W_4, a natural analogue of (1.1), defined to be a linear map

$$F : W_1 \otimes W_2 \otimes W_3 \longrightarrow W_4 \tag{1.8}$$

such that

$$
\begin{aligned}
\pi_4(v)F(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \;=\;& F(\pi_1(v)w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) \\
+\;& F(w_{(1)} \otimes \pi_2(v)w_{(2)} \otimes w_{(3)}) \\
+\;& F(w_{(1)} \otimes w_{(2)} \otimes \pi_{(3)}(v)w_3),
\end{aligned}
\tag{1.9}
$$

with the obvious notation. The relation between (1.7) and (1.9) comes directly from the natural linear isomorphism

$$\mathrm{Hom}(W_1 \otimes W_2 \otimes W_3, W_4) \xrightarrow{\sim} \mathrm{Hom}(W_4^*, (W_1 \otimes W_2 \otimes W_3)^*); \tag{1.10}$$

given F, we have

$$
\begin{aligned}
W_4^* &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto \nu \circ F.
\end{aligned}
\tag{1.11}
$$

Under this natural linear isomorphism, the intertwining maps correspond precisely to the V-module maps from W_4^* to $(W_1 \otimes W_2 \otimes W_3)^*$. In the situation for vertex algebras, as was the case for tensor products of two rather than three modules, there are analogues of all of the notions and comments discussed in this paragraph *except that we will not put V-module structure onto the vector space $W_1 \otimes W_2 \otimes W_3$*; as we have emphasized, we will instead base the theory on intertwining maps.

Two important ways of constructing maps of the type (1.8) are as follows: For modules W_1, W_2, W_3, W_4, M_1 and intertwining maps I_1 and I_2 of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, by definition the composition $I_1 \circ (1_{W_1} \otimes I_2)$ is an intertwining map from $W_1 \otimes W_2 \otimes W_3$ to W_4. Analogously, for intertwining maps I^1, I^2 of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, with M_2 also a module, the composition $I^1 \circ (I^2 \otimes 1_{W_3})$ is an intertwining map from $W_1 \otimes W_2 \otimes W_3$ to W_4. Hence we have two V-module homomorphisms

$$
\begin{aligned}
W_4^* &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto \nu \circ F_1,
\end{aligned}
\tag{1.12}
$$

where F_1 is the intertwining map $I_1 \circ (1_{W_1} \otimes I_2)$; and

$$
\begin{aligned}
W_4^* &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto \nu \circ F_2,
\end{aligned}
\tag{1.13}
$$

where F_2 is the intertwining map $I^1 \circ (I^2 \circ 1_{W_3})$.

The special cases in which the modules W_4 are two iterated tensor product modules and the "intermediate" modules M_1 and M_2 are two tensor product modules are particularly

interesting: When $W_4 = W_1 \boxtimes (W_2 \boxtimes W_3)$ and $M_1 = W_2 \boxtimes W_3$, and I_1 and I_2 are the corresponding canonical intertwining maps, (1.12) gives the natural V-module homomorphism

$$
\begin{aligned}
W_1 \boxdotright (W_2 \boxtimes W_3) &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \nu(w_{(1)} \boxtimes (w_{(2)} \boxtimes w_{(3)})));
\end{aligned}
\tag{1.14}
$$

when $W_4 = (W_1 \boxtimes W_2) \boxtimes W_3$ and $M_2 = W_1 \boxtimes W_2$, and I^1 and I^2 are the corresponding canonical intertwining maps, (1.13) gives the natural V-module homomorphism

$$
\begin{aligned}
(W_1 \boxtimes W_2) \boxdotright W_3 &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \nu((w_{(1)} \boxtimes w_{(2)}) \boxtimes w_{(3)})).
\end{aligned}
\tag{1.15}
$$

Clearly, in our Lie algebra case, both of the maps (1.14) and (1.15) are isomorphisms, since they both in fact amount to the identity map on $(W_1 \otimes W_2 \otimes W_3)^*$. However, in the vertex algebra case the analogues of these two maps are only injective homomorphisms, and typically not isomorphisms. (Recall the analogous situation, mentioned above, for double rather than triple tensor products.) These two maps enable us to identify both $W_1 \boxdotright (W_2 \boxtimes W_3)$ and $(W_1 \boxtimes W_2) \boxdotright W_3$ with subspaces of $(W_1 \otimes W_2 \otimes W_3)^*$. In the vertex algebra case we will have certain "compatibility conditions" and "local grading restriction conditions" on elements of $(W_1 \otimes W_2 \otimes W_3)^*$ to describe each of the two subspaces. In either the Lie algebra or the vertex algebra case, the construction of our desired natural associativity isomorphism between the two modules $(W_1 \boxtimes W_2) \boxtimes W_3$ and $W_1 \boxtimes (W_2 \boxtimes W_3)$ follows from showing that the ranges of homomorphisms (1.14) and (1.15) are equal to each other, which is of course obvious in the Lie algebra case since both (1.14) and (1.15) are isomorphisms to $(W_1 \otimes W_2 \otimes W_3)^*$. It turns out that, under this associativity isomorphism, (1.2) holds in both the Lie algebra case and the vertex algebra case; in the Lie algebra case, this is obvious because all the maps are the "tautological" ones.

Now we give the reader a preview of how, in the vertex algebra case, these compatibility and local grading restriction conditions on elements of $(W_1 \otimes W_2 \otimes W_3)^*$ will arise. As we have mentioned, in the Lie algebra case, an intertwining map from $W_1 \otimes W_2 \otimes W_3$ to W_4 corresponds to a module map from W_4^* to $(W_1 \otimes W_2 \otimes W_3)^*$. As was discussed in [H2], for the vertex operator algebra analogue, the image of any $w'_{(4)} \in W'_4$ under such an analogous map satisfies certain "compatibility" and "local grading restriction" conditions, and so these conditions must be satisfied by those elements of $(W_1 \otimes W_2 \otimes W_3)^*$ lying in the ranges of the vertex-operator-algebra analogues of either of the maps (1.14) and (1.15) (or the maps (1.12) and (1.13)).

Besides these two conditions, satisfied by the elements of the ranges of the maps of both types (1.14) and (1.15), the elements of the ranges of the analogues of the homomorphisms (1.14) and (1.15) have their own separate properties. First note that any $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ induces the two maps

$$
\begin{aligned}
\mu_\lambda^{(1)} : W_1 &\rightarrow (W_2 \otimes W_3)^* \\
w_{(1)} &\mapsto \lambda(w_{(1)} \otimes \cdot \otimes \cdot)
\end{aligned}
\tag{1.16}
$$

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and

$$\mu_\lambda^{(2)} : W_3 \;\;\to\;\; (W_1 \otimes W_2)^*$$
$$w_{(3)} \;\;\mapsto\;\; \lambda(\cdot \otimes \cdot \otimes w_{(3)}). \qquad (1.17)$$

In the vertex operator algebra analogue [H2], if λ lies in the range of (1.14), then it must satisfy the condition that the elements $\mu_\lambda^{(1)}(w_{(1)})$ all lie in a suitable completion of the subspace $W_2 \boxtimes W_3$ of $(W_2 \otimes W_3)^*$, and if λ lies in the range of (1.15), then it must satisfy the condition that the elements $\mu_\lambda^{(2)}(w_{(3)})$ all lie in a suitable completion of the subspace $W_1 \boxtimes W_2$ of $(W_1 \otimes W_2)^*$. (Of course in the Lie algebra case, these statements are tautological.) In [H2], these important conditions, that $\mu_\lambda^{(1)}(W_1)$ lies in a suitable completion of $W_2 \boxtimes W_3$ and that $\mu_\lambda^{(2)}(W_3)$ lies in a suitable completion of $W_1 \boxtimes W_2$, are understood as "local grading restriction conditions" with respect to the two different ways of composing intertwining maps.

In the construction of our desired natural associativity isomorphism, since we want the ranges of (1.14) and (1.15) to be the same submodule of $(W_1 \otimes W_2 \otimes W_3)^*$, the ranges of both (1.14) and (1.15) should satisfy both of these conditions. This amounts to a certain "expansion condition" in the vertex algebra case. When all these conditions are satisfied, it can in fact be proved [H2] that the associativity isomorphism does indeed exist and that in addition, the "associativity of intertwining maps" holds; that is, the "product" of two suitable intertwining maps can be written, in a certain sense, as the "iterate" of two suitable intertwining maps, and conversely. This equality of products with iterates, highly nontrivial in the vertex algebra case, amounts in the Lie algebra case to the easy statement that in the notation above, any intertwining map of the form $I_1 \circ (1_{W_1} \otimes I_2)$ can also be written as an intertwining map of the form $I^1 \circ (I^2 \otimes 1_{W_3})$, for a suitable "intermediate module" M_2 and suitable intertwining maps I^1 and I^2, and conversely. The reason why this statement is easy in the Lie algebra case is that in fact *any* intertwining map F of the type (1.8) can be "factored" in either of these two ways; for example, to write F in the form $I_1 \circ (1_{W_1} \otimes I_2)$, take M_1 to be $W_2 \otimes W_3$, I_2 to be the canonical (identity) map and I_1 to be F itself (with the appropriate identifications having been made).

We are now ready to discuss the vertex algebra case.

1.4 The vertex algebra case

In this section, which should be carefully compared with the previous one, we shall lay out our "road map" of the constructions of the tensor product functors and the associativity isomorphisms for a suitable class of vertex algebras, considerably generalizing, but also following the ideas of, the corresponding theory developed in [HL5], [HL6], [HL7] and [H2] for vertex operator algebras. Without yet specifying the precise class of vertex algebras that we shall be using in the body of this work, except to say that our vertex algebras will be $\mathbb{Z}$-graded and our modules will be $\mathbb{C}$-graded at first and then $\mathbb{R}$-graded for the more substantial results, we now discuss the vertex algebra case. What follows applies to both the theory of [HL5], [HL6], [HL7], [H2] and the present new logarithmic theory. In Remark 1.7 below, we comment on the substantial new features of the logarithmic generality.

In the vertex algebra case, the concept of intertwining map involves the moduli space of Riemann spheres with one negatively oriented puncture and two positively oriented punctures and with local coordinates around each puncture; the details of the geometric structures needed in this theory are presented in [H1] and [H5]. For each element of this moduli space there is a notion of intertwining map adapted to the particular element. Let z be a nonzero complex number and let $P(z)$ be the Riemann sphere $\hat{\mathbb{C}}$ with one negatively oriented puncture at ∞ and two positively oriented punctures at z and 0, with local coordinates $1/w$, $w - z$ and w at these three punctures, respectively.

Let V be a vertex algebra (on which appropriate assumptions, including the existence of a suitable $\mathbb{Z}$-grading, will be made later), and let $Y(\cdot, x)$ be the vertex operator map defining the algebra structure (see Section 2 below for a brief summary of basic notions and notation, including the formal delta function). Let W_1, W_2 and W_3 be modules for V, and let $Y_1(\cdot, x)$, $Y_2(\cdot, x)$ and $Y_3(\cdot, x)$ be the corresponding vertex operator maps. (The cases in which some of the W_i are V itself, and some of the Y_i are, correspondingly, Y, are important, but the most interesting cases are those where all three modules are different from V.) A "$P(z)$-intertwining map of type $\binom{W_3}{W_1 W_2}$" is a linear map

$$I : W_1 \otimes W_2 \longrightarrow \overline{W}_3, \tag{1.18}$$

where $\overline{W}_3$ is a certain completion of W_3, related to its $\mathbb{C}$-grading, such that

$$x_0^{-1} \delta\left(\frac{x_1 - z}{x_0}\right) Y_3(v, x_1) I(w_{(1)} \otimes w_{(2)})$$

$$= z^{-1} \delta\left(\frac{x_1 - x_0}{z}\right) I(Y_1(v, x_0) w_{(1)} \otimes w_{(2)})$$

$$+ x_0^{-1} \delta\left(\frac{z - x_1}{-x_0}\right) I(w_{(1)} \otimes Y_2(v, x_1) w_{(2)}) \tag{1.19}$$

for $v \in V$, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, where x_0, x_1 and x_2 are commuting independent formal variables. This notion is motivated in detail in [HL4], [HL5] and [HL7]; we shall recall the motivation below.

Remark 1.3 In this theory, it is crucial to distinguish between formal variables and complex variables. Thus we shall use the following notational convention: *Throughout this work, unless we specify otherwise, the symbols* $\underline{x, x_0, x_1, x_2, \ldots, y, y_0, y_1, y_2, \ldots}$ *will denote commuting independent formal variables, and by contrast, the symbols* $\underline{z, z_0, z_1, z_2, \ldots}$ *will denote complex numbers in specified domains*, not formal variables.

Remark 1.4 Recall from [FHL] the definition of the notion of intertwining operator $\mathcal{Y}(\cdot, x)$ in the theory of vertex (operator) algebras. Given (W_1, Y_1), (W_2, Y_2) and (W_3, Y_3) as above, an intertwining operator of type $\binom{W_3}{W_1 W_2}$ can be viewed as a certain type of linear map $\mathcal{Y}(\cdot, x) \cdot$ from $W_1 \otimes W_2$ to the vector space of formal series in x of the form $\sum_{n \in \mathbb{C}} w(n) x^n$, where the coefficients $w(n)$ lie in W_3, and where we are allowing arbitrary complex powers of x, suitably

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"truncated from below" in this sum. The main property of an intertwining operator is the following "Jacobi identity":

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_3(v, x_1)\mathcal{Y}(w_{(1)}, x_2)w_{(2)}$$

$$-x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) \mathcal{Y}(w_{(1)}, x_2)Y_2(v, x_1)w_{(2)}$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) \mathcal{Y}(Y_1(v, x_0)w_{(1)}, x_2)w_{(2)} \qquad (1.20)$$

for $v \in V$, $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$. (When all three modules W_i are V itself and all four operators Y_i and $\mathcal{Y}$ are Y itself, (1.20) becomes the usual Jacobi identity in the definition of the notion of vertex algebra. When W_1 is V, $W_2 = W_3$ and $\mathcal{Y} = Y_2 = Y_3$, (1.20) becomes the usual Jacobi identity in the definition of the notion of V-module.) The point is that by "substituting z for x_2" in (1.20), we obtain (1.19), where we make the identification

$$I(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, z)w_{(2)}; \qquad (1.21)$$

the resulting complex powers of the complex number z are made precise by a choice of branch of the log function. The nonzero complex number z in the notion of $P(z)$-intertwining map thus "comes from" the substitution of z for x_2 in the Jacobi identity in the definition of the notion of intertwining operator. In fact, this correspondence (given a choice of branch of log) actually defines an isomorphism between the space of $P(z)$-intertwining maps and the space of intertwining operators of the same type ([HL5], [HL7]); this will be discussed.

There is a natural linear injection

$$\text{Hom}(W_1 \otimes W_2, \overline{W}_3) \longrightarrow \text{Hom}(W_3', (W_1 \otimes W_2)^*), \qquad (1.22)$$

where here and below we denote by W' the (suitably defined) contragredient module of a V-module W; we have $W'' = W$. Under this injection, a map $I \in \text{Hom}(W_1 \otimes W_2, \overline{W}_3)$ amounts to a map $I' : W_3' \longrightarrow (W_1 \otimes W_2)^*$:

$$w_{(3)}' \mapsto \langle w_{(3)}', I(\cdot \otimes \cdot)\rangle, \qquad (1.23)$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between the contragredient of a module and its completion. If I is a $P(z)$-intertwining map, then as in the Lie algebra case (see above), where such a map is a module map, the map (1.23) intertwines two natural V-actions on W_3' and $(W_1 \otimes W_2)^*$. We will see that in the present (vertex algebra) case, $(W_1 \otimes W_2)^*$ is typically not a V-module. The images of all the elements $w_{(3)}' \in W_3'$ under this map satisfy certain conditions, called the "$P(z)$-compatibility condition" and the "$P(z)$-local grading restriction condition," as formulated in [HL5] and [HL7]; we shall be discussing these.

Given a category of V-modules and two modules W_1 and W_2 in this category, as in the Lie algebra case, the "$P(z)$-tensor product of W_1 and W_2" is then defined to be a pair (W_0, I_0),

where W_0 is a module in the category and I_0 is a $P(z)$-intertwining map of type $\binom{W_0}{W_1 W_2}$, such that for any pair (W, I) with W a module in the category and I a $P(z)$-intertwining map of type $\binom{W}{W_1 W_2}$, there is a unique morphism $\eta : W_0 \longrightarrow W$ such that $I = \bar{\eta} \circ I_0$; here and throughout this work we denote by $\bar{\chi}$ the linear map naturally extending a suitable linear map χ from a graded space to its appropriate completion. This universal property characterizes (W_0, I_0) up to canonical isomorphism, *if it exists.* We will denote the $P(z)$-tensor product of W_1 and W_2, if it exists, by $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$, and we will denote the image of $w_{(1)} \otimes w_{(2)}$ under $\boxtimes_{P(z)}$ by $w_{(1)} \boxtimes_{P(z)} w_{(2)}$, which is an element of $\overline{W_1 \boxtimes_{P(z)} W_2}$, not of $W_1 \boxtimes_{P(z)} W_2$.

From this definition and the natural map (1.22), we will see that if the $P(z)$-tensor product of W_1 and W_2 exists, then its contragredient module can be realized as the union of ranges of all maps of the form (1.23) as W_3' and I vary. Even if the $P(z)$-tensor product of W_1 and W_2 does not exist, we denote this union (which is always a subspace stable under a natural action of V) by $W_1 \boxdot_{P(z)} W_2$. If the tensor product does exist, then

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxdot_{P(z)} W_2)', \tag{1.24}$$

$$W_1 \boxdot_{P(z)} W_2 = (W_1 \boxtimes_{P(z)} W_2)'; \tag{1.25}$$

examining (1.24) will show the reader why the notation $\boxdot$ was chosen in the earlier papers ($\boxtimes = \boxdot'$!). Several critical facts about $W_1 \boxdot_{P(z)} W_2$ were proved in [HL5], [HL6] and [HL7], notably, $W_1 \boxdot_{P(z)} W_2$ is equal to the subspace of $(W_1 \otimes W_2)^*$ consisting of all the elements satisfying the $P(z)$-compatibility condition and the $P(z)$-local grading restriction condition, and in particular, this subspace is V-stable; and the condition that $W_1 \boxdot_{P(z)} W_2$ is a module is equivalent to the existence of the $P(z)$-tensor product $W_1 \boxtimes_{P(z)} W_2$. All these facts will be proved.

In order to construct vertex tensor category structure, we need to construct appropriate natural associativity isomorphisms. Assuming the existence of the relevant tensor products, we in fact need to construct an appropriate natural isomorphism from $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ for complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Note that we are using two distinct nonzero complex numbers, and that certain inequalities hold. This situation corresponds to the fact that a Riemann sphere with one negatively oriented puncture and three positively oriented punctures can be seen in two different ways as the "product" of two Riemann spheres each with one negatively oriented puncture and two positively oriented punctures; the detailed geometric motivation is presented in [H1], [H5], [HL4] and [H2].

To construct this natural isomorphism, we first consider compositions of certain intertwining maps. As we have mentioned, a $P(z)$-intertwining map I of type $\binom{W_3}{W_1 W_2}$ maps into $\overline{W_3}$ rather than W_3. Thus the existence of compositions of suitable intertwining maps always entails certain convergence. In particular, the existence of the composition $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ when $|z_1| > |z_2| > 0$ and the existence of the composition $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ when $|z_2| > |z_1 - z_2| > 0$, for general elements $w_{(i)}$ of W_i, $i = 1, 2, 3$, requires the proof of certain convergence conditions. These conditions will be discussed in detail.

