$$F^p = \frac{w_p n_p v}{A_p} \quad , \quad F^\delta = \frac{w_\delta (n_\delta - \alpha n_s)}{A_\delta}$$

then, we have:

$$\langle F^\delta(r) F^{p*}(r') \rangle = \left\langle \frac{w_\delta(r)[n_\delta(r) - \alpha n_s(r)]}{A_\delta(r)} \frac{w_p(r') n_p(r') v(r')}{A_p(r')} \right\rangle$$

$$= \frac{1}{A_\delta(r) A_p(r')} \left\langle w_\delta(r) w_p(r') \big(n_\delta(r) - \alpha n_s(r) \big) n_p(r') v(r') \right\rangle$$

$$= \frac{1}{A_\delta(r) A_p(r')} w_\delta(r) w_p(r') \left\langle n_\delta(r) n_p(r') v(r') - \alpha\, n_s(r) n_p(r') v(r') \right\rangle$$

$$= \frac{1}{A_\delta(r) A_p(r')} w_\delta(r) w_p(r') \Big(\langle n_\delta(r) n_p(r') v(r') \rangle - \alpha \langle n_s(r) n_p(r') v(r') \rangle \Big)$$

the $\alpha \langle n_s(r) n_p(r') v(r') \rangle = 0$ since the cross-correlation beteeween randoms and velocities is 0.

$$\langle F^\delta(r) F^{p*}(r') \rangle = \frac{1}{A_\delta(r) A_p(r')} w_\delta(r) w_p(r') \langle n_\delta(r) n_p(r') v(r') \rangle$$

Following Feldman 1994 and Park 2006, we have

$$\langle n_\delta(r) n_p(r') v(r') \rangle = \bar{n}_\delta(r) \bar{n}_p(r') \xi_{\delta p}(r-r') + \min\{\bar{n}_\delta, \bar{n}_p\} \langle v(r') \rangle \delta^D(r-r')$$

Thus, we have:

$$\langle F^\delta(r) F^{p*}(r') \rangle = \frac{1}{A_\delta(r) A_p(r')} \Big(w_\delta(r) w_p(r') \bar{n}_\delta(r) \bar{n}_p(r') \xi_{\delta p}(r-r')$$
$$+ w_\delta(r) w_p(r') \min\{\bar{n}_\delta, \bar{n}_p\} \langle v(r') \rangle \delta^D(r-r') \Big)$$

(2)

Some definations:

Fourier Transform $F(\vec{k}) \equiv \frac{1}{V}\int F(\vec{r})\, e^{i\vec{k}\cdot\vec{r}}\, d^3r$

Inverse Fourier Transform $F(\vec{r}) \equiv \frac{1}{(2\pi)^3}\int F(\vec{k})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3r$

Dirac δ-Function: $\delta^D(\vec{r}-\vec{r}') \equiv \frac{1}{(2\pi)^3}\int e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}\, d^3k$

$$F(\vec{r}) = \frac{1}{V}\int F(\vec{r}')\, \delta^D(\vec{r}-\vec{r}')\, d^3r' \quad ③$$

Legendre Transform: $F_l(\vec{r}) = (2l+1)\int F(\vec{r})\, L_l(\hat{k}\cdot\hat{r})\, \frac{d\Omega_k}{4\pi}$ ①

$$|F^\delta(k) F^{p*}_l(k)| = \frac{1}{V}\int F^\delta(r) e^{ikr} d^3r \; \frac{1}{V'}\int \underbrace{F^{p*}_l(r')}_{\text{Eq. ①}} e^{-ikr'} d^3r'$$

$$= \frac{2l+1}{V}\frac{1}{V'}\int\frac{d\Omega_k}{4\pi}\int F^\delta(r) e^{ikr} d^3r \int F^{p*}(r') L^*_l(\hat{k}\hat{r}') e^{-ikr'} d^3r'$$

$$= \frac{2l+1}{V}\frac{1}{V'}\int\frac{d\Omega_k}{4\pi}\int d^3r\int d^3r'\; F^\delta(r) F^{p*}(r')\; L^*_l(\hat{k}\hat{r}')\, e^{ik(r-r')} \quad ⑥$$

