Chapter 3

Constitutive Theory

Section 3.1 Viscoplasticity

We saw in the preceding chapter that while "yielding" is the most striking feature of plastic behavior, the existence of a well-defined yield stress is the exception rather than the rule. It so happens, however, that mild steel, which belongs to this exceptional class, is one of the most commonly used of metals, and attempts at a theoretical description of its behavior preceded those for other metals; such attempts naturally incorporated a yield criterion as an essential feature of what came to be known as plasticity theory, as well as of a later development, known as viscoplasticity theory, which takes rate sensitivity into account.

It should be pointed out that while most workers in solid mechanics use "viscoplasticity" in its classical meaning (see Prager [1961]), that is, to denote the description of rate-dependent behavior with a well-defined yield criterion, this usage is not universal. Others, following Bodner [1968], use the term for models of highly nonlinear viscoelastic behavior, without any elastic range, that is characteristic of metals, especially at higher temperatures. Such models are discussed in 3.1.3. 3.1.1 is limited to models of classical viscoplasticity. Both classes of models are subclasses of the internal-variable models presented in Section 1.5. In 3.1.2, rate-independent plasticity, the foundation for most of the remainder of this book, is derived as a limiting case of classical viscoplasticity.

3.1.1. Internal-Variable Theory of Viscoplasticity

Yield Surface

As in Section 1.5, let $\boldsymbol{\xi}$ denote the array of internal variables $\xi_1, ..., \xi_n$. If there is a continuous function $f(\boldsymbol{\sigma}, T, \boldsymbol{\xi})$ such that there exists a region in the space of the stress components in which (at given values of $T, \boldsymbol{\xi}$)

$f(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) < 0$, and such that the inelastic strain-rate tensor $\dot{\boldsymbol{\varepsilon}}^i$ vanishes in that region but not outside it, then this region constitutes the aforementioned elastic range, and $f(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) = 0$ defines the yield surface in stress space; the orientation of the yield surface is defined in such a way that the elastic range forms its interior. A material having such a *yield function* $f(\cdot)$ is viscoplastic in the stricter sense. This definition, it should be noted, does not entail the simultaneous vanishing of *all* the internal-variable rates $\dot{\xi}_\alpha$ in the elastic region; if such were the case, strain-aging as described in the preceding chapter would not be possible, since it requires an evolution of the local structure while the material is stress-free. However, this proviso is of significance only for processes whose time scale is of the order of magnitude of the relaxation time for strain-aging, which for mild steel at ordinary temperatures is of the order of hours. Thus, for a process lasting a few minutes or less, the internal variables governing strain-aging are essentially constant and their rates may be ignored. For the sake of simplicity, we adopt a somewhat more restricted definition of viscoplasticity, according to which all the internal-variable rates vanish in the elastic region, that is, the functions $g_\alpha(\boldsymbol{\sigma}, T, \boldsymbol{\xi})$ constituting the right-hand sides of the rate equations (1.5.1) are assumed to vanish whenever $f(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) \le 0$. In particular, this definition includes all those models (such as that of Perzyna [1971]) in which the rates of the internal variables depend linearly on $\dot{\boldsymbol{\varepsilon}}^i$.

In view of this definition it now becomes convenient to redefine the g_α as $g_\alpha = \phi h_\alpha$, where ϕ is a scalar function that embodies the rate and yielding characteristics of the material, with the property that $\phi = 0$ when $f \le 0$ and $\phi > 0$ when $f > 0$. Such a function was introduced by Perzyna [1963] in the form $\gamma(T){<}\Phi(f){>}$, where $\gamma(T)$ is a temperature-dependent "viscosity coefficient" (actually an inverse viscosity, or fluidity), and the notation ${<}\Phi(f){>}$ is defined — somewhat misleadingly — as

$$<\Phi(f)> = \begin{cases} 0 & \text{for } f \le 0 \\ \Phi(f) & \text{for } f > 0 \end{cases}$$

(the more usual definition of the operator $< \cdot >$ is given below). Note that our definition of ϕ is determinate only to within a multiplicative scalar; that is, if λ is a positive continuous function of the state variables, then ϕ may be replaced by ϕ/λ and the h_α by λh_α without changing the rate equations.

Hardening

The dependence of the yield function f on the internal variables ξ_α describes what are usually called the *hardening* properties of the material. The relationship between this dependence and the behavior of the material can be understood by considering a stress $\boldsymbol{\sigma}$ that is close to the yield surface but outside it, that is, $f(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) > 0$. In particular, let us look at a case of uniaxial stress in a specimen of a material whose static stress-strain curve is given by the solid curve of Figure 3.1.1, which shows both rising ("hardening") and falling ("softening") portions. If the material is viscoplastic, then its behavior is elastic at points below the curve, and viscoelastic at points above the curve — that is, the curve represents the yield surface.

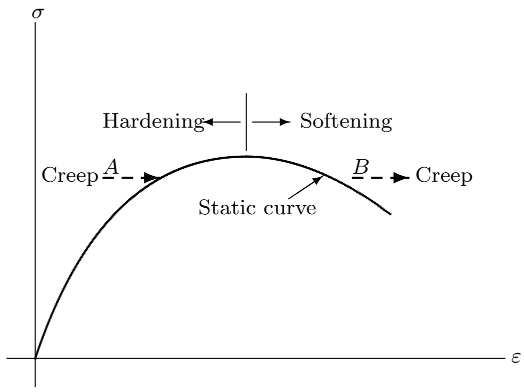


Figure 3.1.1. Hardening and softening in viscoplasticity: relation to creep and static curve.

If the stress is held constant at a value above the static curve, creep occurs, resulting in increasing strain as shown by the dashed horizontal lines. If the initial point is, like A, above the rising portion of the static stress-strain curve, then the creep tends toward the static curve and is bounded, while if it is, like B, above the falling portion, then the creep tends away from the static curve and is unbounded. Since the points on the static stress-strain curve are in effect those on the yield surface, we may generalize from the uniaxial case as follows: creep *toward* the yield surface, characterizing hardening, means that at constant stress and temperature, the yield function f decreases from a positive value toward zero, that is, $\dot{f} < 0$. Similarly, softening is characterized by $\dot{f} > 0$. But

$$\begin{aligned}\dot{f}\Big|_{\boldsymbol{\sigma}=\text{const},T=\text{const}} &= \sum_\alpha \frac{\partial f}{\partial \xi_\alpha}\dot{\xi}_\alpha = \phi \sum_\alpha \frac{\partial f}{\partial \xi_\alpha} h_\alpha \\ &= -\phi H,\end{aligned}$$

where, by definition,

$$H = -\sum_\alpha \frac{\partial f}{\partial \xi_\alpha} h_\alpha. \tag{3.1.1}$$

Thus $H > 0$ and $H < 0$ for hardening and softening materials (or hardening and softening ranges of the same material), respectively. The limiting case $H = 0$, which in particular occurs when f is independent of the ξ_α, describes a *perfectly plastic* material.

Viscoplastic Potential

If a viscoplastic material has a flow potential in the sense of 1.5.3 (not necessarily in the stricter sense of Rice or Moreau), then it may also be shown to have a viscoplastic potential in the following sense. Let h_{ij} be defined by

$$h_{ij} = \sum_\alpha \frac{\partial \varepsilon_{ij}^i}{\partial \xi_\alpha} h_\alpha.$$

The flow equations then are

$$\dot{\varepsilon}_{ij}^i = \phi h_{ij}. \tag{3.1.2}$$

If there exists a function $g(\boldsymbol{\sigma}, T, \boldsymbol{\xi})$, continuously differentiable with respect to $\boldsymbol{\sigma}$ wherever $f(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) > 0$, such that

$$h_{ij} = \frac{\partial g}{\partial \sigma_{ij}},$$

then g is called a *viscoplastic potential*. (The relation $h_{ij} = \lambda \partial g / \partial \sigma_{ij}$ is not more general, since the factor λ can be absorbed in ϕ.) Perzyna [1963] and many others have assumed the existence of a viscoplastic potential identical with the yield function f, or at least such that $\partial g / \sigma_{ij} \propto \partial f / \sigma_{ij}$; this identity is of no great significance in viscoplasticity, but becomes highly important after the transition to rate-independent plasticity.

Specific Models Based on J_2 Flow Potential

In 1.5.3 a flow potential was discussed that depends on the stress only through J_2, leading to the flow equation $\dot{\varepsilon}_{ij}^i = \phi s_{ij}$. A yield criterion having the same stress dependence, that is, one that can be represented by the equation

$$\sqrt{J_2} - k = 0$$

(where k depends on T and $\boldsymbol{\xi}$ and equals the yield stress in shear) is known as the *Mises* (sometimes *Huber–Mises*) yield criterion. A model of viscoplasticity incorporating this yield criterion and a J_2 flow potential was first proposed by Hohenemser and Prager [1932] as a generalization to three-dimensional behavior of the Bingham model described in 2.1.3. The flow equation is

$$\dot{\varepsilon}_{ij}^i = \frac{1}{2\eta} <1 - \frac{k}{\sqrt{J_2}}> s_{ij}, \tag{3.1.3}$$

where η is a temperature-dependent viscosity, and the *Macauley bracket* $<\cdot>$ is defined by $<x> = xH(x)$, where $H(\cdot)$ is the Heaviside step function:

$$H(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}$$

In other words,

$$<x> = \begin{cases} 0, & x \le 0, \\ x, & x > 0. \end{cases}$$

The previously discussed model of Perzyna [1963] is a generalization of the Hohenemser–Prager model in which $<f>$ is replaced by $H(f)\Phi(f)$, or $<\Phi(f)>$ in Perzyna's notation. It will be noted that as $k \to 0$, the Hohenemser–Prager and Perzyna models reduce to the Maxwell model of linear viscoelasticity discussed in 1.5.1.

A generalized potential Ω, as discussed in 1.5.4, may be associated with the Hohenemser–Prager model if it takes the form $\Omega(\boldsymbol{\sigma}) = <f>^2/(2\eta)$, where $f = \sqrt{J_2} - k$, and with the Perzyna model if it is $\Omega(\boldsymbol{\sigma}) = H(f)\Omega_0(f)$. Hardening can be included in a simple manner by letting k be a variable. If the generalized potential is viewed as a function $\Omega(\boldsymbol{\sigma}, k)$, then the effective inelastic strain $\bar{\varepsilon}^i$ defined by (1.5.7) can easily be shown to obey the rate equation

$$\dot{\bar{\varepsilon}}^i = -\frac{1}{\sqrt{3}}\frac{\partial \Omega}{\partial k}.$$

It is convenient to let $k = k_0 + R/\sqrt{3}$, where k_0 is the initial value of k, and to treat Ω as a function of $(\boldsymbol{\sigma}, R)$. Then $\dot{\bar{\varepsilon}}^i = -\partial\Omega/\partial R$, and $-R$ may be regarded as the thermodynamic force conjugate to the internal variable $\bar{\varepsilon}^i$.

A more sophisticated model developed by Chaboche [1977] uses as internal variables $\bar{\varepsilon}^i$ and a strain-like symmetric second-rank tensor $\boldsymbol{\alpha}$. The thermodynamic forces conjugate to these variables are the stress-like variables $-R$ and $-\boldsymbol{\rho}$, respectively, and the yield surface is assumed to be given by

$$f(\boldsymbol{\sigma}, \boldsymbol{\rho}, R) = \sqrt{\bar{J}_2} - \frac{R}{\sqrt{3}} - k_0 = 0,$$

where

$$\bar{J}_2 = \frac{1}{2}(s_{ij} - \rho'_{ij})(s_{ij} - \rho'_{ij}),$$

$\boldsymbol{\rho}'$ being the deviator of $\boldsymbol{\rho}$. The yield surface is thus again of the Mises type, but capable not only of expansion (as measured by R) but also of translation (as shown by $\boldsymbol{\rho}'$, which locates the center of the elastic region). The hardening described by the expansion of the yield surface is called *isotropic*, while that described by the translation is called *kinematic*. The significance of the terms is discussed in Section 3.2.

If a generalized potential is again assumed in the Perzyna form, $\Omega(\boldsymbol{\sigma}, R, \boldsymbol{\rho}) = H(f)\Omega_0(f)$, then $\dot{\varepsilon}^i_{ij} = \partial\Omega/\partial\sigma_{ij}$ and $\dot{\bar{\varepsilon}}^i = -\partial\Omega/\partial R$ as before, the flow equations being

$$\dot{\varepsilon}^i_{ij} = \frac{\partial \Omega}{\partial \sigma_{ij}} = H(f)\Omega_0'(f)\frac{s_{ij} - \rho'_{ij}}{2\sqrt{\bar{J}_2}}.$$

In addition, $\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\varepsilon}}^i$, so that the kinematic-hardening variable $\boldsymbol{\alpha}$, though it must be treated as a distinct variable, coincides with the inelastic strain.

Chaboche, however, assumes the generalized potential in the form $\Omega(\boldsymbol{\sigma}, R, \boldsymbol{\rho}) = H(f)\Omega_0(f) + \Omega_r(\boldsymbol{\rho})$, where the second term represents recovery (see 2.1.2). Moreover, Chaboche abandons the generalized normality hypothesis for $\boldsymbol{\alpha}$ by introducing an additional term representing a concept called *fading strain memory*, due to Il'iushin [1954], the better to describe the Bauschinger effect. The rate equation for $\boldsymbol{\alpha}$ is therefore taken as

$$\dot{\alpha}_{ij} = -\frac{\partial \Omega}{\partial \rho_{ij}} - F(\bar{\varepsilon}^i)\dot{\bar{\varepsilon}}^i \rho_{ij} = \dot{\varepsilon}^i_{ij} - F(\bar{\varepsilon}^i)\dot{\bar{\varepsilon}}^i \rho_{ij} - \frac{\partial \Omega_r}{\partial \rho_{ij}},$$

where $F(\bar{\varepsilon}^i)$ is a function to be specified, along with $\Omega_0(f)$, $\Omega_r(\boldsymbol{\rho})$, and the free-energy density $\psi(T, \boldsymbol{\varepsilon}, \bar{\varepsilon}^i, \boldsymbol{\alpha})$, from which R and $\boldsymbol{\rho}$ can be derived in accordance with Equation (1.5.4): $R = \rho \partial\psi/\partial\bar{\varepsilon}^i$, $\rho_{ij} = \rho\partial\psi/\partial\alpha_{ij}$.

3.1.2. Transition to Rate-Independent Plasticity

Aside from the previously discussed limit of the Hohenemser–Prager model as the yield stress goes to zero, another limiting case is of great interest, namely, as the viscosity η goes to zero. Obviously, if $\mathbf{s} \neq 0$ then the inelastic strain rate would become infinite, unless $\sqrt{J_2}$ simultaneously tends to k, in which case the quantity $(1/\eta)<1 - k/\sqrt{J_2}>$ becomes indeterminate but may remain finite and positive.

Supposing for simplicity that k in Equation (3.1.3) is constant, for a given input of stress we can solve this equation for $\boldsymbol{\varepsilon}^i$ as a function of time, and the dependence on time is through the variable t/η. In other words, decreasing the viscosity is equivalent to slowing down the process of inelastic deformation, and the limit of zero viscosity is equivalent to the limit of "infinitely slow" processes. Thus a slow process can take place if J_2 is slightly larger than k^2. We can also see this result by forming the scalar product $\dot{\varepsilon}^i_{ij}\dot{\varepsilon}^i_{ij}$ from Equation (3.1.3), from which we obtain

$$\sqrt{J_2} = k + \eta\sqrt{2\dot{\varepsilon}^i_{ij}\dot{\varepsilon}^i_{ij}}, \qquad \dot{\boldsymbol{\varepsilon}}^i \neq \mathbf{0},$$

an equation that is sometimes interpreted as a *rate-dependent yield criterion*.

Let us return to the more general model of viscoplasticity considered above, and particularly one in which ϕ increases with f. The rate equations (3.1.2) indicate that, in the same sense as in the Hohenemser–Prager model, the rate of a process in which inelastic deformation takes place increases with distance from the yield surface. If such a process is *very slow*, then it takes place *very near but just outside* the yield surface, so that ϕ is very small. In the limit as $f \to 0+$ we can eliminate ϕ (and thus no longer need to concern ourselves with the actual rate at which the process takes place) as follows: if f remains equal to zero (or a very small positive constant), then

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \sum_\alpha \frac{\partial f}{\partial \xi_\alpha}\phi h_\alpha = 0.$$

We define

$$\mathring{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \tag{3.1.4}$$

and assume $H > 0$ (i.e., hardening), with H as defined by Equation (3.1.1); then the condition $\dot{f} = \mathring{f} - \phi H = 0$ is possible together with $\phi > 0$ only if $\mathring{f} > 0$; this last condition is called *loading*. Thus we have the result

$$\phi = \frac{1}{H} < \mathring{f} >,$$

and therefore

$$\dot{\xi}_\alpha = \frac{1}{H} < \mathring{f} > h_\alpha. \tag{3.1.5}$$

Note that both sides of Equation (3.1.5) are derivatives with respect to time, so that a change in the time scale does not affect the equation. Such an equation is called *rate-independent*. If it is assumed that this equation describes material behavior over a sufficiently wide range of loading rates, then the behavior is called *rate-independent plasticity*, also called *inviscid plasticity* (since it corresponds to the zero-viscosity limit of the Hohenemser–Prager model), or just plain *plasticity*. Rate-independent plasticity constitutes the principal topic of the remainder of this book. The inelastic strain occurring in rate-independent plasticity is usually denoted $\boldsymbol{\varepsilon}^p$ rather than $\boldsymbol{\varepsilon}^i$, and is called the plastic strain. The flow equation for the plastic strain may be written as

$$\dot{\varepsilon}^p_{ij} = \frac{1}{H} < \mathring{f} > h_{ij}. \tag{3.1.6}$$

For purposes of computation, however, it is sometimes advantageous to remain within the framework of viscoplasticity without making the full transition, even when the problem to be solved is regarded as rate-independent. In other words, a fictitious viscoplastic material of very low viscosity is "associated" with a given rate-independent plastic material, with rate equations given, for example (Nguyen and Bui [1974]), by

$$\dot{\xi}_\alpha = \frac{< f >}{\eta} h_\alpha, \tag{3.1.7}$$

with the viscosity η taken as constant. Computations are then performed under time-independent loads and boundary conditions until all strain rates vanish. It was shown by Zienkiewicz and Cormeau [1974], among others, that the results are equivalent to those of rate-independent plasticity.

Combined Viscoplasticity and Rate-Independent Plasticity

At extremely high rates of deformation or loading, the internal variables do not have enough time to change and consequently the deformation can be only elastic. However, the various rate processes responsible for plastic deformation, corresponding to the generation of dislocations and the many different kinds of obstacles that dislocations must overcome, may have very different characteristic times. This means that not only do metals differ greatly among one another in their rate-sensitivity, but different mechanisms in the same metal may respond with very different speeds. Thus, those mechanisms whose characteristic times are very short compared with a typical loading time produce what appears to be instantaneous inelastic deformation, while the others produce rate-dependent deformation as discussed so far in this section. If both phenomena occur in a metal over a certain range of loading times, then the total inelastic strain $\boldsymbol{\varepsilon}^i$ may be decomposed as

$$\boldsymbol{\varepsilon}^i = \boldsymbol{\varepsilon}^{vp} + \boldsymbol{\varepsilon}^p, \tag{3.1.8}$$

where $\boldsymbol{\varepsilon}^{vp}$ is the viscoplastic strain equivalent to that governed by Equation (3.1.2), and $\boldsymbol{\varepsilon}^p$ is the apparently rate-independent plastic strain, governed by Equation (3.1.6). It is important to note that the yield functions f and flow tensors h_{ij} are, in general, different for the two inelastic strain tensors. In particular, the viscoplastic yield surface is always assumed to be inside the rate-independent plastic (or "dynamic") yield surface.

3.1.3. Viscoplasticity Without a Yield Surface

As we have seen, in both classical viscoplasticity and rate-independent plasticity the yield surface is a central ingredient; in the latter it is indispensable. The significance of the yield surface has, however, repeatedly been questioned. Consider the following remarks by Bell [1973]: "Among the many matters pertaining to the plastic deformation of crystalline solids, yield surfaces and failure criteria early became subjects of overemphasis... Indeed most of the outstanding 19th century experimentists doubted that such a phenomenon as an elastic limit, let alone a yield surface, existed... well over a half-century of experiment, and the study of restricted plasticity theories for the 'ideal solid,' have not disposed of most of the original questions."

"Unified" Viscoplasticity Models

According to Bodner [1968], "yielding is not a separate and independent criterion but is a consequence of a general constitutive law of the material behavior." Since the 1970s several constitutive models for the rate-dependent inelastic behavior of metals have been formulated without a formal hypothesis of a yield surface, but with the feature that at sufficiently low rates the resulting stress-strain curves may resemble those of materials with fairly well defined yield stresses. In fact, with yield based on the offset definition (see 2.1.1), these models can *predict* yield surfaces in accordance with Bodner's dictum, particularly if offset strains of the order of 10^{-6} to 10^{-5} are used, in contrast to the conventional 10^{-3} to 10^{-2}.

In addition to describing the behavior traditionally called plasticity, in both monotonic and cyclic loading, these models also aim to describe creep, especially at higher temperatures, without a decomposition such as (3.1.8). They have consequently come to be known as "unified" viscoplasticity models, and are particularly useful for the description of bodies undergoing significant temperature changes — for example, spacecraft. Perhaps the simplest such model is due to Bodner and Partom [1972, 1975], in which the flow equations are given by Equation (3.1.2) with $h_{ij} = s_{ij}$ and ϕ a function of J_2 and (in order to describe hardening) of the inelastic work W_i defined by Equation (1.5.6) as the only internal variable. The rate equation is obviously

$$\dot{W}_i = 2J_2\phi(W_i,\, J_2).$$

The hardening in this case is purely isotropic, since $\sqrt{3J_2}$ is the value of the effective stress necessary to maintain a given inelastic work rate $\dot{W}_i$.[1]

More sophisticated "unified" viscoplasticity models, that describe many features of the behavior of metals at elevated temperatures, have been developed since 1975 by, among others, Miller [1976], Hart [1976], Krieg, Swearengen, and Jones [1978], Walker [1981], and Krieg, Swearengen, and Rohde [1987] (see reviews by Chan, Bodner, Walker, and Lindholm [1984], Krempl [1987], and Bammann and Krieg [1987]). The essential internal variables in these models are the *equilibrium stress* tensor $\boldsymbol{\rho}$, and the scalar *drag stress* or *friction stress* σ_D;[2] the terminology is loosely related to that of dislocation theory, and is an example of "physical" nomenclature for phenomenological internal variables.

In the "unified" models the stress-like variables σ_D and $\boldsymbol{\rho}$ are used directly as internal variables, rather than as conjugate thermodynamic forces. The equilibrium stress $\boldsymbol{\rho}$, like its counterpart in the Chaboche model, describes kinematic hardening. Some writers, following Kochendörfer [1938], relate it to the back stress due to stuck dislocations (see 2.2.3), and consequently the equilibrium stress is also termed *back stress*; see Krempl [1987] for a discussion (the relationship between constitutive theory and crystal behavior has also been discussed by Kocks [1987]). For isotropic behavior $\boldsymbol{\rho}$ is assumed as purely deviatoric, and the rate equation for inelastic strain takes the form

$$\dot{\varepsilon}^i_{ij} = \dot{e}^i_{ij} = \frac{3}{2}\frac{\phi(\Gamma/\sigma_D)}{\Gamma}(s_{ij} - \rho_{ij}), \tag{3.1.9}$$

[1] Or, equivalently, a given effective inelastic strain rate, since in this model $\dot{W}_i = \sqrt{3J_2}\dot{\bar{\varepsilon}}^i$ is an increasing function of W_i (or of $\bar{\varepsilon}^i$).

