Coding Up the Power Spectrum

1: Grid of galaxies

Supposing that there are N_{gal} galaxies obtained from a survey. To measure the power spectrum of these galaxies, we need to assign them onto a 3D grid under the Cartesian coordinates. The observer is located at the center of the cubic box which is used to create the grid. The side length of the cubic box are L_x, L_y and L_z, respectively, therefore the volume of the cubic box is $V_{cub} = L_x L_y L_z$. The grid is made of N_x, N_y and N_z voxels along each direction, the total number of voxels is $N_{cell} = N_x N_y N_z$, and the size of each voxel is

$$dx = \frac{L_x}{N_x}\ ,\ \ dy = \frac{L_y}{N_y}\ ,\ \ dz = \frac{L_z}{N_z}\ .$$

To measure power spectrum we also need a random catalogue which contains N_{rand} random points. Then $\alpha \equiv \frac{N_{gal}}{N_{rand}}$ denotes the ratio between the galaxies and random points. The random points should be grided using the same grid as the galaxies. The random catalogue is not required for momentum power spectrum and cross-power spectrum.

2: The definition of the field function

Assuming the Cartesian coordinate position of a given voxel is $\vec{r} = [r_x, r_y, r_z]$, the distance between this voxel and the observer is

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2}\ .$$

The observer is located at the origin of the Cartesian coordinate. If there are C_{gal} galaxies and C_{rand} random points located inside the given voxel, the definition of the density value for this voxel is given by:

$$D(\vec{r}) = \sum_{i=1}^{C_{gal}} w_i(\vec{r}) - \sum_{j=1}^{C_{rand}} \alpha\, w_{r,j}(\vec{r})\ ,$$

while the definition of the momentum value for this voxel is given by:

$$D(\vec{r}) = \sum_{i=1}^{C_{gal}} w_i(\vec{r})\, v_i(\vec{r})\ ,$$

where $w_i(\vec{r})$ is the weight factor of the *i*-th galaxy of that voxel, $v_i(\vec{r})$ is the line-of-sight velocity of the *i*-th galaxy, $w_{r,j}(\vec{r})$ is the weight factor of the *j*-th random point of that voxel.

The expressions of the weight factors are different for density and momentum power spectrum, given by:

$$w_i \to \begin{cases} w_i^{\delta} = \dfrac{1}{1 + \overline{n}^{\delta} \times FKPD} \\ w_i^{p} = \dfrac{1}{\sigma_{v,i}^2 + \overline{n}^{p} \times FKPP} \end{cases}$$

respectively, where $\overline{n}^{\delta,p}$ are the average numbers of galaxies of the redshift and peculiar velocity catalogues respectively. $\sigma_{v,i}$ is the velocity error of the galaxy. The number of galaxies of the redshift is usually larger than it of the peculiar velocity catalogue.

3: The shot noise of the field function

The shot noise for the density field is given by

$$P_{gal,\,noise} = \sum_{i=1}^{N_{gal}} w_i^2 \ ,$$

while the shot noise for the random points is given by

$$P_{rand,\,noise} = \alpha^2 \sum_{i=1}^{N_{rand}} w_{r,\,i}^2 \ ,$$

The total shot noise for the density power spectrum is then given by

$$P_{noise} = P_{gal,\,noise} + P_{rand,\,noise}$$

The shot noise for the momentum field is given by

$$P_{noise} = \sum_{i=1}^{N_{gal}} w_i^2 v_i^2 \ .$$

The shot noise for the density-momentum cross power spectrum is given by

$$P_{noiseC} = \sum_{i=1}^{N_{gal}} w_i^\delta w_i^p \, v_i.$$

where N_{gal} should be the number of galaxies in the peculiar velocity catalogue.

