Modeling damage in finite elasto-plasticity

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ABSTRACT: In this paper we describe models of damaged materials within the constitutive framework of finite, multiplicative elasto-plasticity. The anisotropic damage is characterized by a second order invertible tensor, $\mathbf{F}^d-$ the damage deformation tensor, whose existence is related to an undamaged (fictitious) stress free configuration. The existence of the damage deformation tensor leads to a modified multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}^e\mathbf{F}^d\mathbf{F}^p$, where the plastic part of deformation $\mathbf{F}^p$ can only affect the undamaged material structure. The behavior of the material is elastic and dependent on the damage deformation tensor $\mathbf{F}^d$, and we adopt the concepts of damage surface and yield surface to describe the irreversible behaviour by the appropriate evolution equations.

KEYWORDS: Anisotropic damage, effective stress, undamaged (fictitious) stress free configuration, finite elasto-plasticity

1 INTRODUCTION

In this paper we describe anisotropic damage of ductile materials within the constitutive framework of finite, multiplicative elasto-plasticity, proposed by Cleja-Ţigoiu in [3], Cleja-Tigoiu and Sóos [6]. On the background of the theory there is the existence of local stress free configuration, the so called relaxed configuration, in which the internal variables preserve their values, reached during the deformation process. The physical nature of the mechanical variables which describe the damage state of materials is an important issue in constitutive models. The anisotropic damage can be characterized by a second order invertible tensor, $\mathbf{F}^d-$ the damage deformation tensor, whose existence is related to an undamaged (fictitious) stress free configuration, say $\mathcal{K}_t$. In this case the role of the relaxed configuration is played by the current stress free, but damaged configuration, say $\widetilde{\mathcal{K}}_t$. $\mathbf{F}^d$ characterizes the passage from the undamaged stress free configuration to $\widetilde{\mathcal{K}}_t$. The behavior of the material is elastic and dependent on the damage deformation tensor $\mathbf{F}^d$, and the elastic constitutive equation expresses the symmetric Piola-Kirchhoff stress tensor, $\widetilde{\mathbf{T}}$, in terms of elastic strain $\mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e$. Here $\mathbf{F}^e$, the elastic part of the deformation gradient characterizes the local deformation from $\widetilde{\mathcal{K}}_t$ to the deformed configuration of the body. The existence of the damage deformation tensor leads to a modified multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}^e\mathbf{F}^d\mathbf{F}^p$, where the plastic part of deformation $\mathbf{F}^p$ can only affect the undamaged material structure, see Brünig and Ricci (2005). The time evolution equation for plastic part of deformation will be written with respect to $\mathcal{K}_t$, in terms of the effective stress, $\bar{\mathbf{T}}$. $\widetilde{\mathbf{T}}$ and $\bar{\mathbf{T}}$ are related by the standard relationship between the stress tensors, through the local deformation $\mathbf{F}^d$.

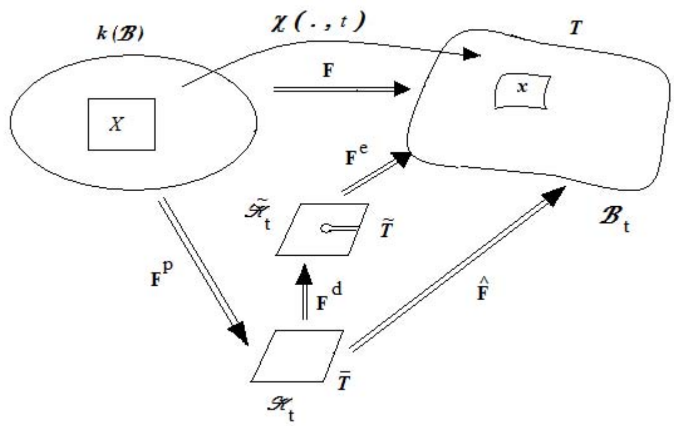

Figure 1: Configurations, deformation measures and stress measures.

