Eur. J. Mech. A/Solids 19 (2000) 187–208

S0997-7538(00)00161-3/FLA

Anisotropic damage law of evolution

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(Received 19 August 1999; revised and accepted 2 November 1999)

Abstract – A formulation for anisotropic damage is established in the framework of the principle of strain equivalence. The damage variable is still related to the surface density of microcracks and microvoids and, as its evolution is governed by the plastic strain, it is represented by a second order tensor and is orthotropic. The coupling of damage with elasticity is written through a tensor on the deviatoric part of the energy and through a scalar taken as its trace on the hydrostatic part. The kinetic law of damage evolution is an extension of the isotropic case. Here, the principal components of the damage rate tensor are proportional to the absolute value of principal components of the plastic strain rate tensor and are a nonlinear function of the effective elastic strain energy. The proposed damage evolution law does not introduce any other material parameter. Several series of experiments on metals give a good validation of this theory. The coupling of damage with plasticity and the quasi-unilateral conditions of partial closure of microcracks naturally derive from the concept of effective stress. Finally, a study of strain localization makes it possible to determine the critical value of the damage at mesocrack initiation. © 2000 Éditions scientifiques et médicales Elsevier SAS

damage / anisotropy thermodinamique / effective stress / constitutive equations

1. Introduction

The formulation of anisotropic damage is already a long story with several not fully satisfactory theories. The present one also has a price to pay in the introduction of an extra-parameter η although it disappears in the final constitutive equation.

The extension of isotropic damage theory to anisotropy is not a straight forward task in the coupling between elasticity and damage. In the case of isotropic damage represented by the scalar variable D, the effective stress concept associated to the principle of strain equivalence, $\tilde{\sigma}_{ij} = \sigma_{ij}/(1-D) = E_{ijkl}\varepsilon^e_{kl}$ for elasticity solves the problem (Lemaitre, 1992). In the case of general anisotropy, the damage variable is represented by a fourth order tensor (Kracinovic, 1985; Leckie and Onat, 1981; Lemaitre and Chaboche, 1985). Here we consider a second order tensor $\underline{\underline{D}}$ (Murakami and Ohno, 1978; Murakami, 1988) which corresponds to orthotropy justified by the mechanism of damage essentially related to plasticity represented by the second order plastic strain tensor $\underline{\underline{\varepsilon}}^p$. In these conditions the direct generalization of the concept of effective stress and the principle of strain equivalence generally does not yield existence of an elastic potential (Cordebois and Sidoroff, 1982).

Using the concept of effective stress only on the principal stress components (Hayhurst and Leckie, 1973) does not solve the problem. Replacing the strain equivalence principle by an energy equivalence (Cordebois and Sidoroff, 1982; Cordebois, 1983) is a possible solution which has been followed by several authors (Chow and Wang, 1987; Ju, 1989). It yields existence of the elastic potential but the physical definition of the damage variable is lost. The damage is no longer directly related to the surface density of defects but is a mathematical

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variable defined by its coupling with elasticity. Then, no more rules exist to describe the coupling with plasticity or the conditions of partial closure of microcracks (quasi-unilateral effect) (Horii and Nemat-Nasser, 1983; Ladevèze, 1983; Ladevèze and Lemaitre, 1984; Liénard, 1989; Mazars et al., 1990; Chaboche, 1993; Dragon and Halm, 1995) nor to define the critical value of the damage at mesocrack initiation. Micromechanics has also been used (Kachanov, 1993; Dragon et al., 1994; Fichant, 1996) but restricted to brittle failure it does not yield a closed form constitutive equation.

2. Elastic potential

This problem of incompatibility between physics and thermodynamics is avoided if two damage variables corresponding to two mechanisms are introduced as in composites for delamination and matrix cracking (Ladevèze, 1983). The Gibbs elastic potential is split into its deviatoric part affected by a tensorial damage variable $\underline{\underline{D}}$ and its hydrostatic part affected by another (scalar) damage variable d_H.

For monolithic materials subjected to one damage mechanism, only one damage variable is considered, the second order tensor $\underline{\underline{D}}$. The same decomposition of the Gibbs energy may be kept if the coupling of the damage with the hydrostatic part of the energy is written through a function of the mean damage D_H, the trace of $\underline{\underline{D}}$

$$d_H = g(D_H), \quad D_H = \frac{1}{3} D_{kk}, \tag{1}$$

E and ν being the Young's modulus and the Poisson's ratio of the virgin material, $\underline{\underline{\sigma}}^D$ being the stress deviator and σ_H the hydrostatic stress

$$\sigma_{ij}^D = \sigma_{ij} - \sigma_H \delta_{ij}, \quad \sigma_H = \frac{1}{3}\sigma_{kk} \tag{2}$$

the Gibbs energy for linear and initially isotropic elasticity is

$$\rho\psi^* = \frac{1+\nu}{2E} H_{ij}\sigma_{jk}^D H_{kl}\sigma_{li}^D + \frac{3(1-2\nu)}{2E}\frac{\sigma_H^2}{1-d_H}, \tag{3}$$

where

$$\underline{\underline{H}} = (\underline{\underline{1}} - \underline{\underline{D}})^{-1/2}. \tag{4}$$

A procedure to identify the function g is given in the next section and a good approximation consists in taking

$$d_H = \eta D_H, \tag{5}$$

η is a necessary parameter for a correct representation of experiments concerning the variations in the Poisson's ratio with damage. It depends upon materials and characterizes their sensibility to hydrostatic stress. It is one more parameter to identify. The particular value $\eta = 1$ corresponds to isotropic damage, but often for metals $\eta \approx 3$. Furthermore, it does not appear in the damage law of evolution proposed in Section 4.

The law of elasticity derives from the Gibbs potential

$$\varepsilon_{ij}^e = \rho\frac{\partial\psi^*}{\partial\sigma_{ij}} = \frac{1+\nu}{E}\tilde{\sigma}_{ij} - \frac{\nu}{E}\tilde{\sigma}_{kk}\delta_{ij}. \tag{6}$$

It naturally introduces a symmetric effective stress tensor which does not depend upon the elasticity parameters and which can be used for the coupling with plasticity (Section 6)

$$\tilde{\sigma}_{ij} = \left(H_{ik}\sigma^D_{kl}H_{lj}\right)^D + \frac{\sigma_H}{1-\eta D_H}\delta_{ij}. \tag{7}$$

The associated variable to $\underline{D}$ is the strain energy density release rate tensor $\underline{Y}$ which also derives from the Gibbs energy, $Y_{ij} = \rho\partial\psi^*/\partial D_{ij}$. The final derivative needs some care owing to the possible variation of the principal direction of the damage. With $\underline{H}d\underline{H} + d\underline{H}\,\underline{H} = \underline{H}^2 d\underline{D}\,\underline{H}^2$ written

$$A_{ijkl}\dot{H}_{kl} = H^2_{ip}\dot{D}_{pq}H^2_{qj}, \quad A_{ijkl} = \frac{1}{2}(H_{ik}\delta_{jl} + H_{jl}\delta_{ik} + H_{il}\delta_{jk} + H_{jk}\delta_{il}) \tag{8}$$

$$Y_{ij} = \frac{1+\nu}{E}\sigma^D_{kp}H_{pq}\sigma^D_{ql}A^{-1}_{klmn}H^2_{mi}H^2_{jn} + \frac{\eta(1-2\nu)}{2E}\frac{\sigma^2_H}{(1-\eta D_H)^2}\delta_{ij}. \tag{9}$$

By chance the law of damage evolution will not be a function of $\underline{Y}$. Nevertheless it is possible to verify that the dissipation $Y_{ij}\dot{D}_{ij}$ is positive or zero at least for all practical cases.

3. Damage measurements

As for the isotropic case (Lemaitre and Dufailly, 1987) the values of the components of the damage may be obtained from the changes in elasticity characteristics.