[2] A model developed by Krempl and coworkers, known as **viscoplasticity based on overstress**, dispenses with drag stress as a variable (see Yao and Krempl [1985] and Krempl, McMahon, and Yao [1986]). In another model called **viscoplasticity based on total strain** (Cernocky and Krempl [1979]), the equilibrium stress is not an internal variable but a function of total strain; this model is therefore a nonlinear version of the "standard solid" model of linear viscoelasticity (see 1.5.1).

where $\Gamma = \sqrt{3\bar{J}_2}$, with $\bar{J}_2$ as defined in 3.1.1, and ϕ is a function (whose values have the dimensions of inverse time) that increases rapidly with its argument. The evolution of the equivalent inelastic strain is given by

$$\dot{\bar{\varepsilon}}^i = \phi(\Gamma/\sigma_D), \tag{3.1.10}$$

and, in uniaxial stress,

$$\dot{\varepsilon}^i = \phi(|\sigma - \rho|/\sigma_D).$$

Typical forms of $\phi(x)$ are Ax^n, $A(e^x - 1)$, and $A[\sinh(x^m)]^n$, where A, m, and n are constants, n in particular being a large exponent. For an extension to initially anisotropic behavior, see, for example, Helling and Miller [1987].

A variety of forms has been proposed for the rate equations for $\boldsymbol{\rho}$ and σ_D; a typical set is due to Walker [1981]:

$$\dot{\rho}_{ij} = a_1\dot{\varepsilon}^i_{ij} - [a_2\dot{\bar{\varepsilon}}^i + a_3(2\rho_{kl}\rho_{kl}/3)^{(m-1)/2}]\rho_{ij},$$

$$\dot{\sigma}_D = [a_4 - a_5(\sigma_D - \sigma_{D0})]\dot{\bar{\varepsilon}}^i - a_6(\sigma_D - \sigma_{D0})^p,$$

where $\dot{\varepsilon}^i_{ij}$ and $\dot{\bar{\varepsilon}}^i$ are substituted from (3.1.9)–(3.1.10), and a_1, ..., a_6, m, p and σ_{D0} are constants.

Endochronic Theory

A different "theory of viscoplasticity without a yield surface" is the **endochronic theory** of Valanis [1971], originally formulated by him (Valanis [1971]) for "application to the mechanical behavior of metals," though its range of application has recently been extended to other materials, such as concrete (Bažant [1978]). The basic concept in the theory is that of an *intrinsic time* (hence the name) that is related to the deformation history of the material point, the relation itself being a material property. An *intrinsic time measure* ζ is defined, for example, by

$$d\zeta^2 = A_{ijkl}\, d\varepsilon^i_{ij}\, d\varepsilon^i_{kl} + B^2\, dt^2,$$

where the tensor $\mathbf{A}$ and scalar B may depend on temperature. (In the original theory of Valanis [1971], the total strain $\boldsymbol{\varepsilon}$ rather than the inelastic strain $\boldsymbol{\varepsilon}^i$ appeared in the definition.) A model in which $B = 0$ describes rate-independent behavior and thus defines the **endochronic theory of plasticity**.

An *intrinsic time scale* is next defined as $z(\zeta)$, a monotonically increasing function, and the behavior of the material is assumed to be governed by constitutive relations having the same structure as those of linear viscoelasticity, as described in 1.5.2, but with z replacing the real time t. As in linear viscoelasticity, the internal variables can be eliminated, and the stress can be related to the strain history by means of a pseudo-relaxation function. The uniaxial relation is

$$\sigma = \int_0^z R(z - z')\frac{d\varepsilon}{dz'}dz', \tag{3.1.11}$$

while the multiaxial relation describing isotropic behavior is

$$\sigma_{ij} = \int_0^z \left[R_1(z - z')\delta_{ij}\frac{d\varepsilon_{kk}}{dz'} + 2R_2(z - z')\frac{d\varepsilon_{ij}}{dz'}\right] dz'.$$

With a pseudo-relaxation function analogous to that of the "standard solid," that is, $R(z) = E_1 + E_2 e^{-\alpha z}$, and with $z(\zeta)$ given by

$$z = \frac{1}{\beta}\ln(1 + \beta\zeta),$$

where α and β are positive constants, Valanis [1971] was able to fit many experimental data on repetitive uniaxial loading-unloading cycles and on coupling between tension and shear.

More recently, Valanis [1980] showed that Equation (3.1.11) can be replaced by

$$\sigma = \sigma_0\frac{d\varepsilon^i}{dz} + \int_0^z \rho(z - z')\frac{d\varepsilon^i}{dz'}dz'. \tag{3.1.12}$$

For rate-independent uniaxial behavior, $d\zeta = |d\varepsilon^i|$ with no loss in generality. If the last integral is called α, then the stress must satisfy

$$|\sigma - \alpha| = \sigma_0 h(z), \tag{3.1.13}$$

where $h(z) = d\zeta/dz$. Equation (3.1.12) can be used to construct stress-strain curves showing hardening depending both on the effective inelastic strain [through $h(z)$] and on the strain path (through α). The equation has a natural extension to multiaxial stress states, which for isotropic materials is

$$(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij}) = [s_0 h(z)]^2.$$

This equation represents a yield surface capable of both expansion and translation in stress space, thus exhibiting both isotropic and kinematic hardening. An endochronic model unifying viscoplasticity and plasticity was presented by Watanabe and Atluri [1986].

Exercises: Section 3.1

1. Suppose that the yield function has the form $f(\boldsymbol{\sigma}, T, \boldsymbol{\xi}) = F(\boldsymbol{\sigma}) - k(T, \kappa)$, where κ is the hardening variable defined by either (1.5.6) or (1.5.7), and the flow equations are assumed as in the form (3.1.2). What is the "hardening modulus" H, defined by Equation (3.1.1)?

2. If the only stress components are $\sigma_{12} = \sigma_{21} = \tau$, with $\tau > 0$, write the equation for the shear rate $\dot{\gamma} = 2\dot{\varepsilon}_{12}$ given by the Hohenemser–Prager model (3.1.3). Discuss the special case $k = 0$.

3. Generalize the Hohenemser–Prager model to include isotropic and kinematic hardening. Compare with the Chaboche model.

4. Find the flow equation for a viscoplastic solid with a rate-dependent yield criterion given by

$$\sqrt{J_2} = k + \eta(2\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij})^{\frac{1}{2m}}.$$

5. Construct a simple model for combined viscoplasticity and plasticity, with a perfectly plastic Mises yield criterion and associated flow rule in both.

6. Derive (3.1.12) from (3.1.11).

Section 3.2 Rate-Independent Plasticity

3.2.1. Flow Rule and Work-Hardening

Flow Rule

In keeping with the formulation of rate-independent plasticity as the limit of classical viscoplasticity for infinitely slow processes, we henceforth consider all processes to be "infinitely" slow (compared with the material relaxation time τ), and correspondingly regard the material as "inviscid plastic," "rate-independent plastic," or simply *plastic*. The inelastic strain $\boldsymbol{\varepsilon}^i$ will from now on be called the *plastic strain* and denoted $\boldsymbol{\varepsilon}^p$ instead of $\boldsymbol{\varepsilon}^i$. The flow equations (3.1.6) may be written as

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda} h_{ij}, \tag{3.2.1}$$

where

$$\dot{\lambda} = \begin{cases} \dfrac{1}{H} <\overset{\circ}{f}>, & f = 0, \\ 0, & f < 0, \end{cases} \tag{3.2.2}$$

with H as defined by Equation (3.1.1). The rate equations (3.1.5) analogously become

$$\dot{\xi}_\alpha = \dot{\lambda} h_\alpha.$$

If $\partial f / \partial \xi_\alpha \equiv 0$, then, as mentioned before, the material is called *perfectly plastic*. In this case $H = 0$, but $\overset{\circ}{f} = \dot{f}$, and therefore the condition $\overset{\circ}{f} > 0$

is impossible. Plastic deformation then occurs only if $(\partial f/\partial \sigma_{ij})\dot{\sigma}_{ij} = 0$ (*neutral loading*), and the definition (3.2.2) of $\dot{\lambda}$ cannot be used. Instead, $\dot{\lambda}$ is an indeterminate positive quantity when $f = 0$ and $(\partial f/\partial \sigma_{ij})\dot{\sigma}_{ij} = 0$, and is zero otherwise.

In either case, $\dot{\lambda}$ and f can easily be seen to obey the **Kuhn–Tucker conditions** of optimization theory:

$$\dot{\lambda} f = 0, \quad \dot{\lambda} \geq 0, \quad f \leq 0.$$

The specification of the tensor function **h** in Equation (3.2.1), at least to within a multiplicative scalar, is known as the **flow rule**, and if there exists a function g (analogous to a viscoplastic potential) such that $h_{ij} = \partial g/\partial \sigma_{ij}$, then such a function is called a *plastic potential*.

Deformation Theory

The plasticity theory in which the plastic strain is governed by rate equations such as (3.2.1) is known as the *incremental* or *flow* theory of plasticity. A *deformation* or *total-strain* theory was proposed by Hencky [1924]. In this theory the plastic strain tensor itself is assumed to be determined by the stress tensor, provided that the yield criterion is met. Elastic unloading from and reloading to the yield surface are in principle provided for, although a contradiction is seen as soon as one considers reloading to a stress other than the one from which unloading took place, but located on the same yield surface; clearly, no plastic deformation could have occurred during the unloading-reloading process, yet the theory requires different values of the plastic strain at the two stress states. The deformation theory, which is mathematically much simpler than the flow theory, gives results that coincide with those of the latter only under highly restricted circumstances. An obvious example is the uniaxial case, provided that no reverse plastic deformation occurs; the equivalence is implicit in the use of relations such as (2.1.2).

A more general case is that of a material element subject to *proportional* or *radial loading*, that is, loading in which the ratios among the stress components remain constant, provided the yield criterion and flow rule are sufficiently simple (for example, the Mises yield criterion and the flow rule with $h_{ij} = s_{ij}$). A rough definition of "nearly proportional" loading, for which the deformation theory gives satisfactory results, is discussed by Rabotnov [1969]. It was shown by Kachanov [1954] (see also Kachanov [1971]) that the stress states derived from the two theories converge if the deformation develops in a definite direction.

Another example of a range of validity of the deformation theory, discussed in separate developments by Budiansky [1959] and by Kliushnikov [1959], concerns a material whose yield surface has a singular point or corner, with the stress point remaining at the corner in the course of loading.

For a simplified discussion, see Chakrabarty [1987], pp. 91–94.

The deformation theory has recently been the subject of far-reaching mathematical developments (Temam [1985]). It has also been found to give better results than the incremental theory in the study of the plastic buckling of elements under multiaxial stress, as is shown in Section 5.3.

Work-Hardening

The hardening criterion $H > 0$, and the corresponding criteria $H = 0$ for perfect plasticity and $H < 0$ for softening, were formulated in 3.1.1 for viscoplastic materials on the basis of rate-dependent behavior at states outside the yield surface. An alternative derivation can be given entirely in the context of rate-independent plasticity.

For given $\boldsymbol{\xi}$, $f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = 0$ is the equation describing the yield surface in stress space. If $f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = 0$ and $\dot{f}|_{\boldsymbol{\sigma}=\text{const}} < 0$ (i.e. $H > 0$) at a given time t, then at a slightly later time $t + \Delta t$ we have $f(\boldsymbol{\sigma}, \boldsymbol{\xi} + \dot{\boldsymbol{\xi}}\Delta t) < 0$; the yield surface is seen to have moved so that $\boldsymbol{\sigma}$ is now inside it. In other words, $H > 0$ implies that, at least locally, *the yield surface is expanding* in stress space. The expansion of the yield surface is equivalent, in uniaxial stress, to a rising stress-strain curve (see Figure 3.2.1).

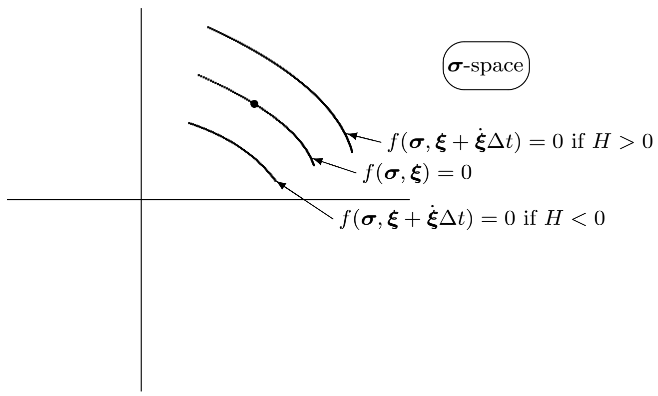


Figure 3.2.1. Hardening and softening in rate-independent plasticity: motion of yield surface in stress space

Conversely, a contracting yield surface denotes work-softening, and a stationary yield surface perfect plasticity. The description of work-softening materials is best achieved in strain space rather than stress space, and is discussed later. For now, we treat work-hardening materials only, with perfectly plastic materials as a limiting case.

In the simplest models of plasticity the internal variables are taken as (1) the plastic strain components ε_{ij}^p themselves, and (2) the hardening variable κ, defined by either Equation (1.5.6) or (1.5.7) (in rate-independent plasticity, $\bar{\varepsilon}^p$ is written in place of $\bar{\varepsilon}^i$). When the yield function is taken to have the form

$$f(\boldsymbol{\sigma},\, \boldsymbol{\varepsilon}^p,\, \kappa) = F(\boldsymbol{\sigma} - \boldsymbol{\rho}(\boldsymbol{\varepsilon}^p)) - k(\kappa),$$

both isotropic and kinematic hardening, as discussed in Section 3.1, can be described; the hardening is isotropic if $\boldsymbol{\rho} \equiv 0$ and $dk/d\kappa > 0$, and purely kinematic if $dk/d\kappa \equiv 0$ and $\boldsymbol{\rho} \neq 0$. The condition $dk/d\kappa \equiv 0$ *and* $\boldsymbol{\rho} \equiv 0$ represents perfect plasticity. The simplest model of kinematic hardening — that of Melan [1938] — has $\boldsymbol{\rho}(\boldsymbol{\varepsilon}^p) = c\boldsymbol{\varepsilon}^p$, with c a constant. More sophisticated hardening models are discussed in Section 3.3.

Drucker's Postulate

A more restricted definition of work-hardening was formulated by Drucker [1950, 1951] by generalizing the characteristics of uniaxial stress-strain curves. With a single stress component σ, the conjugate plastic strain rate $\dot{\varepsilon}^p$ clearly satisfies [see Figure 3.2.2(a)]

$$\dot{\sigma}\dot{\varepsilon}^p \begin{cases} \geq 0, & \text{hardening material,} \\ = 0, & \text{perfectly plastic material,} \\ \leq 0, & \text{softening material.} \end{cases}$$

The inequalities are unchanged if the stress and plastic-strain rates are multiplied by the infinitesimal time increment dt, so that they hold equally well for $d\sigma\, d\varepsilon^p$. This product has the dimensions of work per unit volume, and was given by Drucker the following interpretation: *if a unit volume of an elastic–plastic specimen under uniaxial stress is initially at stress* σ *and plastic strain* ε^p*, and if an "external agency"* (one that is independent of whatever has produced the current loads) *slowly applies an incremental load resulting in a stress increment* $d\sigma$ (which causes the elastic and plastic strain increments $d\varepsilon^e$ and $d\varepsilon^p$, respectively) *and subsequently slowly removes it, then* $d\sigma\, d\varepsilon = d\sigma\,(d\varepsilon^e + d\varepsilon^p)$ *is the work*[1] *performed by the external agency in the course of incremental loading, and* $d\sigma\, d\varepsilon^p$ *is the work performed in the course of the cycle consisting of the application and removal of the incremental stress.* (Note that for $d\varepsilon^p \neq 0$, σ must be the current yield stress.)

Since $d\sigma\, d\varepsilon^e$ is always positive, and for a work-hardening material $d\sigma\, d\varepsilon^p \geq 0$, it follows that for such a material $d\sigma d\varepsilon > 0$. Drucker accordingly *defines* a work-hardening (or "stable") plastic material as one in which the work done during incremental loading is positive, and the work done in the loading-unloading cycle is nonnegative; this definition is generally known in the literature as **Drucker's postulate** (see also Drucker [1959]).

Having defined hardening in terms of work, Drucker naturally extends the definition to general three-dimensional states of stress and strain, such that

$$d\sigma_{ij}\, d\varepsilon_{ij} > 0 \quad \textit{and} \quad d\sigma_{ij}\, d\varepsilon^p_{ij} \geq 0,$$

[1] Actually it is *twice* the work.

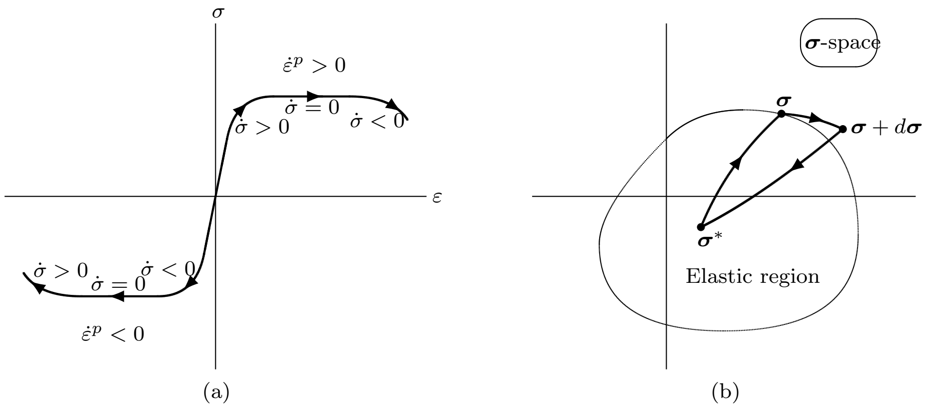


Figure 3.2.2. Drucker's postulate: (a) illustration in the uniaxial stress-strain plane; (b) illustration in stress space.

the equality holding only if $d\boldsymbol{\varepsilon}^p = 0$. For perfectly plastic ("neutrally stable") materials Drucker's inequalities are $d\sigma_{ij}\, d\varepsilon_{ij} \geq 0$ and $d\sigma_{ij}\, d\varepsilon^p_{ij} = 0$. It can be seen that the inequality

$$\dot{\sigma}_{ij}\dot{\varepsilon}^p_{ij} \geq 0, \tag{3.2.3}$$

sometimes known simply as **Drucker's inequality**, is valid for both work-hardening and perfectly plastic materials.

Because it uses the concept of work, Drucker's postulate is often referred to as a quasi-thermodynamic postulate, although it is quite independent of the basic laws of thermodynamics. Drucker's inequality (3.2.3) may also be given an interpretation that is free of any considerations of incremental work: the left-hand side represents the scalar product $\dot{\boldsymbol{\sigma}} \cdot \dot{\boldsymbol{\varepsilon}}^p$, and the inequality therefore expresses the hypothesis that *the plastic strain rate cannot oppose the stress rate*.

We should note, lastly, that Drucker's definition of work-hardening is in a sense circular. The definition *assumes* an external agency that is capable of applying arbitrary stress increments. But as can readily be seen from stress-strain diagrams, this assumption is not valid for softening or perfectly plastic materials; for example, in a tension test no increase in stress is possible. In other words, such materials are *unstable under stress control*. On the other hand, they are stable under strain control (or displacement control[1]), since arbitrary strain increments that do not violate internal constraints may, in principle, be applied. This fact points to the applicability of strain-space plasticity, to be discussed later, to a wider class of materials.

Drucker's statement of his work-hardening postulate is broader than summarized above, in that the additional stress produced by the external agency

[1]Stability under strain control and displacement control are equivalent when deformations are infinitesimal, but not when they are finite.

need not be a small increment. In particular, the initial stress, say $\boldsymbol{\sigma}^*$, may be inside the elastic region, or at a point on the yield surface far away from $\boldsymbol{\sigma}$, and the process followed by the external agency may consist of elastic loading to a stress $\boldsymbol{\sigma}$ on the current yield surface, a small stress increment $d\boldsymbol{\sigma}$ producing an incremental plastic strain $d\boldsymbol{\varepsilon}$, and finally, elastic unloading back to $\boldsymbol{\sigma}^*$; the path is illustrated in Figure 3.2.2(b). With $d\boldsymbol{\sigma}$ neglected alongside $\boldsymbol{\sigma} - \boldsymbol{\sigma}^*$, the work per unit volume done by the external agency is $(\sigma_{ij} - \sigma_{ij}^*)\, d\varepsilon_{ij}^p$. Drucker's postulate, consequently, implies

$$(\sigma_{ij} - \sigma_{ij}^*)\, \dot{\varepsilon}_{ij}^p \geq 0. \tag{3.2.4}$$

3.2.2. Maximum-Dissipation Postulate and Normality

Maximum-Dissipation Postulate

Inequality (3.2.4) is, as we have just seen, a necessary condition for Drucker's postulate, but it is not a sufficient one. In other words, its validity is not limited to materials that are work-hardening in Drucker's sense. Its significance may best be understood when we consider its uniaxial counterpart,

$$(\sigma - \sigma^*)\dot{\varepsilon}^p \geq 0.$$

As is seen in Figure 3.2.3, the inequality expresses the property that the

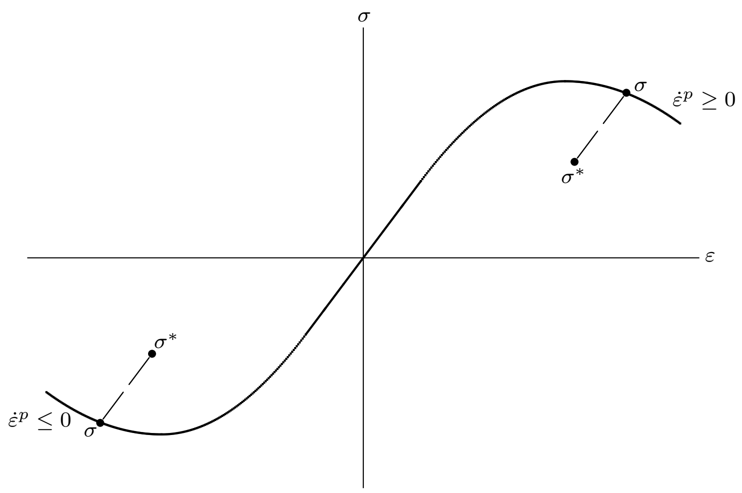


Figure 3.2.3. Maximum-plastic-dissipation postulate: (a) illustration in the uniaxial stress-strain plane.

plastic strain rate is positive (negative) only if the current stress σ is not less than (not greater than) any stress σ^* in the current elastic range — in other words, if σ equals the current tensile (compressive) yield stress. Clearly, work-softening and perfectly plastic materials have this property as well. Inequality (3.2.4) thus constitutes a postulate in its own right, called the **postulate of maximum plastic dissipation**. It was proposed independently by Mises [1928], Taylor [1947] and Hill [1948a]; it was derived from considerations of crystal plasticity by Bishop and Hill [1951], and is shown later to follow also from Il'iushin's postulate of plasticity in strain space.

