Introduction to Fracture and Damage Mechanics

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Contents

Concepts of Fracture Mechanics

Part I: Linear Elastic Fracture Mechanics

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Overview

I. **Linear elastic fracture mechanics (LEFM)**
II. Small Scale Yielding
III. Elastic-plastic fracture mechanics (EPFM)

LEFM

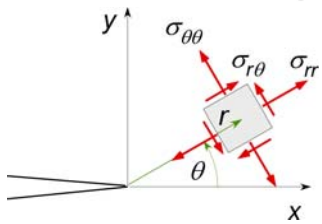
- Stress Field at a crack tip
- Stress intensity approach (Irwin)
- Energy approach (Griffith)
- *J*-integral
- Fracture criteria – fracture toughness
- Terminology

Stress Field at a Crack Tip

boundary conditions

$$\sigma_{yy}(x \le 0, y = 0) = \sigma_{\theta\theta}(r, \theta = \pi) = 0$$

$$\sigma_{xy}(x \le 0, y = 0) = \sigma_{r\theta}(r, \theta = \pi) = 0$$



Hooke's law of linear elasticity

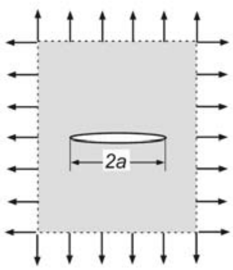
$$\sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

Inglis [1913], **Westergaard [1939]**, **Sneddon [1946]**, ...

Williams' series **[1957]**

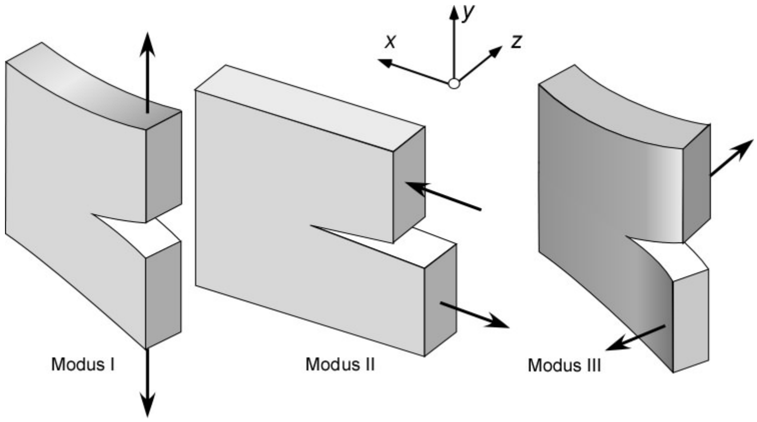
Airy's stress function $\quad \Delta\Delta\Phi = 0$

$$\Phi = r^{\lambda+2}\left[A\cos\lambda\theta + B\cos(\lambda+2)\theta + \right.$$
$$\left. + C\sin\lambda\theta + D\sin(\lambda+2)\theta\right]$$



$$\sigma_{rr} = \frac{A_{-1}}{\sqrt{r}}\left[\tfrac{5}{4}\cos\tfrac{\theta}{2} - \tfrac{1}{4}\cos\tfrac{3\theta}{2}\right] + \frac{C_{-1}}{\sqrt{r}}\left[-\tfrac{5}{4}\sin\tfrac{\theta}{2} + \tfrac{3}{4}\cos\tfrac{3\theta}{2}\right] + 4A_0\cos^2\theta +$$
$$+ A_1\sqrt{r}\left[\tfrac{9}{4}\cos\tfrac{\theta}{2} + \tfrac{3}{4}\cos\tfrac{5\theta}{2}\right] + C_1\sqrt{r}\left[\tfrac{9}{4}\sin\tfrac{\theta}{2} + \tfrac{15}{4}\cos\tfrac{5\theta}{2}\right] + \mathcal{O}(r)$$

Fracture Modes



LEFM: Stress Intensity Approach

Irwin [1957]: mode I

$$\sigma_{ij}(r,\theta) = \frac{K_{\mathrm{I}}}{\sqrt{2\pi r}} f_{ij}^{\mathrm{I}}(\theta) + T\delta_{i1}\delta_{1j}$$

K_{I} = stress intensity factor

$$K_{\mathrm{I}} = \sigma_{\infty}\sqrt{\pi a}\, Y(\text{geometry})$$

T = non-singular T-stress

Rice [1974]: effect on plastic zone

General asymptotic solution

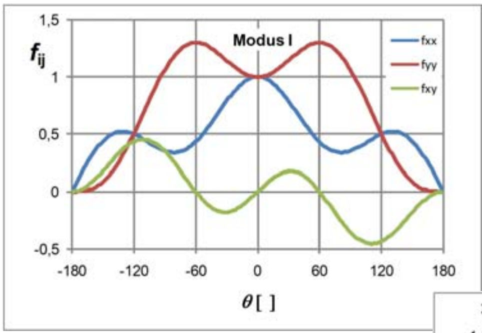
stresses

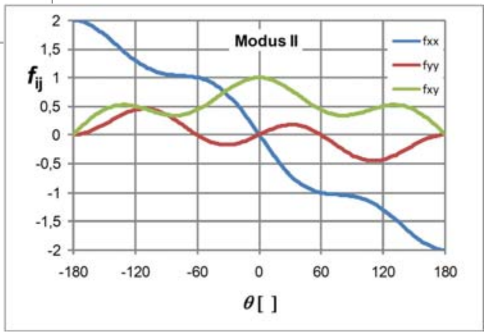
$$\sigma_{ij}(r,\theta) = \frac{1}{\sqrt{2\pi r}}\left[K_{\mathrm{I}} f_{ij}^{\mathrm{I}}(\theta) + K_{\mathrm{II}} f_{ij}^{\mathrm{II}}(\theta) + K_{\mathrm{III}} f_{ij}^{\mathrm{III}}(\theta)\right]$$

displacements

$$u_i(r,\theta) = \frac{1}{2G}\sqrt{\frac{r}{2\pi}}\left[K_{\mathrm{I}} g_i^{\mathrm{I}}(\theta) + K_{\mathrm{II}} g_i^{\mathrm{II}}(\theta) + K_{\mathrm{III}} g_i^{\mathrm{III}}(\theta)\right]$$

Angular Functions in LEFM





LEFM: Energy Approach

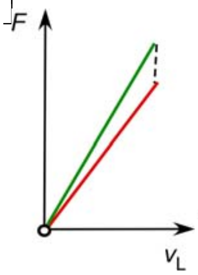
Elastic strain energy of a panel of thickness *B* under biaxial tension in a circular domain of radius *r*

$$U_0^{\mathrm{e}} = \frac{\pi B r^2 \sigma_\infty^2}{16G}\left[(\kappa-1)(1+\lambda)^2 + 2(1-\lambda)^2\right]$$

insert hole: fixed grips

> energy release

$$U^{\mathrm{e}} = U_0^{\mathrm{e}} - U_{\mathrm{rel}}^{\mathrm{e}}$$

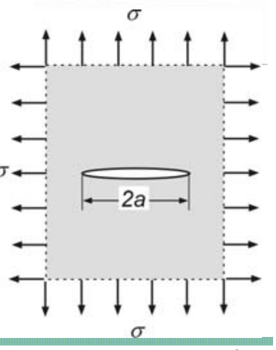


A.A. Griffith [1920]

elliptical hole, axes a,b

$$U_{\mathrm{rel}}^{\mathrm{e}} = \frac{\pi B \sigma_\infty^2}{32G}(1+\kappa)\Big[(1-\lambda)^2(a+b)^2 + 2(1-\lambda^2)(a^2-b^2) + (1+\lambda)^2(a^2+b^2)\Big]$$

crack $$U_{\mathrm{rel}}^{\mathrm{e}} = \frac{\pi a^2 B \sigma_\infty^2}{8G}(1+\kappa)$$



Fracture of “Brittle” Materials

Crack extends if $$\frac{\partial}{B\partial(2a)}\left(U^{e}_{rel} - U_{sep}\right) \geq 0$$

Energy release rate $$\mathcal{G}^{e} = -\left(\frac{\partial U^{e}}{B\partial(2a)}\right)_{v_L} = \frac{\partial U^{e}_{rel}}{B\partial(2a)}$$

Separation energy (SE)
(energy per area) $$\frac{\partial U_{sep}}{B\partial(2a)} = 2\gamma = \Gamma_c$$

fracture criterion	$\mathcal{G}^{e}(a) = \Gamma_c = 2\gamma$
fracture stress	$\sigma_c = \sqrt{\dfrac{E'\Gamma_c}{\pi a}}$

Irwin [1957]: $$\mathcal{G}^{e} = \frac{K_I^2}{E'}$$

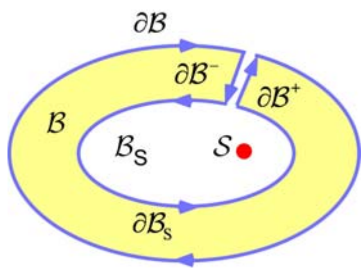
Path-Independent Integrals

$\varphi(x_i)$ is some (scalar, vector, tensor) field quantity being steadily differentiable in domain $\mathcal{B}$ and divergence free

$$\varphi_{,i} := \frac{\partial \varphi}{\partial x_i} = 0 \quad \text{in } \mathcal{B}$$

Gauß' theorem $$\int_{\mathcal{B}} \varphi_{,i} dv = \int_{\partial\mathcal{B}} \varphi n_i \, da = 0$$

singularity $\mathcal{S}$ in $\mathcal{B}$: $\mathcal{B}_0 = \mathcal{B} - \mathcal{B}_{\mathcal{S}}$



$$\int_{\partial\mathcal{B}_0} (.) = \oint_{\partial\mathcal{B}} (.) + \oint_{\partial\mathcal{B}^-} (.) + \oint_{\partial\mathcal{B}_{\mathcal{S}}} (.) + \oint_{\partial\mathcal{B}^+} (.) = 0$$

$$\oint_{\partial\mathcal{B}} (.) = \oint_{\partial\mathcal{B}} (.) \quad \text{and} \quad \oint_{\partial\mathcal{B}^+} (.) = -\oint_{\partial\mathcal{B}^-} (.)$$

path independence $$\boxed{\oint_{\partial\mathcal{B}} \varphi n_i \, da = \oint_{\partial\mathcal{B}_{\mathcal{S}}} \varphi n_i \, da}$$

Energy Momentum Tensor

Eshelby [1965]: energy momentum tensor

$$P_{ij} = \overline{w}\delta_{ij} - \frac{\partial \overline{w}}{\partial u_{k,j}} u_{k,i} \quad \text{with} \quad P_{ij,j} = 0$$

$\overline{w}$	energy density
$u_i(x_j)$	displacement field

Material forces acting on singularities (defects) in the continuum, e.g. dislocations, inclusions, ...

$$F_i = \oint_{\partial \mathcal{B}} P_{ij} n_j \, da$$

and

the *J*-vector

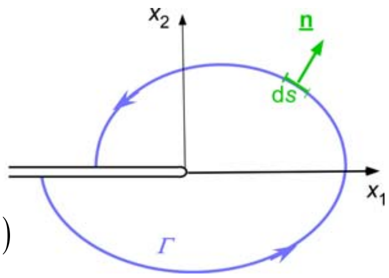
$$J_i = \oint_{\Gamma} \left[\overline{w} n_i - \sigma_{jk} n_k u_{j,i} \right] ds$$

$$\overline{w} = \int_{\tau=0}^{t} \sigma_{ij} \dot{\varepsilon}_{ij} \, d\tau$$

J-Integral

The *J*-integral of **Cherepanov [1967]** and **Rice [1986]** is the 1st component of the *J*-vector

$$J = \oint_{\Gamma} \left[\overline{w} \, dx_2 - \sigma_{ij} n_j u_{i,1} \, ds \right]$$



Conditions:

- Equilibrium $\quad \sigma_{ij,i} = 0$
- Small (linear) strains $\quad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$
- Hyperelastic material $\quad \sigma_{ij} = \dfrac{\partial \overline{w}}{\partial \varepsilon_{ij}}$

✓ time independent processes
✓ no volume forces
✓ homogeneous material
✓ plane stress and strain fields, no dependence on x_3
✓ straight and stress free crack faces parallel to x_1

J as Energy Release Rate

$$J = \mathcal{G}^{\mathrm{e}} = -\left(\frac{\partial U^{\mathrm{e}}}{\partial A_{\mathrm{crack}}}\right)_{v_{\mathrm{L}}}$$

mode I

$$\frac{\partial}{\partial A_{\mathrm{crack}}} = \begin{cases} \dfrac{\partial}{B\partial a} & \text{C(T), SE(B)} \\ \dfrac{\partial}{2B\partial a} & \text{M(T), DE(T)} \end{cases}$$

mixed mode

$$J = \mathcal{G}^{\mathrm{e}} = \mathcal{G}_{\mathrm{I}}^{\mathrm{e}} + \mathcal{G}_{\mathrm{II}}^{\mathrm{e}} + \mathcal{G}_{\mathrm{III}}^{\mathrm{e}} = \frac{K_{\mathrm{I}}^2}{E'} + \frac{K_{\mathrm{II}}^2}{E'} + \frac{K_{\mathrm{III}}^2}{2G}$$

Fracture Criteria

Brittle fracture:
predominantly elastic – “catastrophic” failure
Mode I

stress Intensity factor	$K_{\mathrm{I}} = K_{\mathrm{Ic}}$	
energy release rate	$\mathcal{G}_{\mathrm{I}}^{\mathrm{e}} = \mathcal{G}_{\mathrm{Ic}}$	$\mathcal{G}_{\mathrm{Ic}} = \Gamma_{\mathrm{c}}^{\mathrm{e}} = \dfrac{K_{\mathrm{Ic}}^2}{E'}$
J-integral	$J = J_{\mathrm{Ic}}$	$J_{\mathrm{Ic}} = \mathcal{G}_{\mathrm{Ic}} = \dfrac{K_{\mathrm{Ic}}^2}{E'}$

$$E' = \begin{cases} E & \text{plane stress} \\ \dfrac{E}{1-\nu^2} & \text{plane strain} \end{cases}$$

ASTM E 1823

Standard Terminology Relating to Fatigue and Fracture Testing

Crack extension, Δa – an increase in crack size.

Crack-extension force, $\mathcal{G}$– the elastic energy per unit of new separation area that is made available at the front of an ideal crack in an elastic solid during a virtual increment of forward crack extension.

Crack-tip plane strain – a stress-strain field (near the crack tip) that approaches plane strain to a degree required by an empirical criterion.

Crack-tip plane stress – a stress-strain field (near the crack tip) that is not in plane strain.

Fracture toughness – a generic term for measures of resistance to extension of a crack.

ASTM E 1823 ctd.

