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SOME FUNDAMENTAL ISSUES IN RATE THEORY OF DAMAGE-ELASTOPLASTICITY

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Abstract — The paper elaborates on some fundamental constitutive issues in the rate theory of damage-elastoplasticity. The analysis combines the constitutive theories of elastoplastic and progressively damaged solids. After defining needed kinematic and kinetic preliminaries, the anisotropic elastic response is analyzed by introducing a set of damage tensors which represent material degradation and induced elastic anisotropy. Decomposition of the rate of stress and deformation tensors into their elastic and inelastic parts is then defined in a manner analogous to the corresponding decomposition used in large-deformation elastoplasticity theory. The procedure is further developed to partition the inelastic stress and strain rates into the damage and plastic parts, which takes into account the physics of these deformation processes. The energy dissipation rate is derived and the thermodynamic forces conjugate to elastic stiffness and compliance tensors are identified, based on a thermodynamic analysis of isothermal deformation process. The damage potentials for the corresponding fluxes are introduced and the constitutive expressions for the damage stress and strain rates are established. The concept of a damage surface is used to define the onset and evolution of damage. A constitutive analysis for inelastic stress and strain rates is then presented. The inelastic potential function and the yield surface are introduced. A dual formulation is constructed in both the stress and strain spaces. The two limiting cases, one involving plasticity without damage, and the other involving damage without plasticity, are deduced from the developed and more general constitutive framework of damage-elastoplasticity.

I. INTRODUCTION

This paper addresses some of the fundamental issues in the formulation of rate-type constitutive theory of time-independent inelastic solids, which are deformed at finite strain under isothermal conditions. Inelasticity may either result from the crystalline slip through dislocation motion, which is a dominant mechanism of plastic deformation in ductile metals, or from the evolution of a large number of internal micro-cracks and other micro-defects in brittle materials like rocks and concrete. The rate theory of elastoplastic deformation of crystalline materials has been thoroughly studied in the past. The essential structure of the elastoplastic constitutive relations has been established. The plasticity postulates and constitutive inequalities were developed to classify material response and define the normality of the plastic stress and strain rate tensors. Evolution of the yield surface during the course of plastic deformation was studied both analytically and experimentally. Conditions for the existence of plastic potential functions, variational principles and uniqueness were formulated and used to study the strain localizations and constitutive instabilities. The relationship between single crystal and polycrystalline plasticity, along with the transition from the micromechanically based to continuum phenomenological studies were given. The rate theory of plasticity was further extended to describe the constitutive behavior of geotechnical materials by

accounting for the microscopic frictional effects, plastic dilatancy and strong pressure sensitivity. Since a comprehensive list of corresponding references would be too long to present, only several recent review articles are here cited, i.e. HILL [1978]; NEALE [1981]; ASARO [1983]; NAGHDI [1990]; and NEMAT-NASSER [1992].

Brittle and quasi-brittle deformation processes in materials such as rocks, concrete and ceramics have lately attracted a great deal of attention from the continuum damage mechanics point of view. The continuum damage mechanics approach is used to model the progressive degradation of mechanical properties attributable to evolution of many internal microcracks or other defects, prior to the initiation and growth of a major macrocrack. Developed micromechanical models and the self-consistent methods were used to determine the effective elastic properties in the presence of the microdefect distributions (NEMAT-NASSER & HORI [1993]). In the wake of the original damage model, in which material degradation was represented by a scalar quantity, various vectorial, second-order, fourth-order and higher-order tensor variables have been subsequently introduced to characterize the deterioration of the elastic properties and the resulting damage response (KRAJCINOVIC [1989]). The rate-type constitutive theory, analogous to the rate-type plasticity theory, were developed since during a typical deformation process the macroresponse becomes in general anisotropic and path-dependent. Initially, the suggested theory attempted to describe the nonlinear behavior of progressively fracturing solids solely due to the degradation of elastic stiffness (DOUGILL [1975,1983]). This theory was later modified and extended to include the residual-plastic components of strain by BAŽANT and KIM [1979]; DRAGON and MRÓZ [1979]; ORTIZ [1985]; SIMO and JU [1987]; VOYIADJIS and KATTAN [1992], and others. The resulting theory is referred to as damage-elastoplastic constitutive theory. The development of such a theory is of great physical significance. Damage can accompany plastic deformation in metals due to the void growth, or the grain rotations can cause the crystallographic texture to develop which induces elastic anisotropy. Residual-plastic deformation can accompany damage in brittle materials like rocks, due to irreversible frictional sliding and uplifting at asperities over the internal crack surfaces. The purpose of the present paper is to systematically extend the rate-type elastoplastic constitutive theory, and address some of the fundamental issues that arise in the formulation of a more general damage-elastoplastic constitutive theory.

The contents of the present paper is as follows. Kinematic and kinetic preliminaries are described in Section II. Different stress and strain measures and their rates are introduced, along with some useful relationships which are used later in the body of the paper. The derivation is based on the multiplicative decomposition of the deformation gradient into its elastic and plastic parts (LEE [1969]). Section III addresses the description of the anisotropic elastic response. The elastic strain energy is assumed to be a function of the strain tensor and the introduced set of damage tensors, which account for material degradation and induced elastic anisotropy. The constitutive elaboration is focussed on a physically important special case for which the strain energy is a quadratic function of the strain components. The rate-type elasticity equations, and the relationships between the introduced tensors of the material and special elasticities, are provided in Section IV. Decomposition of the rate of stress and deformation tensors into their elastic and inelastic parts is presented in Section V. This decomposition is analogous to that used in the framework of large-strain elastoplasticity by HILL and RICE [1973]; and HILL [1978]. A procedure developed in Section VI to partition the inelastic stress and strain rates into their damage and plastic parts follows

from the rate-type elasticity relationship by taking into account the physics of the damage and plastic deformation processes. Remarkable relationships between various measures of the rate of deformation tensors, such as eqns (105) and (114), are derived in the process. The connections between the damage and plastic stress and strain rates, and other quantities, corresponding to the spatial and referential formulations are systematically given. Section VII is devoted to thermodynamic analysis of the inelastic deformation process, assuming isothermal conditions and negligible temperature gradients. Expression for the energy dissipation rate is derived from the constructed representation of the Helmholtz free energy. Macroscopically non-recoverable parts of the rate of work expanded on the damage and plastic processes are identified. A dual analysis employing the Gibbs energy and stress as an independent variable is also given. Section VIII provides definition of the thermodynamic forces, conjugate to elastic stiffness and compliance tensors selected as the macroscopic damage variables. The constitutive expressions for the damage stress and strain rates are derived, introducing the damage potentials for the stiffness and compliance fluxes. The damage surface is used to designate the onset and evolution of damage. Finally, Section IX provides the constitutive analysis for the inelastic stress and strain rates. This analysis utilizes the introduced inelastic potential and inelastic yield surface. A dual formulation is constructed in both the stress or strain spaces. Derived constitutive relations are applied to describe the two limiting cases of inelastic response, one involving plasticity without damage, and the other involving damage without plasticity.