Consequences of Maximum-Dissipation Postulate

Inequality (3.2.4) has consequences of the highest importance in plasticity theory. To examine them, we represent symmetric second-rank tensors as vectors in a six-dimensional space, as in 1.3.5, but using boldface rather than underline notation, and using the dot-product notation for the scalar product. Our inequality may thus be written as

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) \cdot \dot{\boldsymbol{\varepsilon}}^p \geq 0.$$

We suppose at first that the yield surface is everywhere smooth, so that a well-defined tangent hyperplane and normal direction exist at every point. It is clear from the two-dimensional representation in Figure 3.2.4(a) that if (3.2.4) is to be valid for all $\boldsymbol{\sigma}^*$ to the inward side of the tangent to the yield surface at $\boldsymbol{\sigma}$, then $\dot{\boldsymbol{\varepsilon}}^p$ *must be directed along the outward normal there*; this consequence is known as the **normality rule**. But as can be seen in Figure 3.2.4(b), if there are any $\boldsymbol{\sigma}^*$ lying to the outward side of the tangent, the inequality is violated. In other words, the entire elastic region must lie to one side of the tangent. As a result, **the yield surface is convex**.

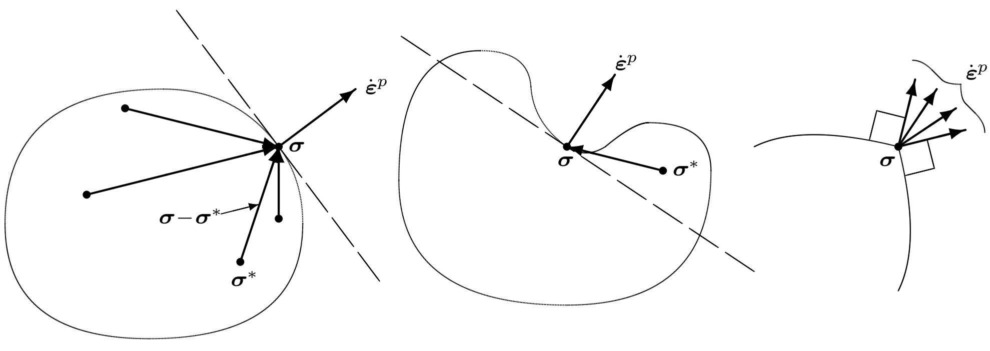


Figure 3.2.4. Properties of yield surface with associated flow rule: (a) normality; (b) convexity; (c) corner.

Let us define $D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi})$ by

$$D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}) = \max_{\boldsymbol{\sigma}^*} \sigma_{ij}^* \dot{\varepsilon}_{ij}^p,$$

the maximum being taken over all $\boldsymbol{\sigma}^*$ such that $f(\boldsymbol{\sigma}^*, \boldsymbol{\xi}) \le 0$. It follows from (3.2.4) that

$$\sigma_{ij}\dot{\varepsilon}^p_{ij} = D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}). \tag{3.2.5}$$

To make it clear that $D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi})$ depends only on $\dot{\boldsymbol{\varepsilon}}^p$ and $\boldsymbol{\xi}$ and not on $\boldsymbol{\sigma}$, we note that, if the yield surface is strictly convex at $\boldsymbol{\sigma}$ (whether this point is regular or singular), then this is the only stress that corresponds to a given normal direction in stress space and hence to a given $\dot{\boldsymbol{\varepsilon}}^p$. If the yield surface has a flat portion, then all points on this portion have the same normal, that is, different stresses correspond to the same $\dot{\boldsymbol{\varepsilon}}^p$, but the scalar product $\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p = \sigma_{ij}\dot{\varepsilon}^p_{ij}$ is the same for all of them. $D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi})$ will be called simply the plastic dissipation. Inequality (3.2.4) may now be rewritten as

$$D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}) \ge \sigma^*_{ij}\dot{\varepsilon}^p_{ij}, \tag{3.2.6}$$

giving explicit meaning to the name "principle of maximum plastic dissipation."

Normality

The normality rule is now discussed in more detail. At any point of the yield surface $f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = 0$ where the surface is smooth, the outward normal vector is proportional to the gradient of f (in stress space), and therefore, reverting to indicial notation, we may express the normality rule as

$$h_{ij} = \frac{\partial f}{\partial \sigma_{ij}}, \tag{3.2.7}$$

where h_{ij} is the tensor function appearing in the flow equation (3.2.1). Equation (3.2.7) expresses the result that the function f defining the yield surface is itself a plastic potential, and therefore the normality rule is also called a flow rule that is *associated with the yield criterion*, or, briefly, an **associated** (sometimes **associative**) **flow rule**. A flow rule derivable from a plastic potential g that is distinct from f (more precisely, such that $\partial g/\partial\sigma_{ij}$ is not proportional to $\partial f/\partial\sigma_{ij}$) is accordingly called a **nonassociated** flow rule. In the French literature, materials obeying an associated flow rule are usually called *standard* materials, and this term will often be used here.

We are now in a position to say that Drucker's postulate applies to *standard work-hardening* (or, in the limit, perfectly plastic) materials. The frequently expressed notion that Drucker's postulate is required for the convexity of the yield surface and for the normality rule is clearly erroneous, as is the idea that work-hardening materials are necessarily standard.

If the yield surface is not everywhere smooth but has one or more singular points (corners) at which the normal direction is not unique, then at such a point $\dot{\boldsymbol{\varepsilon}}^p$ must lie in the cone formed by the normal vectors meeting there [see Figure 3.2.4(c)]. The argument leading to the convexity of the yield surface is not affected by this generalization. As will be seen, Equation (3.2.7) can still be formally used in this case, provided that the partial derivatives are properly interpreted. In a rigorous treatment, the concept of gradient must be replaced by that of *subgradient*, due to Moreau [1963]; its application in plasticity theory was formulated by Moreau [1976].

Another treatment of singular yield surfaces was proposed by Koiter [1953a], who supposed the yield surface to be made of a number — say n — of smooth surfaces, each defined by an equation $f_k(\boldsymbol{\sigma}, \boldsymbol{\xi}) = 0$ $(k = 1, ..., n)$; the elastic region is the intersection of the regions defined by $f_k(\boldsymbol{\sigma}, \boldsymbol{\xi}) < 0$, and $\boldsymbol{\sigma}$ is on the yield surface if at least one of the f_k vanishes there, it being a singular point only if two or more of the f_k vanish. Equation (3.2.7) is replaced by

$$h_{ij} = \sum_k \alpha_k \frac{\partial f_k}{\partial \sigma_{ij}},$$

the summation being over those k for which $f_k(\boldsymbol{\sigma}, \boldsymbol{\xi}) = 0$, and the α_k are nonnegative numbers that may, with no loss of generality, be constrained so that $\sum_k \alpha_k = 1$.

3.2.3. Strain-Space Plasticity

As we noted above, it is only in work-hardening materials, which are stable under stress control, that we may consider processes with arbitrary stress increments, and therefore it is only for such materials — with perfect plasticity as a limiting case — that a theory in which stress is an independent variable may be expected to work. No such limitation applies to theories using strain as an independent variable. Surprisingly, such theories were not proposed until the 1960s, beginning with the pioneering work of Il'iushin [1961], followed by papers by Pipkin and Rivlin [1965], Owen [1968], Lubliner [1974], Nguyen and Bui [1974], Naghdi and Trapp [1975], and others.

To see that strain-space yield surfaces have the same character whether the material is work-hardening or work-softening, let us consider one whose stress-strain diagram in tension/compression or in shear is as shown in Figure 3.2.5(a); such a material exemplifies Melan's linear kinematic hardening if $E' > 0$, and is perfectly plastic if $E' = 0$. The stress σ is on the yield surface if $\sigma = E'\varepsilon \pm (1 - E'/E)\sigma_E$, and therefore the condition on the strain ε is $E(\varepsilon - \varepsilon^p) = E'\varepsilon \pm (1 - E'/E)\sigma_E$, or, equivalently, $\varepsilon = [E/(E - E')]\varepsilon^p \pm \varepsilon_E$, where $\varepsilon_E = \sigma_E/E$ [see Figure 3.2.5(b)]. A yield surface in ε-space is thus given by the pair of points corresponding to a given value of ε^p, and the ε-ε^p diagram has a "stable" form (i.e., a positive slope) for all $E' < E$, even if negative.

Yield Criterion and Flow Rule in Strain Space

To formulate the three-dimensional yield criterion in strain space, let C

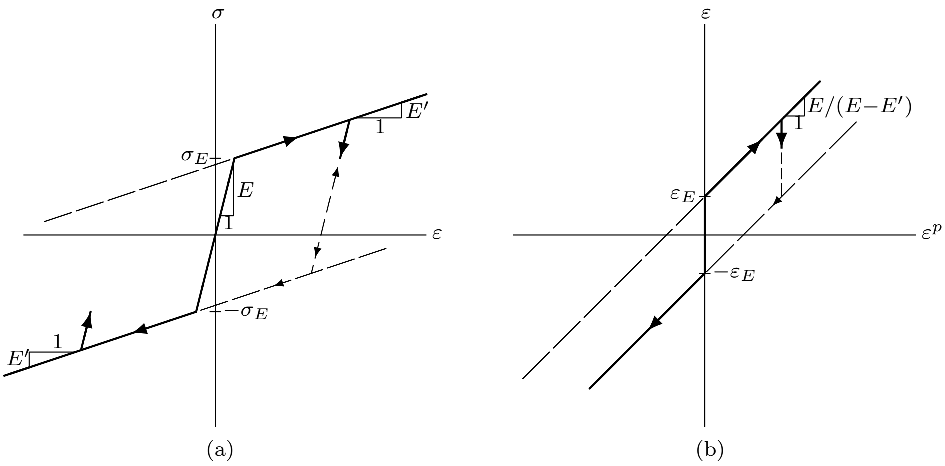


Figure 3.2.5. Material with linear hardening: (a) stress-strain diagram; (b) ε-ε^p diagram.

denote the elastic modulus tensor, so that the $\boldsymbol{\sigma}$-$\boldsymbol{\varepsilon}$-$\boldsymbol{\varepsilon}^p$ relation may be written $\boldsymbol{\sigma} = \mathsf{C} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathsf{p}})$ [i.e., $\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^p)$]. If $\hat{f}(\boldsymbol{\varepsilon}, \boldsymbol{\xi}) \stackrel{\text{def}}{=} f(\mathsf{C} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathsf{p}}), \boldsymbol{\xi})$, then the strain-space yield criterion is just

$$\hat{f}(\boldsymbol{\varepsilon}, \boldsymbol{\xi}) = 0.$$

Since

$$\frac{\partial \hat{f}}{\partial \varepsilon_{ij}} = C_{ijkl} \frac{\partial f}{\partial \sigma_{kl}}\bigg|_{\boldsymbol{\sigma}=\mathsf{C}\cdot(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathsf{p}})},$$

the same logic that led to (3.2.1)–(3.2.2) produces the flow equations

$$\dot{\varepsilon}_{ij}^p = \begin{cases} \dfrac{1}{L} h_{ij} < C_{ijkl} \dfrac{\partial f}{\partial \sigma_{ij}} \dot{\varepsilon}_{kl} >, & f = 0, \\ 0, & f < 0, \end{cases} \tag{3.2.8}$$

where

$$L = -\sum_{\alpha} \frac{\partial \hat{f}}{\partial \xi_\alpha} h_\alpha = H + C_{ijkl} \frac{\partial f}{\partial \sigma_{ij}} h_{kl}. \tag{3.2.9}$$

The normality rule (3.2.7), when translated into the strain-space formulation, takes the form

$$h_{ij} = C_{ijkl}^{-1} \frac{\partial \hat{f}}{\partial \varepsilon_{kl}}.$$

Note that L may very well be, and normally may be assumed to be, positive even when H is zero or negative, that is, for perfectly plastic or work-softening materials. It is thus not necessary to distinguish between these material types, the only restriction being $L > 0$. This condition describes stability under strain control in the same sense that the work-hardening criterion $H > 0$ describes stability under stress control; it will here be called *kinematic stability*.

The flow equation given by (3.2.8)–(3.2.9), when combined with the relation $\dot{\sigma}_{ij} = C_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl})$, yields

$$\dot{\sigma}_{ij} = C_{ijkl}\dot{\varepsilon}_{kl} - \begin{cases} \frac{1}{L} C_{ijmn} h_{mn} < C_{pqkl} \frac{\partial f}{\partial \sigma_{pq}} \dot{\varepsilon}_{kl} >, & f = 0, \\ 0, & f < 0. \end{cases} \tag{3.2.10}$$

This is an explicit expression for $\dot{\boldsymbol{\sigma}}$ in terms of $\dot{\boldsymbol{\varepsilon}}$, which may be regarded as an inversion of $\dot{\varepsilon}_{ij} = C^{-1}_{ijkl}\dot{\sigma}_{kl} + \dot{\varepsilon}^p_{ij}$ with $\dot{\boldsymbol{\varepsilon}}^p$ given by Equation (3.2.1). In this sense the result, which was first derived by Hill [1958] [for standard materials, i.e. with $\mathbf{h}$ given by (3.2.7)], is not necessarily based on strain-space plasticity.

Plastic Modulus

It is easy to show that when $\dot{\boldsymbol{\varepsilon}}^p \neq 0$,

$$H = (C_{ijkl} f_{ij} h_{kl}) \frac{f_{ij}\dot{\sigma}_{ij}}{C_{ijkl} f_{ij} \dot{\varepsilon}_{kl}},$$

where $f_{ij} = \partial f / \sigma_{ij}$. Thus H may be related to the so-called *plastic modulus* or *work-hardening modulus* $d\sigma/d\varepsilon^p$ obtained in a simple tension test. If the material has (a) elastic isotropy, (b) plastic incompressibility, and (c) sufficient plastic symmetry so that $\sigma_{ij} = \sigma\delta_{i1}\delta_{j1}$ implies that $\dot{\varepsilon}^p_{22} = \dot{\varepsilon}^p_{33} = -\frac{1}{2}\dot{\varepsilon}^p_{11}$ and $\dot{\varepsilon}^p_{ij} = 0$ for $i \neq j$, then (with $\varepsilon^p = \varepsilon^p_{11}$)

$$H = h_{11} f_{11} \frac{d\sigma}{d\varepsilon^p}.$$

Il'iushin's Postulate

It can also be shown that the normality rule follows from a "postulate of plasticity" in strain space first proposed by Il'iushin [1961], namely, that in any cycle that is closed in strain space,

$$\oint \sigma_{ij}\, d\varepsilon_{ij} \geq 0, \tag{3.2.11}$$

where the equality holds only if the process is elastic; we show this by proving that (3.2.11) implies the maximum-dissipation postulate (3.2.4).

Consider a state $(\boldsymbol{\varepsilon}^1, \boldsymbol{\xi}^1)$ with $\boldsymbol{\varepsilon}^1$ on the yield surface, and any strain $\boldsymbol{\varepsilon}^*$ on or inside both the current yield surface and the subsequent yield surface obtained after a brief plastic process of duration Δt from $(\boldsymbol{\varepsilon}^1, \boldsymbol{\xi}^1)$ to $(\boldsymbol{\varepsilon}^1 + \dot{\boldsymbol{\varepsilon}}\Delta t, \boldsymbol{\xi}^1 + \dot{\boldsymbol{\xi}}\Delta t)$, that is,

$$\hat{f}(\boldsymbol{\varepsilon}^1, \boldsymbol{\xi}^1) = 0, \quad \hat{f}(\boldsymbol{\varepsilon}^1 + \dot{\boldsymbol{\varepsilon}}\Delta t, \boldsymbol{\xi}^1 + \dot{\boldsymbol{\xi}}\Delta t) = 0,$$

and

$$\hat{f}(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1) \le 0, \quad \hat{f}(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1 + \dot{\boldsymbol{\xi}}\Delta t) \le 0.$$

In the cycle

$$(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1) \ \stackrel{1}{\to}\ (\boldsymbol{\varepsilon}^1, \boldsymbol{\xi}^1) \ \stackrel{2}{\to}\ (\boldsymbol{\varepsilon}^1 + \dot{\boldsymbol{\varepsilon}}\Delta t, \boldsymbol{\xi}^1 + \dot{\boldsymbol{\xi}}\Delta t) \ \stackrel{3}{\to}\ (\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1 + \dot{\boldsymbol{\xi}}\Delta t),$$

segments 1 and 3 are elastic, so that, if the process is isothermal,

$$\sigma_{ij}\dot{\varepsilon}_{ij} = \begin{cases} \rho\dot{\psi} & \text{in 1 and 3} \\ \rho\dot{\psi} + D & \text{in 2,} \end{cases}$$

where $\psi(\boldsymbol{\varepsilon}, \boldsymbol{\xi})$ is the free energy per unit mass at the given temperature and $D(\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = -\rho\sum_\alpha (\partial\psi/\partial\boldsymbol{\xi}_\alpha)\,\dot{\xi}_\alpha$ is the dissipation per unit volume. Now

$$\oint \sigma_{ij}\, d\varepsilon_{ij} = \oint \rho\, d\psi + \int_2 D\, dt;$$

but

$$\oint \rho\, d\psi = \rho\psi(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1 + \dot{\boldsymbol{\xi}}\Delta t) - \rho\psi(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1) \doteq -D(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1, \dot{\boldsymbol{\xi}})\Delta t,$$

and $size - 1int_2 D\, dt \doteq D(\boldsymbol{\varepsilon}^1, \boldsymbol{\xi}^1, \dot{\boldsymbol{\xi}})\Delta t$, the approximations being to within $o(\Delta t)$. It follows that

$$D(\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) \ge D(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) \tag{3.2.12}$$

(the superscripts 1 can now be dropped) if $(\boldsymbol{\varepsilon}, \boldsymbol{\xi})$ is a state with $\boldsymbol{\varepsilon}$ on the current yield surface with corresponding $\dot{\boldsymbol{\xi}}$, and $\boldsymbol{\varepsilon}^*$ is any strain on or inside the yield surface.

With the free-energy density decomposed as in Equation (1.5.5) (and $\boldsymbol{\varepsilon}^p$ written in place of $\boldsymbol{\varepsilon}^i$), $D = \sigma_{ij}\dot{\varepsilon}^p_{ij} - \rho\dot{\psi}^i$ and therefore, if $\boldsymbol{\sigma}^* = \mathsf{C}\cdot(\boldsymbol{\varepsilon}^* - \boldsymbol{\varepsilon}^{\mathsf{p}})$ is any stress on or inside the yield surface, inequality (3.2.12) is equivalent to (3.2.4), which is thus seen to be a consequence of Il'iushin's postulate.

It remains to be investigated whether the converse holds. Consider an arbitrary process that is closed in strain space, going from $(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^1)$ to $(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}^2)$, with $\boldsymbol{\xi}^2$ not necessarily infinitesimally close to $\boldsymbol{\xi}^1$. At any state of the process,

$$\sigma_{ij}\dot{\varepsilon}_{ij} = \rho\dot{\psi} + D(\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}),$$

with $D = 0$ whenever the process is instantaneously elastic, and therefore

$$\oint \sigma_{ij}\, d\varepsilon_{ij} = \oint [D(\boldsymbol{\varepsilon}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) - D(\boldsymbol{\varepsilon}^*, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}})]\, dt.$$

According to the principle of maximum plastic dissipation, the integrand is nonnegative whenever $\boldsymbol{\varepsilon}^*$ is on or inside the strain-space yield surface at the current value of $\boldsymbol{\xi}$, and therefore Il'iushin's postulate is satisfied for processes in which the original yield surface is inside all subsequent yield surfaces. The last condition is satisfied in materials with isotropic hardening, but not in general. Consequently Il'iushin's postulate is a stronger (i.e. less general) hypothesis than the principle of maximum plastic dissipation.

Nguyen–Bui Inequality

On the other hand, an inequality first explicitly stated by Nguyen and Bui [1974] may be shown to be weaker than the maximum-dissipation principle. It is readily seen that this principle, as expressed in the form (3.2.12), is equivalent to

$$C_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl})\dot{\varepsilon}^p_{ij} \geq 0 \tag{3.2.13}$$

for any strain $\boldsymbol{\varepsilon}^*$ that is on or inside the current yield surface in strain space. Suppose, in particular, that $\boldsymbol{\varepsilon}^*$ is close to $\boldsymbol{\varepsilon}$ and is given by $\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon} \pm \dot{\boldsymbol{\varepsilon}}\, dt$, with $dt > 0$ and $\dot{\boldsymbol{\varepsilon}}$ the strain-rate tensor in a possible process. With the plus sign chosen, the process goes from $\boldsymbol{\varepsilon}$ to $\boldsymbol{\varepsilon}^*$ and is necessarily elastic, so that $\dot{\boldsymbol{\varepsilon}}^p = 0$ and therefore Inequality (3.2.13) is satisfied as an equality. Thus $\dot{\boldsymbol{\varepsilon}} \neq 0$ only if the minus sign is taken, and therefore (3.2.13) takes the local form

$$C_{ijkl}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}_{kl} \geq 0, \tag{3.2.14}$$

or the equivalent form given by Nguyen and Bui,

$$\dot{\sigma}_{ij}\dot{\varepsilon}^p_{ij} \geq C_{ijkl}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{kl}.$$

Inequality (3.2.14), like Drucker's inequality (3.2.3), may be interpreted as a stability postulate, this time in strain space: if we take $C_{ijkl}a_{ij}b_{kl}$ as defining a scalar product between two tensors $\mathbf{a}$ and $\mathbf{b}$ in strain-increment space, then (3.2.14) expresses the notion that *the plastic strain rate cannot oppose the total strain rate*. Inequality (3.2.14) is by itself sufficient for the associated flow rule to follow, and consequently describes *standard kinematically stable* materials.

Exercises: Section 3.2

1. A work-hardening plastic solid is assumed to obey the Mises yield criterion with isotropic hardening, that is, $f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \sqrt{J_2} - k(\bar{\varepsilon}^p)$, and the flow rule $h_{ij} = s_{ij}$. Show that

$$\dot{\varepsilon}^p_{ij} = \frac{\sqrt{3}s_{ij}{<}s_{kl}\dot{s}_{kl}{>}}{4k^2k'(\bar{\varepsilon}^p)}.$$

2. Show that the solid described in Exercise 1 obeys Drucker's inequality (3.2.3) if and only if $k'(\bar{\varepsilon}^p) > 0$.

3. If $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ are stresses such that $J_2 = k^2$ and $J_2^* \le k^2$, show that $(s_{ij} - s_{ij}^*)s_{ij} \ge 0$, and hence that the solid of Exercise 1 obeys the maximum-plastic-dissipation postulate (3.2.4) independently of $k'(\bar{\varepsilon}^p)$, that is, whether the solid hardens or softens.

4. For a work-hardening solid with the yield criterion of Exercise 1, but with the nonassociated flow rule $h_{ij} = s_{ij} + t_{ij}$, where $s_{ij}t_{ij} = 0$, show that for some $\dot{\boldsymbol{\sigma}}$ Drucker's inequality (3.2.3) is violated.

5. For the standard isotropically hardening Mises solid of Exercise 1, show that
$$\dot{\varepsilon}_{ij}^p = \frac{s_{ij} \langle s_{kl}\dot{\varepsilon}_{kl} \rangle}{2k^2\left(1 + \dfrac{k'(\bar{\varepsilon}^p)}{\sqrt{3}G}\right)}.$$

6. Show that the standard Mises solid obeys the Nguyen–Bui inequality (3.2.14) whether it hardens or softens.

Section 3.3 Yield Criteria, Flow Rules and Hardening Rules

3.3.1. Introduction

The yield function f in stress space may be written with no loss of generality in terms of the stress deviator and the first invariant of stress, that is, $f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \bar{f}(\mathbf{s}, I_1, \boldsymbol{\xi})$, where $I_1 = \sigma_{kk} = \delta_{ij}\sigma_{ij}$, so that $\partial I_1/\partial\sigma_{ij} = \delta_{ij}$. Since
$$s_{kl} = \sigma_{kl} - \frac{1}{3}I_1\delta_{kl} = \left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}\right)\sigma_{ij},$$
it follows that $\partial s_{kl}/\partial\sigma_{ij} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$. Consequently,
$$\frac{\partial f}{\partial\sigma_{ij}} = \frac{\partial\bar{f}}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial\sigma_{ij}} + \frac{\partial\bar{f}}{\partial I_1}\frac{\partial I_1}{\partial\sigma_{ij}} = (\bar{f}_{ij} - \tfrac{1}{3}\delta_{ij}\bar{f}_{kk}) + \frac{\partial f}{\partial I_1}\delta_{ij},$$
where $\bar{f}_{ij} = \partial\bar{f}/\partial s_{ij}$. Accordingly, in a standard material plastic volume change ("dilatancy") occurs if and only if the yield criterion depends on I_1, i.e. on the mean stress, and, conversely, plastic incompressibility obtains if and only if the yield criterion depends on $\mathbf{s}$ but not on I_1. If the yield criterion of a plastically incompressible material is significantly affected by mean stress, then the material is necessarily nonstandard.

