Available online at www.sciencedirect.com
SCIENCE DIRECT®
International Journal of Engineering Science 43 (2005) 1313–1322
International Journal of Engineering Science
www.elsevier.com/locate/ijengsci

Equivalence principles in continuum damage mechanics

René Souchet

Association Française de Mécanique, Ecole Nationale Superieure de Mecanique et d'Aerotechnique, 13 rue Johann Strauss, 86180 Buxerolles, France

Received 26 March 2005; accepted 4 June 2005
Available online 18 October 2005

Abstract

In this paper, as usual in continuum damage mechanics, an effective continuum is introduced, but this continuum is here considered as an auxiliary body. The main purpose is then to connect together these two materials for a possible comparison, i.e. to propose geometrical and mechanical constraints between these materials. In this paper, we recall briefly the three-terms multiplicative decomposition of the deformation gradient by using a natural geometrical constraint, and we propose a new theoretical method available for obtaining mechanical constraints between the two materials. The proposed approach is then applied to generalize the hypothesis of strain equivalence and the hypothesis of energy equivalence. In this approach, new equivalence principles are obtained, and a new mechanical constraint based on reciprocity is analysed. This paper is restricted to mechanical processes and time-independant plasticity, but large strains are considered.

Keywords: Damage; Effective stress; Equivalence principles; Reciprocity

1. Introduction

In this paper, we are concerned with the coupling of two non-linear material behaviours, viz *plasticity* and *damage* modelised by internal variables, noted respectively (a_i) and (b_a). The first one is caused by the dislocation motion (e.g. in ductile metals), and the second one by the evolution of micro-defects, such micro-cracks (e.g. in brittle materials). Both of them are irreversible and dissipative phenomena, and are well described in micro-mechanics. If the phenomenological elasto-plasticity without damage is well modelised, the elasto-plasticity in presence of damage has not received a complete satisfactory framework. Moreover, many studies are restricted to small strains. We refer to the bibliography [1–4], for some extended developments on damage mechanics.

A first and general approach consists in the exclusive use of the *Thermodynamics of Irreversible Processes* in order to formulate constitutive laws of elasticity, plasticity and damage. This method introduces an a priori free energy, often decomposed in several parts, generally an elastic part, a plastic part and a damage part like in Refs. [5–7]. The description is completed by dissipation potentials connected to plasticity and damage.

E-mail address: souchet.rene@wanadoo.fr

0020-7225/$ - see front matter
doi:10.1016/j.ijengsci.2005.06.003

Although we recognize the ability of this method to treat the actual problem of damage and plasticity, we will not consider this point of view in this paper.

A second approach, not so general, but originated from mechanical basis, may be developed from the concept of *effective stress* [8], first used by Kachanov in1958 and then by many authors. The main idea of this theory is to introduce a fictitious undamaged state, said an *effective continuum*, for a comparison with the real damaged state. Concerning this comparison, the authors give generally an a priori linear relation between the stress tensors of the two states, and complete by a principle linking the elastic strains, except if they use the energy equivalence principle where the stress and strain relations are obtained simultaneously [9]. Only in the paper [10], the kinematics of damage is considered. In the framework of effective stress, we note also the difficulty to take account of plasticity, and, as it is recognised [11–13], the damage mechanics is not still completed.

So it may be useful to bring together the various results concerning the effective stress method and to propose a consistent and unified framework. In this paper, an attempt is made to realize this purpose, by using an auxiliary body having its own kinematics and internal variables. The problem is to connect together the auxiliary and the real bodies [14]. In a first step, kinematics variables are introduced to link the bodies, so defining a kinematic variable of damage. In a second step, a mechanical property is chosen to link the two bodies, so defining constraints between the stress tensors on one part, and the elastic strains on an other part. As it will be clear on examples, this procedure gives the damaged elasticity law from the given virgin law and damage parameters. In this paper, we do not consider the construction of the plasticity laws that may be deduced from the effective plasticity [15,16]. Naturally the evolution laws of damage variables must be obtained from specific hypotheses.

2. Physical and mechanical backgrounds

Let us consider a body B performing some deformation $x = \chi(X, t)$ under mechanical forces and given displacements only, i.e. without thermal effects.

2.1. Micromechanics

Due to the breaking of atomic bondings, microdefects appear after some interval of time, inducing damage throughout the body [17]. The Fig. 1 shows the elasticity degradation in an unidimensional sample; in the second case, permanent strains appear under zero stress.

The three following main physical facts appear in these damaging processes and are the basic concepts used by workers in CDM:

(I) Microdefects are continuously distributed throughout the body B. They are embedded in the matrix material that constitutes the resisting, or effective, body [3].