II. PRELIMINARIES

Let $\mathcal{B}_0$ be the initial, undeformed configuration of a considered body, and $\mathcal{B}_t$ its deformed configuration obtained by a specified loading program from the initial to current time t. Consider a loading for which the inelastic deformation processes take place, pertinent to internal structure and composition of the considered material. In the case of a ductile metal, inelasticity is caused by the dislocation motion and related micromechanisms occurring within a polycrystalline metal structure. In a brittle material, such as rock or concrete, inelastic deformation is a consequence of the evolution of internal crack structure, i.e. the nucleation and propagation of microfractures within the sample. Regardless of the cause of inelasticity, let F be the deformation gradient that maps the infinitesimal material element $\mathrm{d}\mathbf{X}$ from its initial configuration to its current configuration $\mathrm{d}\mathbf{x}$, i.e. $\mathrm{d}\mathbf{x} = F\mathrm{d}\mathbf{X}$. Both, the initial $\mathbf{X}$ and the current $\mathbf{x}$ locations of the material particle are referred to the same, fixed set of the rectangular coordinate axes. Introduce next the intermediate reference configuration $\mathcal{P}_t$ by elastic distressing the current configuration $\mathcal{B}_t$ to zero stress. If $\mathrm{d}\mathbf{p}$ is the material element in $\mathcal{P}_t$, corresponding to its configuration $\mathrm{d}\mathbf{x}$ in $\mathcal{B}_t$, then $\mathrm{d}\mathbf{x} = F_e\mathrm{d}\mathbf{p}$, where F_e denotes the deformation gradient associated with elastic loading from $\mathcal{P}_t$ to $\mathcal{B}_t$. Introducing also the deformation gradient of the transformation $\mathcal{B}_0 \to \mathcal{P}_t$, by $\mathrm{d}\mathbf{p} = F_p\mathrm{d}\mathbf{X}$, the multiplicative decomposition of the deformation gradient follows (LEE [1969])

$$F = F_e F_p\,, \tag{1}$$

where F_e is the elastic, and F_p the plastic part of the total deformation gradient F. For inhomogeneous deformations, only F is the true deformation gradient, whose components are the partial derivatives $\partial\mathbf{x}/\partial\mathbf{X}$. In contrast, the mappings $\mathcal{P}_t \to \mathcal{B}_t$ and

$\mathcal{B}_0 \to \mathcal{P}_t$ are not in general a continuous one-to-one mappings, so that F_e and F_p are not defined as the gradients of the respective mappings (which may not exist), but as the point functions (local deformation gradients). Elastic distressing to zero stress ($\mathcal{B}_t \to \mathcal{P}_t$) is not always physically achievable, since the reverse inelastic deformation may occur before the zero stress is reached (which often occurs at advanced stages of deformation due to anisotropic hardening and strong Bauschinger effects in ductile metals, or due to the incomplete frictional back-sliding of the crack faces in brittle rocks). In this case the intermediate configuration can be conceptually introduced by virtual distressing to zero stress, during which all inelastic structural changes that would occur during the actual distressing are locked. The use and the extent of utility of the multiplicative decomposition of deformation gradient in large deformation elastoplasticity has been studied in the past by many authors. The issues regarding nonuniqueness of intermediate configuration, appropriate invariance requirements, and additive decomposition of the rate of deformation tensor into its elastic and plastic parts were, in particular, debated. Some of these topics were recently discussed by NAGHDI [1990]; and LUBARDA and SHIH [1994], where a reference to other related work can also be found.

II.1. *Kinematics*

The Lagrangian strains corresponding to deformation gradients F_e and F_p are

$$E_e = \frac{1}{2}(F_e^T F_e - \mathbf{I}) , \quad E_p = \frac{1}{2}(F_p^T F_p - \mathbf{I}) , \tag{2}$$

where $\mathbf{I}$ denotes the second-order identity tensor and $(\,)^T$ the transpose. The total Lagrangian strain can consequently be expressed as

$$E = \frac{1}{2}(F^T F - \mathbf{I}) = E_p + F_p^T E_e F_p . \tag{3}$$

The total strain E is not equal to the sum of the elastic and plastic strains E_e and E_p, since E and E_p are defined relative to the initial configuration $\mathcal{B}_0$ as a reference configuration, while E_e is defined relative to the intermediate configuration $\mathcal{P}_t$ as a reference configuration. Instead, the total strain E is the sum of the strain $F_p{}^T E_e F_p$, induced from elastic strain E_e by plastic deformation F_p, and the plastic strain E_p.

The Eulerian strains corresponding to deformation gradients F_e and F_p are

$$\epsilon_e = \frac{1}{2}(\mathbf{I} - F_e^{-T} F_e^{-1}) , \quad \epsilon_p = \frac{1}{2}(\mathbf{I} - F_p^{-T} F_p^{-1}) , \tag{4}$$

where $(\,)^{-1}$ designates the inverse, while $(\,)^{-T}$ is the transpose of the inverse. The total Eulerian strain can consequently be expressed as

$$\epsilon = \frac{1}{2}(\mathbf{I} - F^{-T} F^{-1}) = \epsilon_e + F_e^{-T} \epsilon_p F_e^{-1} . \tag{5}$$

The strains ϵ and ϵ_e are defined relative to current configuration, while the strain ϵ_p is defined relative to intermediate configuration as a reference. Thus, the total strain ϵ is obtained by superimposing the strain $F_e^{-T} \epsilon_p F_e^{-1}$ on the elastic strain ϵ_e. The transition from eqn (3) to eqn (5) can be obtained directly by using eqn (1) and the well-known relationship between the Lagrangian and Eulerian strains

$$E = F^{\mathrm{T}} \epsilon F . \tag{6}$$

A useful relationship

$$E - E_{\mathrm{p}} = F^{\mathrm{T}} \epsilon_{\mathrm{e}} F , \tag{7}$$

can also be derived, which shows that the difference between the total and plastic Lagrangian strain tensors is equal to the strain induced from the Eulerian elastic strain ϵ_{e} by the total deformation F.

Consider the velocity gradient in the current configuration at time t, defined by

$$\mathrm{L} = \dot{F} F^{-1} = \mathrm{D} + \mathrm{W} , \tag{8}$$

where the dot designates the material time derivative, and where D and W are the symmetric and antisymmetric parts of L, i.e. the velocity strain and the material spin tensor, respectively. By introducing the multiplicative decomposition (1) of the deformation gradient F, the velocity gradient becomes

$$\mathrm{L} = \dot{F}_{\mathrm{e}} F_{\mathrm{e}}^{-1} + F_{\mathrm{e}} (\dot{F}_{\mathrm{p}} F_{\mathrm{p}}^{-1}) F_{\mathrm{e}}^{-1} . \tag{9}$$

The velocity strain (also called the rate of deformation, or the strain rate tensor) D is given by the symmetric part of eqn (9), i.e.

$$\mathrm{D} = (\dot{F}_{\mathrm{e}} F_{\mathrm{e}}^{-1})_{\mathrm{s}} + \left[F_{\mathrm{e}} (\dot{F}_{\mathrm{p}} F_{\mathrm{p}}^{-1}) F_{\mathrm{e}}^{-1} \right]_{\mathrm{s}} . \tag{10}$$

The spin (or vorticity) tensor W is equal to the antisymmetric part of eqn (9), i.e.

$$\mathrm{W} = (\dot{F}_{\mathrm{e}} F_{\mathrm{e}}^{-1})_{\mathrm{a}} + \left[F_{\mathrm{e}} (\dot{F}_{\mathrm{p}} F_{\mathrm{p}}^{-1}) F_{\mathrm{e}}^{-1} \right]_{\mathrm{a}} . \tag{11}$$

The rate of the Lagrangian strain is related to the rate of deformation tensor D by

$$\dot{E} = F^{\mathrm{T}} \, \mathrm{D} \, F , \tag{12}$$

while the Cotter–Rivlin convected rate of the Eulerian strain is identically equal to D, i.e.

$$\overset{*}{\epsilon} = \mathrm{D} , \quad \overset{*}{\epsilon} = \dot{\epsilon} + \epsilon L + L^{\mathrm{T}} \epsilon . \tag{13}$$

The convected derivative appearing in eqn (13) can also be interpreted as the Lie derivative of the Eulerian strain (MARSDEN & HUGHES [1983]).