Plane-strain fracture toughness, K_{Ic} – the crack-extension resistance under conditions of crack-tip plane strain in Mode I for slow rates of loading under predominantly linear-elastic conditions and negligible plastic-zone adjustment. The stress intensity factor, KIc, is measured using the operational procedure (and satisfying all of the validity requirements) specified in Test Method E 399, that provides for the measurement of crack-extension resistance at the onset (2% or less) of crack extension and provides operational definitions of crack-tip sharpness, onset of crack-extension, and crack-tip plane strain.

Plane-strain fracture toughness, J_{Ic} – the crack-extension resistance under conditions of crack-tip plane strain in Mode I with slow rates of loading and substantial plastic deformation. The J-integral, JIc, is measured using the operational procedure (and satisfying all of the validity requirements) specified in Test Method E 1820, that provides for the measurement of crack-extension resistance near the onset of stable crack extension.

ASTM E 399

Standard Test Method for Linear-Elastic Plane-Strain Fracture Toughness K_{Ic} of Metallic Materials

Characterizes the resistance of a material to fracture in a neutral environment in the presence of a sharp crack under essentially linear-elastic stress and severe tensile constraint, such that (1) the state of stress near the crack front approaches tritensile plane strain, and (2) the crack-tip plastic zone is small compared to the crack size, specimen thickness, and ligament ahead of the crack;

Is believed to represent a lower limiting value of fracture toughness;

May be used to estimate the relation between failure stress and crack size for a material in service wherein the conditions of high constraint described above would be expected;

Only if the dimensions of the product are sufficient to provide specimens of the size required for valid K_{Ic} determination.

Specimen size $W - a \geq 2.5\left(\frac{K_{\mathrm{Ic}}}{\sigma_{\mathrm{YS}}}\right)^2$ σ_{YS} 0.2 % offset yield strength

ASTM E 399 ctd.

Specimen configurations

SE(B): Single-edge-notched and fatigue precracked beam loaded in three-point bending; support span *S* = 4 *W*, thickness *B* = *W*/2, *W* = width;

C(T): Compact specimen, single-edge-notched and fatigue precracked plate loaded in tension; thickness *B* = *W*/2;

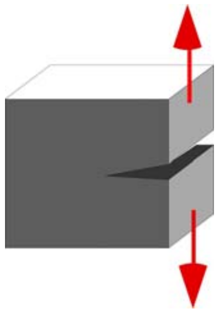
DC(T): Disk-shaped compact specimen, single-edge-notched and fatigue precracked disc segment loaded in tension; thickness *B* = *W*/2;

A(T): Arc-shaped tension specimen, single-edge-notched and fatigue precracked ring segment loaded in tension; radius ratio unspecified;

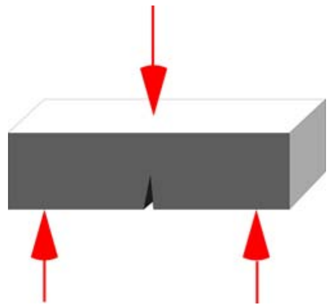
A(B): Arc-shaped bend specimen, single-edge-notched and fatigue precracked ring segment loaded in bending; radius ratio for *S*/*W* = 4 and for *S*/*W* = 3;

Specimen Configurations

Bend Type



C(T)



SE(B)

W	width
B	thickness
a	crack length
b = *W*-*a*	ligament width

Inelastic Deformation and Damage

Part I: Plasticity

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Material Mechanics

Outline

- Fundamentals of incremental plasticity
- Finite plasticity (deformation theory)
- Plasticity, damage, fracture
- Porous Metal Plasticity (GTN Model)

Plasticity

- Inelastic deformation of metals at “low” temperatures and “slow” (= quasistatic) loading, i. e. time and rate independent material behaviour
- Microscopic mechanisms: motion of dislocations, twinning.
- Phenomenological theory on macro-scale in the framework of continuum mechanics.

- Material behaviour is non-linear and plastic (permanent) deformations depend on loading history.
- Constitutive equations are established “incrementally” for small changes of loading and deformation

$$\Delta\sigma_{ij} = \dot{\sigma}_{ij}\Delta t\,, \quad \Delta\varepsilon_{ij} = \dot{\varepsilon}_{ij}\Delta t$$

Incremental Theory of Plasticity

$t > 0$ is no physical time but a scalar loading parameter, and hence

$\dot{\sigma}_{ij}$ and $\dot{\varepsilon}_{ij}$ are no ”velocities” but “**rates**”

Uniaxial Tensile Test

$\sigma \le R_0 : \quad \sigma = E\varepsilon$ linear elasticity: Hooke

$\sigma > R_0 :$ plasticity: nonlinear σ-ε-curve

permanent strain

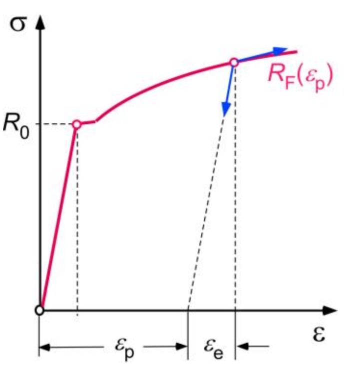
$$\varepsilon = \varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{p}} = \frac{\sigma}{E} + \varepsilon_{\mathrm{p}}$$

yield condition $\quad \sigma \le R_{\mathrm{F}}(\varepsilon_{\mathrm{p}}) \quad , \quad R_{\mathrm{F}}(0) = R_0$

R_0 yield strength (σ_0, σ_Y)

$R_F(\varepsilon_p)$ uniaxial yield curve

“true” stress-strain curve



loading / unloading

$$\dot{\sigma} > 0\,, \quad \dot{\varepsilon}_{\mathrm{p}} > 0 \quad \text{loading}$$
$$\dot{\sigma} < 0\,, \quad \dot{\varepsilon}_{\mathrm{p}} = 0 \quad \text{unloading}$$

3D Generalisation

Additive decomposition of strain rates $\quad \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{\mathrm{e}} + \dot{\varepsilon}_{ij}^{\mathrm{p}}$

Total plastic strains $\quad \varepsilon_{ij}^{\mathrm{p}} = \int_{\tau=0}^{t} \dot{\varepsilon}_{ij}^{\mathrm{p}} d\tau$

Plastic incompressibility:

- plastic deformations are isochoric $\quad \dot{\varepsilon}_{kk}^{\mathrm{p}} = 0$
- plastic yielding is not affected by hydrostatic stress

$\sigma'_{ij} = \sigma_{ij} - \sigma_{\mathrm{h}}\delta_{ij}$ deviatoric stress

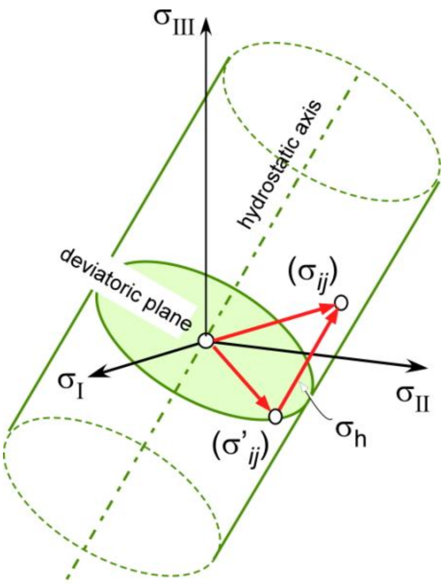
$\sigma_{\mathrm{h}} = \frac{1}{3}\sigma_{kk}$ hydrostatic stress

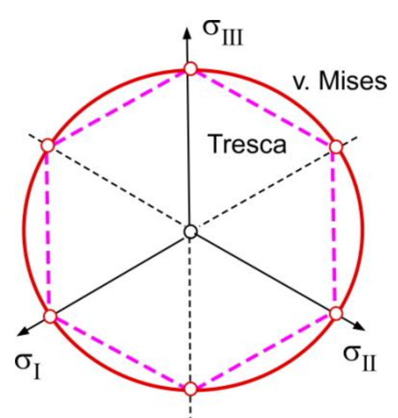
Yield condition $\quad \varphi(\sigma_{ij}, \varepsilon_{ij}^{\mathrm{p}}) = \left(\sigma'_{ij} - \alpha_{ij}\right)\left(\sigma'_{ij} - \alpha_{ij}\right) - \kappa^2(\bar{\varepsilon}^{\mathrm{p}}) = 0$

Hardening:

- **kinematic**: tensorial variable back stresses $\quad \alpha_{ij}$
- **isotropic**: scalar variable accumulated plastic strain $\quad \bar{\varepsilon}^{\mathrm{p}}$

Yield Surface





$\varphi < 0$ elastic

$\varphi = 0$ elastic

$\varphi > 0$ inadmissible

Elasto-Plasticity

Hooke's law of elasticity
$$\varepsilon_{ij}^{\mathrm{e}} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right]$$

Associated **flow rule** – normality rule
$$\dot{\varepsilon}_{ij}^{\mathrm{p}} = \dot{\lambda}\frac{\partial\varphi}{\partial\sigma_{ij}}$$

Loading condition

$$\frac{\partial\varphi}{\partial\sigma_{ij}}\dot{\sigma}_{ij} \geq 0\,, \quad \dot{\varepsilon}_{ij}^{\mathrm{p}} \neq 0 \qquad \text{loading}$$
$$\frac{\partial\varphi}{\partial\sigma_{ij}}\dot{\sigma}_{ij} < 0, \quad \dot{\varepsilon}_{ij}^{\mathrm{p}} = 0 \qquad \text{unloading}$$

Consistency condition
$$\dot{\varphi} = \frac{\partial\varphi}{\partial\sigma_{ij}}\sigma'_{ij} + \frac{\partial\varphi}{\partial\varepsilon_{ij}^{\mathrm{p}}}\dot{\varepsilon}_{ij}^{\mathrm{p}} = 0$$

Equivalence of dissipation rates
$$\dot{\sigma}'_{ij}\,\dot{\varepsilon}_{ij}^{\mathrm{p}} = \dot{\bar{\sigma}}\,\dot{\bar{\varepsilon}}^{\mathrm{p}} \geq 0$$

Stability
Drucker [1964]

von Mises Yield Condition

Yield condition - isotropic hardening
$$\varphi(\sigma_{ij}, \varepsilon_{ij}^{\mathrm{p}}) = \bar{\sigma}^2 - R_{\mathrm{F}}^2(\varepsilon_{\mathrm{p}}) = 0$$

von Mises [1913, 1928]
equivalent stress
$$\bar{\sigma} = \sqrt{3J_2(\sigma'_{ij})} = \sqrt{\tfrac{3}{2}\sigma'_{ij}\sigma'_{ij}}$$

"J_2-theory"

$$\bar{\sigma} = \sqrt{\tfrac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2\right)}$$

Loading condition

$$\sigma'_{ij}\dot{\sigma}'_{ij} \geq 0 \qquad \text{loading}$$
$$\sigma'_{ij}\dot{\sigma}'_{ij} < 0 \qquad \text{unloading}$$

Flow rule
$$\dot{\varepsilon}_{ij}^{\mathrm{p}} = \dot{\lambda}\sigma'_{ij}$$

$\dot{\lambda} =$ plastic multiplier from uniaxial test

Prandtl-Reuss

Hooke's law of elasticity + associated flow rule $\qquad \dot\varepsilon_{ij} = \dot\varepsilon^{\mathrm{e}}_{ij} + \dot\varepsilon^{\mathrm{p}}_{ij}$

Equivalence of dissipation rates $\qquad \bar\sigma\,\dot{\bar\varepsilon}^{\mathrm{p}} = \sigma'_{ij}\,\dot\varepsilon^{\mathrm{p}}_{ij} = \dot\lambda\,\sigma'_{ij}\sigma'_{ij}$

- equivalent plastic strain $\qquad \bar\varepsilon^{\mathrm{p}} = \int_{\tau=0}^{t} \sqrt{\tfrac{2}{3}\,\dot\varepsilon^{\mathrm{p}}_{ij}\dot\varepsilon^{\mathrm{p}}_{ij}}\,d\tau = \varepsilon_{\mathrm{p}}$
- plastic multiplier $\qquad \dot\lambda = \frac{3}{2}\frac{\dot{\bar\varepsilon}^{\mathrm{p}}}{\bar\sigma} = \frac{3}{2}\frac{\dot\varepsilon_{\mathrm{p}}}{R_{\mathrm{F}}(\varepsilon_{\mathrm{p}})}$

Total strain rates (**Prandtl [1924], Reuß [1930]**)

$$\dot\varepsilon_{ij} = \dot\varepsilon^{\mathrm{e}}_{ij} + \dot\varepsilon^{\mathrm{p}}_{ij} = \dot\varepsilon'^{\mathrm{e}}_{ij} + \tfrac{1}{3}\dot\varepsilon^{\mathrm{e}}_{kk}\delta_{ij} + \dot\varepsilon^{\mathrm{p}}_{ij} = \frac{1}{2G}\dot\sigma'_{ij} + \frac{1}{3K}\sigma_{\mathrm{h}}\delta_{ij} + \frac{3}{2T_{\mathrm{p}}}\frac{\dot{\bar\sigma}}{R_{\mathrm{F}}}\sigma'_{ij}$$

Finite Plasticity

additive decomposition of total strains $\qquad \varepsilon_{ij} = \varepsilon^{\mathrm{e}}_{ij} + \varepsilon^{\mathrm{p}}_{ij} \qquad$ **Hencky [1924]**

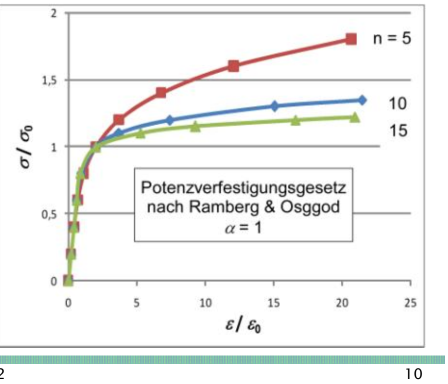
plastic strains $\qquad \varepsilon^{\mathrm{p}}_{ij} = \lambda\sigma'_{ij}$

total strains $\qquad \varepsilon_{ij} = \left(\frac{1}{2G} + \frac{3}{2S_{\mathrm{p}}}\right)\sigma'_{ij} + \frac{1}{3K}\sigma_{\mathrm{h}}\delta_{ij}$

power law of **Ramberg & Osgood [1945]**

uniaxial $\qquad \frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$

3D $\qquad \frac{\varepsilon^{\mathrm{p}}_{ij}}{\varepsilon_0} = \frac{3}{2}\alpha\left(\frac{\bar\sigma}{\sigma_0}\right)^{1-n}\frac{\sigma'_{ij}}{\sigma_0}$



“Deformation Theory”

For “radial” (proportional) loading, $\sigma_{ij}(t) = \phi(t)\sigma_{ij}^0$
the Hencky equations can be derived by integration of the Prandtl-Reuß equations.
This has do hold for every point of the continuum and excludes

- stress redistribution,
- unloading.