Then, we have:

$$\langle|F^\delta(k) F^{p*}_l(k)|\rangle = \frac{2l+1}{V}\frac{1}{V'}\int\frac{d\Omega_k}{4\pi}\int d^3r\int d^3r'\; \underbrace{\langle F^\delta(r) F^{p*}(r')\rangle}_{\text{Eq. ②}}\; L^*_l(\hat{k}\hat{r}')\, e^{ik(r-r')}$$

$$= \frac{2l+1}{V}\frac{1}{V'}\int\frac{d\Omega_k}{4\pi}\int d^3r\int d^3r'\; \frac{1}{A_\delta A_p}\Big(w_\delta(r) w_p(r') \bar{n}_\delta(r)\bar{n}_p(r')\, \xi_{\delta p}(r-r') + w_\delta(r) w_p(r') \min\{\bar{n}_\delta, \bar{n}_p\} \langle v(r')\rangle \delta^D(r-r')\Big)$$
$$\times L^*_l(\hat{k}\hat{r}')\, e^{ik(r-r')}$$

$$= \frac{2l+1}{V}\frac{1}{A_\delta A_p}\int\frac{d\Omega_k}{4\pi}\int d^3r\Big[\frac{1}{V'}\int d^3r'\, w_\delta(r) w_p(r') \bar{n}_\delta(r)\bar{n}_p(r')\, \xi_{\delta p}(r-r')\, L^*_l(\hat{k}\hat{r}')\, e^{ik(r-r')}$$
$$+ \underbrace{\frac{1}{V'}\int d^3r'\, w_\delta(r) w_p(r') \min(\bar{n}_\delta, \bar{n}_p) \langle v(r')\rangle L^*_l(\hat{k}\hat{r}')\, e^{ik(r-r')}\, \delta^D(r-r')}_{\text{Integral over } \delta^D \text{ for } r' \text{ using Eq. ③}}\Big]$$

$$= \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\int d^3r\Big[\frac{1}{V'}\int d^3r'\, w_\delta(r) w_p(r') \bar{n}_\delta(r)\bar{n}_p(r')\, \xi_{\delta p}(r-r')\, L^*_l(\hat{k}\hat{r}')\, e^{ik(r-r')}$$
$$+ w_\delta(r) w_p(r) \min\{\bar{n}_\delta, \bar{n}_p\} \langle v(r)\rangle L^*_l(\hat{k}\hat{r})\Big]$$

$$= \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\Big[\int d^3r\, \frac{1}{V'}\int d^3r'\, w_\delta(r) w_p(r') \bar{n}_\delta(r)\bar{n}_p(r')\, \xi_{\delta p}(r-r')\, L^*_l(\hat{k}\hat{r}')\, e^{ik(r-r')}$$
$$+ \underbrace{\int d^3r\, w_\delta(r) w_p(r) \min\{\bar{n}_\delta, \bar{n}_p\} \langle v(r)\rangle L^*_l(\hat{k}\hat{r})}_{\text{shot-noise } N^{\delta p}_l}\Big]$$

Define $N^{\delta p}_l \equiv \int w_\delta(r) w_p(r) \min\{\bar{n}_\delta, \bar{n}_p\} \langle v(r)\rangle L^*_l(\hat{k}\hat{r})\, d^3r$

Then we have:

$$\langle |F^{\delta}(k)\, F_l^{p*}(k)|\rangle = \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\left[\int d^3r\,\frac{1}{V'}\int d^3r'\, w_\delta(r)\, w_p(r')\, \bar{n}_\delta(r)\, \bar{n}_p(r')\, \underbrace{\xi_{\delta p}(r-r')}_{\Downarrow}\, L_l^*(\hat{k}\hat{r}')\, e^{ik(r-r')} + N_l^{\delta p}\right]$$

$$\xi_{\delta p}(r-r') = \frac{1}{(2\pi)^3}\int P^{\delta p}(k)\, e^{-ik(r-r')}\, d^3k$$