II.2. *Kinetics*

The work conjugate to the Lagrangian strain E is the symmetric Piola–Kirchhoff stress

$$S = F^{-1} \tau F^{-\mathrm{T}} , \tag{14}$$

where $\tau = |F| \sigma$ is the Kirchhoff stress, defined as the Cauchy stress σ multiplied by the determinant of deformation gradient F. In view of eqns (6), (12) and (13) if follows that

$$S : E = \tau : \epsilon \tag{15}$$

$$S : \dot{E} = \tau : \overset{*}{\dot{\epsilon}}, \tag{16}$$

where (:) denotes the inner (trace) product of the two second-order tensors. The rate of the Piola-Kirchhoff stress eqn (14) is

$$\dot{S} = F^{-1} \overset{*}{\tau} F^{-\mathrm{T}}, \tag{17}$$

where

$$\overset{*}{\tau} = \dot{\tau} - \mathrm{L}\tau - \tau \mathrm{L}^{\mathrm{T}} \tag{18}$$

is the Oldroyd convected rate of the Kirchhoff stress. The relationship dual to eqn (16) is easily obtained as

$$E : \dot{S} = \epsilon : \overset{*}{\tau}. \tag{19}$$

For subsequent analysis it will be convenient to introduce the stress tensor

$$S_{\mathrm{e}} = F_{\mathrm{e}}^{-1} \tau F_{\mathrm{e}}^{-\mathrm{T}}, \tag{20}$$

related to the Piola–Kirchhoff stress S by

$$S_{\mathrm{e}} = F_{\mathrm{p}} S F_{\mathrm{p}}^{\mathrm{T}}. \tag{21}$$

The following relationship also holds

$$S : E = S_{\mathrm{e}} : E_{\mathrm{e}} + S : E_{\mathrm{p}}. \tag{22}$$

The rate of the stress tensor eqn (20) is

$$\dot{S}_{\mathrm{e}} = F_{\mathrm{e}}^{-1} \overset{*\mathrm{e}}{\tau} F_{\mathrm{e}}^{-\mathrm{T}}, \tag{23}$$

where the convected derivative of the Kirchhoff stress $\overset{*\mathrm{e}}{\tau}$ is defined relative to the velocity gradient $\dot{F}_{\mathrm{e}} F_{\mathrm{e}}^{-1}$ by

$$\overset{*\mathrm{e}}{\tau} = \dot{\tau} - (\dot{F}_{\mathrm{e}} F_{\mathrm{e}}^{-1})\tau - \tau(\dot{F}_{\mathrm{e}} F_{\mathrm{e}}^{-1})^{\mathrm{T}}. \tag{24}$$

The convected derivative (24) can be expressed in terms of the convected derivative (18) as

$$\overset{*\mathrm{e}}{\tau} = \overset{*}{\tau} + l_{\mathrm{p}}\tau + \tau l_{\mathrm{p}}^{\mathrm{T}}, \tag{25}$$

where, in view of eqn (9),

$$l_{\mathrm{p}} = F_{\mathrm{e}}(\dot{F}_{\mathrm{p}} F_{\mathrm{p}}^{-1}) F_{\mathrm{e}}^{-1} = \mathrm{L} - \dot{F}_{\mathrm{e}} F_{\mathrm{e}}^{-1}. \tag{26}$$

Denoting by d_{p} the symmetric part of l_{p}, the following useful relationships can be further established

$$S : \dot{E}_p = \tau : d_p - \epsilon_e : (l_p \tau + \tau l_p^T) \tag{27}$$

$$\dot{S} : E_p = \overset{*}{\tau} : (\epsilon - \epsilon_e) \tag{28}$$

$$S_e : \dot{E}_e = \tau : D - \tau : d_p \tag{29}$$

$$E_e : \dot{S}_e = \epsilon_e : \overset{*e}{\tau} \cdot \tag{30}$$

III. ANISOTROPIC ELASTIC RESPONSE

The deformation gradients F_e and F_p are not uniquely defined, because arbitrary local material element rotations superimposed to unstressed state produce different intermediate configurations. However, if a material is elastically isotropic and remains such during inelastic deformation preserving its elastic properties, the elastic strain energy per unit initial volume ψ is an isotropic function of the Lagrangian strain E_e, i.e. $\psi(QE_eQ^T) = \psi(E_e)$, where Q is an orthogonal second-order tensor corresponding to an arbitrary rigid-body rotation superimposed to the unstressed state. The elastic stress response from $\mathcal{P}_t \to \mathcal{B}_t$ is, therefore, not influenced by the non-uniqueness of intermediate configuration and is given by the well-known constitutive expression of the Green- or hyper-elasticity type

$$S_e = \frac{\partial \psi(E_e)}{\partial E_e} \cdot \tag{31}$$

Note that $\psi = |F_p|\psi_e$, where ψ_e is the elastic strain energy per unit unstressed volume in the intermediate configuration. Whenever a plastic deformation is compressible, i.e. $|F_p| \neq 1$, the unit of unstressed volume carries a variable amount of mass proportional to the current mass density, whereas the unit of initial volume has the fixed amount of mass, equal to the initial mass density multiplied by the unit of volume.

The elastic constitutive structure equivalent to eqn (31) was used in a series of papers devoted to elastoplastic constitutive analysis of elastically isotropic materials, by LEE [1969]; LUBARDA and LEE [1981]; LUBARDA [1991a]; and LUBARDA and SHIH [1994]. The generalization of the analysis to include the initial and induced elastic anisotropy has been discussed by LUBARDA [1991b,1994a]. Some aspects of this analysis, which are important for the constitutive models presented in this paper, are reviewed in the following subsection.

III.1. *Representation of elastic anisotropy*

Consider an intermediate configuration $\mathcal{P}_t$ obtained by distressing the current configuration $\mathcal{B}_t$ to zero stress. Assume that the material in configuration $\mathcal{P}_t$ is elastically anisotropic, either because it was initially anisotropic, or because it has developed elastic anisotropy during a previous inelastic deformation (for example, due to the grain rotations in a polycrystalline metal sample and the consequent crystallographic texture, or due to anisotropic crack distribution within a brittle rock specimen). Let $\mathcal{D}$ denote a set of the symmetric tensor variables of various orders (scalars, second-order, fourth-order tensors, etc.), attached to the current configuration $\mathcal{B}_t$, which appropriately account for the degradation of elastic material properties and their directional changes, accumulated during the previous inelastic deformation. The variables $\mathcal{D}$ will

be referred to as the damage variables. For example, in modeling inelastic behavior with infinitesimal elastic component of strain, the current (degraded) fourth-order elastic stiffness tensor can be selected as an appropriate damage tensor (DOUGILL [1975,1983]). Similarly, ORTIZ [1985] has used the current elastic compliance tensor as the damage tensor in his study of inelastic behavior of concrete. Various order damage tensors and their relationship to some typical two- and three-dimensional crack density distributions were analyzed by LUBARDA and KRAJCINOVIC [1993].

Even if there is no degradation of the elastic material properties, the tensor variables $\mathcal{D}$ can be introduced to properly and conveniently describe the state of initial elastic anisotropy of the material (structural tensors, BOEHLER [1977,1987]). For example, in the case of transverse isotropy with the axis of isotropy in the current configuration $\mathcal{B}_t$ coincident with the direction $\mathbf{n}$, the structural tensor is the second-order tensor $\mathcal{D} = \mathbf{n} \otimes \mathbf{n}$. For orthotropic material, with the principal directions of orthotropy coincident with the directions $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, structural tensors are $\mathbf{n}_1 \otimes \mathbf{n}_1$, $\mathbf{n}_2 \otimes \mathbf{n}_2$, and $\mathbf{n}_3 \otimes \mathbf{n}_3 = \mathrm{I} - \mathbf{n}_1 \otimes \mathbf{n}_1 - \mathbf{n}_2 \otimes \mathbf{n}_2$ (I denotes the second-order unit tensor, and $\otimes$ the outside tensor product). The structural tensors given in BOEHLER [1987] which correspond to a general elastic anisotropy can be similarly formed.