Isotropic Yield Criteria

If the yield criterion is initially isotropic, then the dependence of f on $\boldsymbol{\sigma}$ must be through the stress invariants I_1, I_2, and I_3, or, equivalently, on the principal stresses σ_I ($I = 1, 2, 3$), provided this dependence is symmetric, that is, invariant under any change of the index I. Similarly, the dependence of $\overline{f}$ on $\mathbf{s}$ must be through the stress-deviator invariants J_2 and J_3; the equivalent dependence on the principal stresses may be exhibited in the so-called principal stress-deviator plane or π-plane, namely, the plane in $\sigma_1\sigma_2\sigma_3$-space given by $\sigma_1 + \sigma_2 + \sigma_3 = 0$, shown in Figure 3.3.1.

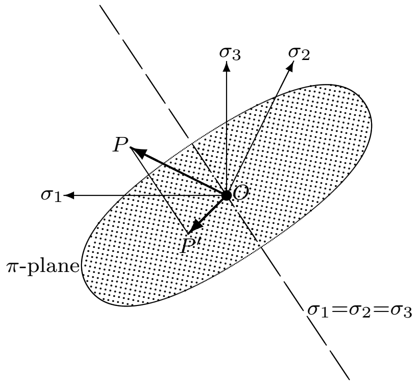


Figure 3.3.1. π-plane.

Indeed, if a point $P(\sigma_1, \sigma_2, \sigma_3)$ in $\sigma_1\sigma_2\sigma_3$-space (the *Haigh–Westergaard space*) is represented by the vector $\overrightarrow{OP}$, then its projection $\overrightarrow{OP'}$ onto the π-plane is the vector whose components are the principal stress deviators s_1, s_2, s_3. The magnitude of this projection — that is, the distance from P to the axis $\sigma_1 = \sigma_2 = \sigma_3$ — is just $\sqrt{2J_2}$. A yield surface that is independent of I_1 has, in this space, the form of a cylinder perpendicular to the π-plane, and therefore may be specified by a single curve in this plane. A yield surface that depends on I_1 may be described by a family of curves in the π-plane, each corresponding to a different value of I_1 and forming the intersection of the yield surface in $\sigma_1\sigma_2\sigma_3$-space with a plane I_1 = constant, that is, a plane parallel to the π-plane.

A curve in the π-plane can also be described in terms of the polar coordinates $(\sqrt{2J_2}, \theta)$, where the polar angle θ may defined as that measured from the projection of the σ_1-axis toward the projection of the σ_2-axis, and can be shown to be given by

$$\tan\theta = \frac{\sqrt{3}(\sigma_2 - \sigma_3)}{2\sigma_1 - \sigma_2 - \sigma_3} = \frac{s_2 - s_3}{\sqrt{3}s_1}.$$

Using some trigonometric identities and the fact that $s_1 + s_2 + s_3 = 0$, it is also possible to define θ in terms of the deviatoric stress invariants J_2 and J_3:

$$\cos 3\theta = \frac{3\sqrt{3}J_3}{2J_2^{3/2}}.$$

A point with $\theta = 0$ corresponds to $\sigma_1 > \sigma_2 = \sigma_3$; the locus of such points on the yield surface is said to represent one of the three *tensile meridians* of the surface. A point with $\theta = \pi/3$ corresponds to $\sigma_1 = \sigma_2 > \sigma_3$, and lies on a *compressive meridian*.

3.3.2. Yield Criteria Independent of the Mean Stress

Since the concept of plasticity was first applied to metals, in which the influence of mean stress on yielding is generally negligible (Bridgman [1923, 1950]), the oldest and most commonly used yield criteria are those that are independent of I_1. Such criteria have an alternative two-dimensional representation: since their dependence on the principal stresses must be through the differences $\sigma_1 - \sigma_2$, $\sigma_1 - \sigma_3$ and $\sigma_2 - \sigma_3$, and since $\sigma_1 - \sigma_2 = (\sigma_1 - \sigma_3) - (\sigma_2 - \sigma_3)$, the yield criterion can be plotted in a plane with $\sigma_1 - \sigma_3$ and $\sigma_2 - \sigma_3$ as coordinate axes.

Tresca Criterion

The **Tresca** yield criterion is historically the oldest; it embodies the assumption that plastic deformation occurs when the maximum shear stress over all planes attains a critical value, namely, the value of the current yield stress in shear, denoted $k(\boldsymbol{\xi})$. Because of Equation (1.3.11), this criterion may be represented by the yield function

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \frac{1}{2} \max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right) - k(\boldsymbol{\xi}), \tag{3.3.1}$$

or, equivalently,

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \frac{1}{4}(|\sigma_1 - \sigma_2| + |\sigma_2 - \sigma_3| + |\sigma_3 - \sigma_1|) - k(\boldsymbol{\xi}). \tag{3.3.2}$$

The projection of the Tresca yield surface in the π-plane is a regular hexagon, shown in Figure 3.3.2(a), whose vertices lie on the projections of the positive and negative σ_1, σ_2 and σ_3-axes, while in the $(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)$-plane it takes the form of the irregular hexagon shown in Figure 3.3.2(b).

Of course, the forms (3.3.1) and (3.3.2) for the Tresca yield function are not unique. The form

$$f(\boldsymbol{\sigma}) = [(\sigma_1 - \sigma_2)^2 - 4k^2][(\sigma_2 - \sigma_3)^2 - 4k^2][(\sigma_1 - \sigma_3)^2 - 4k^2]$$

(with the dependence on $\boldsymbol{\xi}$ not indicated) has the advantage of being analytic and, moreover, expressible in terms of the principal stress-deviator invariants J_2 and J_3:

$$f(\boldsymbol{\sigma}) = 4{J_2}^3 - 27J_3^2 - 36k^2{J_2}^2 + 96k^4J_2 - 64k^6.$$

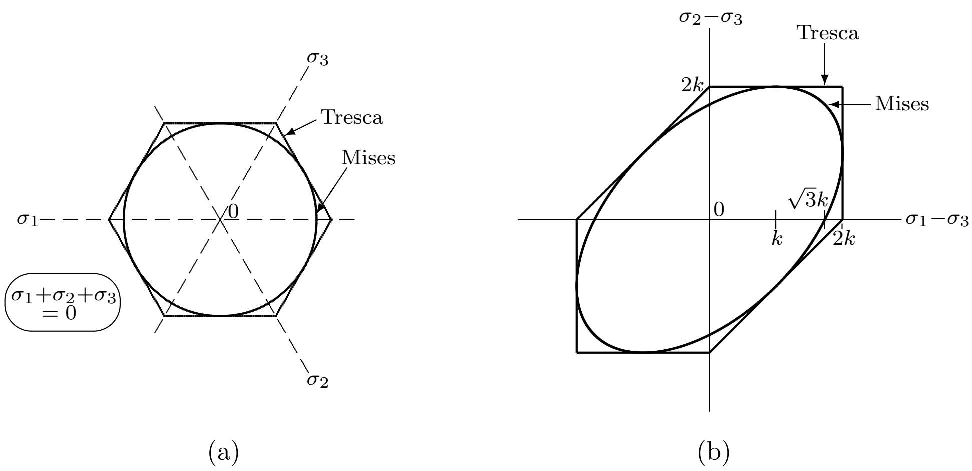


Figure 3.3.2. Projections of Tresca and Mises yield surfaces: (a) π-plane; (b) $\sigma_1-\sigma_3$-$\sigma_2-\sigma_3$ plane.

Tresca Criterion: Associated Flow Rule

Although the Tresca yield surface is singular, we can nonetheless derive its associated flow rule by means of a formal application of Equation (3.2.7) to the second form, Equation (3.3.2). We write

$$\frac{d}{dx}|x| = \operatorname{sgn} x, \tag{3.3.3}$$

where

$$\operatorname{sgn} x = 2H(x) - 1 = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$$

defines the *signum* function for $x \neq 0$. It is conventional to define $\operatorname{sgn} 0 = 0$, but in the present context it is more convenient if $\operatorname{sgn} x$ does not have a unique value at 0 but can have any value between -1 and 1. Strictly speaking, then, it is not a function in the usual sense but a *set-valued function* or *multifunction*, a concept with considerable use in convex analysis.[1] In this way we obtain

$$\dot{\varepsilon}_1^p = \frac{1}{4}\dot{\lambda}[\operatorname{sgn}(\sigma_1 - \sigma_2) + \operatorname{sgn}(\sigma_1 - \sigma_3)],$$

where, in accordance with Equations (3.2.1)–(3.2.2), for work-hardening materials $\dot{\lambda} = <\overset{\circ}{f}>/H$, with

$$H = \sum_\alpha \frac{\partial k}{\partial \xi_\alpha} h_\alpha,$$

[1]In fact, our use of Equation (3.3.3) comes rather close to the subdifferential calculus of Moreau.

while for the perfectly plastic material $\dot{\lambda}$ is indeterminate. Similar expressions are obtained for $\dot{\varepsilon}_2^p$ and $\dot{\varepsilon}_3^p$. Thus, if the principal stresses are all distinct, and ordered such that $\sigma_1 > \sigma_2 > \sigma_3$, then $\dot{\varepsilon}_1^p = \frac{1}{2}\dot{\lambda}$, $\dot{\varepsilon}_2^p = 0$, $\dot{\varepsilon}_3^p = -\frac{1}{2}\dot{\lambda}$. If, on the other hand, $\sigma_1 = \sigma_2 > \sigma_3$, then $\dot{\varepsilon}_1^p = \frac{1}{4}\dot{\lambda}(1+\beta)$, $\dot{\varepsilon}_2^p = \frac{1}{4}\dot{\lambda}(1-\beta)$, $\dot{\varepsilon}_3^p = -\frac{1}{2}\dot{\lambda}$, where β is any real number between -1 and 1. Analogous expressions can be obtained for all other combinations of principal stresses.

The Tresca flow rule can also be obtained by the method due to Koiter, discussed in 3.2.2.

It can be seen that for every combination of principal stresses, $|\dot{\varepsilon}_1^p| + |\dot{\varepsilon}_2^p| + |\dot{\varepsilon}_3^p| = \dot{\lambda}$, and therefore the plastic dissipation is given by

$$D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}) = \dot{\lambda} k(\boldsymbol{\xi}) = k(\boldsymbol{\xi})(|\dot{\varepsilon}_1^p| + |\dot{\varepsilon}_2^p| + |\dot{\varepsilon}_3^p|).$$

If it is desired to use an effective plastic strain $\bar{\varepsilon}^p$ as an internal variable in conjunction with the Tresca criterion and its associated flow rule, then the definition

$$\bar{\varepsilon}^p = \frac{1}{2}\int (|\dot{\varepsilon}_1^p| + |\dot{\varepsilon}_2^p| + |\dot{\varepsilon}_3^p|)\, dt$$

is more appropriate than (1.5.7). In fact, if it is assumed that k is a function of $\bar{\varepsilon}^p$ as thus defined, then $W_p = 2\int k\, d\bar{\varepsilon}^p$, so that a one-to-one correspondence can be established between W_p and $\bar{\varepsilon}^p$.

Lévy Flow Rule and Mises Yield Criterion

In the nineteenth century Saint-Venant and others used the Tresca yield criterion together with the (nonassociated) flow rule derived from the J_2 potential (see 1.5.4) whose general form was first proposed (for total rather than plastic strain) by Lévy, namely,

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda} s_{ij},$$

with $\dot{\lambda}$ defined as above. As seen in Section 3.1 in connection with viscoplasticity, and as first pointed out by Mises [1913], the yield criterion with which this flow rule is associated is the **Mises criterion**, represented by the yield function

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = \sqrt{J_2} - k(\boldsymbol{\xi}),$$

where $k(\boldsymbol{\xi})$ is again the yield stress in shear at the current values of $\boldsymbol{\xi}$. In view of the relation (1.3.5) between J_2 and the octahedral shear stress, the Mises criterion is also known as the *maximum-octahedral-shear-stress* criterion, and as a result of Equation (1.4.17), which shows the complementary energy of an isotropic, linearly elastic material to be uncoupled into volumetric and distortional parts, it is also called the *maximum-distortional-energy* criterion.

An alternative — and analytic — form of the Mises yield function (with the dependence on $\boldsymbol{\xi}$ not shown explicitly) is

$$f(\boldsymbol{\sigma}) = J_2 - k^2.$$

Expressing J_2 in terms of the principal stresses (see Section 1.3), we may formulate the Mises yield criterion in the form

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 6k^2$$

or

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 - \sigma_1\sigma_2 = 3k^2.$$

The form taken by the Mises yield surface in the π-plane is that of a circle of radius $\sqrt{2}k$, and in the $(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)$-plane that of an ellipse. Both forms are shown, along with those for the Tresca criterion, in Figure 3.3.2 (page 138).

The plastic dissipation for the Mises criterion and associated flow rule is given by

$$\begin{aligned} D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}) &= \sigma_{ij}\dot{\varepsilon}_{ij}^p = \dot{\lambda} s_{ij} s_{ij} = \sqrt{2J_2}\sqrt{\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p} \\ &= k(\boldsymbol{\xi})\sqrt{2\dot{\varepsilon}_{ij}^p \dot{\varepsilon}_{ij}^p}. \end{aligned}$$

If k is a function of $\bar{\varepsilon}^p$ as defined by (1.5.7), then W_p is in one-to-one correspondence with $\bar{\varepsilon}_p$.

As mentioned above, Lévy and Mises formulated the flow rule bearing their name for the total, rather than merely the plastic, strain rate; in this form it is valid as an approximation for problems in which elastic strains are vanishingly small, or, equivalently, for materials whose elastic moduli are infinite — the so-called *rigid–plastic* materials (see Section 3.4). The generalization allowing for nonvanishing elastic strains is due to Prandtl [1924] and Reuss [1930]; expressed in terms of total strain rate, with isotropic linear elasticity, the result is known as the **Prandtl–Reuss equations**:

$$\begin{aligned} \dot{\varepsilon}_{kk} &= \frac{1}{3K}\dot{\sigma}_{kk}, \\ \dot{e}_{ij} &= \frac{1}{2G}\dot{s}_{ij} + \dot{\lambda} s_{ij}. \end{aligned} \tag{3.3.4}$$

Some generalizations of the Mises yield function have been proposed so that dependence on J_3 is included. A typical form is

$$f(\boldsymbol{\sigma}) = \left(1 - c\frac{J_3^2}{J_2^3}\right)^{\alpha} J_2 - k^2.$$

The exponent α has variously been taken as $\frac{1}{3}$ and 1, k is as usual the yield stress in simple shear, and c is a parameter that is to be determined so as to optimize the fit with experimental data.

Anisotropic Yield Criteria

Anisotropy in yielding may be of two types: *initial* anisotropy and *induced* anisotropy. The former exists in materials that are structurally anisotropic, even before any plastic deformation has taken place; the latter appears, even in initially isotropic materials, as a result of work-hardening (Section 2.1). An example of an initially anisotropic yield criterion is Schmid's law (Section 2.2), according to which yielding in single crystals occurs when the shear stress on certain preferred planes (the slip planes) reaches a critical value; in the special case when every plane is a slip plane, Schmid's law reduces to the Tresca criterion.

An anisotropic generalization of the Mises criterion is due to Hill [1950]; it replaces J_2 with a general quadratic function of $\boldsymbol{\sigma}$ and therefore has the form

$$\tfrac{1}{2}A_{ijkl}\sigma_{ij}\sigma_{kl} = k^2,$$

where $\mathbf{A}$ is a fourth-rank tensor which has the same symmetries as the elasticity tensors ($A_{ijkl} = A_{jikl} = A_{klij}$). If the yield criterion is independent of mean stress, then $\mathbf{A}$ also obeys $A_{ijkk} = 0$, so that it has at most fifteen independent components (like a symmetric 5×5 matrix); the isotropic (Mises) case corresponds to $A_{ijkl} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$. A special case considered by Hill [1948b] refers to a material with three mutually perpendicular planes of symmetry; if a Cartesian basis is chosen so that the coordinate planes are parallel to the planes of symmetry, then in this basis the components of $\mathbf{A}$ coupling normal stresses with shear stresses (e.g. A_{1112}, A_{1123}, etc. — nine independent components altogether) are zero, and $\mathbf{A}$ is given by

$$\begin{aligned} A_{ijkl}\sigma_{ij}\sigma_{kl} \;=\; & A(\sigma_{22}-\sigma_{33})^2 + B(\sigma_{11}-\sigma_{33})^2 + C(\sigma_{11}-\sigma_{22})^2 \\ & + 4D\sigma_{23}^2 + 4E\sigma_{13}^2 + 4F\sigma_{12}^2, \end{aligned}$$

where $A, \ldots, F$ are constants; clearly $A_{1111} = B + C$, $A_{1122} = -C$, $A_{1133} = -B$, $A_{1212} = F$, and so on.

3.3.3. Yield Criteria Dependent on the Mean Stress

A yield criterion depending on the mean stress becomes necessary when it is desired to apply plasticity theory to soils, rocks, and concrete, as discussed in Section 2.3. One such criterion has its origin in the **Mohr theory of rupture**, according to which failure (rupture) occurs on a plane in a body if the shear stress and normal stress on that plane achieve a critical combination. Since the strength properties of an isotropic material are unchanged when the direction of the shear stress is reversed, the critical combination may be expressed by the functional equation $\tau = \pm g(\sigma)$. This equation represents a pair of curves (each being the other's reflection through the σ-axis) in the Mohr plane, and a state of stress, as determined by the three Mohr's circles, is safe if all three circles lie between the curves, while it is a critical state if one of the three is tangent to the curves. These curves are thus the envelopes of the Mohr's circles representing failure and are therefore called the *Mohr* *failure (rupture)* envelopes. The point (σ, τ) is a point of tangency — say the upper one — if it obeys (1) the equation $\tau = g(\sigma)$, (2) the equation of the Mohr's circle [centered at $(\sigma_m, 0)$ and of radius τ_m], and (3) the tangency condition. If σ and τ are eliminated between these three equations, there remains an equation in terms of σ_m and τ_m, which constitutes the failure criterion.[1] Concretely, if a point in principal-stress space is located in the sextant $\sigma_1 > \sigma_2 > \sigma_3$, then $\sigma_m = \frac{1}{2}(\sigma_1 + \sigma_3)$ and $\tau_m = \frac{1}{2}|\sigma_1 - \sigma_3|$; the equation consequently represents a cylindrical surface parallel to the σ_2-axis, and the failure surface is formed by six such surfaces.

Mohr–Coulomb Criterion

The equations can be reduced explicitly if the Mohr envelopes are straight lines, that is, if

$$g(\sigma) = c - \mu\sigma.$$

This is just Equation (2.3.3), with the sign of σ changed to the usual convention whereby it is positive in tension; c is the cohesion, and $\mu = \tan\phi$ is the coefficient of internal friction in the sense of the Coulomb model of friction. The resulting criterion is consequently known as the **Mohr–Coulomb criterion**. It is convenient to represent the Mohr's circle parametrically:

$$\sigma = \sigma_m + \tau_m \cos 2\alpha, \quad \tau = \tau_m \sin 2\alpha,$$

where α is the angle between the failure plane and the axis of the least tensile (greatest compressive) stress. The tangency condition is then $\mu = \cot 2\alpha$, so that $\alpha = \frac{1}{4}\pi - \frac{1}{2}\phi$, $\sin 2\alpha = \cos\phi$, and $\cos 2\alpha = \sin\phi$. The equation in terms of σ_m and τ_m becomes

$$\tau_m + \sigma_m \sin\phi = c\cos\phi,$$

from which it is seen that the failure stress in simple shear is $c\cos\phi$. (Needless to say, when $\phi = 0$ the Mohr–Coulomb criterion reduces to that of Tresca.) In terms of the principal stresses the criterion takes the form

$$\max_{i \neq j}[|\sigma_i - \sigma_j| + (\sigma_i + \sigma_j)\sin\phi] = 2c\cos\phi,$$

so that the yield stresses in tension and compression are respectively $2c\cos\phi/(1+\sin\phi) = 2c\tan\alpha$ and $2c\cos\phi/(1-\sin\phi) = 2c\cot\alpha$. The associated plastic dissipation was shown by Drucker [1953] to be

$$D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}) = c\cot\phi(\dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p + \dot{\varepsilon}_3^p).$$

[1]It is pointed out by Hill [1950] that tangency between the Mohr envelope and Mohr's circle does not necessarily occur at real (σ, τ), and that it is the failure criterion and not the envelope that is fundamental.

The failure surfaces in $\sigma_1\sigma_2\sigma_3$-space are obviously planes that intersect to form a hexagonal pyramid; the plane in the sextant $\sigma_1 > \sigma_2 > \sigma_3$, for example, is described by

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi = 2c\cos\phi.$$

A form valid in all six sextants is

$$\sigma_{\max} - \sigma_{\min} + (\sigma_{\max} + \sigma_{\min})\sin\phi = 2c\cos\phi,$$

where $\sigma_{\max}$ and $\sigma_{\min}$ denote respectively the (algebraically) largest and smallest principal stresses.

The last equation may be rewritten as

$$\sigma_{\max} - \sigma_{\min} + \frac{1}{3}[(\sigma_{\max} - \sigma_{\text{int}}) - (\sigma_{\text{int}} - \sigma_{\min})]\sin\phi = 2c\cos\phi - \frac{2}{3}I_1\sin\phi,$$

where σ_{int} denotes the intermediate principal stress. The left-hand side, being an isotropic function of the stress deviator, is therefore a function of J_2 and J_3. The Mohr–Coulomb criterion is therefore seen to be a special case of the family of criteria based on Coulomb friction and described by equations of the form

$$\bar{F}(J_2,\, J_3) = c - \lambda I_1,$$

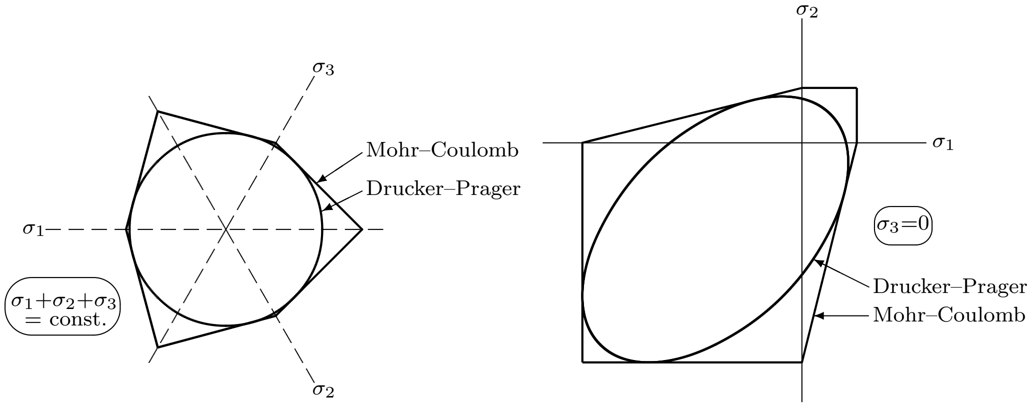
where c and λ are constants.