Let us now assume these convergence conditions and let z_1, z_2 satisfy $|z_1| > |z_2| > |z_1 -$

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$z_2| > 0$. To construct the desired associativity isomorphism from $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$, it is equivalent (by duality) to give a suitable natural isomorphism from $W_1 \boxdot_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ to $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxdot_{P(z_2)} W_3$. As we mentioned in the previous section, instead of constructing this isomorphism directly, we shall embed both of these spaces, separately, into the single space $(W_1 \otimes W_2 \otimes W_3)^*$.

We will see that $(W_1 \otimes W_2 \otimes W_3)^*$ carries a natural V-action analogous to the contragredient of the diagonal action in the Lie algebra case (recall the similar action of V on $(W_1 \otimes W_2)^*$ mentioned above). Also, for four V-modules W_1, W_2, W_3 and W_4, we have a canonical notion of "$P(z_1, z_2)$-intertwining map from $W_1 \otimes W_2 \otimes W_3$ to $\overline{W}_4$" given by a vertex-algebraic analogue of (1.9); for this notion, we need only that z_1 and z_2 are nonzero and distinct. The relation between these two concepts comes from the natural linear injection

$$
\begin{aligned}
\text{Hom}(W_1 \otimes W_2 \otimes W_3, \overline{W}_4) &\longrightarrow \text{Hom}(W_4', (W_1 \otimes W_2 \otimes W_3)^*) \\
F &\mapsto F',
\end{aligned}
\tag{1.26}
$$

where $F' : W_4' \longrightarrow (W_1 \otimes W_2 \otimes W_3)^*$ is given by

$$
\nu \mapsto \nu \circ F,
\tag{1.27}
$$

which is indeed well defined. Under this natural map, the $P(z_1, z_2)$-intertwining maps correspond precisely to the maps from W_4' to $(W_1 \otimes W_2 \otimes W_3)^*$ that intertwine the two natural V-actions on W_4' and $(W_1 \otimes W_2 \otimes W_3)^*$.

Now for modules W_1, W_2, W_3, W_4, M_1, and a $P(z_1)$-intertwining map I_1 and a $P(z_2)$-intertwining map I_2 of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, it turns out that the composition $I_1 \circ (1_{W_1} \otimes I_2)$ exists and is a $P(z_1, z_2)$-intertwining map when $|z_1| > |z_2| > 0$. Analogously, for a $P(z_2)$-intertwining map I^1 and a $P(z_1 - z_2)$-intertwining map I^2 of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, where M_2 is also a module, the composition $I^1 \circ (I^2 \otimes 1_{W_3})$ is a $P(z_1, z_2)$-intertwining map when $|z_2| > |z_1 - z_2| > 0$. Hence we have two maps intertwining the V-actions:

$$
\begin{aligned}
W_4' &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto \nu \circ F_1,
\end{aligned}
\tag{1.28}
$$

where F_1 is the intertwining map $I_1 \circ (1_{W_1} \otimes I_2)$, and

$$
\begin{aligned}
W_4' &\longrightarrow (W_1 \otimes W_2 \otimes W_3)^* \\
\nu &\mapsto \nu \circ F_2,
\end{aligned}
\tag{1.29}
$$

where F_2 is the intertwining map $I^1 \circ (I^2 \circ 1_{W_3})$.

It is important to note that we can express these compositions $I_1 \circ (1_{W_1} \otimes I_2)$ and $I^1 \circ (I^2 \otimes 1_{W_3})$ in terms of intertwining operators, as discussed in Remark 1.4. Let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and $\mathcal{Y}^2$ be the intertwining operators corresponding to I_1, I_2, I^1 and I^2, respectively. Then the compositions $I_1 \circ (1_{W_1} \otimes I_2)$ and $I^1 \circ (I^2 \otimes 1_{W_3})$ correspond to the "product" $\mathcal{Y}_1(\cdot, x_1)\mathcal{Y}_2(\cdot, x_2)\cdot$ and "iterate" $\mathcal{Y}^1(\mathcal{Y}^2(\cdot, x_0)\cdot, x_2)\cdot$ of intertwining operators, respectively, and we make the

"substitutions" (in the sense of Remark 1.4) $x_1 \mapsto z_1$, $x_2 \mapsto z_2$ and $x_0 \mapsto z_1 - z_2$ in order to express the two compositions of intertwining maps as the "product" $\mathcal{Y}_1(\cdot, z_1)\mathcal{Y}_2(\cdot, z_2)\cdot$ and "iterate" $\mathcal{Y}^1(\mathcal{Y}^2(\cdot, z_1 - z_2)\cdot, z_2)\cdot$ of intertwining maps, respectively. (These products and iterates involve a branch of the log function and also certain convergence.)

Just as in the Lie algebra case, the special cases in which the modules W_4 are two iterated tensor product modules and the "intermediate" modules M_1 and M_2 are two tensor product modules are particularly interesting: When $W_4 = W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and $M_1 = W_2 \boxtimes_{P(z_2)} W_3$, and I_1 and I_2 are the corresponding canonical intertwining maps, (1.28) gives the natural V-homomorphism

$$
\begin{aligned}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \quad &\longrightarrow \quad (W_1 \otimes W_2 \otimes W_3)^* \\
\nu \quad &\mapsto \quad (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \\
&\qquad \nu(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})));
\end{aligned}
$$

(1.30)

when $W_4 = (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3$ and $M_2 = W_1 \boxtimes_{P(z_1 - z_2)} W_2$, and I^1 and I^2 are the corresponding canonical intertwining maps, (1.29) gives the natural V-homomorphism

$$
\begin{aligned}
(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3 \quad &\longrightarrow \quad (W_1 \otimes W_2 \otimes W_3)^* \\
\nu \quad &\mapsto \quad (w_{(1)} \otimes w_{(2)} \otimes w_{(3)} \mapsto \\
&\qquad \nu((w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)})).
\end{aligned}
$$

(1.31)

It turns out that both of these maps are injections, as in [H2] (as we shall prove), so that we are embedding the spaces $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ and $(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3$ into the space $(W_1 \otimes W_2 \otimes W_3)^*$. Following the ideas in [H2], we shall give a precise description of the ranges of these two maps, and under suitable conditions, prove that the two ranges are the same; this will establish the associativity isomorphism.

More precisely, as in [H2], we prove that for any $P(z_1, z_2)$-intertwining map F, the image of any $\nu \in W_4'$ under F' (recall (1.27)) satisfies certain conditions that we call the "$P(z_1, z_2)$-compatibility condition" and the "$P(z_1, z_2)$-local grading restriction condition." Hence, as special cases, the elements of $(W_1 \otimes W_2 \otimes W_3)^*$ in the ranges of either of the maps (1.28) or (1.29), and in particular, of (1.30) or (1.31), satisfy these conditions.

In addition, any $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ induces two maps $\mu_\lambda^{(1)}$ and $\mu_\lambda^{(2)}$ as in (1.16) and (1.17). We will see that any element λ of the range of (1.28), and in particular, of (1.30), must satisfy the condition that the elements $\mu_\lambda^{(1)}(w_{(1)})$ all lie, roughly speaking, in a suitable completion of the subspace $W_2 \boxtimes_{P(z_2)} W_3$ of $(W_2 \otimes W_3)^*$, and any element λ of the range of (1.29), and in particular, of (1.31), must satisfy the condition that the elements $\mu_\lambda^{(2)}(w_{(3)})$ all lie, again roughly speaking, in a suitable completion of the subspace $W_1 \boxtimes_{P(z_1 - z_2)} W_2$ of $(W_1 \otimes W_2)^*$. These conditions will be called the "$P^{(1)}(z)$-local grading restriction condition" and the "$P^{(2)}(z)$-local grading restriction condition," respectively.

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It turns out that the construction of the desired natural associativity isomorphism follows from showing that the ranges of both of (1.30) and (1.31) satisfy both of these conditions. This amounts to a certain "expansion condition" on our module category. When this expansion condition and a suitable convergence condition are satisfied, we show that the desired associativity isomorphisms do exist, and that in addition, the associativity of intertwining maps holds. That is, let z_1 and z_2 be complex numbers satisfying the inequalities $|z_1| > |z_2| > |z_1 - z_2| > 0$. Then for any $P(z_1)$-intertwining map I_1 and $P(z_2)$-intertwining map I_2 of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively, there is a suitable module M_2, and a $P(z_2)$-intertwining map I^1 and a $P(z_1 - z_2)$-intertwining map I^2 of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively, such that

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle = \langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}) \rangle \tag{1.32}$$

for $w_{(1)} \in W_1, w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$; and conversely, given I^1 and I^2 as indicated, there exist a suitable module M_1 and maps I_1 and I_2 with the indicated properties. In terms of intertwining operators (recall the comments above), the equality (1.32) reads

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)} \rangle|_{x_1=z_1,\, x_2=z_2}$$
$$= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)} \rangle|_{x_0=z_1-z_2,\, x_2=z_2}, \tag{1.33}$$

where $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}^1$ and $\mathcal{Y}^2$ are the intertwining operators corresponding to I_1, I_2, I^1 and I^2, respectively. (As we have been mentioning, the substitution of complex numbers for formal variables involves a branch of the log function and also certain convergence.) In this sense, the associativity asserts that the "product" of two suitable intertwining maps can be written as the "iterate" of two suitable intertwining maps, and conversely.

From this construction of the natural associativity isomorphisms we will see, by analogy with (1.2), that $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ is mapped naturally to $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ under the natural extension of the corresponding associativity isomorphism (these elements in general lying in the algebraic completions of the corresponding tensor product modules). In fact, this property

$$(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \mapsto w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \tag{1.34}$$

for $w_{(1)} \in W_1, w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ characterizes the associativity isomorphism

$$(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \to W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \tag{1.35}$$

(cf. (1.2)). The coherence property of the associativity isomorphisms will follow from this fact. We will of course have mutually inverse associativity isomorphisms.

Remark 1.5 Note that equation (1.33) can be written as

$$\mathcal{Y}_1(w_{(1)}, z_1)\mathcal{Y}_2(w_{(2)}, z_2) = \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, z_1 - z_2)w_{(2)}, z_2), \tag{1.36}$$

with the appearance of the complex numbers being understood as substitutions in the sense mentioned above, and with the "generic" vectors $w_{(3)}$ and $w'_{(4)}$ being implicit. This (rigorous)

equation amounts to the "operator product expansion" in the physics literature on conformal field theory; indeed, in our language, if we expand the right-hand side of (1.36) in powers of $z_1 - z_2$, we find that a product of intertwining maps is expressed as an expansion in powers of $z_1 - z_2$, with coefficients that are again intertwining maps, of the form $\mathcal{Y}^1(w, z_2)$. When all three modules are the vertex algebra itself, and all the intertwining operators are the canonical vertex operator $Y(\cdot, x)$ itself, this "operator product expansion" follows easily from the Jacobi identity. But for intertwining operators in general, it is a deep matter to prove the operator product expansion, that is, to prove the assertions involving (1.32) and (1.33) above. This was proved in [H2] in the finitely reductive setting and is considerably generalized in the present work to the logarithmic setting.

Remark 1.6 The constructions of the tensor product modules and of the associativity isomorphisms previewed above for suitably general vertex algebras follow those in [HL5], [HL6], [HL7] and [H2]. Alternative constructions are certainly possible. For example, an alternative construction of the tensor product modules was given in [Li]. However, no matter what construction is used for the tensor product modules of suitably general vertex algebras, one cannot avoid constructing structures and proving results equivalent to what is carried out in this work. The constructions in this work of the tensor product functors and of the natural associativity isomorphisms are crucial in the deeper part of the theory of vertex tensor categories.

Remark 1.7 We have outlined the construction of the tensor product functors and the associativity isomorphisms without getting into the technical details. On the other hand, though the general ideas of the constructions are the same for both the semisimple theory developed in [HL5], [HL6], [HL7] and [H2] and the nonsemisimple logarithmic theory carried out in the present work, many of the proofs of the results in the present work involve substantial new ideas and techniques, making the nonsemisimple logarithmic theory vastly more difficult technically than the semisimple theory. First, we have had to further develop formal calculus beyond what had been developed in [FLM2], [FHL], [HL5]–[HL7], [H2], [LL] and many other works. We have had to study new kinds of combinations of formal delta function expressions in several *formal and complex* variables. Second, we have extended formal calculus to include logarithms of formal variables. In formal calculus, logarithms of formal variables are in fact additional independent formal variables. We develop our "logarithmic formal calculus" in a much more general setting than what we need for the main results in this work. In particular, we at first allow the formal series in a formal variable and its logarithm to involve *infinitely many arbitrary complex powers* of the logarithm. This study of logarithmic formal calculus has surprising connections with various classes of combinatorial identities and has been extended and exploited by Robinson [Ro1], [Ro2], [Ro3]. Third, to construct the natural associativity isomorphisms and other data for the tensor categories and to prove the coherence property, it is necessary to use complex analysis. We wanted to carry out our theory under the most general natural sets of assumptions that would indeed *yield* a theory. This required us to work with series involving *arbitrary real* powers of the complex variables, *with the powers not even being lower bounded*. We have in fact

extended a number of classical results in complex analysis to results that can be applied to such series. In particular, we have had to prove many results that allow us to switch orders of infinite sums, by either proving the multiconvergence of the corresponding multisums or by using Taylor expansion for analytic functions. Fourth, since our theory also involves logarithms of complex variables, we have also had to extend those same classical results in complex analysis to results that can be applied still further to series involving logarithms of the complex variables. In particular, we prove that when the powers of the logarithm of a complex variable are bounded above in a series involving *arbitrary real powers* of the variable and nonnegative integral powers of its logarithm, the convergence of suitable *iterated* sums implies absolute convergence of the corresponding *double* sums. We also prove what we call the "unique expansion property" for the set $\mathbb{R} \times \{0, \dots, N\}$ (see Proposition 7.8), which says that the coefficients of an absolutely convergent series of the form just indicated are determined uniquely by its sum. One important difference from the logarithmic *formal* calculus is that when we use complex analysis, it is necessary for the powers of the logarithms to be bounded from above, essentially because a complex variable z can also be expressed as the sum of the series $z = \sum_{n \in \mathbb{N}} \frac{(\log z)^n}{n!}$. Fifth, we have had to combine our results on formal calculus, on logarithmic formal calculus, and on complex analysis for series with both arbitrary real powers and also logarithms to prove our main results on the construction of the tensor category structures. In many proofs, we encounter expressions involving both formal variables and complex variables, and thus we have had to develop new and delicate methods exploiting both the formal and complex analysis methods that we have just mentioned. The proofs, which are not short (and cannot be), accomplish the necessary interchanges of order of summations.

Remark 1.8 The operator product expansion and resulting braided tensor category structure constructed by the theory in [HL5], [HL6], [HL7], [H2] were originally structures whose existence was conjectured: It was in their important study of conformal field theory that Moore and Seiberg [MS1] [MS2] first discovered a set of polynomial equations from a suitable axiom system for a "rational conformal field theory." Inspired by a comment of Witten, they observed an analogy between the theory of these polynomial equations and the theory of tensor categories. The structures given by these Moore-Seiberg equations were called "modular tensor categories" by I. Frenkel. However, in the work of Moore and Seiberg, as they commented, neither tensor product structure nor other related structures were either formulated or constructed mathematically. Later, Turaev formulated a precise notion of modular tensor category in [T1] and [T2] and gave examples of such tensor categories from representations of quantum groups at roots of unity, based on results obtained by many people on quantum groups and their representations, especially those in the pioneering work [ReT1] and [ReT2] by Reshetikhin and Turaev on the construction of knot and 3-manifold invariants from representations of quantum groups. On the other hand, on the "rational conformal field theory" side, a modular tensor category structure in this sense on certain module categories for affine Lie algebras, and much more generally, on certain module categories for "chiral algebras" associated with rational conformal field theories, was then believed to exist by both physicists and mathematicians, but such structure was not in fact constructed at that time. Moore

and Seiberg observed the analogy mentioned above based on the assumption of the existence of a suitable tensor product functor (including a tensor product module) and derived their polynomial equations based on the assumption of the existence of a suitable operator product expansion for chiral vertex operators, which is essentially equivalent to assuming the associativity of intertwining maps, as we have expressed it above. As we have discussed, the desired tensor product modules and functors were constructed under suitable conditions in the series of papers [HL5], [HL6] and [HL7], and in [H2] the appropriate natural associativity isomorphisms among tensor products of triples of modules were constructed, and it was shown that this is equivalent to the desired associativity of intertwining maps (and thus the existence of a suitable operator product expansion). In particular, this work [HL5], [HL6], [HL7] and [H2] served to construct the desired braided tensor category structure in the generality of suitable vertex operator algebras, including those associated with affine Lie algebras and the Virasoro algebra as a very special case; see [HL8] and [H3], respectively. (For a discussion of the remaining parts of the modular tensor category structure in this generality, see below and [H15].) The results in these papers will be generalized in this work. In the special case of affine Lie algebras and also in the special case of Virasoro-algebraic structures, using the work of Tsuchiya-Ueno-Yamada [TUY] and Beilinson-Feigin-Mazur [BFM] combined with a formulation of braided tensor category structure by Deligne [De], one can obtain the braided tensor category structure discussed above (but not the modular tensor category structure).

1.5   Some recent applications and related literature

We begin with a discussion concerning the "rational" case, with semisimple module categories. We also refer the reader to the recent review by Fuchs, Runkel and Schweigert [FRS] on rational conformal field theory, which also in fact briefly discusses nonrational conformal field theories, including in particular logarithmic conformal field theories.

After the important work [MS1] and [MS2] of Moore and Seiberg, it was widely believed that the category of modules for a suitable vertex operator algebra must have a structure of braided tensor category satisfying additional properties related to the modular invariance of the vertex operator algebra. As is mentioned in Remark 1.8, for a suitable vertex operator algebra, the work [HL5], [HL6], [HL7] and [H2] constructed a structure of braided tensor category on the category of modules for the vertex operator algebra; see also [H3] and [HL8]. On the other hand, the precise and conceptual formulation of the notion of modular tensor category by Turaev [T1] led to a mathematical conjecture that the category of modules for a suitable vertex operator algebra can be endowed in a natural way with modular tensor category structure in this sense. It was in 2005 that this conjecture was finally proved by the first author in [H13] (see also the announcement [H10] and the exposition [H11]). The hardest part of the proof of this conjecture was the proof of the rigidity property of the braided tensor category constructed in [HL5], [HL6], [HL7] and [H2].