(II) Elastic properties are weakened, since the Cauchy stress vectors act on degraded surfaces. In small strains, the elasticity coefficients depend on damage parameters [18].

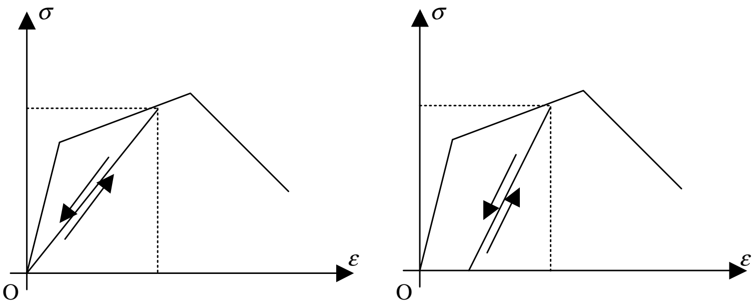

Fig. 1. Unidimensional unloading–reloading (ε, σ) of damage and elasticity.

(III) Movements of dislocations can appear in the matrix material. So plasticity evolves in the virgin parts surrounding the present microdefects [17].

2.2. Real and effective bodies

The first point (I) in Section 2.1 allows to treat the damaged body B as a classical continuum. The deformation gradient F maps the representative material element (RME) $B(X)$ onto its actual position $B(x)$. We recall that F (with $J = \det F > 0$) depends on reversible and irreversible deformations, respectively denoted L and P_{i}, in such a way that the Lee multiplicative decomposition [19,20] showed in Fig. 2

$$F = LP_{\mathrm{i}}, \quad \det L > 0,$$

holds, this decomposition being obtained by an eventually virtual unloading process. We note that the internal variable P_{i} is relevant of both damage and plasticity.

Furthermore this first point (I) puts forward the physical existence of a matrix body B_{r}, collection of all the virgin parts surrounding microdefects. This body B_{r} accompanies the body B, but the two bodies are linked together. So necessarily B_{r} is not a classical continuum, but a fictitious body introduced for mathematical purpose only. In the following we will be concerned by the RME's of the bodies, not by the bodies themselves. At initial time, the two bodies are identical and we can consider that, at the point X, the RME's $B(X)$ and $B_{\mathrm{r}}(X)$ coincides. The free-damage RME $B_{\mathrm{r}}(X)$ possesses its own deformation F_{r} that can be decomposed as (see Fig. 2)

$$F_{\mathrm{r}} = L_{\mathrm{r}}P_{\mathrm{r}}$$

as it was made for F, where the internal variable P_{r} is relevant of plasticity only. However we note that F_{r} is not necessarily a gradient defined on whole the body B_{r}.

The fictitious character of $B_{\mathrm{r}}(X)$ allows to imagine and to draw the two RME's occupying the same place, as it is represented on the Fig. 2, where the unloaded configurations $B(X^*)$ and $B_{\mathrm{r}}(X^*)$ are also drawn.

The concerned displacements in the present study are referring to the RME's at X, so that we take account of local motions only, viz

$$y = x + F(Y - X), \quad y_{\mathrm{r}} = x + F_{\mathrm{r}}(Y - X)$$

where X is a fixed particle. Until now, there exists neither mathematical nor mechanical coupling between the kinematic variables. It will be the main aim of the Section 3 to define such coupling variables.

2.3. Constraints between the two bodies

The RME $B_{\mathrm{r}}(X)$ (i.e. F_{r}) was introduced to take account of the properties of the matrix material. This material represents the initial virgin material, and, so, is equipped of known laws of elasticity and plasticity.

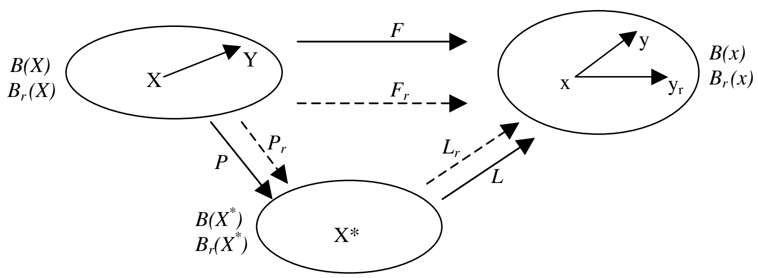

Fig. 2. Lee decompositions and the various configurations.

