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Strain-based anisotropic damage modelling and unilateral effects

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Abstract

A new continuum damage mechanics model is developed to describe the behaviour of quasi-brittle materials under general path loading. The induced damage is represented by a second rank symmetric tensor. The constitutive equations are based on irreversible thermodynamics theory. The strain-based model covers in an unified way the unsymmetrical behaviour in tension and in compression and the unilateral response due to crack closure effect. Uniaxial stress tests (in tension as in compression) show realistic non-linear responses in the stress–strain space. The different behaviour in both domains is covered by a single set of equations. A significant volume dilatation is noticed in compression. The model can be generalised to time-dependent phenomena.

Keywords: Continuum damage mechanics; Unilateral effects; Anisotropic damage; Quasi-brittle materials; Thermodynamic modelling

1. Introduction

Anisotropic damage modelling is a difficult topic, especially when unilateral damage effects are taken into account [1]. A simple way to deal properly with the quasi-unilateral conditions is to

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expose a Gibbs potential function of the positive part and the negative part of the stress tensor (also known as spectral decomposition of the stress tensor). This technique has been successfully applied to uniaxial stress state (see for instance the isotropic Mazars model [2] for concrete material). Extension of such a model to tensorial damage framework has also been considered ([1,3] or more recently [4]) although such stress decomposition seems quite arbitrarily and not physically motivated. In particular, the difficulty is to recognise what is compression and what is tension in a three-dimensional state of stress. As damage is mainly a strain-controlled phenomenon, most of the existing anisotropic damage models have been devoted to a strain-based formulation. The main properties required for a consistent damage model applied to quasi-brittle materials are [5]:

- continuity of stress–strain relation,
- consistent thermodynamic framework,
- unsymmetrical behaviour in tension and compression (strength but also distinct softening),
- crack-closure effects included in the unilateral properties,
- anisotropic damage behaviour: after a prior damaging history, the actual elastic behaviour of the material is anisotropic.

Nowadays, it seems that most of the anisotropic damage models do not comply with all these conditions [5,6]. The efficient and robust Mazars model [2,7] cannot capture anisotropic damage effect. As shown by Chaboche [5], the model of Ju [8] does not verify the continuity of the stress–strain relation. This pathology is also encountered for other common models referenced in [5]. Models of Chaboche [9] or Halm and Dragon [10] are thermodynamically inconsistent (as analysed in [6]): the correction associated to the unilateral damage effect generates a loss of uniqueness of the free Helmholtz energy for constant internal variables. The model of Fichant et al. [11] recently extended to unilateral effects [12] has not been developed in the thermodynamic framework of irreversible processes. The model of Murakami and Kamiya [13] is probably one of the best compromise between all the above referenced constraints. Nevertheless, the ratio between the uniaxial compressive strength and uniaxial tension strength is unrealistic for quasi-brittle materials (the ratio is about 4 in the model). Moreover, the unilateral aspect is not completely covered in the sense that a tension path has an incidence on the stiffness of a compression path for uniaxial stress tests. We suggest the distinction between complete unilateral effect (a unidimensional tension path has no incidence on the stiffness of the unidimensional compression path and conversely, as in [1,2]) and weak unilateral effect (the elastic stiffness in the tension domain can be affected by the history of the material in the compression domain, and conversely, as in [13] for instance).

A new anisotropic damage model is presented in this paper devoted to quasi-brittle materials. The model meets the five enunciated features of satisfying constitutive damage modelling. The induced damage is represented by a second rank symmetric tensor. The strain-based model, based on irreversible thermodynamics, covers in an unified way the unsymmetrical behaviour in tension and in compression and the unilateral response due to crack closure effect. Complete unilateral effect is described in the model, but this assumption can be eventually relaxed. No frictional effects, leading to plasticity-like constitutive formulation coupled with damage are taken into account. Uniaxial stress tests (in tension as in compression) show realistic non-linear responses in the stress–strain space. The different behaviour in both domains is covered by a single set of equations, using identical material constants. A time-dependent generalisation is finally considered.

2. Thermodynamical background

State variables are reduced to two second rank symmetric tensorial variables $\underline{\underline{\varepsilon}}$ (strain variable) and $\underline{\underline{D}}$ (damage variable). Let us notice that no additional internal variable are introduced, on the contrary to the models of Cordebois and Sidoroff [14], Chow and Lu [15] or Murakami and Kamiya [13]. The structure of the model at this level is similar to the vectorial damage model of Krajcinovic [16]. For isothermal processes considered in the study, the free Helmholtz energy depends on both variables:

$$\psi = \psi(\underline{\underline{\varepsilon}}, \underline{\underline{D}}). \tag{1}$$

The thermodynamic forces $\underline{\underline{\sigma}}$ (stress variable) and $\underline{\underline{Y}}$ (damage energy release rate) are classically introduced:

$$\underline{\underline{\sigma}} = \rho \frac{\partial \psi}{\partial \underline{\underline{\varepsilon}}}, \quad \underline{\underline{Y}} = -\rho \frac{\partial \psi}{\partial \underline{\underline{D}}}. \tag{2}$$

In case of isothermal processes, the intrinsic dissipation density is reduced to

$$\Phi = \underline{\underline{Y}} : \underline{\underline{\dot{D}}} \geqslant 0 \tag{3}$$