4: The normalization factor of the power spectrum

The normalization factor of the power spectrum of galaxies is given by

$$I_{gal} = \sum_{i=1}^{N_{gal}} \overline{n} w_i^2 \ .$$

The normalization factor of the power spectrum of random points is given by

$$I_{rand} = \alpha \sum_{i=1}^{N_{rand}} \overline{n}_r w_{r,\,i}^2 \ .$$

We should have

$$I \approx I_{rand} \approx I_{gal}$$

The normalization factor of the cross power spectrum of galaxies is given by

$$I_{gal} = \sqrt{\sum_{i=1}^{N_{gal}} \overline{n}^\delta w_i^{\delta,\,2}} \times \sqrt{\sum_{i=1}^{N_{pv\ of\ gal}} \overline{n}^p w_i^{p,\,2}}$$

The number of galaxies of the redshift catalogue is N_{gal}, usually it is larger than the number of galaxies in the peculiar velocity catalogue $N_{pv\ of\ gal}$.

5: Wave number

The frequency corresponding to the given grid cell $\vec{r} = [r_x, r_y, r_z]$ is defined by

$$f_x = \begin{cases} \dfrac{j-1}{N_x dx}, & j = 1, \ldots, \dfrac{N_x}{2} \\ \dfrac{j-1-N_x}{N_x dx}, & j = \dfrac{N_x}{2} + 1, \ldots, N_x \end{cases}$$

$$f_y = \begin{cases} \dfrac{k-1}{N_y dy}, & k = 1, \ldots, \dfrac{N_y}{2} \\ \dfrac{k-1-N_y}{N_y dy}, & k = \dfrac{N_y}{2} + 1, \ldots, N_y \end{cases}$$

$$f_z = \frac{i-1}{N_z dz}, \quad i = 1, \ldots, \frac{N_z}{2}.$$

The corresponding wave vector and wave number for the given grid cell are given by

$$\vec{k} = [2\pi f_x, 2\pi f_y, 2\pi f_z],$$

and

$$k = 2\pi \sqrt{f_x^2 + f_y^2 + f_z^2}.$$

6: Binning the wave number

For each cell, we can calculate a wave number, therefore we will have plenty of wave numbers. To measure the histogram of wave number in a interval $[k_{min}, k_{max}]$, we firstly divide this interval into N_k bins and the bin size is

$$dk = \frac{k_{max} - k_{min}}{N_k}$$

Then, we can find which bin the wave number *k* of the given cell is in:

$$\text{ibin} = \frac{k - k_{min}}{dk}$$

The center of ibin is denoted by

$$k_{ibin} = k_{min} + (\text{ibin} - 1 + 0.5) \times dk$$

The above equations are only for one cell. We should loop over all the cells , i.e.

$$j = 1, \ldots, N_x, \qquad k = 1, \ldots, N_y, \qquad i = 1, \ldots, \frac{N_z}{2}$$

For each set of (j, k, i), we need to calculate a wave number k and ibin, if ibin obey the following condition:

$$\text{ibin} > 0 \;\; \& \;\; \text{ibin} < K_{nqu} \;\; \& \;\; \text{ibin} < dk$$

we will account it into the number of wave number of that bin

$$B_{ibin}(k_{ibin}) = B_{ibin}(k_{ibin}) + 1$$

Finally, we will have a histogram, the *x*-axis is the bin values $k_{ibin} \in [k_{min}, k_{max}]$, the *y*-axis is B_{ibin} for each bin. This histogram has not been normalize yet.

The Nyquist frequency is given by

$$K_{nqu} = \min\left\{\frac{\pi}{dx}, \frac{\pi}{dy}, \frac{\pi}{dz}\right\}$$

7: The grid smooth correction

We need correction for gridding. In this effect, we are convolving the field values with a top-hat function in each direction, so we are multiplying each Fourier mode by a sinc function. To correct this, we therefore divide by the sinc functions. The correction factor for the given cell, which is located at $\vec{r}$ with wave number *k*, is given by

$$G_d = \frac{1}{\frac{sin(f_x\, dx\, \pi)}{f_x\, dx\, \pi}\ \frac{sin(f_y\, dy\, \pi)}{f_y\, dy\, \pi}\ \frac{sin(f_z\, dz\, \pi)}{f_z\, dz\, \pi}}\ .$$