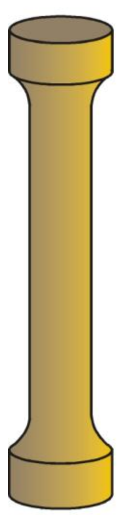
Finite plasticity actually describes a hyperelastic material having a strain energy density
$$\overline{w} = \int_{\tau=0}^{t} \sigma_{ij}\dot{\varepsilon}_{ij}\,d\tau$$

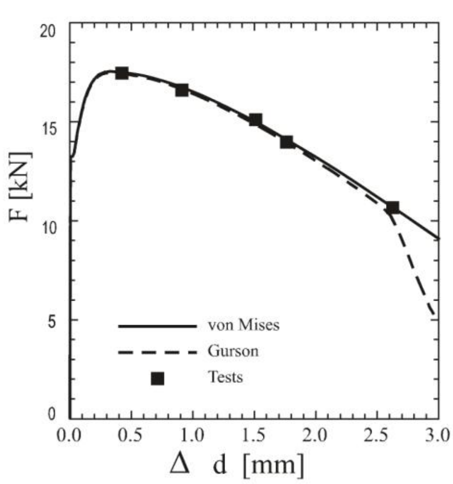
so that $$\sigma_{ij} = \frac{\partial \overline{w}}{\partial \varepsilon_{ij}}$$

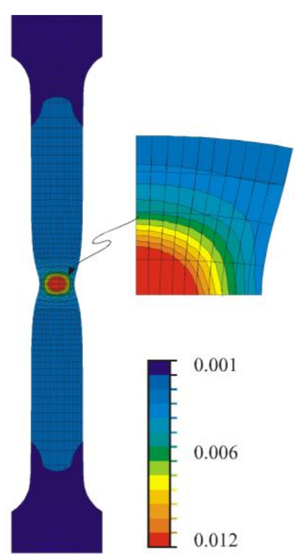
In elastic-plastic fracture mechanics (EPFM), finite plasticity + Ramberg-Osgood Power law are adressed as “**Deformation Theory**” of plasticity.

Plasticity and Fracture

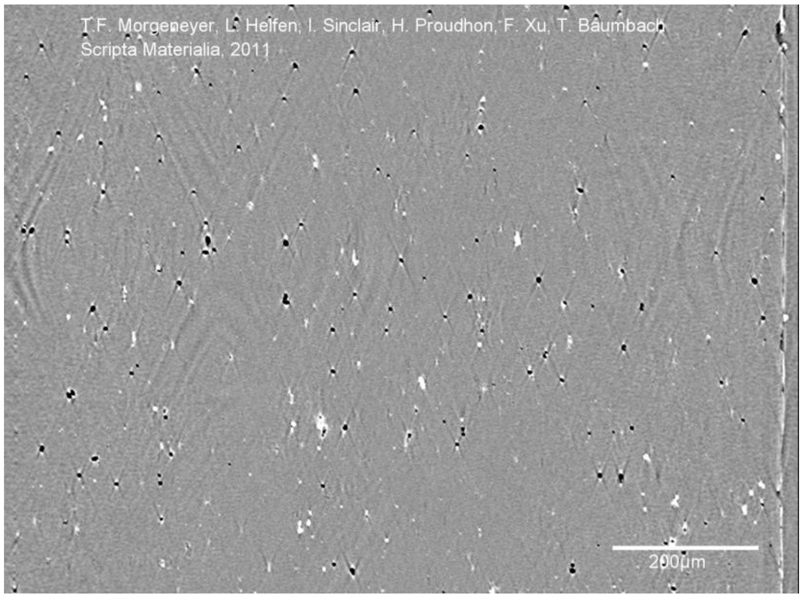
uniaxial tensile test



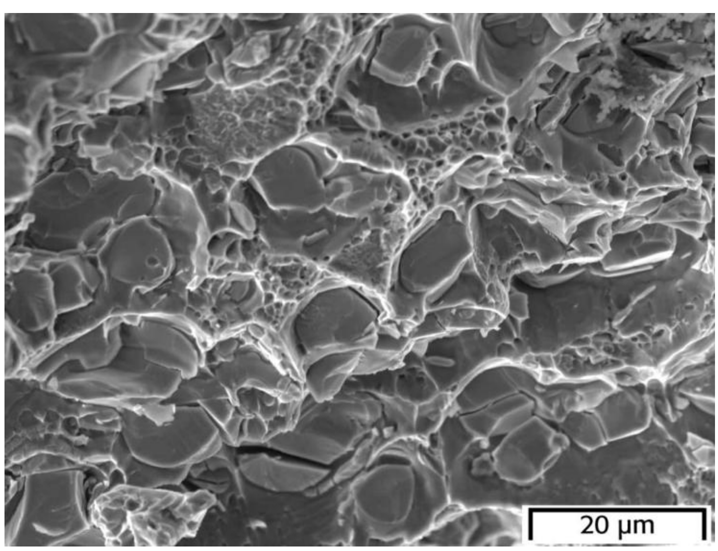




Fracture of a Tensile Bar



Fracture surface



round tensile specimen of Al 2024 T 351

Deformation, Damage, Fracture

- **Deformation**

  Cohesion of matter is conserved

  Elastic: atomic scale

  Reversible change of atomic distances

  Plastic: crystalline scale

  Irreversible shift of atoms, dislocation movement

- **Damage**

  Laminar or volumetric discontinuities on the micro scale (micro-cracks, microvoids, micro-cavities)

  Damage evolution is an irreversible process, whose micromechanical causes are very similar to deformation processes but whose macroscopic implications are much different

- **Fracture**

  Laminar discontinuities on the macro scale leading to global failure (cleavage fracture, ductile rupture)

Damage Models

Damage models describe evolution of degradation phenomena on the microscale from initial (undamaged or predamaged) state up to creation of a crack on the mesoscale (material element)

Damage is described by means of internal variables in the framework of continuum mechanics.

- **Phenomenological models**

  Change of macroscopically observable properties are interpreted by means of the internal variable(s);

  Concept of "effective stress": Kachanov [1958, 1986], Lemaitre & Chaboche [1992], Lemaitre [1992].

- **Micromechanical models**

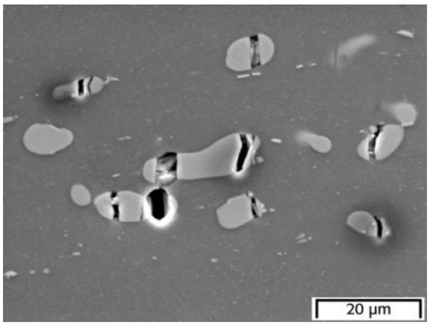
  The mechanical behaviour of a representative volume element (RVE) with defect(s) is studied;

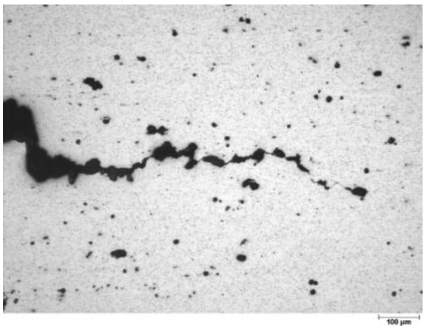
  Constitutive equations are formulated on a mesoscale by homogenisation of local stresses and strains in the RVE.

Ductile Damage

Nucleation, growth and coalescence of microvoids at inclusions or second-phase particles

Void growth is strain controlled, and depends on hydrostatic stress





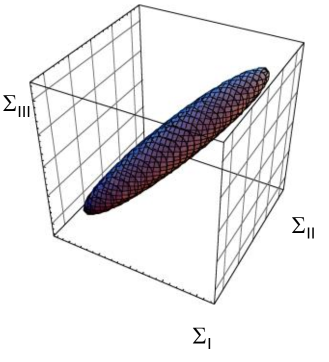
Porous Metal Plasticity

Plastic Potential of **Gurson, Tvergaard & Needleman** (GTN model) including scalar **damage variable** $f^*(f)$, (f = void volume fraction)

$$\varphi(\sigma_{ij}, \varepsilon_\mathrm{p}, f^*) = \frac{\bar{\sigma}^2}{R_\mathrm{F}^2(\varepsilon_\mathrm{p})} + 2 q_1 f^* \cosh\left(\frac{3}{2} q_2 \frac{\sigma_\mathrm{h}}{R_\mathrm{F}(\varepsilon_\mathrm{p})}\right) - 1 - q_3 f^{*2} = 0$$

Pores are assumed

- to be present from the beginning, f_0,
- or nucleate as a function of plastic equivalent strain, f_n, ε_n, s_n



Evolution equation of damage $\dot{f} = (1 - f)\,\dot{\varepsilon}^\mathrm{p}_{kk}$

Volume dilatation $\varepsilon^\mathrm{p}_{kk}$ caused by void growth

Concepts of Fracture Mechanics

Part II: Small Scale Yielding

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Material Mechanics

Overview

I. Linear elastic fracture mechanics (LEFM)
II. Small Scale Yielding (SSY)
III. Elastic-plastic fracture mechanics (EPFM)

SSY

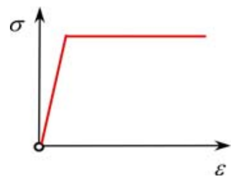
- Plastic zone at the crack tip
- Effective crack length (Irwin)
- Effective SIF
- Dugdale model
- Crack tip opening displacement (CTOD)
- Standards: ASTM E 399, ASTM E 561

Yielding in the Ligament (I)

Irwin [1964]: extension of LEFM to small plastic zones
Small Scale Yielding, SSY
K dominated stress field – mode I

Yielding in the ligament $\theta = 0$

perfectly plastic material $R_F(\varepsilon_p) = R_0$



(a) **plane stress** $\sigma_3 = \sigma_{zz} = 0$

$$\sigma_1 = \sigma_{xx} = \sigma_2 = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

Yield condition (both: von Mises and Tresca) $\sigma_{yy} = R_0\,, \quad 0 \le r \le r_p$

$$\Rightarrow \boxed{r_p = \frac{1}{2\pi}\left(\frac{K_I}{R_0}\right)^2}$$

r_p = "radius" of plastic zone

Yielding in the Ligament (II)

(b) **plane strain** $\varepsilon_3 = \varepsilon_{zz} = 0$

$$\sigma_3 = \sigma_{zz} = \nu(\sigma_1 + \sigma_2) = 2\nu\sigma_{yy} = \frac{2\nu K_\mathrm{I}}{\sqrt{2\pi r}}$$

Yield condition (both: von Mises and Tresca)

$$(1-2\nu)\sigma_{yy} = R_0\,, \quad 0 \le r \le r_\mathrm{p}$$

$$\Rightarrow \quad r_\mathrm{p} = \frac{(1-2\nu)^2}{2\pi}\left(\frac{K_\mathrm{I}}{R_0}\right)^2$$

smaller plastic zone due to triaxiality of stress state

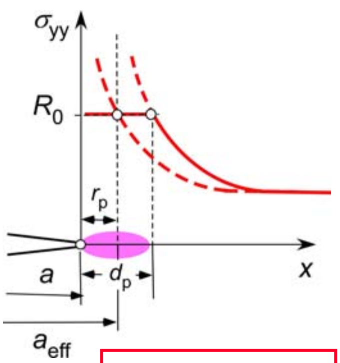
Cut-off of largest principle stress at R_0 (plane stress)

$$\int_0^{r_\mathrm{p}} \sigma_{yy}(r)\,dr = \int_0^{r_\mathrm{p}} \frac{K_\mathrm{I}}{\sqrt{2\pi r}}\,dr = \sqrt{\frac{2}{\pi}}\sqrt{r_\mathrm{p}} = 2R_0 r_\mathrm{p}$$

equilibrium ?

Effective Crack Length

$a_\mathrm{eff} = a + r_\mathrm{p}$ $\qquad r_\mathrm{p} \ll a$



effective SIF

$$K_\mathrm{Ieff} = K_\mathrm{I}(a_\mathrm{eff}) = \sigma_\infty\sqrt{\pi a_\mathrm{eff}}\,Y\left(\frac{a_\mathrm{eff}}{W}\right)$$

effective J $\qquad J_\mathrm{ssy} = \mathcal{G}_\mathrm{ssy} = \frac{K_\mathrm{Ieff}^2}{E'}$

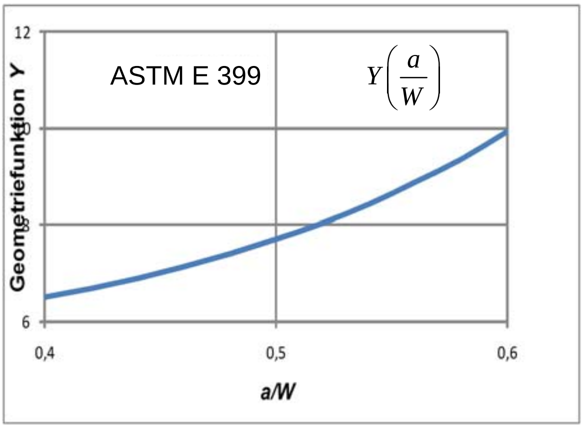
no singularity at the crack tip

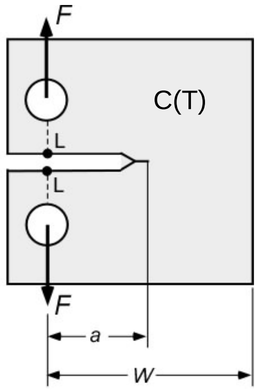
total “diameter” of plastic zone

$$d_\mathrm{p} = 2r_\mathrm{p} = \frac{\beta}{2\pi}\left(\frac{K_\mathrm{I}}{R_0}\right)^2 \quad , \quad \beta = \begin{cases} 1 & \text{plane stress} \\ (1-2\nu)^2 & \text{plane strain} \end{cases}$$

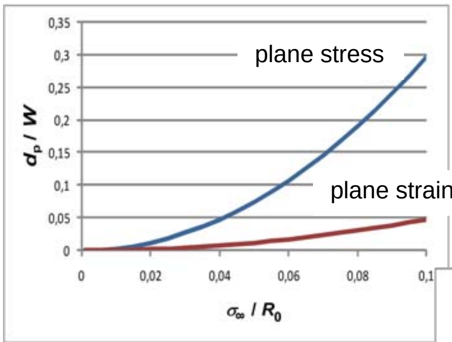
Example SSY (I)

$$K_I = \sigma_\infty \sqrt{\pi a}\, Y\left(\frac{a}{W}\right) \qquad \sigma_\infty = \frac{F}{BW}$$



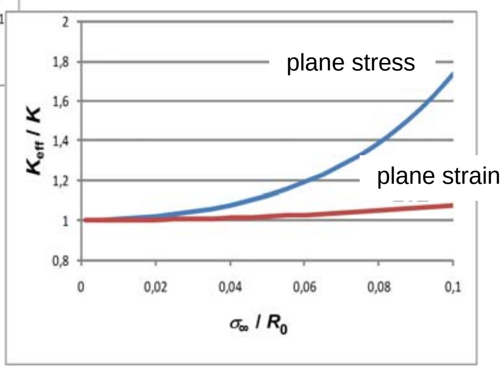


Example SSY (II)