3.1. Uniaxial tension (TU)

Consider a representative volume element (RVE) damaged in the orthotropic frame $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ of *figure 1*

$$\underline{D} = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix}. \tag{10}$$

The elastic strains in this frame are

$$\begin{bmatrix} \varepsilon^e_1 & 0 & 0 \\ 0 & \varepsilon^e_2 & 0 \\ 0 & 0 & \varepsilon^e_3 \end{bmatrix} = \frac{1-2\nu}{3E}\,\frac{\sigma_1}{1-\eta D_H}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+\frac{1+\nu}{E}\left(\begin{bmatrix} \frac{1}{\sqrt{1-D_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-D_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-D_3}} \end{bmatrix}\begin{bmatrix} \frac{2\sigma_1}{3} & 0 & 0 \\ 0 & \frac{-\sigma_1}{3} & 0 \\ 0 & 0 & \frac{-\sigma_1}{3} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{1-D_1}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-D_2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{1-D_3}} \end{bmatrix}\right)^D. \tag{11}$$

Defining the damaged elasticity modulus in direction 1 and the associated contraction ratios by

$$\widetilde{E}_1 = \frac{\sigma_1}{\varepsilon^e_1}, \qquad \tilde{\nu}_{12} = -\frac{\varepsilon^e_2}{\varepsilon^e_1}, \qquad \tilde{\nu}_{13} = -\frac{\varepsilon^e_3}{\varepsilon^e_1}. \tag{12}$$

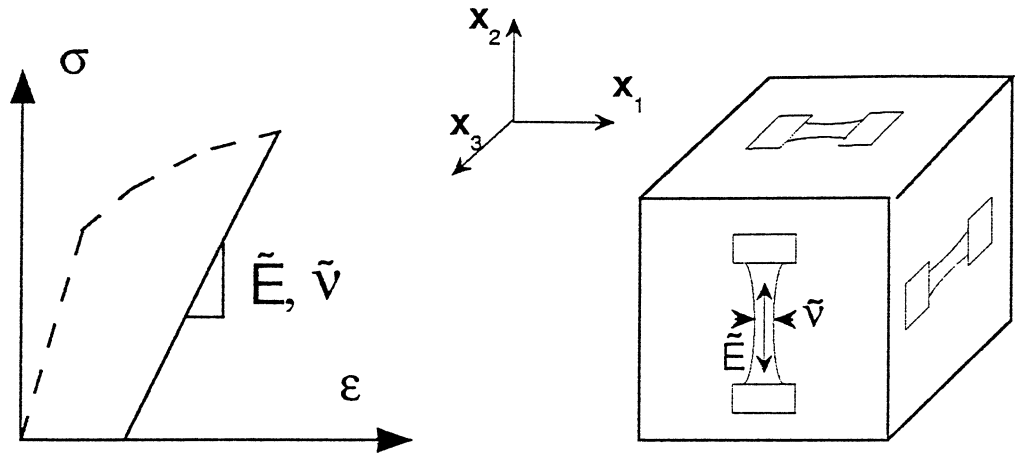

Figure 1. Damaged elasticity in the orthotropic frame.

Then

$$\frac{E}{\widetilde{E}_1} = \frac{1+\nu}{9}\left(\frac{4}{1-D_1}+\frac{1}{1-D_2}+\frac{1}{1-D_3}\right)+\frac{1-2\nu}{3(1-\eta D_H)}, \tag{13}$$

$$\tilde{\nu}_{12}\frac{E}{\widetilde{E}_1} = \frac{1+\nu}{9}\left(\frac{2}{1-D_1}+\frac{2}{1-D_2}-\frac{1}{1-D_3}\right)-\frac{1-2\nu}{3(1-\eta D_H)}, \tag{14}$$

$$\tilde{\nu}_{13}\frac{E}{\widetilde{E}_1} = \frac{1+\nu}{9}\left(\frac{2}{1-D_1}-\frac{1}{1-D_2}+\frac{2}{1-D_3}\right)-\frac{1-2\nu}{3(1-\eta D_H)}. \tag{15}$$

The same operation for directions 2 and 3 gives nine equations to determine the three components of the damage $D_1,\ D_2,\ D_3$ and the coefficient η.

Practically, most of the time, it is not possible to obtain three specimens in three orthogonal directions but four measures are sufficient if the orthotropic directions are known. In the case of a damaged plane sheet, the only possible directions for measures are $\vec{x}_1$ and $\vec{x}_2$ to give $\widetilde{E}_1,\ \widetilde{E}_2$ and $\tilde{\nu}_{12}$ then

$$D_1 = 1-\frac{\widetilde{E}_1}{E}(1+\nu)\left[2+\tilde{\nu}_{12}-\frac{\widetilde{E}_1}{\widetilde{E}_2}\right]^{-1}, \tag{16}$$

$$D_2 = 1-\frac{\widetilde{E}_2}{E}(1+\nu)\left[2-(1-\tilde{\nu}_{12})\frac{\widetilde{E}_2}{\widetilde{E}_1}\right]^{-1}, \tag{17}$$

$$\eta D_H = 1-\frac{\widetilde{E}_1}{E}\frac{1-2\nu}{1-2\tilde{\nu}_{12}}. \tag{18}$$

If tension is applied in direction 1, D_1 and D_2 are determined by equations (16) and (17), $D_3 = D_2$ for a material initially isotropic and $D_H = (D_1+2D_2)/3$. Then, η is obtained from equation (18). Examples are given in Section 5.

3.2. Plane tension (TP)

In the case of plane sheets, complementary tests may be performed on large specimens loaded in tension ensuring a plane strain state in the middle of each sheet. This induces damage under a different stress triaxiality.

If $\vec{x}_1$ is the direction of tension, $\vec{x}_2$ the direction in the width and $\vec{x}_3$ the direction in the thickness, the strain ε_{22} vanishes owing to the plane strain condition. Furthermore, if during loading elastic strains are neglected then $\varepsilon_{22}^p \approx 0$ and

$$\underline{\varepsilon}^p = \begin{bmatrix} \varepsilon_{11}^p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\varepsilon_{11}^p \end{bmatrix}. \tag{19}$$

During loading and unloading the states of stress are, respectively,

$$\underline{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \dfrac{\sigma}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \underline{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

The plane strain condition is then $\dot{\varepsilon}_{22} = \dot{\varepsilon}_{22}^e = 0$. The elastic strain is calculated from the law of elasticity (6) with a diagonal damage matrix $\underline{D} = \mathrm{diag}[D_1, D_2, D_3]$. Avoiding the calculations, let us just recall that

$$\tilde{\sigma}_{11}^D = \frac{\sigma_1}{9}\left(\frac{4}{1-D_1} + \frac{1}{1-D_2} + \frac{1}{1-D_3}\right) - \frac{\sigma_2}{9}\left(\frac{4}{1-D_1} - \frac{2}{1-D_2} + \frac{1}{1-D_3}\right), \tag{21}$$

$$\tilde{\sigma}_{22}^D = -\frac{\sigma_1}{9}\left(-\frac{2}{1-D_1} + \frac{4}{1-D_2} + \frac{1}{1-D_3}\right) + \frac{\sigma_2}{9}\left(\frac{1}{1-D_1} + \frac{4}{1-D_2} + \frac{1}{1-D_3}\right), \tag{22}$$

$$\tilde{\sigma}_{33}^D = -\frac{\sigma_1}{9}\left(-\frac{1}{1-D_1} + \frac{2}{1-D_2} + \frac{2}{1-D_3}\right) - \frac{\sigma_2}{9}\left(-\frac{1}{1-D_1} + \frac{2}{1-D_2} + \frac{2}{1-D_3}\right), \tag{23}$$

$$\tilde{\sigma}_H = \frac{\sigma_1 + \sigma_2}{3(1-d_H)}. \tag{24}$$

For the unloading, the relation between σ_2 and σ_1 as well as the determination of the damaged modulus are still obtained by consideration of the plane strain condition.

In order to measure the damage tensor, small uniaxial tension specimens are then cut in the homogeneous deformation zones of the large sheets to obtain $\widetilde{E}_1,\ \widetilde{E}_2,\ \tilde{\nu}_{12}$ from which $D_1,\ D_2,\ \eta$ (and D_3) are derived from equations (16)–(18).

4. Kinetic law of damage evolution

It is possible to formulate a law which generalizes the isotropic damage law of evolution (Lemaitre, 1992) in the framework of generalized standard materials (Halphen and Nguyen, 1975; Germain et al., 1983)

$$\dot{D} = \left(\frac{\overline{Y}}{S}\right)^s \dot{p}, \tag{25}$$

where $\overline{Y} = \frac{1}{2} E_{ijkl} \varepsilon_{kl}^e \varepsilon_{ij}^e$ is the strain energy release rate density. For the anisotropic case $\overline{Y}$ is also the trace of $\underline{Y}$ (equation (9)) when $D_{ij} = 0$ and $\eta = 1$. $\dot{p} = (\frac{2}{3}\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p)^{1/2}$ is the accumulated plastic strain rate, S and s are two parameters characteristic of each material. This law considers the damage governed by the elastic energy and the plastic strain as shown by many observations (Lemaitre et al., 1998).