Drucker–Prager Criterion

Another yield criterion of this family, combining Coulomb friction with the Mises yield criterion, was proposed by Drucker and Prager [1952] and has become known as the **Drucker–Prager criterion**. With the Mises criterion interpreted in terms of the octahedral shear stress, it may be postulated that yielding occurs on the octahedral planes when $\tau_{\text{oct}} = \sqrt{\frac{2}{3}}k - \frac{1}{3}\mu I_1$, so that, in view of Equation (1.3.5), the criterion may be represented by the yield function $\overline{f}(\mathbf{s},\, I_1) = \sqrt{J_2} + \mu I_1/\sqrt{6} - k$. The yield surface in Haigh–Westergaard space is a right circular cone about the mean-stress axis, subtending the angle $\tan^{-1}(\sqrt{3}\mu)$. The yield stresses in simple shear, tension, and compression are respectively k, $\sqrt{3}k/(1 + \mu/\sqrt{2})$ and $\sqrt{3}k/(1 - \mu/\sqrt{2})$; note that for this criterion to be physically meaningful, μ must be less than $\sqrt{2}$. The associated plastic dissipation is

$$D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}) = \frac{k\sqrt{2\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}}}{\sqrt{1+\mu^2}}.$$

Projections of the yield surfaces corresponding to the Mohr–Coulomb and Drucker–Prager criteria onto a plane parallel to the π-plane (i.e. one with $\sigma_1 + \sigma_2 + \sigma_3 = \text{constant}$) are shown in Figure 3.3.3(a).



(a) (b)

Figure 3.3.3. Mohr–Coulomb and Drucker–Prager criteria: (a) plane parallel to π-plane; (b) plane stress.

Mises–Schleicher Criterion

A yield criterion that takes into account the difference between the yield strengths in tension and compression was discussed by Mises [1926] and Schleicher [1926]. If σ_T and σ_C denote, respectively, the tensile and compressive yield stresses, then the criterion may be expressed in the form

$$3J_2 + (\sigma_C - \sigma_T)I_1 - \sigma_T\sigma_C = 0.$$

The associated plastic dissipation is

$$D_p(\dot{\boldsymbol{\varepsilon}}^p; \boldsymbol{\xi}) = \frac{\sigma_C - \sigma_T}{2}\frac{\dot{e}^p_{ij}\dot{e}^p_{ij}}{\dot{\varepsilon}^p_{kk}} + \frac{\sigma_C\sigma_T}{3(\sigma_C - \sigma_T)}\dot{\varepsilon}^p_{kk}.$$

3.3.4. Yield Criteria Under Special States of Stress or Deformation

Equibiaxial Stress

A state of stress is called *equibiaxial* if two of the principal stresses are equal, as, for example, in the triaxial soil test described in Section 2.3. If $\sigma_2 = \sigma_3$, then

$$\sqrt{J_2} = \sqrt{\tfrac{1}{3}}|\sigma_1 - \sigma_3| = 2\sqrt{\tfrac{1}{3}}\tau_m,$$

so that the Mises and Tresca yield criteria are formally equivalent, as are the Mohr–Coulomb and Drucker–Prager criteria.

Plane Stress

The criteria for *plane stress* are obtained simply by setting $\sigma_3 = 0$. Thus the Tresca and Mises criteria are just as they appear in Figure 3.3.2(b) (page

138). In a state of plane stress in the x_1x_2-plane with $\sigma_{22} = 0$ (i.e., a state of stress that may be represented as a superposition of simple tension or compression and shear), it can further be shown that both the Mises and the Tresca yield criteria can be expressed in the form

$$\left(\frac{\sigma}{\sigma_Y}\right)^2 + \left(\frac{\tau}{\tau_Y}\right)^2 = 1, \tag{3.3.5}$$

where $\sigma = \sigma_{11}$, $\tau = \sigma_{12}$, and σ_Y and τ_Y are respectively the yield stresses in simple tension or compression and in shear, that is, $\tau_Y = k$, and $\sigma_Y = \sqrt{3}k$ or $2k$, depending on the criterion. The Mohr–Coulomb and Drucker–Prager criteria in plane stress are shown in Figure 3.3.3(b).

In general, an isotropic yield criterion with $\sigma_3 = 0$ may be written (with dependence on $\boldsymbol{\xi}$ not indicated explicitly) as

$$f_0(\sigma_1,\, \sigma_2) = 0,$$

or equivalently, upon transforming the independent variables, as

$$f_1[n,\, \tfrac{1}{2}(\sigma_1 - \sigma_2)] = 0,$$

where $n \stackrel{\text{def}}{=} \frac{1}{2}(\sigma_1 + \sigma_2)$. Because of isotropy, the dependence of f_1 on its second argument must be through the absolute value $r = \frac{1}{2}|\sigma_1 - \sigma_2|$. The preceding equation can then be solved for r as a function of n:

$$r = h(n). \tag{3.3.6}$$

In particular, $h(n)$ takes the form

$$h(n) = \sqrt{k^2 - \tfrac{1}{3}n^2}$$

for the Mises criterion and

$$h(n) = \begin{cases} k, & |n| < k \\ 2k - |n|, & |n| > k \end{cases}$$

for the Tresca criterion.

Plane Strain

In plane strain, as defined, for example, by $\dot{\varepsilon}_3 = 0$, the situation is more complicated, since the plane-strain condition requires $\dot{\varepsilon}_3^p = -\dot{\varepsilon}_3^e$, in turn involving the stress rates. If, however, the elastic strain rates may be equated to zero (the condition for this is discussed later), then we have $\dot{\varepsilon}_3 = \dot{\varepsilon}_3^p = 0$. Assuming a plastic potential $g(\sigma_1,\, \sigma_2,\, \sigma_3)$, we obtain the equation

$$\frac{\partial}{\partial \sigma_3} g(\sigma_1,\, \sigma_2,\, \sigma_3) = 0,$$

which when combined with the yield criterion in terms of the principal stresses, permits the elimination of σ_3 and hence the formulation of a yield criterion in terms of σ_1 and σ_2, leading once more to Equation (3.3.6).

Consider, for example, the Mises criterion with its associated flow rule $\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\mathbf{s}$, which requires $s_3 = \frac{2}{3}[\sigma_3 - \frac{1}{2}(\sigma_1+\sigma_2)] = 0$, that is, $\sigma_3 = \frac{1}{2}(\sigma_1+\sigma_2)$. Substituting this into the Mises yield criterion yields $\frac{3}{4}(\sigma_1-\sigma_2)^2 = 3k^2$, or $|\sigma_1-\sigma_2| = 2k$. According to the Tresca flow rule, on the other hand, for $\dot{\varepsilon}_3^p$ to be zero, σ_3 must be the intermediate principal stress, that is, either $\sigma_1 > \sigma_3 > \sigma_2$ or $\sigma_1 < \sigma_3 < \sigma_2$, so that

$$\max\left(|\sigma_1-\sigma_3|,\, |\sigma_2-\sigma_3|,\, |\sigma_1-\sigma_2|\right) = |\sigma_1-\sigma_2| = 2k.$$

Consequently the two criteria coincide, and may be expressed by Equation (3.3.6) with $h(n) = k$.

Consider next the Mohr–Coulomb criterion, with a nonassociated flow rule that is governed by a plastic potential having the same form as the yield function, that is,

$$g(\boldsymbol{\sigma}) = \sigma_{\max} - \sigma_{\min} + (\sigma_{\max} + \sigma_{\min})\sin\psi,$$

where ψ is known as the angle of dilatancy, since $\psi = 0$ describes a plastically incompressible solid; the special case $\psi = \phi$ represents the associated flow rule. The plane-strain condition $\dot{\varepsilon}_3^p = 0$ again requires that σ_3 be the intermediate principal stress. The criterion therefore may be described by Equation (3.3.6) with

$$h(n) = c\cos\phi - n\sin\phi.$$

3.3.5. Hardening Rules

A specification of the dependence of the yield criterion on the internal variables, along with the rate equations for these variables, is called a *hardening rule*. In this subsection we first review in more detail the significance of the two models of hardening — isotropic and kinematic — previously discussed for viscoplasticity in Section 3.1. Afterwards we look at some more general hardening rules.

Isotropic Hardening

The yield functions that we have studied so far in this section are all reducible to the form

$$f(\boldsymbol{\sigma},\, \boldsymbol{\xi}) = F(\boldsymbol{\sigma}) - k(\boldsymbol{\xi}).$$

Since it is only the yield stress that is affected by the internal variables, no generality is lost if it is assumed to depend on only one internal variable, say ξ_1, and this is invariably identified with the hardening variable κ, defined as either the plastic work W_p by Equation (1.5.6) or as the effective plastic strain $\bar{\varepsilon}^p$ by Equation (1.5.7). The function h_1 corresponding to ξ_1 [see Equation (3.1.5)] is given by $\sigma_{ij}h_{ij}$ or $\sqrt{\frac{2}{3}h_{ij}h_{ij}}$, respectively, for each of the two definitions of κ, so that the plastic modulus H is

$$H = \begin{cases} k'(W_p)\sigma_{ij}h_{ij}, \\ k'(\bar{\varepsilon}^p)\sqrt{\frac{2}{3}h_{ij}h_{ij}}. \end{cases}$$

As was pointed out in Section 3.2, work-hardening in rate-independent plasticity corresponds to a local expansion of the yield surface. The present behavior model (which, as we said in Section 3.1, is called isotropic hardening) represents a *global* expansion, with no change in shape. Thus for a given yield criterion and flow rule, hardening behavior in any process can be predicted from the knowledge of the function $k(\kappa)$, and this function may, in principle, be determined from a single test (such as a tension test).

The most attractive feature of the isotropic hardening model, which was introduced by Odqvist [1933], is its simplicity. However, its usefulness in approximating real behavior is limited. In uniaxial stressing it predicts that when a certain yield stress σ has been attained as a result of work-hardening, the yield stress encountered on stress reversal is just $-\sigma$, a result clearly at odds with the Bauschinger effect (Section 2.1). Furthermore, if $F(\boldsymbol{\sigma})$ is an isotropic function, then the yield criterion remains isotropic even after plastic deformation has taken place, so that the model cannot describe induced anisotropy.

Kinematic Hardening

In Sections 3.1 and 3.2 we saw, however, that if f can be written in the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = F(\boldsymbol{\sigma} - \boldsymbol{\rho}) - k(\boldsymbol{\xi}), \tag{3.3.7}$$

then more general hardening behavior can be described. Isotropic hardening is a special case of (3.3.7) if $\boldsymbol{\rho} \equiv 0$ and if k depends only on κ, while purely kinematic hardening corresponds to constant k but nonvanishing variable $\boldsymbol{\rho}$. Kinematic hardening represents a translation of the yield surface in stress space by shifting its reference point from the origin to $\boldsymbol{\rho}$, and with uniaxial stressing this means that the the length of the stress interval representing the elastic region (i.e., the difference between the current yield stress and the one found on reversal) remains constant. This is in fairly good agreement with the Bauschinger effect for those materials whose stress-strain curve in the work-hardening range can be approximated by a straight line ("linear hardening"), and it is for such materials that Melan [1938] proposed the model in which $\boldsymbol{\rho} = c\boldsymbol{\varepsilon}^p$, with c a constant. A similar idea was also proposed by Ishlinskii [1954], and a generalization of it is due to Prager [1955a, 1956a], who coined the term "kinematic hardening" on the basis of his use of a mechanical model in explaining the hardening rule (Figure 3.3.4). A kinematic

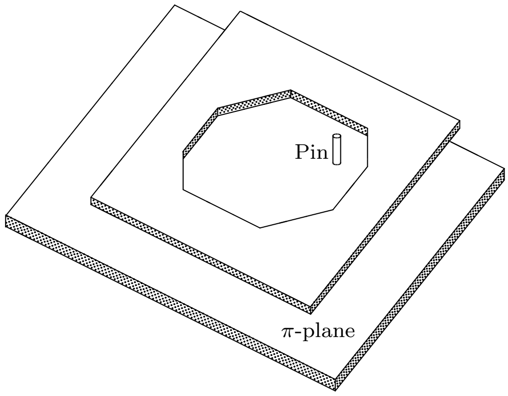


Figure 3.3.4. Prager's mechanical model of kinematic hardening.

hardening model is also capable of representing induced anisotropy, since a function $F(\boldsymbol{\sigma} - \boldsymbol{\rho})$ that depends only on the invariants of its argument stops being an isotropic function of the stress tensor as soon as $\boldsymbol{\rho}$ differs from zero.

It should be pointed out that, since $\boldsymbol{\rho}$ is a tensor in stress space (sometimes called the *back stress*, as discussed in 3.1.3), the equation $\rho_{ij} = c\varepsilon_{ij}^p$ does not imply proportionality between the vectors representing $\boldsymbol{\rho}$ and $\boldsymbol{\varepsilon}^p$ in any space other than the nine-dimensional space of second-rank tensors, and particularly not in the six-dimensional space in which symmetric tensors are represented, since the mappings of stress and strain into this space must be different [see Equations (1.4.9)] in order to preserve the scalar product $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$; consequently, as was pointed out by Hodge [1957a], the translation of the yield surface for a material with an associated flow rule is not necessarily in the direction of the normal to the yield surface, as was assumed by Prager in constructing his model.

In more sophisticated kinematic hardening models, internal variables other than $\boldsymbol{\varepsilon}^p$ and κ are included; in particular, the back stress $\boldsymbol{\rho}$ may be treated as a tensorial internal variable with its own rate equation. Indeed, the Melan–Prager model falls into this category when its equation is rewritten as

$$\dot{\rho}_{ij} = c\dot{\varepsilon}_{ij}^p; \tag{3.3.8}$$

here c need not be a constant but may itself depend on other internal variables. In the model described by Backhaus [1968], for example, c depends on the effective plastic strain $\bar{\varepsilon}^p$. Lehmann [1972] replaces the isotropic relation (3.3.8) between $\dot{\boldsymbol{\rho}}$ and $\dot{\boldsymbol{\varepsilon}}^p$ by a more general one,

$$\dot{\rho}_{ij} = c_{ijkl}(\boldsymbol{\sigma}, \boldsymbol{\rho})\dot{\varepsilon}_{kl}^p.$$

Another example of a kinematic hardening model is that due to Ziegler [1959],

$$\dot{\rho}_{ij} = \dot{\mu}(\sigma_{ij} - \rho_{ij}),$$

where

$$\dot{\mu} = \frac{<\frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl}>}{\partial f/\partial \sigma_{mn}(\sigma_{mn} - \rho_{mn})}$$

in order to satisfy the consistency condition $\dot{f} = 0$.

A modification of Equation (3.3.8) that better reproduces the real Bauschinger effect consists of including in its right-hand side a term representing "fading strain memory," so that the rate equation takes the form

$$\dot{\rho}_{ij} = c\dot{\varepsilon}^p_{ij} - a\rho_{ij}\dot{\bar{\varepsilon}}^p.$$

The more general kinematic hardening models can be similarly modified.

Generalized Hardening Rules

The hardening represented by Equation (3.3.7) with both $\boldsymbol{\rho}$ and k variable seems to have been first studied by Kadashevich and Novozhilov [1952]; it is called *combined hardening* by Hodge [1957a]. The combined hardening model proposed for viscoplasticity by Chaboche [1977], presented in Section 3.1, has been applied by Chaboche and his collaborators to rate-independent plasticity as well.

A model with a family of back stresses $\boldsymbol{\rho}_{(l)}$ ($l = 1, 2, ..., n$) is due to Mróz [1967]; a similar model is due to Iwan [1967]. Both models describe materials whose stress-strain curves are piecewise linear. For materials whose stress-strain curves in the work-hardening range are smooth with straight-line asymptotes, a class of models known as *two-surface* models have been proposed by Dafalias [1975] (see also Dafalias and Popov [1975]), Krieg [1975], and others. In these models the yield surface in stress space is constrained to move inside an outer surface, known variously as *bounding surface, loading surface*, or *memory surface*, given by, say, $\overline{f}(\boldsymbol{\sigma}, \boldsymbol{\xi}) = 0$. The plastic modulus H at a given state is assumed to be an increasing function of a suitably defined distance, in stress space, between the current stress $\boldsymbol{\sigma}$ and a stress $\overline{\boldsymbol{\sigma}}$ on the outer surface, called the *image stress* of $\boldsymbol{\sigma}$. When this distance vanishes, the plastic modulus attains its minimum value, and further hardening proceeds linearly, with the two surfaces remaining in contact at $\boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}}$.

The various two-surface models differ from one another in the definition of the bounding surface, in the way the image stress depends on the current state, and in the variation of plastic modulus. In the model of Dafalias and Popov, both surfaces are given similar combined-hardening structures, with a "back stress" $\boldsymbol{\beta}$ playing the same role for the outer surface that $\boldsymbol{\rho}$ plays for the yield surface, and $\overline{\boldsymbol{\sigma}} = c(\boldsymbol{\sigma} - \boldsymbol{\rho}) + \boldsymbol{\beta}$, where c is a constant. H is assumed to depend on $\delta = \sqrt{(\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) : (\overline{\boldsymbol{\sigma}} - \boldsymbol{\sigma})}$ in such a way that $H = \infty$ at initial yield, producing a smooth hardening curve.

Experiments by Phillips and Moon [1977] showed that when yield surfaces are defined on the basis of a very small offset strain, they undergo considerable distortion, in addition to the expansion and translation considered thus far. In order to describe such distortion in initially isotropic materials, Equation (3.3.7) must be modified to

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = F(\boldsymbol{\sigma} - \boldsymbol{\rho}, \boldsymbol{\xi}) - k(\boldsymbol{\xi}),$$

where F is initially an isotropic function of its first argument but becomes anisotropic as plastic deformation takes place. An example of such a function is that proposed by Baltov and Sawczuk [1965] for a Mises-type yield surface:

$$F(\boldsymbol{\sigma} - \boldsymbol{\rho}, \boldsymbol{\xi}) = \frac{1}{2} A_{ijkl}(\boldsymbol{\xi})(\sigma_{ij} - \rho_{ij})(\sigma_{kl} - \rho_{kl}),$$

where

$$A_{ijkl}(\boldsymbol{\xi}) = \delta_{ik}\delta_{ij} - \frac{1}{3}\delta_{ij}\delta_{kl} + A\varepsilon_{ij}^p\varepsilon_{kl}^p,$$

A being a constant. Other proposals are reviewed by Bergander [1980]. Extensive experimental investigation of the hardening of metals were carried out by Phillips and co-workers; their work, along with that of others, is reviewed by Phillips [1986].

Exercises: Section 3.3

1. Show that the forms (3.3.1) and (3.3.2) for the Tresca yield function are equivalent.

2. Derive the associated flow rule for the Tresca yield criterion by means of Koiter's method (see 3.2.2).

3. Show that for any combination of principal stresses, the associated flow rule for the Tresca yield criterion gives $|\dot{\varepsilon}_1^p| + |\dot{\varepsilon}_2^p| + |\dot{\varepsilon}_3^p| = \phi$.

4. An elastic–perfectly plastic solid with a uniaxial yield stress of 300 MPa is assumed to obey the Tresca yield criterion and its associated flow rule. If the rate of plastic work per unit volume is 1.2 MW/m^3, find the principal plastic strain-rate components when

   (a) $\sigma_1 = 300$ MPa, $\sigma_2 = 100$ MPa, $\sigma_3 = 0$,

   (b) $\sigma_1 = 200$ MPa, $\sigma_2 = -100$ MPa, $\sigma_3 = 0$,

   (c) $\sigma_1 = 200$ MPa, $\sigma_2 = -100$ MPa, $\sigma_3 = -100$ MPa.

5. Derive the associated flow rule for the general isotropic yield criterion given by $F(J_2, J_3) - k^2 = 0$, and in particular (a) for the one given by the equation following (3.3.4) and (b) for the analytic form of the Tresca criterion.

6. A work-hardening elastic–plastic solid is assumed to obey the Mises criterion with the associated flow rule and isotropic hardening. If the virgin curve in uniaxial tension can be described in the small-deformation range by $\sigma = F(\varepsilon^p)$, state the rate equations (3.2.1)–(3.2.2) explicitly when k is assumed to depend (a) on $\bar{\varepsilon}^p$ and (b) on W_p.

7. Derive the Mohr-Coulomb criterion as follows.

   (a) Using the theory of Mohr's circles in plane stress, in particular Equations (1.3.9)–(1.3.10), find the direction θ such that $\tau_\theta - \mu(-\sigma_\theta)$ is maximum.

   (b) Show that this maximum value is $\sqrt{1+\mu^2}|\sigma_1 - \sigma_2|/2 + \mu(\sigma_1 + \sigma_2)/2$, and that the Mohr-Coulomb criterion results when this value is equated to the cohesion c, with $\mu = \tan\phi$.

   (c) Show that the Mohr circles whose parameters σ_1, σ_2 are governed by this criterion are bounded by the lines $\pm\tau_\theta = \sigma_\theta \tan\phi - c$.

8. Derive the associated flow rule and plastic dissipation for the Drucker-Prager yield criterion.

9. Given the yield stresses σ_T and σ_C in uniaxial tension and compression, respectively, find the yield stress in shear resulting from the following yield criteria: (a) Mohr–Coulomb, (b) Drucker–Prager, (c) Mises–Schleicher.

10. Show that in a state of plane stress with $\sigma_{11} = \sigma$, $\sigma_{12} = \tau$ and $\sigma_{22} = 0$, both the Tresca and the Mises yield criteria can be expressed in the form (3.3.5).

11. Derive the form of Equation (3.3.6) for the Mohr-Coulomb criterion in plane stress.

12. If the function F in Equation (3.3.7) equals $\sqrt{\bar{J}_2}$, with $\bar{J}_2$ defined as in 3.1.1, while k is constant, and if the evolution of $\boldsymbol{\rho}$ is governed by (3.3.8), show that the rate equation of $\boldsymbol{\rho}$ is

$$\dot{\rho}_{ij} = (s_{ij} - \rho_{ij})\frac{(s_{kl} - \rho_{kl})\dot{s}_{kl}}{2k^2}.$$

13. Generalize the preceding result to the case where k depends on $\bar{\varepsilon}^p$, obtaining rate equations for both $\boldsymbol{\rho}$ and $\boldsymbol{\varepsilon}^p$.