":" denotes the doubly contracted tensorial product. ψ is a scalar function of both symmetrical tensors $\underline{\underline{\varepsilon}}$ and $\underline{\underline{D}}$. According to the representation theory of tensorial functions ([17,18] or [19]), the most general form of such a scalar isotropic function can be expressed by a combination of 10 basic invariants:

$$\{tr\ \underline{\underline{\varepsilon}}, tr(\underline{\underline{\varepsilon}}^2), tr(\underline{\underline{\varepsilon}}^3), tr\,\underline{\underline{D}}, tr(\underline{\underline{D}}^2), tr(\underline{\underline{D}}^3), tr(\underline{\underline{\varepsilon}}\,.\,\underline{\underline{D}}), tr(\underline{\underline{\varepsilon}}\,.\underline{\underline{D}}^2), tr(\underline{\underline{\varepsilon}}^2.\,\underline{\underline{D}}), tr(\underline{\underline{\varepsilon}}^2.\underline{\underline{D}}^2)\}. \tag{4}$$

As a strong hypothesis, ψ is assumed to be decomposed in two terms

$$\psi(\underline{\underline{\varepsilon}}, \underline{\underline{D}}) = \psi_0(\underline{\underline{\varepsilon}}, \underline{\underline{D}}) + \psi_1(\underline{\underline{D}}) \quad \text{with} \left| \begin{array}{l} \psi_0(\underline{\underline{\varepsilon}} = \underline{\underline{O}}, \underline{\underline{D}}) = 0, \\ \psi_1(\underline{\underline{D}} = \underline{\underline{O}}) = 0. \end{array} \right. \tag{5}$$

The first term ψ_0 is assumed to be quadratic in $\underline{\underline{\varepsilon}}$ (the material is linear elastic for constant damage values) and linear in $\underline{\underline{D}}$. With this restriction, the most general form of the function ψ_0 can be expressed from seven invariants of the symmetrical tensors $\underline{\underline{\varepsilon}}$ and $\underline{\underline{D}}$:

$$\begin{aligned} \rho\psi_0(\underline{\underline{\varepsilon}}, \underline{\underline{D}}) = {} & \tfrac{1}{2}\lambda(tr\,\underline{\underline{\varepsilon}})^2 + \mu\ tr(\underline{\underline{\varepsilon}}^2) + \eta_1\ tr\,\underline{\underline{D}}(tr\,\underline{\underline{\varepsilon}})^2 \\ & + \eta_2\ tr\,\underline{\underline{D}}\,tr(\underline{\underline{\varepsilon}}^2) + \eta_3\ tr\,\underline{\underline{\varepsilon}}\,tr(\underline{\underline{\varepsilon}}\,.\,\underline{\underline{D}}) + \eta_4\ tr(\underline{\underline{\varepsilon}}^2.\,\underline{\underline{D}}) + \eta_5\ tr\,\underline{\underline{D}}, \end{aligned} \tag{6}$$

where λ and μ are the Lamé coefficient of the undamaged material and η_i; $i \in \{1, \ldots, 5\}$ are material constants of the model. Moreover, (5) decreases the number of material coefficients:

$$\psi_0(\underline{\underline{\varepsilon}} = \underline{\underline{0}}, \underline{\underline{D}}) = 0 \quad \Rightarrow \quad \eta_5 = 0. \tag{7}$$

The positive strain is obtained from the positive part of the principal strain. A fourth order tensor can be introduced as in [20]:

$$\underline{\underline{\varepsilon}}^{+} = \underline{\underline{\underline{\underline{P}}}}^{+}(\underline{\underline{\varepsilon}}) : \underline{\underline{\varepsilon}}. \tag{8}$$

One of the basis of the model is the coupling between the active part of the tensor strain $\underline{\underline{\varepsilon}}^{+}$ and the damage tensor $\underline{\underline{D}}$. Mode I of crack opening is mainly covered by this formalism. The free Helmholtz energy is then corrected by

$$\rho\psi_0(\underline{\underline{\varepsilon}}, \underline{\underline{D}}) = \tfrac{1}{2}\lambda(tr\,\underline{\underline{\varepsilon}})^2 + \mu\, tr(\underline{\underline{\varepsilon}}^2) + \eta_1\, tr\,\underline{\underline{D}}(tr\,\underline{\underline{\varepsilon}}^{+})^2 + \eta_2\, tr\,\underline{\underline{D}}\, tr(\underline{\underline{\varepsilon}}^{+^2}) + \eta_3\, tr\,\underline{\underline{\varepsilon}}^{+} tr(\underline{\underline{\varepsilon}}^{+}.\underline{\underline{D}}) + \eta_4\, tr(\underline{\underline{\varepsilon}}^{+^2}.\underline{\underline{D}}). \tag{9}$$

Introducing this active variable generates a loss of symmetry of the free Helmholtz energy:

$$\psi(-\underline{\underline{\varepsilon}}, \underline{\underline{D}}) \neq \psi(\underline{\underline{\varepsilon}}, \underline{\underline{D}}). \tag{10}$$

It can be remarked that the symmetry is also lost in the Gibbs free energy χ of the unilateral scalar model of Mazars ([1,2] or [21]).

$$\chi(-\underline{\underline{\sigma}}, d) \neq \chi(\underline{\underline{\sigma}}, d), \tag{11}$$

where d are internal scalar variables.