Even in the case of a vertex operator algebra associated to an affine Lie algebra or the Virasoro algebra, there was no proof of rigidity for the braided tensor category of modules in the literature, before the proof discovered in [H13]. The works of Tsuchiya-Ueno-Yamada [TUY] and Beilinson-Feigin-Mazur [BFM] can be used to construct a structure of braided

tensor category on the category of modules for such a vertex operator algebra, but neither the rigidity property nor the other main axiom, called the nondegeneracy property, of these braided tensor categories has ever been proved using the results or methods in those works. Under the assumption that the braided tensor category structure on the category of integrable highest weight (standard) modules of a fixed positive integral level for an affine Lie algebra was already known to have the rigidity property, Finkelberg [Fi1] [Fi2] showed that this braided tensor category structure could be recovered by transporting to this category the corresponding rigid braided tensor category structure previously constructed for negative levels by Kazhdan and Lusztig [KL1]–[KL5]. But since the rigidity was an assumption needed in the proof, the work [Fi1], [Fi2] did not actually serve to give a construction of the braided tensor category structure at positive integral level. The book [BK] asserted that one had a construction of the structure of modular tensor category on the category of modules for a vertex operator algebra associated to an affine Lie algebra at positive integral level, and while a construction of the structure of braided tensor category was indeed given, there was no proof of the rigidity property, so that even in the cases of affine Lie algebras and the Virasoro algebra, the construction of the corresponding modular tensor category structures was still an unsolved open problem before 2005.

Under the assumption of the rigidity for positive integral level, the work [Fi1], [Fi2] of Finkelberg combined with the work [KL1]–[KL5] of Kazhdan and Lusztig established the important equivalence between the braided tensor category of a semisimple subquotient of the category of modules for a quantum group at a root of unity and the braided tensor category of integrable highest weight modules of a positive integral level for an affine Lie algebra. The proof of the rigidity of the braided tensor category of integrable highest weight modules of a positive integral level for an affine Lie algebra, as a special case in [H13], based on the the braided tensor category structure constructed in [HL8], as a special case in [HL5], [HL6], [HL7] and [H2], thus in fact provided the completion of the proof of the equivalence theorem that was the goal in [Fi1] and [Fi2] above. As we have mentioned, the only known proof of this rigidity requires the work [HL5], [HL6], [HL7] and [H2], and in particular, in the affine Lie algebra case, uses the work [HL8].

The proof of the rigidity in [H13] is highly nontrivial. The reason why the rigidity was so hard is that one needed to prove the Verlinde conjecture for suitable vertex operator algebras in order to prove the rigidity, and the Verlinde conjecture requires the consideration of genus-one as opposed to genus-zero conformal field theory. The nondegeneracy property of the modular tensor category also follows from the truth of the Verlinde conjecture. The Verlinde conjecture was discovered by E. Verlinde [V] in 1987, and as was demonstrated by Moore and Seiberg [MS1] [MS2] in 1988, the validity of the conjecture follows from their axiom system for a rational conformal field theory. However, the construction of rational conformal field theories is much harder than the construction of modular tensor categories, and this in turn requires the proof of the Verlinde conjecture without the assumption of the axioms for a rational conformal field theory. The Verlinde conjecture for suitable vertex operator algebras was proved in 2004 by the first author in [H12] (without the assumption of the axioms for a rational conformal field theory), and its proof in turn depended on

the aspects of the theory of intertwining operators (the genus-zero theory) developed in [H8] and on the aspects of the theory of q-traces of products or iterates of intertwining operators and their modular invariance (the genus-one theory) developed in [H9]. (These works in turn depended on [HL5], [HL6], [HL7] and [H2].) The modular invariance theorem proved in the pioneering work [Zhu1], [Zhu2] of Zhu actually turned out to be only a very special case of the stronger necessary result proved in [H9], and was far from enough for the purpose of establishing either the required rigidity property or the required nondegeneracy property of the modular tensor category structure. The paper [H9] established the most general modular invariance result in the semisimple case and also constructed all genus-one correlation functions of the corresponding chiral rational conformal field theories. After Zhu's modular invariance was proved in 1990, the modular invariance for products or iterates of more than one intertwining operator was an open problem for a long time. In the case of products or iterates of at most one intertwining operator and any number of vertex operators for modules, a straightforward generalization of Zhu's result using his same method gives the modular invariance (see [Miy1]). But for products or iterates of more than one intertwining operator, Zhu's method is not sufficient because the commutator formula that he used to derive his recurrence formula in his proof has no generalization for intertwining operators. This was one of the main reasons that for about 15 years after 1990, there had been not much progress toward the proof of the rigidity and nondegeneracy properties. In [H9], this difficulty was overcome by means of a proof that q-traces of products or iterates of intertwining operators satisfy modular invariant differential equations with regular singular points; the need for a recurrence formula was thus bypassed.

We have been discussing the case of rational conformal field theories. The present work includes as a special case a complete treatment of the work [HL5], [HL6], [HL7] and [H2], with much stronger results added as well; this work is required for the results that we have just discussed. The main theme of the present work being the logarithmic generalization of this theory, allowing categories of modules that are not completely reducible, we would now like to comment on some recent applications and related literature in the (much greater) logarithmic generality, and also, in this generality we are in addition able to replace vertex operator algebras by much more general vertex algebras equipped with a suitable additional grading by an abelian group. (Allowing logarithmic structures and allowing vertex algebras with a grading by an abelian group are "unrelated" generalizations of the context of [HL5], [HL6], [HL7] and [H2]; in the present work we are able to carry out both generalizations simultaneously.)

Triplet $\mathcal{W}$-algebras are a class of vertex operator algebras of central charge $1 - 6\frac{(p-1)^2}{p}$ which in recent years have attracted a lot of attention from physicists and mathematicians. As we mentioned above, these algebras were introduced by Kausch [K1] and have been studied extensively by Flohr [Fi1] [Fi2], Gaberdiel-Kausch [GK1] [GK2], Kausch [K2], Fuchs-Hwang-Semikhatov-Tipunin [FHST], Abe [Ab], Feigin-Gaĭnutdinov-Semikhatov-Tipunin [FGST1] [FGST2] [FGST3], Carqueville-Flohr [CF], Flohr-Gaberdiel [FG], Fuchs [Fu], Adamović-Milas [AM1] [AM2] [AM5] [AM7], Flohr-Grabow-Koehn [FGK], Flohr-Knuth [FK], Gaberdiel-Runkel [GR1] [GR2], Gaĭnutdinov-Tipunin [GT], Pearce-Rasmussen-Ruelle [PRR1] [PRR2],

Nagatomo-Tsuchiya [NT2] and Rasmussen [Ra4]. A triplet $\mathcal{W}$-algebra $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ satisfies the positive energy condition ($V_{(0)} = \mathbb{C}\mathbf{1}$ and $V_{(n)} = 0$ for $n < 0$) and the C_2-cofiniteness condition (the quotient space $V/C_2(V)$ is finite dimensional, where $C_2(V)$ is the subspace of V spanned by the elements of the form $u_{-2}v$ for $u, v \in V$). The C_2-cofiniteness condition was proved by Abe [Ab] in the simplest $p = 2$ case and by Carqueville-Flohr [CF] and Adamović-Milas [AM2] in the general case.

In [H14], the first author proved that for a vertex operator algebra V satisfying the positive energy condition and the C_2-cofiniteness condition, the category of grading-restricted generalized V-modules satisfies the assumptions needed to invoke the theory carried out in the present work. The present work, combined with [H14] (for proving the assumptions of the present work), thus establishes the logarithmic operator product expansion and constructs the logarithmic tensor category theory for any vertex operator algebra satisfying the positive energy condition and the C_2-cofiniteness condition. For example, the logarithmic tensor products used heavily in the papers [Miy3], [Miy4] and [Miy5] of Miyamoto are in fact constructed in the present work together with [H14]. In particular, for a triplet $\mathcal{W}$-algebra V discussed above, the category of grading-restricted generalized V-modules indeed has the natural braided tensor category structure constructed in the present work. Many of the assertions involving a logarithmic operator product expansion and a logarithmic tensor category theory in the works on triplet $\mathcal{W}$-algebras mentioned above are mathematically formulated and established in the present work together with the paper [H14], so that now, they do not have to be taken as unproved assumptions in those works.

Based on the results of Feigin-Gaĭnutdinov-Semikhatov-Tipunin [FGST1] and of Fuchs-Hwang-Semikhatov-Tipunin [FHST], Feigin, Gaĭnutdinov, Semikhatov and Tipunin conjectured [FGST2] an equivalence between the braided finite tensor category of grading-restricted generalized modules for a triplet $\mathcal{W}$-algebra and the braided finite tensor category of suitable modules for a restricted quantum group. Their formulation of the conjecture also includes the statement that the categories of grading-restricted generalized modules for the triplet $\mathcal{W}$-algebras considered in their paper are indeed braided tensor categories. Assuming the existence of the braided tensor category structure on the triplet $\mathcal{W}$-algebra with $p = 2$, Feigin, Gaĭnutdinov, Semikhatov and Tipunin gave a proof of their conjecture. However, in the case $p \neq 2$, Kondo and Saito [KS] showed that the tensor category of modules for the corresponding restricted quantum group is not braided. Thus, the conjecture in the case $p \neq 2$ cannot be true as it is stated. It is believed that the correct formulation of the conjecture and the proof will be possible only after the conformal-field-theoretic aspects of the representations of triplet $\mathcal{W}$-algebras are studied thoroughly. As we mentioned above, the present work, the paper [H14] and the papers [Ab], [CF] and [AM2] provide a proof of the assumption in their conjecture that the categories of grading-restricted generalized modules for the triplet $\mathcal{W}$-algebras are indeed braided tensor categories. We expect that further studies of the tensor-categorical structures and conformal-field-theoretic properties for triplet $\mathcal{W}$-algebras will provide a correct formulation and proof of suitable equivalence between categories of suitable modules for triplet $\mathcal{W}$-algebras and for restricted quantum groups.

In [H16], the first author introduced a notion of generalized twisted module associated to a general automorphism of a vertex operator algebra, including an automorphism of infinite order. The first author in [H16] also gave a construction of such generalized twisted modules associated to the automorphisms obtained by exponentiating weight 1 elements of the vertex operator algebra. If the automorphism of the vertex operator algebra does not act semisimply, the twisted vertex operators for these generalized twisted modules must involve the logarithm of a formal or complex variable, and we need additional $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-gradings on these generalized twisted modules. As was noticed by Milas, the triplet $\mathcal{W}$-algebras are fixed-point subalgebras of suitable vertex operator algebras constructed from a one-dimensional lattice under an automorphism obtained by exponentiating a weight 1 element. In particular, some logarithmic intertwining operators constructed in [AM4] are in fact twisted vertex operators. Thus the paper [H16] provided an orbifold approach to the representation theory of triplet $\mathcal{W}$-algebras. (This orbifold point of view is one of the analogues of the orbifold point of view for vertex operator algebras introduced in [FLM2].) Since the automorphisms involved indeed do not act on the vertex operator algebra semisimply, the twisted vertex operators for the generalized twisted modules associated to these automorphisms must involve the logarithm of the variables, and we also need additional $\mathbb{C}/\mathbb{Z}$- or $\mathbb{C}$-gradings on these generalized twisted modules. Here $\mathbb{C}/\mathbb{Z}$ or $\mathbb{C}$ are instances of the additional grading abelian group in the present work. Thus we need the general framework and results in the present work, including both the logarithmic generality and also the additional abelian-group gradings, for the study of these generalized twisted modules.

Many of the results on the representation theory of triplet $\mathcal{W}$-algebras have been also generalized to the more general case of $\mathcal{W}(p, q)$-algebras of central charge $1 - 6\frac{(p-q)^2}{pq}$ (see for example [PRZ], [EF], [AM2], [Se], [RP1], [RP2], [Ra1], [Ra2], [Ra3], [Ra5], [GRW], [AM6] and [Wo]) and to $N = 1$ triplet vertex operator superalgebras (see [AM3], [AM5] and [AM7]). The C_2-cofiniteness of the $\mathcal{W}(2, q)$-algebras has been proved by Adamović and Milas in [AM6]. Thus using the results obtained in [H16], the theory developed in the present work applies to these $\mathcal{W}(2, q)$-algebras, yielding braided tensor categories. The $N = 1$ triplet vertex operator superalgebras introduced by Adamović and Milas in [AM3] are also proved by these authors in [AM5] to be C_2-cofinite. As was mentioned above in Section 1.2, the theory developed in this work also applies to vertex superalgebras. The same remarks apply to the results in [H16]. Thus the theory developed in the present work applies to these $N = 1$ triplet vertex operator superalgebras, producing the corresponding braided tensor categories.

Finally, we would like to emphasize that it is interesting that the methods developed and used in the present work, even in the special case of categories of modules for an affine Lie algebra at negative levels, are very different from those developed and used by Kazhdan and Lusztig in [KL1]–[KL5], and are much more general. The methods used in [KL1]–[KL5], closely related to algebraic geometry, depended heavily on the Knizhnik-Zamolodchikov equations. In the present work, we use and develop the general theory of vertex (operator) algebras (and generalizations), requiring both formal calculus theory and complex analysis, and we do not use algebraic geometry. Also, in the present work and in the work [Zha1]

and [Zha2], which verified the assumptions needed for the application of the present theory, although we need to show that products of intertwining operators satisfy certain differential equations with regular singular points, no explicit form of the equations, such as the explicit form of the Knizhnik-Zamolodchikov equations, is needed. In fact, because for a general vertex (operator) algebra satisfying those assumptions in the present work or in [H14] no explicit form of the differential equations such as the form of the Knizhnik-Zamolodchikov equations exists, it was crucial that in the work [HL5], [HL6], [HL7], [H2], [H8], the present work and [H14], we have developed methods that are independent of the explicit form of the differential equations. Another interesting difference between the present general theory and this work of Kazhdan and Lusztig is that logarithmic structures (necessarily) pervade our theory, starting from the vertex-algebraic foundations, while the logarithmic nature of solutions of the Knizhnik-Zamolodchikov equations involved in [KL1]–[KL5] did not have to be emphasized there.

1.6 Main results of the present work

In this section, we state the main results of the present work, numbered as in the main text. The reader is referred to the relevant sections for definitions, notations and details.

Let A be an abelian group and $\tilde{A}$ an abelian group containing A as a subgroup. Let V be a strongly A-graded Möbius or conformal vertex algebra, as defined in Section 2. Let $\mathcal{C}$ be a full subcategory of the category $\mathcal{M}_{sg}$ of strongly $\tilde{A}$-graded (ordinary) V-modules or the category $\mathcal{GM}_{sg}$ of strongly $\tilde{A}$-graded generalized V-modules, closed under the contragredient functor and under taking finite direct sums; see Section 2 and Assumptions 4.1 and 5.30.

In Section 4, the notions of $P(z)$- and $Q(z)$-tensor product functor are defined in terms of $P(z)$- and $Q(z)$-intertwining maps and $P(z)$- and $Q(z)$-products; intertwining maps are related to logarithmic intertwining operators, defined and studied in Section 3. The symbols $P(z)$ and $Q(z)$ refer to the moduli space elements described in Remarks 4.3 and 4.37, respectively. In Section 5, we give a construction of the $P(z)$-tensor product of two objects of $\mathcal{C}$, when this structure exists. For $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$, define the subset

$$W_1 \boxdot_{P(z)} W_2 \subset (W_1 \otimes W_2)^*$$

of $(W_1 \otimes W_2)^*$ to be the union, or equivalently, the sum, of the images

$$I'(W') \subset (W_1 \otimes W_2)^*$$

as $(W; I)$ ranges through all the $P(z)$-products of W_1 and W_2 with $W \in \mathrm{ob}\,\mathcal{C}$, where I' is a map corresponding naturally to the $P(z)$-intertwining map I and where W' is the contragredient (generalized) module of W.

The following two results give the construction of the $P(z)$-tensor product:

Proposition 5.37 *Let $W_1, W_2 \in \mathrm{ob}\,\mathcal{C}$. If $(W_1 \boxdot_{P(z)} W_2, Y'_{P(z)})$ is an object of $\mathcal{C}$ (where $Y'_{P(z)}$ is the natural action of V), denote by $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ its contragredient (generalized) module:*

$$W_1 \boxtimes_{P(z)} W_2 = (W_1 \boxdot_{P(z)} W_2)'.$$

Then the $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ exists and is

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; i'),$$

where i is the natural inclusion from $W_1\boxtimes_{P(z)}W_2$ to $(W_1 \otimes W_2)^*$. Conversely, let us assume that $\mathcal{C}$ is closed under images. If the $P(z)$-tensor product of W_1 and W_2 in $\mathcal{C}$ exists, then $(W_1\boxtimes_{P(z)}W_2, Y'_{P(z)})$ is an object of $\mathcal{C}$.

For

$$\lambda \in (W_1 \otimes W_2)^*,$$

let W_λ be the smallest doubly graded subspace of $((W_1 \otimes W_2)^*)^{(\tilde{A})}_{[\mathbb{C}]}$ (the direct sum of the homogeneous subspaces with respect to the gradings both by conformal generalized weights and by $\tilde{A}$) containing λ and stable under the component operators of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$ (to handle the Möbius but non-conformal case). Let

$$\mathrm{COMP}_{P(z)}((W_1 \otimes W_2)^*),$$

$$\mathrm{LGR}_{[\mathbb{C}];P(z)}((W_1 \otimes W_2)^*)$$

and

$$\mathrm{LGR}_{(\mathbb{C});P(z)}((W_1 \otimes W_2)^*)$$

be the spaces of elements of $(W_1\otimes W_2)^*$ satisfying the $P(z)$-compatibility condition, the $P(z)$-local grading restriction condition and the $L(0)$-semisimple $P(z)$-local grading restriction condition, respectively, as defined in Section 4; the subscript $(\mathbb{C})$ refers to the semisimplicity of the action of $L(0)$ in this case, so that generalized weights are weights.

Theorem 5.50 *Suppose that for every element*

$$\lambda \in \mathrm{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \mathrm{LGR}_{[\mathbb{C}];P(z)}((W_1 \otimes W_2)^*)$$

the space W_λ (which is a (strongly-graded) generalized module) is a generalized submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$ (this holds vacuously if $\mathcal{C} = \mathcal{GM}_{sg}$). Then

$$W_1\boxtimes_{P(z)}W_2 = \mathrm{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \mathrm{LGR}_{[\mathbb{C}];P(z)}((W_1 \otimes W_2)^*).$$

Suppose that $\mathcal{C}$ is a category of strongly-graded V-modules (that is, $\mathcal{C} \subset \mathcal{M}_{sg}$) and that for every element

$$\lambda \in \mathrm{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \mathrm{LGR}_{(\mathbb{C});P(z)}((W_1 \otimes W_2)^*)$$

the space W_λ (which is a (strongly-graded) V-module) is a submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$ (which holds vacuously if $\mathcal{C} = \mathcal{M}_{sg}$). Then

$$W_1\boxtimes_{P(z)}W_2 = \mathrm{COMP}_{P(z)}((W_1 \otimes W_2)^*) \cap \mathrm{LGR}_{(\mathbb{C});P(z)}((W_1 \otimes W_2)^*).$$

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The hard parts of the proof of Theorem 5.50 are given in Section 6.

We also give an analogous construction of $Q(z)$-tensor products in these sections.

For the construction of the natural associativity isomorphism between suitable pairs of triple tensor product functors, we assume that for any object of $\mathcal{C}$, all the (generalized) weights are real numbers and in addition there exists $K \in \mathbb{Z}_+$ such that

$$(L(0) - L(0)_s)^K = 0$$

on the module, $L(0)_s$ being the semisimple part of $L(0)$ (the latter condition holding vacuously when $\mathcal{C}$ is in $\mathcal{M}_{sg}$); see Assumption 7.11.