Due to their definitions, the RME's are strongly associate and this coupling must be defined by some mathematical constraints between the two RME's. Concerning plasticity, the point (III) of the Section 2.1 states that the plasticity laws to be used must be issued, in a way to be defined, from the effective material with the associate variables. This hypothesis is called "*effective stress space plasticity*" [21] and, for the present time, we take account of this hypothesis by assuming that an only family (P, a_k) defines the plastic variables on the two RME's; in particular, we have $P_r = P$, this hypothesis being the first constraint between the two materials.

Concerning elasticity, the mechanical interactions between the real body and the effective body are expressed by coupling relations as it is noted in the point (II) of the Section 2.1. So a mathematical relation between the stress tensors Λ^*, Λ_r^*, necessary for a comparison between stresses, expresses a mechanical relation between the matrix material and the microdefects. This relation will be written as

$$h(\Lambda^*, \Lambda_r^*; b_a) = 0, \quad \sigma = L\Lambda^* L^T$$

where σ is the Cauchy stress tensor and b_a the collective notation for the internal damage parameters. We assume that this constraint must be satisfied whatever elastic evolutions, i.e. the above function h does not depend on elastic kinematic variables.

From the above constraint and the elasticity laws of the materials, independant on plastic variables, written as

$$\Lambda^* = \hat{\Lambda}(C_e; b_a), \quad \Lambda_r^* = \hat{\Lambda}_r((C_r)_e)$$

where $C_e = L^T L$ is the elastic strain, it is seen that the above constraint between stresses may be transformed into a constraint between the elastic evolutions, viz

$$g(C_e, (C_r)_e; b_a) = 0, \text{ or } \quad g(L, L_r; b_a) = 0$$

The functions h and g express mechanical constraints between the matrix material and the real material, so defining some class of materials undergoing appropriate evolutions, but not for all materials and all possible evolutions : the couple (g, h) will be said *functions of correlations* or *constraints* between the two materials.

2.4. Inverse problem

In the above section we have seen that, under the existence of a virtual body, the relation between elastic strains is obtained from the constraint between stresses and the two elasticity laws. But generally the damaged elasticity law is not known. However, if by inverse methods we know the two constraints h and g, then the damaged elasticity law is known by eliminating the variable $(C_r)_e$.

As an example, if we have both the a priori constraints

$$(C_r)_e = C_e, \quad \Lambda_r^* = \mathbb{M}(b_a) : \Lambda^*$$

respectively said the "*strain equivalence principle*" and the "*projection of stresses by the damage effect tensor*", then the damaged elasticity law is given by

$$\Lambda^* = \mathbb{M}(b_a)^{-1} : \hat{\Lambda}_r(C_e)$$

In these formulae, $\mathbb{M}$ is a fourth order tensor and (:) signifies the double contraction $(M_{ijkl}C_{ij})$. These hypotheses are often used by the workers in CDM to obtain the damaged elasticity law [3,22].

In practical applications, such constraints (g, h) will be obtained as consequences of various mechanical properties connecting the two materials. In each case that will be considered, it will be necessary to prove that these properties allow the effective construction of the functions of correlations (g, h). Before that, we recall briefly the kinematic variables that correlate the two RME's.

3. Geometric damage parameter

In this part, we recall briefly the construction of a multiplicative decomposition of the deformation gradient F, proposed in the paper [24], whatever the conditions of the mechanical coupling between the two bodies.

3.1. Geometric parameter of comparison

A comparison between the local deformations F and F_r is naturally made by the difference of the transformed vectors $(y - x)$ and $(y_r - x)$ from the same initial material line $(Y - X)$, viz

$$(y-x)-(y_r-x)=(F-F_r)(Y-X)$$

However F and F_r are regular maps, $J = \det F > 0$ and $J_r = \det F_r > 0$, and it results

$$F - F_r = F_r(D-I) = (d-I)F_r, \quad D = F_r^{-1}F, \quad d = FF_r^{-1}$$

so introducing the coupling variables D and d [24] as they are showed on Fig. 3. Note that D and d are not internal variables since they take into account both reversible and irreversible processes.

We examine some remarkable properties of D or d, illustrating the ability of this parameter to couple the bodies. First, we have between the Cauchy-Green strain tensors

$$C = D^T C_r D, \quad D = U_r^{-1}(R_r^T R)U$$

where in the last equality use was made of the classical polar decomposition $F = RU$.