The complete unilateral properties mean that a tension path has no incidence on the initial stiffness of a compression path. A necessary condition to achieve this goal is that all coupling terms $(\underline{\underline{\varepsilon}}^{+}, \underline{\underline{D}})$ including $tr\,\underline{\underline{D}}$ vanish in the free Helmholtz energy:

$$\eta_1 = \eta_2 = 0. \tag{12}$$

The stress is deduced from application of (2). The continuity of the stress–strain relation imposes:

$$\eta_1 = \eta_3 = 0. \tag{13}$$

On the opposite case, non-diagonal terms of the stiffness tensor can be affected by the deactivation condition: the stress–strain relation could not remain continuous [5]. It can be noticed that this assumption can be relaxed in some specific cases, as pointed out in [22]. As a first conclusion, the first part of the free Helmholtz energy finally depends on three parameters, the two elastic coefficients of virgin material and one adding coefficient η_4 also denoted as α.

$$\alpha = -\eta_4 \geqslant 0. \tag{14}$$

The non-linear term ψ_1 is chosen to be expressed by

$$\rho\psi_1(\underline{\underline{D}}) = \beta\, tr\lfloor(\underline{\underline{1}} - \underline{\underline{D}})^{-p}\rfloor - p\beta\, tr\,\underline{\underline{D}} - 3\beta \quad \text{with } \beta \geqslant 0 \text{ and } p > 0. \tag{15}$$

This non-linear tensorial function depends on two additional material parameters p and β. Despite this equation is not built on physical arguments (no micro-mechanic approach), this type of function allows to obtain realistic results, qualitatively acceptable regarding quasi-brittle behaviour (unsymmetrical behaviour in tension and compression and dilatancy in compression). The function parameters will be adjusted to fit realistic quasi-brittle materials such as concrete or rock materials.

The stress $\underline{\underline{\sigma}}$ is then reduced to

$$\underline{\underline{\sigma}} = [\lambda(tr\ \underline{\underline{\varepsilon}})]\,\underline{\underline{1}} + 2\mu\,\underline{\underline{\varepsilon}} - \alpha\,\frac{\partial \underline{\underline{\varepsilon}}^{+}}{\partial\,\underline{\underline{\varepsilon}}} : (\underline{\underline{\varepsilon}}^{+}.\,\underline{\underline{D}} + \underline{\underline{D}}\,.\underline{\underline{\varepsilon}}^{+}) \tag{16}$$

$\partial \underline{\underline{\varepsilon}}^{+}/\partial\,\underline{\underline{\varepsilon}}$ is a complex fourth-rank tensor. This tensor has to be distinguished from $\underline{\underline{\underline{\underline{P}}}}^{+}$ (see the Appendix A). The thermodynamic force is then expressed by

$$\underline{\underline{Y}} = \alpha \underline{\underline{\varepsilon}}^{+}.\underline{\underline{\varepsilon}}^{+} - p\beta(\underline{\underline{1}} - \underline{\underline{D}})^{-(p+1)} + p\beta\,\underline{\underline{1}}\,. \tag{17}$$

The thermodynamic force verifies the properties

$$\underline{\underline{Y}}(\underline{\underline{\varepsilon}}, \underline{\underline{D}} = \underline{\underline{0}}) = \alpha \underline{\underline{\varepsilon}}^{+}.\underline{\underline{\varepsilon}}^{+} \quad \text{and} \quad \underline{\underline{Y}}(\underline{\underline{\varepsilon}} = \underline{\underline{0}}, \underline{\underline{D}} = \underline{\underline{0}}) = \underline{\underline{0}}\,. \tag{18}$$

Function (15) permits to exhibit a thermodynamic force uniquely related to the positive strain for the undamaged material. The free Helmholtz energy ψ depends on five parameters: the undamaged elastic parameters λ and μ and three other positive parameters α, β and p.

3. Damage evolution equations

The elastic domain is defined from the loading function:

$$f(\underline{\underline{Y}}) = \sqrt{tr(\underline{\underline{Y}}^{+^2})} - Y_{\mathrm{e}} \leqslant 0 \quad \text{with } \underline{\underline{Y}}^{+} = \underline{\underline{\underline{\underline{P}}}}^{+}(\underline{\underline{Y}}) : \underline{\underline{Y}}\,. \tag{19}$$

The elastic domain is defined in the space of thermodynamic force, expressed in terms of strain and damage variables. Y_e is a positive parameter associated to the size of the elastic domain. The undamaged domain (initial damage vanishing) correspond to a strain domain, defined in the principal strain axes I, II and III:

$$\sqrt{tr(\underline{\underline{\varepsilon}}^{+^4})} \leqslant \frac{Y_e}{\alpha}. \tag{20}$$

The initial elastic domain (20) corresponds to an open domain (in the contraction zone) in the space of principal strain. A characteristic extension strain is naturally introduced:

$$\varepsilon_t = \sqrt{\frac{Y_e}{\alpha}}. \tag{21}$$

The elastic domain is illustrated in Fig. 1 in cases of plane strain ($\varepsilon_{\mathrm{III}} = 0$).

As in [16], the damage evolution equation is derived from standard "flow" rule (see [23] or [24]):

$$\dot{\underline{\underline{D}}} = \dot{\lambda}\,\frac{\partial f}{\partial\,\underline{\underline{Y}}} \quad \text{with } \dot{\lambda} \geqslant 0 \tag{22}$$

$\dot{\lambda}$ is the damage inelastic multiplier. The loading–unloading conditions are summarised in

$$\dot{\lambda} \geqslant 0, \quad f(\underline{\underline{Y}}) \leqslant 0, \quad \dot{\lambda} f(\underline{\underline{Y}}) = 0. \tag{23}$$

Eqs. (22) and (23) can be viewed as the so-called optimality Kuhn–Tucker conditions [25].