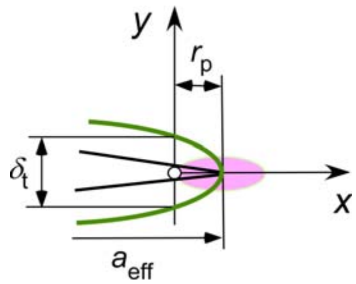
$$\frac{d_p}{W} = \frac{\beta}{2}\left(\frac{a}{W}\right)\left(\frac{\sigma_\infty}{R_0}\right)^2 Y\left(\frac{a}{W}\right)$$

$$K_{Ieff} = \sigma_\infty \sqrt{\pi a_{eff}}\, Y\left(\frac{a_{eff}}{W}\right)$$



CTOD (Irwin)

$$u_y(r,\pi) = 4\frac{K_\mathrm{I}}{E}\sqrt{\frac{r}{2\pi}}\begin{cases} 1 & \text{plane stress} \\ 1-\nu^2 & \text{plane strain} \end{cases}$$



Wells [1961] $\delta_\mathrm{t} = 2u_y(r_\mathrm{p},\pi)$

$$\delta_\mathrm{t}^\mathrm{Irwin} = \frac{4}{\pi}\frac{K_\mathrm{I}^2}{ER_0}\begin{cases} 1 & \text{plane stress} \\ (1-\nu^2)(1-2\nu) & \text{plane strain} \end{cases}$$

$$(1-\nu^2)(1-2\nu) \approx 0.36$$

$$\delta_\mathrm{t}^\text{plane strain} \approx 0{,}36\,\delta_\mathrm{t}^\text{plane stress}$$

Criterion for crack initiation: $\boxed{\delta_\mathrm{t} = \delta_\mathrm{c}}$

Shape of Plastic Zone

LEFM: $\sigma_{ij}(r,\theta) = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} f_{ij}^\mathrm{I}(\theta)$

$$\sigma_{33} = \begin{cases} 0 & \text{plane stress} \\ \nu(\sigma_{11}+\sigma_{22}) & \text{plane strain} \end{cases}$$

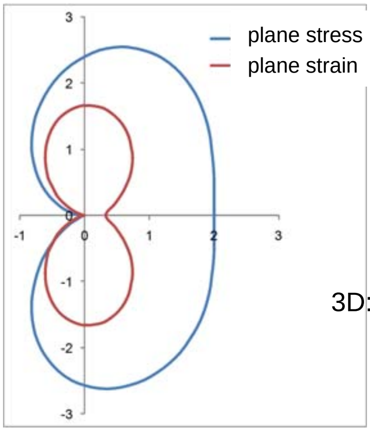
Yield condition (von Mises)

$$\bar{\sigma}^2 = \tfrac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + 3\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2\right)$$

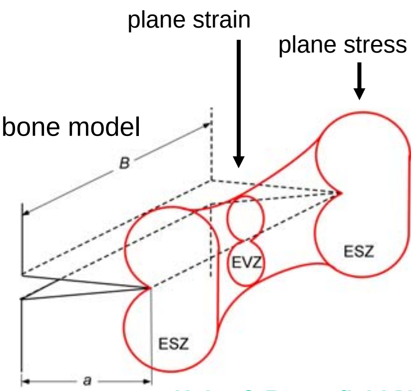
$$\bar{\sigma}\big|_{r_\mathrm{p}} = R_0$$

$$d_\mathrm{p}(\theta) = \frac{1}{2\pi}\left(\frac{K_\mathrm{I}}{R_0}\right)^2\begin{cases} 1+\frac{3}{2}\sin^2\theta+\cos\theta & \text{plane stress} \\ \frac{3}{2}\sin^2\theta+(1-2\nu)^2(1+\cos\theta) & \text{plane strain} \end{cases}$$

Plastic Zone

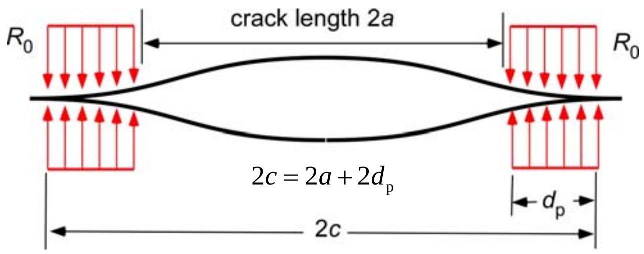


3D: dog bone model



Hahn & Rosenfield [1965]

Dugdale Model



strip yield model

$$\sigma_{yy}(r,0) = R_0 , \quad 0 \le r \le d_\mathrm{p}$$

Superposition $\quad K_\mathrm{I}^{(1)} = \sigma_\infty \sqrt{\pi c} \quad , \quad K_\mathrm{I}^{(2)} = -\frac{2}{\pi} R_0 \sqrt{\pi c} \arccos \frac{a}{c}$

no singularity $\quad K_\mathrm{I}^{(1)} + K_\mathrm{I}^{(2)} \overset{!}{=} 0 \quad \Rightarrow \quad \frac{a}{c} = \cos\left(\frac{\pi}{2} \frac{\sigma_\infty}{R_0} \right)$

$$d_\mathrm{p} = c\left[1 - \cos\left(\frac{\pi}{2} \frac{\sigma_\infty}{R_0} \right) \right] = a\left[\sec\left(\frac{\pi}{2} \frac{\sigma_\infty}{R_0} \right) - 1 \right]$$

no restriction with respect to plastic zone size!

$\sigma_\infty / R_0 \ll 1 : \quad d_\mathrm{p} \approx \frac{\pi^2}{8} c \left(\frac{\sigma_\infty}{R_0} \right)^2 \approx 1.23 a \left(\frac{\sigma_\infty}{R_0} \right)^2 = 1.23 d_\mathrm{p}^\mathrm{Irwin}$ $\quad$ plane stress!

CTOD (Dugdale)

Crack opening profile $u_y(x=a, y=0) = 4\frac{R_0}{E\pi} a \ln\sec\left(\frac{\pi}{2}\frac{\sigma_\infty}{R_0}\right)$

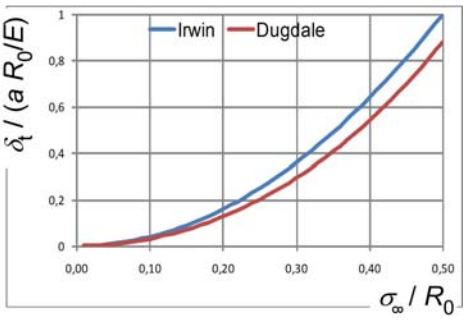
Definition of CTOD $\delta_t = 2u_y(x=a, y=0)$

$$\delta_t^{\mathrm{Dugdale}} = \frac{8}{\pi}\frac{R_0}{E} a \ln\sec\left(\frac{\pi}{2}\frac{\sigma_\infty}{R_0}\right)$$

no dependence on geometry!

$$\delta_t^{\mathrm{Irwin}} = \frac{4a\sigma_\infty^2}{ER_0}$$

for Griffith crack, plane stress

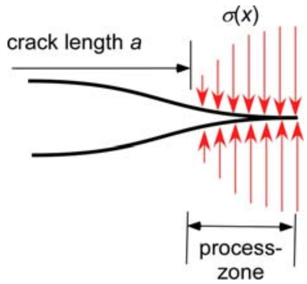


Barenblatt Model

Idea:

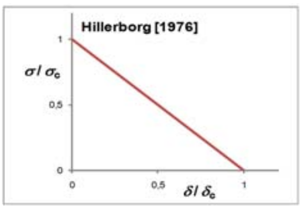
Singularity at the crack tip is unphysical

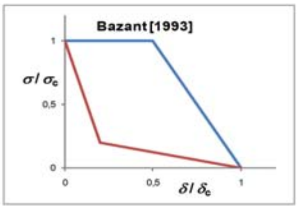
- **Griffith [1920]**: Energy approach
- **Irwin [1964]**: Effective crack length
- **Dugdale [1960]**: Strip yield model
- **Barenblatt [1959]**: **Cohesive zone**

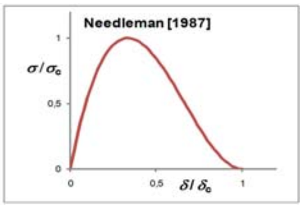


Stress distribution $\sigma(x)$ is unknown and cannot be measured

Cohesive model: traction-separation law $\sigma(\delta)$







Energy Release Rate

$$\mathcal{G} = -\left(\frac{\partial U}{B\partial a}\right)_{v_L} = \frac{\partial U_{rel}}{B\partial a}$$

$$= \mathcal{G}_{ssy} = \frac{K_{Ieff}^2}{E'} \approx \frac{a + r_p}{a}\frac{K_I^2}{E'} = \mathcal{G}^e + \mathcal{G}^p$$

$$\frac{\partial U_{sep}}{B\partial a} = \Gamma_c > \Gamma_c^e = 2\gamma$$

Fracture criterion: $\boxed{\mathcal{G} = \Gamma_c}$

Cohesive model: $\Gamma_c = \int_0^{\delta_c} \sigma(\delta)\, d\delta$ **Separation energy**

local criterion!

ASTM E 399: Size Condition

Standard Test Method for Linear-Elastic Plane-Strain Fracture Toughness K_{Ic} of Metallic Materials

Characterizes the resistance of a material to fracture in a neutral environment in the presence of a sharp crack under essentially linear-elastic stress and severe tensile constraint, such that

(1) the state of stress near the crack front approaches tritensile plane strain, and

(2) the crack-tip plastic zone is small compared to the crack size, specimen thickness, and ligament ahead of the crack;

Specimen size $W - a \geq 2.5\left(\frac{K_{Ic}}{\sigma_{YS}}\right)^2$ σ_{YS} 0.2 % offset yield strength

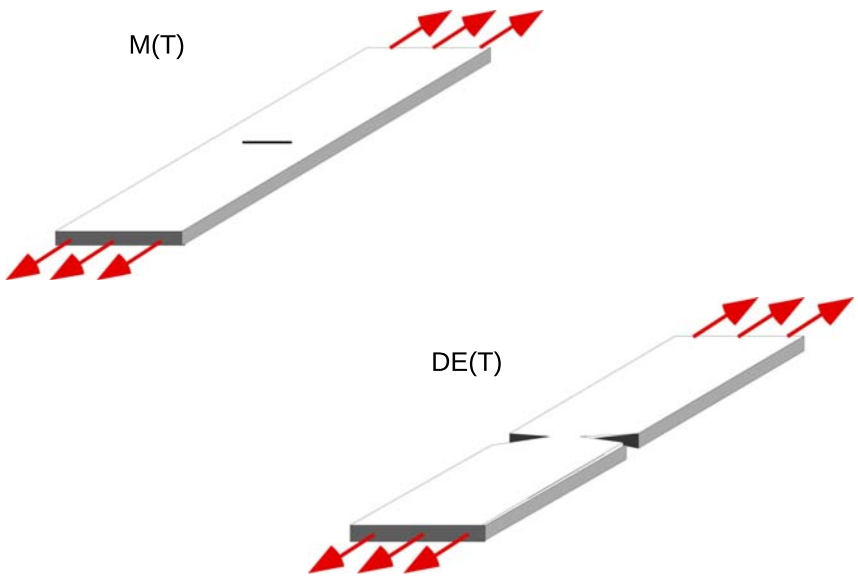
$$d_p = \frac{(1-2\nu)^2}{\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 \qquad \Rightarrow \frac{d_p}{W-a} \leq \frac{(1-2\nu)^2}{2.5\pi}\frac{B}{(W-a)} \approx 0.02$$

ASTM E 561

Standard Test Method for K-R Curve Determination

- Determination of the resistance to fracture under Mode I loading … using **M(T)**, C(T), or crack-line wedge-loaded C(W) specimen; continuous record of toughness development in terms of K_R plotted against crack extension.
- Materials are not limited by strength, thickness or toughness, so long as specimens are of sufficient size to remain predominantly elastic.
- Plot of **crack extension resistance K_R as a function of effective crack extension Δa_e**.
- Measurement of physical crack size by direct observation and then **calculating the effective crack size a_e by adding the plastic zone size**;
- Measurement of physical crack size by unloading compliance and then calculating the effective crack size a_e by adding the plastic zone size;
- Measurement of the effective crack size a_e directly by loading compliance.

FM Test Specimens (Tension Type)



Concepts of Fracture Mechanics

Part III: Elastic-Plastic Fracture

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Material Mechanics

Overview

I. Linear elastic fracture mechanics (LEFM)

II. Small Scale Yielding

III. Elastic-plastic fracture mechanics (EPFM)

EPFM

- Deformation theory of plasticity
- *J* as energy release rate
- HRR field, CTOD
- Deformation vs incremental theory of plasticity
- R curves
- Energy dissipation rate

EPFM

Analytical solutions and analyses in

Elastic-Plastic Fracture Mechanics,

i.e. fracture under large scale yielding (LSY)conditions

are based on "**Deformation Theory of Plasticity**"

which actually describes hyperelastic materials

$$\sigma_{ij} = \frac{\partial \overline{w}}{\partial \varepsilon_{ij}} \qquad \overline{w} = \int_{\tau=0}^{t} \sigma_{ij} \dot{\varepsilon}_{ij} d\tau \approx \int_{\tau=0}^{t} \sigma_{ij} \dot{\varepsilon}_{ij}^{\mathrm{p}} d\tau$$

in the following, the superscripts "e" stands for "linear elastic"
"p" stands for "nonlinear"

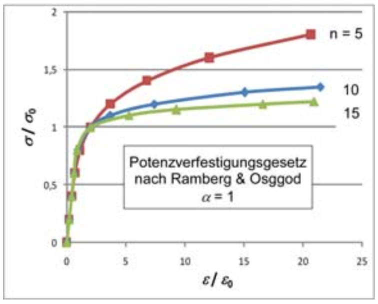
$$U = U^{\mathrm{e}} + U^{\mathrm{p}} = \int F\, dv_{\mathrm{L}}^{\mathrm{e}} + \int F\, dv_{\mathrm{L}}^{\mathrm{p}} \qquad J = J^{\mathrm{e}} + J^{\mathrm{p}} = \frac{K_{\mathrm{I}}^2}{E'} - \left(\frac{\partial U^{\mathrm{p}}}{\partial A_{\mathrm{crack}}} \right)_{v_{\mathrm{L}}}$$

J as Stress Intensity Factor

Power law of **Ramberg & Osgood [1945]**

uniaxial $$\frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon^{\mathrm{e}}}{\varepsilon_0} + \frac{\varepsilon^{\mathrm{p}}}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha\left(\frac{\sigma}{\sigma_0}\right)^n$$