Now we use the Nanson formula for the surface element $N\mathrm{d}A$ of the body B undergoing the motion $x = \chi(X,t)$, viz

$$n\,\mathrm{d}a = [F]N\,\mathrm{d}A, \quad [F] = (\det F)F^{-T}$$

using the notation $F^{-T} = (F^{-1})^T$ and the Nanson projector $[\cdot]$. In an analogous way, the Nanson formula may be extended to the surface element $N_r\mathrm{d}A_r$ of the RME $B_r(X)$ undergoing the local motion F_r, viz

$$n_r\,\mathrm{d}a_r = [F_r]N_r\,\mathrm{d}A_r$$

As a second property, let $N\mathrm{d}A$, $N_r\mathrm{d}A_r$ be two initial surface elements of the RME's, and $n\,\mathrm{d}a$, $n_r\,\mathrm{d}a_r$ the respective actual surface elements. If we consider a same initial element $N\mathrm{d}A = N_r\mathrm{d}A_r$, then the transformed "*associate elements*" $n\,\mathrm{d}a$ and $n_r\,\mathrm{d}a_r$ are connected by

$$n\,\mathrm{d}a = [d]n_r\,\mathrm{d}a_r, \quad n\,\mathrm{d}a - n_r\,\mathrm{d}a_r = ([d]-I)n_r\,\mathrm{d}a_r$$

The last difference may be considered as the overall area of microvoids due to both damage and applied stresses.

A third property concerns the void volume fraction. Since the initial volume of the RME's are the same, we have

$$\mathrm{vol}.B_r^t(x)/\mathrm{vol}.B^t(x) = J_r/J = (\det d)^{-1}$$

The void volume fraction $f = 1 - J_r/J$ is then given by $(1-f) = (\det d)^{-1}$. The three above properties show that d or D is a significant parameter of comparison between the two evolutions F and F_r. These properties will be also used in the comparison of the unloaded RME's as it is seen in the next sections.

3.2. Geometric damage parameter

As we have said, the above parameter d (or D) takes account of both reversible and irreversible strains. In order to define internal variables, it is necessary to remove the elastic deformations L and L_r. This is made by using the unloaded configurations $B(X^*)$ and $B_r(X^*)$, i.e.P_i and P_r

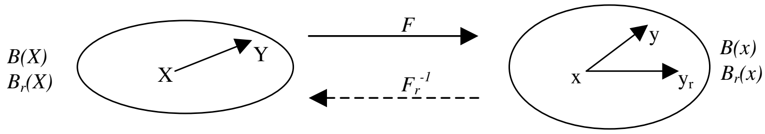

Fig. 3. Local maps $D = F_r^{-1}F$ and $d = FF_r^{-1}$.

$$P_{\mathrm{i}} = L^{-1}F, \quad P_{\mathrm{r}} = L_{\mathrm{r}}^{-1}F_{\mathrm{r}}$$

obtained by the Lee distressing processes (see Fig. 2).

As it was made for F an F_{r} in Section 3.1, a comparison between the two intermediate unloaded configurations P_{i} and P_{r} can be made by the objective geometric parameters

$$D^* = P_{\mathrm{r}}^{-1}P_{\mathrm{i}} \quad \text{or} \quad d^* = P_{\mathrm{i}}P_{\mathrm{r}}^{-1}$$

Naturally these parameters are internal variables like P_{i} and P_{r}. Now $P_{\mathrm{i}} = d^*P_{\mathrm{r}}$ contains both plasticity and damage effects, and we must separate the damage effects from the plasticity effects. But, as a consequence of the point (III) in Section 2.1, P_{r} is also the plastic evolution P in the real material ($P = P_{\mathrm{r}}$) as it was said in Section 2.3. So, as a consequence, the internal variable d^* is related to damage only. Due to its geometric definition, d^* may be viewed as linked to the irreversible non-closing of microdefects and appears as a main effect in damage mechanics. Often this geometric variable d^* is the only considered damage parameter.

It results a multiplicative decomposition of the true deformation gradient F (Souchet, 2004)

$$F = LPD^* = Ld^*P \quad (\text{and } F_{\mathrm{r}} = L_{\mathrm{r}}P,\ d = Ld^*L_{\mathrm{r}}^{-1})$$

involving two materials, not a single one as in the Lee decomposition. The velocity gradient field may be decomposed under the form of three terms

$$u' = \dot{F}F^{-1} = \dot{L}L^{-1} + L(\dot{P}P^{-1})L^{-1} + (LP)(\dot{D}^*D^{*^{-1}})(LP)^{-1}$$

the meaning of each of them being clear in damage mechanics. Such decomposition was first obtained in the paper [10] by an other way.

A simplification arises if the geometric parameter d^* is not the main damage variable. Then we may assume that $F = L\ P$(i.e. $D^* = I$) as in the Lee decomposition.

4. Constraints based on the stress vector

In this part we consider materials where damage involves the geometric parameter d^* only, under some classes of eligible evolutions. It is the more popular example of constraints to make known the damaged elasticity law [13,23,25].