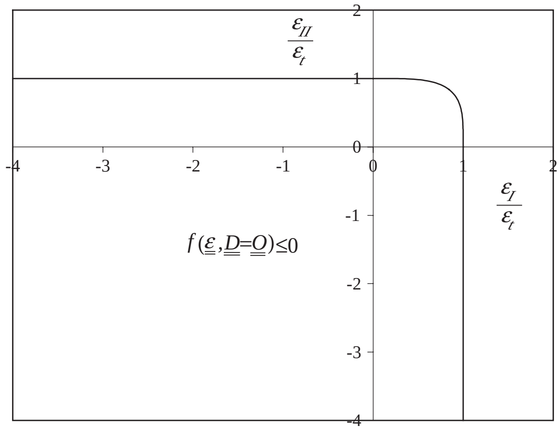

Fig. 1. Representation of the initial elastic domain in plane strain ($\varepsilon_{\mathrm{III}} = 0$).

The damage rate is calculated as (see the Appendix A):

$$\dot{\underline{\underline{D}}} = \frac{\dot{\lambda}}{Y_e}\underline{\underline{Y}}^{+}. \tag{24}$$

As the model is standard, the evolution law (24) is compatible with an acceptable thermodynamic framework (in Eq. (3)). It appears that $\dot{\underline{\underline{D}}}$ and $\underline{\underline{Y}}$ have the same principal directions. As a synthesis, the time-independent damage model depends on six parameters, two classical virgin elastic parameters and four additional positive parameters α, β, p and Y_e whose meaning is now detailed.

4. Uniaxial stress tests—tension test

The study is presented for three-dimensional problems. The uniaxial monotonic tension test is characterised by the simple state of stress:

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{\mathrm{I}} = \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with } \sigma \geqslant 0 \tag{25}$$

The strain and damage tensor are written as

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{\mathrm{I}} = \varepsilon & 0 & 0 \\ 0 & \varepsilon_{\mathrm{II}} & 0 \\ 0 & 0 & \varepsilon_{\mathrm{III}} \end{pmatrix} \quad \text{and} \quad \underline{\underline{D}} = \begin{pmatrix} D_{\mathrm{I}} & 0 & 0 \\ 0 & D_{\mathrm{II}} & 0 \\ 0 & 0 & D_{\mathrm{III}} \end{pmatrix}. \tag{26}$$

In this case, $\underline{\underline{D}}$ and $\underline{\underline{\varepsilon}}$ have the same principal directions. The damage elastic behaviour is obtained from (16):

$$\begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu - 2\alpha D_{\mathrm{I}} & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{pmatrix} \begin{pmatrix} \varepsilon \\ \varepsilon_{\mathrm{II}} \\ \varepsilon_{\mathrm{III}} \end{pmatrix}. \tag{27}$$

In case of monotonic strain increasing, the loading function (19) leads to the strain-damage relation:

$$\alpha\varepsilon^2 - \frac{p\beta}{(1 - D_{\mathrm{I}})^{p+1}} + p\beta = Y_e \Rightarrow D_{\mathrm{I}} = 1 - \left(\frac{p\beta}{\alpha\varepsilon^2 + p\beta - Y_e}\right)^{1/(p+1)} \quad \text{for } \varepsilon \geqslant \varepsilon_t, \tag{28}$$

where the characteristic extension strain ε_t is defined in (21). Moreover, the transversal strains are related to the axial positive strain by

$$\varepsilon_{\mathrm{II}} = \varepsilon_{\mathrm{III}} \quad \text{and} \quad \varepsilon_{\mathrm{II}} = -\frac{\lambda}{2(\lambda + \mu)}\varepsilon = -\upsilon\varepsilon, \tag{29}$$

where υ is the undamaged Poisson's ratio. The non-linear relation is finally obtained for the inelastic branch:

$$\sigma = E\varepsilon - 2\alpha D_{\mathrm{I}}(\varepsilon)\varepsilon, \tag{30}$$

where E is the undamaged Young modulus. This relation does not depend on the damage values D_{II} and D_{III} in the transversal directions.

5. Uniaxial stress tests—compression test

The numerical uniaxial compression test is also studied:

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{\mathrm{I}} = \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with } \sigma \leqslant 0 \quad \text{and} \quad \underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{\mathrm{I}} = \varepsilon & 0 & 0 \\ 0 & \varepsilon_{\mathrm{II}} & 0 \\ 0 & 0 & \varepsilon_{\mathrm{III}} \end{pmatrix}. \tag{31}$$

As in the previous case, $\underline{\underline{D}}$ and $\underline{\underline{\varepsilon}}$ have the same principal directions. The stress–strain relation is written as

$$\begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu - 2\alpha D_{\mathrm{II}} & \lambda \\ \lambda & \lambda & \lambda + 2\mu - 2\alpha D_{\mathrm{III}} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \varepsilon_{\mathrm{II}} \\ \varepsilon_{\mathrm{III}} \end{pmatrix}. \tag{32}$$

Moreover, symmetry considerations simplify the resolution as

$$\varepsilon_{\mathrm{II}} = \varepsilon_{\mathrm{III}} \quad \text{and} \quad D_{\mathrm{II}} = D_{\mathrm{III}}. \tag{33}$$