4.1. Damage evolution

In order to obtain the above isotropic law as a particular case of the anisotropic law the simplest choice of the dissipative potential is

$$F = f + F_D = f + \left(\frac{\overline{Y}(\underline{\varepsilon}^e)}{S}\right)^s Y_{ij} \left|\frac{\mathrm{d}\underline{\varepsilon}^p}{\mathrm{d}r}\right|_{ij}, \tag{26}$$

where $|\cdot|$ applied to a tensor means the absolute value of the principal values, r is the internal variable associated to the isotropic hardening R, $\underline{X}$ is the kinematic hardening back stress variable

$$r = -\rho \frac{\partial \psi^*(\underline{\sigma}, \underline{D}, R, \underline{X})}{\partial R}. \tag{27}$$

$\overline{Y}(\underline{\varepsilon}^e)$ is also the effective strain energy $\int \tilde{\sigma}_{ij}\, \mathrm{d}\varepsilon^e_{ij}$ which can be written as a function of the effective stress

$$\overline{Y} = \frac{1}{2} E_{ijkl} \varepsilon^e_{kl} \varepsilon^e_{ij} = \frac{1}{2} \tilde{\sigma}_{ij} \varepsilon^e_{ij} = \frac{\tilde{\sigma}^2_{eq} R_v}{2E}. \tag{28}$$

This introduces the triaxiality function as

$$R_v = \frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\tilde{\sigma}_H}{\tilde{\sigma}_{eq}}\right)^2 \tag{29}$$

with

$$\tilde{\sigma}_{eq} = (\underline{H}\,\underline{\sigma}^D\,\underline{H})_{eq}, \qquad \tilde{\sigma}_H = \frac{\sigma_H}{1-\eta D_H}. \tag{30}$$

f is the von Mises plastic loading function in which σ_y is the yield stress

$$f = (\underline{\tilde{\sigma}} - \underline{X})_{eq} - R - \sigma_y. \tag{31}$$

The law of plasticity coupled to damage derives from the potential F or f if linear kinematic hardening is assumed (if not refer to Section 6)

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}} = \dot{\lambda}\frac{3}{2}\frac{[H_{ik}(\tilde{\sigma}^D_{kl} - X_{kl})H_{lj}]^D}{(\underline{\tilde{\sigma}} - \underline{X})_{eq}}, \tag{32}$$

$$\dot{r} = -\dot{\lambda}\frac{\partial f}{\partial R} = \dot{\lambda} = \dot{p}\frac{(\underline{\tilde{\sigma}} - \underline{X})_{eq}}{[\underline{H}(\underline{\tilde{\sigma}}^D - \underline{X})\underline{H}]_{eq}}. \tag{33}$$

The damage law also derives from the damage potential F or F_D

$$\dot{D}_{ij} = \dot{\lambda}\frac{\partial F}{\partial Y_{ij}} = \frac{\mathrm{d}r}{\mathrm{d}t}\left(\frac{\overline{Y}}{S}\right)^s \left|\frac{\mathrm{d}\underline{\varepsilon}^p}{\mathrm{d}r}\right|_{ij}. \tag{34}$$

And finally

$$\dot{D}_{ij} = \left(\frac{\overline{Y}}{S}\right)^s |\underline{\dot{\varepsilon}}^p|_{ij}. \tag{35}$$

The principal directions of the damage rate coincide with those of the plastic strain rate. Furthermore, the damage exists only above a threshold written in terms of the accumulated plastic strain

$$\dot{D}_{ij} = 0 \quad \text{if } p \leqslant p_D. \tag{36}$$

4.2. Stored energy based damage threshold

Several attempts have been made to determine the damage threshold as a function of the loading. The simplest is to consider p_D having for any loading a constant value ε_{pD} which is the measured plastic strain threshold in pure tension. This is a rough approximation valid only for monotonic loadings. In fact this threshold is much (up to 1 000 times!) larger in cyclic loadings inducing fatigue because it is related to the stored energy (Lemaitre, 1992) related to the partial irreversibility of the crystalline plastic slips (Tanaka and Mura, 1981). In order to fit this last statement, let us consider that damage occurs at a given amount ϕ_D (material dependent) of energy stored in the material ϕ_s, i.e.

$$\dot{D}_{ij} = 0 \quad \text{if } \phi_s \leqslant \phi_D, \tag{37}$$

$$\phi_s = \int_0^t (X_{ij}\dot{\alpha}_{ij} + R\dot{r})\,\mathrm{d}t, \quad R = R_\infty\big(1 - \mathrm{e}^{-\gamma_0 r}\big). \tag{38}$$

Equation (38) is the standard expression of the stored energy with an exponetial isotropic hardening R. The parameter $R_\infty = \sigma_u - \sigma_y$ is the saturated value of the hardening and γ_0 the nonlinearity exponent. Please note that as long as there is no damage, the equality $r = p$ stands.

In the further developments we choose to omit the kinematic hardening contribution which is small in fatigue and which does not change the fact that the classical standard material framework greatly overestimates the value of the stored energy. Usual standard models lead to an increasing stored energy with the accumulated plastic strain, when very elegant experiments made on several materials (Chrysochoos, 1987) have shown that the rate $\dot{\phi}_s$ becomes negligible at large p. This means that the correct expression of the stored energy should be

$$\phi_s = \int_0^p R(p)z(p)\,\mathrm{d}p \quad \text{with} \quad z(\infty) = 0. \tag{39}$$

Chrysochoos (1987) proposes a new set of thermodynamical variables $(Q_0,\ q_0)$ from which $\phi_s(p) = \int_0^t Q_0\dot{q}_0\,\mathrm{d}t$ is proportional to the exponential hardening R

$$q_0 = q_\infty\big(1 - \mathrm{e}^{-(b+\gamma_0)p}\big), \qquad Q_0 = \frac{\sigma_y}{q_\infty(b+\gamma_0)}\left(1 - \frac{R}{R_\infty}\right)^{-b/\gamma_0}. \tag{40}$$

The expression (but not the value) of the yield criterion is changed, $f = \sigma_{eq} - R(Q) - \sigma_y$ which modifies the plastic constitutive equations. It is possible to avoid this problem by the introduction of new plasticity variables Q and q such as

$$Q(q) = R(p), \qquad \mathrm{d}q = z(p)\,\mathrm{d}p. \tag{41}$$

As the previous $(Q_0,\ q_0)$ they generally differ from the standard couple $(R,\ p)$ but lead to the more correct expression of the stored energy (39). The law $Q(q)$ derives from the Gibbs energy corrected in consequence $(q = -\rho\partial\psi^*/\partial Q)$.

Several choices of $z(p)$ are possible (Sermage, 1998) according to several series of tests. We consider the power law

$$z(p) = \frac{A}{m} p^{(1-m)/m}, \quad \text{with } m > 1, \tag{42}$$

where A and m are material dependent parameters. The expression of the stored energy (39) calculated in the general case and in the uniaxial tension case allows for the determination of the corresponding plastic strain threshold p_D as a function of the threshold in monotonic tension ε_{pD}

$$\phi_D = A \int_0^{p_D} \langle \sigma_{eq} - \sigma_y \rangle \frac{1}{m} p^{(1-m)/m} \, \mathrm{d}p = A(\sigma_u - \sigma_y)\varepsilon_{pD}^{1/m}, \tag{43}$$

where $\langle \cdot \rangle$ denotes the positive part.

If monotonic or symmetric periodic loading are considered together with perfectly plastic behaviour for the maximum stress $\sigma_{eq} = \text{const} = \sigma_{eq}^{\text{Max}}$ then we obtain the formula

$$p_D = \varepsilon_{pD} \left(\frac{\sigma_u - \sigma_y}{\sigma_{eq}^{\text{Max}} - \sigma_y} \right)^m \tag{44}$$

to be used in equation (36). At least a few specific tests are needed to identify the parameter m for each material.

4.3. Crack initiation criterion

Last, a mesocrack is initiated when the damage reaches a critical value D_c which corresponds to the instability under loading in the plane (of normal **n**) in which the density of microcracks is maximum. According to the principle of strain equivalence it is defined by the norm of the damage vector $D_{ij}n_j$ or by the larger principal value of the damage

$$\sup_I D_I = D_c \rightarrow \text{ mesocrack initiation}. \tag{45}$$

5. Some verifications

5.1. Uniaxial measurements

Five sets of experiments were used to check that the measured damage obeys the theory and that the η coefficient may be taken as a constant. They are experiments of ductile damage in uniaxial tension for which the elasticity modulus $\tilde{E}_1$ and the contraction coefficient $\tilde{\nu}_{12}$ are measured. The materials under consideration are

- an aluminium alloy 2024 T4 (Cordebois and Sidoroff, 1982);
- a pure copper CUA1 (Cordebois and Sidoroff, 1982);
- a steel XC38 (Nouailhas, 1980);
- an other aluminum alloy 2024 T3 (Chow and Wang, 1987);
- a dispersoid steel SOLDUR 355 studied at LEDEPP (SOLLAC) and LMT Cachan.

In principle two measurements $\widetilde{E}_1$ and $\tilde{\nu}_{12}$ are not sufficient to identify the principal components of the damage $D_1,\ D_2,\ D_3$ and coefficient η but the symmetry with respect to the direction 1 allows us to consider $D_3 = D_2$

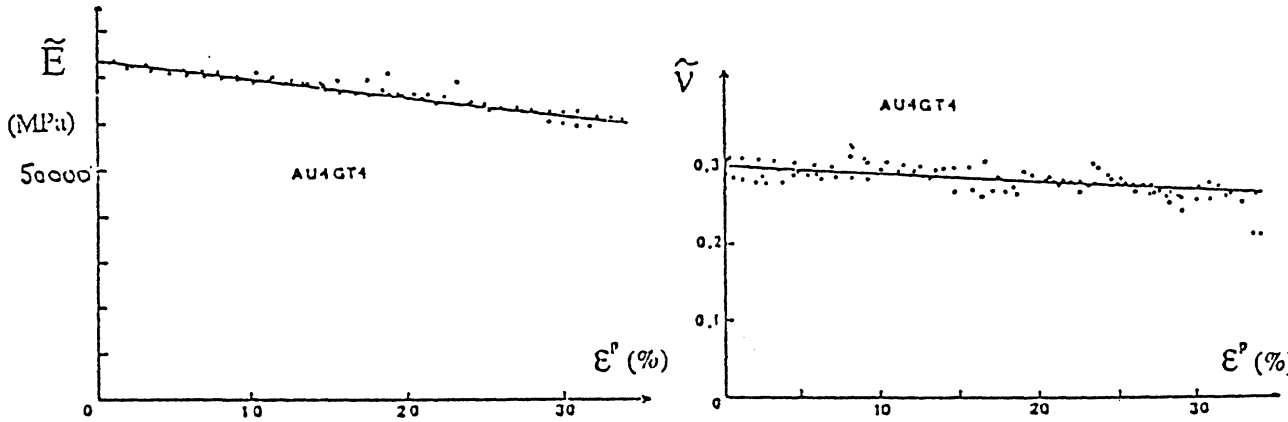

Figure 2. Elasticity modulus and contraction ratio changes for 2024 T4.