The introduced damage variables can change only during a continuing inelastic deformation, but remain unaltered during elastic unloading, or reverse elastic loading, except for elastic embedding which convects them together with the material. Therefore, the damage variables $\mathcal{D}$ in the current configuration $\mathcal{B}_t$ become the variables $\hat{\mathcal{D}}$ in the intermediate configuration, induced from $\mathcal{D}$ by elastic deformation $\mathrm{F_e}$. For example, for the second-order damage tensor, the induced tensor can be defined by a transformation of the weighted covariant or contravariant type, i.e.

$$\hat{\mathcal{D}} = |F_\mathrm{e}|^{-m} F_\mathrm{e}^\mathrm{T} \mathcal{D} F_\mathrm{e} , \quad \text{or} \quad \hat{\mathcal{D}} = |F_\mathrm{e}|^{m} F_\mathrm{e}^{-1} \mathcal{D} F_\mathrm{e}^{-\mathrm{T}} , \tag{32}$$

where m is the weight. For the fourth-order damage tensor, the corresponding induced damage tensor is

$$\hat{\mathcal{D}} = |F_\mathrm{e}|^{-m} F_\mathrm{e}^\mathrm{T} \otimes F_\mathrm{e}^\mathrm{T} \mathcal{D} F_\mathrm{e} \otimes F_\mathrm{e} , \quad \text{or} \quad \hat{\mathcal{D}} = |F_\mathrm{e}|^{m} F_\mathrm{e}^{-1} \otimes F_\mathrm{e}^{-1} \mathcal{D} F_\mathrm{e}^{-\mathrm{T}} \otimes F_\mathrm{e}^{-\mathrm{T}} . \tag{33}$$

The tensor product $\otimes$ is defined such that, for example, the first of the fourth-order tensors on the right-hand side of eqn (33) has the rectangular Cartesian components

$$\hat{\mathcal{D}}_{ijkl} = |F_\mathrm{e}|^{-m} (F^\mathrm{e}_{i\alpha})^\mathrm{T} (F^\mathrm{e}_{j\beta})^\mathrm{T} \mathcal{D}_{\alpha\beta\gamma\delta} F^\mathrm{e}_{\gamma k} F^\mathrm{e}_{\delta l} . \tag{34}$$

To describe elastic response of an anisotropic material, the elastic strain energy at the current state of the deformation and material damage is assumed to be given by

$$\psi = \psi(E_\mathrm{e}, \hat{\mathcal{D}}) . \tag{35}$$

Since material response is independent of a superimposed rotation on the intermediate-unstressed configuration, expression (35) must be an isotropic scalar function of the set of all its arguments, i.e. E_e and $\hat{\mathcal{D}}$. For example, if the damage tensors are the fourth-order tensors, the isotropy of ψ requires that for every orthogonal transformation Q

$$\psi(\mathrm{Q} E_\mathrm{e} \mathrm{Q}^\mathrm{T} , \ \mathrm{Q} \otimes \mathrm{Q} \hat{\mathcal{D}} \mathrm{Q}^\mathrm{T} \otimes \mathrm{Q}^\mathrm{T}) = \psi(E_\mathrm{e}, \hat{\mathcal{D}}) . \tag{36}$$

Note that the rotation Q superimposed to the intermediate configuration does not change $\mathcal{D}$, which is defined with respect to the current configuration. Since F_e changes to $F_e Q^T$, from eqn (33) it follows that $\hat{\mathcal{D}}$ changes to $Q \otimes Q\hat{\mathcal{D}}Q^T \otimes Q^T$.

The theory of isotropic scalar and tensor functions of several tensor arguments has been extensively studied in the past (SPENCER [1971]; BOEHLER [1987]). The integrity basis for the considered functions are derived mainly for the vector and second-order tensor arguments. BETTEN [1982,1992] also considered the functions that depend on the second- and fourth-order tensors, and construction of their individual and joint invariants. For example, if $\hat{\mathcal{D}}$ in eqn (35) is a single second-order symmetric tensor, ψ can be represented as a polynomial of its irreducible integrity basis consisting of the following invariants:

$$\begin{gathered}(E_e : I),\ (E_e : E_e),\ (E_e^2 : E_e),\ (\hat{\mathcal{D}} : I),\ (\hat{\mathcal{D}} : \hat{\mathcal{D}}) \\ (\hat{\mathcal{D}}^2 : \hat{\mathcal{D}}),\ (E_e : \hat{\mathcal{D}}),\ (E_e : \hat{\mathcal{D}}^2),\ (E_e^2 : \hat{\mathcal{D}}),\ (E_e^2 : \hat{\mathcal{D}}^2)\ .\end{gathered} \tag{37}$$

The integrity basis can be written for any finite set of second-order tensors. SPENCER [1971] provides a list of invariants and integrity bases for the polynomial scalar functions dependent on one up to six, second-order tensor arguments. For general (not necessarily polynomial) functions, the integrity bases are replaced by the function bases, which in general contain fewer terms than the corresponding integrity bases. For example, the function bases of the general scalar function dependent on an arbitrary number of second-order tensors are composed of the traces of the products of all unordered combinations of one, two and three tensorial arguments.

The construction of the integrity bases for second- and fourth-order symmetric tensors is a more difficult task. Some of the individual and joint invariants are listed in LUBARDA [1994a], while a more complete list is available in BETTEN [1992].

III.2. *Elastic stress–strain relationships*

The stress response for the transition from an intermediate to the current configuration is given by the hyperelastic constitutive equation

$$S_e = \frac{\partial \psi(E_e, \hat{\mathcal{D}})}{\partial E_e}\ . \tag{38}$$

Expression (38) can be rewritten in the form $\tau = \partial\psi/\partial\epsilon_e$, which provides the Kirchhoff stress clearly independent of any rigid-body rotation superimposed to the intermediate configuration.

Introducing the complementary elastic strain energy by a Legendre transformation

$$\phi(S_e, \hat{\mathcal{D}}) = S_e : E_e - \psi(E_e, \hat{\mathcal{D}})\ , \tag{39}$$

the strain response is given by

$$E_e = \frac{\partial \phi(S_e, \hat{\mathcal{D}})}{\partial S_e}\ . \tag{40}$$

It is useful to further introduce the potential functions ψ_0 and ϕ_0 by the following expressions:

$$\psi_0 = \psi(E_e, \hat{\mathcal{D}}) = \psi\left[F_p^{-T}(E - E_p)F_p^{-1}, \hat{\mathcal{D}}\right] \tag{41}$$

$$\phi_0 = \phi(S_e, \hat{\mathcal{D}}) + S : E_p = \phi\left(F_p S F_p^T, \hat{\mathcal{D}}\right) + S : E_p \,. \tag{42}$$

It can be easily verified that ϕ_0 is the Legendre transform of ψ_0, i.e.

$$\phi_0(S, F_p, \hat{\mathcal{D}}) = S : E - \psi_0(E, F_p, \hat{\mathcal{D}}) \,. \tag{43}$$

The function ψ_0 is the potential for the Piola–Kirchhoff stress S, such that

$$S = \frac{\partial \psi_0(E, F_p, \hat{\mathcal{D}})}{\partial E} \,, \tag{44}$$

while the function ϕ_0 is the potential for the Lagrangian strain E, i.e.

$$E = \frac{\partial \phi_0(S, F_p, \hat{\mathcal{D}})}{\partial S} \,. \tag{45}$$

The subsequent constitutive elaboration will focus on a physically important special case for which the strain energy ψ can be assumed to be a quadratic function of the Lagrangian elastic strain components

$$\psi = \frac{1}{2}\Lambda :: (E_e \otimes E_e) \,, \tag{46}$$

where the fourth-order tensor Λ is the elastic stiffness tensor, and (::) denotes the double-trace product. The stress response (38) is accordingly

$$S_e = \Lambda : E_e \,. \tag{47}$$

This can be rewritten in terms of the Kirchhoff stress and Eulerian elastic strain as

$$\tau = \mathcal{L} : \epsilon_e, \tag{48}$$

where

$$\mathcal{L} = F_e \otimes F_e \Lambda F_e^T \otimes F_e^T \tag{49}$$

is the corresponding elastic stiffness tensor. The expression (49) indicates that the tensor Λ can be induced from the tensor $\mathcal{L}$ by a transformation of the contravariant type (33b) (of weight $m = 0$). The tensor Λ plays the role of the fourth-order damage tensor $\hat{\mathcal{D}}$ in the intermediate configuration, whereas the tensor $\mathcal{L}$ is the corresponding damage tensor $\mathcal{D}$ in the current configuration. Note that the tensor Λ depends on the rotation of the intermediate configuration, while the tensor $\mathcal{L}$ is independent of any such rotation. Similarly, both S_e and E_e are dependent on the rotation of the intermediate configuration, while neither τ nor ϵ_e depend on such rotation.