3D $$\frac{\varepsilon^{\mathrm{p}}_{ij}}{\varepsilon_0} = \frac{3}{2}\alpha\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{1-n}\frac{\sigma'_{ij}}{\sigma_0}$$



Hutchinson [1868],
Rice & Rosengren [1968]

singular stress and strain fields at the crack tip (**HRR field**) – mode I

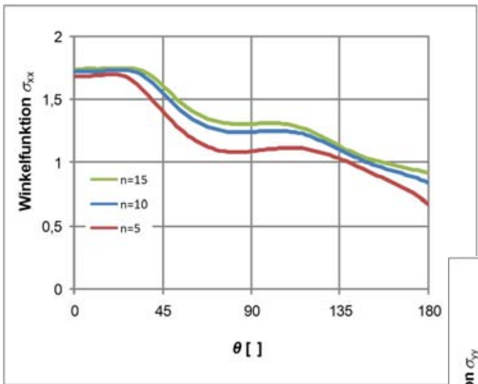
$$\sigma_{ij} = K_\sigma r^{-\frac{1}{n+1}}\tilde{\sigma}_{ij}(\theta)$$

$$K_\sigma = \sigma_0\left(\frac{J}{\alpha\sigma_0\,\varepsilon_0 I_n}\right)^{\frac{1}{n+1}}$$

$$\varepsilon_{ij} \approx \varepsilon^{\mathrm{p}}_{ij} = \alpha\varepsilon_0\left(\frac{K_\sigma}{\sigma_0}\right)^n r^{-\frac{n}{n+1}}\tilde{\varepsilon}_{ij}(\theta)$$

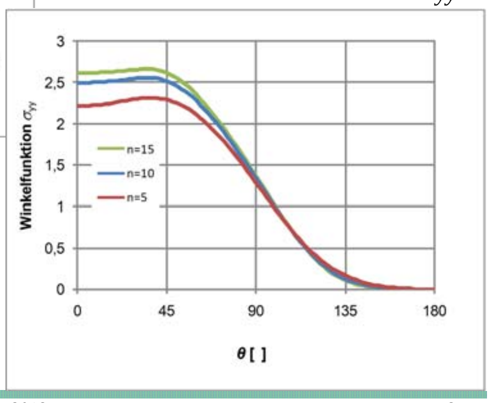
$$\boxed{\sigma_{ij}\varepsilon^{\mathrm{p}}_{ij} = \mathcal{O}(r^{-1})}$$

HRR Angular Functions



$\tilde{\sigma}_{xx}$

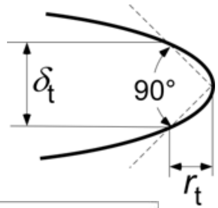
$\tilde{\sigma}_{yy}$



CTOD

HRR displacement field $$u_i = \alpha\varepsilon_0 \left(\frac{J}{\alpha\sigma_0\,\varepsilon_0 I_n} \right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}} \tilde{u}_i(\theta)$$

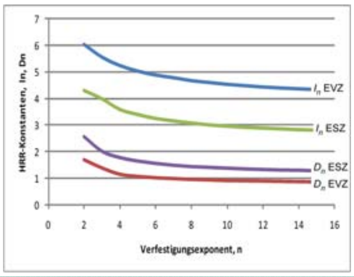
Crack Tip Opening Displacement, δ_t, Shih [1981]



$$\delta_t = 2u_y(r_t,\pi) \quad , \quad r - u_x(r_t,\pi) = u_y(r_t,\pi)$$

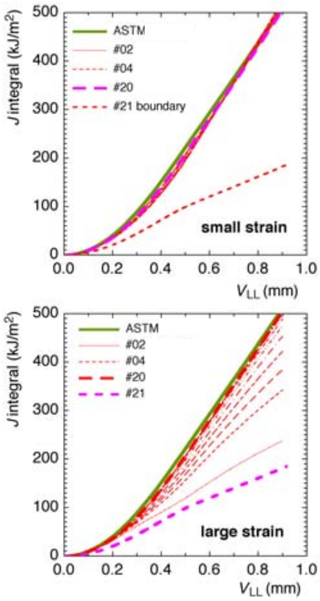
$$\boxed{\delta_t = d_n \frac{J}{\sigma_0}}$$

$$d_n = \left(\alpha\varepsilon_0\right)^{\frac{1}{n}} D_n$$

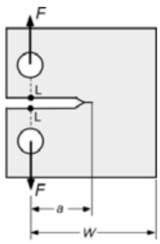


$J = J^{el} + J^{pl}$

Path Dependence of J



FE simulation
C(T) specimen
plane strain
stationary crack
incremental theory of plasticity



ASTM E 1820: reference value
"far field" value

$$J = J^e + J^p \qquad J^e = \frac{K_I^2}{E'}$$

$$K_I = \sigma_\infty \sqrt{\pi a}\, Y\left(a/W\right)$$

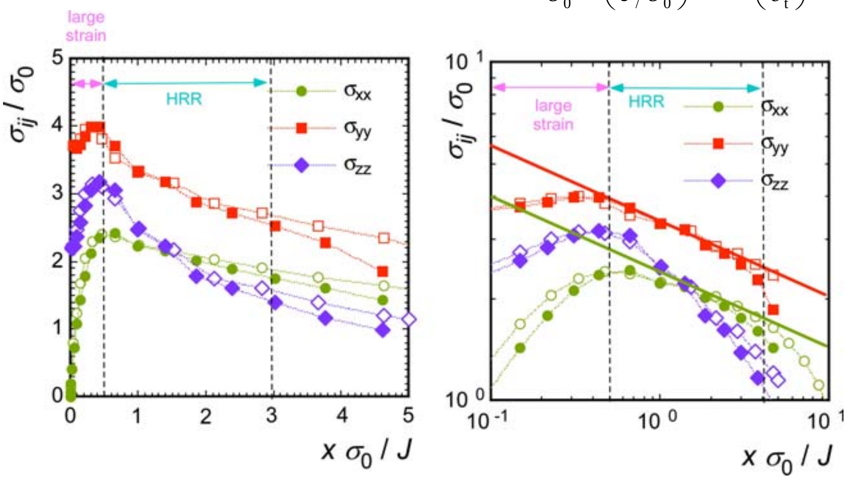
$$J^p = \frac{\eta U^p}{B\left(W - a\right)}$$

Stresses at Crack Tip

- incremental theory of plasticity
- large strain analysis

HRR

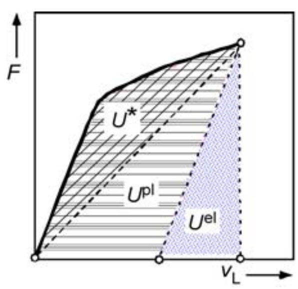
$$\frac{\sigma_{ij}}{\sigma_0} \simeq \left(\frac{r}{J/\sigma_0}\right)^{-\frac{1}{n+1}} \simeq \left(\frac{r}{\delta_t}\right)^{-\frac{1}{n+1}}$$



R-Curves in FM

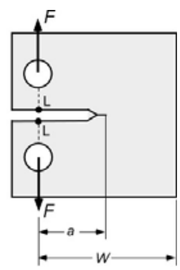
Different from quasi-brittle fracture, **ductile crack extension** is deformation controlled: **R-curves $J(\Delta a)$, $\delta(\Delta a)$**

R curve – a plot of crack-extension resistance as a function of stable crack extension (ASTM E 1820)



J_R-curve: $J(\Delta a)$

measure F, V_L, Δa



$$J(a_{(i)}) = J^e + J^p_{(i)} = \frac{K_I^2}{E'} + \left(J^p_{(i-1)} + \frac{\eta_{(i-1)}}{b_{(i-1)}} \frac{\Delta U^p_{(i)}}{B}\right)\left(1 - \gamma_{(i-1)} \frac{\Delta a_{(i)}}{b_{(i-1)}}\right)$$

recursion formula

ASTM E 1823

Standard Terminology Relating to Fatigue and Fracture Testing

Crack extension, Δa – an increase in crack size.

Crack-extension resistance, K_R, $\mathcal{G}_R$ of J_R – a measure of the resistance of a material to crack extension expressed in terms of the stress-intensity factor, K; crack-extension force, $\mathcal{G}$; or values of J derived using the J-integral concept.

Crack-tip opening displacement (CTOD), δ – the crack displacement resulting from the total deformation (elastic plus plastic) at variously defined locations near the original (prior to force application) crack tip.

***J*-R curve** – a plot of resistance to stable crack extension, Δa_p.

R curve – a plot of crack-extension resistance as a function of stable crack extension, Δa_p or Δa_e.

Stable crack extension – a displacement-controlled crack extension beyond the stretch-zone width. The extension stops when the applied displacement is held constant.

ASTM E 1820

Standard Test Method for Measurement of Fracture Toughness

Determination of fracture toughness of metallic materials using the parameters K, J, and CTOD (δ).

Assuming the existence of a preexisting, sharp, fatigue crack, the material fracture toughness values identified by this test method characterize its resistance to

(1) Fracture of a stationary crack

(2) Fracture after some stable tearing

(3) Stable tearing onset

(4) Sustained stable tearing

This test method is particularly useful when the material response cannot be anticipated before the test.

Serve as a basis for material comparison, selection and quality assurance; rank materials within a similar yield strength range;

Serve as a basis for structural flaw tolerance assessment; awareness of differences that may exist between laboratory test and field conditions is required.

ASTM E 1820 (ctd.)

Cautionary statements

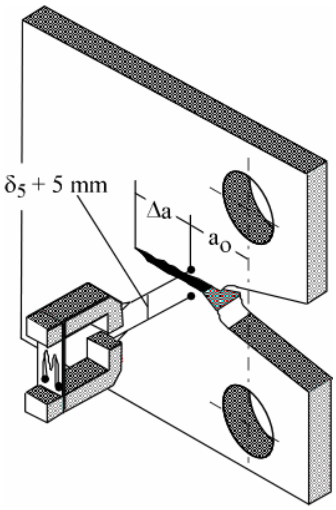
- Fracture after some stable tearing is sensitive to material inhomogeneity and to constraint variations that may be induced to planar geometry, thickness differences, mode of loading, and structural details;
- *J*-R curve from bend-type specimens, SE(B), C(T), DC(T), has been observed to be conservative with respect to results from tensile loading configurations;
- The values of δ_c, δ_u, J_c, and J_u, may be affected by specimen dimensions

CTOD R-Curve

Schwalbe [1995]: $\delta_5(\Delta a)$

ASTM E 2472

particularly for thin panels



ASTM E 2472

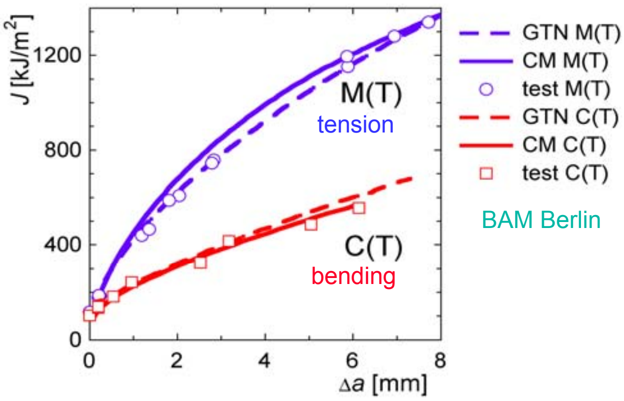
Standard Test Method for Determination of Resistance to Stable Crack Extension under Low-Constraint Conditions

- Determination of the resistance against stable crack extension of metallic materials under Mode I loading in terms of critical **crack-tip opening angle** (CTOA) and/or **crack opening displacement** (COD) as δ_5 resistance curve.
- Materials are not limited by strength, thickness or toughness, as long as and , ensuring low constraint conditions in M(T) and C(T) specimens.

Limitations

For extending crack

- J becomes significantly path dependent
- J looses its property of being an energy release rate
- J is a cumulated quantity of global dissipation
- J_R curves are geometry dependent



Dissipation Rate

Energy balance

$$\frac{\partial W_{ex}}{B\partial a} = \frac{\partial}{B\partial a}\left(U^e + U^p + U_{sep}\right)$$

Dissipation rate
Turner (1990)

$$R = \frac{\partial U_{diss}}{B\partial a} = \frac{\partial U^p}{B\partial a} + \frac{\partial U_{sep}}{B\partial a} = R^p + \Gamma_c$$

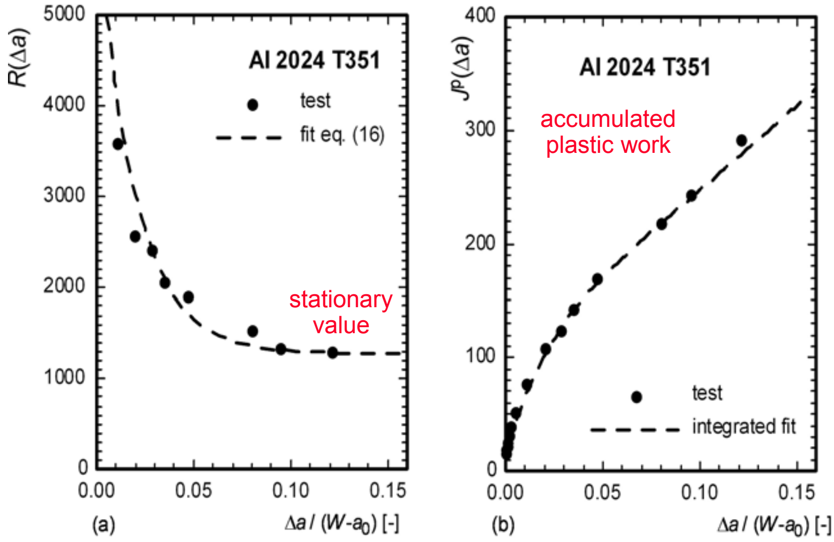
$$R^p = \frac{\partial U^p}{B\partial a}$$ global plastic dissipation rate

$$\Gamma_c = \frac{\partial U_{sep}}{B\partial a}$$ local separation rate

commonly: $R^p \gg \Gamma_c \quad \Rightarrow$ geometry dependence of J_R curves

$$R(\Delta a) = \begin{cases} (W-a)\dfrac{dJ^p}{da} & \text{M(T), DE(T)} \\[2ex] \dfrac{(W-a)}{\eta}\dfrac{dJ^p}{da} + J^p\dfrac{\gamma}{\eta} & \text{C(T), SE(B)} \end{cases}$$

$R(\Delta a)$



M(T) specimen, W = 100mm, B = 5mm, a_0/W = 0.5,

Inelastic Deformation and Damage

Part II: Damage Mechanics

W. Brocks

Christian Albrecht University
Material Mechanics

Outline

- Deformation, Damage and Fracture
- Crack Tip and Process Zone
- Damage Models
- Micromechanisms of Ductile Fracture
- Micromechanical Models
- Porous Metal Plasticity (GTN Model)

Damage - Definition

l'endommagement, comme le diable, invisible mais redoutable

- Surface or volume-like discontinuities on the materials micro-level (microcracks, microvoids)
- Damage evolution is irreversible (dissipation!)
- Damage causes degradation (reduction of performance)
- Examples for processes involving damage phenomena: **ductile damage in metals**, creep damage, fiber cracking or fiber-matrix delamination in reinforced composites, corrosion, fatigue

Observable Effects

Physical Appearance of Damage

- volume defects (microvoids, microcavities)
- surface defects (microcracks)

Mascroscopic Effects of Damage

- decreases elasticity modulus
- decreases yield stress
- decreases hardness
- increases creep strain rate
- decreases sound-propagation velocity
- decreases density
- increases electrical resistance

Damage Models

Damage models describe evolution of degradation phenomena on the microscale from initial (undamaged or predamaged) state up to creation of a crack on the mesoscale (material element)

Damage is described by means of internal variables in the framework of continuum mechanics.