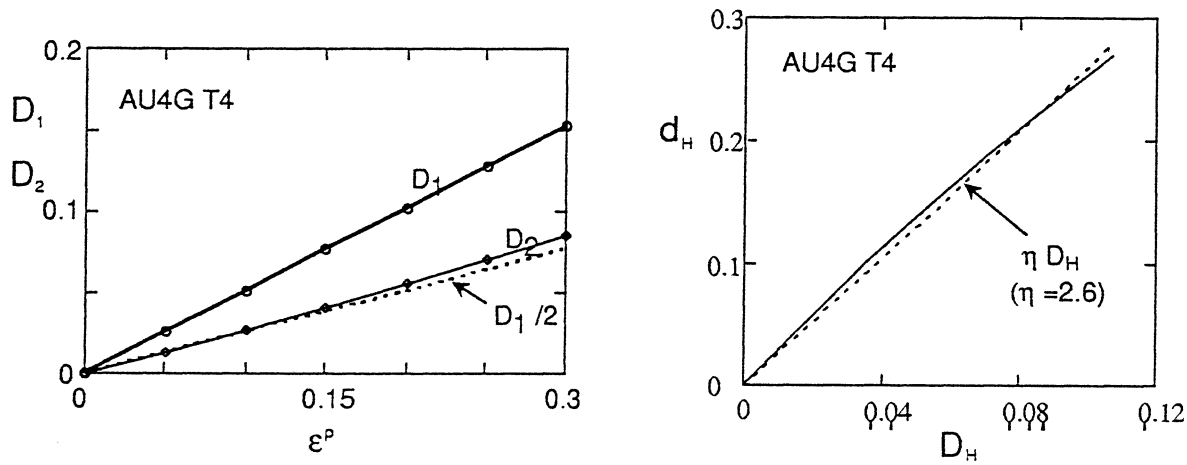

Figure 3. Damage evolutions (2024 T4).

and to make the approximation

$$D_1 \approx 1 - \frac{\widetilde{E}_1}{E}. \tag{46}$$

For the extremal values $\frac{1}{2} \leqslant \frac{\tilde{\nu}}{\nu} \leqslant 1$ and $\frac{3}{4} \leqslant \frac{\widetilde{E}}{E} \leqslant 1$, this corresponds to a maximal relative error of 15% on D_1 with respect to equation (46). With this approximation,

$$D_2 \approx 1 - \frac{\widetilde{E}_1}{E} \frac{1+\nu}{1+3\tilde{\nu}_{12}-2\nu}. \tag{47}$$

The parameter η is determined by the slope of the best straight line fitting the experimental points in the graph d_H fonction of D_H (equation (5)). As an example *figure 3* shows the evolution of D_1, D_2 and the value of η from experimental data of *figure 2* for the aluminium alloy 2024 T4.

For all the cases the following results are obtained:

– 2024 T4: $\eta = 2.6$;

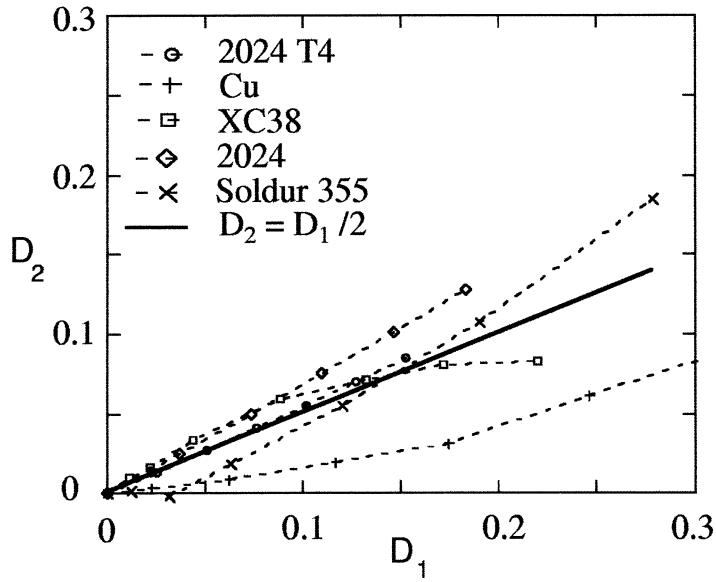

Figure 4. Checking $D_1 = 2D_2$.

– CUA1: $\eta = 3.5$;
– XC 38: $\eta = 2.6$;
– 2024 T3: $\eta = 2.1$;
– SOLDUR 355: $\eta = 2.8$.

Figure 3 and similar results for other materials (*figure 4*) show that $D_1 = 2D_2$ is verified with an accuracy of the order of magnitude of the scatter which is always large in this kind of experiments (Dufailly, 1995). $D_1 = 2D_2$ is precisely the result given by the damage evolution law as in uniaxial tension $\dot{\underline{\varepsilon}}^p = \dot{p}[1, -1/2, -1/2]$ from which $|\dot{\underline{\varepsilon}}^p|_{11} = 2|\dot{\underline{\varepsilon}}^p|_{22}$ and from equation (35), $D_1 = 2D_2$.

5.2. Multiaxial measurements

We have performed tension experiments of ductile damage on large plates made of SOLDUR 355 steel (TU and TP as explained in Section 3.2). These experiments are used to induce the two different plastic strain fields $\underline{\varepsilon}^p = p[1, -1/2, -1/2]$ (TU) and $\underline{\varepsilon}^p = [\varepsilon_{11}^p, 0, \varepsilon_{11}^p]$ (TP).

For each tension TU and TP, three small specimens have then been cut in three directions (0, 90 and 45 degrees with respect to the axis of the previous tension, *figure 5*) of three sheets previously damaged at three levels of plastic strains: $\varepsilon_{11}^p = 7\,10^{-2}$, $\varepsilon_{11}^p = 15\,10^{-2}$ and $\varepsilon_{11}^p = 33\,10^{-2}$, for uniaxial tension, $\varepsilon_{11}^p = 5\,10^{-2}$, $\varepsilon_{11}^p = 6.7\,10^{-2}$ and $\varepsilon_{11}^p = 11\,10^{-2}$ for plane tension. They have then been tested in uniaxial tension and the elasticity modulus as well as the contraction coefficient have been measured for each of them. The Young's moduli of the small specimens coincide of course with the plates moduli $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{E}(45)$.

Only specimens cut in the 0 degree direction of the large TU and TP sheets have been used to identify the constitutive equation of the damage (equation (35)). The damage law thus identified was then applied to all other tests in order to calculate the components D_{ij} and then $\widetilde{E}_2$, $\widetilde{E}(45)$, $\tilde{\nu}_{12}$ to compare with experiments.

The measurements of strains have been made by strain gauges glued axialy and transversaly. The scatter is important.

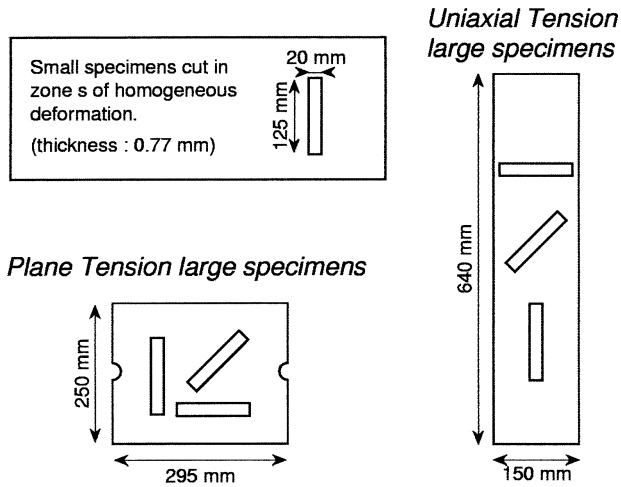

Figure 5. Small specimens in large specimens.

5.2.1. Identification

The material coefficients to be identified are $S,\ s,\ D_c$ for the evolution law (35) and $\varepsilon_{pD},\ \sigma_u,\ \sigma_y,\ m$ for the threshold (44) (with in uniaxial tension $p_D = \varepsilon_{pD}$). We also need the elasticity coefficients E and ν of the virgin material. One of the main values of the law (35) stands in its identification procedure identical to the one for isotropic law: the non-isotropic character is given by the plastic strain tensor alone. Furthermore, the additional parameter η may be ignored as it only becomes influential for large damages.