Section 3.4 Uniqueness and Extremum Theorems

3.4.1. Uniqueness Theorems

Uniqueness Theorems in Elastic Bodies

Consider a body made of a linearly elastic material with no internal constraints, occupying a region R and subject to prescribed tractions $\mathbf{t}^a$ on ∂R_t, prescribed displacements $\mathbf{u}^a$ on ∂R_u, and a prescribed body-force field $\mathbf{b}$ in R. For convenience, the body force per unit volume is defined as $\mathbf{f} = \rho\mathbf{b}$. We suppose that a stress field $\boldsymbol{\sigma}$ and a displacement field $\mathbf{u}$ in R have been found such that $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ (where $\boldsymbol{\varepsilon}$ is the strain field derived from $\mathbf{u}$) and $\sigma_{ij},_j + f_i = 0$ in R, $n_j\sigma_{ij} = t_i^a$ on ∂R_t, and $\mathbf{u} = \mathbf{u}^a$ on ∂R_u. In other words, $(\boldsymbol{\sigma}, \mathbf{u})$ constitutes a solution of the static boundary-value problem. Is this solution unique? As a result of the classical uniqueness theorem due to Kirchhoff, the answer is "yes" as regards the stress field and "almost" as regards the displacement field. For, if $(\boldsymbol{\sigma}^{(1)}, \mathbf{u}^{(1)})$ and $(\boldsymbol{\sigma}^{(2)}, \mathbf{u}^{(2)})$ are two different solutions, and if we write $\overline{\phi} = \phi^{(1)} - \phi^{(2)}$ for any ϕ, then $\overline{\sigma}_{ij} = C_{ijkl}\overline{\varepsilon}_{kl}$, $\overline{\sigma}_{ij},_j = 0$ in R, and $n_j\overline{\sigma}_{ij}\bar{u}_i = 0$ on ∂R. Consequently, by the divergence theorem,

$$\begin{aligned} 0 &= \int_R (\overline{\sigma}_{ij}\bar{u}_i),_j \, dV \\ &= \int_R (\overline{\sigma}_{ij},_j \bar{u}_i + \overline{\sigma}_{ij}\bar{u}_i,_j\,)\, dV \\ &= \int_R \overline{\sigma}_{ij}\overline{\varepsilon}_{ij}\, dV \\ &= \int_R C_{ijkl}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{kl}\, dV. \end{aligned}$$

It follows from the positive-definiteness of C (see 1.4.3) that $\overline{\boldsymbol{\varepsilon}}$ must vanish throughout R. Consequently, $\overline{\boldsymbol{\sigma}}$ must vanish as well, while $\bar{\mathbf{u}}$ may have at most the form of a rigid-body displacement; full uniqueness of the displacement field depends on having sufficient external constraints.

If the material were nonlinearly elastic, the same method could be applied, but *incrementally*. Suppose that the stress field $\boldsymbol{\sigma}$ and displacement field $\mathbf{u}$ have been found under the current $\mathbf{f}$, $\mathbf{t}^a$ and $\mathbf{u}^a$. We may then prove the uniqueness of *infinitesimal increments* $d\boldsymbol{\sigma}$ resulting from increments $d\mathbf{f}$, $d\mathbf{t}^a$ and $d\mathbf{u}^a$, provided that C is interpreted as the tangent elastic modulus tensor as defined by Equation (1.4.8), so that

$$d\sigma_{ij} = C_{ijkl}\, d\varepsilon_{kl}.$$

Incremental uniqueness implies global uniqueness, since any state of loading can be attained by the successive imposition of small incremental loads. Consequently, the stress and strain fields are uniquely determined (and the displacement field determined to within a rigid-body displacement) so long as C is positive definite.

Uniqueness of Stress Field in an Elastic–Plastic Body

The positive-definiteness of C means that, for an elastic material, $d\sigma_{ij}\, d\varepsilon_{ij} > 0$ whenever $d\boldsymbol{\sigma} \neq 0$. The last inequality is equivalent to Drucker's first inequality for a work-hardening plastic material (see Section 3.2). Indeed, we know that as long as no unloading occurs, no distinction can be made between plastic and nonlinearly elastic materials. It was shown by Melan [1938] that incremental uniqueness of stress and strain can be established for work-hardening standard materials when unloading has taken place, provided that the hypothesis of infinitesimal strains is valid. The reason for the proviso is that, with finite deformations, a distinction must be made between increments at a fixed material point and those at a fixed point in space (see Hill [1950] and Chapter 8 of the present book).

By analogy with the proof for elastic bodies, it can be shown that a sufficient condition for the uniqueness of the stress field in a plastic body is that if $d\boldsymbol{\sigma}^{(1)}$ and $d\boldsymbol{\sigma}^{(2)}$ are two possible incremental stress fields and $d\boldsymbol{\varepsilon}^{(1)}$ and $d\boldsymbol{\varepsilon}^{(2)}$ are the associated incremental strain fields, then

$$(d\sigma_{ij}^{(1)} - d\sigma_{ij}^{(2)})(d\varepsilon_{ij}^{(1)} - d\varepsilon_{ij}^{(2)}) > 0 \tag{3.4.1}$$

unless $d\boldsymbol{\sigma}^{(1)} = d\boldsymbol{\sigma}^{(2)}$. It was shown by Valanis [1985] that condition (3.4.1) applies in dynamic as well as in quasi-static problems.

Consider next an elastic–plastic body made of standard material, occupying the region R. Let R_e and R_p denote the parts of R where $f < 0$ and $f = 0$, respectively. With linear elasticity assumed, the inequality is clearly satisfied in R_e, while in R_p we may use the general flow equation (3.2.1) together with (3.2.2) and the normality rule (3.2.7). Converting rates into increments by multiplying them by the infinitesimal time increment dt, we obtain, at any point in R_p,

$$d\varepsilon_{ij}^{(\alpha)} = C_{ijkl}^{-1} d\sigma_{kl}^{(\alpha)} + \frac{<df^{(\alpha)}>}{H}\frac{\partial f}{\partial \sigma_{ij}}, \quad \alpha = 1,\, 2, \tag{3.4.2}$$

where

$$df^{(\alpha)} = \frac{\partial f}{\partial \sigma_{kl}} d\sigma_{kl}^{(\alpha)}.$$

Hence

$$(d\sigma_{ij}^{(1)} - d\sigma_{ij}^{(2)})(d\varepsilon_{ij}^{(1)} - d\varepsilon_{ij}^{(2)}) = C_{ijkl}^{-1}(d\sigma_{ij}^{(1)} - d\sigma_{ij}^{(2)})(d\sigma_{kl}^{(1)} - d\sigma_{kl}^{(2)}) + \frac{1}{H}(df^{(1)} - df^{(2)})(<df^{(1)}> - <df^{(2)}>).$$

It is easy to see that any two real numbers a, b satisfy $<a> - <b> = \beta(a - b)$ for some β, $0 \leq \beta \leq 1$, and that therefore the second term on the right-hand side is never negative. Since the first term is positive unless $d\boldsymbol{\sigma}^{(1)} = d\boldsymbol{\sigma}^{(2)}$, the uniqueness of $d\boldsymbol{\sigma}$ (and therefore of $d\boldsymbol{\varepsilon}$) is proved, and hence the uniqueness of the stress and strain fields under a given *history* of $\mathbf{f}$, $\mathbf{t}^a$, and $\mathbf{u}^a$.

If the material is perfectly plastic, then Equation (3.4.2) for the strain increments at points where $f = 0$ must be replaced by

$$d\varepsilon_{ij}^{(\alpha)} = C_{ijkl}^{-1}\, d\sigma_{kl}^{(\alpha)} + d\lambda^{(\alpha)} \frac{\partial f}{\partial \sigma_{ij}}, \quad \alpha = 1,\, 2, \tag{3.4.3}$$

where $d\lambda^{(\alpha)} = 0$ if $df^{(\alpha)} < 0$, and $d\lambda^{(\alpha)} > 0$ only if $df^{(\alpha)} = 0$ but is otherwise undetermined. Thus

$$(d\sigma_{ij}^{(1)} - d\sigma_{ij}^{(2)})(d\varepsilon_{ij}^{(1)} - d\varepsilon_{ij}^{(2)}) = C_{ijkl}^{-1}(d\sigma_{ij}^{(1)} - d\sigma_{ij}^{(2)})(d\sigma_{kl}^{(1)} - d\sigma_{kl}^{(2)})$$
$$+(d\lambda^{(1)} - d\lambda^{(2)})(df^{(1)} - df^{(2)}).$$

The second term evidently vanishes either if both $df^{(1)}$ and $df^{(2)}$ vanish, or if both are negative (leading to $d\lambda^{(1)} = d\lambda^{(2)} = 0$). If $df^{(1)} = 0$ and $df^{(2)} < 0$, then the term equals $-d\lambda^{(1)}\, df^{(2)}$ and is positive, as it is when (1) and (2) are interchanged. The uniqueness of $d\boldsymbol{\sigma}$ — and hence that of the stress field — follows, but not that of $d\boldsymbol{\varepsilon}$.

If normality is not obeyed, then work-hardening (i.e. the positiveness of H) is not sufficient for uniqueness. The corresponding sufficient condition is, instead,

$$H > H_{cr},$$

where H_{cr} is a critical value of the hardening modulus given by Raniecki [1979] as

$$H_{cr} = \frac{1}{2}\left(\sqrt{C_{ijkl}\frac{\partial f}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}}}\sqrt{C_{ijkl}h_{ij}h_{kl}} - C_{ijkl}h_{ij}\frac{\partial f}{\partial \sigma_{kl}}\right).$$

Clearly, $H_{cr} = 0$ when the normality rule (3.2.7) is obeyed.

3.4.2. Extremum and Variational Principles

In Section 1.4 we derived the two fundamental variational principles of elastostatics, which form a dual pair. The principle of minimum potential energy teaches that in a body in stable equilibrium the correct displacement field — that is, the one that, with its associated stress field, forms the solution of the boundary-value problem — is the one which, among all the kinematically admissible displacement fields that are close to it, minimizes the total potential energy Π. Similarly, the principle of minimum complementary energy asserts that the correct stress field is the one which, among all the neighboring stress fields that are statically admissible, minimizes the total complementary energy Π^c.

If the material is linear, then unless large displacements come into play, the respective energies depend at most quadratically on their variables, and it is easy to see that the restriction to admissible fields that are close to the correct one may be removed. Indeed, it can be shown directly that, if Π^* is the total potential energy evaluated at the arbitrary kinematically admissible displacement field $\mathbf{u}^*$, and if $\Pi = \Pi^*|_{\mathbf{u}^*=\mathbf{u}}$, where $\mathbf{u}$ is the correct displacement field, then

$$\Pi^* > \Pi \quad \text{unless} \quad \mathbf{u}^* = \mathbf{u}.$$

For

$$\Pi^* - \Pi = \frac{1}{2}\int_R C_{ijkl}(\varepsilon_{ij}^*\varepsilon_{kl}^* - \varepsilon_{ij}\varepsilon_{kl})\,dV - \int_R f_i(u_i^* - u_i)\,dV - \int_{\partial R_t} t_i^a(u_i^* - u_i)\,dS;$$

but the surface integral may be changed into one over all of ∂R, since $\mathbf{u}^* = \mathbf{u}$ on ∂R_u, and consequently, by the divergence theorem, into

$$\int_R [\sigma_{ij},_j(u_i^* - u_i) + \sigma_{ij}(\varepsilon_{ij}^* - \varepsilon_{ij})]\,dV,$$

where $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ is the correct stress field. If we define $\sigma_{ij}^* = C_{ijkl}\varepsilon_{kl}^*$ then with the help of the equilibrium equations, we obtain

$$\Pi^* - \Pi = \frac{1}{2}\int_R (\sigma_{ij}^*\varepsilon_{ij}^* + \sigma_{ij}\varepsilon_{ij} - 2\sigma_{ij}\varepsilon_{ij}^*)\,dV.$$

The integrand, however, is

$$C_{ijkl}(\varepsilon_{ij}\varepsilon_{kl} + \varepsilon_{ij}^*\varepsilon_{kl}^* - 2\varepsilon_{ij}^*\varepsilon_{kl}) = C_{ijkl}(\varepsilon_{ij}^* - \varepsilon_{ij})(\varepsilon_{kl}^* - \varepsilon_{kl}),$$

and is positive except when $\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon}$.

Similarly,

$$\Pi^{c*} - \Pi^c = \frac{1}{2}\int_R C_{ijkl}^{-1}(\sigma_{ij}^*\sigma_{kl}^* - \sigma_{ij}\sigma_{kl}) - \int_{\partial R_u} n_j(\sigma_{ij}^* - \sigma_{ij})u_i^a\,dS,$$

and with the help of analogous transformations, we obtain

$$\Pi^{c*} - \Pi^c = \frac{1}{2}\int_R C_{ijkl}^{-1}[\sigma_{ij}^*\sigma_{kl}^* - \sigma_{ij}\sigma_{kl} - 2(\sigma_{ij}^* - \sigma_{ij})\varepsilon_{ij}]\,dV,$$

where $\varepsilon_{ij} = C_{ijkl}^{-1}\sigma_{kl}$, so that the integrand is

$$C_{ijkl}^{-1}(\sigma_{ij}^* - \sigma_{ij})(\sigma_{kl}^* - \sigma_{kl}),$$

also positive except when $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}$.

In view of the resemblance between the uniqueness proof and the proofs of the extremum principles, it appears reasonable that such principles may be derived for displacement and stress increments (or, equivalently, velocities and stress rates) in elastic–plastic bodies that are either work-hardening or perfectly plastic. Such is indeed the case. The theorems to be presented have been derived by Handelman [1944], Markov [1947], Greenberg [1949], and Hill [1950];[1] the proofs are Hill's.

Extremum Principle for Displacement

Let $d\mathbf{u}^*$ denote a kinematically admissible displacement increment, that is, one which obeys the internal constraints, if any, and which satisfies $d\mathbf{u}^* = d\mathbf{u}^a$ on ∂R_u. The corresponding incremental strain and stress field are $d\boldsymbol{\varepsilon}^*$ and $d\boldsymbol{\sigma}^*$, where $d\boldsymbol{\sigma}^*$ is not in general statically admissible, but is related to $d\boldsymbol{\varepsilon}^*$ through the associated flow rule; we thus use Equation (3.2.10) with $h_{ij} = \partial f/\partial \sigma_{ij}$, to obtain Hill's [1958] result

$$\dot{\sigma}_{ij} = \begin{cases} C_{ijkl}\dot{\varepsilon}_{kl}, & f < 0, \\ C_{ijkl}\left(\dot{\varepsilon}_{kl} - \dfrac{1}{L} < C_{pqmn}\dfrac{\partial f}{\partial \sigma_{pq}}\dot{\varepsilon}_{mn} > \dfrac{\partial f}{\partial \sigma_{kl}}\right), & f = 0. \end{cases}$$

Here L is given by Equation (3.2.9) for a standard material, namely,

$$L = H + C_{ijkl}\frac{\partial f}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}},$$

and the perfectly plastic case corresponds to $H = 0$.

Let the functional Λ^* be defined by

$$\Lambda^* = \frac{1}{2}\int_R d\sigma_{ij}^* d\varepsilon_{ij}^*\, dV - \int_R df_i\, du_i^*\, dV - \int_{\partial R_t} dt_i^a\, du_i^*\, dS,$$

with

$$\Lambda = \Lambda^*|_{d\mathbf{u}^* = d\mathbf{u}}\,.$$

Note that the form of Λ is essentially that of the potential energy Π of the linear elastic body, with incremental rather than total displacements and with the total (elastic–plastic) tangent modulus tensor in place of C. By means of the transformations used in the elastic case, we can therefore show that

$$\Lambda^* - \Lambda = \frac{1}{2}\int_R (d\sigma_{ij}^*\, d\varepsilon_{ij}^* + d\sigma_{ij}\, d\varepsilon_{ij} - 2d\sigma_{ij}\, d\varepsilon_{ij}^*)\, dV,$$

and it remains to be shown that the integrand is positive unless $d\boldsymbol{\sigma}^* = d\boldsymbol{\sigma}$. With the decomposition $d\boldsymbol{\varepsilon} = \mathsf{C}^{-1}d\boldsymbol{\sigma} + d\boldsymbol{\varepsilon}^p$, the integrand becomes

$$C_{ijkl}^{-1}(d\sigma_{ij}^* - d\sigma_{ij})(d\sigma_{kl}^* - d\sigma_{kl}) + d\sigma_{ij}^* d\varepsilon_{ij}^{p*} + d\sigma_{ij} d\varepsilon_{ij} - 2d\sigma_{ij} d\varepsilon_{ij}^{p*}.$$

[1] Some extensions have been proposed by, among others, Ceradini [1966], Maier [1969, 1970] and Martin [1975].

The term in C^{-1} is clearly nonnegative, and vanishes only if $d\boldsymbol{\sigma}^* = d\boldsymbol{\sigma}$.

As for the remaining terms, in the case of a work-hardening material their sum becomes

$$\phi = \frac{1}{H}(df^* <df^*> + df <df> - 2df <df^*>),$$

where $df = (\partial f/\partial \sigma_{ij})\, d\sigma_{ij}$ and $df^* = (\partial f/\partial \sigma_{ij})\, d\sigma^*_{ij}$. Clearly, ϕ vanishes if df^* and df are both nonpositive, and equals $H^{-1}(df^* - df)^2$ if both are positive. If $df^* > 0$ and $df \le 0$, $\phi = H^{-1}df^*(df^* - 2df) > 0$, while in the opposite case $\phi = H^{-1}(df)^2$. For the perfectly plastic material, we replace $H^{-1}<df>$ by $d\lambda$, with $d\lambda$ related to df as in the uniqueness proof, and similarly $H^{-1}<df^*>$ by $d\lambda^*$, so that

$$\phi = df^*\, d\lambda^* + df\, d\lambda - 2df\, d\lambda^*.$$

The first two terms always vanish, as does the last term except when $df < 0$ and $df^* = 0$, in which case $d\lambda^* > 0$, so that $\phi = -2df\, d\lambda^* > 0$. Consequently ϕ is never negative, so that

$$\Lambda^* > \Lambda \tag{3.4.4}$$

except when $d\boldsymbol{\sigma}^* = d\boldsymbol{\sigma}$. For the work-hardening (but not the perfectly plastic) material this also means that $d\boldsymbol{\varepsilon}^* = d\boldsymbol{\varepsilon}$ and therefore $d\mathbf{u}^* = d\mathbf{u}$ if sufficient constraints exist.

Extremum Principle for Stress

The complementary extremum principle concerns a statically admissible incremental stress field $d\boldsymbol{\sigma}^*$. In the case of a work-hardening material this determines an incremental strain field $d\boldsymbol{\varepsilon}^*$, related to $d\boldsymbol{\sigma}^*$ through the associated flow rule — Equation (3.4.2) with the superscript (α) replaced by $*$ — but not, in general, derivable from a continuous displacement field. Clearly,

$$d\sigma^*_{ij}\, d\varepsilon^*_{ij} = C^{-1}_{ijkl}\, d\sigma^*_{ij}\, d\sigma^*_{kl} + \frac{1}{H}<df^*>^2.$$

If the material is perfectly plastic, on the other hand, then $d\varepsilon^*_{ij} = C^{-1}_{ijkl}\, d\sigma^*_{kl} + d\varepsilon^{p*}_{ij}$, with $d\boldsymbol{\varepsilon}^{p*}$ not determined by $d\boldsymbol{\sigma}^*$. However, in such a material $d\sigma^*_{ij} d\varepsilon^{p*}_{ij} = 0$, and therefore

$$d\sigma^*_{ij}\, d\varepsilon^*_{ij} = C^{-1}_{ijkl}\, d\sigma^*_{ij}\, d\sigma^*_{kl}.$$

We now define the functional

$$\Omega^* = \frac{1}{2}\int_R d\sigma^*_{ij}\, d\varepsilon^*_{ij}\, dV - \int_{\partial R_u} n_j\, d\sigma^*_{ij}\, du^a_i\, dS,$$

with $d\sigma^*_{ij}\, d\varepsilon^*_{ij}$ defined by the appropriate formula above. We also define, as before,

$$\Omega = \Omega^*|_{d\boldsymbol{\sigma}^* = d\boldsymbol{\sigma}}\,.$$

Note that the form of Ω is, *mutatis mutandis*, that of Π^c, and also that

$$\Omega = -\Lambda.$$

The surface integral in Ω^* is transformed into

$$\int_R d\sigma_{ij}^* \, d\varepsilon_{ij} \, dV,$$

and therefore

$$\Omega^* - \Omega = \frac{1}{2} \int_R (d\sigma_{ij}^* \, d\varepsilon_{ij}^* + d\sigma_{ij} \, d\varepsilon_{ij} - 2 d\sigma_{ij}^* \, d\varepsilon_{ij}) \, dV.$$

The integrand differs from the one in $\Lambda^* - \Lambda$ only in the interchange of starred and unstarred quantities, and may by the same method be shown to be nonnegative and to vanish only if $d\boldsymbol{\sigma}^* = d\boldsymbol{\sigma}$, so that

$$\Omega^* > \Omega \quad \text{unless} \quad d\boldsymbol{\sigma}^* = d\boldsymbol{\sigma}. \tag{3.4.5}$$

We furthermore have the double inclusion

$$-\Lambda^* \le -\Lambda = \Omega \le \Omega^*.$$

Variational Principles

As mentioned earlier, an extremum principle is stronger than a variational principle because it asserts an extremum over all admissible functions of a certain class, not only over those that are infinitesimally close to the extremal. Going further, we see that a variational principle need not assert an extremum at all, even a local one, but only the condition that the functional obeying it is stationary. For example, the function $f(x) = x^3$ is stationary at $x = 0$, since $f'(0) = 0$, but has neither a minimum nor a maximum there. Likewise, $f(x, y) = x^2 - y^2$ is stationary at $(0, 0)$; the point is a *saddle point* — a maximum when viewed along the x-axis and a minimum along the y-axis.

Extremum principles are useful for many reasons, one of them being that they allow us to evaluate approximate solutions when the exact solution is unknown: between two incremental displacement fields $d\mathbf{u}^*$, in the absence of other information we choose the one that produces the smaller value of Λ^*. But variational principles are useful even when they are not extremum principles: they permit compact statements of boundary-value problems, and they are useful in formulating the "weak form" of such problems, which is necessary for consistent discretization, as shown in 1.3.5 and as further shown in Section 4.5 which deals with numerical methods.

The extremum principles derived above clearly imply the corresponding variational principles:

$$\delta\Lambda = 0, \quad \delta\Omega = 0.$$

We should remember, however, that the extremum principles rely on the positive-definiteness of certain local quantities, whose proof requires that the material be nonsoftening. Consider, on the other hand, the following functional:

$$\Theta = \frac{1}{2}\int_R C_{ijkl}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{kl}\,dV - \frac{1}{2}\int_{R_p}\frac{1}{L}{<}C_{ijkl}\frac{\partial f}{\partial\sigma_{kl}}\dot{\varepsilon}_{ij}{>}^2\,dV$$
$$-\int_R \dot{b}_i\dot{u}_i\,dV - \int_{\partial R_t}\dot{t}_i^a\dot{u}_i\,dV.$$

It is easy to see that Θ is just Λ with rates in place of increments; the change is made to avoid mixing the differential operator d with the variation operator δ. Clearly, with $\dot{\boldsymbol{\sigma}}$ given in terms of $\dot{\boldsymbol{\varepsilon}}$ by the strain-space flow rule,

$$\delta\Theta = \int_R \dot{\sigma}_{ij}\,\delta\dot{\varepsilon}_{ij}\,dV - \int_R \dot{b}_i\,\delta\dot{u}_i\,dV - \int_{\partial R_t}\dot{t}_i^a\,\delta\dot{u}_i\,dV,$$

and $\delta\Theta = 0$ for an arbitrary kinematically admissible $\delta\dot{\mathbf{u}}$ only if $\dot{\boldsymbol{\sigma}}$ is statically admissible.