We adopt here the concepts of damage surface and yield surface to describe the appropriate evolution equations. We pay attention to the full description of the model, not only on the damage and yield criteria. Based on experimental, theoretical and numerical studies existing in the field, realistic macroscopic damage evolution conditions can be introduced. We make reference to other existing theories.

2 A CONSTITUITVE FRAMEWORK FOR ANISOTROPIC DAMAGE

Within the constitutive framework of continuum mechanics we describe the behaviour of the damaged body $\mathcal{B}$. Let $\mathbf{X} \in \mathcal{B}$ be a material point of the body and let k be an initial reference configuration. The motion of the body, defined on a given neighbor of the material particle $\mathbf{X}$, $\chi : \mathcal{N}_{\mathbf{X}} \times \mathbf{R} \longrightarrow \mathcal{E}$. Here $\mathcal{E}$ the Euclidean physic space and $\mathcal{V}$ denotes the vector space of the translations of $\mathcal{E}$.

We introduce the axioms which mathematically describe the ductile damage of the body.

Ax.1 For any motion χ, for any material particle $\mathbf{X}$ and at any moment t, there exist *the invertible, tensorial fields* $\mathbf{F}^p, \mathbf{F}^d, \mathbf{F}^e$, called plastic, damage and elastic parts of the deformation gradient, such that the *multiplicative decomposition* of the deformation gradient $\mathbf{F}$ into its components

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^d\mathbf{F}^p, \tag{1}$$

holds.

In our model $\mathbf{F}^{pd} := \mathbf{F}^d\mathbf{F}^p$ define the irreversible behaviour of the body. $\mathbf{F}^d$ appears as a *kinematic damage variable.* Note that the tensorial fields are defined for any material particle $\mathbf{Y} \in \mathcal{N}_{\mathbf{X}}$ and at any time t. For any fixed material points the fields $\mathbf{F}^p, \mathbf{F}^d, \mathbf{F}^e$ are defined for different vectorial domains, denoted by $\mathcal{V}$, $\mathcal{V}_{\mathcal{K}_t} := \mathbf{F}^p(\mathcal{V})$, and $\mathcal{V}_{\widetilde{\mathcal{K}}_t} := \mathbf{F}^d(\mathcal{V})$, respectively.

Ax.2 The tensorial fields $\mathbf{F}^d, \mathbf{F}^p$ do not satisfy the *integrability condition*, in the sens that they could not be represented through the gradient of certain potentials, like the deformation gradient, i.e the non-integrability condition, written for instance for $\mathbf{F}^p$, $\exists \ \ \mathbf{u}, \mathbf{v} \in \mathcal{V}$ so that $(\nabla\mathbf{F}^p(\mathbf{X},t)\mathbf{u})\mathbf{v} - (\nabla\mathbf{F}^p(\mathbf{X},t)\mathbf{v})\mathbf{u} \neq 0, \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}$, holds.

The following stress fields can be defined

$\mathbf{T}(\mathbf{x},t)-$ Cauchy stress in the particle $\mathbf{X}$, in the deformed configuration, where $\mathbf{x} = \chi\mathbf{X}, t$,

$\widetilde{\mathbf{T}}(\mathbf{x},t)-$ a Piola-Kirchoff like tensor, associated with the non-local stress free, but damaged configuration (or the so-called intermediate configuration), denoted by $\widetilde{\mathcal{K}}_t$.

$\bar{\mathbf{T}}(\mathbf{x},t)-$ a Piola-Kirchoff stress tensor, associated with the stress free and undamaged configuration, i.e. in $\mathcal{K}_t$, called fictitious configuration by Menzel et al. [11]. $\alpha-$ internal state variable, a *stress type* variable.