The main hard parts of the construction of the associativity isomorphisms are presented in Section 9, after necessary preparation in Section 8. To discuss these results, we need the important $P^{(1)}(z)$- and $P^{(2)}(z)$-local grading restriction conditions on

$$\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$$

(where W_1, W_2, and W_3 are objects of $\mathcal{C}$) and their $L(0)$-semisimple versions. Here we state the (two-part) $P^{(2)}(z)$-local grading restriction condition, the other conditions being analogous:

The $P^{(2)}(z)$-local grading restriction condition

(a) The $P^{(2)}(z)$-*grading condition*: For any $w_{(3)} \in W_3$, there exists a formal series $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ with

$$\lambda_n^{(2)} \in \coprod_{\beta \in \tilde{A}} ((W_1 \otimes W_2)^*)_{[n]}^{(\beta)}$$

for $n \in \mathbb{R}$, an open neighborhood of $z' = 0$, and $N \in \mathbb{N}$ such that for $w_{(1)} \in W_1$ and $w_{(2)} \in W_2$, the series

$$\sum_{n \in \mathbb{R}} (e^{z' L'_{P(z)}(0)} \lambda_n^{(2)})(w_{(1)} \otimes w_{(2)})$$

has the following properties:

(i) It can be written as the iterated series

$$\sum_{n \in \mathbb{R}} e^{nz'} \left(\left(\sum_{i=0}^{N} \frac{(z')^i}{i!} (L'_{P(z)}(0) - n)^i \lambda_n^{(2)} \right) (w_{(1)} \otimes w_{(2)}) \right).$$

(ii) It is absolutely convergent for $z' \in \mathbb{C}$ in the neighborhood of $z' = 0$ above.

(iii) It is absolutely convergent to $\mu_{\lambda, w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)})$ when $z' = 0$:

$$\sum_{n \in \mathbb{R}} \lambda_n^{(2)}(w_{(1)} \otimes w_{(2)}) = \mu_{\lambda, w_{(3)}}^{(2)}(w_{(1)} \otimes w_{(2)}) = \lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)})$$

(the last equality being the definition of $\mu_{\lambda, w_{(3)}}^{(2)}$).

(b) For any $w_{(3)} \in W_3$, let $W^{(2)}_{\lambda, w_{(3)}}$ be the smallest doubly graded subspace of $((W_1 \otimes W_2)^*)^{(\tilde{A})}_{[\mathbb{R}]}$ containing all the terms $\lambda^{(2)}_n$ in the formal series in (a) and stable under the component operators of the operators $Y'_{P(z)}(v, x)$ for $v \in V$, $m \in \mathbb{Z}$, and under the operators $L'_{P(z)}(-1)$, $L'_{P(z)}(0)$ and $L'_{P(z)}(1)$. Then $W^{(2)}_{\lambda, w_{(3)}}$ has the properties

$$\dim(W^{(2)}_{\lambda, w_{(3)}})^{(\beta)}_{[n]} < \infty,$$

$$(W^{(2)}_{\lambda, w_{(3)}})^{(\beta)}_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \text{ sufficiently negative}$$

for any $n \in \mathbb{R}$ and $\beta \in \tilde{A}$, where the subscripts denote the $\mathbb{R}$-grading by $L'_{P(z)}(0)$-(generalized) eigenvalues and the superscripts denote the $\tilde{A}$-grading.

The following result gives, among other things, the deep fact that when λ is obtained from a suitable product of intertwining maps, the elements $\lambda^{(2)}_n$ for $n \in \mathbb{R}$ in the assumed $P^{(2)}(z)$-local grading restriction condition for suitable $z \in \mathbb{C}^\times$ satisfy the $P(z)$-compatibility condition:

Theorem 9.17 *Assume that the convergence condition for intertwining maps in $\mathcal{C}$ (see Section 7) holds and that*

$$|z_1| > |z_2| > |z_1 - z_2| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$ and let I_1, I_2, I^1 and I^2 be $P(z_1)$-, $P(z_2)$-, $P(z_2)$- and $P(z_1 - z_2)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$, $\binom{M_1}{W_2 W_3}$, $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Let $w'_{(4)} \in W'_4$.

1. *Suppose that $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ satisfies Part (a) of the $P^{(2)}(z_1 - z_2)$-local grading restriction condition, that is, the $P^{(2)}(z_1 - z_2)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-grading condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For any $w_{(3)} \in W_3$, let $\sum_{n \in \mathbb{R}} \lambda^{(2)}_n$ be a series weakly absolutely convergent to*

$$\mu^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}} \in (W_1 \otimes W_2)^*$$

as indicated in the $P^{(2)}(z_1 - z_2)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-grading condition), and suppose in addition that the elements $\lambda^{(2)}_n \in (W_1 \otimes W_2)^*$, $n \in \mathbb{R}$, satisfy the $P(z_1 - z_2)$-lower truncation condition (Part (a) of the $P(z_1 - z_2)$-compatibility condition in Section 5). Then each $\lambda^{(2)}_n$ satisfies the (full) $P(z_1 - z_2)$-compatibility condition. Moreover, the corresponding space

$$W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}} \subset (W_1 \otimes W_2)^*,$$

equipped with the vertex operator map given by $Y'_{P(z_1 - z_2)}$ and the operators $L'_{P(z_1 - z_2)}(j)$ for $j = -1, 0, 1$, is a doubly-graded generalized V-module, and when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, a doubly-graded V-module. In particular, if $(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)})$ satisfies the full

$P^{(2)}(z_1 - z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$), then $W^{(2)}_{(I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}), w_{(3)}}$ is an object of $\mathcal{GM}_{sg}$ (or $\mathcal{M}_{sg}$ when $\mathcal{C}$ is in $\mathcal{M}_{sg}$); in this case, the assumption that each $\lambda^{(2)}_n$ satisfies the $P(z_1 - z_2)$-lower truncation condition is redundant.

2. Analogously, suppose that $(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)})$ satisfies Part (a) of the $P^{(1)}(z_2)$-local grading restriction condition, that is, the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For any $w_{(1)} \in W_1$, let $\sum_{n \in \mathbb{R}} \lambda^{(1)}_n$ be a series weakly absolutely convergent to

$$\mu^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}} \in (W_2 \otimes W_3)^*$$

as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition), and suppose in addition that the elements $\lambda^{(1)}_n \in (W_2 \otimes W_3)^*$, $n \in \mathbb{R}$, satisfy the $P(z_2)$-lower truncation condition (Part (a) of the $P(z_2)$-compatibility condition). Then each $\lambda^{(1)}_n$ satisfies the (full) $P(z_2)$-compatibility condition. Moreover, the corresponding space

$$W^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}} \subset (W_2 \otimes W_3)^*,$$

equipped with the vertex operator map given by $Y'_{P(z_2)}$ and the operators $L'_{P(z_2)}(j)$ for $j = -1, 0, 1$, is a doubly-graded generalized V-module, and when $\mathcal{C}$ is in $\mathcal{M}_{sg}$, a doubly-graded V-module. In particular, if $(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)})$ satisfies the full $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$), then $W^{(1)}_{(I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}), w_{(1)}}$ is an object of $\mathcal{GM}_{sg}$ (or $\mathcal{M}_{sg}$ when $\mathcal{C}$ is in $\mathcal{M}_{sg}$); in this case, the assumption that each $\lambda^{(1)}_n$ satisfies the $P(z_2)$-lower truncation condition is redundant.

The following result, based heavily on the previous theorem, establishes the associativity of intertwining maps:

Theorem 9.23 *Assume that $\mathcal{C}$ is closed under images, that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that*

$$|z_1| > |z_2| > |z_1 - z_2| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$. Assume also that $W_1 \boxtimes_{P(z_1 - z_2)} W_2$ and $W_2 \boxtimes_{P(z_2)} W_3$ exist in $\mathcal{C}$.

1. *Let I_1 and I_2 be $P(z_1)$- and $P(z_2)$-intertwining maps of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively. Suppose that for each $w'_{(4)} \in W'_4$,*

$$\lambda = (I_1 \circ (1_{W_1} \otimes I_2))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(2)}(z_1 - z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(2)}$ be the (unique) series weakly absolutely convergent to $\mu^{(2)}_{\lambda, w_{(3)}}$ as indicated in the $P^{(2)}(z_1 - z_2)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, the generalized V-submodule of $W^{(2)}_{\lambda, w_{(3)}}$ generated by $\lambda_n^{(2)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$. Then the product

$$I_1 \circ (1_{W_1} \otimes I_2)$$

can be expressed as an iterate, and in fact, there exists a unique $P(z_2)$-intertwining map I^1 of type $\binom{W_4}{W_1 \boxtimes_{P(z_1 - z_2)} W_2 \ W_3}$ such that

$$\langle w'_{(4)}, I_1(w_{(1)} \otimes I_2(w_{(2)} \otimes w_{(3)})) \rangle = \langle w'_{(4)}, I^1((w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \otimes w_{(3)}) \rangle$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

2. Analogously, let I^1 and I^2 be $P(z_2)$- and $P(z_1 - z_2)$-intertwining maps of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Suppose that for each $w'_{(4)} \in W'_4$,

$$\lambda = (I^1 \circ (I^2 \otimes 1_{W_3}))'(w'_{(4)}) \in (W_1 \otimes W_2 \otimes W_3)^*$$

satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, let $\sum_{n \in \mathbb{R}} \lambda_n^{(1)}$ be the (unique) series weakly absolutely convergent to $\mu^{(1)}_{\lambda, w_{(1)}}$ as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, the generalized V-submodule of $W^{(1)}_{\lambda, w_{(1)}}$ generated by $\lambda_n^{(1)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$. Then the iterate

$$I^1 \circ (I^2 \otimes 1_{W_3})$$

can be expressed as a product, and in fact, there exists a unique $P(z_1)$-intertwining map I_1 of type $\binom{W_4}{W_1 \ W_2 \boxtimes_{P(z_2)} W_3}$ such that

$$\langle w'_{(4)}, I^1(I^2(w_{(1)} \otimes w_{(2)}) \otimes w_{(3)}) \rangle = \langle w'_{(4)}, I_1(w_{(1)} \otimes (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \rangle$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$.

The hard part of the proof of this theorem is the proof of Lemma 9.22. This associativity of intertwining maps immediately gives the following important associativity of logarithmic intertwining operators, which is a strong version of logarithmic operator product expansion:

Corollary 9.24 *Assume that $\mathcal{C}$ is closed under images, that the convergence condition for intertwining maps in $\mathcal{C}$ holds and that*

$$|z_1| > |z_2| > |z_1 - z_2| > 0.$$

Let W_1, W_2, W_3, W_4, M_1 and M_2 be objects of $\mathcal{C}$. Assume also that $W_1 \boxtimes_{P(z_1-z_2)} W_2$ and $W_2 \boxtimes_{P(z_2)} W_3$ exist in $\mathcal{C}$.

1. *Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively. Suppose that for each $w'_{(4)} \in W'_4$, the element $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ given by*

$$\lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\, x_2=z_2}$$

   for $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ satisfies the $P^{(2)}(z_1 - z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, let $\sum_{n\in\mathbb{R}} \lambda_n^{(2)}$ be the (unique) series weakly absolutely convergent to $\mu_{\lambda,w_{(3)}}^{(2)}$ as indicated in the $P^{(2)}(z_1-z_2)$-grading condition (or the $L(0)$-semisimple $P^{(2)}(z_1 - z_2)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(3)} \in W_3$, the generalized V-submodule of $W_{\lambda,w_{(3)}}^{(2)}$ generated by $\lambda_n^{(2)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_1 \otimes W_2)^*$. Then there exists a unique logarithmic intertwining operator (a unique ordinary intertwining operator in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ of type $\binom{W_4}{W_1 \boxtimes_{P(z_1-z_2)} W_2\ W_3}$ such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_2(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\, x_2=z_2}$$
$$= \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}_{\boxtimes_{P(z_1-z_2)},0}(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0=z_1-z_2,\, x_2=z_2}$$

   for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. In particular, the product of the logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ and $\mathcal{Y}_2$ evaluated at z_1 and z_2, respectively, can be expressed as an iterate (with the intermediate generalized V-module $W_1 \boxtimes_{P(z_1-z_2)} W_2$) of logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) evaluated at z_2 and $z_1 - z_2$.

2. *Analogously, let $\mathcal{Y}^1$ and $\mathcal{Y}^2$ be logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) of types $\binom{W_4}{M_2 W_3}$ and $\binom{M_2}{W_1 W_2}$, respectively. Suppose that for each $w'_{(4)} \in W'_4$, the element $\lambda \in (W_1 \otimes W_2 \otimes W_3)^*$ given by*

$$\lambda(w_{(1)} \otimes w_{(2)} \otimes w_{(3)}) = \langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0=z_1-z_2,\, x_2=z_2}$$

satisfies the $P^{(1)}(z_2)$-local grading restriction condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-local grading restriction condition when $\mathcal{C}$ is in $\mathcal{M}_{sg}$). For $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, let $\sum_{n\in\mathbb{R}} \lambda_n^{(1)}$ be the (unique) series weakly absolutely convergent to $\mu^{(1)}_{\lambda,w_{(1)}}$ as indicated in the $P^{(1)}(z_2)$-grading condition (or the $L(0)$-semisimple $P^{(1)}(z_2)$-grading condition). Suppose also that for each $n \in \mathbb{R}$, $w'_{(4)} \in W'_4$ and $w_{(1)} \in W_1$, the generalized V-submodule of $W^{(1)}_{\lambda,w_{(1)}}$ generated by $\lambda_n^{(1)}$ is a generalized V-submodule of some object of $\mathcal{C}$ included in $(W_2 \otimes W_3)^*$. Then there exists a unique logarithmic intertwining operator (a unique ordinary intertwining operator in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}_1$ of type $\binom{W_4}{W_1\,W_2\boxtimes_{P(z_2)}W_3}$ such that

$$
\langle w'_{(4)}, \mathcal{Y}^1(\mathcal{Y}^2(w_{(1)}, x_0)w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_0=z_1-z_2,\, x_2=z_2}
$$
$$
= \langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_1)\mathcal{Y}_{\boxtimes_{P(z_2)},0}(w_{(2)}, x_2)w_{(3)}\rangle\Big|_{x_1=z_1,\, x_2=z_2}
$$

for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$. In particular, the iterate of the logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) $\mathcal{Y}^1$ and $\mathcal{Y}^2$ evaluated at z_2 and $z_1 - z_2$, respectively, can be expressed as a product (with the intermediate generalized V-module $W_2 \boxtimes_{P(z_2)} W_3$) of logarithmic intertwining operators (ordinary intertwining operators in the case that $\mathcal{C}$ is in $\mathcal{M}_{sg}$) evaluated at z_1 and z_2.

In Section 10, we construct the associativity isomorphisms, under certain assumptions: In addition to the assumptions above, we assume that $\mathcal{C}$ is closed under images and that for some $z \in \mathbb{C}^\times$ (and hence for every $z \in \mathbb{C}^\times$), $\mathcal{C}$ is closed under $P(z)$-tensor products; see Assumption 10.1. Besides the convergence condition (Section 7), at the end of Section 9 we introduce what we call the "expansion condition," which, roughly speaking, states that an element of $(W_1 \otimes W_2 \otimes W_3)^*$ obtained from a product or an iterate of intertwining maps satisfies the $P^{(2)}(z)$- or $P^{(1)}(z)$-local grading restriction condition, respectively, for suitable $z \in \mathbb{C}^\times$, along with certain other "minor" conditions. Then we have:

Theorem 10.3 *Assume that the convergence condition and the expansion condition for intertwining maps in $\mathcal{C}$ both hold. Let z_1, z_2 be complex numbers satisfying*

$$
|z_1| > |z_2| > |z_1 - z_2| > 0
$$

(so that in particular $z_1 \neq 0$, $z_2 \neq 0$ and $z_1 \neq z_2$). Then there exists a unique natural isomorphism

$$
\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : \boxtimes_{P(z_1)} \circ (1 \times \boxtimes_{P(z_2)}) \to \boxtimes_{P(z_2)} \circ (\boxtimes_{P(z_1-z_2)} \times 1)
$$

such that for all $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$, with W_j objects of $\mathcal{C}$,

$$
\overline{\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) = (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)},
$$

where for simplicity we use the same notation $\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)}$ *to denote the isomorphism of generalized modules*

$$\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \longrightarrow (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3.$$

Here we are using the notation

$$\overline{\eta} : \overline{W_1} \to \overline{W_2}$$

to denote the natural extension of a map $\eta : W_1 \to W_2$ of generalized modules to the (suitably defined) formal completions; such natural extensions enter into many of the constructions in this work.

In Section 11, we give results which will allow us to verify the convergence and expansion conditions. We need the "convergence and extension property" for products or iterates and the "convergence and extension property without logarithms" for products or iterates. Here we only give the convergence and extension property for products:

Given objects W_1, W_2, W_3, W_4 and M_1 of the category $\mathcal{C}$, let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be logarithmic intertwining operators of types $\binom{W_4}{W_1 M_1}$ and $\binom{M_1}{W_2 W_3}$, respectively.

Convergence and extension property for products For any $\beta \in \tilde{A}$, there exists an integer N_β depending only on $\mathcal{Y}_1$, $\mathcal{Y}_2$ and β, and for any weight-homogeneous elements $w_{(1)} \in (W_1)^{(\beta_1)}$ and $w_{(2)} \in (W_2)^{(\beta_2)}$ $(\beta_1, \beta_2 \in \tilde{A})$ and any $w_{(3)} \in W_3$ and $w'_{(4)} \in W'_4$ such that

$$\beta_1 + \beta_2 = -\beta,$$

there exist $M \in \mathbb{N}$, $r_k, s_k \in \mathbb{R}$, $i_k, j_k \in \mathbb{N}$, $k = 1, \ldots, M$; $K \in \mathbb{Z}_+$ independent of $w_{(1)}$ and $w_{(2)}$ such that each $i_k < K$; and analytic functions $f_k(z)$ on $|z| < 1$, $k = 1, \ldots, M$, satisfying

$$\text{wt } w_{(1)} + \text{wt } w_{(2)} + s_k > N_\beta, \quad k = 1, \ldots, M,$$

such that

$$\langle w'_{(4)}, \mathcal{Y}_1(w_{(1)}, x_2) \mathcal{Y}_2(w_{(2)}, x_2) w_{(3)} \rangle_{W_4} \Big|_{x_1 = z_1,\ x_2 = z_2}$$

is absolutely convergent when $|z_1| > |z_2| > 0$ and can be analytically extended to the multivalued analytic function

$$\sum_{k=1}^{M} z_2^{r_k} (z_1 - z_2)^{s_k} (\log z_2)^{i_k} (\log(z_1 - z_2))^{j_k} f_k \left(\frac{z_1 - z_2}{z_2} \right)$$

(here $\log(z_1 - z_2)$ and $\log z_2$, and in particular, the powers of the variables, mean the multivalued functions, not the particular branch we have been using) in the region $|z_2| > |z_1 - z_2| > 0$.

Theorem 11.4 *Suppose that the following two conditions are satisfied:*

1. *Every finitely-generated lower bounded doubly-graded (as defined in Section 11) generalized V-module is an object of $\mathcal{C}$ (or every finitely-generated lower bounded doubly-graded V-module is an object of $\mathcal{C}$, when $\mathcal{C}$ is in $\mathcal{M}_{sg}$).*

2. *The convergence and extension property for either products or iterates holds in $\mathcal{C}$ (or the convergence and extension property without logarithms for either products or iterates holds in $\mathcal{C}$, when $\mathcal{C}$ is in $\mathcal{M}_{sg}$).*

Then the convergence and expansion conditions for intertwining maps in $\mathcal{C}$ both hold.