$$= \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\left[\int d^3r\,\frac{1}{V'}\int d^3r'\, w_\delta(r)\, w_p(r')\, \bar{n}_\delta(r)\, \bar{n}_p(r')\int P^{\delta p}(k')\, e^{-ik'(r-r')}\frac{d^3k'}{(2\pi)^3}\, L_l^*(\hat{k}\hat{r}')\, e^{ik(r-r')} + N_l^{\delta p}\right]$$

$$e^{-ik'(r-r')}\, e^{ik(r-r')} = e^{-ik'r + ik'r' + ikr - ikr'}$$

$$= e^{ir(k-k')}\, e^{-ir'(k-k')}$$

$$= \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\left[\underbrace{\int w_\delta(r)\, \bar{n}_\delta(r)\, e^{ir(k-k')}\, d^3r}_{\Downarrow\ G^\delta \text{ by definition}}\ \frac{1}{V'}\int d^3r'\, w_p(r')\, \bar{n}_p(r')\int P^{\delta p}(k')\, e^{-ir'(k-k')}\, L_l^*(\hat{k}\hat{r}')\frac{d^3k'}{(2\pi)^3} + N_l^{\delta p}\right]$$

$$= \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\left[\frac{1}{V'}\int d^3r'\, w_p(r')\, \bar{n}_p(r')\int\frac{d^3k'}{(2\pi)^3}\, \underbrace{P^{\delta p}(k')}_{\hookrightarrow\ P^{\delta p}(k') = \sum_{l'} P_{l'}^{\delta p}(k')\, L_{l'}(\hat{k}'\hat{r}')}\, L_l^*(\hat{k}\hat{r}')\, e^{-ir'(k-k')}\, G^\delta + N_l^{\delta p}\right]$$

$$= \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\left[\frac{1}{V'}\int d^3r'\, w_p(r')\, \bar{n}_p(r')\int\frac{d^3k'}{(2\pi)^3}\sum_{l'} P_{l'}^{\delta p}(k')\, L_{l'}(\hat{k}'\hat{r}')\, L_l^*(\hat{k}\hat{r}')\, e^{-ir'(k-k')}\, G^\delta + N_l^{\delta p}\right]$$

$$= \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\left[\sum_{l'}\int\frac{d^3k'}{(2\pi)^3}\, P_{l'}^{\delta p}(k')\, G^\delta\, \underbrace{\frac{1}{V'}\int d^3r'\, w_p(r')\, \bar{n}_p(r')\, L_{l'}(\hat{k}'\hat{r}')\, L_l^*(\hat{k}\hat{r}')\, e^{-ir'(k-k')}}_{\Downarrow\ S_{ll'}^{p}} + N_l^{\delta p}\right]$$

Define:

$$S_{ll'}^{p} = \frac{1}{V}\int w_p(r)\, \bar{n}_p(r)\, L_{l'}^*(\hat{k}'\hat{r})\, L_l(\hat{k}\hat{r})\, e^{ir(k-k')}\, d^3r \quad (4)$$

$$S_{ll'}^{p*} = \frac{1}{V}\int w_p(r)\, \bar{n}_p(r)\, L_{l'}(\hat{k}'\hat{r})\, L_l^*(\hat{k}\hat{r})\, e^{-ir(k-k')}\, d^3r$$

Then we have:

$$\langle |F^{\delta}(k)\, F_l^{p*}(k)|\rangle = \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\left[\sum_{l'}\int\frac{d^3k'}{(2\pi)^3}\, G^\delta\, S_{ll'}^{p*}\, P_{l'}^{\delta p}(k') + N_l^{\delta p}\right]$$

Define window function:

$$W = \frac{1}{A_\delta A_p}\frac{2l+1}{V}\int\frac{d\Omega_k}{4\pi}\sum_{l'}\int\frac{d^3k'}{(2\pi)^3}\, G^\delta\, S_{ll'}^{p*} \quad (5)$$

Define: $L_l(\hat{k}\cdot\hat{r}) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_m^{l*}(\hat{k})\, Y_m^{l}(\hat{r})$

$$L_{l'}^{*}(\hat{k}'\cdot\hat{r}) = \frac{4\pi}{2l'+1}\sum_{m'=-l'}^{l'} Y_{m'}^{l'}(\hat{k}')\, Y_{m'}^{l'*}(\hat{r})$$