The stress measures are related by

$$\begin{aligned} &\widetilde{\mathbf{T}}(\mathbf{x},t) = \det(\mathbf{F}^e)(\mathbf{F}^e)^{-1}\mathbf{T}(\mathbf{F}^e)^{-T}, \\ &\rho \ \det\mathbf{F}^e = \rho^d, \end{aligned} \tag{2}$$

and

$$\bar{\mathbf{T}} = (\det\mathbf{F}^d)(\mathbf{F}^d)^{-1}\widetilde{\mathbf{T}}(\mathbf{F}^d)^{-T}, \quad \rho^p\det\mathbf{F}^d = \rho^d. \tag{3}$$

The stress tensor $\bar{\mathbf{T}}$ represents the effective stress, being related to the stress free and undamaged configuration, $\mathcal{K}_t$. The mass densities have been denoted by ρ^d, ρ^p, ρ for the fixed material particle, in the stress free damaged and undamaged configurations, and in the actual configuration, respectively. In the elasto-plastic material irreversible strains are mainly caused by damage, plastic volumetric change are negligible, i.e. $\rho^p = \rho_0$, the density in reference configuration, see [1].

Ax.3 The behaviour of the material is *elastic*, with respect to the stress free and damaged configuration

$$\begin{aligned} &\widetilde{\mathbf{T}}(\mathbf{x},t) = \rho_d\mathbf{h}_{\widetilde{\mathcal{K}}}(\Delta^e(\mathbf{X},t), \alpha(\mathbf{X},t)), \\ &\Delta^e(\mathbf{X},t) = \frac{1}{2}((\mathbf{F}^e)^T\mathbf{F}^e - \mathbf{I}), \quad \text{or equivalently} \\ &\mathbf{T} = \rho\mathbf{F}^e\mathbf{h}_{\widetilde{\mathcal{K}}}(\Delta^e, \alpha)(\mathbf{F}^e)^T, \end{aligned} \tag{4}$$

and the elastic type constitutive function satisfys the *stress free or relaxed restriction*

$$\mathbf{h}_{\widetilde{\mathcal{K}}}(\mathbf{0}, \alpha) = 0, \tag{5}$$

when we combine the constitutive 4 with (2). The material point $\mathbf{X}$ as well as the current moment of time t, will be ommited.

$\mathbf{F}^d$ which represents the local deformation from the *stress free and undamaged configuration*, $\mathcal{K}$, to the *stress free but damaged configuration*, $\widetilde{\mathcal{K}}$.

We introduce an assumption, which adapt to our description, the equivalence of the free energy in the fictitious configuration and in the intermediate configuration, see Menzel et al. [11].

Ax.4 The elastic type behaviour of the material can be represented by the constitutive equation with respect to the stress free and undamaged configuration in terms of Cauchy stress, as it follows

$$\begin{aligned}&\mathbf{T}(\mathbf{x},t)=\rho\mathbf{F}^e\hat{\mathbf{h}}_{\mathcal{K}}(\hat{\mathbf{C}},\alpha)(\mathbf{F}^e)^T,\\&\hat{\mathbf{C}}:=(\mathbf{F}^e\mathbf{F}^d)^T\mathbf{F}^e\mathbf{F}^d\equiv\hat{\mathbf{C}}:=(\mathbf{F}^d)^T\mathbf{C}^e\mathbf{F}^d,\end{aligned}\tag{6}$$

with the restriction

$$\text{for}\quad \mathbf{C}^e:=(\mathbf{F}^e)^T\mathbf{F}^e=\mathbf{I}, i.e$$

$$\text{for}\quad \hat{\mathbf{C}}:=(\mathbf{F}^d)^T\mathbf{F}^d\equiv\mathbf{C}^p,\text{to have}\quad \hat{\mathbf{h}}_{\mathcal{K}}(\mathbf{C}^p,\alpha)=0.$$

Let us calculate the relative strain measure and introduce a new elastic type constitutive function

$$\begin{aligned}&\hat{\mathbf{C}}-\mathbf{C}^d=2(\mathbf{F}^d)^T(\Delta^e)\mathbf{F}^d,\\&\mathbf{h}_{\mathcal{K}}((\mathbf{F}^d)^T(\Delta^e)\mathbf{F}^d,\delta^p,\alpha):=\hat{\mathbf{h}}_{\mathcal{K}}(\hat{\mathbf{C}},\alpha).\end{aligned}\tag{7}$$

Consequently, from (4), (6), (7) we derived the elastic type constitutive equation in terms of Cauchy stress, with respect to the stress free and undamaged configuration,

$$\mathbf{T}=\rho_d\mathbf{F}^e\mathbf{h}_{\mathcal{K}}((\mathbf{F}^d)^T\Delta^e\mathbf{F}^d,\alpha)(\mathbf{F}^e)^T\tag{8}$$

the damaged part of the deformation gradient $\mathbf{F}^d$, being involved like an internal variable.