The complementary energy can be written as

$$\phi = \frac{1}{2}\mathbf{M} :: (S_e \otimes S_e) , \tag{50}$$

where the fourth-order tensor M is the elastic compliance tensor, being the inverse of the elastic stiffness tensor Λ. The Lagrangian elastic strain is

$$E_e = \mathbf{M} : S_e , \tag{51}$$

while the Eulerian elastic strain is given by

$$\epsilon_e = \mathcal{M} : \tau . \tag{52}$$

In eqn (52), the fourth-order compliance tensor $\mathcal{M}$ is the inverse of the stiffness tensor $\mathcal{L}$, and is given by

$$\mathcal{M} = F_e^{-T} \otimes F_e^{-T} \mathbf{M} F_e^{-1} \otimes F_e^{-1} . \tag{53}$$

Therefore, if $\mathcal{M}$ is considered to be the damage tensor $\mathcal{D}$ in the current configuration, the corresponding damage tensor $\hat{\mathcal{D}}$ in the intermediate configuration is the tensor M, induced from the tensor $\mathcal{M}$ by a transformation of the covariant type eqn (33a) (of weight $m = 0$).

Furthermore, by using the potential function ψ_0 defined in eqn (41), i.e.

$$\psi_0 = \frac{1}{2}\Lambda_0 :: \left[(E - E_p) \otimes (E - E_p)\right] , \tag{54}$$

where

$$\Lambda_0 = F_p^{-1} \otimes F_p^{-1} \Lambda F_p^{-T} \otimes F_p^{-T} \tag{55}$$

is the corresponding stiffness tensor, the Piola–Kirchhoff stress S becomes

$$S = \Lambda_0 : (E - E_p) . \tag{56}$$

Similarly, by using the potential function ϕ_0 defined in eqn (42), i.e.

$$\phi_0 = \frac{1}{2}\mathbf{M}_0 :: (S \otimes S) + S : E_p , \tag{57}$$

where

$$\mathbf{M}_0 = F_p^T \otimes F_p^T \mathbf{M} F_p \otimes F_p \tag{58}$$

is the corresponding compliance tensor, the inverse of the stiffness tensor Λ_0, the Lagrangian strain is given by

$$E = \mathbf{M}_0 : S + E_p . \tag{59}$$

Observe finally that

$$\frac{\partial^2 \psi}{\partial E_e \otimes \partial E_e} = \Lambda , \quad \frac{\partial^2 \phi}{\partial S_e \otimes \partial S_e} = \mathbf{M} \tag{60}$$

and

$$\frac{\partial^2 \psi_0}{\partial E \otimes \partial E} = \Lambda_0 , \quad \frac{\partial^2 \phi_0}{\partial S \otimes \partial S} = \mathbf{M}_0 . \tag{61}$$

IV. RATE-TYPE ELASTICITY

If the current increment of deformation is purely elastic, as during elastic unloading or loading within the elastic range, the tensor of elastic moduli Λ does not change and differentiation of eqn (47) gives

$$\dot{S}_e = \Lambda : \dot{E}_e . \tag{62}$$

Since the plastic deformation gradient F_p does not change during a purely elastic deformation, by differentiating eqn (3) and using eqn (12), it follows that

$$\dot{E}_e = F_e^T \mathbf{D} F_e . \tag{63}$$

Furthermore, eqn (25) gives that during purely elastic deformation $\overset{*e}{\tau} = \overset{*}{\tau}$, because l_p vanishes, and eqn (23) becomes

$$\dot{S}_e = F_e^{-1} \overset{*}{\tau} F_e^{-T} . \tag{64}$$

Substituting eqns (63) and (64) into eqn (62) consequently gives

$$\overset{*}{\tau} = \mathcal{L} : \mathbf{D} , \tag{65}$$

where the elastic stiffness tensor $\mathcal{L}$ is defined by eqn (49). The inverted form of eqn (65) is

$$\mathbf{D} = \mathcal{M} : \overset{*}{\tau} , \tag{66}$$

where the elastic compliance tensor $\mathcal{M}$, defined by eqn (53), is the inverse of the tensor $\mathcal{L}$.

The expression (65) can alternatively be obtained directly from eqn (48) by applying to both sides the convected derivative with respect to deformation gradient $\dot{F}_e F_e^{-1} \equiv \dot{F} F^{-1}$, and using the relationship

$$\overset{*e}{\epsilon}_e = (\dot{F}_e F_e^{-1})_s \equiv \mathbf{D} . \tag{67}$$

If eqn (56) or (59) are used rather than eqns (47) and (51), one has by differentiation the rate-type elasticity equations

$$\dot{S} = \Lambda_0 : \dot{E} , \quad \text{or} \quad \dot{E} = \mathbf{M}_0 : \dot{S} . \tag{68}$$

Transition from the constitutive structures (68) to eqns (65) and (66), derived by using eqns (12) and (17), provides the following relationships between the stiffness and compliance tensors:

$$\mathcal{L} = F \otimes F \, \Lambda_0 \, F^T \otimes F^T , \quad \mathcal{M} = F^{-T} \otimes F^{-T} \, \mathbf{M}_0 \, F^{-1} \otimes F^{-1} . \tag{69}$$

The relation (69a) is the well-known relationship between the tensors of material and spatial elasticities (Λ_0 and $\mathcal{L}$). The tensor Λ_0 induced from the tensor $\mathcal{L}$ by a transformation of the contravariant type, relative to total deformation gradient F. Various alternative rate form representations of hyperplastic constitutive equations, discussed more recently by SIMO and MARSDEN [1984], have been frequently studied in the past.

V. PARTITION OF STRESS AND STRAIN RATES

If the current deformation involves both elastic and inelastic processes, the rate of deformation tensor is decomposed into its elastic and inelastic part as

$$\mathrm{D} = \mathrm{D}^{\mathrm{e}} + \mathrm{D}^{\mathrm{i}} , \tag{70}$$

where the elastic part is defined by

$$\mathrm{D}^{\mathrm{e}} = \mathcal{M} : \overset{*}{\tau} . \tag{71}$$

The rate of the Lagrangian strain can also be partitioned into elastic and inelastic part, i.e.

$$\dot{E} = (\dot{E})^{\mathrm{e}} + (\dot{E})^{\mathrm{i}} , \tag{72}$$

with the elastic part given by

$$(\dot{E})^{\mathrm{e}} = \mathrm{M}_0 : \dot{S} . \tag{73}$$

The inelastic part of the strain rate $(\dot{E})^{\mathrm{i}}$ in a hardening material physically corresponds to a residual strain increment after an infinitesimal loading-unloading stress cycle associated with $\dot{S}$.

The convected rate of the Kirchhoff stress is similarly decomposed into its elastic and inelastic part

$$\overset{*}{\tau} = (\overset{*}{\tau})^{\mathrm{e}} + (\overset{*}{\tau})^{\mathrm{i}} , \tag{74}$$

where

$$(\overset{*}{\tau})^{\mathrm{e}} = \mathcal{L} : \mathrm{D} . \tag{75}$$

The rate of the Piola–Kirchhoff stress can be decomposed as

$$\dot{S} = (\dot{S})^{\mathrm{e}} + (\dot{S})^{\mathrm{i}}, \tag{76}$$

where the elastic part is defined by

$$(\dot{S})^{\mathrm{e}} = \Lambda_0 : \dot{E} . \tag{77}$$

Physically, the inelastic part of the stress rate $(\dot{S})^{\mathrm{i}}$ corresponds to the residual stress decrement in an infinitesimal strain cycle associated with $\dot{E}$.

An analogous partitioning of the stress and strain rates into their elastic and plastic constituents have been used within the context of large-strain elastoplasticity by HILL and RICE [1973]; and HILL [1978].