In the following two results, we assume that that the grading abelian groups A and $\tilde{A}$ are trivial. Set

$$V_+ = \coprod_{n > 0} V_{(n)}.$$

Let W be a generalized V-module and let

$$C_1(W) = \mathrm{span}\{u_{-1}w \mid u \in V_+,\ w \in W\}.$$

If $W/C_1(W)$ is finite dimensional, we say that W is C_1-cofinite or satisfies the C_1-cofiniteness condition. If for any $N \in \mathbb{R}$, $\coprod_{n<N} W_{[n]}$ is finite dimensional, we say that W is quasi-finite dimensional or satisfies the quasi-finite-dimensionality condition. The following result in Section 11 allows us to verify the convergence and extension properties and thus the convergence and expansion conditions:

Theorem 11.6 *Let W_i for $i = 0, \ldots, n+1$ be generalized V-modules satisfying the C_1-cofiniteness condition and the quasi-finite-dimensionality condition. Then for any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \ldots, w_{(n+1)} \in W_{n+1}$, there exist*

$$a_{k,\,l}(z_1, \ldots, z_n) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}, (z_1 - z_2)^{-1}, (z_1 - z_3)^{-1}, \ldots, (z_{n-1} - z_n)^{-1}],$$

for $k = 1, \ldots, m$ and $l = 1, \ldots, n$, such that the following holds: For any generalized V-modules $\widetilde{W}_1, \ldots, \widetilde{W}_{n-1}$, and any logarithmic intertwining operators

$$\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

of types

$$\binom{W_0}{W_1 \widetilde{W}_1}, \binom{\widetilde{W}_1}{W_2 \widetilde{W}_2}, \ldots, \binom{\widetilde{W}_{n-2}}{W_{n-1} \widetilde{W}_{n-1}}, \binom{\widetilde{W}_{n-1}}{W_n W_{n+1}},$$

respectively, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle$$

satisfies the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^{m} \iota_{|z_1| > \cdots > |z_n| > 0}(a_{k,\,l}(z_1, \ldots, z_n)) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \ldots, n$$

in the region $|z_1| > \cdots > |z_n| > 0$, where

$$\iota_{|z_1| > \cdots > |z_n| > 0}(a_{k,\,l}(z_1, \ldots, z_n))$$

for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ are the (unique) Laurent expansions of $a_{k,l}(z_1, \ldots, z_n)$ in the region $|z_1| > \cdots > |z_n| > 0$. Moreover, for any set of possible singular points of the system

$$\frac{\partial^m \varphi}{\partial z_l^m} + \sum_{k=1}^{m} a_{k,l}(z_1, \ldots, z_n) \frac{\partial^{m-k} \varphi}{\partial z_l^{m-k}} = 0, \quad l = 1, \ldots, n$$

such that either $z_i = 0$ or $z_i = \infty$ for some i or $z_i = z_j$ for some $i \neq j$, the $a_{k,l}(z_1, \ldots, z_n)$ can be chosen for $k = 1, \ldots, m$ and $l = 1, \ldots, n$ so that these singular points are regular.

Using this result, we prove the following:

Theorem 11.8 *Suppose that all generalized V-modules in $\mathcal{C}$ satisfy the C_1-cofiniteness condition and the quasi-finite-dimensionality condition. Then:*

1. *The convergence and extension properties for products and iterates hold in $\mathcal{C}$. If $\mathcal{C}$ is in $\mathcal{M}_{sg}$ and if every object of $\mathcal{C}$ is a direct sum of irreducible objects of $\mathcal{C}$ and there are only finitely many irreducible objects of $\mathcal{C}$ (up to equivalence), then the convergence and extension properties without logarithms for products and iterates hold in $\mathcal{C}$.*

2. *For any $n \in \mathbb{Z}_+$, any objects $W_0, \ldots, W_{n+1}$ and $\widetilde{W}_1, \ldots, \widetilde{W}_{n-1}$ of $\mathcal{C}$, any logarithmic intertwining operators*

$$\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$$

   of types

$$\begin{pmatrix} W_0 \\ W_1 \widetilde{W}_1 \end{pmatrix}, \begin{pmatrix} \widetilde{W}_1 \\ W_2 \widetilde{W}_2 \end{pmatrix}, \ldots, \begin{pmatrix} \widetilde{W}_{n-2} \\ W_{n-1} \widetilde{W}_{n-1} \end{pmatrix}, \begin{pmatrix} \widetilde{W}_{n-1} \\ W_n W_{n+1} \end{pmatrix},$$

   respectively, and any $w'_{(0)} \in W'_0$, $w_{(1)} \in W_1, \ldots, w_{(n+1)} \in W_{n+1}$, the series

$$\langle w'_{(0)}, \mathcal{Y}_1(w_{(1)}, z_1) \cdots \mathcal{Y}_n(w_{(n)}, z_n) w_{(n+1)} \rangle \tag{11.37}$$

   is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically extended to a multivalued analytic function on the region given by $z_i \neq 0$, $i = 1, \ldots, n$, $z_i \neq z_j$, $i \neq j$, such that for any set of possible singular points with either $z_i = 0$, $z_i = \infty$ or $z_i = z_j$ for $i \neq j$, this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points.

We now return to the assumptions before Theorem 11.6, that is, we do not assume that A and $\tilde{A}$ are trivial. To construct the braided tensor category structure, we need more assumptions in addition to those mentioned above, which are collected in Assumption 10.1. We assume in addition that the Möbius or conformal vertex algebra V, viewed as a V-module, is an object of $\mathcal{C}$; and also that the product of three logarithmic intertwining operators is absolutely convergent in a suitable region and can be analytically extended to a multivalued analytic function, admitting suitable expansions as series in powers of the variables and their

logarithms near its singularities (expansions that hold for solutions of systems of differential equations with regular singularities), on a suitable largest possible region containing the original region for the convergence of the product. See Assumptions 12.1 and 12.2 for the precise statements. Under these assumptions, we construct, in addition to the tensor product bifunctor $\boxtimes = \boxtimes_{P(1)}$, a braiding isomorphism $\mathcal{R}$, an associativity isomorphism $\mathcal{A}$ (for the braided tensor category structure, different from the associativity isomorphisms above), a left unit isomorphism l and a right unit isomorphism r. The following main results of this work are given in Section 12:

Theorem 12.15 *Let V be a Möbius or conformal vertex algebra and $\mathcal{C}$ a full subcategory of $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$ satisfying Assumptions 10.1, 12.1 and 12.2. Then the category $\mathcal{C}$, equipped with the tensor product bifunctor $\boxtimes$, the unit object V, the braiding isomorphism $\mathcal{R}$, the associativity isomorphism $\mathcal{A}$, and the left and right unit isomorphisms l and r, is an additive braided monoidal category.*

Corollary 12.16 *If the category $\mathcal{C}$ is an abelian category, then $\mathcal{C}$, equipped with the tensor product bifunctor $\boxtimes$, the unit object V, the braiding isomorphism $\mathcal{R}$, the associativity isomorphism $\mathcal{A}$, and the left and right unit isomorphisms l and r, is a braided tensor category.*

2 The setting: strongly graded conformal and Möbius vertex algebras and their generalized modules

In this section we define and discuss the basic structures and introduce some notation that will be used in this work. More specifically, we first introduce the notions of "conformal vertex algebra" and "Möbius vertex algebra." A conformal vertex algebra is just a vertex algebra equipped with a conformal vector satisfying the usual axioms; a Möbius vertex algebra is a variant of a "quasi-vertex operator algebra" as in [FHL], with the difference that the two grading restriction conditions in the definition of vertex operator algebra are not required. We then define the notion of module for each of these types of vertex algebra. Relaxing the $L(0)$-semisimplicity in the definition of module we obtain the notion of "generalized module." Finally, we notice that in order to have a contragredient functor on the module category under consideration, we need to impose a stronger grading condition. This leads to the notions of "strong gradedness" of Möbius vertex algebras and their generalized modules. In this work we are mainly interested in certain full subcategories of the category of strongly graded generalized modules for certain strongly graded Möbius vertex algebras.

Throughout the work we shall assume some familiarity with the material in [B], [FLM2], [FHL], [DL] and [LL].

In particular, we recall the necessary basic material on "formal calculus," starting with the "formal delta function." Formal calculus will be needed throughout this work, and in fact, the theory of formal calculus will be considerably developed, whenever new formal-calculus ideas are needed for the formulations and for the proofs of the results.

Throughout, we shall use the notation $\mathbb{N}$ for the nonnegative integers and $\mathbb{Z}_+$ for the positive integers.

We shall continue to use the notational convention concerning formal variables and complex variables given in Remark 1.3. Recall from [FLM2], [FHL] or [LL] that the formal delta function is defined as the formal series

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

in the formal variable x. We will consistently use the *binomial expansion convention*: For any complex number λ, $(x+y)^\lambda$ is to be expanded as a formal series in nonnegative integral powers of the second variable, i.e.,

$$(x+y)^\lambda = \sum_{n \in \mathbb{N}} \binom{\lambda}{n} x^{\lambda-n} y^n.$$

Here x or y might be something other than a formal variable (or a nonzero complex multiple of a formal variable); for instance, x or y (but not both; this expansion is understood to be formal) might be a nonzero complex number, or x or y might be some more complicated object. The use of the binomial expansion convention will be clear in context.

Objects like $\delta(x)$ and $(x + y)^\lambda$ lie in spaces of formal series. Some of the spaces that we will use are, with W a vector space (over $\mathbb{C}$) and x a formal variable:

$$W[x] = \left\{ \sum_{n \in \mathbb{N}} a_n x^n \,|\, a_n \in W, \text{ all but finitely many } a_n = 0 \right\}$$

(the space of formal polynomials with coefficients in W),

$$W[x, x^{-1}] = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \,|\, a_n \in W, \text{ all but finitely many } a_n = 0 \right\}$$

(the formal Laurent polynomials),

$$W[[x]] = \left\{ \sum_{n \in \mathbb{N}} a_n x^n \,|\, a_n \in W \text{ (with possibly infinitely many } a_n \text{ not } 0) \right\}$$

(the formal power series),

$$W((x)) = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \,|\, a_n \in W, \ a_n = 0 \text{ for sufficiently small } n \right\}$$

(the truncated formal Laurent series), and

$$W[[x, x^{-1}]] = \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \,|\, a_n \in W \text{ (with possibly infinitely many } a_n \text{ not } 0) \right\}$$

(the formal Laurent series). We will also need the space

$$W\{x\} = \left\{ \sum_{n \in \mathbb{C}} a_n x^n \,|\, a_n \in W \text{ for } n \in \mathbb{C} \right\} \tag{2.1}$$

as in [FLM2]; here the powers of the formal variable are complex, and the coefficients may all be nonzero. We will also use analogues of these spaces involving two or more formal variables. Note that for us, a "formal power series" involves only nonnegative integral powers of the formal variable(s), and a "formal Laurent series" can involve all the integral powers of the formal variable(s).

The following formal version of Taylor's theorem is easily verified by direct expansion (see Proposition 8.3.1 of [FLM2]): For $f(x) \in W\{x\}$,

$$e^{y \frac{d}{dx}} f(x) = f(x + y), \tag{2.2}$$

where the exponential denotes the formal exponential series, and where we are using the binomial expansion convention on the right-hand side. It is important to note that this formula holds for arbitrary formal series $f(x)$ with complex powers of x, where $f(x)$ need not be an expansion in any sense of an analytic function (again, see Proposition 8.3.1 of [FLM2]).

The formal delta function $\delta(x)$ has the following simple and fundamental property: For any $f(x) \in W[x, x^{-1}]$,

$$f(x)\delta(x) = f(1)\delta(x). \tag{2.3}$$

(Here we are taking the liberty of writing complex numbers to the right of vectors in W.) This is proved immediately by observing its truth for $f(x) = x^n$ and then using linearity. This property has many important variants; in general, whenever an expression is multiplied by the formal delta function, we may formally set the argument appearing in the delta function equal to 1, provided that the relevant algebraic expressions make sense. For example, for any

$$X(x_1, x_2) \in (\text{End } W)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$$

such that

$$\lim_{x_1 \to x_2} X(x_1, x_2) = X(x_1, x_2)\Big|_{x_1 = x_2} \tag{2.4}$$

exists, we have

$$X(x_1, x_2)\delta\left(\frac{x_1}{x_2}\right) = X(x_2, x_2)\delta\left(\frac{x_1}{x_2}\right). \tag{2.5}$$

The existence of the "algebraic limit" defined in (2.4) means that for an arbitrary vector $w \in W$, the coefficient of each power of x_2 in the formal expansion $X(x_1, x_2)w\Big|_{x_1 = x_2}$ is a finite sum. In general, the existence of such "algebraic limits," and also such products of formal sums, always means that the coefficient of each monomial in the relevant formal variables gives a finite sum. Often, proving the existence of the relevant algebraic limits (or products) is a much more subtle matter than computing such limits (or products), just as in analysis. (In this work, we will typically use "substitution notation" like $\Big|_{x_1 = x_2}$ or $X(x_2, x_2)$ rather than the formal limit notation on the left-hand side of (2.4).) Below, we will give a more sophisticated analogue of the delta-function substitution principle (2.5), an analogue that we will need in this work.

This analogue, and in fact, many fundamental principles of vertex operator algebra theory, are based on certain delta-function expressions of the following type, involving three (commuting and independent, as usual) formal variables:

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m \in \mathbb{N},\, n \in \mathbb{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^{n-m} x_2^m;$$

here the binomial expansion convention is of course being used.

The following important identities involving such three-variable delta-function expressions are easily proved (see [FLM2] or [LL], where extensive motivation for these formulas is also given):

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) = x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right), \tag{2.6}$$

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right). \tag{2.7}$$

Note that the three terms in (2.7) involve nonnegative integral powers of x_2, x_1 and x_0, respectively. In particular, the two terms on the left-hand side of (2.7) are unequal formal Laurent series in three variables, even though they might appear equal at first glance. We shall use these two identities extensively.

Remark 2.1 Here is the useful analogue, mentioned above, of the delta-function substitution principle (2.5): Let

$$f(x_1, x_2, y) \in (\text{End } W)[[x_1, x_1^{-1}, x_2, x_2^{-1}, y, y^{-1}]] \qquad (2.8)$$

be such that

$$\lim_{x_1 \to x_2} f(x_1, x_2, y) \text{ exists} \qquad (2.9)$$

and such that for any $w \in W$,

$$f(x_1, x_2, y)w \in W[[x_1, x_1^{-1}, x_2, x_2^{-1}]]((y)). \qquad (2.10)$$

Then

$$x_1^{-1} \delta \left(\frac{x_2 - y}{x_1} \right) f(x_1, x_2, y) = x_1^{-1} \delta \left(\frac{x_2 - y}{x_1} \right) f(x_2 - y, x_2, y). \qquad (2.11)$$

For this principle, see Remark 2.3.25 of [LL], where the proof is also presented.

The following formal residue notation will be useful: For

$$f(x) = \sum_{n \in \mathbb{C}} a_n x^n \in W\{x\}$$

(note that the powers of x need not be integral),

$$\text{Res}_x f(x) = a_{-1}.$$

For instance, for the expression in (2.6),

$$\text{Res}_{x_2} x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) = 1. \qquad (2.12)$$

For a vector space W, we will denote its vector space dual by W^* ($= \text{Hom}_{\mathbb{C}}(W, \mathbb{C})$), and we will use the notation $\langle \cdot, \cdot \rangle_W$, or $\langle \cdot, \cdot \rangle$ if the underlying space W is clear, for the canonical pairing between W^* and W.

We will use the following version of the notion of "conformal vertex algebra": A conformal vertex algebra is a vertex algebra (in the sense of Borcherds [B]; see [LL]) equipped with a $\mathbb{Z}$-grading and with a conformal vector satisfying the usual compatibility conditions. Specifically:

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Definition 2.2 A *conformal vertex algebra* is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)} \tag{2.13}$$

(for $v \in V_{(n)}$, we say the *weight* of v is n and we write $\operatorname{wt} v = n$) equipped with a linear map $V \otimes V \to V[[x, x^{-1}]]$, or equivalently,

$$
\begin{aligned}
V &\to (\operatorname{End} V)[[x, x^{-1}]] \\
v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \operatorname{End} V),
\end{aligned}
\tag{2.14}
$$

$Y(v, x)$ denoting the *vertex operator associated with* v, and equipped also with two distinguished vectors $\mathbf{1} \in V_{(0)}$ (the *vacuum vector*) and $\omega \in V_{(2)}$ (the *conformal vector*), satisfying the following conditions for $u, v \in V$: the *lower truncation condition*:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large} \tag{2.15}$$

(or equivalently, $Y(u, x)v \in V((x))$); the *vacuum property*:

$$Y(\mathbf{1}, x) = 1_V; \tag{2.16}$$

the *creation property*:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v \tag{2.17}$$

(that is, $Y(v, x)\mathbf{1}$ involves only nonnegative integral powers of x and the constant term is v); the *Jacobi identity* (the main axiom):

$$
x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1)
$$
$$
= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) \tag{2.18}
$$

(note that when each expression in (2.18) is applied to any element of V, the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation $Y(\cdot, x_2)$ is understood to be extended in the obvious way to $V[[x_0, x_0^{-1}]]$); the *Virasoro algebra relations*:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c \tag{2.19}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}, \tag{2.20}$$

$$c \in \mathbb{C} \tag{2.21}$$

(the *central charge* or *rank* of V);

$$\frac{d}{dx}Y(v,x) = Y(L(-1)v, x) \tag{2.22}$$

(the $L(-1)$-*derivative property*); and

$$L(0)v = nv = (\text{wt } v)v \quad \text{for} \quad n \in \mathbb{Z} \text{ and } v \in V_{(n)}. \tag{2.23}$$

This completes the definition of the notion of conformal vertex algebra. We will denote such a conformal vertex algebra by $(V, Y, \mathbf{1}, \omega)$ or simply by V.

The only difference between the definition of conformal vertex algebra and the definition of *vertex operator algebra* (in the sense of [FLM2] and [FHL]) is that a vertex operator algebra V also satisfies the two *grading restriction conditions*

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently negative,} \tag{2.24}$$

and

$$\dim V_{(n)} < \infty \quad \text{for } n \in \mathbb{Z}. \tag{2.25}$$

(As we mentioned above, a *vertex algebra* is the same thing as a conformal vertex algebra but without the assumptions of a grading or a conformal vector, or, of course, the $L(n)$'s.)

Remark 2.3 Of course, not every vertex algebra is conformal. For example, it is well known [B] that any commutative associative algebra A with unit 1, together with a derivation $D : A \to A$ can be equipped with a vertex algebra structure, by:

$$Y(\cdot, x)\cdot : A \times A \to A[[x]], \quad Y(a, x)b = (e^{xD}a)b,$$

and $\mathbf{1} = 1$. In particular, $u_n = 0$ for any $u \in A$ and $n \geq 0$. If ω is a conformal vector for such a vertex algebra, then for any $u \in A$, $Du = u_{-2}\mathbf{1} = L(-1)u$ from (2.17) and (2.22), so $D = L(-1) = \omega_0$, which equals 0 because $\omega = L(0)\omega/2 = \omega_1\omega/2 = 0$. Thus a vertex algebra constructed from a commutative associative algebra with nonzero derivation in this way cannot be conformal.