The strain and the damage variable are linked by

$$D_{\mathrm{II}} = 1 - \left(\frac{p\beta}{\alpha\varepsilon_{\mathrm{II}}^2 + p\beta - \dfrac{Y_e}{\sqrt{2}}} \right)^{1/(p+1)} \quad \text{for } \varepsilon_{\mathrm{II}} \geqslant \frac{\varepsilon_t}{\sqrt[4]{2}}. \tag{34}$$

The following non-linear system is finally obtained

$$\begin{aligned} &\sigma = (\lambda + 2\mu)\varepsilon + 2\lambda\varepsilon_{\mathrm{II}}, \\ &2(\lambda + \mu - \alpha D_{\mathrm{II}}(\varepsilon_{\mathrm{II}}))\varepsilon_{\mathrm{II}} + \lambda\varepsilon = 0. \end{aligned} \tag{35}$$

As in the tension test, the stress–strain curve does not depend on the damage values in the passive direction (direction I for the compression test). The following undamaged elastic material parameters are chosen for a concrete material:

$$\begin{aligned} &E = 3 \times 10^4\,\mathrm{MPa}; \quad \upsilon = 0.2 \;\Rightarrow\; \lambda = 8.33 \times 10^3\,\mathrm{MPa}; \\ &\mu = 1.25 \times 10^4\,\mathrm{MPa}. \end{aligned} \tag{36}$$

The non-linear strain-damage parameters are chosen as

$$\alpha = 1.95 \times 10^4\,\mathrm{MPa}; \quad \beta = 95\,\mathrm{Pa}; \quad p = 2; \quad Y_e = 400\,\mathrm{Pa} \tag{37}$$

in order to obtain the following strength in tension and in compression:

$$\sigma_c = -30\,\mathrm{MPa}; \quad \sigma_t = 4.3\,\mathrm{MPa}, \tag{38}$$

where σ_c is the uniaxial compressive strength and σ_t is the uniaxial tension strength. The response in tension and compression is represented in Fig. 2 with the parameters given in (36) and (37). The ratio of uniaxial compressive strength between uniaxial tension strength is realistic (about 7 for the parameters chosen). Moreover, the shape of the monotonic uniaxial stress curves is typical of classical quasi-brittle materials. Brittle behaviour is observed in tension domain whereas the softening behaviour in compression is more ductile. This is qualitatively analogous to the results of Mazars model [2]. Moreover, the apparent Poisson's ratio is increasing under uniaxial

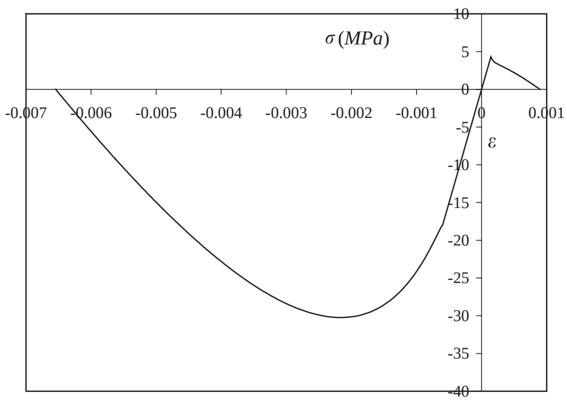

Fig. 2. Numerical test of monotonic uniaxial tension and compression.

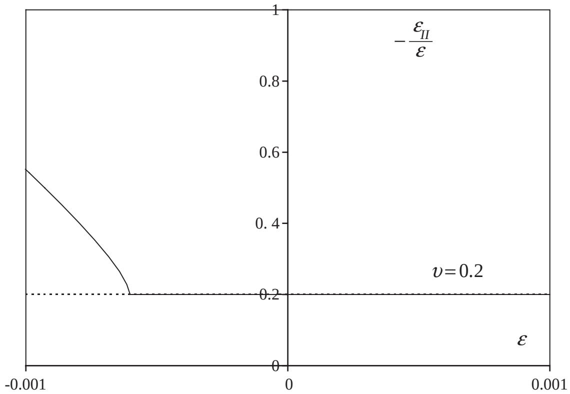

Fig. 3. Evolution of the apparent Poisson's ratio.

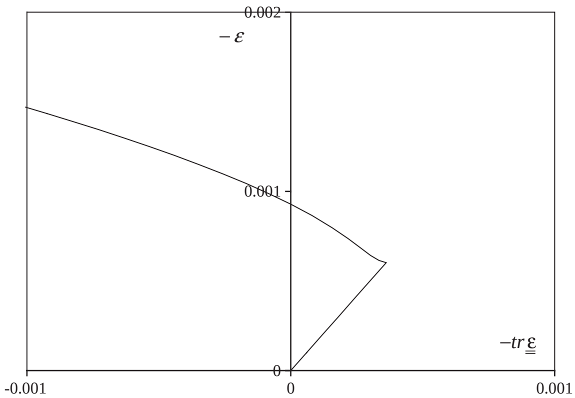

Fig. 4. Dilatance modelling in compression.