In order to determine the damage evolution D_1 the plastic criterion function $f = 0$ must be considered (equation (31)). If the damage occurs upon saturated strain hardening a good approximation of the function criterion is $\tilde{\sigma}_{eq} - \sigma_u = 0$ where σ_u is the ultimate stress. Then, from equations (28)–(31) and (35) the evolution of $\underline{D}$ is obtained as an implicit function of the plastic strain and the triaxiality

$$\int_0^{D_1} R_\nu^{-s}\,\mathrm{d}D_1 = \int_{p_D}^{p} \left(\frac{\sigma_u^2}{2ES}\right)^s \mathrm{d}\varepsilon_{11}^p. \tag{48}$$

For uniaxial tension the R_ν triaxiality function is a function of D_1 alone because $D_2 = D_3 = D_1/2$ and $D_H = 2D_1/3$

$$R_{\nu TU} = \frac{2}{3}(1+\nu) + 3(1-2\nu)\left[\left(1 - \frac{2\eta}{3}D_1\right)\left(\frac{2}{1-D_1} + \frac{1}{1-\frac{D_1}{2}}\right)\right]^{-2}. \tag{49}$$

As $p = \varepsilon_{11}^p$ the relation betweeen D_1 and the plastic strain is

$$\varepsilon_{11}^p = \varepsilon_{11}^p(D_1) = \varepsilon_{pD} + \left(\frac{2ES}{\sigma_u^2}\right)^s \int_0^{D_1} R_{\nu TU}^{-s}\,\mathrm{d}D_1. \tag{50}$$

For plane tension $p = 2\varepsilon_{11}^p/\sqrt{3}$. The damage law (35) gives $D_3 = D_1,\ D_2 = 0$ and

$$R_{\nu TP} = \frac{2}{3}(1+\nu) + (1-2\nu)\left(\frac{1-D_1}{1-\frac{2\eta}{3}D_1}\right)^2, \tag{51}$$

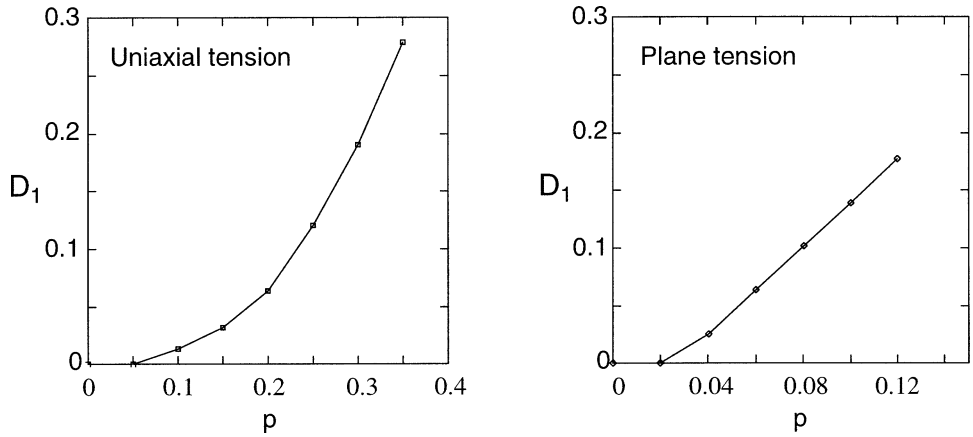

Figure 6. Damage evolutions (SOLDUR 355).

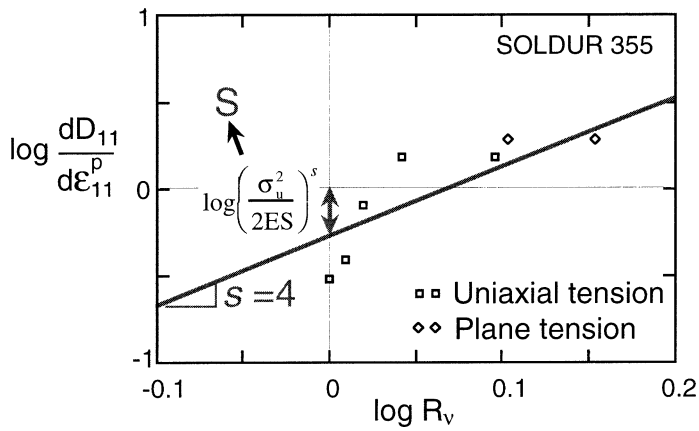

Figure 7. Determination of parameters S and s.

$$\varepsilon_{11}^{p} = \frac{\sqrt{3}}{2}\varepsilon_{pD} + \left(\frac{2ES}{\sigma_u^2}\right)^{s} \int_0^{D_1} R_{vTP}^{-s}\, \mathrm{d}D_1. \tag{52}$$

The results are reported in *figure 6* where the damage is calculated by equation (46) and the triaxiality functions by equations (49)–(51).

Knowing the ultimate stress from the uniaxial tension test, the S and s parameters are deduced from the graph $\mathrm{Log}\frac{\mathrm{d}D_1}{\mathrm{d}\varepsilon_{11}^{p}}$ function of $\mathrm{Log}(R_v)$

$$\mathrm{Log}\frac{\mathrm{d}D_1}{\mathrm{d}\varepsilon_{11}^{p}} = \mathrm{Log}\left(\frac{\sigma_u^2}{2ES}\right)^{s} + s\mathrm{Log}R_v. \tag{53}$$

The corresponding points are reported in *figure 7*. Due to the scatter and the small variation of R_v, the accuracy is poor.

Nevertheless identification of the steel SOLDUR 355 is as follows: $S = 0.57$ MPa, $s = 4$, $\varepsilon_{pD} = 2.5\,10^{-2}$, $D_c = 0.3$ with $E = 230$ GPa, $\nu = 0.3$, $\sigma_u = 474$ MPa, $\eta = 2.8$.

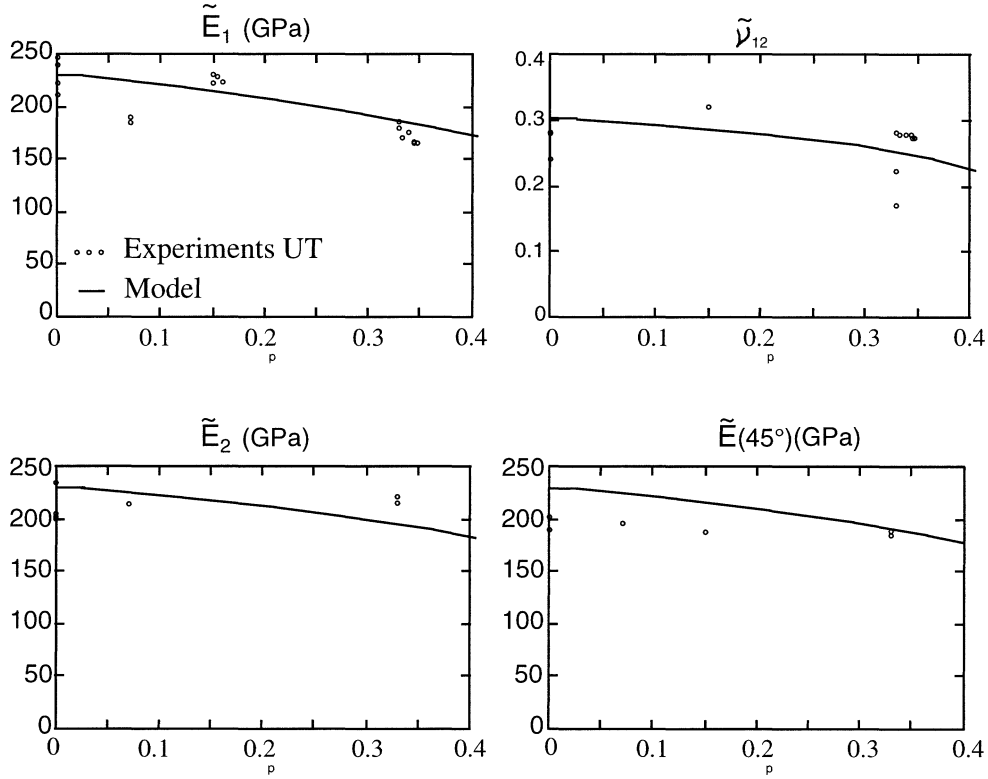

Figure 8. Comparison theory–experiments for uniaxial tension.