We mention that in [9] the strain measure on the intermediate configuration has been defined as $\hat{\Gamma}=\frac{1}{2}(\mathbf{C}^e-\mathbf{c}^p)$, with $\mathbf{C}^e=(\mathbf{F}^e)^T\mathbf{F}^e$ and $\mathbf{c}^p=(\mathbf{F}^p)^{-T}(\mathbf{F}^e)^{-1}$ in our notation.

We make some comments relative to the elastic type constitutive equation presented by a Brünig and Ricci in [2]

$$\begin{aligned}\mathbf{T}\ &=2(G+\eta_2\mathrm{tr}\mathbf{A}^{da})\mathbf{A}^{el}+[(K-\frac{2}{3}G+\\&+2\eta_1\mathrm{tr}\mathbf{A}^{da})\mathrm{tr}\mathbf{A}^{el}+\eta_3(\mathbf{A}^{da}\cdot\mathrm{tr}\mathbf{A}^{el})]\mathbf{I}+\\&+\eta_3(\mathrm{tr}\mathbf{A}^{el})\mathbf{A}^{da}+\eta_4(\mathbf{A}^{da}\mathbf{A}^{el}+\mathbf{A}^{el}\mathbf{A}^{da})\end{aligned}\tag{9}$$

with the elastic and damage strain measures defined by

$$\begin{aligned}&\mathbf{A}^{el}=\frac{1}{2}(\mathbf{I}-(\mathbf{F}^e)^{-T}(\mathbf{F}^e)^{-1}),\\&\mathbf{A}^{da}=\frac{1}{2}(\mathbf{I}-(\mathbf{F}^d)^{-T}(\mathbf{F}^d)^{-1}).\end{aligned}\tag{10}$$

Let us remark that the elastic type constitutive equation (9) *satisfies the principle of the objectivity*, if it is assumed that under a change of frame in the actual configuration, characterized by an orthogonal mapping $\mathbf{Q}$, $\mathbf{F}^e$ as well as $\mathbf{F}^d$ sustain the transformation $\mathbf{F}^{*e}=\mathbf{Q}\mathbf{F}^e$, $\mathbf{F}^{*d}=\mathbf{Q}\mathbf{F}^d$. On the other *the stress free condition* written in (8), which means the stress is zero, $\mathbf{T}=0$, if the elastic strain is zero, i.e. $\mathbf{A}^{el}=0$, is satisfied.

Under the supposition that $\mathbf{F}^e$ is transformed as $\mathbf{F}^{*e}=\mathbf{Q}\mathbf{F}^e$, while $\mathbf{F}^d$, considered to be an internal state variable, remains invariant under a change of frame in the actual configuration characterized by $\mathbf{Q}$, the constitutive equation proposed in (8) satisfies the objectivity principle.

Remark. The constitutive equation (9) which characterizes an elastic behaviour is not written in an appropriate manner, since it contains two measure of deformations with respect to different configurations, the elastic strain $\mathbf{A}^{el}$, with respect to the deformed configuration, while $\mathbf{A}^{da}$ is defined on the stress free and damaged configuration. As a counterpart we proposed (8).