- **Phenomenological models**

  Change of macroscopically observable properties are interpreted by means of the internal variable(s);

  Concept of “effective stress”: Kachanov [1958, 1986], Lemaitre & Chaboche [1992], Lemaitre [1992].

- **Micromechanical models**

  The mechanical behaviour of a representative volume element (RVE) with defect(s) is studied;

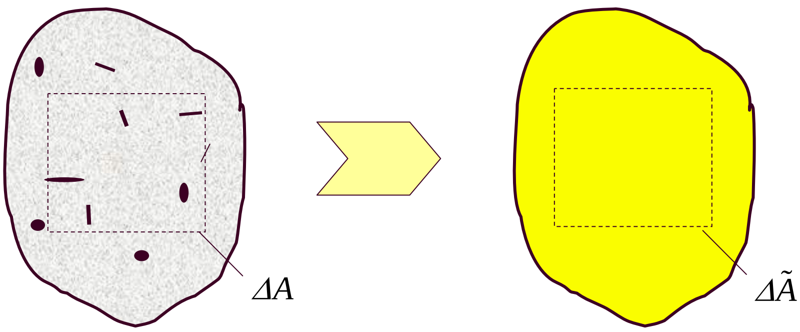
  Constitutive equations are formulated on a mesoscale by homogenisation of local stresses and strains in the RVE.

Continuum Damage Mechanics (CDM)

Kachanov [1958]
Hult [1972]
Lemaitre [1971]
Lemaitre & Chaboche [1976]

J. Lemaitre, R. Desmorat
Engineering Damage Mechanics
Springer, 2005

D. Krajcinovic
Damage Mechanics
Elsevier, 1996



Effective Area

Volume density of microvoids $D_V = \dfrac{\Delta V_{\text{voids}}}{\Delta V_{\text{RVE}}} = f_V$

Surface density of microcracks or intersections of microvoids with plane of normal **n** $D_{(\mathbf{n})} = \dfrac{\Delta A_{\text{cracks}}}{\Delta A_{\text{RVE}}}$

"Effective" area $\Delta\tilde{A} = \Delta A - \Delta A_D$

Isotropic Damage $\Rightarrow$ **Scalar Damage Variable** $D = \dfrac{\Delta A_D}{\Delta A}$

if $D_{(\mathbf{n})}$ does not depend on **n**

$$\boxed{\Delta\tilde{A} = (1 - D)\,\Delta A}$$

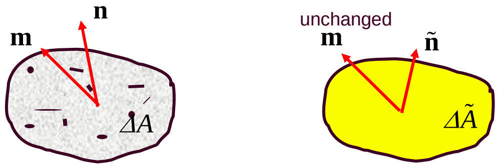
Anisotropic Damage

Tensorial Damage Variables

- rank 2 tensor $\mathcal{D}$ $\quad \tilde{\mathbf{n}}\,\Delta\tilde{A} = (\mathbf{1} - \mathcal{D})\cdot\mathbf{n}\,\Delta A$
- rank 4 tensor $\mathbb{D}$ $\quad (\mathbf{m}\tilde{\mathbf{n}})\,\Delta\tilde{A} = (\mathbb{I} - \mathbb{D})\cdot\cdot(\mathbf{m}\mathbf{n})\,\Delta A$

with symmetries $D_{ijkl} = D_{ijlk} = D_{jikl} = D_{klij}$, most general case

metric tensor (**mn**) defines the reference configuration



Effective Stress (I)

Isotropic Damage

1D $$\tilde{\sigma} = \frac{\sigma}{1-D}$$

3D $$\tilde{\mathbf{S}} = \frac{\mathbf{S}}{1-D} \quad \text{or} \quad \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{1-D}$$

Anisotropic Damage

rank 4 tensor $\mathbb{D}$

$$\mathbf{m}\cdot\tilde{\mathbf{S}}\cdot\tilde{\mathbf{n}}\,\Delta\tilde{A} = \mathbf{m}\cdot\mathbf{S}\cdot\mathbf{n}\,\Delta A$$

projection of stress vector on **m**

$$\tilde{\mathbf{S}}\cdot\cdot\,\mathbf{m}\tilde{\mathbf{n}}\,\Delta\tilde{A} = \mathbf{S}\cdot\cdot\,\mathbf{mn}\,\Delta A = \tilde{\mathbf{S}}\cdot\cdot\left(\mathbb{I}-\mathbb{D}\right)\cdot\cdot\,\mathbf{mn}\,\Delta A$$

$$\tilde{\mathbf{S}} = \mathbf{S}\cdot\cdot\left(\mathbb{I}-\mathbb{D}\right)^{-1} \quad \text{or} \quad \tilde{\sigma}_{ij} = \left(\delta_{ik}\delta_{jl} - D_{ijkl}\right)^{-1}\sigma_{kl}$$

Effective Stress (II)

Rank 2 tensor $\mathcal{D}$ requires additional conditions:

- Symmetry of the effective stress:

$$\tilde{\mathbf{S}} = \mathbf{S}\cdot\left(\mathbf{1}-\mathcal{D}\right)^{-1}$$ is not symmetric!

- Compatibility with the thermodynamics framework: existence of strain potentials and principle of strain equivalence,
- Symmetrisation (not derived from a potential)

$$\tilde{\mathbf{S}} = \tfrac{1}{2}\left[\mathbf{S}\cdot\left(\mathbf{1}-\mathcal{D}\right)^{-1} + \left(\mathbf{1}-\mathcal{D}\right)^{-1}\cdot\mathbf{S}\right]$$

- Different effect of the damage on the hydrostatic and deviatoric stress

$$\tilde{\mathbf{S}} = \left(\mathcal{H}\cdot\mathbf{S}'\cdot\mathcal{H}\right)' + \frac{\sigma_{\mathrm{h}}}{1-\eta D_{\mathrm{h}}}\mathbf{1} \quad \text{with} \quad \mathcal{H} = \left(1-\mathcal{D}\right)^{-1/2}$$

Example: Rank 2 Tensor

Transversal isotropic damage (2=3):

Uniaxial loading σ_1,

$$\mathcal{D} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_2 \end{pmatrix} \mathbf{e}_i \mathbf{e}_j ; \quad \mathcal{H} = \begin{pmatrix} \sqrt{\frac{1}{1-D_1}} & 0 & 0 \\ 0 & \sqrt{\frac{1}{1-D_2}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{1-D_2}} \end{pmatrix} \mathbf{e}_i \mathbf{e}_j$$

$$\tilde{\sigma}_1 = \frac{4}{9}\frac{\sigma_1}{1-D_1} + \frac{2}{9}\frac{\sigma_1}{1-D_2} + \frac{1}{3}\frac{\sigma_1}{\eta D_h}$$

$$\tilde{\tilde{\sigma}} = \frac{2}{3}\frac{\sigma_1}{1-D_1} + \frac{1}{3}\frac{\sigma_1}{1-D_2} \geq \sigma_1$$

effective von Mises stress

Thermodynamics of Damage

1. Definition of **state variables**, the actual value of each defining the present state of the corresponding mechanism involved
2. Definition of a **state potential** from which derive the state laws such as thermo-elasticity and the definition of the variables associated with the internal state variables
3. Definition of a **dissipation potential** from which derive the laws of evolution of the state variables associated with the dissipative mechanism

Check 2nd Principle of Thermodynamics !

Variables

Mechanism	State variable		conjugate variable
	observable	internal	
Thermoelasticity	$\mathbf{E}$		$\mathbf{S}$
Temperature/Entropy	θ		s
Plasticity		$\mathbf{E}^{\mathrm{p}}$	$-\mathbf{S}$
Isotropic hardening		p	R
Kinematic hardening		$\mathbf{A}$	$\mathbf{X}$
Damage isotropic		D	$-Y$
Damage anisotropic		$\mathcal{D}$	$-\mathbf{Y}$

$$p \equiv \bar{\varepsilon}^{\mathrm{p}}, \quad R \equiv \bar{\sigma}$$

State Potential

Helmholtz specific free energy

$$\psi\left(\mathbf{E}^{\mathrm{e}}, D \text{ or } \mathcal{D}, p, \mathbf{A}, \theta\right) = \psi^{\mathrm{e}} + \psi^{\mathrm{p}} + \psi^{\theta}$$

Gibbs specific free enthalpy taken as **state potential**

$$\psi^{*} = \sup_{\mathbf{E}}\left[\frac{1}{\rho}\mathbf{S}\cdot\cdot\mathbf{E} - \psi\right]$$

$$= \sup_{\mathbf{E}^{\mathrm{e}}}\left[\frac{1}{\rho}\mathbf{S}\cdot\cdot\mathbf{E}^{\mathrm{e}} - \psi^{\mathrm{e}}\right] + \frac{1}{\rho}\mathbf{S}\cdot\cdot\mathbf{E}^{\mathrm{p}} - \psi^{\mathrm{p}} - \psi^{\theta}$$

State laws of thermoelasticity can be deducted

$$\mathbf{E} = \rho\frac{\partial\psi^{*}}{\partial\mathbf{S}} = \rho\frac{\partial\psi_{\mathrm{e}}^{*}}{\partial\mathbf{S}} + \mathbf{E}^{\mathrm{p}} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}}$$

$$s = \frac{\partial\psi^{*}}{\partial\theta}$$

Dissipation Potential

Definition of **conjugate variables**

$$R = -\rho \frac{\partial \psi^*}{\partial p}$$
$$\mathbf{X} = -\rho \frac{\partial \psi^*}{\partial \mathbf{A}}$$
$$-Y = -\rho \frac{\partial \psi^*}{\partial D} \quad \text{or} \quad -\mathbf{Y} = -\rho \frac{\partial \psi^*}{\partial \boldsymbol{\mathcal{D}}}$$

2nd Principle of Thermodynamics (**Clausius-Duhem** inequality)

$$\mathbf{S} \cdot\cdot \dot{\mathbf{E}}^{\mathrm{p}} - \left(R\, \dot{p} + \mathbf{X} \cdot\cdot \dot{\mathbf{A}} \right) + \mathbf{Y} \cdot\cdot \dot{\boldsymbol{\mathcal{D}}} - \frac{\mathbf{q} \cdot \operatorname{grad} \theta}{\theta} \geq 0$$

Evolution equations for internal variables (kinetic laws) are derived from a **dissipation potential** Φ, which is a convex function of the conjugate variables

$$\Phi\left(\mathbf{S}, R, \mathbf{X}, Y \text{ or } \mathbf{Y}, \theta\right)$$

Normality Rule

Normality rule of “generalised standard materials”

$$\dot{\mathbf{E}}^{\mathrm{p}} = -\dot{\lambda} \frac{\partial \Phi}{\partial(-\mathbf{S})} = \dot{\lambda} \frac{\partial \Phi}{\partial \mathbf{S}} \qquad \text{flow rule}$$
$$\dot{p} = -\dot{\lambda} \frac{\partial \Phi}{\partial R}$$
$$\dot{\mathbf{A}} = -\dot{\lambda} \frac{\partial \Phi}{\partial \mathbf{X}}$$
$$\dot{D} = -\dot{\lambda} \frac{\partial \Phi}{\partial(-Y)} = \dot{\lambda} \frac{\partial \Phi}{\partial Y} \quad \text{or} \quad \dot{\boldsymbol{\mathcal{D}}} = -\dot{\lambda} \frac{\partial \Phi}{\partial(-\mathbf{Y})} = \dot{\lambda} \frac{\partial \Phi}{\partial \mathbf{Y}}$$

Nice and consistent theoretical framework – but wherefrom to get the dissipation potential Φ?

Principle of Strain Equivalence

Strain constitutive equations of a damage material are derived from the same formalism as for a non-damaged material except that the **stress is replaced by the effective stress**

Example: State potential for linear isotropic elasticity

$$\rho\psi_e^* = \frac{(1+\nu)}{2E}\frac{\sigma_{ij}\sigma_{ij}}{(1-D)} - \frac{\nu}{2E}\frac{\sigma_{kk}^2}{(1-D)}$$

➢ Elastic strain

$$\varepsilon_{ij}^e = \rho\frac{\partial\psi_e^*}{\partial\sigma_{ij}} = \frac{1+\nu}{E}\tilde{\sigma}_{ij} - \frac{\nu}{E}\tilde{\sigma}_{kk}\delta_{ij}$$

➢ Energy density release rate *Y*

$$Y = \rho\frac{\partial\psi_e^*}{\partial D} = \frac{\tilde{\bar{\sigma}}}{2E}\left[\frac{2}{3}(1+\nu)+3(1-2\nu)\left(\frac{\sigma_h}{\bar{\sigma}}\right)^2\right]$$

$$\sigma_h = \tfrac{1}{3}\sigma_{kk}$$

$$\bar{\sigma} = \sqrt{\tfrac{2}{3}\sigma'_{ij}\sigma'_{ij}}$$

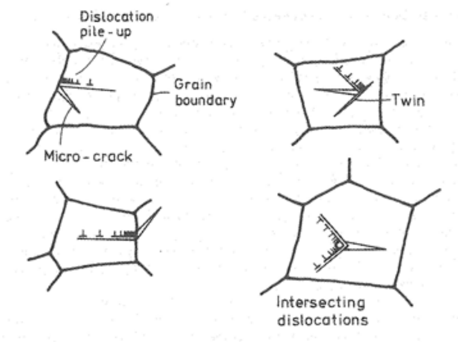
Local and Micromechanical Approaches

➢ **Cleavage** (“brittle” fracture)
Microcrack formation and coalescence
Stress controlled
Ritchie, Knott & Rice [1973]: RKR model, Beremin [1983]

➢ **Ductile tearing**
Nucleation, growth and coalescence of microvoids at inclusions or second-phase particles
Strain controlled, void growth dependent on hydrostatic stress
Rice & Tracey [1973], Gurson [1977], Beremin [1983], Tvergaard & Needleman [1982, 1984, ...], Thomason [1985, 1990], Rousselier [1987]