Then Eq. (4) $\Rightarrow$

$$S_{ll'}^{p} = \frac{1}{V}\int w_p \bar{n}_p \frac{4\pi}{2l'+1}\sum_{m'} Y_{m'}^{l'}(\hat{k}') Y_{m'}^{l'*}(\hat{r}) \frac{4\pi}{2l+1}\sum_{m} Y_m^{l*}(\hat{k}) Y_m^{l}(\hat{r}) e^{ir(k-k')} d^3r$$

$$= \frac{(4\pi)^2}{(2l+1)(2l'+1)}\sum_m Y_m^{l*}(\hat{k})\sum_{m'} Y_{m'}^{l'}(\hat{k}') \underbrace{\frac{1}{V}\int w_p\bar{n}_p Y_m^{l}(\hat{r}) Y_{m'}^{l'*}(\hat{r}) e^{ir(k-k')} d^3r}_{S_{mm'}^{p,ll'}}$$

$$= \frac{(4\pi)^2}{(2l+1)(2l'+1)}\sum_m Y_m^{l*}(\hat{k})\sum_{m'} Y_{m'}^{l'}(\hat{k}')\, S_{mm'}^{p,ll'}$$

Then Eq. (5) $\Rightarrow$

$$W = \frac{1}{A_\delta A_p V}\int\frac{d\Omega_k}{4\pi}\sum_{m=-l}^{l} Y_m^{l}(\hat{k}) \sum_{l'}\frac{(4\pi)^2}{2l'+1}\sum_{m'=-l'}^{l'}\int\frac{d^3k'}{(2\pi)^3} Y_{m'}^{l'*}(\hat{k}')\, G^{\delta}\, S_{mm'}^{p,ll'*}$$

Eq. ⑥ $\Rightarrow$

$$|F^{\delta}(k)\, F_4^{\delta *}(k)| = \frac{9}{V}\frac{1}{V'}\int \frac{d\Omega_k}{4\pi}\int d^3r \int d^3r'\, F^{\delta}(r)\, F^{\delta *}(r')\, L_4(\hat{k}\hat{r}')\, e^{ik(r-r')}$$

$$= \frac{9}{V}\frac{1}{V'}\int \frac{d\Omega_k}{4\pi}\int d^3r \int d^3r'\, F^{\delta}(r)\, F^{\delta *}(r')\, \frac{1}{8}\left(35(\hat{k}\hat{r}')^4 - 30(\hat{k}\hat{r}')^2 + 3(\hat{k}\hat{r}')^0\right) e^{ik(r-r')}$$

$$= \frac{9}{8}\int \frac{d\Omega_k}{4\pi}\, \underbrace{\frac{1}{V}\int F^{\delta}(r)\, e^{ikr}\, d^3r}_{F^{\delta}(k)}\, \frac{1}{V'}\int F^{\delta *}(r')\left[35(\hat{k}\hat{r}')^4 - 30(\hat{k}\hat{r}')^2 + 3(\hat{k}\hat{r}')^0\right] e^{-ikr'}\, d^3r'$$

$$= \frac{9}{8}\int \frac{d\Omega_k}{4\pi}\, F^{\delta}(k)\left(\frac{1}{V'}\int 35\,(\hat{k}\hat{r}')^4 F^{\delta *}(r')\, e^{-ikr'}\, d^3r' - \frac{1}{V'}\int 30\,(\hat{k}\hat{r}')^2 F^{\delta *}(r')\, e^{-ikr'}\, d^3r' + \frac{1}{V'}\int 3\,(\hat{k}\hat{r}')^0 F^{\delta *}(r')\, e^{-ikr'}\, d^3r'\right)$$

Define $T_\ell(k) = \int (\hat{k}\hat{r})^\ell F(r)\, e^{ikr}\, d^3r$

Then, we have

$$|F^{\delta}(k)\, F_4^{\delta *}(k)| = \frac{9}{8V}\int \frac{d\Omega_k}{4\pi}\, F^{\delta}(k)\left[35\, T_4^{*} - 30\, T_2^{*} + 3\, T_0^{*}\right]$$