5.2.2. Comparison between theory and experiments

Once the damage law is identified it is possible to calculate the components of the damage for any history of loading. D_1 is obtained as a function of ε_{11}^p by equations (49)–(52). Then, $D_2 = D_3 = D_1/2$ for uniaxial tension and $D_2 = 0,\ D_3 = D_1$ for plane tension. $\widetilde{E}_1(\varepsilon_{11}^p),\ \widetilde{E}_2(\varepsilon_{11}^p),\ \tilde{\nu}_{12}(\varepsilon_{11}^p)$ are calculated to be compared with experimental data in *figures 8* and *9*. Note that the curves $\widetilde{E}_1(\varepsilon_{11}^p)$ for TU and TP only verify the identification. The other curves verify the validity of the model. In *figures 8* and *9* the result concerning the specimens cut at 45 degrees are also reported. For this direction

$$\widetilde{E}(45) = 4\left[\frac{1}{\widetilde{E}_1} + \frac{1}{\widetilde{E}_2} + \frac{1}{\widetilde{G}_{12}} - 2\frac{\tilde{\nu}_{12}}{\widetilde{E}_1}\right]^{-1}, \tag{54}$$

$$\tilde{\nu}(45) = \frac{\widetilde{E}_{45}}{4}\left[2\frac{\tilde{\nu}_{12}}{\widetilde{E}_1} + \frac{1}{\widetilde{G}_{12}} - \frac{1}{\widetilde{E}_1} - \frac{1}{\widetilde{E}_2}\right]. \tag{55}$$

The shear modulus may also be obtained as

$$\widetilde{G}_{12} = \widetilde{G}_{12}(\varepsilon_{11}^p) = G\sqrt{(1-D_1)(1-D_2)}. \tag{56}$$

Due to experimental discrepancy it is not possible to conclude that the proposed theory exactly fits the experimental results. Nevertheless no contradiction exists and, at least qualitatively, the agreement is good. The

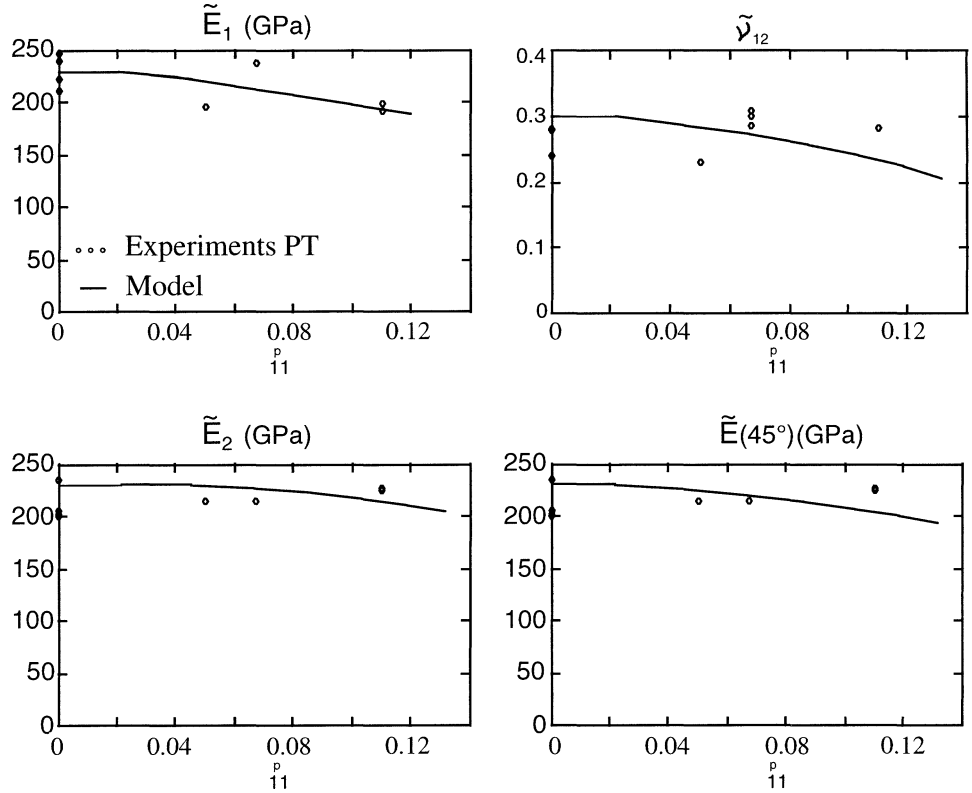

Figure 9. Comparison theory–experiments for plane tension.

differences between experiments and numerical simulations partially come from a small initial anisotropy of the material not taken into account here.

6. Coupling between damage and plasticity

The greatest value of the effective stress being associated with the principle of strain equivalence is to naturally allow any mechanical coupling. For coupled plasticity, the only change in comparison with natural plasticity is to replace the stress $\underline{\sigma}$ by the effective stress $\underline{\tilde{\sigma}}$ (equation (7)) in the criterion function. The non-linear kinematic hardening or back stress is introduced by considering the non-associated potential $f + F_X$

$$f = (\underline{\tilde{\sigma}} - \underline{X})_{eq} - R - \sigma_y, \qquad F_X = \frac{3\gamma}{4C}\underline{X} : \underline{X}, \tag{57}$$

C and γ are the non-linear kinematic hardening material parameters. The associated variable to $\underline{X}$ is $\underline{\alpha} = 3\underline{X}/2C$ whose evolution law is given by

$$\dot{\alpha}_{ij} = -\dot{\lambda}\frac{\partial(f + F_X)}{\partial X_{ij}}. \tag{58}$$

The normality rule gives the following set of equations

$$
\begin{aligned}
&\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij},\\
&\varepsilon^e_{ij} = \frac{1+\nu}{E}\tilde{\sigma}_{ij} - \frac{\nu}{E}\tilde{\sigma}_{kk}\delta_{ij},\\
&R = R_\infty\left(1 - \mathrm{e}^{-\gamma_0 r}\right),\\
&\dot{\varepsilon}^p_{ij} = \frac{3}{2}\frac{[H_{ik}(\tilde{\sigma}^D_{kl} - X_{kl})H_{lj}]^D}{(\tilde{\underline{\sigma}} - \underline{X})_{eq}}\dot{r},\\
&\dot{X}_{ij} = \frac{2}{3}C\dot{\beta}_{ij} - \gamma X_{ij}\dot{r}, \quad \dot{\beta}_{ij} = \frac{3}{2}\frac{(\tilde{\sigma}^D_{ij} - X_{ij})}{(\tilde{\underline{\sigma}} - \underline{X})_{eq}}\dot{r},\\
&\dot{D}_{ij} = \left(\frac{\overline{Y}}{S}\right)^s \left|\dot{\underline{\varepsilon}}^p\right|_{ij} \quad \text{if } r > p_D
\end{aligned}
\tag{59}
$$

in which the plastic multiplier $\dot{\lambda}$ is given by the consistancy condition $f = 0$, $\dot{f} = 0$ and $\dot{r} = \dot{\lambda}$, the effective stress tensor is defined by equation (7), the effective elastic energy density is

$$
\overline{Y} = \frac{E}{2}\left(\frac{\varepsilon^{eD}_{ij}\varepsilon^{eD}_{ij}}{1+\nu} + \frac{{\varepsilon^e_{kk}}^2}{3(1-2\nu)}\right) = \frac{\tilde{\sigma}^2_{eq}R_v}{2E} \tag{60}
$$

and the damage threshold is given by equation (43). To write the previous set of equations as driven by the variable r instead of the accumulated plastic strain p makes the equations simpler. Let us finally note that

$$
\dot{p} = \frac{[\underline{H}(\tilde{\underline{\sigma}}^D - \underline{X})\underline{H}]_{eq}}{(\tilde{\underline{\sigma}} - \underline{X})_{eq}}\dot{r}. \tag{61}
$$

7. Quasi-unilateral conditions

The partial closure of microcracks loaded in compression often induces a damage rate much smaller in compression than in tension (Horii and Nemat-Nasser, 1983; Ladevèze, 1983; Ladevèze and Lemaitre, 1984; Chaboche, 1993; Dragon and Halm, 1995). The difficulty is to recognize what is compression and what is tension in a three dimensional state of stress and to write a Gibbs energy able to be differentiated. In the isotropic case the problem has been solved by the introduction of a parameter h ($0 \leqslant h \leqslant 1$, often $h = 0.2$) which operates by $(1 - hD)^{-1}$ on the negative part of the principal stresses, the positive part being the classical effective stresses $\sigma_I/(1-D)$ (Ladevèze and Lemaitre, 1984; Liénard, 1989).