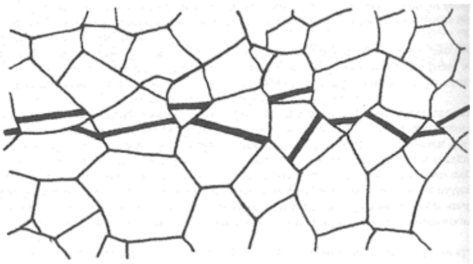
➢ **Creep damage**
Nucleation, growth and coalescence of micropores at grain boundaries
Stress or strain controlled
Hutchinson [1983], Rodin & Parks [1988], Sester & Riedel [1995]

Cleavage

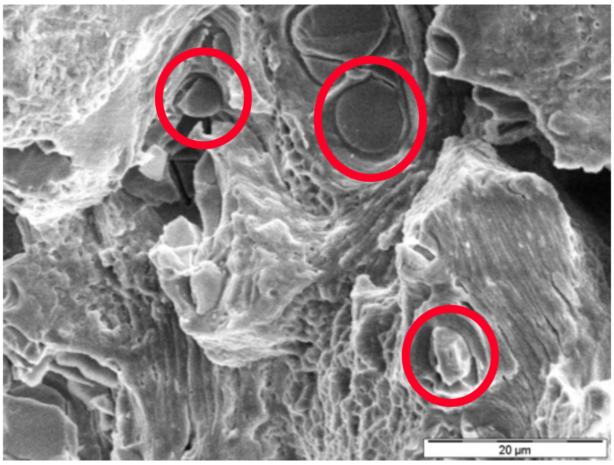


Mechanisms of microcrack initiation
Broberg [1999]

Coalescence of microcracks



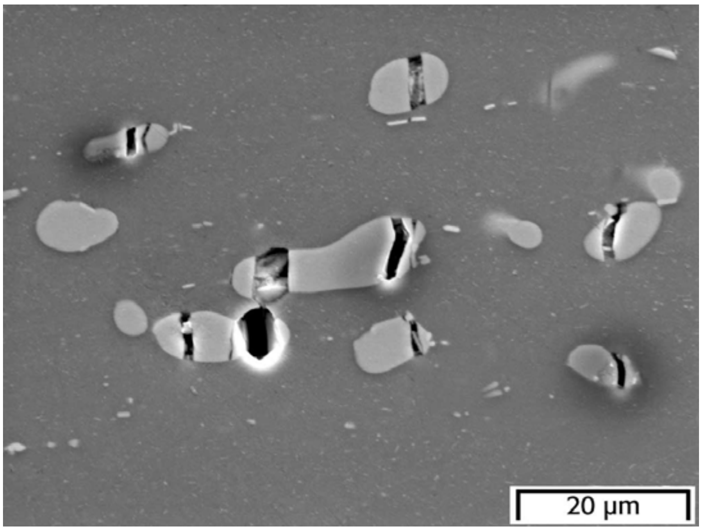
Ductile Fracture



fracture surface of Al 2024

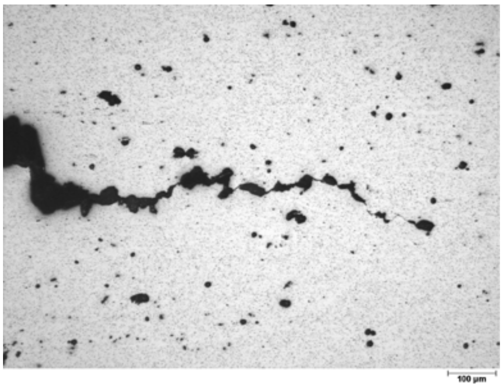
- Failure mechanisms: **Nucleation, growth and coalescence of voids**
- Voids nucleate at secondary phase particles due to particle/matrix debonding and/or particle fracture
- Localisation of plastic deformation is prior to failure

Void Nucleation

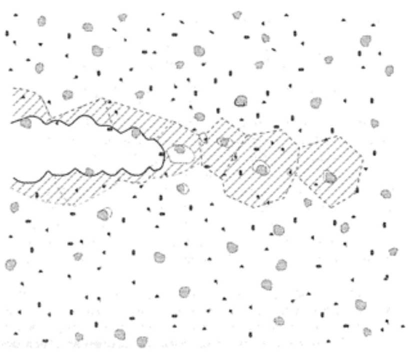


Void nucleation at coarse particles in Al 2024 T 351

Ductile Crack Extension (I)



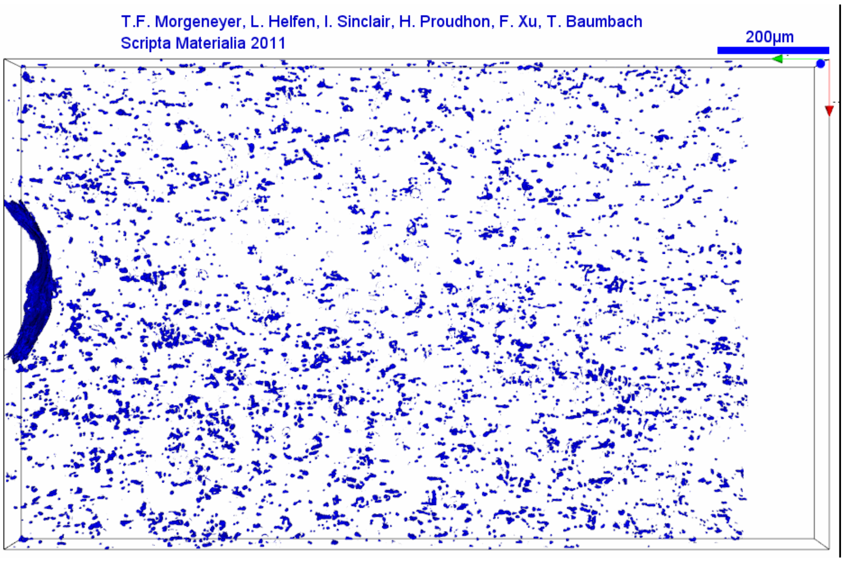
Ductile crack extension in an Al alloy



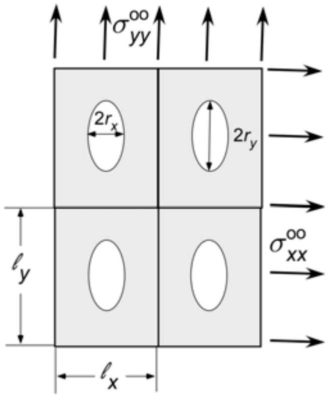
Schematic view of process zone with "unit cells"

Broberg [1999]

Ductile Crack Extension (I)



Models of Void Growth (I)



McClintock [1968]

damage

$$\eta_{zx} = \int d\eta_{zx} = \int \frac{d[\ln(r_x/\ell_x)]}{\ln\left(\ell_x^0/r_x^0\right)} \le 1$$

coalescence $\quad 2r_x = \ell_x$

power law $\quad \bar{\sigma} = \bar{\varepsilon}^n$

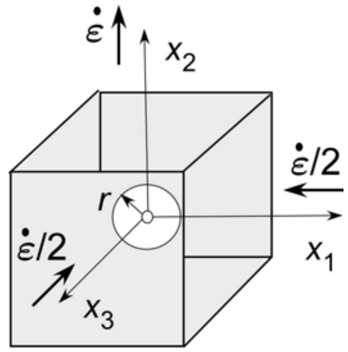
void growth

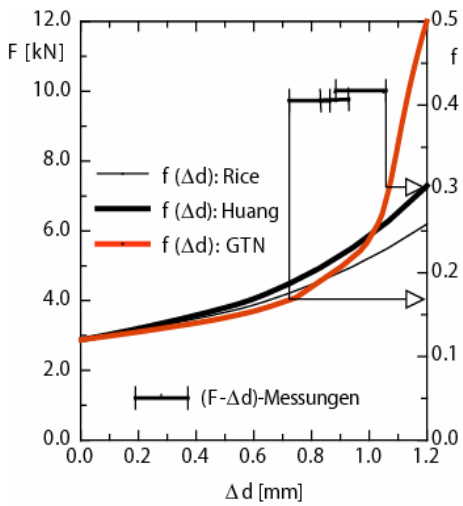
$$\frac{d\eta_{zx}}{d\bar{\varepsilon}^{\infty}} = \frac{1}{\ln\left(\ell_x^0/r_x^0\right)}\left[\frac{\sqrt{3}}{2(1-n)}\sinh\left(\frac{\sqrt{3}(1-n)}{2}\frac{\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)}{\bar{\sigma}}\right)+\frac{3}{4}\frac{\left(\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty}\right)}{\bar{\sigma}}\right]$$

fracture strain

$$\varepsilon_f = \frac{(1-n)\ln\left(\ell_x^0/r_x^0\right)}{\sinh\left(\left[(1-n)\left(\sigma_{xx}^{\infty}+\sigma_{yy}^{\infty}\right)/\left(2\bar{\sigma}/\sqrt{3}\right)\right]\right)}$$

Models of Void Growth (II)





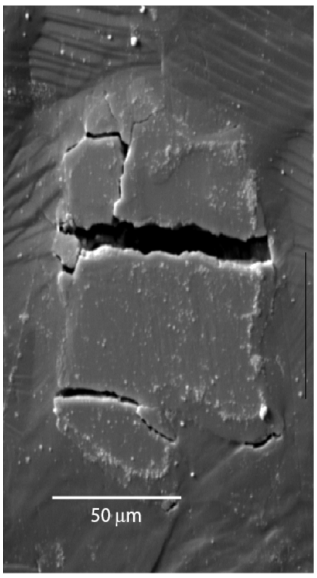
Rice & Tracey [1969]

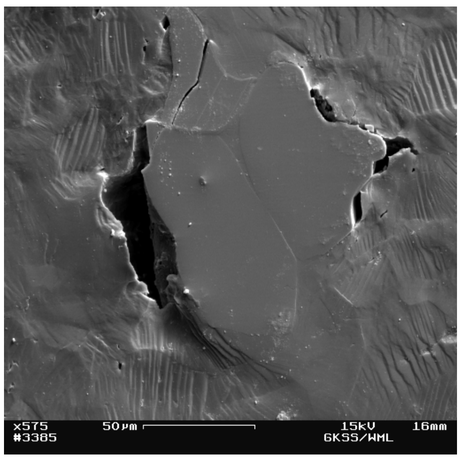
$$\dot{D} = \frac{\dot{r}}{\dot{\varepsilon} r} = 0.283 \exp\left(\frac{2\sigma_h}{3\overline{\sigma}}\right)$$

$$\frac{\sigma_h}{\overline{\sigma}} = T$$ triaxiality

void-volume fraction for tensile test

Void Nucleation

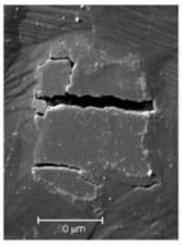


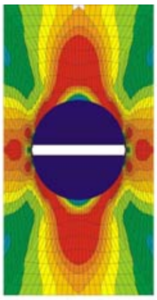


- Particle cracking
- Particle-matrix debonding

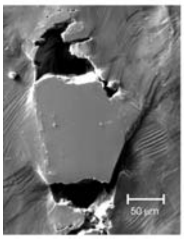
in Al-TiAl MMC

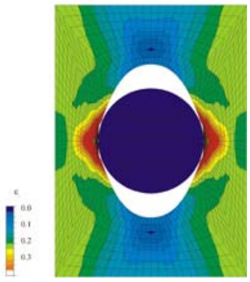
Representative Volume Element (Unit Cell)



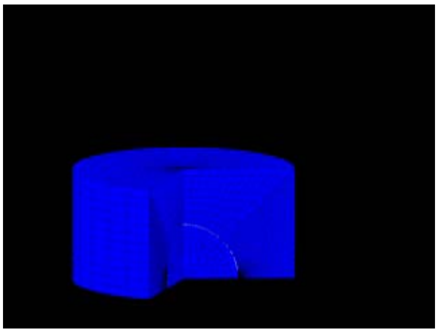


In-situ observation and FE simulation of void nucleation by particle decohesion and fracture



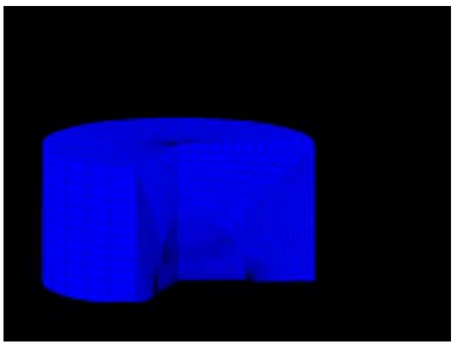


Unit Cell Simulations

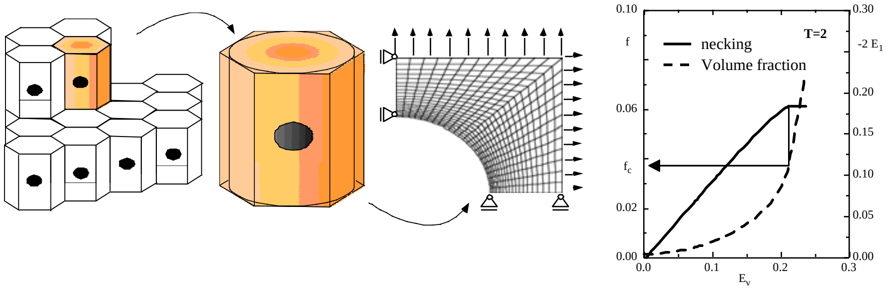


Debonding of matrix at a particle

Cracking of particle



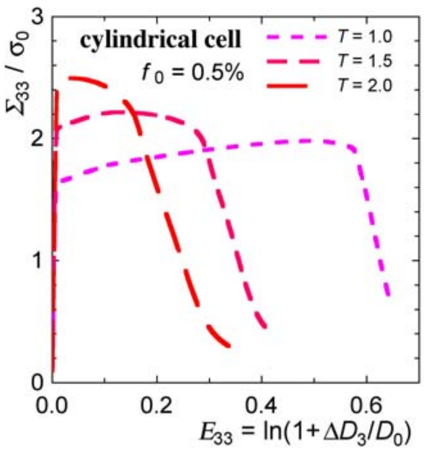
Representative Volume Element (Unit Cell)

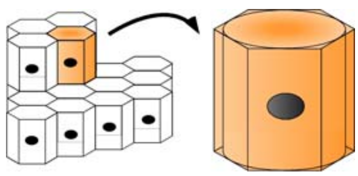


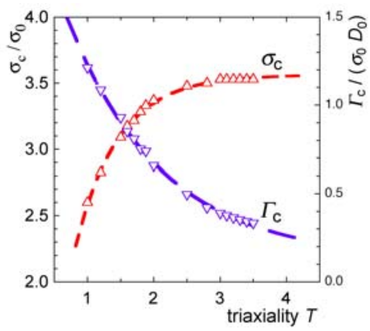
- Evolution of void volume fraction can be computed from simple geometrical RVEs
- Critical volume fractions can be obtained from plastic collapse of the cell (function of triaxiality!)
- Procedure can be applied independently of the aggregate (void, particle, evolving object…)