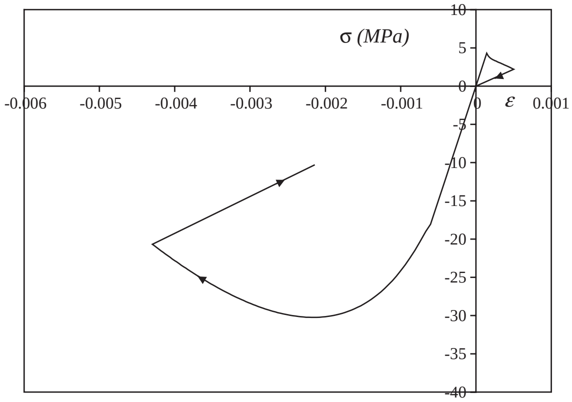

Fig. 5. Complete unilateral effects.

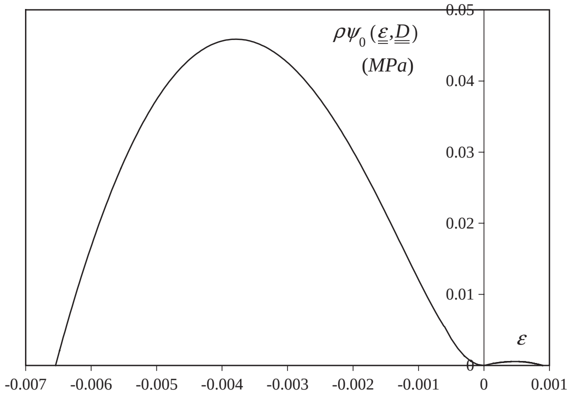

Fig. 6. Anisotropic energy dissipation.

compression (Fig. 3) and has close relation with the dilatancy of quasi-brittle materials (Fig. 4), as observed in common experimental results. This effect is probably amplified in the present simulation. Finally, the unilateral effects is clearly observed in Fig. 5: a tension path has no incidence on a compression path and conversely. Weak unilateral effects can be modelled by loss of condition (12) i.e. with ($\eta_1 = 0$ and $\eta_2 \neq 0$).

The definition of failure condition is difficult for tensorial damage models. Using a similar tensorial damage model, Murakami and Kamiya [13] show that critical damage values depend on the test considered. For isotropic damage models using the single damage variable D, failure occurs when the magnitude of damage D is equal to 1. It also means that the first term of the free Helmholtz energy vanishes. This condition is also chosen in the tensorial framework:

$$\rho\psi_0(\underline{\underline{\varepsilon}},\underline{\underline{D}}) = \tfrac{1}{2}\lambda(tr\ \underline{\underline{\varepsilon}})^2 + \mu\ tr(\underline{\underline{\varepsilon}}^2) - \alpha\ tr(\underline{\underline{\varepsilon}}^{+^2}.\underline{\underline{D}}) = 0. \tag{39}$$

Eq. (39) can be associated to a failure condition. For both tests (uniaxial tension test and uniaxial compression test), (39) is verified when the stress vanishes, as for isotropic damage models. At failure, the principal damage value D_{I} is equal to 0.77 for the monotonic tension test ($D_{\mathrm{II}} = D_{\mathrm{III}} = 0$) whereas D_{II} is equal to 0.96 for the monotonic compression test ($D_{\mathrm{II}} = D_{\mathrm{III}}; D_{\mathrm{I}} = 0$). These characteristic damage values depend on the previous chosen material parameters. It is then confirmed that the critical damage values are strain-dependent (as in [13]). The physical energy criterion can also explain the unsymmetrical cracking mechanism in compression and in traction. $\rho\psi_0$ as an energy per volume is plotted versus the axial strain for both uniaxial tension and compression tests (Fig. 6). The unsymmetrical curves in tension (positive axial strain) and compression (negative axial strain) confirms the predominant tension-mode failure for general path-loading.

6. Time-dependent phenomena

Time-dependent damage models have necessarily to be considered when time-dependent phenomena as creep failure have to be taken into account. The present model is also developed in order to include these time-dependent effects (rate-dependent models, relaxation phenomena, creep failure). The generalisation to time-dependent brittle damage models has first been proposed by Krajcinovic [26], later applied to dynamical loading ([8,21] or [27]). The framework of standard materials is also chosen, in order to verify the second principle of thermodynamics. The creep potential Ω is written as

$$\Omega = \frac{1}{\eta}\frac{\beta}{m+2}\left\langle \frac{f(\underline{\underline{Y}})}{\beta} \right\rangle^{m+2}; \quad \langle\rangle \text{ is the Macauley bracket.} \tag{40}$$

The potential Ω depends on two additional parameters: a dimensionless parameter m and a viscosity constant η. The normality rule is written as

$$\dot{\underline{\underline{D}}} = \frac{\partial \Omega}{\partial \underline{\underline{Y}}}. \tag{41}$$

The damage rate is calculated as (see the Appendix A):

$$\dot{\underline{\underline{D}}} = \frac{1}{\eta}\left\langle \frac{f(\underline{\underline{Y}})}{\beta} \right\rangle^{m} \frac{\underline{\underline{Y}}^{+}}{\beta}. \tag{42}$$

It can been shown that time-independent damage model is a particular case of time-dependent damage models for infinitely slow loading [28,29]. The same analogy has been noticed between viscoplasticity and plasticity theory when the viscosity vanishes [30]. As time-independent damage models can be viewed as the limit of time-dependent damage models when the rate of loading vanishes, (42) can also be used as a particular case of classical time-independent damage models (also-called visco-damage regularisation method). Previous uniaxial compression test and uniaxial tension test (Figs. 2–5) can then be obtained for infinitely slow loading. For such numerical tests, the visco-damage parameters m and η are not mobilised.