Let the RME's $B(X)$ and $B_{\mathrm{r}}(X)$ evolve such that (F, σ) and $(F_{\mathrm{r}}, \sigma_{\mathrm{r}})$ represent their actual states. As a matter of fact, there is no reason to have the equality of the Cauchy vectors on associate area elements whatever the materials and the damaging evolutions of the two RME's, i.e. generally we have

$$\sigma_{\mathrm{r}}n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}} \neq \sigma n\,\mathrm{d}a \quad (\text{if } n\,\mathrm{d}a = [d]n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}})$$

when no restrictions are imposed on materials and damaging evolutions.

Now the question is: What are the conditions for connecting the two RME's, i.e. for determining constraints (g, h), in order to have

$$\sigma_{\mathrm{r}}(n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}}) = \sigma n\,\mathrm{d}a \quad (\text{if } n\,\mathrm{d}a = [d]n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}})?$$

On a mechanical point of view, if the above inequality is evident, the existence of coupling such that the associate area elements $n\,\mathrm{d}a$ and $n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}}$ are submitted to the same Cauchy force may be justified as in the pioneering work of Kachanov. In fact, we can consider that there exists some materials and damaging evolutions, where the resisting area is the area $n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}}$ without voids. By definition this area is loaded by the applied force $\sigma n\,\mathrm{d}a$. Since the Cauchy force on $n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}}$ is defined by $\sigma_{\mathrm{r}}(n_{\mathrm{r}}\,\mathrm{d}a_{\mathrm{r}})$, we have the above required equality.

Now we solve the above mathematical problem. First, by using the condition between the associate area elements and the arbitrariness of these elements, we obtain

$$\sigma_{\mathrm{r}} - \mathrm{sym}(\sigma[d]) = 0, \quad \sigma - \mathrm{sym}(\sigma_{\mathrm{r}}[d]^{-1}) = 0$$

for the symmetric part of the effective stress tensor. Now, this constraint is written by using the stress tensors defined on the intermediate configurations, so obtaining

$$\Lambda_{\mathrm{r}}^* - \mathrm{sym}((\det D_{\mathrm{e}})(\det d^*)D_{\mathrm{e}}\Lambda^*d^{*-T}) = 0, \quad D_{\mathrm{e}} = L_{\mathrm{r}}^{-1}L$$

or equivalently

$$\Lambda^* - \mathrm{sym}((\det D_e)^{-1}(\det d^*)^{-1} \quad D_e^{-1}\Lambda_r^* d^{*T}) = 0$$

But, as a constraint between stresses defined in Section 2.3, this relation does not depend on evolutions. So it results

$$(\det D_e)D_e = [(\det d^*)K^*]^{-1}, \text{ or } D_e = (\det d^*)^{-1/4}(\det K^*)^{1/4}K^{*-1},$$
$$\Lambda_r^* = \mathrm{sym}(K^{*-1}\Lambda^* d^{*-T}), \quad \Lambda^* = \mathrm{sym}(K^*\Lambda_r^* d^{*T})$$

where K^* is some second order tensor depending on damage parameters only. Furthermore, since $C_e = D_e^T(C_r)_e D_e$ (Section 3.1), we can write explicitly

$$(C_r)_e = (\det d^*)^{1/2}(\det K^*)^{-1/2}K^{*T}C_e K^*$$

So the net stress tensor scheme gives simultaneously the two relations defining the functions g and h introduced in Section 2.3. Therefore we have proved the existence of a virtual body associate to the damaged body by the proposed mechanical coupling, i.e. the existence of Kachanov materials and damaging evolutions.

Then the damaged elasticity law $\hat{\Lambda}$ is given by $\Lambda^* = \mathrm{sym}(K^*\hat{\Lambda}_r((C_r)_e d^{*T})$ where $(C_r)_e$ is given by the above formula as a function of (d^*, K^*, C_e), recalling that the elasticity law $\hat{\Lambda}_r$ of the virgin matrix material is known.

This general result is now applied to the more popular example that consists in the choice of $K^* = (\det d^*)^{-1}I$, so that $(C_r)_e = C_e$. This condition is the well-known ‘‘*strain equivalence principle*’’ (Lemaître, 1992). In this case we have the damaged elasticity law

$$\Lambda^* = \mathrm{sym}((\det d^*)^{-1}\hat{\Lambda}_r(C_e)d^{*T})$$

a real simplification for the forthcoming calculus, in spite of the presence of the symmetrisation operator.

As a new particular result, the second example consists in the choice of $K^* = d^*$, so that

$$(C_r)_e = d^{*T}C_e d^*, \quad \Lambda^* = d^*\Lambda_r^* d^{*T}$$

relations being satisfied without any condition of symmetrisation, but containing a damage condition on elastic strains.