Remark 2.4 The theory of vertex tensor categories inherently uses the whole moduli space of spheres with two positively oriented punctures and one negatively oriented puncture (and in fact, more generally, with arbitrary numbers of positively oriented punctures and one negatively oriented puncture) equipped with general (analytic) local coordinates vanishing at the punctures. Because of the analytic local coordinates, our constructions require certain conditions on the Virasoro algebra operators. However, recalling the definition of the moduli space elements $P(z)$ from Section 1.4, we point out that if we restrict our attention to elements of the moduli space of only the type $P(z)$, then the relevant operations of sewing and subsequently decomposing Riemann spheres continue to yield spheres of the same type, and rather than general conformal transformations around the punctures, only Möbius (projective) transformations around the punctures are needed. This makes it possible to develop

the essential structure of our tensor product theory by working entirely with spheres of this special type; the general vertex tensor category theory then follows from the structure thus developed. This is why, in the present work, we are focusing on the theory of $P(z)$-tensor products. Correspondingly, it turns out that it is very natural for us to consider, along with the notion of conformal vertex algebra (Definition 2.2), a weaker notion of vertex algebra involving only the three-dimensional subalgebra of the Virasoro algebra corresponding to the group of Möbius transformations. That is, instead of requiring an action of the whole Virasoro algebra, we use only the action of the Lie algebra $\mathfrak{sl}(2)$ generated by $L(-1)$, $L(0)$ and $L(1)$. Thus we get a notion essentially identical to the notion of "quasi-vertex operator algebra" in [FHL]; the reason for focusing on this notion here is the same as the reason why it was considered in [FHL]. Here we designate this notion by the term "Möbius vertex algebra"; the only difference between the definition of Möbius vertex algebra and the definition of quasi-vertex operator algebra [FHL] is that a quasi-vertex operator algebra V also satisfies the two grading restriction conditions (2.24) and (2.25).

Thus we formulate:

Definition 2.5 The notion of *Möbius vertex algebra* is defined in the same way as that of conformal vertex algebra except that in addition to the data and axioms concerning V, Y and $\mathbf{1}$ (through (2.18) in Definition 2.2), we assume (in place of the existence of the conformal vector ω and the Virasoro algebra conditions (2.19), (2.20) and (2.21)) the following: We have a representation ρ of $\mathfrak{sl}(2)$ on V given by

$$L(j) = \rho(L_j), \quad j = 0, \pm 1, \tag{2.26}$$

where $\{L_{-1}, L_0, L_1\}$ is a basis of $\mathfrak{sl}(2)$ with Lie brackets

$$[L_0, L_{-1}] = L_{-1}, \quad [L_0, L_1] = -L_1, \quad \text{and} \quad [L_{-1}, L_1] = -2L_0, \tag{2.27}$$

and the following conditions hold for $v \in V$:

$$[L(-1), Y(v, x)] = Y(L(-1)v, x), \tag{2.28}$$

$$[L(0), Y(v, x)] = Y(L(0)v, x) + xY(L(-1)v, x), \tag{2.29}$$

$$[L(1), Y(v, x)] = Y(L(1)v, x) + 2xY(L(0)v, x) + x^2 Y(L(-1)v, x), \tag{2.30}$$

and also, (2.22) and (2.23). Of course, (2.28)–(2.30) can be written as

$$
\begin{aligned}
[L(j), Y(v, x)] &= \sum_{k=0}^{j+1} \binom{j+1}{k} x^k Y(L(j-k)v, x) \\
&= \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} Y(L(k-1)v, x)
\end{aligned}
\tag{2.31}
$$

for $j = 0, \pm 1$.

We will denote such a Möbius vertex algebra by $(V, Y, \mathbf{1}, \rho)$ or simply by V. Note that there is no notion of central charge (or rank) for a Möbius vertex algebra. Also, a conformal vertex algebra can certainly be viewed as a Möbius vertex algebra in the obvious way. (Of course, a conformal vertex algebra could have other $\mathfrak{sl}(2)$-structures making it a Möbius vertex algebra in a different way.)

Remark 2.6 By (2.26) and (2.27) we have $[L(0), L(j)] = -jL(j)$ for $j = 0, \pm 1$. Hence

$$L(j)V_{(n)} \subset V_{(n-j)}, \quad \text{for} \ \ j = 0, \pm 1. \tag{2.32}$$

Moreover, from (2.28), (2.29) and (2.30) with $v = \mathbf{1}$ we get, by (2.16) and (2.17),

$$L(j)\mathbf{1} = 0 \quad \text{for} \ \ j = 0, \pm 1.$$

Remark 2.7 Not every Möbius vertex algebra is conformal. As an example, take the commutative associative algebra $\mathbb{C}[t]$ with derivation $D = -d/dt$, and form a vertex algebra as in Remark 2.3. By Remark 2.3, this vertex algebra is not conformal. However, define linear operators

$$L(-1) = D, \quad L(0) = tD, \quad L(1) = t^2 D$$

on $\mathbb{C}[t]$. Then it is straightforward to verify that $\mathbb{C}[t]$ becomes a Möbius vertex algebra with these operators giving a representation of $\mathfrak{sl}(2)$ having the desired properties and with the $\mathbb{Z}$-grading (by nonpositive integers) given by the eigenspace decomposition with respect to $L(0)$.

Remark 2.8 It is also easy to see that not every vertex algebra is Möbius. For example, take the two-dimensional commutative associative algebra $A = \mathbb{C}1 \oplus \mathbb{C}a$ with 1 as identity and $a^2 = 0$. The linear operator D defined by $D(1) = 0$, $D(a) = a$ is a nonzero derivation of A. Hence A has a vertex algebra structure by Remark 2.3. Now if it is a module for $\mathfrak{sl}(2)$ as in Definition 2.5, since A is two-dimensional and $L(0)1 = 0$, $L(0)$ must act as 0. But then $D = L(-1) = [L(0), L(-1)] = 0$, a contradiction.

A module for a conformal vertex algebra V is a module for V viewed as a vertex algebra such that the conformal element acts in the same way as in the definition of vertex operator algebra. More precisely:

Definition 2.9 Given a conformal vertex algebra $(V, Y, \mathbf{1}, \omega)$, a *module* for V is a $\mathbb{C}$-graded vector space

$$W = \coprod_{n \in \mathbb{C}} W_{(n)} \tag{2.33}$$

(graded by *weights*) equipped with a linear map $V \otimes W \to W[[x, x^{-1}]]$, or equivalently,

$$
\begin{aligned}
V & \to (\text{End } W)[[x, x^{-1}]] \\
v & \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where} \ \ v_n \in \text{End } W)
\end{aligned}
\tag{2.34}
$$

(note that the sum is over $\mathbb{Z}$, not $\mathbb{C}$), $Y(v,x)$ denoting the *vertex operator on W associated with v*, such that all the defining properties of a conformal vertex algebra that make sense hold. That is, the following conditions are satisfied: the lower truncation condition: for $v \in V$ and $w \in W$,

$$v_n w = 0 \quad \text{for} \ n \ \text{sufficiently large} \tag{2.35}$$

(or equivalently, $Y(v,x)w \in W((x))$); the vacuum property:

$$Y(\mathbf{1}, x) = 1_W; \tag{2.36}$$

the Jacobi identity for vertex operators on W: for $u, v \in V$,

$$x_0^{-1}\delta\Big(\frac{x_1 - x_2}{x_0}\Big)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\Big(\frac{x_2 - x_1}{-x_0}\Big)Y(v, x_2)Y(u, x_1)$$

$$= x_2^{-1}\delta\Big(\frac{x_1 - x_0}{x_2}\Big)Y(Y(u, x_0)v, x_2) \tag{2.37}$$

(note that on the right-hand side, $Y(u, x_0)$ is the operator on V associated with u); the Virasoro algebra relations on W with scalar c equal to the central charge of V:

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}c \tag{2.38}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \ \text{i.e.,} \ \ Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}; \tag{2.39}$$

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x) \tag{2.40}$$

(the $L(-1)$-derivative property); and

$$(L(0) - n)w = 0 \quad \text{for} \ \ n \in \mathbb{C} \ \text{ and } \ w \in W_{(n)}. \tag{2.41}$$

This completes the definition of the notion of module for a conformal vertex algebra.

Remark 2.10 The Virasoro algebra relations (2.38) for a module action follow from the corresponding relations (2.19) for V together with the Jacobi identities (2.18) and (2.37) and the $L(-1)$-derivative properties (2.22) and (2.40), as we recall from (for example) [FHL] or [LL].

We also have:

Definition 2.11 The notion of *module* for a Möbius vertex algebra is defined in the same way as that of module for a conformal vertex algebra except that in addition to the data and axioms concerning W and Y (through (2.37) in Definition 2.9), we assume (in place of the Virasoro algebra conditions (2.38) and (2.39)) a representation ρ of $\mathfrak{sl}(2)$ on W given by (2.26) and the conditions (2.28), (2.29) and (2.30), for operators acting on W, and also, (2.40) and (2.41).

In addition to modules, we have the following notion of *generalized module* (or *logarithmic module*, as in, for example, [Mil1]):

Definition 2.12 A *generalized module* for a conformal (respectively, Möbius) vertex algebra is defined in the same way as a module for a conformal (respectively, Möbius) vertex algebra except that in the grading (2.33), each space $W_{(n)}$ is replaced by $W_{[n]}$, where $W_{[n]}$ is the generalized $L(0)$-eigenspace corresponding to the (generalized) eigenvalue $n \in \mathbb{C}$; that is, (2.33) and (2.41) in the definition are replaced by

$$W = \coprod_{n \in \mathbb{C}} W_{[n]} \tag{2.42}$$

and

$$\text{for } n \in \mathbb{C} \text{ and } w \in W_{[n]}, \ (L(0) - n)^m w = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large,} \tag{2.43}$$

respectively. For $w \in W_{[n]}$, we still write wt $w = n$ for the (generalized) weight of w.

We will denote such a module or generalized module just defined by (W, Y), or sometimes by (W, Y_W) or simply by W. We will use the notation

$$\pi_n : W \to W_{[n]} \tag{2.44}$$

for the projection from W to its subspace of (generalized) weight n, and for its natural extensions to spaces of formal series with coefficients in W. In either the conformal or Möbius case, a module is of course a generalized module.

Remark 2.13 For any vector space U on which an operator, say, $L(0)$, acts in such a way that

$$U = \coprod_{n \in \mathbb{C}} U_{[n]} \tag{2.45}$$

where for $n \in \mathbb{C}$,

$$U_{[n]} = \{u \in U | (L(0) - n)^m u = 0 \text{ for } m \in \mathbb{N} \text{ sufficiently large}\},$$

we shall typically use the same projection notation

$$\pi_n : U \to U_{[n]} \tag{2.46}$$

as in (2.44). If instead of (2.45) we have only

$$U = \sum_{n \in \mathbb{C}} U_{[n]},$$

then in fact this sum is indeed direct, and for any $L(0)$-stable subspace T of U, we have

$$T = \coprod_{n \in \mathbb{C}} T_{[n]}$$

(as with ordinary rather than generalized eigenspaces).

Remark 2.14 A module for a conformal vertex algebra V is obviously again a module for V viewed as a Möbius vertex algebra, and conversely, a module for V viewed as a Möbius vertex algebra is a module for V viewed as a conformal vertex algebra, by Remark 2.10. Similarly, the generalized modules for a conformal vertex algebra V are exactly the generalized modules for V viewed as a Möbius vertex algebra.

Remark 2.15 A conformal or Möbius vertex algebra is a module for itself (and in particular, a generalized module for itself).

Remark 2.16 In either the conformal or Möbius vertex algebra case, we have the obvious notions of *V-module homomorphism, submodule, quotient module*, and so on; in particular, homomorphisms are understood to be grading-preserving. We sometimes write the vector space of (generalized-) module maps (homomorphisms) $W_1 \to W_2$ for (generalized) V-modules W_1 and W_2 as $\mathrm{Hom}_V(W_1, W_2)$.

Remark 2.17 We have chosen the name "generalized module" here because the vector space underlying the module is graded by generalized eigenvalues. (This notion is different from the notion of "generalized module" used in [HL5]. A generalized module for a vertex operator algebra V as defined in, for example, Definition 2.11 of [HL5] is precisely a module for V viewed as a conformal vertex algebra.)

We will use the following notion of (formal algebraic) completion of a generalized module:

Definition 2.18 Let $W = \coprod_{n \in \mathbb{C}} W_{[n]}$ be a generalized module for a Möbius (or conformal) vertex algebra. We denote by $\overline{W}$ the (formal) completion of W with respect to the $\mathbb{C}$-grading, that is,

$$\overline{W} = \prod_{n \in \mathbb{C}} W_{[n]}. \tag{2.47}$$

We will use the same notation $\overline{U}$ for any $\mathbb{C}$-graded subspace U of W. We will continue to use the notation π_n for the projection from $\overline{W}$ to $W_{[n]}$:

$$\pi_n : \overline{W} \to W_{[n]}.$$

We will also continue to use the notation $\langle \cdot, \cdot \rangle_W$, or $\langle \cdot, \cdot \rangle$ if the underlying space is clear, for the canonical pairing between the subspace $\coprod_{n \in \mathbb{C}} (W_{[n]})^*$ of W^*, and $\overline{W}$. We are of course viewing $(W_{[n]})^*$ as embedded in W^* in the natural way, that is, for $w^* \in (W_{[n]})^*$,

$$\langle w^*, w \rangle_W = \langle w^*, w_n \rangle_{W_{[n]}} \tag{2.48}$$

for any $w = \sum_{m \in \mathbb{C}} w_m$ (finite sum) in W, where $w_m \in W_{[m]}$.

The following weight formula holds for generalized modules, generalizing the corresponding formula in the module case (cf. [Mil1]):

Proposition 2.19 *Let W be a generalized module for a Möbius (or conformal) vertex algebra V. Let both $v \in V$ and $w \in W$ be homogeneous. Then*

$$\text{wt } (v_n w) = \text{wt } v + \text{wt } w - n - 1 \quad \text{for any } n \in \mathbb{Z}, \tag{2.49}$$

$$\text{wt } (L(j)w) = \text{wt } w - j \quad \text{for } j = 0, \pm 1. \tag{2.50}$$

Proof Applying the $L(-1)$-derivative property (2.40) to formula (2.29), with the operators acting on W, and extracting the coefficient of x^{-n-1}, we obtain:

$$[L(0), v_n] = (L(0)\, v)_n + (-n - 1)v_n. \tag{2.51}$$

This can be written as

$$(L(0) - (\text{wt } v - n - 1))v_n = v_n L(0),$$

and so we have

$$(L(0) - (\text{wt } v + m - n - 1))v_n = v_n(L(0) - m)$$

for any $m \in \mathbb{C}$. Applying this repeatedly we get

$$(L(0) - (\text{wt } v + m - n - 1))^t v_n = v_n(L(0) - m)^t$$

for any $t \in \mathbb{N}$, $m \in \mathbb{C}$, and (2.49) follows.

For (2.50), since as operators acting on W we have

$$[L(0), L(j)] = -jL(j) \tag{2.52}$$

for $j = 0, \pm 1$, we get $(L(0) + j)L(j) = L(j)L(0)$ so that

$$(L(0) - m + j)L(j) = L(j)(L(0) - m)$$

for any $m \in \mathbb{C}$. Thus

$$(L(0) - m + j)^t L(j) = L(j)(L(0) - m)^t$$

for any $t \in \mathbb{N}$, $m \in \mathbb{C}$, and (2.50) follows. $\square$

Remark 2.20 From Proposition 2.19 we see that a generalized V-module W decomposes into submodules corresponding to the congruence classes of its weights modulo $\mathbb{Z}$: For $\mu \in \mathbb{C}/\mathbb{Z}$, let

$$W_{[\mu]} = \coprod_{\bar{n} = \mu} W_{[n]}, \tag{2.53}$$

where $\bar{n}$ denotes the equivalence class of $n \in \mathbb{C}$ in $\mathbb{C}/\mathbb{Z}$. Then

$$W = \coprod_{\mu \in \mathbb{C}/\mathbb{Z}} W_{[\mu]} \tag{2.54}$$

and each $W_{[\mu]}$ is a V-submodule of W. Thus if a generalized module W is indecomposable (in particular, if it is irreducible), then all complex numbers n for which $W_{[n]} \neq 0$ are congruent modulo $\mathbb{Z}$ to each other.

Remark 2.21 Let W be a generalized module for a Möbius (or conformal) vertex algebra V. We consider the "semisimple part" $L(0)_s \in \text{End } W$ of the operator $L(0)$:

$$L(0)_s w = n w \quad \text{for} \quad w \in W_{[n]}, \ n \in \mathbb{C}.$$

Then on W we have

$$[L(0)_s, v_n] = [L(0), v_n] \quad \text{for all} \ v \in V \ \text{and} \ n \in \mathbb{Z}; \tag{2.55}$$

$$[L(0)_s, L(j)] = [L(0), L(j)] \quad \text{for} \ j = 0, \pm 1. \tag{2.56}$$

Indeed, for homogeneous elements $v \in V$ and $w \in W$, (2.49) and (2.51) imply that

$$
\begin{aligned}
{[L(0)_s, v_n]w} &= L(0)_s(v_n w) - v_n(L(0)_s w) \\
&= (\text{wt } v + \text{wt } w - n - 1)v_n w - (\text{wt } w)v_n w \\
&= (\text{wt } v)v_n w + (-n-1)v_n w \\
&= (L(0)v)_n w + (-n-1)v_n w \\
&= [L(0), v_n]w.
\end{aligned}
$$

Similarly, for any homogeneous element $w \in W$ and $j = 0, \pm 1$, (2.50) and (2.52) imply that

$$
\begin{aligned}
{[L(0)_s, L(j)]w} &= L(0)_s(L(j)w) - L(j)(L(0)_s w) \\
&= (\text{wt } w - j)L(j)w - (\text{wt } w)L(j)w \\
&= -jL(j)w \\
&= [L(0), L(j)]w.
\end{aligned}
$$

Thus the "locally nilpotent part" $L(0) - L(0)_s$ of $L(0)$ commutes with the action of V and of $\mathfrak{sl}(2)$ on W. In other words, $L(0) - L(0)_s$ is a V-homomorphism from W to itself.

Now suppose that $L(1)$ acts locally nilpotently on a Möbius (or conformal) vertex algebra V, that is, for any $v \in V$, there is $m \in \mathbb{N}$ such that $L(1)^m v = 0$. Then generalizing formula (3.20) in [HL5] (the case of ordinary modules for a vertex operator algebra), we define the *opposite vertex operator* on a generalized V-module (W, Y_W) associated to $v \in V$ by

$$Y_W^o(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}), \tag{2.57}$$

that is, for $k \in \mathbb{Z}$ and $v \in V_{(k)}$,

$$
\begin{aligned}
Y_W^o(v, x) &= \sum_{n \in \mathbb{Z}} v_n^o x^{-n-1} \\
&= \sum_{n \in \mathbb{Z}} \left((-1)^k \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{-n-m-2+2k} \right) x^{-n-1},
\end{aligned} \tag{2.58}
$$

as in [HL5]. (In the present work, we are replacing the symbol $*$ used in [HL5] for opposite vertex operators by the symbol o; see also Section 5.1 below.) Here we are defining the component operators

$$v_n^o = (-1)^k \sum_{m \in \mathbb{N}} \frac{1}{m!} (L(1)^m v)_{-n-m-2+2k} \tag{2.59}$$

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for $v \in V_{(k)}$ and $n, k \in \mathbb{Z}$. Note that the $L(1)$-local nilpotence ensures well-definedness here. Clearly, $v \mapsto Y_W^o(v, x)$ is a linear map $V \to (\text{End } W)[[x, x^{-1}]]$ such that $V \otimes W \to W((x^{-1}))$ $(v \otimes w \mapsto Y_W^o(v, x)w)$.