7. Conclusions

A consistent damage model applied to quasi-brittle materials has been presented in this paper. The proposed model has been developed in the framework of irreversible thermodynamic theory. Internal variable is reduced to one damage second rank symmetric tensor. The model covers the anisotropic and unilateral aspects, by introducing an active part of the strain-damage process in the free Helmholtz energy. Complete unilateral effect has been modelled (a unidimensional tension path has no incidence on the stiffness of the unidimensional compression path, and conversely), although this strong assumption can be relaxed. The damage law is derived directly from the "normal dissipative mechanism". Uniaxial monotonic tension and compression tests show realistic unsymmetrical behaviour, both for the strength and for the softening damage branch. As experimentally noticed, the behaviour is more brittle in tension than in compression. A time-dependent generalisation has been also suggested in the framework of standard materials and has to be implemented in order to compute phenomena as creep failure (as in [28,29] for much simpler damage models).

It is now well admitted that softening media as quasi-brittle materials exhibit a pathological mesh-dependence of Finite-Element computations. Non-local damage models have been developed in order to regularise the induced strain localisation [31]. Mesh-dependence have been also observed when modelling creep fracture with local models [32]. Non-local implementation of the presented tensorial damage model has then to be implemented for simulations at the structural scale.

Appendix A. On the determination of the derivative operator

The derivative operator acts for both the stress–strain relation (16) and the damage evolution rules (24) or (42). As we will see, the second case can be considered in fact as a particular case of the first one. This derivative operator is given for a two-dimensional system. The part $\rho\psi_0$ of the free Helmholtz energy takes the following form:

$$\rho\psi_0(\underline{\underline{\varepsilon}}, \underline{\underline{D}}) = \tfrac{1}{2}\lambda(tr\ \underline{\underline{\varepsilon}})^2 + \mu\ tr(\underline{\underline{\varepsilon}}^2) - \alpha\ tr(\underline{\underline{\varepsilon}}^{+^2}.\underline{\underline{D}}). \tag{A.1}$$

This energy is explicitly written for a two-dimensional system as

$$\begin{aligned}\rho\psi_0(\underline{\underline{\varepsilon}}, \underline{\underline{D}}) = {} & \tfrac{1}{2}\lambda(\varepsilon_{11}+\varepsilon_{22})^2 + \mu\ tr\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \frac{\gamma_{12}^2}{2}\right) - \alpha\left[D_{11}\left(\varepsilon_{11}^{+^2} + \frac{\gamma_{12}^{+^2}}{4}\right)\right. \\ & \left. + D_{21}\gamma_{12}^{+}(\varepsilon_{11}^{+}+\varepsilon_{22}^{+}) + D_{22}\left(\varepsilon_{22}^{+^2} + \frac{\gamma_{12}^{+^2}}{4}\right)\right] \quad \text{with } \gamma_{12} = \varepsilon_{12}+\varepsilon_{21}.\end{aligned} \tag{A.2}$$

Components of the positive part of the strain tensor can be expressed from the strain tensor itself by the non-linear relation:

$$\begin{aligned}\varepsilon_{11}^{+} &= \frac{h_{\mathrm{I}}\varepsilon_{\mathrm{I}} + h_{\mathrm{II}}\varepsilon_{\mathrm{II}}}{2} + \frac{h_{\mathrm{I}}\varepsilon_{\mathrm{I}} - h_{\mathrm{II}}\varepsilon_{\mathrm{II}}}{2}\cos 2\theta, \\ \varepsilon_{22}^{+} &= \frac{h_{\mathrm{I}}\varepsilon_{\mathrm{I}} + h_{\mathrm{II}}\varepsilon_{\mathrm{II}}}{2} - \frac{h_{\mathrm{I}}\varepsilon_{\mathrm{I}} - h_{\mathrm{II}}\varepsilon_{\mathrm{II}}}{2}\cos 2\theta \quad \text{with } \theta = \tfrac{1}{2}\arctan\frac{\gamma_{12}}{\varepsilon_{11}-\varepsilon_{22}}, \\ \gamma_{12}^{+} &= (h_{\mathrm{I}}\varepsilon_{\mathrm{I}} - h_{\mathrm{II}}\varepsilon_{\mathrm{II}})\sin 2\theta.\end{aligned} \tag{A.3}$$

h_{I} and h_{II} are two heaviside functions of the principal strains:

$$\begin{aligned}h_{\mathrm{I}} &= H\left(\frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \frac{\varepsilon_{11}-\varepsilon_{22}}{2\cos 2\theta}\right), \\ h_{\mathrm{II}} &= H\left(\frac{\varepsilon_{11}+\varepsilon_{22}}{2} - \frac{\varepsilon_{11}-\varepsilon_{22}}{2\cos 2\theta}\right).\end{aligned} \tag{A.4}$$