It is easily verified that the inelastic parts of the stress and strain rates are related through

$$(\overset{*}{\tau})^{\mathrm{i}} = -\mathcal{L} : \mathbf{D}^{\mathrm{i}} \tag{78}$$

$$(\dot{S})^{\mathrm{i}} = -\Lambda_0 : (\dot{E})^{\mathrm{i}} . \tag{79}$$

The following relationships also hold:

$$(\dot{E})^{\mathrm{i}} = F^{\mathrm{T}}\mathbf{D}^{\mathrm{i}}F \tag{80}$$

$$(\dot{S})^{\mathrm{i}} = F^{-1}(\overset{*}{\tau})^{\mathrm{i}}F^{-\mathrm{T}} . \tag{81}$$

Therefore, the inelastic part of the rate of the Lagrangian strain $(\dot{E})^{\mathrm{i}}$ is the strain rate induced from the rate of deformation tensor $\mathbf{D}^{\mathrm{i}}$ by the deformation F. Analogous remark applies to the inelastic stress rates $(\dot{S})^{\mathrm{i}}$ and $(\overset{*}{\tau})^{\mathrm{i}}$. It can finally be shown that

$$\tau : \mathbf{D}^{\mathrm{i}} = -\epsilon_{\mathrm{e}} : (\overset{*}{\tau})^{\mathrm{i}} \tag{82}$$

$$S : (\dot{E})^{\mathrm{i}} = -(E - E_{\mathrm{p}}) : (\dot{S})^{\mathrm{i}} . \tag{83}$$

VI. DAMAGE AND PLASTIC STRESS AND STRAIN RATES

Consider an arbitrary instant of a deformation process which involves degradation of elastic properties. Differentiation of eqn (47) in this case gives

$$\dot{S}_{\mathrm{e}} = \Lambda : \dot{E}_{\mathrm{e}} + \dot{\Lambda} : E_{\mathrm{e}} . \tag{84}$$

The above can be rewritten by using eqn (23) as

$$\overset{*\mathrm{e}}{\tau} = \mathcal{L} : (\dot{F}_{\mathrm{e}}F_{\mathrm{e}}^{-1})_{\mathrm{s}} + (F_{\mathrm{e}} \otimes F_{\mathrm{e}}\dot{\Lambda}F_{\mathrm{e}}^{\mathrm{T}} \otimes F_{\mathrm{e}}^{\mathrm{T}}) : \epsilon_{\mathrm{e}} . \tag{85}$$

On the other hand, from eqn (49) it follows that

$$\overset{*\mathrm{e}}{\mathcal{L}} = F_{\mathrm{e}} \otimes F_{\mathrm{e}}\dot{\Lambda}F_{\mathrm{e}}^{\mathrm{T}} \otimes F_{\mathrm{e}}^{\mathrm{T}} . \tag{86}$$

The convected derivative $\overset{*\mathrm{e}}{\mathcal{L}}$ of the stiffness tensor $\mathcal{L}$ has the components given by

$$\overset{*\mathrm{e}}{\mathcal{L}}_{ijkl} = \dot{\mathcal{L}}_{ijkl} - l^{\mathrm{e}}_{im}\mathcal{L}_{mjkl} - l^{\mathrm{e}}_{jm}\mathcal{L}_{imkl} - \mathcal{L}_{ijml}l^{\mathrm{e}}_{km} - \mathcal{L}_{ijkm}l^{\mathrm{e}}_{lm} , \tag{87}$$

where l^{e}_{ij} denotes the components of the velocity gradient $l_{\mathrm{e}} = \dot{F}_{\mathrm{e}}F_{\mathrm{e}}^{-1}$. Substitution of eqn (86) into eqn (85), therefore, gives

$$\overset{*\mathrm{e}}{\tau} = \mathcal{L} : (\dot{F}_{\mathrm{e}}F_{\mathrm{e}}^{-1})_{\mathrm{s}} + \overset{*\mathrm{e}}{\mathcal{L}} : \epsilon_{\mathrm{e}} . \tag{88}$$

The relationship (88) can also be derived directly by applying the convected derivative associated with elastic deformation ($\overset{*\mathrm{e}}{\ }$) to the constitutive relation (48).

The expression (88) can be further written in a more convenient manner by using eqn (25) to express the convected derivative $\overset{*e}{\tau}$ in terms the convected derivative $\overset{*}{\tau}$, and by using eqn (10) to eliminate $(\dot{\mathrm{F}}_\mathrm{e}\mathrm{F}_\mathrm{e}^{-1})_\mathrm{s}$. This gives

$$\overset{*}{\tau} = \mathcal{L} : \mathrm{D} + \overset{*e}{\mathcal{L}} : \epsilon_\mathrm{e} - \mathcal{L} : d_\mathrm{p} - l_\mathrm{p}\tau - \tau l_\mathrm{p}^\mathrm{T} . \tag{89}$$

The first term on the right-hand side of eqn (89) represents the elastic part eqn (75) of the stress rate $\overset{*}{\tau}$. The remaining part

$$(\overset{*}{\tau})^\mathrm{i} = \overset{*e}{\mathcal{L}} : \epsilon_\mathrm{e} - \mathcal{L} : \left[d_\mathrm{p} + \mathcal{M} : (l_\mathrm{p}\tau + \tau l_\mathrm{p}^\mathrm{T})\right] \tag{90}$$

is the inelastic part of the stress rate $\overset{*}{\tau}$. This can be further partitioned as

$$(\overset{*}{\tau})^\mathrm{i} = (\overset{*}{\tau})^\mathrm{d} + (\overset{*}{\tau})^\mathrm{p} , \tag{91}$$

where

$$(\overset{*}{\tau})^\mathrm{d} = \overset{*e}{\mathcal{L}} : \epsilon_\mathrm{e} \tag{92}$$

represents the damage part of the stress rate $\overset{*}{\tau}$, while

$$(\overset{*}{\tau})^\mathrm{p} = -\mathcal{L} : \left[\mathrm{d}_\mathrm{p} + \mathcal{M} : (l_\mathrm{p}\tau + \tau l_\mathrm{p}^\mathrm{T})\right] \tag{93}$$

represents the plastic part of the stress rate $\overset{*}{\tau}$.

The inelastic part of the rate of deformation tensor can be expressed by inverting eqn (78) and using eqn (91) as

$$\mathrm{D}^\mathrm{i} = -\mathcal{M} : (\overset{*}{\tau})^\mathrm{i} = -\mathcal{M} : (\overset{*}{\tau})^\mathrm{d} - \mathcal{M} : (\overset{*}{\tau})^\mathrm{p} . \tag{94}$$

Hence, by decomposing inelastic rate of deformation tensor into its damage and plastic parts

$$\mathrm{D}^\mathrm{i} = \mathrm{D}^\mathrm{d} + \mathrm{D}^\mathrm{p} , \tag{95}$$

comparison with eqn (94) establishes the relationships

$$\mathrm{D}^\mathrm{d} = -\mathcal{M} : (\overset{*}{\tau})^\mathrm{d} = \overset{*e}{\mathcal{M}} : \tau , \tag{96}$$

and

$$\mathrm{D}^\mathrm{p} = -\mathcal{M} : (\overset{*}{\tau})^\mathrm{p} = \mathrm{d}_\mathrm{p} + \mathcal{M} : (l_\mathrm{p}\tau + \tau l_\mathrm{p}^\mathrm{T}) . \tag{97}$$

It is also important to note the following relations:

$$\tau : \mathrm{D}^\mathrm{d} = -\epsilon_\mathrm{e} : (\overset{*}{\tau})^\mathrm{d} , \quad \tau : \mathrm{D}^\mathrm{p} = -\epsilon_\mathrm{e} : (\overset{*}{\tau})^\mathrm{p} . \tag{98}$$

VI.1. *An alternative partition*

The introduced convected derivatives $(\overset{*e}{\ })$, such as eqn (86), are not uniquely defined, since the unstressed intermediate configuration is specified only to within an arbitrary

rigid-body rotation. Hence, the velocity gradient $\dot{F}_e F_e^{-1}$, used to define $(\overset{*e}{\ })$, is not uniquely defined either. However, in some applications it may be convenient to specify the intermediate configuration uniquely, on the basis of some additional physical structure, explicitly introduced in the considered material model and pertinent to its internal structure and the deformation modes. For example, in crystal plasticity (ASARO [1983]), the rotation of the intermediate configuration is uniquely specified by requiring that the basic crystalline-lattice structure retain the orientation relative to the fixed reference frame (isoclinic intermediate configuration, in the terminology of MANDEL [1973]). The velocity gradient $\dot{F}_e F_e^{-1}$ is in this case uniquely defined, and represents the sum of the lattice strain rate and the lattice spin. Physically, it is the discontinuous slip of the material over the active slip planes that causes the lattice orientation to be convected by the lattice and not by the material itself.