8: The mode function A_0

The mode function $A_0(k)$ for the given cell (located at $\vec{r}$) is simply calculated from the Fourier transform of the field function $D(\vec{r})$, given by:

$$A_0(k) = FFT\{D\} = \int D(\vec{r})e^{i\vec{k}\cdot\vec{r}}d^3r$$

9: The mode function A_1

For the given cell, which located at $\vec{r}$ with wave number *k*, firstly, we need to calculate the following terms:

$$R = \left[\ \frac{r_x}{r}, \frac{r_y}{r}, \frac{r_z}{r}\ \right] \quad \text{and} \quad K = \left[\ \frac{k_x}{k}, \frac{k_y}{k}, \frac{k_z}{k}\ \right]$$

Then, for a given R_a (where $a = 1,2,3$), we should project the field value D into the R_a, i.e. calculate product DR_a, then we calculate the FFT of the product DR_a, i.e.

$$F_a\ = FFT\{\, D\, R_a\, \},\ \ a = 1,2,3$$

Finally, we should sum over a for the product F_aK_a:

$$A_1(k) = \sum_{a=1}^{3} F_aK_a$$

10: The mode function A_2

For the given cell, which located at $\vec{r}$ with wave number *k*, firstly, we need to calculate the following terms:

$$R = \left[\ \frac{r_xr_x}{r^2}, \frac{r_yr_y}{r^2}, \frac{r_zr_z}{r^2}, \frac{r_xr_y}{r^2}, \frac{r_xr_z}{r^2}, \frac{r_yr_z}{r^2}\ \right]$$

$$K = \left[\ \frac{k_xk_x}{k^2}, \frac{k_yk_y}{k^2}, \frac{k_zk_z}{k^2}, 2\frac{k_xk_y}{k^2}, 2\frac{k_xk_z}{k^2}, 2\frac{k_yk_z}{k^2}\ \right]$$

Then, for a given R_a (where $a = 1,2,\dots,6$), we should project the field value D into the R_a, i.e. calculate product DR_a, then we calculate the FFT of the product DR_a, i.e.

$$F_a = FFT\{ D\, R_a \}, \ \ a = 1,2,\ldots,6$$

Finally, we should sum over a for the product $F_a K_a$:

$$A_2(k) = \sum_{a=1}^{6} F_a K_a$$

11: The mode function A_3

For the given cell, which located at $\vec{r}$ with wave number k, firstly, we need to calculate the following terms:

$$R = \left[\frac{r_x^3}{r^3}, \frac{r_y^3}{r^3}, \frac{r_z^3}{r^3}, \frac{r_x^2 r_y}{r^3}, \frac{r_x^2 r_z}{r^3}, \frac{r_y^2 r_x}{r^3}, \frac{r_y^2 r_z}{r^3}, \frac{r_z^2 r_x}{r^3}, \frac{r_z^2 r_y}{r^3}, \frac{r_x r_y r_z}{r^3}\right]$$

$$K = \left[\frac{k_x^3}{k^3}, \frac{k_y^3}{k^3}, \frac{k_z^3}{k^3}, 3\frac{k_x^2 k_y}{k^3}, 3\frac{k_x^2 k_z}{k^3}, 3\frac{k_y^2 k_x}{k^3}, 3\frac{k_y^2 k_z}{k^3}, 3\frac{k_z^2 k_x}{k^3}, 3\frac{k_z^2 k_y}{k^3}, 6\frac{k_x k_y k_z}{k^3}\right]$$

Then, for a given R_a (where $a = 1,2,\ldots,10$), we should project the field value D into the R_a, i.e. calculate product DR_a, then we calculate the FFT of the product DR_a, i.e.