3 EVOLUTION EQUATIONS FOR IRREVERSIBLE BEHAVIOUR

We adopt the point of view formulated by Brünig [1]: *By combining plasticity and damage it seems to be natural that plasticity can only affect the undamaged material skeleton.*

(Ev.1) Following the constitutive framework of finite elasto-plasticity, the evolution equation for the plastic part of deformation $\mathbf{F}^p$, will be written with respect to the stress free (undamaged), in terms of $\bar{\mathbf{T}}$,

$$\begin{aligned}&\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}=\mu_1\mathcal{B}_{\mathcal{K}}(\bar{\mathbf{T}}),\\&\text{associated with the yield conditions}\\&\bar{f}(\bar{\mathbf{T}},\alpha)\leq 0,\quad \mu_1\geq 0,\\&\mu_1\bar{f}(\bar{\mathbf{T}},\alpha)=0,\quad \mu_1\dot{\bar{f}}(\bar{\mathbf{T}},\alpha)=0,\end{aligned}\tag{11}$$

following the standard procedure developed for rate-independent elasto-plastic materials. Here $\bar{\mathbf{T}}$ is defined through the constitutive representation

$$\begin{aligned}&\bar{\mathbf{T}}(\mathbf{X},t)=\\&=\det(\mathbf{F}^d)\{(\mathbf{F}^d)^{-1}\widetilde{\mathbf{h}}_{\mathcal{K}}(\Delta^e(\mathbf{X},t),\mathbf{F}^d)(\mathbf{F}^d)^{-T}\},\end{aligned}\tag{12}$$

with $\mathbf{F}^d$ viewed as an internal variable.

(**Ev.2**) The evolution equation for tensorial, kinematic variable $\mathbf{F}^d$ will be written in terms of the stress measure $\widetilde{\mathbf{T}}$. The damage variable $\mathbf{F}^d$ is considered an internal state variable given by

$$\begin{aligned} &\dot{\mathbf{F}}^d(\mathbf{F}^d)^{-1} = \mu_2 \mathcal{D}_{\widetilde{\mathcal{K}}_t}(\widetilde{\mathbf{T}}), \\ \text{or} \quad &\dot{\mathbf{F}}^d(\mathbf{F}^d)^{-1} = \mu_2 \mathcal{D}_{\mathcal{K}_t}(\widetilde{\mathbf{T}}, (\mathbf{F}^d)^{-1}). \end{aligned} \tag{13}$$

Here we pass from the stress free (damaged) configuration $\widetilde{\mathcal{K}_t}$ to the stress free and undamaged configuration $\mathcal{K}_t$. Note that $\widetilde{\mathbf{T}}$ is represented through the constitutive representation (8), together with (2), and is associated with the damage criterion

$$g_{\widetilde{\mathcal{K}}}(\widetilde{\mathbf{T}}, \alpha) = 0, \quad \text{or} \quad g_{\mathcal{K}}(\bar{\mathbf{T}}, (\mathbf{F}^d)^{-1}, \alpha) = 0. \tag{14}$$

The damage factor μ_2 has to be subjected to the similar conditions which whose postulated for μ_1, plastic factor.

The key point of the model is to provide the consistency condition, related to the *yield condition and damage criterion*.

Finally we consider the isotropic damage, when a scalar field replaces the tensorial damage variable, and the multiplicative decomposition of the deformation gradient is reduced to $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$. The scalar damage variable is viewed as a scalar internal variable, with its variation in time described by evolution equation.

Discussions about how to couple damage and elasto-plasticity, and different type of the models describing anisotropic as well as isotropic damage, can be found in [1], [2], [11], [7] . Models, involving stress like damage variables can be found in the papers by [9], and by Lubarda si Krajcinovic, [10]. We can introduce in the modelisation the non-local effects, following the procedure proposed by Cleja-Tigoiu in [4], [5], or by Gurtin in [8].

4 CONCLUSIONS

The proposed framework for anisotropic damage coupled to finite elasto-plasticity allows the description of the material behaviour with respect to the stress free, but damaged configuration, as well as with respect to the stress free (fictitious) undamaged configuration. The different physically motivated assumptions have to be added in order to define appropriate internal variables which can be involved in the damage potential and in the damage criterion, taking into account that the damage only influence the elastic response.

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