Eq. (A.3) can be simplified into:

$$\begin{aligned}\varepsilon_{11}^{+} &= \varepsilon_{11}\left(\frac{h_{\mathrm{I}}+h_{\mathrm{II}}}{2} + \frac{h_{\mathrm{I}}-h_{\mathrm{II}}}{2}\cos 2\theta\right) + \frac{\gamma_{12}}{2}\left(\frac{h_{\mathrm{I}}-h_{\mathrm{II}}}{2}\right)\sin 2\theta, \\ \varepsilon_{22}^{+} &= \varepsilon_{22}\left(\frac{h_{\mathrm{I}}+h_{\mathrm{II}}}{2} + \frac{h_{\mathrm{II}}-h_{\mathrm{I}}}{2}\cos 2\theta\right) + \frac{\gamma_{12}}{2}\left(\frac{h_{\mathrm{I}}-h_{\mathrm{II}}}{2}\right)\sin 2\theta, \\ \gamma_{12}^{+} &= \varepsilon_{11}\left(\frac{h_{\mathrm{I}}-h_{\mathrm{II}}}{2}\right)\sin 2\theta + \varepsilon_{22}\left(\frac{h_{\mathrm{I}}-h_{\mathrm{II}}}{2}\right)\sin 2\theta + \gamma_{12}\left(\frac{h_{\mathrm{I}}+h_{\mathrm{II}}}{2}\right).\end{aligned} \tag{A.5}$$

The stress tensor is derived as follows:

$$\sigma_{11} = \frac{\partial \rho \psi_0}{\partial \varepsilon_{11}},\quad \sigma_{22} = \frac{\partial \rho \psi_0}{\partial \varepsilon_{22}},\quad \sigma_{12} = \frac{\partial \rho \psi_0}{\partial \gamma_{12}}. \tag{A.6}$$

The non-linear elastic stiffness is finally obtained from

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{pmatrix} - \alpha(J(\varepsilon_{11}, \varepsilon_{22}, \gamma_{12})) \begin{pmatrix} 2D_{11}\varepsilon_{11}^{+} + D_{12}\gamma_{12}^{+} \\ 2D_{22}\varepsilon_{22}^{+} + D_{12}\gamma_{12}^{+} \\ D_{11}\dfrac{\gamma_{12}^{+}}{2} + D_{12}\varepsilon_{11}^{+} + D_{22}\dfrac{\gamma_{12}^{+}}{2} + D_{12}\varepsilon_{22}^{+} \end{pmatrix}. \tag{A.7}$$

(J) is the Jacobian matrix, defined by

$$(J(\varepsilon_{11}, \varepsilon_{22}, \gamma_{12})) = \begin{pmatrix} \dfrac{\partial \varepsilon_{11}^{+}}{\partial \varepsilon_{11}} & \dfrac{\partial \varepsilon_{22}^{+}}{\partial \varepsilon_{11}} & \dfrac{\partial \gamma_{12}^{+}}{\partial \varepsilon_{11}} \\ \dfrac{\partial \varepsilon_{11}^{+}}{\partial \varepsilon_{22}} & \dfrac{\partial \varepsilon_{22}^{+}}{\partial \varepsilon_{22}} & \dfrac{\partial \gamma_{12}^{+}}{\partial \varepsilon_{22}} \\ \dfrac{\partial \varepsilon_{11}^{+}}{\partial \gamma_{12}} & \dfrac{\partial \varepsilon_{22}^{+}}{\partial \gamma_{12}} & \dfrac{\partial \gamma_{12}^{+}}{\partial \gamma_{12}} \end{pmatrix}. \tag{A.8}$$

One can calculate for instance the Jacobian in the principal basis ($\theta = 0$):

$$(J(\theta = 0)) = \begin{pmatrix} h_{\mathrm{I}} & 0 & 0 \\ 0 & h_{\mathrm{II}} & 0 \\ 0 & 0 & \dfrac{h_{\mathrm{I}}\varepsilon_{\mathrm{I}} - h_{\mathrm{II}}\varepsilon_{\mathrm{II}}}{(\varepsilon_{\mathrm{I}} - \varepsilon_{\mathrm{II}})} \end{pmatrix}. \tag{A.9}$$

As a consequence,

$$D_{12} = 0 \;\Rightarrow\; \sigma_{12} = 0. \tag{A.10}$$

It means that the principal directions of the stress tensor are the same than the ones of the strain tensor, when the principal directions of the strain and the damage tensor coincide. Moreover, in this last case, the constitutive law can be simplified, using

$$\frac{\partial \underline{\underline{\varepsilon}}^{+}}{\partial \underline{\underline{\varepsilon}}} : (\underline{\underline{D}} . \underline{\underline{\varepsilon}}^{+} + \underline{\underline{\varepsilon}}^{+} . \underline{\underline{D}}) = (\underline{\underline{D}} . \underline{\underline{\varepsilon}}^{+} + \underline{\underline{\varepsilon}}^{+} . \underline{\underline{D}}) = 2\, \underline{\underline{D}} . \underline{\underline{\varepsilon}}^{+}. \tag{A.11}$$

Of course, this relation is not true in the general case. As a particular case of (A.11), for an isotropic damage tensor, the following relation holds:

$$\frac{\partial \underline{\underline{\varepsilon}}^{+}}{\partial \underline{\underline{\varepsilon}}} : \underline{\underline{\varepsilon}}^{+} = \underline{\underline{\varepsilon}}^{+}. \tag{A.12}$$

This result is implicitly used in most isotropic or anisotropic damage models including unilateral effects. This also concerns the damage rate equations (24) or (42) in the paper:

$$\frac{\partial \underline{\underline{Y}}^{+}}{\partial \underline{\underline{Y}}} : \underline{\underline{Y}}^{+} = \underline{\underline{Y}}^{+}. \tag{A.13}$$

The proof for a three-dimensional system is more difficult.

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