In some applications, however, it may be more appropriate to introduce the convected derivative as the derivative observed in the reference frame that deforms with the material, i.e. relative to the material velocity gradient $\mathrm{L} = \dot{F}F^{-1}$. For instance, in brittle materials like brittle rocks, the change of elastic properties occurs due to propagation of the crack-like defects through the material, which are convected with the material during the deformation process. Therefore, by using eqn (9) to eliminate $(\dot{F}_e F_e^{-1})$, the first term on the right-hand side of eqn (90) can be rewritten as

$$\overset{*e}{\mathcal{L}} : \epsilon_e = \overset{*}{\mathcal{L}} : \epsilon_e + l_p \tau + \tau l_p^{\mathrm{T}} + \mathcal{L} : (l_p^{\mathrm{T}} \epsilon_e + \epsilon_e l_p) \,. \tag{99}$$

In eqn (99), $\overset{*}{\mathcal{L}}$ is the convected derivative of $\mathcal{L}$ relative to total velocity gradient L, defined by eqn (8). Consequently, eqn (90) becomes

$$(\overset{*}{\tau})^{\mathrm{i}} = \overset{*}{\mathcal{L}} : \epsilon_e - \mathcal{L} : (\mathrm{d}_p - \epsilon_e l_p - l_p^{\mathrm{T}} \epsilon_e) \,. \tag{100}$$

The group of terms within the parentheses on the right-hand side of eqn (100) can be conveniently written as

$$\mathrm{d}_p - \epsilon_e l_p - l_p^{\mathrm{T}} \epsilon_e = F_e^{-\mathrm{T}} (\dot{F}_p F_p^{-1})_s F_e^{-1} \,. \tag{101}$$

The first term on the right-hand side of eqn (100) is the damage part of the stress rate $\overset{*}{\tau}$, corresponding to the rate of stiffness tensor $\overset{*}{\mathcal{L}}$, i.e.

$$(\overset{*}{\tau})^{\mathrm{d}} = \overset{*}{\mathcal{L}} : \epsilon_e \,. \tag{102}$$

The plastic part of the stress rate is the remaining part on the right-hand side of eqn (100), which can be written in view of eqn (101) as

$$(\overset{*}{\tau})^{\mathrm{p}} = -\mathcal{L} : [F_e^{-\mathrm{T}} (\dot{F}_p F_p^{-1})_s F_e^{-1}] \,. \tag{103}$$

The corresponding damage and plastic strain rates are

$$\mathrm{D}^{\mathrm{d}} = \overset{*}{\mathcal{M}} : \tau \tag{104}$$

and

$$\mathrm{D}^{\mathrm{p}} = F_e^{-\mathrm{T}} (\dot{F}_p F_p^{-1})_s F_e^{-1} \,. \tag{105}$$

The relationship (105) shows that the rate of deformation tensor $(\dot{F}_p F_p^{-1})_s$ is induced from the plastic strain rate D^p by elastic deformation F_e.

VI.2. *Partition of inelastic rates of the Lagrangian strain and Piola–Kirchhoff stress*

By differentiating eqn (56) it follows:

$$\dot{S} = \Lambda_0 : \dot{E} + \dot{\Lambda}_0 : (E - E_p) - \Lambda_0 : \dot{E}_p \,. \tag{106}$$

The first term on the right-hand side of eqn (106) is according to eqn (77) the elastic part of the stress rate $\dot{S}$. The remaining part

$$(\dot{S})^i = \dot{\Lambda}_0 : (E - E_p) - \Lambda_0 : \dot{E}_p \tag{107}$$

represents the inelastic part of the stress rate $\dot{S}$. This can be further partitioned as

$$(\dot{S})^i = (\dot{S})^d + (\dot{S})^p \,, \tag{108}$$

where

$$(\dot{S})^d = \dot{\Lambda}_0 : (E - E_p) \tag{109}$$

represents the damage part, while

$$(\dot{S})^p = -\Lambda_0 : \dot{E}_p \tag{110}$$

represents the plastic part of the stress rate $\dot{S}$.

On the other hand, inverting eqn (79), substituting eqn (107) for $(\dot{S})^i$, and using eqn (59), it follows that:

$$(\dot{E})^i = \dot{M}_0 : S + \dot{E}_p \,. \tag{111}$$

Hence, the inelastic part of the rate of the Lagrangian strain can be partitioned as

$$(\dot{E})^i = (\dot{E})^d + (\dot{E})^p \,, \tag{112}$$

where

$$(\dot{E})^d = \dot{M}_0 : S \tag{113}$$

represents the damage part, while

$$(\dot{E})^p = \dot{E}_p \tag{114}$$

is the plastic part of the strain rate $\dot{E}$.

The transition from eqn (109) to eqn (102) can be established directly by using the kinematic relationship (7), with

$$(\dot{S})^d = F^{-1}(\overset{*}{\tau})^d F^{-T} \,, \tag{115}$$

and the relationship between the rates of the stiffness tensors Λ_0 and $\mathcal{L}$, i.e.

$$\overset{*}{\mathcal{L}} = F \otimes F \, \dot{\Lambda}_0 \, F^T \otimes F^T \,. \tag{116}$$

The transition from the expression (110) to eqn (103) can be derived by using eqn (69a) to express the stiffness Λ_0 in terms of $\mathcal{L}$, and by using the relationships

$$(\dot{S})^{\mathrm{p}} = F^{-1}(\overset{*}{\tau})^{\mathrm{p}}F^{-\mathrm{T}}, \tag{117}$$

and

$$\dot{E}_{\mathrm{p}} = F_{\mathrm{p}}^{\mathrm{T}}(\dot{F}_{\mathrm{p}}F_{\mathrm{p}}^{-1})_{\mathrm{s}}F_{\mathrm{p}}. \tag{118}$$

The connections between the expressions for the rates of strain are also readily obtained. From eqns (104), (105), (113) and (114) it follows that:

$$(\dot{E})^{\mathrm{d}} = F^{\mathrm{T}}\mathrm{D}^{\mathrm{d}}F, \quad (\dot{E})^{\mathrm{p}} = F^{\mathrm{T}}\mathrm{D}^{\mathrm{p}}F. \tag{119}$$

VII. THERMODYNAMIC ANALYSIS

Within the framework of thermodynamics with internal state variables, the inelastic deformation of a considered material sample can be viewed as a sequence of constrained equilibrium states. These are created by a conceptual constraining of the internal variables at their current values through imposed appropriate thermodynamic forces or constraints (KESTIN & RICE [1970]; RICE [1971]; SCHAPERY [1990]). Let Υ be the strain energy associated with a specific microstructural rearrangements produced during the inelastic deformation process. This can include the surface energy of the newly created internal crack and pore surfaces, residual elastic energy associated with a non-reversible relative displacement (sliding) of frictional crack surfaces, locked-in energy due to elastic stress and strain fields around dislocations or other imperfections, etc. For instance, in the case of progressively microfracturing solids the corresponding contribution to Υ is the product of the total area of all crack surfaces within the representative volume element and the material specific surface energy (RICE [1978]). In general, an explicit representation for the energy Υ or its rate $\dot{\Upsilon}$ is a more difficult task, and depends on the considered material model and selected variables introduced to account for its pertinent microstructural rearrangements.