Mesoscopic Response

FE simulation of void growth:
Mesoscopic stress-strain curves



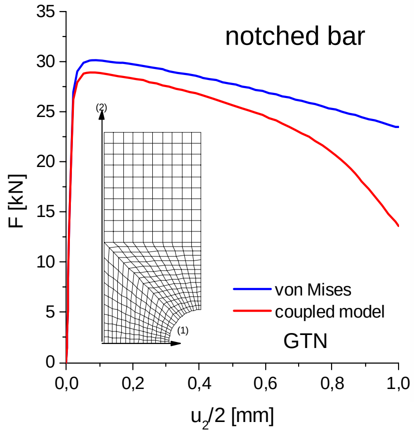


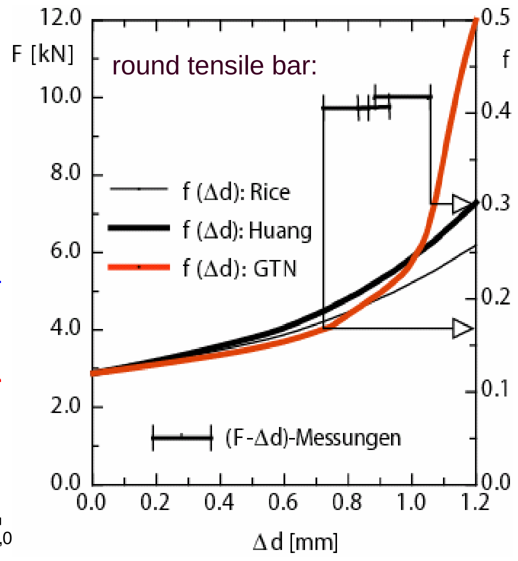


Coupled / Uncoupled Models

Rice & Tracey [1969]

$$\frac{dr}{r} = 0.283\, d\bar{\varepsilon}^{\mathrm{p}} \exp\left(\frac{3}{2} T\right)$$





Porous Metal Plasticity

Additional scalar internal variable in the **yield potential**, which is a function of **porosity** *f*

$$\Phi\left(\sigma_{ij}, \varepsilon^{\mathrm{p}}\right) = 0 \qquad \Rightarrow \qquad \boxed{\Phi\left(\Sigma_{ij}, E^{\mathrm{p}}, f\right) = 0}$$

Porosity equals the **void volume fraction** in an RVE:

$$f = \frac{\Delta V_{\mathrm{voids}}}{\Delta V_{\mathrm{RVE}}}$$

Yield potential formulation is obtained from homogenisation

$$\Sigma_{ij} = \frac{1}{\Delta V_{\mathrm{RVE}}} \iiint_{\Delta V_{\mathrm{RVE}}} \sigma_{ij} dV = \frac{1}{\partial\left(\Delta V_{\mathrm{RVE}}\right)} \iint_{\partial(\Delta V_{\mathrm{RVE}})} \sigma_{ij} n_j dS$$

$$E_{ij} = \frac{1}{\Delta V_{\mathrm{RVE}}} \iiint_{\Delta V} \varepsilon_{ij} dV = \frac{1}{2\Delta V_{\mathrm{RVE}}} \iiint_{\Delta V} \left(u_{i,j} + u_{j,i}\right) dV$$

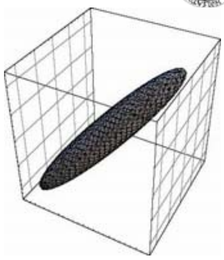
"mesoscopic" stresses and strains

Evolution equation of void growth is derived from plastic incompressibility of "matrix"

$$\dot{f} = (1 - f)\, \dot{E}^{\mathrm{p}}_{kk} \qquad \dot{E}^{\mathrm{p}}_{kk} \neq 0$$

volume dilatation due to void growth

Gurson and Rousselier Model

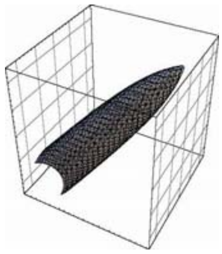


Gurson [1977], Tvergaard & Needleman [1984]

$$\Phi = \frac{\bar{\Sigma}^2}{R(\varepsilon_p)} + 2 q_1 f^* \cosh\left(\frac{3}{2} q_2 \frac{\Sigma_h}{R(\varepsilon_p)}\right) - 1 - q_3 f^{*2} = 0$$

damage variable $f^*(f)$

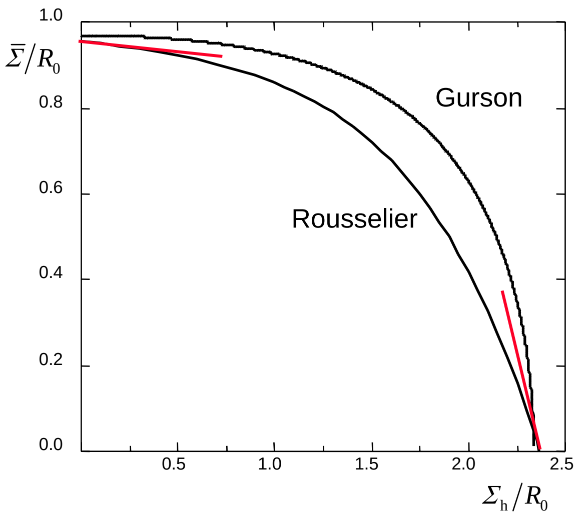
$$\varepsilon_p = \bar{E}^p = \sqrt{\tfrac{2}{3} E'^p_{ij} E'^p_{ij}}$$



Rousselier [1987]

$$\Phi = \frac{\bar{\Sigma}}{(1-f)R(\varepsilon_p)} + \frac{\sigma_1}{R(\varepsilon_p)} D f \exp\left(\frac{\Sigma_h}{(1-f)\sigma_1}\right) - 1 = 0$$

Comparison



Extensions

Tvergaard & Needleman

damage function

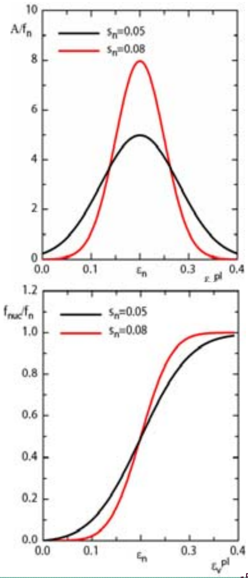
$$f^* = \begin{cases} f & \text{for} \quad f \le f_c \\ f_c + \kappa (f - f_c) & \text{for} \quad f \ge f_c \end{cases}$$

Chu & Needleman [1980]

void nucleation

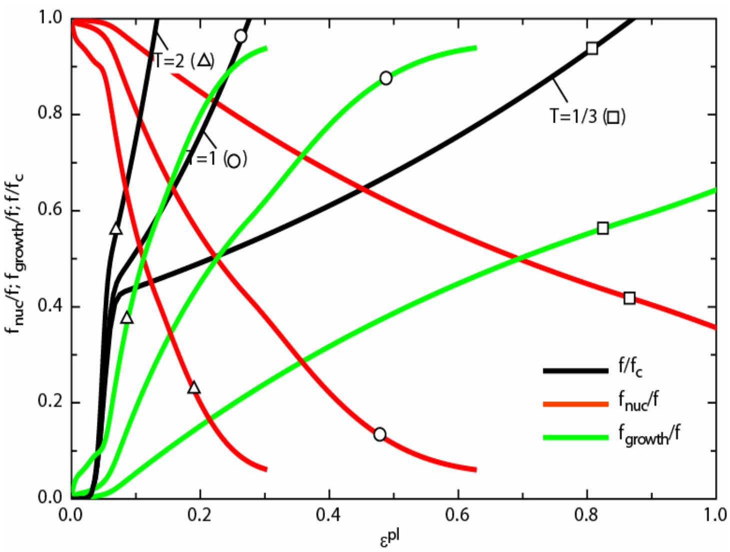
$$\dot{f} = \dot{f}_{\text{growth}} + \dot{f}_{\text{nucl}} = (1 - f)\dot{E}^{\text{p}}_{kk} + A_{\text{n}} \dot{\varepsilon}_{\text{p}}$$

$$A_{\text{n}} = \frac{f_{\text{n}}}{s_{\text{n}}\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{\varepsilon_{\text{p}} - \varepsilon_{\text{n}}}{s_{\text{n}}} \right)^2 \right)$$

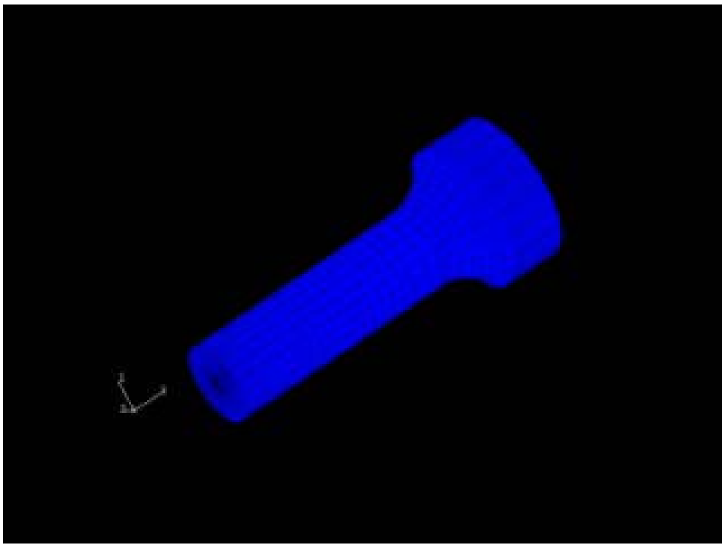


Effect of Triaxiality

$f_0 = 0$
$f_c = 0.12$
$f_n = 0.05$
$\varepsilon_n = 0.05$
$s_n = 0.15$
$q_1 = 1.5$
$q_2 = 1.0$
$q_3 = q_1^2 = 2.25$

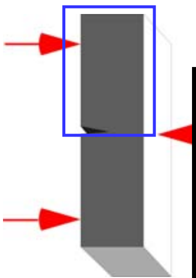


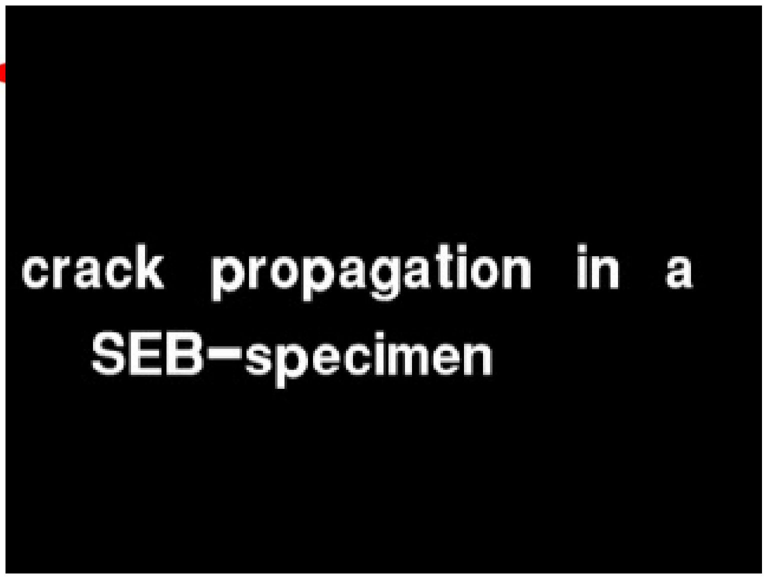
Tensile Test: GTN model



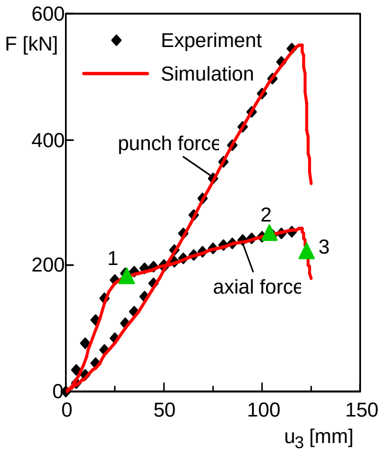
Simulation of deformation and damage in a round tensile bar

SE(B): GTN model



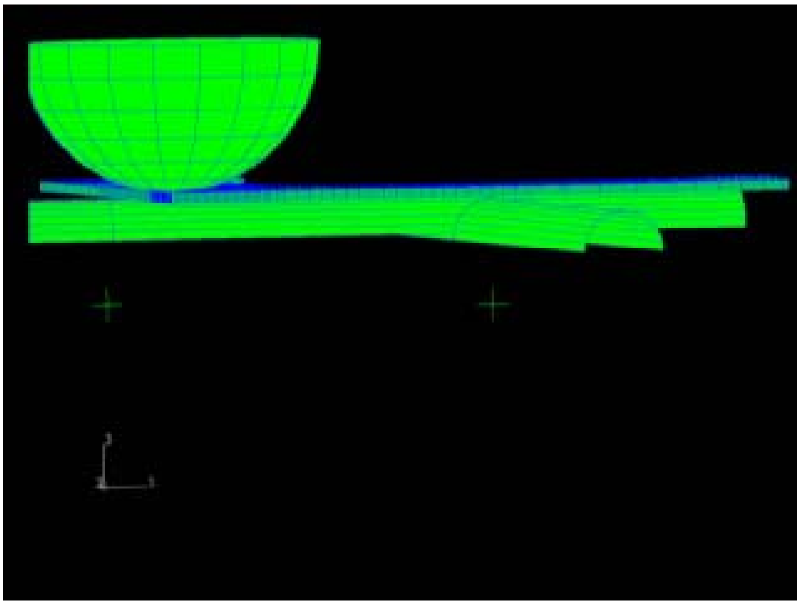


Punch Test



Simulation with the GTN model

Punch Test



Summary (I)

Ductile crack extension and fracture can be modelled on various **length scales**:

(1) Micromechanics: void nucleation, growth and coalescence

(2) Continuum mechanics: constitutive equations with damage

(3) Cohesive surfaces: traction-separation law

(4) Elastic-plastic FM: R-curves for *J* or CTOD

The models require determination of respective **parameters**:

(1) Microstructural characteristics: volume fraction, shape, distance of particles, ...

(2) Initiation: f_0, f_n, ε_n, s_n, coalescence: f_c, final fracture: f_f, ...

(3) Shape of TSL, cohesive strength σ_c , separation energy Γ_c

(4) $J(\Delta a)$ or $\delta(\Delta a)$

Summary (II)

The models have specific favourable and preferential **applications**:

(1) Effects of nucleation mechanism, stress triaxiality, void / particle shape, void / particle spacing, ...

(2) Constraint effects, inhomogeneous materials, damage evolution, ...

(3) Large crack growth, residual strength of structures

(4) Standard FM assessment of engineering structures

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