5. Constraints based on energy

In this second example, the constraints (g, h) are obtained from elastic energy considerations [9,26], leading to a symmetric effective stress tensor. Although the relevant results were often used by workers in damage mechanics, there exists no satisfying mechanical interpretation as in the above net stress tensor scheme. Here we propose the following method.

Let the two RME's evolve such that (u'_e, σ) and $((u'_r)_e, \sigma_r)$ represent their actual elastic states where u' denote the strain rate. Naturally, there is no reason to have the equality of the incremental elastic strain energies, i.e. generally

$$\int_{B(x)} \sigma : u'_e \,\mathrm{d}y \neq \int_{B_r(x)} \sigma_r : (u'_r)_e \,\mathrm{d}y_r, \quad \text{if} \quad y - x = \mathrm{d}(y_r - x), \quad u'_e = \dot{L}L^{-1}$$

if no restrictions are imposed on materials and damaging evolutions.

Now the question is: what are the conditions for connecting the two bodies, i.e. for obtaining constraints (g, h), in order to have

$$\int_{B(x)} \sigma : u'_e \,\mathrm{d}y = \int_{B_r(x)} \sigma_r : (u'_r)_e \,\mathrm{d}y_r \quad \text{if} \quad y - x = \mathrm{d}(y_r - x)?$$

In order to find the associate constraints (g, h), we suppose first that all the internal variables are kept constant. By using the reference configuration $B(X)$, this relation is written

$$\sigma : u'_e(\det F) = \sigma_r : (u'_r)_e(\det F_r)$$

But $u'_{\rm e} = \dot{L}L^{-1}$ and $\sigma = L\Lambda^* L^T$, so that we have by taking account of the symmetry of the stress tensors

$$(\det d)\Lambda^* : \dot{C}_{\rm e} = \Lambda^*_{\rm r} : (\dot{C}_{\rm r})_{\rm e}$$

Now we use the identities

$$\det d = (\det d^*)(\det d_{\rm e}), \quad \det d_{\rm e} = (\det C_{\rm e})^{1/2}/(\det (C_{\rm r})_{\rm e})^{1/2}$$

to obtain

$$\Lambda^* : ((\det d^*)(\det C_{\rm e})^{1/2}\dot{C}_{\rm e}) = \Lambda^*_{\rm r} : (\det(C_{\rm r})_{\rm e}^{1/2}(\dot{C}_{\rm r})_{\rm e})$$

This last relation is a constraint between stress tensors whatever the elastic evolutions if we have separately for the functions (g, h)

$$(\det(C_{\rm r})_{\rm e})^{1/2}(\dot{C}_{\rm r})_{\rm e} = (\det d^*)\mathbb{K}^* : ((\det C_{\rm e})^{1/2}\dot{C}_{\rm e}$$

$$\Lambda^* = \mathbb{K}^{*T} : \Lambda^*_{\rm r}$$

where $\mathbb{K}^*$ is a fourth order operator depending on damage variables including d^* (actually constant). Naturally this fourth order tensor must respect some symmetry properties. The above result generalizes the well-known "*energy equivalence principle*" (Cordebois, Sidoroff, 1983).

Untill now the damage variables were kept constant. But we assume that these relations may be extended to the general case, so obtaining the existence of a virtual body associate to the damaged body.

These relations have been used exclusively for small elastic strains, i.e.

$$L = I + N_{\rm e}, \quad C_{\rm e} = I + 2N_{\rm e}, \quad \det C_{\rm e} = 1 + 2{\rm tr}N_{\rm e}$$

where $N_{\rm e}$ is an infinitesimal term. Since under this approximation we can write

$$(\det C_{\rm e})^{1/2}\dot{C}_{\rm e} = 2\dot{N}_{\rm e}$$

we obtain by integrating the constraints (g, h) under an integral form, viz

$$(N_{\rm r})_{\rm e} = (\det d^*)\mathbb{K}^* : N_{\rm e}$$

$$\Lambda^* = \mathbb{K}^{*T} : \Lambda^*_{\rm r}$$

(often $d^* = I$) recalling that $\mathbb{K}^*$ is constant for purely elastic evolutions. Using these relations we have the following property concerning the strain energies

$$\Lambda^*_{\rm r} : (N_{\rm r})_{\rm e} = (\det d^*)\Lambda^* : N_{\rm e}, \qquad \text{or} \qquad \Lambda^*_{\rm r} : (N_{\rm r})_{\rm e} = \Lambda^* : N_{\rm e}$$

the second form (Cordebois and Sidoroff, 1983) if the geometric variable d^* is not considered. Then, the elasticity law can be given under the explicit form

$$\Lambda^* = (\det d^*)(\mathbb{K}^{*T}\mathbb{\Lambda}_{\rm r}\mathbb{K}^*) : N_{\rm e}$$

where $\mathbb{\Lambda}_{\rm r}$ is the stiffness tensor of the matrix material (known constant stiffnesses).