By (2.59), (2.32) and (2.49), we see that for $n, k \in \mathbb{Z}$ and $v \in V_{(k)}$, the operator v_n^o is of generalized weight $n + 1 - k \ (= n + 1 - \text{wt } v)$, in the sense that

$$v_n^o W_{[m]} \subset W_{[m+n+1-k]} \quad \text{for any } m \in \mathbb{C}. \tag{2.60}$$

As mentioned in [HL5] (see (3.23) in [HL5]), the proof of the Jacobi identity in Theorem 5.2.1 of [FHL] proves the following *opposite Jacobi identity* for Y_W^o in the case where V is a vertex operator algebra and W is a V-module:

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W^o(v, x_2) Y_W^o(u, x_1)$$
$$- x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W^o(u, x_1) Y_W^o(v, x_2)$$
$$= x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W^o(Y(u, x_0)v, x_2) \tag{2.61}$$

for $u, v \in V$, and taking Res_{x_0} gives us the *opposite commutator formula*. Similarly, the proof of the $L(-1)$-derivative property in Theorem 5.2.1 of [FHL] proves the following $L(-1)$-*derivative property* for Y_W^o in the same case:

$$\frac{d}{dx} Y_W^o(v, x) = Y_W^o(L(-1)v, x). \tag{2.62}$$

The same proofs carry over and prove the opposite Jacobi identity and the $L(-1)$-derivative property for Y_W^o in the present case, where V is a Möbius (or conformal) vertex algebra with $L(1)$ acting locally nilpotently and where W is a generalized V-module. In the case in which V is a conformal vertex algebra, we have

$$Y_W^o(\omega, x) = Y_W(x^{-4}\omega, x^{-1}) = \sum_{n \in \mathbb{Z}} L(n) x^{n-2} \tag{2.63}$$

since $L(1)\omega = 0$.

For opposite vertex operators, we have the following analogues of (2.28)–(2.31) in the Möbius case:

Lemma 2.22 *For $v \in V$,*

$$[Y_W^o(v, x), L(1)] = Y_W^o(L(-1)v, x), \tag{2.64}$$

$$[Y_W^o(v, x), L(0)] = Y_W^o(L(0)v, x) + x Y_W^o(L(-1)v, x), \tag{2.65}$$

$$[Y_W^o(v, x), L(-1)] = Y_W^o(L(1)v, x) + 2x Y_W^o(L(0)v, x) + x^2 Y_W^o(L(-1)v, x). \tag{2.66}$$

Equivalently,

$$[Y_W^o(v,x), L(-j)] = \sum_{k=0}^{j+1} \binom{j+1}{k} x^k Y_W^o(L(j-k)v, x)$$

$$= \sum_{k=0}^{j+1} \binom{j+1}{k} x^{j+1-k} Y_W^o(L(k-1)v, x) \qquad (2.67)$$

for $j = 0, \pm 1$.

Proof For $j = 0, \pm 1$, by definition and (2.31) we have

$$[Y_W^o(v,x), L(j)] = -[L(j), Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})]$$

$$= -\sum_{k=0}^{j+1} \binom{j+1}{k} x^{-k} Y_W(L(j-k)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}). \quad (2.68)$$

By (5.2.14) in [FHL] and the fact that

$$x^{L(0)} L(j) x^{-L(0)} = x^{-j} L(j) \qquad (2.69)$$

(easily proved by applying to a homogeneous vector),

$$L(-1)e^{xL(1)}(-x^{-2})^{L(0)}$$
$$= e^{xL(1)} L(-1)(-x^{-2})^{L(0)} - 2xe^{xL(1)} L(0)(-x^{-2})^{L(0)} + x^2 e^{xL(1)} L(1)(-x^{-2})^{L(0)}$$
$$= -x^2 e^{xL(1)}(-x^{-2})^{L(0)} L(-1) - 2xe^{xL(1)}(-x^{-2})^{L(0)} L(0) - e^{xL(1)}(-x^{-2})^{L(0)} L(1)$$
$$= -e^{xL(1)}(-x^{-2})^{L(0)}(x^2 L(-1) + 2xL(0) + L(1)). \qquad (2.70)$$

We also have

$$L(1)e^{xL(1)}(-x^{-2})^{L(0)} = e^{xL(1)} L(1)(-x^{-2})^{L(0)}$$
$$= -x^{-2} e^{xL(1)}(-x^{-2})^{L(0)} L(1). \qquad (2.71)$$

By (2.70), (2.71), $L(0) = \frac{1}{2}[L(1), L(-1)]$ and $[L(1), L(0)] = L(1)$, we have

$$L(0)e^{xL(1)}(-x^{-2})^{L(0)}$$
$$= \frac{1}{2} L(1)L(-1)e^{xL(1)}(-x^{-2})^{L(0)} - \frac{1}{2} L(-1)L(1)e^{xL(1)}(-x^{-2})^{L(0)}$$
$$= -\frac{1}{2} L(1)e^{xL(1)}(-x^{-2})^{L(0)}(x^2 L(-1) + 2xL(0) + L(1))$$
$$\quad + \frac{1}{2} x^{-2} L(-1)e^{xL(1)}(-x^{-2})^{L(0)} L(1)$$
$$= \frac{1}{2} x^{-2} e^{xL(1)}(-x^{-2})^{L(0)} L(1)(x^2 L(-1) + 2xL(0) + L(1))$$
$$\quad - \frac{1}{2} x^{-2} e^{xL(1)}(-x^{-2})^{L(0)}(x^2 L(-1) + 2xL(0) + L(1))L(1)$$
$$= e^{xL(1)}(-x^{-2})^{L(0)} L(0) + x^{-1} e^{xL(1)}(-x^{-2})^{L(0)} L(1)$$
$$= e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + x^{-1} L(1)). \qquad (2.72)$$

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Thus we obtain

$$[Y_W^o(v, x), L(1)]$$
$$= -\sum_{k=0}^{2} \binom{2}{k} x^{-k} Y_W(L(1-k)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})$$
$$= -Y_W(L(1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) - 2x^{-1}Y_W(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})$$
$$\quad -x^{-2}Y_W(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})$$
$$= x^{-2}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}L(1)v, x^{-1})$$
$$\quad -2x^{-1}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + x^{-1}L(1))v, x^{-1})$$
$$\quad +x^{-2}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1) + 2xL(0) + L(1))v, x^{-1})$$
$$= Y_W(e^{xL(1)}(-x^{-2})^{L(0)}L(-1)v, x^{-1})$$
$$= Y_W^o(L(-1)v, x),$$

$$[Y_W^o(v, x), L(0)]$$
$$= -\sum_{k=0}^{1} \binom{1}{k} x^{-k} Y_W(L(-k)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})$$
$$= -Y_W(L(0)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) - x^{-1}Y_W(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})$$
$$= -Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(L(0) + x^{-1}L(1))v, x^{-1})$$
$$\quad +x^{-1}Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1) + 2xL(0) + L(1))v, x^{-1})$$
$$= Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(xL(-1) + L(0))v, x^{-1})$$
$$= Y_W^o(L(0)v, x) + xY_W^o(L(-1)v, x)$$

and

$$
\begin{aligned}
[Y_W^o(v, x), L(-1)] &= -Y_W(L(-1)e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1}) \\
&= Y_W(e^{xL(1)}(-x^{-2})^{L(0)}(x^2L(-1) + 2xL(0) + L(1))v, x^{-1}) \\
&= Y_W^o(L(1)v, x) + 2xY_W^o(L(0)v, x) + x^2Y_W^o(L(-1)v, x),
\end{aligned}
$$

proving the lemma.   □

As in Section 5.2 of [FHL], we can define a V-action on W^* as follows:

$$\langle Y'(v, x)w', w \rangle = \langle w', Y_W^o(v, x)w \rangle \tag{2.73}$$

for $v \in V$, $w' \in W^*$ and $w \in W$; the correspondence $v \mapsto Y'(v, x)$ is a linear map from V to $(\operatorname{End} W^*)[[x, x^{-1}]]$. Writing

$$Y'(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}$$

$(v_n \in \operatorname{End} W^*)$, we have

$$\langle v_n w', w \rangle = \langle w', v_n^o w \rangle \tag{2.74}$$

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for $v \in V$, $w' \in W^*$ and $w \in W$. (Actually, in [FHL] this V-action was defined on a space smaller than W^*, but this definition holds without change on all of W^*.) In the case in which V is a conformal vertex algebra we define the operators $L'(n)$ ($n \in \mathbb{Z}$) by

$$Y'(\omega, x) = \sum_{n \in \mathbb{Z}} L'(n) x^{-n-2};$$

then, by extracting the coefficient of x^{-n-2} in (2.73) with $v = \omega$ and using the fact that $L(1)\omega = 0$ we have

$$\langle L'(n)w', w \rangle = \langle w', L(-n)w \rangle \quad \text{for} \ \ n \in \mathbb{Z} \tag{2.75}$$

(see (2.63)), as in Section 5.2 of [FHL]. In the case where V is only a Möbius vertex algebra, we define operators $L'(-1)$, $L'(0)$ and $L'(1)$ on W^* by formula (2.75) for $n = 0, \pm 1$. It follows from (2.50) that

$$L'(j)(W_{[m]})^* \subset (W_{[m-j]})^* \tag{2.76}$$

for $m \in \mathbb{C}$ and $j = 0, \pm 1$. By combining (2.74) with (2.60) we get

$$v_n (W_{[m]})^* \subset (W_{[m+k-n-1]})^* \tag{2.77}$$

for any $n, k \in \mathbb{Z}$, $v \in V_{(k)}$ and $m \in \mathbb{C}$.

We have just seen that the $L(1)$-local nilpotence condition enables us to define a natural vertex operator action on the vector space dual of a generalized module for a Möbius (or conformal) vertex algebra. This condition is satisfied by all vertex operator algebras, due to (2.32) and the grading restriction condition (2.24). However, the functor $W \mapsto W^*$ is certainly not involutive, and W^* is not in general a generalized module. In this work we will need certain module categories equipped with an involutive "contragredient functor" $W \mapsto W'$ which generalizes the contragredient functor for the category of modules for vertex operator algebras. For this purpose, we introduce the following:

Definition 2.23 Let A be an abelian group. A Möbius (or conformal) vertex algebra

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}$$

is said to be *strongly graded with respect to A* (or *strongly A-graded*, or just *strongly graded* if the abelian group A is understood) if V is equipped with a second gradation, by A,

$$V = \coprod_{\alpha \in A} V^{(\alpha)},$$

such that the following conditions are satisfied: the two gradations are compatible, that is,

$$V^{(\alpha)} = \coprod_{n \in \mathbb{Z}} V_{(n)}^{(\alpha)} \quad (\text{where } V_{(n)}^{(\alpha)} = V_{(n)} \cap V^{(\alpha)}) \ \ \text{for any} \ \ \alpha \in A;$$

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for any $\alpha, \beta \in A$ and $n \in \mathbb{Z}$,

$$V_{(n)}^{(\alpha)} = 0 \quad \text{for} \ \ n \ \ \text{sufficiently negative;} \tag{2.78}$$

$$\dim V_{(n)}^{(\alpha)} < \infty; \tag{2.79}$$

$$\mathbf{1} \in V_{(0)}^{(0)}; \tag{2.80}$$

$$v_l V^{(\beta)} \subset V^{(\alpha+\beta)} \quad \text{for any} \ \ v \in V^{(\alpha)}, \ l \in \mathbb{Z}; \tag{2.81}$$

and

$$L(j) V^{(\alpha)} \subset V^{(\alpha)} \quad \text{for} \ \ j = 0, \pm 1. \tag{2.82}$$

If V is in fact a conformal vertex algebra, we in addition require that

$$\omega \in V_{(2)}^{(0)}, \tag{2.83}$$

so that for all $j \in \mathbb{Z}$, (2.82) follows from (2.81).

Remark 2.24 Note that the notion of conformal vertex algebra strongly graded with respect to the trivial group is exactly the notion of vertex operator algebra. Also note that (2.32), (2.78) and (2.82) imply the local nilpotence of $L(1)$ acting on V, and hence we have the construction and properties of opposite vertex operators on a generalized module for a strongly graded Möbius (or conformal) vertex algebra.

For (generalized) modules for a strongly graded algebra we will also have a second grading by an abelian group, and it is natural to allow this group to be larger than the second grading group A for the algebra. (Note that this already occurs for the *first* grading group, which is $\mathbb{Z}$ for algebras and $\mathbb{C}$ for (generalized) modules.) We now define the notions of strongly graded module and generalized module, and also, at the end of this definition, the notions of lower bounded such structures.

Definition 2.25 Let A be an abelian group and V a strongly A-graded Möbius (or conformal) vertex algebra. Let $\tilde{A}$ be an abelian group containing A as a subgroup. A V-module (respectively, generalized V-module)

$$W = \coprod_{n \in \mathbb{C}} W_{(n)} \quad (\text{respectively,} \ \ W = \coprod_{n \in \mathbb{C}} W_{[n]})$$

is said to be *strongly graded with respect to* $\tilde{A}$ (or *strongly $\tilde{A}$-graded*, or just *strongly graded*) if the abelian group $\tilde{A}$ is understood) if W is equipped with a second gradation, by $\tilde{A}$,

$$W = \coprod_{\beta \in \tilde{A}} W^{(\beta)}, \tag{2.84}$$

such that the following conditions are satisfied: the two gradations are compatible, that is, for any $\beta \in \tilde{A}$,

$$W^{(\beta)} = \coprod_{n \in \mathbb{C}} W_{(n)}^{(\beta)} \quad (\text{where} \ \ W_{(n)}^{(\beta)} = W_{(n)} \cap W^{(\beta)})$$

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$$\text{(respectively, } W^{(\beta)} = \coprod_{n \in \mathbb{C}} W^{(\beta)}_{[n]} \text{ (where } W^{(\beta)}_{[n]} = W_{[n]} \cap W^{(\beta)} \text{));}$$

for any $\alpha \in A$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$,

$$W^{(\beta)}_{(n+k)} = 0 \text{ (respectively, } W^{(\beta)}_{[n+k]} = 0) \quad \text{for} \quad k \in \mathbb{Z} \text{ sufficiently negative;} \qquad (2.85)$$

$$\dim W^{(\beta)}_{(n)} < \infty \text{ (respectively, } \dim W^{(\beta)}_{[n]} < \infty); \qquad (2.86)$$

$$v_l W^{(\beta)} \subset W^{(\alpha+\beta)} \quad \text{for any} \quad v \in V^{(\alpha)}, \, l \in \mathbb{Z}; \qquad (2.87)$$

and

$$L(j) W^{(\beta)} \subset W^{(\beta)} \quad \text{for} \quad j = 0, \pm 1. \qquad (2.88)$$

(Note that if V is in fact a conformal vertex algebra, then for all $j \in \mathbb{Z}$, (2.88) follows from (2.83) and (2.87).) A strongly $\tilde{A}$-graded (generalized) V-module W is said to be *lower bounded* if instead of (2.85), it satisfies the stronger condition that for any $\beta \in \tilde{A}$,

$$W^{(\beta)}_{(n)} = 0 \text{ (respectively, } W^{(\beta)}_{[n]} = 0) \quad \text{for} \quad \Re(n) \text{ sufficiently negative} \qquad (2.89)$$

$(n \in \mathbb{C})$.

Remark 2.26 A strongly A-graded conformal or Möbius vertex algebra is a strongly A-graded module for itself (and in particular, a strongly A-graded generalized module for itself), and is in fact lower bounded.

Remark 2.27 Let V be a vertex operator algebra, viewed (equivalently) as a conformal vertex algebra strongly graded with respect to the trivial group (recall Remark 2.24). Then the V-modules that are strongly graded with respect to the trivial group (in the sense of Definition 2.25) are exactly the ($\mathbb{C}$-graded) modules for V as a vertex operator algebra, with the grading restrictions as follows: For $n \in \mathbb{C}$,

$$W_{(n+k)} = 0 \quad \text{for} \quad k \in \mathbb{Z} \text{ sufficiently negative} \qquad (2.90)$$

and

$$\dim W_{(n)} < \infty, \qquad (2.91)$$

and the lower bounded such structures have (2.90) replaced by:

$$W_{(n)} = 0 \quad \text{for} \quad \Re(n) \text{ sufficiently negative.} \qquad (2.92)$$

Also, the generalized V-modules that are strongly graded with respect to the trivial group are exactly the generalized V-modules (in the sense of Definition 2.12) such that for $n \in \mathbb{C}$,

$$W_{[n+k]} = 0 \quad \text{for} \quad k \in \mathbb{Z} \text{ sufficiently negative} \qquad (2.93)$$

and

$$\dim W_{[n]} < \infty, \qquad (2.94)$$

and the lower bounded ones have (2.90) replaced by:

$$W_{[n]} = 0 \quad \text{for} \quad \Re(n) \text{ sufficiently negative.} \qquad (2.95)$$

Remark 2.28 In the strongly graded case, algebra and module homomorphisms are of course understood to preserve the grading by A or $\tilde{A}$.

Example 2.29 An important source of examples of strongly graded conformal vertex algebras and modules comes from the vertex algebras and modules associated with even lattices. Let L be an even lattice, i.e., a finite-rank free abelian group equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, not necessarily positive definite, such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. Then there is a natural structure of conformal vertex algebra on a certain vector space V_L; see [B] and Chapter 8 of [FLM2]. If the form $\langle \cdot, \cdot \rangle$ on L is also positive definite, then V_L is a vertex operator algebra (that is, the grading restrictions hold). If L is not necessarily positive definite, then V_L is equipped with a natural second grading given by L itself, making V_L a strongly L-graded conformal vertex algebra in the sense of Definition 2.23. Any (rational) sublattice M of the "dual lattice" L° of L containing L gives rise to a lower bounded strongly M-graded module for the strongly L-graded conformal vertex algebra (see Chapter 8 of [FLM2]; cf. [LL]).

In the next two remarks, we mention certain important properties of compositions of two or more vertex operators, properties that will also be important in the further generality of logarithmic intertwining operators in the future.