Using the same concept in the present anisotropic damage theory consists in writing

$$
\underline{H}^+ = (\underline{1} - \underline{D})^{1/2}, \qquad \underline{H}^- = (\underline{1} - h\underline{D})^{1/2}, \tag{62}
$$

$$
d^+_H = \eta D_H, \qquad d^-_H = h\eta D_H, \tag{63}
$$

$$
\rho\psi^* = \frac{1+\nu}{2E}\mathrm{Tr}(\underline{H}^+\underline{\sigma}^D_+\underline{H}^+\underline{\sigma}^D_+ + \underline{H}^-\underline{\sigma}^D_-\underline{H}^-\underline{\sigma}^D_-) + \frac{3(1-2\nu)}{2E}\left(\frac{\langle\sigma_H\rangle^2_+}{1-d^+_H} + \frac{\langle\sigma_H\rangle^2_-}{1-d^-_H}\right), \tag{64}
$$

$\langle\cdot\rangle_+$ and $\langle\cdot\rangle_-$ are the positive and the negative parts, $\underline{\sigma}^D_+$ (resp. $\underline{\sigma}^D_-$) is built with the positive (resp. negative) eigenvalues and the corresponding eigenvectors of $(\underline{H}^+\underline{\sigma}^D)$ (resp. $(\underline{H}^-\underline{\sigma}^D)$) (Ladevèze, 1983, 1995). We finally obtain the effective stress taking into account the quasi-unilateral effect

$$
\tilde{\underline{\sigma}} = \left(\underline{H}^+\underline{\sigma}^D_+\underline{H}^+\right)^D + \left(\underline{H}^-\underline{\sigma}^D_-\underline{H}^-\right)^D + \left[\frac{\langle\sigma_H\rangle_+}{1-d^+_H} + \frac{\langle\sigma_H\rangle_-}{1-d^-_H}\right]\underline{1}. \tag{65}
$$

The energy function $\overline{Y}$ to be introduce in the damage law is:

$$\overline{Y} = \frac{1+\nu}{3E}\left[\left(\underline{H}^{+}\underline{\sigma}_{+}^{D}\underline{H}^{+}\right)_{eq}^{2} + h\left(\underline{H}^{-}\underline{\sigma}_{-}^{D}\underline{H}^{-}\right)_{eq}^{2}\right] + \frac{3(1-2\nu)}{2E}\left[\frac{\langle\sigma_H\rangle_{+}^{2}}{(1-d_H^{+})^2} + h\frac{\langle\sigma_H\rangle_{-}^{2}}{(1-d_H^{-})^2}\right]. \tag{66}$$

8. Strain localization

The classical conditions of localization (Rice and Rudnicki, 1980; Borre and Mayer, 1989; Benallal et al., 1992)

$$\det\left(\mathbf{n}\underline{\underline{L}}\mathbf{n}\right) = 0 \tag{67}$$

are studied here in the framework of elasto-plasticity coupled to anisotropic damage. In this section, we prefer intrinsic notations (second order tensors are underlined one time when fourth order tensors are underlined twice), $\underline{\underline{L}}$ is the tangent operator defined by

$$\dot{\underline{\sigma}} = \underline{\underline{L}} : \dot{\underline{\varepsilon}}, \tag{68}$$

n is the vector normal to the localization plane. Condition (67) makes it possible to determine the critical value of the damage D_c by considering that a mesocrack initiation is the final stage of the strain localization.

In order to calculate $\underline{\underline{L}}$ we write the law of elasticity as

$$\dot{\tilde{\underline{\sigma}}} = \underline{\underline{E}} : \left(\dot{\underline{\varepsilon}} - \dot{\underline{\varepsilon}}^{p}\right), \tag{69}$$

where the plastic strain is given by equations (32)–(33). The effective stress $\tilde{\underline{\sigma}}$ (equation (7)) defines the fourth order tensor $M_{ijkl} = H_{ik}H_{lj} - \frac{1}{3}[H_{kl}^2\delta_{ij} + H_{ij}^2\delta_{kl}] + \frac{1}{9}H_{pp}^2\delta_{ij}\delta_{kl}$ such that

$$\tilde{\underline{\sigma}} = \underline{\underline{M}} : \underline{\sigma}. \tag{70}$$

Then

$$\dot{\tilde{\underline{\sigma}}} = \dot{\tilde{\underline{\sigma}}} + \frac{\partial\tilde{\underline{\sigma}}}{\partial\underline{D}} : \dot{\underline{D}}, \quad \dot{\tilde{\underline{\sigma}}} = \underline{\underline{M}} : \dot{\underline{\sigma}}. \tag{71}$$

The calculation of $\partial\tilde{\underline{\sigma}}/\partial\underline{D}$ is similar to the calculation of $\underline{Y}$ (equation (9)) as

$$\frac{\partial\tilde{\sigma}_{ij}}{\partial D_{rs}} = \frac{\partial}{\partial D_{rs}}E_{ijkl}\varepsilon_{kl}^{e} = E_{ijkl}\frac{\partial^2\rho\psi^{*}}{\partial D_{rs}\partial\sigma_{kl}} = E_{ijkl}\frac{\partial Y_{rs}}{\partial\sigma_{kl}}. \tag{72}$$

For commodity no kinematic hardening is considered ($\underline{X} = \underline{0}$). The laws governing the evolution of the plastic strain, of the isotropic strain hardening and of the damage are written as

$$\dot{\underline{\varepsilon}}^{p} = \underline{a}\,\dot{p} \quad \text{with } \underline{a} = \frac{3}{2}\frac{(\underline{H}\,\tilde{\underline{\sigma}}^{D}\,\underline{H})^{D}}{(\underline{H}\,\tilde{\underline{\sigma}}^{D}\,\underline{H})_{eq}}, \tag{73}$$

$$\dot{r} = \Delta\,\dot{p} \quad \text{with } \Delta = \frac{\tilde{\sigma}_{eq}}{(\underline{H}\,\tilde{\underline{\sigma}}^{D}\,\underline{H})_{eq}}, \tag{74}$$

$$\dot{\underline{D}} = \underline{d}\,\dot{p} \quad \text{with } \underline{d} = \left(\frac{\overline{Y}}{S}\right)^{s}|\underline{a}| \tag{75}$$

and the accumulated plastic strain rate is determined by the consistancy condition $\dot{f}=0$

$$\dot{p}=\frac{\underline{b}:\dot{\underline{\tilde{\sigma}}}}{k} \quad \text{with } k=\Delta R' \text{ and } \underline{b}=\frac{3}{2}\frac{\underline{\tilde{\sigma}}^D}{\tilde{\sigma}_{eq}}. \tag{76}$$

With the notations just introduced

$$\dot{\underline{\tilde{\sigma}}}=\underline{\underline{E}}:\dot{\underline{\varepsilon}}-\frac{\underline{b}:\dot{\underline{\tilde{\sigma}}}}{k}2G\underline{a}, \tag{77}$$

$$\frac{\underline{b}:\dot{\underline{\tilde{\sigma}}}}{k}=\frac{2G}{k+2G\underline{a}:\underline{b}}\,\underline{b}:\dot{\underline{\varepsilon}} \tag{78}$$

and with equation (71)

$$\dot{\underline{\tilde{\sigma}}}=\underline{\underline{E}}:\dot{\underline{\varepsilon}}-\frac{2G}{k+2G\underline{a}:\underline{b}}\left(2G\underline{a}+\frac{\partial\tilde{\underline{\sigma}}}{\partial\underline{D}}:\underline{d}\right)\underline{b}:\dot{\underline{\varepsilon}}. \tag{79}$$

Finally the expression for the tangent operator is for any loading

$$\underline{\underline{L}}=\underline{\underline{\tilde{E}}}-\frac{2G}{k+2G\underline{a}:\underline{b}}\left[(\underline{\underline{\tilde{E}}}:\underline{a})\otimes\underline{b}+\left(\underline{\underline{M}}^{-1}:\frac{\partial\tilde{\underline{\sigma}}}{\partial\underline{D}}:\underline{d}\right)\otimes\underline{b}\right], \tag{80}$$

where $\underline{\underline{\widetilde{E}}}=\underline{\underline{M}}^{-1}:\underline{\underline{E}}$ is the elastic effective tensor.

We derive here the conditions of localization in tension for isotropic as well as anisotropic damages. When dealing with anisotropy, we consider an hydrostatic sensibility parameter η larger than 1 (usually $2<\eta<3$).

If a tension loading $\underline{\sigma}=\sigma\vec{x}_1\otimes\vec{x}_1$ is applied parallel to the $\vec{x}_1$ direction, then $\underline{a}=\underline{b}=\text{diag}[1,-1/2,-1/2]$. The vector normal to the strain localization plane is

$$\mathbf{n}=\begin{pmatrix}c\\ s\\ 0\end{pmatrix}, \quad c=\cos\theta,\ s=\sin\theta. \tag{81}$$

For isotropic damage we introduce Lamé coefficients λ and μ such that

$$\mathbf{n}\underline{\underline{\widetilde{E}}}\mathbf{n}=(1-D)\begin{bmatrix}\mu+(\lambda+\mu)c^2 & (\lambda+\mu)sc & 0\\ (\lambda+\mu)sc & \mu+(\lambda+\mu)s^2 & 0\\ 0 & 0 & \mu\end{bmatrix}. \tag{82}$$

The strain localization conditions (67)–(80) are

$$\begin{vmatrix}(1-D)\left[\mu+\left(\lambda-\dfrac{\mu}{3}\right)c^2\right]-Z\mu c^2 & (1-D)\left(\lambda+\dfrac{5\mu}{3}\right)sc+\dfrac{1}{2}Z\mu sc\\ (1-D)\left(\lambda+\dfrac{5\mu}{3}\right)sc & (1-D)\left[\mu+\left(\lambda+\dfrac{2\mu}{3}\right)s^2\right]\end{vmatrix}=0 \tag{83}$$

in which

$$Z = \frac{2}{3}\left(\frac{\sigma_u^2}{2ES}\right)^s \frac{\sigma_u}{\mu}. \tag{84}$$

Further derivations give the closed form expression for the localization conditions as a critical damage D_c depending on the normal **n**,

$$D_c(\mathbf{n}) = 1 - g(\theta)\left(\frac{\sigma_u^2}{2ES}\right)^s \frac{\sigma_u}{\mu}, \quad g(\theta) = \frac{2 - 4\nu + (5 - 4\nu)s^2c^2}{3 - 6s^2c^2 + (1 - 2\nu)(2s^2 - c^2)}. \tag{85}$$

It can be verified that g is of the order of magnitude of unity. Furthermore, the ratio σ_u/μ is very small for any material. We obtain the classical result of Benallal et al. (1992) for which the condition of localization in tension is close to $D_c \approx 1$.