6. Constraints based on reciprocity

In this third example, use is made of reciprocity considerations. Let (F, σ) and $(F_{\rm r}, \sigma_{\rm r})$ be the actual states of the two RME's. We assume that all the internal variables are kept constant, and we note by $\Delta_{\rm e}$,$(\Delta_{\rm r})_{\rm e}$ the respective elastic increments of the two RME's. As in the precedent cases, and for the same reasons, we have

$$\Delta_{\rm e} \cdot \sigma_{\rm r} n_{\rm r}\, {\rm d}a_{\rm r} \neq (\Delta_{\rm r})_{\rm e} \cdot \sigma n\, {\rm d}a \quad (\text{if } n\, {\rm d}a = [d]n_{\rm r}\, {\rm d}a_{\rm r})$$

if no restrictions are imposed on materials and damaging evolutions.

Now the question is: what are the conditions for connecting the two RME's, i.e. for obtaining constraints (g, h), in order to have

$$\Delta_{\rm e} \cdot \sigma_{\rm r} n_{\rm r}\, {\rm d}a_{\rm r} = (\Delta_{\rm r})_{\rm e} \cdot \sigma n\, {\rm d}a \quad (\text{if } n\, {\rm d}a = [d]n_{\rm r}\, {\rm d}a_{\rm r})?$$

On a mechanical point of view the above inequality is evident. We recall that the classical reciprocity theorem concerns two states of a same elastic body, a condition not satisfied here, even if the two RME's may be viewed as two different physical states concerning the same initial body. Like in the energy equivalence principle, no mechanical justification appears a priori. To solve the problem, let the elastic increments written under the form

$$\Delta_e/\Delta t = (\dot{L}L^{-1})(y - x)$$

and an analogous expression for the RME $B_r(X)$. The above reciprocity condition is written, since $(y - x) = [d]\,(y_r - x)$,

$$\sigma_r(\dot{L}L^{-1})d = [d]^T\sigma(\dot{L}_r L_r^{-1}), \quad \text{or} \quad \sigma_r = \mathrm{sym}([d]^T\sigma(\dot{L}_r d^{*-1}\dot{L}^{-1})$$

This relation is rewritten as

$$\Lambda_r^* = \mathrm{sym}((\det d)d^{*-1}\Lambda^*(L^T\dot{L}_r)d^{*-1}(L_r^T\dot{L})^{-1})$$

But since this constraint is independent on elastic evolutions, we have for the functions (g, h)

$$(\det d)(L^T\dot{L}_r)d^{*-1} = K^*(L_r^T\dot{L})$$
$$\Lambda_r^* = \mathrm{sym}(d^{*-1}\Lambda^*K^*)$$

where K^* is a second order tensor depending on damage variables. The two above relations show the existence of a virtual body coupled to the real body such that the reciprocity condition is satisfied.

Now we examine the case of small elastic strains recalling that

$$L = I + N_e, \quad C_e = I + 2N_e, \quad \det C_e = 1 + 2\mathrm{tr}N_e$$
$$(\det d)L^T\dot{L}_r = (\det d^*)\dot{(N_r)}_e, \quad L_r^T\dot{L} = \dot{N}_e$$

After integration, the condition on evolutions is written under the form

$$(\det d^*)(N_r)_e d^{*-1} = K^*N_e$$

A particular case is obtained if we choose $K^* = (\det d^*)d^{*-T}$. Then, we obtained the constraints (g, h) under the forms

$$\Lambda_r^* = d^{*-1}\Lambda^*d^{*-T}$$
$$(N_r)_e d^{*-1} = d^{*-T}N_e$$

We note the a priori symmetry of the effective stress tensor, unlike the effective stress tensor in the net stress scheme.

Under this form, this last relation is compatible with the strain equivalence principle, since $N_e = (N_r)_e$ leads to impose the symmetry of $N_e d^{*-1}$. We note the a priori symmetry of the effective stress tensor. This last form of the effective stress tensor was partially used in particular in (Lemaître et al., 2000).

7. Conclusion

In this paper, we have proposed a consistent and unified framework in damage mechanics, based on the presence of an undamaged fictitious body accompanying the damaged real body. A kinematic coupling between both the bodies was realized by means of an internal damage parameter, analogous to the damage tensor of Murakami. But the coupling between stresses on one part, and the coupling between elastic strains on an other part, were obtained from a single mechanical constraint. The two classical principles of strain equivalence and energy equivalence were reexamined and generalised. Several examples were given in order to illustrate the method and a new constraint based on reciprocity was proposed.