3.4.3. Rigid–Plastic Materials

In discussing the Saint-Venant–Lévy–Mises flow rule in the preceding section, we mentioned in passing that these authors equated $\dot{\boldsymbol{\varepsilon}}$ with $\dot{\boldsymbol{\varepsilon}}^p$, in effect neglecting the elastic strain rate — a treatment tantamount to treating the nonvanishing elastic moduli as infinite. Any solutions obtained on this basis are, theoretically, valid for idealized materials called *rigid–plastic*.[1] Practically, however, they are useful approximations for two classes, not mutually exclusive, of problems: (a) those in which the elastic strain rates may be neglected, and (b) those in which the elastic strains are significantly smaller than the plastic ones. Problems of class (a) include those of *impending collapse* or *incipient plastic flow* of elastic–perfectly plastic bodies, for which it is shown later in this section that, when deformation proceeds at constant loads, the elastic strain rates vanish identically. In order to accommodate problems of class (b), the hypothesis of infinitesimal deformations should be abandoned in formulating any general theorems for this theory. Since the problems are frequently those of flow, it appears natural to use an Eulerian formulation, in which R is the region currently occupied by the body and $\boldsymbol{\sigma}$ is the true (Cauchy) stress tensor. No reference configuration is used in

[1]In many references, the theory of rigid–plastic materials obeying the Mises yield criterion and the associated flow rule is called the **Mises theory**, while the corresponding theory of elastic–plastic materials is called the **Prandtl–Reuss theory**.

this approach, and therefore no displacement field $\mathbf{u}$ appears; in its place we have the velocity field $\mathbf{v}$. The boundary ∂R is accordingly partitioned into ∂R_t and ∂R_v, with $\mathbf{v} = \mathbf{v}^a$ on ∂R_v. No strain tensor is introduced, but rather the Eulerian deformation-rate tensor (also called *stretching tensor*) $\mathbf{d}$, defined by

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}).$$

With infinitesimal deformations, of course, $\mathbf{d}$ approximates $\dot{\boldsymbol{\varepsilon}}$.

Uniqueness of Stress Field

In general plasticity theory, the decomposition of the deformation rate into elastic and plastic parts is far from unequivocal, as is shown in Chapter 8. In a rigid–plastic material, however, $\mathbf{d}$ may be identified with the plastic strain rate, and thus the plastic dissipation per unit volume, D_p, defined in Section 3.2, satisfies

$$D_p(\mathbf{d}) = \sigma_{ij} d_{ij}. \tag{3.4.6}$$

The maximum-plastic-dissipation principle may therefore be written

$$(\sigma_{ij} - \sigma_{ij}^*) d_{ij} \geq 0. \tag{3.4.7}$$

If $\mathbf{d} \neq 0$, then the equality holds only if $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ are *plastically equivalent*, that is, if $\mathbf{d}$ is related to both of them through the associated flow rule. In a Mises material, plastically equivalent stresses differ at most by a hydrostatic pressure, but in a Tresca material two stresses are plastically equivalent if they lie on the same facet of the hexagonal cylinder in principal-stress space.

If, now, $(\boldsymbol{\sigma}^{(\alpha)}, \mathbf{v}^{(\alpha)})$, $\alpha = 1, 2$, represent two admissible states of a rigid–plastic body, corresponding to the same body force and boundary conditions, then

$$n_j(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})(v_i^{(1)} - v_i^{(2)}) = 0 \quad \text{on } \partial R,$$

and therefore

$$\int_{\partial R} n_j(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})(v_i^{(1)} - v_i^{(2)})\, dS = \int_R (\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})(d_{ij}^{(1)} - d_{ij}^{(2)})\, dV = 0.$$

But the last integrand may be written as

$$(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}) d_{ij}^{(1)} + (\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}) d_{ij}^{(2)},$$

and this is positive unless either $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ both vanish, or $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ are plastically equivalent. The general conclusion is thus that two admissible stress fields $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ must be plastically equivalent everywhere except in their common rigid region, that is, at points where $f(\boldsymbol{\sigma}^{(\alpha)}) < 0$, $\alpha = 1, 2$. If the body is made of a Mises material (or any other material whose yield surface in stress-deviator space is strictly convex) and deforms plastically in its entirety, then the two stress fields can differ at most by a hydrostatic pressure field, which must be uniform in order to satisfy equilibrium, and must vanish if a surface traction is prescribed anywhere on ∂R. Thus there is not more than one admissible stress field for which the whole body is plastic (Hill [1948a]), unless $\partial R = \partial R_v$, in which case the stress field is determined only to within a uniform hydrostatic pressure. On the other hand, uniqueness of the velocity field is not established.

Extremum Principle for Velocity

The degree to which the velocity field is determined may be learned from the kinematic extremum principle to be shown next, first proposed by Markov [1947] for a Mises material. Given a kinematically admissible velocity field $\mathbf{v}^*$, we define a functional Γ^* by

$$\Gamma^* = \int_R D_p(\mathbf{d}^*)\, dV - \int_R f_i v_i^*\, dV - \int_{\partial R_t} t_i^a v_i^*\, dS,$$

and, as usual,

$$\Gamma = \Gamma^*|_{\mathbf{v}^*=\mathbf{v}}.$$

Using the standard transformations we can show that

$$\Gamma = \int_{\partial R_v} n_j \sigma_{ij} v_i^a\, dS \tag{3.4.8}$$

and that

$$\Gamma^* - \Gamma = \int_R [D_p(\mathbf{d}^*) - D_p(\mathbf{d}) - \sigma_{ij}(d_{ij}^* - d_{ij})]\, dV.$$

Because of (3.4.6), the integrand is just $D_p(\mathbf{d}^*) - \sigma_{ij} d_{ij}^*$, and this is nonnegative as a result of the maximum-plastic-dissipation principle, Equation (3.4.7), since the actual stress $\boldsymbol{\sigma}$ necessarily obeys the yield criterion. Consequently,

$$\Gamma^* \geq \Gamma. \tag{3.4.9}$$

It is not possible, in general, to strengthen the inequality by asserting that the equality holds only when $\mathbf{v}^* = \mathbf{v}$. The most we can say is that if $\Gamma^* = \Gamma$, then $\mathbf{v}^*$ is kinematically admissible, and $\mathbf{d}^*$ is associated with a stress field $\boldsymbol{\sigma}^*$ that is statically admissible and obeys the yield criterion everywhere. More particularly, however, if the body is one for which the stress field is unique (see above) and if the stress determines the deformation rate to within a scale factor (as is true of the Mises material, or any other material with a smooth yield surface), then the entire deformation-rate field is determined to within a scale factor. The indeterminacy may be eliminated if a nonzero velocity is prescribed anywhere on ∂R_v.

Principle of Maximum Plastic Work

Given a statically admissible stress field $\boldsymbol{\sigma}^*$ which nowhere violates the yield criterion, then, if $\boldsymbol{\sigma}$ and $\mathbf{v}$ are the actual stress and velocity fields, respectively, we clearly have

$$\int_R (\sigma_{ij} - \sigma_{ij}^*) d_{ij}\, dV = \int_{\partial R_v} n_j(\sigma_{ij} - \sigma_{ij}^*) v_i^a\, dS.$$

From the principle of maximum plastic dissipation there immediately follows the result

$$\int_{\partial R_v} n_j \sigma_{ij} v_i^a\, dS \geq \int_{\partial R_v} n_j \sigma_{ij}^* v_i^a\, dS,$$

due to Hill [1948a] and dubbed by him the **principle of maximum plastic work**. Here, again, the equality holds only if $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}$ are plastically equivalent, and therefore only if $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}$ whenever $\boldsymbol{\sigma}$ is unique. Note that the left-hand side of the inequality is equal to Γ, Equation (3.4.8), and is therefore bounded both above and below.

Many results relating to variational principles in both elastic–plastic and rigid–plastic solids are contained in Washizu [1975].

Exercises: Section 3.4

1. Show that for any two real numbers a, b,

$$<a> - <b> = \beta(a - b)$$

for some β, $0 \leq \beta \leq 1$.

2. Show that the Hu–Washizu principle (Exercise 14 of Section 1.4) may be extended to elastic–plastic materials if $W(\boldsymbol{\varepsilon})$ is replaced by $W(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$.

3. Derive Equation (3.4.8) and the one following it.

Section 3.5 Limit-Analysis and Shakedown Theorems

3.5.1. Standard Limit-Analysis Theorems

The extremum principles for standard rigid–plastic materials that were discussed in the preceding section can be reformulated as the **theorems of limit analysis**, which give upper and lower bounds on the loads under which a body that may be approximately modeled as elastic–perfectly plastic reaches a critical state. By a critical state we mean one in which large increases in plastic deformation — considerably greater than the elastic deformation — become possible with little if any increase in load. In the case of perfectly plastic bodies this state is called *unrestricted plastic flow*,[1] and the loading state at which it becomes possible is called *ultimate* or *limit loading*. It will be shown that, in a state of unrestricted plastic flow, elasticity may be ignored, and therefore a theory based on rigid–plastic behavior is valid for elastic–plastic bodies.

The proof of the limit-analysis theorems is based on the principle of maximum plastic dissipation, and consequently they are valid only for standard materials; a limited extension to nonstandard materials is discussed in 3.5.2.

It should be noted that the "loads" in the present context include not only the prescribed surface tractions $\mathbf{t}^a$ but all the surface tractions $\mathbf{t}$ operating at points at which the displacement (or velocity) is not constrained to be zero. In other words, the loads include *reactions that do work*; the definition of ∂R_t is accordingly extended. The reason for the extension is that the velocity fields used in the limit-analysis theorems are kinematically admissible velocity fields, not virtual velocity fields. The latter is, as we recall, the difference between two kinematically admissible velocity fields, and must therefore vanish wherever the velocity is prescribed, whereas a kinematically admissible velocity field takes on the values of the prescribed velocity.

We begin by defining a state of impending plastic collapse or incipient plastic flow as one in which a nonvanishing strain rate ($\dot{\boldsymbol{\varepsilon}} \neq 0$) occurs under constant loads ($\dot{\mathbf{f}} = 0$, $\dot{\mathbf{t}} = 0$). The qualification "impending" or "incipient" is important: we are looking at the very beginning of such a state, which means that (1) all prior deformation has been of the same order of magnitude as elastic deformation, so that changes of geometry can be neglected, and (2) acceleration can be neglected and the problem can be treated as quasi-static.

Vanishing of Elastic Strain Rates

In addition to the preceding assumptions, it is assumed that the equations of equilibrium and the traction boundary conditions can be differentiated with respect to time with no change in form; consequently, the principle of virtual work is valid with $\dot{\boldsymbol{\sigma}}$, $\dot{\mathbf{f}}$, and $\dot{\mathbf{t}}$ replacing $\boldsymbol{\sigma}$, $\mathbf{f}$, and $\mathbf{t}$. For a virtual displacement field we take $\mathbf{v}\,\delta t$, where $\mathbf{v}$ is the actual velocity field and δt is a small time increment. The virtual strain field is, accordingly, $\dot{\varepsilon}_{ij}\,\delta t$, where $\dot{\varepsilon}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$. At impending collapse, then,

$$0 = \int_R \dot{\mathbf{f}} \cdot \mathbf{v}\, dV + \int_{\partial R_t} \dot{\mathbf{t}} \cdot \mathbf{v}\, dS = \int_R \dot{\sigma}_{ij}\dot{\varepsilon}_{ij}\, dV = \int_R \dot{\sigma}_{ij}(\dot{\varepsilon}^p_{ij} + C^{-1}_{ijkl}\dot{\sigma}_{kl})\, dV.$$

[1] A rigorous formulation of the theorems in the context of convex analysis is due to Frémond and Friaâ [1982], who show that the concept of unrestricted plastic flow is too general for materials whose elastic range is unbounded in the appropriate stress space, and must be replaced by the weaker concept of "almost unrestricted" plastic flow (*écoulement presque libre*).

The positive definiteness of the elastic complementary energy implies $C^{-1}_{ijkl}\dot\sigma_{ij}\dot\sigma_{kl} \ge 0$ unless $\dot{\boldsymbol{\sigma}} = 0$. This fact, combined with Drucker's inequality (3.2.3), implies that at impending collapse or incipient plastic flow the stress rates vanish, so that $\dot{\boldsymbol{\varepsilon}}^e = 0$ and $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^p$. In other words, a body experiencing plastic collapse or flow behaves as though it were rigid–plastic rather than elastic–plastic. This result, first noted by Drucker, Greenberg, and Prager [1951], makes possible the rigorous application to elastic–plastic bodies of the theorems of limit analysis that had previously been formulated for rigid–plastic bodies. The following presentation of the theorems follows Drucker, Greenberg and Prager.[1]

Lower-Bound Theorem

Suppose that at collapse the actual loads are $\mathbf{f}$, $\mathbf{t}$ and the actual stress, velocity and strain-rate fields (in general unknown) are $\boldsymbol{\sigma}$, $\mathbf{v}$ and $\dot{\boldsymbol{\varepsilon}}$. Suppose further that we have somehow determined a stress field $\boldsymbol{\sigma}^*$ which does not violate the yield criterion anywhere and which is in equilibrium with the loads $\mathbf{f}^* = (1/s)\mathbf{f}$, $\mathbf{t}^* = (1/s)\mathbf{t}$, where s is a numerical factor. By virtual work, we have

$$\begin{aligned}\int_R \sigma^*_{ij}\dot\varepsilon_{ij}\,dV &= \frac{1}{s}\left(\int_{\partial R}\mathbf{t}\cdot\mathbf{v}\,dS + \int_R \mathbf{f}\cdot\mathbf{v}\,dV\right)\\ &= \frac{1}{s}\int_R \sigma_{ij}\dot\varepsilon_{ij}\,dV = \frac{1}{s}\int_R D_p(\dot{\boldsymbol{\varepsilon}})\,dV.\end{aligned}$$

But, by the principle of maximum plastic dissipation, $D_p(\dot{\boldsymbol{\varepsilon}}) \ge \sigma^*_{ij}\dot\varepsilon_{ij}$, so that $s \ge 1$. In other words, the factor s (the so-called "static multiplier") is in fact a **safety factor**.

Upper-Bound Theorem

Let us suppose next that instead of $\boldsymbol{\sigma}^*$, we somehow determine a velocity field $\mathbf{v}^*$ (a *collapse mechanism*), with the corresponding strain-rate field $\dot{\boldsymbol{\varepsilon}}^*$, and loads $\mathbf{f}^* = c\mathbf{f}$, $\mathbf{t}^* = c\mathbf{t}$ that satisfy

$$\int_{\partial R_t}\mathbf{t}^*\cdot\mathbf{v}^*\,dS + \int_R \mathbf{f}^*\cdot\mathbf{v}^*\,dV = \int_R D_p(\dot{\boldsymbol{\varepsilon}}^*)\,dV,$$

provided the right-hand side (the total plastic dissipation) is positive;[2] then, again by virtual work,

$$\int_R D_p(\dot{\boldsymbol{\varepsilon}}^*)\,dV = c\int_R \sigma_{ij}\dot\varepsilon^*_{ij}\,dV,$$

where $\boldsymbol{\sigma}$ is, as before, the actual stress field at collapse. The principle of maximum plastic dissipation, however, also implies that $D_p(\dot{\boldsymbol{\varepsilon}}^*) \ge \sigma_{ij}\dot\varepsilon^*_{ij}$.

[1] See also Drucker, Prager and Greenberg [1952], Hill [1951, 1952], and Lee [1952].

[2] If the total plastic dissipation is negative, $\mathbf{v}^*$ can be replaced by $-\mathbf{v}^*$; if it is zero, $\mathbf{v}^*$ does not represent a collapse mechanism.

Consequently $c \geq 1$, that is, c (the "kinematic multiplier") is an **overload factor**.

Alternative Formulation for One-Parameter Loading

Rather than using multipliers, the theorems can also be expressed in terms of a single reference load to which all the loads on the body are proportional. Let this reference load be denoted P, and let the loading (consisting of applied loads and working reactions) be expressed as

$$\mathbf{f} = P\tilde{\mathbf{f}}, \quad \mathbf{t} = P\tilde{\mathbf{t}},$$

where $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{t}}$ are known functions of position in R and on ∂R_t, respectively. Let, further, P_U denote the value of P at collapse. If a plastically admissible stress field $\boldsymbol{\sigma}^*$ is in equilibrium with $P\tilde{\mathbf{f}}$ and $P\tilde{\mathbf{t}}$ for some value of P, say ${P_{LB}}^*$, then this value is a lower bound, that is, ${P_{LB}}^* \leq P_U$. An upper bound ${P_{UB}}^*$ can be found explicitly for a kinematically admissible velocity field $\mathbf{v}^*$:

$$P_U \leq {P_{UB}}^* = \frac{\int_R D_p(\dot{\boldsymbol{\varepsilon}}^*)dV}{\int_R \tilde{\mathbf{f}} \cdot \mathbf{v}^* \, dV + \int_{\partial R_t} \tilde{\mathbf{t}} \cdot \mathbf{v}^* \, dS}. \tag{3.5.1}$$

For loading governed by a single parameter, therefore, the two theorems may also be expressed as follows: *The loads that are in equilibrium with a stress field that nowhere violates the yield criterion do not exceed the collapse loads, while the loads that do positive work on a kinematically admissible velocity field at a rate equal to the total plastic dissipation are at least equal to the collapse loads*. If the loads produced by the application of the two theorems are equal to each other, then they equal the collapse loads.

In particular, if one has succeeded in finding both (a) a statically and plastically admissible stress field, and (b) a kinematically admissible velocity field such that the strain rate produced by it is everywhere[1] associated to the stress, then a *complete solution* is said to have been found. This solution is not necessarily unique and hence cannot be called an exact solution, but, as a result of the theorems of limit analysis, it predicts the correct collapse load. Some applications of this concept are given in Chapter 5. In Chapter 6 the theorems are used to obtain estimates of collapse loads in problems for which no complete solutions have been found.

Multiparameter Loadings

We now consider loadings that are governed by several parameters that can vary independently. These parameters will be called *generalized loads*; some of them may be applied loads, and others may be reactions that do

[1] If the velocity field involves regions that move as rigid bodies (*rigid regions*), then the strain rate there is of course zero and the question of association does not arise.

work. Let them be denoted P_I ($I = 1, ..., N$), so that

$$\mathbf{f} = \sum_{I=1}^{N} P_I \tilde{\mathbf{f}}^{(I)}, \quad \mathbf{t} = \sum_{I=1}^{N} P_I \tilde{\mathbf{t}}^{(I)},$$

the $\tilde{\mathbf{f}}^{(I)}$ and $\tilde{\mathbf{t}}^{(I)}$ again being known functions. For a kinematically admissible velocity field $\mathbf{v}$, generalized velocities $\dot{p}_I$ can be defined by

$$\dot{p}_I = \int_R \tilde{\mathbf{f}}^{(I)} \cdot \mathbf{v} dR + \int_{\partial R_t} \tilde{\mathbf{t}}^{(I)} \cdot \mathbf{v} dS,$$

so that, by virtual work,

$$\int_R \sigma_{ij} \dot{\varepsilon}_{ij} \, dV = \sum_{I=1}^{N} P_I \dot{p}_I \stackrel{\text{def}}{=} \mathbf{P} \cdot \dot{\mathbf{p}}, \tag{3.5.2}$$

where $\mathbf{P}$ and $\dot{\mathbf{p}}$ are the N-dimensional vectors representing the P_I and $\dot{p}_I$. Any combination of generalized loads P_I thus represents a point in $\mathbf{P}$-space (a *load point*), while a fixed proportion among the P_I represents a direction in this space (a *loading direction*).

A combination of P_I at which unrestricted plastic flow occurs represents a *limit point* (or *flow point* or *yield point* — the last designation makes sense only in terms of a rigid–plastic material), and the set of all such points is the *limit locus* (or *flow locus* or *yield locus*), given by, say,

$$\Phi(\mathbf{P}) = 0.$$

It follows from Equation (3.5.2) and the principle of maximum plastic dissipation that if $\mathbf{P}$ and $\mathbf{P}^*$ are on or inside the limit locus, and if $\dot{\mathbf{p}}$ is a generalized velocity vector that is possible under a load vector $\mathbf{P}$, then

$$(\mathbf{P} - \mathbf{P}^*) \cdot \dot{\mathbf{p}} \geq 0. \tag{3.5.3}$$

By arguments identical with those of 3.2.2 for the maximum-plastic-dissipation principle in terms of stresses and strain rates, it follows from inequality (3.5.3) that the limit locus is convex, and that the generalized velocity vector is normal to the limit locus, that is,

$$\dot{p}_I = \dot{\lambda} \frac{\partial \Phi}{\partial P_I}$$

wherever $\Phi(\mathbf{P})$ is regular. The appropriate generalization, following either the Koiter or the Moreau formalism, may be formed for singular limit loci.

A fixed loading direction is described by $N - 1$ parameters, and keeping these constant is equivalent to one-parameter loading. Consequently, along any direction lower-bound and upper-bound load points can be found, and since they depend on the $N-1$ parameters, they form parametric representations of upper-bound and lower-bound loci.

A simple illustration of multi-parameter loading is provided by an "ideal sandwich beam," composed of two equal, very thin flanges of cross-sectional area A, separated by a distance h that is spanned by a web of negligible area [see Figure 3.5.1(a)]. The beam is subject to an axial force P, whose line of

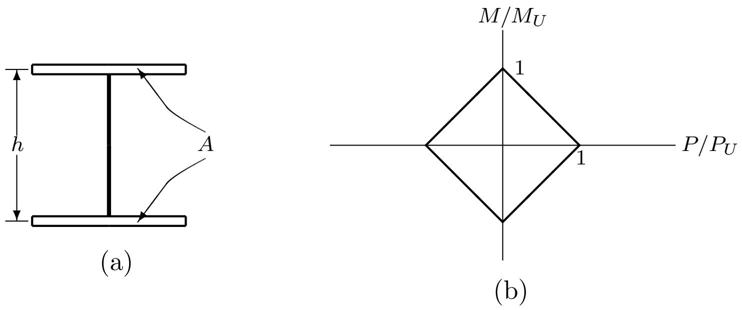


Figure 3.5.1. Ideal sandwich beam: (a) cross-section; (b) flow locus.

action is defined as midway between the flanges, and a bending moment M; P and M may be treated as generalized loads, and the conjugate generalized velocities are respectively the average elongation rate $\dot{\Delta}$ and the rotation rate $\dot{\theta}$. The stresses in the flanges may be assumed to be purely axial, and equilibrium requires that they be

$$\sigma = \frac{P}{2A} \pm \frac{M}{Ah}.$$

If the yield stress is σ_Y, then the yield inequality $|\sigma| \leq \sigma_Y$ in each flange is equivalent to

$$\left|\frac{P}{P_U} \pm \frac{M}{M_U}\right| \leq 1,$$

where $P_U = 2\sigma_Y A$, $M_U = \sigma_Y Ah$. The limit locus, shown in Figure 3.5.1(b), is thus given by

$$\Phi(M,\, P) = \max\left(\left|\frac{P}{P_U} + \frac{M}{M_U}\right|, \left|\frac{P}{P_U} - \frac{M}{M_U}\right|\right) - 1 = 0.$$

Since this locus was found on the basis of a stress distribution, it is strictly speaking a lower-bound locus. The stress distribution is, of course, unique, and therefore the lower bound must in fact give the true limit locus. It can easily be shown that the same limit locus results from an application of the upper-bound theorem. Consider a velocity field in which only one of the flanges elongates while the other remains rigid. In order for the mean elongation rate of the beam to be $\dot{\Delta}$, that of the deforming flange must be $2\dot{\Delta}$, and the rotation rate, to within a sign, is $\dot{\theta} = 2\dot{\Delta}/h$. For the sake of definiteness, let us take both $\dot{\Delta}$ and $\dot{\theta}$ as positive. The strain rate in the deforming flange is $2\dot{\Delta}/L$, so that the total plastic dissipation — the numerator on the right-hand side of (3.5.1) — is $\sigma_Y(2\dot{\Delta}/L)(AL) = P_U\dot{\Delta}$. We take P as the reference load, and pick a loading direction by letting $M = \alpha Ph/2$. The denominator in (3.5.1) is thus $\dot{\Delta} + (\alpha h/2)(2\dot{\Delta}/h) = (1+\alpha)\dot{\Delta}$, and the upper bound for P is $P_U/(1+\alpha)$. Since $M_U = P_U h/2$, the upper bound for M is $\alpha M_U/(1+\alpha)$. The upper-bound values satisfy $M/M_U + P/P_U = 1$, an equation describing the first quadrant of the previously found limit locus. The remaining quadrants are found by varying the signs of $\dot{\Delta}$ and $\dot{\theta}$.