For anisotropic damage, the same result (a critical damage almost equal to 1) is obtained but in terms of hydrosatic damage: the condition of strain localization is close to

$$\eta D_{Hc} \approx 1 \quad \text{or} \quad D_{Hc} \approx 1/\eta \quad \text{or} \quad D_{1c} = 2D_{2c} = 2D_{3c} \approx \frac{3}{2\eta}. \tag{86}$$

With $\eta = 3$, we obtain $D_{1c} \approx 0.5$. This value is in accordance with the critical damages often observed in experiments with a poor accuracy owing to the difficulty in defining in practice what is the initiation of a crack!

The use of a strain localization criterion to model a mesocrack initiation is made possible owing to the consideration of anisotropic damage. Such a criterion depends on the value of the parameter η which could finally be identified by strain localization or mesocrack initiation measurements.

9. Conclusion

As a conclusion let us develop the topic 'How to use anisotropic damage' to predict crack initiation in real structures. In our opinion the main difficulty is the identification of the constitutive equations for each material (at each temperature considered!).

9.1. Identification

To identify the damage law which can be used either in monotonic loadings for ductile fracture or in cyclic loadings for low cycle or high cycle loadings, it is best to consider experiments in the largest possible domain of stresses, strains, time and number of cycles.

The parameters to be identified are defined by the basic equations

$$\dot{D}_{ij} = \left(\frac{\overline{Y}}{S}\right)^s \left|\dot{\underline{\varepsilon}}^p\right|_{ij} \quad \text{if } p > p_D,$$

$$\sup_I D_I = D_c \quad \text{mesocrack initiation}$$

with

$$\overline{Y} = \frac{E}{2}\left(\frac{\varepsilon_{ij}^{eD}\varepsilon_{ij}^{eD}}{1+\nu} + \frac{{\varepsilon_{kk}^e}^2}{3(1-2\nu)}\right) = \frac{\tilde{\sigma}_{eq}^2 R_v}{2E},$$

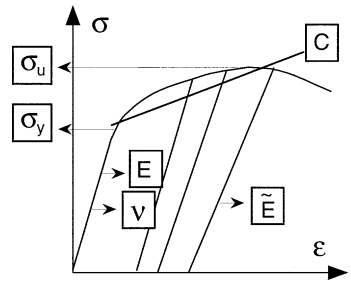

Figure 10. Tension test for identification.

$$p_D = \varepsilon_{pD}\left(\frac{\sigma_u - \sigma_y}{\sigma_{eq}^{\text{Max}} - \sigma_y}\right)^m \tag{87}$$

they are

- $S,\ s,\ \varepsilon_{pD},\ m,\ D_c$ for damage;
- $E,\ \nu$ for elasticity;
- $\sigma_u,\ \sigma_y$ for low cycle fatigue and fracture in tension. Add the fatigue limit σ_f for high cycle fatigue (two scale damage model of Section 9.3).

9.1.1. Uniaxial tension with unloadings

One (at least) experiment in uniaxial tension with unloadings (*figure 11*) defines $E,\ \nu,\ \sigma_u,\ \varepsilon_{pD}$, the isotropic hardening law $R(r)$ and the damage by the simplified equation (46)

$$D_{11} \approx 1 - \frac{\widetilde{E}_1}{E}. \tag{88}$$

For the two scale model of Section 9.3 where only linear kinematic hardening is assumed, the tension experiments allow then for the determination of the plastic sequent modulus C.

9.1.2. Low cycle fatigue tests

About ten very low cycle fatigue tests (at numbers of cycles to failure $N_R = 5,\ 10,\ 50,\ 100,\ 1\,000$ cycles) allow us to calculate the damage evolution by equation (46), to draw the graph $D_{11}(p)$ from which D_c is determined from all tests, and ε_{pD} from monotonic tension/compression tests, p_D from fatigue tests (*figure 11*).

The damage parameters S and s are obtained from the graph $\mathrm{d}D/\mathrm{d}p$ function of the maximum stress σ_{Max} as for fatigue test $\tilde{\sigma}_{eq} \approx \sigma_{\text{Max}}$ for $p > p_D$ and $R_v \approx 1$ as long as damage remains small

$$\frac{\mathrm{d}D_{11}}{\mathrm{d}p} = \left(\frac{\sigma_{\text{Max}}}{2ES}\right)^s. \tag{89}$$

If needed, h comes from the measurements of the elasticity modulus in compression (*figure 11*).

9.1.3. High cycle fatigue tests

High cycle fatigue tests define the asymptotic value of the Woehler curve from which the fatigue limit is appreciated as $\sigma_f = \sigma_{\text{Max}}$ for a number of cycles to failure $N_R \to \infty$ with a zero mean stress.

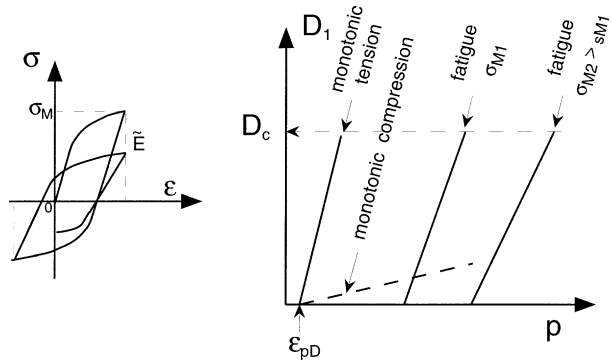

Figure 11. Low cycle tests for identification.

The two scale model of Section 9.3 introduces plasticity and damage at a microscopic scale at which the yield stress is taken equal to the fatigue limit σ_f. The exponent m is then obtained from the graph of the damage threshold at microscale as a function of the macroscopic maximum applied stress σ_{Max}

$$\ln \frac{p_D}{\varepsilon_{pD}} = m \ln \frac{\sigma_u - \sigma_f}{\sigma_M - \sigma_f}. \tag{90}$$

Some two level high cycle fatigue tests help to define the graph for large p_D as for $p < p_D$, $D_{ij} = 0$ (still at microscale).

9.2. Post processing by uncoupled method

An elasto-plastic structure calculation by a finite element analysis or a Neuber plastic correction of an elastic calculation gives the histories of the elastic strains $\varepsilon^e_{ij}(t)$ and of the plastic strain rates $\dot{\varepsilon}^p_{ij}(t)$. The damage evolution is obtained by a simple time integration of the laws (28)–(35) with an eventual initial damage ($D^0_{ij} = D_0 \delta_{ij}$)

$$D_{ij} = D^0_{ij} + \int_{t_O}^{t} \left(\frac{\overline{Y}}{S} \right)^s \left| \underline{\dot{\varepsilon}}^p \right|_{ij} \mathrm{d}t \quad \text{if } p > p_D. \tag{91}$$

Ductile damage and low cycle fatigue may be analysed by this method.

9.3. Post processing by locally coupled method

When the damage is very localized, an elastic structure calculation gives the stresses and strains at the critical point. Here a two-scale model solves the problem of a weak inclusion embedded in an elastic (or elasto-plastic) representative volume element (Lemaitre and Doghri, 1994; Lemaitre et al., 1999). The law of localization (Kroener, 1961; Berveiller and Zaoui, 1980) is used to calculate the microscopic stresses as a function of the mesoscopic stresses applied on the RVE (μ stands for the microscopic values)

$$\sigma^{\mu}_{ij} = \sigma_{ij} - aE\left(\varepsilon^{\mu p}_{ij} - \varepsilon^p_{ij}\right), \quad a \approx 0.4. \tag{92}$$

The constitutive equations at microscale are those of elasto-plasticity coupled to damage with the fatigue limit taken as the yield stress and the strain hardening considered only as linear kinematic. The code

DAMAGE97 (Sermage, 1998) solves the equations in the case of isotropic damage and ‘soon’ in the case of anisotropy.

This method applies with success for high cycle fatigue.

9.4. Fully coupled method

When the damage is not localized and if a high accuracy is needed, the full set of the coupled constitutive equations (59) needs to be solved as field variables. The routines ABAQUS (UMAT) PLASTENDO and VISCOENDO (Benallal et al., 1988; Florez, 1992) solve these equations in case of isotropic damage and ‘soon’ in case of anisotropy but the price to pay is a very important time of calculation.

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