References

[1] J. Betten, Mathematical modeling of materials behavior under creep conditions, Appl. Mech. Rev. 54 (2) (2001) 107–132.
[2] D. Krajcinovic, Damage mechanics: accomplishments, trends and needs, Int. J. Solids Struct. 37 (2000) 267–277.
[3] J. Lemaître, A Course on Damage Mechanics, Springer-Verlag, Berlin, 1992.
[4] G.Z. Voyiadjis, P.I. Kattan, A plasticity-damage theory for large deformation of solids. I. Theoretical formulation, Int. J. Eng. Sci. 30 (1992) 1089–1108.
[5] N.R. Hansen, H.L. Schreyer, A thermodynamically consistent framework for theories of elastoplasticity coupled with damage, Int. J. Solids Struct. 31 (3) (1994) 359–389.
[6] K. Hayakawa, S. Murakami, Y. Liu, An irreversible thermodynamic theory for elastic–plastic materials, Eur. J. Mech. A/Solids 17 (1998) 13–32.
[7] P. Longère, A. Dragon, H. Trumel, T. de Resseguier, X. Deprince, E. Petitpas, Modèle de comportement dynamique d'un matériau en présence de l'endommagement par cisaillement adiabatique, J. Phys. IV France 12 (2002) 327–334.
[8] J.L. Chaboche, Le concept de contrainte effective appliqué à l'élasticité et à la viscoplasticité en présence d'un endommagement isotrope. Colloques intern. du CNRS, 295 (1983) 737–760.
[9] J.P. Cordebois, F. Sidoroff, Damage induced elastic anisotropy, Colloques intern. du CNRS 295 (1983) 761–774.
[10] G.Z. Voyiadjis, T. Park, The kinematics of damage for finite-strain elasto-plastic solids, Int. J. Eng. Sci. 37 (1999) 803–830.
[11] J. Lemaître, R. Desmorat, M. Sauzay, Anisotropic damage law of evolution, Eur. J. Mech. A/Solids 19 (2000) 187–208.
[12] F. Cormery, H. Welemane, A critical review of some damage models with unilateral effect, Mech. Res. Commun. 29 (2002) 391–395.
[13] C.K. Soh, Y. Liu, Y. Yang, Y. Dong, A displacement equivalence-based damage model for brittle materials, J. Appl. Mech. 70 (2003) 681–695.
[14] R. Souchet, Mechanical approach of plasticity in the presence of damage, in: Proceedings of the 7th International Conference on Comp. Struct. Technol., Lisbon, Portugal, 2004.
[15] M. Olsson, M. Ristinmaa, Damage evolution in elasto-plastic materials—material response due to different concepts, Int. J. Damage Mech. 12 (2003) 115–139.
[16] K. Saanouni, K. Nesnas, Y. Hammi, Damage modeling in metal forming processes, Int. J. Damage Mech. 9 (2000) 196–217.
[17] D. François, A. Pineau, A. Zaoui, Comportement Mécanique des Matériaux, Hermes, Paris, France, 1992.
[18] S. Baste, B. Audoin, On internal variables in anisotropic damage, Eur. J. Mech., A/Solids 10 (1991) 587–606.
[19] E.H. Lee, Elastic–plastic deformation at finite strains, J. Appl. Mech. 36 (1968) 1–6.
[20] R. Souchet, Leçons sur les grandes déformations, Cépaduès-Editions, Toulouse, France, 2001.
[21] J.W. Ju, On energy-based coupled elastoplastic damage theories: constitutive modeling and computational aspects, Int. J. Solids Struct. 25 (7) (1989) 803–833.
[22] J.C. Simo, J.W. Ju, Strain- and stress-based continuum damage models. I. Formulation, Int. J. Solids Struct. 23 (1987) 821,840.
[23] L.M. Kachanov, On rupture time under condition of creep, Izvestiya Akademii Nauk. SSSR. Otd. Tekhn, Nauk 8, 1958, pp. 26–31.
[24] R. Souchet, Concerning the fictitious continuum in damage mechanics, Int. J. Eng. Sci. 41 (2003) 1975–1988.
[25] S. Murakami, Mechanical modeling of material damage, J. Appl. Mech. 55 (1988) 280–286.
[26] Q.M. Li, Energy correlations between a damaged macroscopic continuum and its sub-scale, Int. J. Sol. Struct. 37 (2000) 4539–4556.