A velocity field with both flanges deforming leads to an upper-bound load point lying outside the limit locus just found, with two exceptions: one where the elongation rates of the flanges are the same, and one where they are equal and opposite. Details are left to an exercise.

3.5.2. Nonstandard Limit-Analysis Theorems

The theorems of limit analysis can be stated in a form that does not directly refer to any concepts from plasticity theory:

A body will not collapse under a given loading if a possible stress field can be found that is in equilibrium with a loading greater than the given loading.

A body will collapse under a given loading if a velocity field obeying the constraints (or a mechanism) can be found that so that the internal dissipation is less than the rate of work of the given loading.

In this form, the theorems appear intuitively obvious. In fact, the concepts underlying the theorems were used long before the development of plasticity theory. Use of what is essentially the upper-bound theorem goes back to the eighteenth century: it was used in 1741 by a group of Italian mathematicians to design a reinforcement method for the crumbling dome of St. Peter's church, and in 1773 by Coulomb to investigate the collapse strength of soil. The latter problem was also studied by Rankine in the mid-nineteenth century by means of a technique equivalent to the lower-bound theorem.

The simple form of the theorems given above hides the fact that the postulate of maximum plastic dissipation (and therefore the normality of the flow rule) is an essential ingredient of the proof. It was therefore necessary to find a counterexample showing that the theorems are not universally applicable to nonstandard materials. One such counterexample, in which plasticity is combined with Coulomb friction at an interface, was presented by Drucker [1954a]. Another was shown by Salençon [1973].

Radenkovic's Theorems

A theory of limit analysis for nonstandard materials, with a view toward its application to soils, was formulated by Radenkovic [1961, 1962], with modifications by Josselin de Jong [1965, 1974], Palmer [1966], Sacchi and Save [1968], Collins [1969], and Salençon [1972, 1977]. **Radenkovic's first theorem** may be stated simply as follows: *The limit loading for a body made of a nonstandard material is bounded from above by the limit loading for the standard material obeying the same yield criterion.*

The proof is straightforward. Let $\mathbf{v}^*$ denote any kinematically admissible velocity field, and $\mathbf{P}^*$ the upper-bound load point obtained for the standard material on the basis of this velocity field. If $\boldsymbol{\sigma}$ is the actual stress field at collapse in the real material, then, since this stress field is also statically and plastically admissible in the standard material,

$$D_p(\dot{\boldsymbol{\varepsilon}}^*) \geq \sigma_{ij}\dot{\varepsilon}_{ij}{}^*,$$

and therefore, by virtual work,

$$\mathbf{P}^* \cdot \dot{\mathbf{p}}^* \geq \mathbf{P} \cdot \dot{\mathbf{p}}^*.$$

Since $\mathbf{v}^*$ may, as a special case, coincide with the correct collapse velocity field in the fictitious material, $\mathbf{P}^*$ may be the correct collapse loading in this material, and the theorem follows.

Radenkovic's second theorem, as modified by Josselin de Jong [1965], is based on the existence of a function $g(\boldsymbol{\sigma})$ with the following properties:

1. $g(\boldsymbol{\sigma})$ is a convex function (so that any surface $g(\boldsymbol{\sigma}) = \textit{constant}$ is convex);
2. $g(\boldsymbol{\sigma}) = 0$ implies $f(\boldsymbol{\sigma}) \leq 0$ (so that the surface $g(\boldsymbol{\sigma}) = 0$ lies entirely within the yield surface $f(\boldsymbol{\sigma}) = 0$);
3. to any $\boldsymbol{\sigma}$ with $f(\boldsymbol{\sigma}) = 0$ there corresponds a $\boldsymbol{\sigma}'$ such that (a) $\dot{\boldsymbol{\varepsilon}}^p$ is normal to the surface $g(\boldsymbol{\sigma}) = 0$ at $\boldsymbol{\sigma}'$, and (b)

$$(\sigma_{ij} - \sigma'_{ij})\dot{\varepsilon}_{ij} \geq 0. \tag{3.5.4}$$

The theorem may then be stated thus: *The limit loading for a body made of a nonstandard material is bounded from below by the limit loading for the standard material obeying the yield criterion* $g(\boldsymbol{\sigma}) = 0$.

The proof is as follows. Let $\boldsymbol{\sigma}$ denote the actual stress field at collapse, $\mathbf{P}$ the limit loading, $\mathbf{v}$ the actual velocity field at collapse, $\dot{\boldsymbol{\varepsilon}}$ the strain-rate field, and $\dot{\mathbf{p}}$ the generalized velocity vector conjugate to $\mathbf{P}$. Thus, by virtual work,

$$\mathbf{P} \cdot \dot{\mathbf{p}} = \int_R \sigma_{ij}\dot{\varepsilon}_{ij}\, dV.$$

Now, the velocity field $\mathbf{v}$ is kinematically admissible in the fictitious standard material. If $\boldsymbol{\sigma}'$ is the stress field corresponding to $\boldsymbol{\sigma}$ in accordance with the definition of $g(\boldsymbol{\sigma})$, then it is the stress field in the fictitious material that is plastically associated with $\dot{\boldsymbol{\varepsilon}}$, and, if $\mathbf{P}'$ is the loading that is in equilibrium with $\boldsymbol{\sigma}'$, then

$$\mathbf{P}' \cdot \dot{\mathbf{p}} = \int_R \sigma'_{ij} \dot{\varepsilon}_{ij} \, dV.$$

It follows from inequality (3.5.4) that

$$\mathbf{P}' \cdot \dot{\mathbf{p}} \leq \mathbf{P} \cdot \dot{\mathbf{p}}.$$

Again, $\boldsymbol{\sigma}'$ may, as a special case, coincide with the correct stress field at collapse in the standard material, and therefore $\mathbf{P}'$ may be the correct limit loading in this material. The theorem is thus proved.

In the case of a Mohr–Coulomb material, the function $g(\boldsymbol{\sigma})$ may be identified with the plastic potential if this is of the same form as the yield function, but with an angle of dilatation that is less than the angle of internal friction (in fact, the original statement of the theorem by Radenkovic [1962] referred to the plastic potential only). The same is true of the Drucker–Prager material.

It should be noted that neither the function g, nor the assignment of $\boldsymbol{\sigma}'$ to $\boldsymbol{\sigma}$, is unique. In order to achieve the best possible lower bound, g should be chosen so that the surface $g(\boldsymbol{\sigma}) = 0$ is as close as possible to the yield surface $f(\boldsymbol{\sigma}) = 0$, at least in the range of stresses that are expected to be encountered in the problem studied. Since the two surfaces do not coincide, however, it follows that the lower and upper bounds on the limit loading, being based on two different standard materials, cannot be made to coincide. The correct limit loading in the nonstandard material cannot, therefore, be determined in general. This result is consistent with the absence of a uniqueness proof for the stress field in a body made of a nonstandard perfectly plastic material (see 3.4.1).

3.5.3. Shakedown Theorems

The collapse discussed thus far in the present section is known as *static collapse*, since it represents unlimited plastic deformation while the loads remain constant in time. If the loads are applied in a cyclic manner, without ever reaching the static collapse condition, other forms of collapse may occur. If the strain increments change sign in every cycle, with yielding on both sides of the cycle, then *alternating plasticity* is said to occur; the *net* plastic deformation may remain small, but weakening of the material may occur nevertheless — a phenomenon called *low-cycle fatigue* — leading to breaking of the most highly stressed points after a certain number of cycles.

It may also happen that plastic deformation in each cycle accumulates so that after enough cycles, the displacements are large enough to be equivalent to collapse; this is called *incremental collapse*. On the other hand, it may happen that no further plastic deformation occurs after one or a few cycles — that is, all subsequent unloading-reloading cycles are elastic. In that case the body is said to have experienced *shakedown* or *adaptation*. It is obvious that for bodies subject to repeated loading, shakedown is more relevant than static collapse, and that criteria for shakedown are of great importance.

Residual Stresses: Example

If an initially stress-free body has been loaded into the plastic range, but short of collapse, and the loads are then reduced to zero, then the stress field in the unloaded body does not in general vanish. As an example, consider the four-flange sandwich beam shown in Figure 3.5.2, subject to a bending moment M only. If the flanges behave similarly in tension and compression,

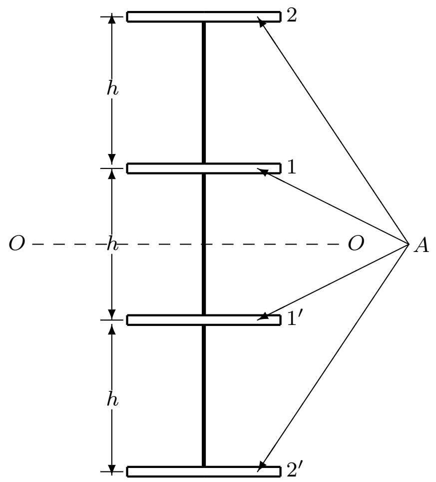


Figure 3.5.2. Ideal four-flange beam: geometry.

then the stresses in the flanges satisfy $\sigma_{1'} = -\sigma_1$ and $\sigma_{2'} = -\sigma_2$, and the moment is

$$M = -\sigma_1 Ah - 3\sigma_2 Ah.$$

In accordance with elementary beam theory (which is discussed further in Section 4.4), the longitudinal strain in the flanges varies linearly with distance from the so-called *neutral plane*, which in the present case may be shown to be the plane OO. Thus $\varepsilon_2 = 3\varepsilon_1$, with $\varepsilon_{1'} = -\varepsilon_1$ and $\varepsilon_{2'} = -\varepsilon_2$. As long as all flanges are elastic, stress is proportional to strain, and therefore $\sigma_2 = 3\sigma_1$. It follows that

$$\sigma_1 = -\frac{M}{10Ah} \stackrel{\text{def}}{=} \sigma_1^e, \quad \sigma_2 = -\frac{3M}{10Ah} \stackrel{\text{def}}{=} \sigma_2^e.$$

By definition, the notation $\boldsymbol{\sigma}^e$ will be used for stresses calculated on the basis of assumed elastic behavior of a body, or *elastic stresses*.

The outer flanges will yield when $|M| = (10/3)\sigma_Y Ah \stackrel{\text{def}}{=} M_E$. Provided that the web holds, however, the deformation will remain contained so long as the inner flanges remain elastic. Collapse occurs when all four flanges yield, that is, when $\sigma_1 = \sigma_2 = -\sigma_Y$, so that $|M| = 4\sigma_Y Ah = M_U$. The range $M_E \le |M| \le M_U$ is called the range of *contained plastic deformation*.

Suppose, for example, that $M = (11/3)\sigma_Y Ah = M^*$. The actual stresses are $\sigma_1 = -(2/3)\sigma_Y$, $\sigma_2 = -\sigma_Y$, while the elastic stresses are $\sigma_1^e = -(11/30)\sigma_Y$, $\sigma_2^e = -(11/10)\sigma_Y$. When the moment is removed, the flanges will unload elastically, a process equivalent to subtracting the elastic stresses from the actual stresses. The resulting *residual stresses* are $\sigma_1^r = -(3/10)\sigma_Y$ and $\sigma_2^r = (1/10)\sigma_Y$. Their resultant moment is, of course, zero. If the beam is now repeatedly reloaded with a bending moment of the same sign as before, the response will be elastic provided that $M \le M^*$. Consequently, for the given load amplitude, shakedown takes place in the first cycle.

Residual Stress and Displacement

A stress field $\boldsymbol{\rho}$ that is in equilibrium with zero body force and zero prescribed surface tractions is called *self-equilibrated*, or, more simply, a field of *self-stress*. Such a field must clearly satisfy

$$\rho_{ij},_j = 0 \text{ in } R, \quad \rho_{ij} n_j = 0 \text{ on } \partial R_t.$$

In an elastic–plastic body under given loads $\mathbf{f}$, $\mathbf{t}^a$, the stress field can always be written as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^e + \boldsymbol{\rho}, \tag{3.5.5}$$

where $\boldsymbol{\sigma}^e$ is the elastic stress field corresponding to the given loads, and $\boldsymbol{\rho}$ is the self-equilibrated field of *residual stresses*. The strain field can accordingly be written as

$$\varepsilon_{ij} = C_{ijkl}^{-1} \sigma_{kl}^e + C_{ijkl}^{-1} \rho_{kl} + \varepsilon_{ij}^p. \tag{3.5.6}$$

The first term, which will be written ε'_{ij}, represents the strain field in the hypothetical elastic body under the prescribed loads. This strain field is compatible with a displacement field that will be denoted $\mathbf{u}^e$, that is,

$$\varepsilon'_{ij} = \frac{1}{2}(u_i^e,_j + u_j^e,_i).$$

Since the total strain field $\boldsymbol{\varepsilon}$ is also compatible, the remaining terms of (3.5.6) are also derivable from a displacement field $\mathbf{u}^r$ (the *residual displacement* field):

$$C_{ijkl}^{-1} \rho_{kl} + \varepsilon_{ij}^p = \frac{1}{2}(u_i^r,_j + u_j^r,_i), \tag{3.5.7}$$

and $\mathbf{u} = \mathbf{u}^e + \mathbf{u}^r$.

It may be shown that the plastic strain field $\boldsymbol{\varepsilon}^p$ uniquely determines the residual stress field $\boldsymbol{\rho}$ and, with sufficient constraints to prevent rigid-body displacement, also the residual displacement field $\mathbf{u}^r$. The method of proof is analogous to that of the elastic uniqueness theorem of 3.4.1. If $\boldsymbol{\rho}$ and $\boldsymbol{\rho}+\overline{\boldsymbol{\rho}}$ are two different residual stress fields, then, from the principle of virtual work,

$$\int_R \rho_{ij}(C^{-1}_{ijkl}\rho_{kl} + \varepsilon^p_{ij})\, dV = \int_R (\rho_{ij} + \overline{\rho}_{ij})(C^{-1}_{ijkl}\rho_{kl} + \varepsilon^p_{ij})\, dV$$

$$= \int_R (\rho_{ij} + \overline{\rho}_{ij})[C^{-1}_{ijkl}(\rho_{kl} + \overline{\rho}_{kl}) + \varepsilon^p_{ij}]\, dV = 0.$$

Rearrangement of terms leads to

$$\int_R \overline{\rho}_{ij} C^{-1}_{ijkl} \overline{\rho}_{kl}\, dV = 0,$$

and hence to $\overline{\boldsymbol{\rho}} = 0$ in view of the positive-definiteness of C^{-1}.

With $\boldsymbol{\rho}$ uniquely determined by $\boldsymbol{\varepsilon}^p$, it follows from (3.5.7) that $\mathbf{u}^r$ is determined to within a rigid-body displacement.

Quasi-Static Shakedown Theorem

Suppose that an elastic–plastic body has already shaken down under a loading that is varying in time (but sufficiently slowly so that inertia may be neglected) within a certain range of the generalized loads. It follows from the definition of shakedown that the plastic strain field $\boldsymbol{\varepsilon}^p$ remains constant in time and defines a time-independent residual stress field $\boldsymbol{\rho}$ such that the total stress field $\boldsymbol{\sigma}$, given by (3.5.5), does not violate the yield criterion anywhere:

$$f(\boldsymbol{\sigma}^e + \boldsymbol{\rho}) \le 0.$$

Clearly, the existence of such a residual stress field is a necessary condition for shakedown.

It was shown by Melan [1938], however, that this is also a sufficient condition: shakedown will occur in the given load range if a time-independent self-stress field $\boldsymbol{\rho}^*$, not necessary equal to the actual residual stress field $\boldsymbol{\rho}$, can be found such that

$$f(\boldsymbol{\sigma}^e + \boldsymbol{\rho}^*) < 0$$

for all elastic stress fields $\boldsymbol{\sigma}^e$ corresponding to loadings within the given range.

To prove the theorem, we consider the nonnegative quantity

$$Y = \frac{1}{2}\int_R C^{-1}_{ijkl}(\rho_{ij} - \rho_{ij}{}^*)(\rho_{kl} - \rho_{kl}{}^*)\, dV.$$

Since the body may not yet have shaken down, the actual residual stress field $\boldsymbol{\rho}$, and hence Y, may be time-dependent, with

$$\dot{Y} = \int_R C^{-1}_{ijkl}(\rho_{ij} - \rho_{ij}{}^*)\dot{\rho}_{kl}\, dV.$$

Since $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$ are both self-equilibrated, and since the left-hand side of (3.5.7) forms a compatible strain field, it follows from the principle of virtual work that

$$\int_R (\rho_{ij} - \rho_{ij}{}^*)(C^{-1}_{ijkl}\dot{\rho}_{kl} + \dot{\varepsilon}^p_{ij})dV = 0.$$

Consequently

$$\dot{Y} = -\int_R (\rho_{ij} - \rho_{ij}{}^*)\dot{\varepsilon}^p_{ij}dV = -\int_R (\sigma_{ij} - \sigma_{ij}{}^*)\dot{\varepsilon}^p_{ij}dV,$$

where $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^e + \boldsymbol{\rho}^*$. By hypothesis, $\boldsymbol{\sigma}^*$ does not violate the yield criterion, and therefore, as a result of the maximum-plastic-dissipation postulate (3.2.4), $\dot{Y} \leq 0$, where the equality holds only in the absence of plastic flow. Since $Y \geq 0$, the condition $\dot{Y} = 0$ must eventually be reached, and this condition corresponds to shakedown.

An extension of the theorem to work-hardening materials is due to Mandel [1976] (see also Mandel, Zarka, and Halphen [1977]). The yield surface is assumed to be of the form

$$f(\boldsymbol{\sigma}, \boldsymbol{\xi}) = F(\boldsymbol{\sigma} - c\boldsymbol{\varepsilon}^p) - k^2(\kappa),$$

where F is a homogeneous quadratic function of its argument, and k is a nondecreasing function of κ. A body made of such a material will shake down if, in addition to a time-independent residual stress field $\boldsymbol{\rho}^*$, there exist time-independent internal-variable fields $\boldsymbol{\xi}^* = (\boldsymbol{\varepsilon}^{p*}, \kappa^*)$ such that

$$f(\boldsymbol{\sigma}^e + \boldsymbol{\rho}^*, \boldsymbol{\xi}^*) < 0$$

everywhere.

Kinematic Shakedown Theorem

A kinematic criterion for shakedown was derived by Koiter [1956, 1960]. A strong version of Koiter's theorem states that shakedown has not taken place if a kinematically admissible velocity field $\mathbf{v}^*$, satisfying $\mathbf{v}^* = 0$ on ∂R_v, can be found so that

$$\int_R \mathbf{f} \cdot \mathbf{v}^* \, dV + \int_{\partial R_t} \mathbf{t}^a \cdot \mathbf{v}^* \, dS > \int_R D_p(\dot{\boldsymbol{\varepsilon}}^*) \, dV.$$

This inequality, with the principle of virtual work applied to its left-hand side, can be transformed into

$$\int_R \sigma^e_{ij}\dot{\varepsilon}_{ij}{}^* \, dV > \int_R D_p(\dot{\boldsymbol{\varepsilon}}^*) \, dV. \tag{3.5.8}$$

Suppose, now, that shakedown has taken place, with a time-independent residual stress field $\boldsymbol{\rho}$. From the maximum-plastic-dissipation postulate,

$$D(\dot{\boldsymbol{\varepsilon}}^*) \geq (\sigma^e_{ij} + \rho_{ij})\dot{\varepsilon}_{ij}{}^*,$$

and therefore

$$\int_R D(\dot{\boldsymbol{\varepsilon}}^*)dV \geq \int_R \sigma_{ij}^e \dot{\varepsilon}_{ij}{}^* \, dV + \int_R \rho_{ij}\dot{\varepsilon}_{ij}{}^* \, dV. \tag{3.5.9}$$

An application of the virtual-work principle to the last integral shows that it vanishes, since $\boldsymbol{\rho}$ is self-equilibrated and $\mathbf{v}^* = 0$ on ∂R_v. Inequalities (3.5.8) and (3.5.9) are therefore in contradiction, that is, shakedown cannot have taken place.

A weaker version of the theorem requires only that a strain rate $\dot{\boldsymbol{\varepsilon}}^*$ be found during a time interval $(0, T)$ such that the strain field

$$\boldsymbol{\varepsilon}^* = \int_0^T \dot{\boldsymbol{\varepsilon}}^* \, dt$$

is compatible with a displacement field $\mathbf{u}^*$ that satisfies $\mathbf{u}^* = 0$ on ∂R_u, and

$$\int_0^T \left(\int_R \mathbf{f} \cdot \dot{\mathbf{u}}^* \, dV + \int_{\partial R_t} \mathbf{t}^a \cdot \dot{\mathbf{u}}^* dS \right) dt > \int_0^T \int_R D_p(\dot{\boldsymbol{\varepsilon}}^*) \, dV \, dt.$$

With these conditions met, the body will not shake down during the interval.

Recent developments in shakedown theory have included taking into account the effects of temperature changes, creep, inertia, and geometric nonlinearities. For a review, see the book by König [1987].

Exercises: Section 3.5

1. Find the ultimate load F_U for the structure shown in Figure 4.1.2(a) (page 185), assuming that all the bars have the same cross-sectional area, are made of the same elastic–perfectly plastic material with uniaxial yield stress σ_Y, and act in simple tension.

2. Using both the lower-bound and the upper-bound theorems, find the limit locus for the beam having the idealized section shown in Figure 3.5.2 (page 171) subject to combined axial force P and bending moment M.

3. In a body made of a standard Mohr–Coulomb material with cohesion c and internal-friction angle ϕ under a load P, lower and upper bounds to the ultimate load P_U have been found as $P_U^- = ch^-(\phi)$ and $P_U^+ = ch^+(\phi)$. Show how the results can be used to find the best bounds on P_U if the material is nonstandard but has a plastic potential of the same form as the yield function, with a dilatation angle $\psi \neq \phi$.

4. For the beam of Figure 3.5.2 subject to a bending moment M only, find the range of M within which shakedown occurs on the basis of the following assumed self-stress distributions.

(a) $\rho_1{}^* = \frac{1}{2}\sigma_Y = -\rho_1'{}^*$, $\rho_2{}^* = -\frac{1}{6}\sigma_Y = -\rho_2'{}^*$

(b) $\rho_1{}^* = -\frac{1}{2}\sigma_Y = -\rho_1'{}^*$, $\rho_2{}^* = \frac{1}{6}\sigma_Y = -\rho_2'{}^*$

(c) $\rho_1{}^* = \frac{3}{4}\sigma_Y = -\rho_1'{}^*$, $\rho_2{}^* = -\frac{1}{4}\sigma_Y = -\rho_2'{}^*$

5. For the beam of Figure 3.5.2 under combined axial force P and bending moment M, (a) find the elastic stresses σ^e in each flange; (b) given the self-stress distribution $\rho_1{}^* = -\frac{3}{5}\sigma_Y$, $\rho_2{}^* = \frac{4}{5}\sigma_Y$, $\rho_1'{}^* = \frac{1}{5}\sigma_Y$, $\rho_2'{}^* = -\frac{2}{5}\sigma_Y$ find the range of P and M moments under which shakedown occurs by ensuring that $|\rho^* + \sigma^e| \leq \sigma_Y$ in each flange.

6. Using the result of Exercise 2 for the beam of Figure 3.5.2 under combined axial force and bending moment, find, if possible, loading cycles between pairs of points on the limit locus such that there occurs (a) incremental plastic deformation, (b) alternating plastic deformation, and (c) shakedown.