$$F_a = FFT\{ D\, R_a \}, \ \ a = 1,2,\ldots,10$$

Finally, we should sum over a for the product $F_a K_a$:

$$A_3(k) = \sum_{a=1}^{10} F_a K_a$$

12: The mode function A_4

For the given cell, which located at $\vec{r}$ with wave number k, firstly, we need to calculate the following terms:

$$R = \left[\frac{r_x^2\, r_x r_x}{r^4}, \frac{r_y^2\, r_y r_y}{r^4}, \frac{r_z^2\, r_z r_z}{r^4}, \frac{r_x^2\, r_x r_y}{r^4}, \frac{r_x^2\, r_x r_z}{r^4}, \frac{r_y^2\, r_y r_x}{r^4}, \frac{r_y^2\, r_y r_z}{r^4}, \frac{r_z^2\, r_z r_x}{r^4}, \frac{r_z^2\, r_z r_y}{r^4},\right.$$
$$\left.\frac{r_x^2\, r_y r_y}{r^4}, \frac{r_x^2\, r_z r_z}{r^4}, \frac{r_y^2\, r_z r_z}{r^4}, \frac{r_x^2\, r_y r_z}{r^4}, \frac{r_y^2\, r_x r_z}{r^4}, \frac{r_z^2\, r_x r_y}{r^4}\right]$$

$$K = \left[k_x^4\frac{1}{k^4}, k_y^4\frac{1}{k^4}, k_z^4\frac{1}{k^4}, 4k_x^3 k_y\frac{1}{k^4}, 4k_x^3 k_z\frac{1}{k^4}, 4k_y^3 k_x\frac{1}{k^4}, 4k_y^3 k_z\frac{1}{k^4}, 4k_z^3 k_x\frac{1}{k^4}, 4k_z^3 k_y\frac{1}{k^4},\right.$$
$$\left.6k_x^2 k_y^2\frac{1}{k^4}, 6k_x^2 k_z^2\frac{1}{k^4}, 6k_y^2 k_z^2\frac{1}{k^4}, 12k_x k_y k_z k_x\frac{1}{k^4}, 12k_x k_y k_z k_y\frac{1}{k^4}, 12k_x k_y k_z k_z\frac{1}{k^4}\right]$$

Then, for a given R_b, (where $b = 1,2,\ldots,15$), we should project the field value D into the R_b, then we should calculate the FFT of $D\, R_b$, i.e.:

$$F_b = FFT\{ D\, R_b \}, \ \ b = 1,2,\ldots,18$$

Finally, we should sum over b for the product $F_b K_b$:

$$A_4(k) = \sum_{b=1}^{15} F_b K_b$$

13: The auto-power spectrum multipole

The power spectrum is the distribution (normalized histogram) of the mode values $A_{0,2,4}(k)$ in wave number bins. Once we obtain all the mode functions for all cells, we can calculate the power spectrum multipole. For each cell, we can calculate a mode value, therefore we will have plenty of mode values. To measure the histogram of mode value $A_{0,2,4}(k)$ in a interval $[k_{min}, k_{max}]$, we firstly divide this interval into N_k bins and the bin size is

$$dk = \frac{k_{max} - k_{min}}{N_k}$$

Then, we can find which bin the wave number *k* is in:

$$\text{ibin} = \frac{k - k_{min}}{dk}$$

The above equation is only for one cell, we should loop over all the cells , i.e.

$$j = 1, \ldots, N_x , \qquad k = 1, \ldots, N_y , \qquad i = 1, \ldots, \frac{N_z}{2}$$

For each set of (j, k, i), we need to calculate a wave number *k* and ibin and $A_{0,2,4}(k)$, if ibin obey the following condition:

$$\text{ibin} > 0 \ \ \& \ \ \text{ibin} < K_{nqu} \ \ \& \ \ \text{ibin} < dk$$

we will sum up the following values of that k_{ibin} bin

$$P_0(k_{ibin}) = P_0(k_{ibin}) + G_g^2 \frac{1}{I}(A_0\, A_0^* \ - \ P_{noise})$$

$$P_1(k_{ibin}) = P_1(k_{ibin}) + G_g^2 \frac{3}{I}(A_0\, A_1^*)$$

$$P_2(k_{ibin}) = P_2(k_{ibin}) + G_g^2 \frac{5}{I}(1.5A_0\, A_2^* - 0.5A_0\, A_0^* \,)$$

$$P_3(k_{ibin}) = P_3(k_{ibin}) + G_g^2 \frac{7}{I}(2.5A_0\, A_3^* - 1.5A_0\, A_1^* \,)$$

$$P_4(k_{ibin}) = P_4(k_{ibin}) + G_g^2 \frac{9}{I}(4.375A_0\, A_4^* - 3.75A_0\, A_2^* + 0.375A_0\, A_0^*)$$