Remark 2.30 As mentioned in Remark 2.24, strong gradedness for a Möbius (or conformal) vertex algebra V implies the local nilpotence of $L(1)$ acting on V. In fact, strong gradedness implies much more that will be important for us: From (2.78), (2.79), (2.81) and (2.82) (and (2.83) in the conformal vertex algebra case), it is clear that strong gradedness for V implies the following *local grading restriction condition on V* (see [H7]):

(i) for any $m > 0$ and $v_{(1)}, \ldots, v_{(m)} \in V$, there exists $r \in \mathbb{Z}$ such that the coefficient of each monomial in $x_1, \ldots, x_{m-1}$ in the formal series

$$Y(v_{(1)}, x_1) \cdots Y(v_{(m-1)}, x_{m-1}) v_{(m)}$$

lies in $\coprod_{n>r} V_{(n)}$;

(ii) in the conformal vertex algebra case: for any element of the conformal vertex algebra V homogeneous with respect to the weight grading, the Virasoro-algebra submodule $M = \coprod_{n \in \mathbb{Z}} M_{(n)}$ (where $M_{(n)} = M \cap V_{(n)}$) of V generated by this element satisfies the following grading restriction conditions: $M_{(n)} = 0$ when n is sufficiently negative and $\dim M_{(n)} < \infty$ for $n \in \mathbb{Z}$

or

(ii′) in the Möbius vertex algebra case: for any element of the Möbius vertex algebra V homogeneous with respect to the weight grading, the $\mathfrak{sl}(2)$-submodule $M = \coprod_{n \in \mathbb{Z}} M_{(n)}$ (where $M_{(n)} = M \cap V_{(n)}$) of V generated by this element satisfies the following grading restriction conditions: $M_{(n)} = 0$ when n is sufficiently negative and $\dim M_{(n)} < \infty$ for $n \in \mathbb{Z}$.

As was pointed out in [H7], Condition (i) above was first stated in [DL] (see formula (9.39), Proposition 9.17 and Theorem 12.33 in [DL]) for generalized vertex algebras and abelian intertwining algebras (certain generalizations of vertex algebras); it guarantees the convergence, rationality and commutativity properties of the matrix coefficients of products of more than two vertex operators. Conditions (i) and (ii) (or (ii')) together ensure that all the essential results involving the Virasoro operators and the geometry of vertex operator algebras in [H1] and [H5] still hold for these algebras.

Remark 2.31 Similarly, from (2.85), (2.86), (2.87) and (2.88) (and (2.83) in the conformal vertex algebra case), it is clear that strong gradedness for (generalized) modules implies the following *local grading restriction condition on a (generalized) module W* for a strongly graded Möbius (or conformal) vertex algebra V:

(i) for any $m > 0$, $v_{(1)}, \ldots, v_{(m-1)} \in V$, $n \in \mathbb{C}$ and $w \in W_{[n]}$, there exists $r \in \mathbb{Z}$ such that the coefficient of each monomial in $x_1, \ldots, x_{m-1}$ in the formal series

$$Y(v_{(1)}, x_1) \cdots Y(v_{(m-1)}, x_{m-1})w$$

lies in $\coprod_{k>r} W_{[n+k]}$;

(ii) in the conformal vertex algebra case: for any $w \in W_{[n]}$ ($n \in \mathbb{C}$), the Virasoro-algebra submodule $M = \coprod_{k \in \mathbb{Z}} M_{[n+k]}$ (where $M_{[n+k]} = M \cap W_{[n+k]}$) of W generated by w satisfies the following grading restriction conditions: $M_{[n+k]} = 0$ when k is sufficiently negative and $\dim M_{[n+k]} < \infty$ for $k \in \mathbb{Z}$

or

(ii') in the Möbius vertex algebra case: for any $w \in W_{[n]}$ ($n \in \mathbb{C}$), the $\mathfrak{sl}(2)$-submodule $M = \coprod_{k \in \mathbb{Z}} M_{[n+k]}$ (where $M_{[n+k]} = M \cap W_{[n+k]}$) of W generated by w satisfies the following grading restriction conditions: $M_{[n+k]} = 0$ when k is sufficiently negative and $\dim M_{[n+k]} < \infty$ for $k \in \mathbb{Z}$.

Note that in the case of ordinary (as opposed to generalized) modules, all the generalized weight spaces such as $W_{[n]}$ mentioned here are ordinary weight spaces $W_{(n)}$. Analogous statments of course hold for lower bounded (generalized) modules.

With the strong gradedness condition on a (generalized) module, we can now define the corresponding notion of contragredient module. First we give:

Definition 2.32 Let $W = \coprod_{\beta \in \tilde{A}, n \in \mathbb{C}} W_{[n]}^{(\beta)}$ be a strongly $\tilde{A}$-graded generalized module for a strongly A-graded Möbius (or conformal) vertex algebra. For each $\beta \in \tilde{A}$ and $n \in \mathbb{C}$, let us identify $(W_{[n]}^{(\beta)})^*$ with the subspace of W^* consisting of the linear functionals on W vanishing on each $W_{[m]}^{(\gamma)}$ with $\gamma \neq \beta$ or $m \neq n$ (cf. (2.48)). We define W' to be the $(\tilde{A} \times \mathbb{C})$-graded vector subspace of W^* given by

$$W' = \coprod_{\beta \in \tilde{A}, n \in \mathbb{C}} (W')_{[n]}^{(\beta)}, \quad \text{where} \quad (W')_{[n]}^{(\beta)} = (W_{[n]}^{(-\beta)})^*; \tag{2.96}$$

we also use the notations

$$(W')^{(\beta)} = \coprod_{n \in \mathbb{C}} (W_{[n]}^{(-\beta)})^* \subset (W^{(-\beta)})^* \subset W^* \tag{2.97}$$

(where $(W^{(\beta)})^*$ consists of the linear functionals on W vanishing on all $W^{(\gamma)}$ with $\gamma \neq \beta$)
and

$$(W')_{[n]} = \coprod_{\beta \in \tilde{A}} (W_{[n]}^{(-\beta)})^* \subset (W_{[n]})^* \subset W^* \tag{2.98}$$

for the homogeneous subspaces of W' with respect to the $\tilde{A}$- and $\mathbb{C}$-grading, respectively
(The reason for the minus signs here will become clear below.) We will still use the notation
$\langle \cdot, \cdot \rangle_W$, or $\langle \cdot, \cdot \rangle$ when the underlying space is clear, for the canonical pairing between W' and

$$\overline{W} \subset \prod_{\beta \in \tilde{A}, \, n \in \mathbb{C}} W_{[n]}^{(\beta)}$$

(recall (2.47)).

Remark 2.33 In the case of ordinary rather than generalized modules, Definition 2.32 still
applies, and all of the generalized weight subspaces $W_{[n]}$ of W are ordinary weight spaces
$W_{(n)}$. In this case, we can write $(W')_{(n)}$ rather than $(W')_{[n]}$ for the corresponding subspace
of W'.

Let W be a strongly graded (generalized) module for a strongly graded Möbius (or
conformal) vertex algebra V. Recall that we have the action (2.73) of V on W^* and that
(2.77) holds. Furthermore, (2.59), (2.74) and (2.87) imply for any $n, k \in \mathbb{Z}$, $\alpha \in A$, $\beta \in \tilde{A}$,
$v \in V_{(k)}^{(\alpha)}$ and $m \in \mathbb{C}$,

$$v_n((W')_{[m]}^{(\beta)}) = v_n((W_{[m]}^{(-\beta)})^*) \subset (W_{[m+k-n-1]}^{(-\alpha-\beta)})^* = (W')_{[m+k-n-1]}^{(\alpha+\beta)}. \tag{2.99}$$

Thus v_n preserves W' for $v \in V$, $n \in \mathbb{Z}$. Similarly (in the Möbius case), (2.75), (2.76) and
(2.88) imply that W' is stable under the operators $L'(-1)$, $L'(0)$ and $L'(1)$, and in fact

$$L'(j)(W')_{[n]}^{(\beta)} \subset (W')_{[n-j]}^{(\beta)}$$

for any $j = 0, \pm 1$, $\beta \in \tilde{A}$ and $n \in \mathbb{C}$. In the case or ordinary rather than generalized modules,
the symbols $(W')_{[n]}^{(\beta)}$, etc., can be replaced by $(W')_{(n)}^{(\beta)}$, etc.

For any fixed $\beta \in \tilde{A}$ and $n \in \mathbb{C}$, by (2.43) and the finite-dimensionality (2.86) of $W_{[n]}^{(-\beta)}$,
there exists $N \in \mathbb{N}$ such that $(L(0) - n)^N W_{[n]}^{(-\beta)} = 0$. But then for any $w' \in (W')_{[n]}^{(\beta)}$,

$$\langle (L'(0) - n)^N w', w \rangle = \langle w', (L(0) - n)^N w \rangle = 0 \tag{2.100}$$

for all $w \in W$. Thus $(L'(0) - n)^N w' = 0$. So (2.43) holds with W replaced by W'. In the
case of ordinary modules, we of course take $N = 1$.

By (2.85) and (2.99) we have the lower truncation condition for the action Y' of V on W':

$$\text{For any } v \in V \text{ and } w' \in W', v_n w' = 0 \text{ for } n \text{ sufficiently large.} \qquad (2.101)$$

As a consequence, the Jacobi identity can now be formulated on W'. In fact, by the above, and using the same proofs as those of Theorems 5.2.1 and 5.3.1 in [FHL], together with Lemma 2.22, we obtain:

Theorem 2.34 *Let $\tilde{A}$ be an abelian group containing A as a subgroup and V a strongly A-graded Möbius (or conformal) vertex algebra. Let (W, Y) be a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module). Then the pair (W', Y') carries a strongly $\tilde{A}$-graded V-module (respectively, generalized V-module) structure, and*

$$(W'', Y'') = (W, Y).$$

If W is lower bounded, then so is W'. □

Definition 2.35 The pair (W', Y') in Theorem 2.34 will be called the *contragredient module* of (W, Y).

Let W_1 and W_2 be strongly $\tilde{A}$-graded (generalized) V-modules and let $f : W_1 \to W_2$ be a module homomorphism (which is of course understood to preserve both the $\mathbb{C}$-grading and the $\tilde{A}$-grading, and to preserve the action of $\mathfrak{sl}(2)$ in the Möbius case). Then by (2.74) and (2.75), the linear map

$$f' : W_2' \to W_1'$$

given by

$$\langle f'(w'_{(2)}), w_{(1)} \rangle = \langle w'_{(2)}, f(w_{(1)}) \rangle \qquad (2.102)$$

for any $w_{(1)} \in W_1$ and $w'_{(2)} \in W_2'$ is well defined and is clearly a module homomorphism from W_2' to W_1'.

Notation 2.36 In this work we will be especially interested in the case where V is strongly A-graded, and we will be focusing on the category of all strongly $\tilde{A}$-graded (ordinary) V-modules, for which we will use the notation

$$\mathcal{M}_{sg},$$

or the category of all strongly $\tilde{A}$-graded generalized V-modules, which we will call

$$\mathcal{GM}_{sg}.$$

From the above we see that in the strongly graded case we have contravariant functors

$$(\cdot)' : (W, Y) \mapsto (W', Y'),$$

the *contragredient functors*, from $\mathcal{M}_{sg}$ to itself and from $\mathcal{GM}_{sg}$ to itself, and also from the full subcategories of lower bounded such structures to themselves. We also know that V itself is a (lower bounded) object of $\mathcal{M}_{sg}$ (and thus of $\mathcal{GM}_{sg}$ as well); recall Remark 2.26. Our main objects of study will be certain full subcategories $\mathcal{C}$ of $\mathcal{M}_{sg}$ or $\mathcal{GM}_{sg}$ that are closed under the contragredient functor and such that $V \in \text{ob}\,\mathcal{C}$.

Remark 2.37 In order to formulate certain results in this work, even in the case when our Möbius or conformal vertex algebra V is strongly graded we will in fact sometimes use the category whose objects are *all* the modules for V and whose morphisms are all the V-module homomorphisms, and also the category of *all* the generalized modules for V. (If V is conformal, then the category of all the V-modules is the same whether V is viewed as either conformal or Möbius, by Remark 2.14, and similarly for the category of all the generalized V-modules.) Note that in view of Remark 2.28, the categories $\mathcal{M}_{sg}$ and $\mathcal{GM}_{sg}$ are not full subcategories of these categories of *all* modules and generalized modules.

We now recall from [FLM2], [FHL], [DL] and [LL] the well-known principles that vertex operator algebras (which are exactly conformal vertex algebras strongly graded with respect to the trivial group; recall Remark 2.24) and their modules have important "rationality," "commutativity" and "associativity" properties, and that these properties can in fact be used as axioms replacing the Jacobi identity in the definition of the notion of vertex operator algebra. (These principles in fact generalize to all vertex algebras, as in [LL].)

In the propositions below,
$$\mathbb{C}[x_1, x_2]_S$$
is the ring of formal rational functions obtained by inverting (localizing with respect to) the products of (zero or more) elements of the set S of nonzero homogeneous linear polynomials in x_1 and x_2. Also, ι_{12} (which might also be written as $\iota_{x_1 x_2}$) is the operation of expanding an element of $\mathbb{C}[x_1, x_2]_S$, that is, a polynomial in x_1 and x_2 divided by a product of homogeneous linear polynomials in x_1 and x_2, as a formal series containing at most finitely many negative powers of x_2 (using binomial expansions for negative powers of linear polynomials involving both x_1 and x_2); similarly for ι_{21} and so on. (The distinction between rational functions and formal Laurent series is crucial.)

Let V be a vertex operator algebra. For W a ($\mathbb{C}$-graded) V-module (including possibly V itself), the space W' is just the "restricted dual space"

$$W' = \coprod_{n \in \mathbb{C}} W_{(n)}^*. \tag{2.103}$$

Proposition 2.38 *We have:*

(a) (rationality of products) For v, v_1, $v_2 \in V$ and $v' \in V'$, the formal series

$$\langle v', Y(v_1, x_1) Y(v_2, x_2) v \rangle, \tag{2.104}$$

which involves only finitely many negative powers of x_2 and only finitely many positive powers of x_1, lies in the image of the map ι_{12}:

$$\langle v', Y(v_1, x_1) Y(v_2, x_2) v \rangle = \iota_{12} f(x_1, x_2), \tag{2.105}$$

where the (uniquely determined) element $f \in \mathbb{C}[x_1, x_2]_S$ is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t} \tag{2.106}$$

for some $g \in \mathbb{C}[x_1, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) (commutativity) We also have

$$\langle v', Y(v_2, x_2)Y(v_1, x_1)v \rangle = \iota_{21} f(x_1, x_2). \tag{2.107}$$

Proposition 2.39 *We have:*

(a) (rationality of iterates) For v, v_1, $v_2 \in V$ and $v' \in V'$, the formal series

$$\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle , \tag{2.108}$$

which involves only finitely many negative powers of x_0 and only finitely many positive powers of x_2, lies in the image of the map ι_{20}:

$$\langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle = \iota_{20} h(x_0, x_2), \tag{2.109}$$

where the (uniquely determined) element $h \in \mathbb{C}[x_0, x_2]_S$ is of the form

$$h(x_0, x_2) = \frac{k(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t} \tag{2.110}$$

for some $k \in \mathbb{C}[x_0, x_2]$ and $r, s, t \in \mathbb{Z}$.

(b) The formal series $\langle v', Y(v_1, x_0 + x_2)Y(v_2, x_2)v \rangle$, which involves only finitely many negative powers of x_2 and only finitely many positive powers of x_0, lies in the image of ι_{02}, and in fact
$$\langle v', Y(v_1, x_0 + x_2)Y(v_2, x_2)v \rangle = \iota_{02} h(x_0, x_2). \tag{2.111}$$

Proposition 2.40 *(associativity) We have the following equality of formal rational functions:*

$$\iota_{12}^{-1} \langle v', Y(v_1, x_1)Y(v_2, x_2)v \rangle = \left. \left(\iota_{20}^{-1} \langle v', Y(Y(v_1, x_0)v_2, x_2)v \rangle \right) \right|_{x_0 = x_1 - x_2}, \tag{2.112}$$

that is,

$$f(x_1, x_2) = h(x_1 - x_2, x_2).$$

Proposition 2.41 *In the presence of the other axioms for the notion of vertex operator algebra, the Jacobi identity follows from the rationality of products and iterates, and commutativity and associativity. In particular, in the definition of vertex operator algebra, the Jacobi identity may be replaced by these properties.*

The rationality, commutativity and associativity properties immediately imply the following result, in which the formal variables x_1 and x_2 are specialized to nonzero complex numbers in suitable domains:

Corollary 2.42 *The formal series obtained by specializing x_1 and x_2 to (nonzero) complex numbers z_1 and z_2, respectively, in (2.104) converges to a rational function of z_1 and z_2 in the domain*

$$|z_1| > |z_2| > 0 \tag{2.113}$$

and the analogous formal series obtained by specializing x_1 and x_2 to z_1 and z_2, respectively, in (2.107) converges to the same rational function of z_1 and z_2 in the (disjoint) domain

$$|z_2| > |z_1| > 0. \tag{2.114}$$

Moreover, the formal series obtained by specializing x_0 and x_2 to $z_1 - z_2$ and z_2, respectively, in (2.108) converges to this same rational function of z_1 and z_2 in the domain

$$|z_2| > |z_1 - z_2| > 0. \tag{2.115}$$

In particular, in the common domain

$$|z_1| > |z_2| > |z_1 - z_2| > 0, \tag{2.116}$$

we have the equality

$$\langle v', Y(v_1, z_1)Y(v_2, z_2)v \rangle = \langle v', Y(Y(v_1, z_1 - z_2)v_2, z_2)v \rangle \tag{2.117}$$

of rational functions of z_1 and z_2.

Remark 2.43 These last five results also hold for modules for a vertex operator algebra V; in the statements, one replaces the vectors v and v' by elements w and w' of a V-module W and its restricted dual W', respectively, and Proposition 2.41 becomes: Given a vertex operator algebra V, in the presence of the other axioms for the notion of V-module, the Jacobi identity follows from the rationality of products and iterates, and commutativity and associativity. In particular, in the definition of V-module, the Jacobi identity may be replaced by these properties.

For either vertex operator algebras or modules, it is sometimes convenient to express the equalities of rational functions in Corollary 2.42 informally as follows:

$$Y(v_1, z_1)Y(v_2, z_2) \sim Y(v_2, z_2)Y(v_1, z_1) \tag{2.118}$$

and

$$Y(v_1, z_1)Y(v_2, z_2) \sim Y(Y(v_1, z_1 - z_2)v_2, z_2), \tag{2.119}$$

meaning that these expressions, defined in the domains indicated in Corollary 2.42 when the "matrix coefficients" of these expressions are taken as in this corollary, agree as operator-valued rational functions, up to analytic continuation.

Remark 2.44 Formulas (2.118) and (2.119) (or more precisely, (2.117)), express the mero-morphic, or single-valued, version of "duality," in the language of conformal field theory. Formulas (2.119) (and (2.117)) express the existence and associativity of the single-valued, or meromorphic, operator product expansion. This is the statement that the product of two (vertex) operators can be expanded as a (suitable, convergent) infinite sum of vertex operators, and that this sum can be expressed in the form of an iterate of vertex operators, parametrized by the complex numbers $z_1 - z_2$ and z_2, in the format indicated; the infinite sum comes from expanding $Y(Y(v_1, z_1 - z_2)v_2, z_2)$ in powers of $z_1 - z_2$. A central goal of this work is to generalize (2.118) and (2.119), or more precisely, (2.117), to logarithmic inter-twining operators in place of the operators $Y(\cdot, z)$. This will give the existence and also the associativity of the general, nonmeromorphic operator product expansion. This was done in the non-logarithmic setting in [HL5]–[HL7] and [H2]. In the next section, we shall develop the concept of logarithmic intertwining operator.

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Department of Mathematics, Rutgers University, Piscataway, NJ 08854 (permanent address)
and
Beijing International Center for Mathematical Research, Peking University, Beijing, China
E-mail address: yzhuang@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: lepowsky@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854
E-mail address: linzhang@math.rutgers.edu