Finally, we will have a histogram, the *x*-axis is the bin values $k_{ibin} \in [k_{min}, k_{max}]$, the *y*-axis is $P_{0,2,4}(k_{ibin})$ for each bin. Finally, This histogram should be normalize using

$$P_0(k_{ibin}) = \frac{P_0(k_{ibin})}{B_{ibin}(k_{ibin})} \ , \ P_1(k_{ibin}) = \frac{P_1(k_{ibin})}{B_{ibin}(k_{ibin})} \ , \ P_2(k_{ibin}) = \frac{P_2(k_{ibin})}{B_{ibin}(k_{ibin})} \ ,$$

$$P_3(k_{ibin}) = \frac{P_3(k_{ibin})}{B_{ibin}(k_{ibin})} \ , \ P_4(ibin) = \frac{P_4(k_{ibin})}{B_{ibin}(k_{ibin})} \ .$$

For the auto-power spectrum, the mode functions， which are basically the FFT of the projected field function, are given by $A_l = Re[A_l] + i\, Im[A_l]$. Therefor we have

$$A_0\, A_0^* = Re[A_0] \times Re[A_0] + Im[A_0] \times Im[A_0]$$
$$A_0\, A_1^* = Re[A_0] \times Im[A_1] - Im[A_0] \times Re[A_1]$$
$$A_0\, A_2^* = Re[A_0] \times Re[A_2] + Im[A_0] \times Im[A_2]$$
$$A_0\, A_3^* = Re[A_0] \times Im[A_3] - Im[A_0] \times Re[A_3]$$
$$A_0\, A_4^* = Re[A_0] \times Re[A_4] + Im[A_0] \times Im[A_4]$$

14: The cross-power spectrum multipole

While For the cross-power spectrumodd-multiple, the mode functions are given by:

$$A_0\, A_l^* = \frac{1}{2} Im[\, -A_0^\delta \times A_l^{p\,*} + A_0^p \times A_l^{\delta\,*}\,]$$

$$= \frac{1}{2}\left(-Re[A_l^\delta] \times Im[A_0^p] + Im[A_l^\delta] \times Re[A_0^p] + Re[A_l^p] \times Im[A_0^\delta] - Im[A_l^p] \times Re[A_0^\delta]\right)$$

where, l=1,3, and δ denotes density field and p denotes momentum field. For the even-multiple, l=0,2,4, the cross power spectrum is zeros. For example, if l=0, $A_0^{\delta,p} = A^{\delta,p}$, we have

$$A_0\, A_0^* = \frac{1}{2} Im[\, A_0^\delta \times A_0^{p\,*} + A_0^p \times A_0^{\delta\,*}\,]$$

where:

$$A_0^\delta \times A_0^{p\,*} + A_0^p \times A_0^{\delta\,*} = (\delta_r + \delta_i i) \times (p_{r,l} - p_{i,l} i) + (p_r + p_i i) \times (\delta_{r,l} - \delta_{i,l} i)$$
$$= (\delta_r p_{r,l} + \delta_i p_{r,l} i - \delta_r p_{i,l} i - \delta_i p_{i,l} ii) + (\delta_{r,l} p_r + \delta_{r,l} p_i i - \delta_{i,l} p_r i - \delta_{i,l} p_i ii)$$
$$= (\delta_r p_{r,l} + \delta_i p_{r,l} i - \delta_r p_{i,l} i + \delta_i p_{i,l}) + (\delta_{r,l} p_r + \delta_{r,l} p_i i - \delta_{i,l} p_r i + \delta_{i,l} p_i) = 2(\delta_r p_r + \delta_i p_i)$$

The imaginary part of the above is 0, i.e. $Im[\, A_0^\delta \times A_0^{p\,*} + A_0^p \times A_0^{\delta\,*}] = 0$. So the l=0 cross power spectrum is zero $A_0\, A_0^* = 0$.

15: The flow chart