VII.1. *Helmholtz free energy*

The Helmholtz free energy per unit initial volume in an isothermal deformation process can be written as

$$\Psi = \psi_0 + \Upsilon, \tag{120}$$

where the strain energy $\psi_0 = \psi$ is defined by eqn (46). The rate of the Helmholtz free energy is then

$$\dot{\Psi} = \frac{1}{2}\dot{\Lambda} :: (E_{\mathrm{e}} \otimes E_{\mathrm{e}}) + (\Lambda : E_{\mathrm{e}}) : \dot{E}_{\mathrm{e}} + \dot{\Upsilon}, \tag{121}$$

where $\dot{\Upsilon}$ denotes the rate of Υ associated with the two neighboring constrained equilibrium states with the two different sets of internal variables. By using eqns (47) and (86), the above expression can be rewritten as

$$\dot{\Psi} = \frac{1}{2}\overset{*e}{\mathcal{L}} :: (\epsilon_e \otimes \epsilon_e) + S_e : \dot{E}_e + \dot{\Upsilon} . \tag{122}$$

In view of eqn (29), this further becomes

$$\dot{\Psi} = \tau : D + \frac{1}{2}\overset{*e}{\mathcal{L}} :: (\epsilon_e \otimes \epsilon_e) - \tau : d_p + \dot{\Upsilon} . \tag{123}$$

Incorporating eqn (92) and using eqns (97) and (98), eqn (123) can be finally written as

$$\dot{\Psi} = \tau : D - \frac{1}{2}\tau : D^d - \tau : D^p + \tau : (\epsilon_e l_p + l_p^T \epsilon_e) + \dot{\Upsilon} . \tag{124}$$

If the damage part of the stress rate is defined by using the convected derivative $\overset{*}{\mathcal{L}}$ rather than $\overset{*e}{\mathcal{L}}$, eqn (92) is replaced by eqn (102), and eqn (124) simplifies to

$$\dot{\Psi} = \tau : D - \frac{1}{2}\tau : D^d - \tau : D^p + \dot{\Upsilon} . \tag{125}$$

If the deformation process is assumed to be isothermal with the negligible temperature non-uniformities, by using the first law of thermodynamics (balance of energy), the rate of the Helmholtz free energy can be written as (see, for example, eqn (5.26) of COLEMAN & GURTIN [1967])

$$\dot{\Psi} = \tau : D - T\zeta , \tag{126}$$

where T is the temperature, and ζ the irreversible entropy production rate. The product $T\zeta$ represents the energy dissipation rate associated with the inelastic (damage and plastic) deformation processes. As a consequence of the second law of thermodynamics, $\zeta \geqslant 0$. Comparing eqns (125) and (126), therefore,

$$T\zeta = \frac{1}{2}\tau : D^d + \tau : D^p - \dot{\Upsilon} . \tag{127}$$

The first two terms on the right-hand side of eqn (127) represent a macroscopically non-recoverable rate of work expanded on inelastic processes. The remaining part of the total rate of work $\tau : D$, i.e.

$$\tau : D^e + \frac{1}{2}\tau : D^d \tag{128}$$

represents the macroscopically recoverable part of the total rate of work.

If the strain energy representation eqn (54) is used rather than eqn (46), the rate of the Helmholtz free energy becomes

$$\dot{\Psi} = \frac{1}{2}\dot{\Lambda}_0 :: [(E - E_p) \otimes (E - E_p)] + [\Lambda_0 : (E - E_p)] : (\dot{E} - \dot{E}_p) + \dot{\Upsilon} . \tag{129}$$

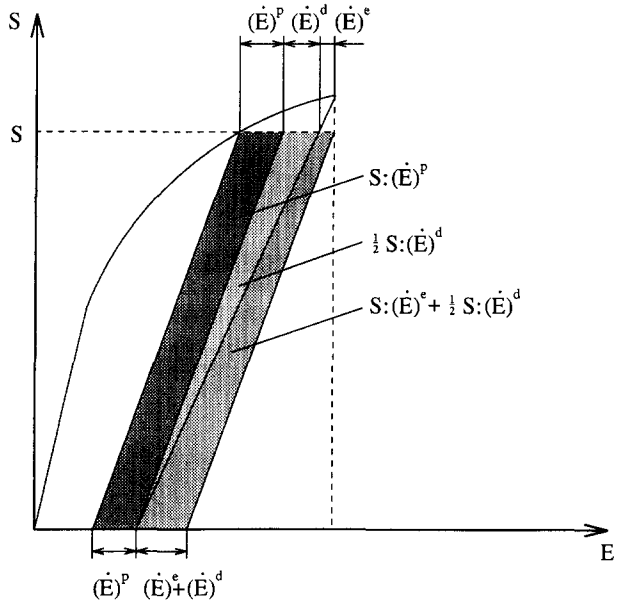

Fig. 1. Schematic representation of the elastic, damage, and plastic *strain* rates (or their increments) in the case of a uniaxial stress–strain curve. The trapezoidal shaded area represents the recoverable part of the total rate of work. The remaining two shaded areas together represent the non-recoverable part of the total rate of work.

Using eqn (56), eqns (113) and (114) above can be written as

$$\dot{\Psi} = S : \dot{E} - \frac{1}{2} S : (\dot{E})^{\mathrm{d}} - S : (\dot{E})^{\mathrm{p}} + \dot{\Upsilon} . \tag{130}$$

The energy dissipation rate is, therefore,

$$T\zeta = \frac{1}{2} S : (\dot{E})^{\mathrm{d}} + S : (\dot{E})^{\mathrm{p}} - \dot{\Upsilon} . \tag{131}$$

In view of the relationships (14) and (119), eqn (131) is equivalent to the previously derived expression (127). A schematic representation of the elastic, damage and plastic strain rates, and the recoverable and non-recoverable parts of the total rate of work $S : \dot{E}$ is shown on the uniaxial stress-strain curve in Fig. 1.

VII.2. *Gibbs energy*

The Legendre transform of the Helmholtz energy is the Gibbs energy

$$\Phi = S : E - \Psi = \phi_0 - \Upsilon , \tag{132}$$

where $\phi_0 = S : E - \psi_0$ is the Legendre transform of the strain energy ψ_0. In view of the expression (57), eqn (132) becomes

$$\Phi = \frac{1}{2}\mathbf{M}_0 :: (S \otimes S) + S : E_p - \Upsilon \,. \tag{133}$$

The rate of the Gibbs energy is, therefore,

$$\dot{\Phi} = \frac{1}{2}\dot{\mathbf{M}}_0 :: (S \otimes S) + (\mathbf{M}_0 : S) : \dot{S} + \dot{S} : E_p + S : \dot{E}_p - \dot{\Upsilon} \,. \tag{134}$$

This can be rewritten by using eqn (59) as

$$\dot{\Phi} = E : \dot{S} - \frac{1}{2}[\dot{\Lambda}_0 : (E - E_p)] : (E - E_p) + S : \dot{E}_p - \dot{\Upsilon} \,. \tag{135}$$

Incorporating the expressions (109) and (110) for the damage and plastic parts of the stress rate $\dot{S}$, the rate of Gibbs energy can be finally expressed as

$$\dot{\Phi} = E : \dot{S} - \frac{1}{2}(E - E_p) : (\dot{S})^d - (E - E_p) : (\dot{S})^p - \dot{\Upsilon} \,. \tag{136}$$

Since in an isothermal process the above must be equal to

$$\dot{\Phi} = E : \dot{S} + T\zeta \,, \tag{137}$$

the comparison with eqn (136) gives the following expression for the energy dissipation rate

$$T\zeta = -\frac{1}{2}(E - E_p) : (\dot{S})^d - (E - E_p) : (\dot{S})^p - \dot{\Upsilon} \,. \tag{138}$$

This is a dual relationship to eqn (131). The transition from one to another expression is evident by using the relationships of the type (83). A schematic representation of the elastic, damage and plastic stress rates, and the recoverable and non-recoverable parts of the total rate of work $(E - E_p) : (\dot{S})^e$ (which is identically equal to $S : \dot{E}$) is shown on the uniaxial stress–strain curve in Fig. 2.

The alternative derivation, employing the convected rate of the Kirchhoff stress $\overset{*}{\tau}$, can be performed in a similar manner. The expression (133) is rewritten as

$$\Phi = \frac{1}{2}\mathbf{M} :: (S_e \otimes S_e) + S : E_p - \Upsilon \,, \tag{139}$$

which gives by differentiation

$$\dot{\Phi} = \frac{1}{2}\dot{\mathbf{M}} :: (S_e \otimes S_e) + (\mathbf{M} : S_e) : \dot{S}_e + \dot{S} : E_p + S : \dot{E}_p - \dot{\Upsilon} \,. \tag{140}$$

Since

$$\dot{\mathbf{M}} :: (S_e \otimes S_e) = \overset{*e}{\mathcal{M}} :: (\tau \otimes \tau) = -\overset{*e}{\mathcal{L}} :: (\epsilon_e \otimes \epsilon_e) \,, \tag{141}$$

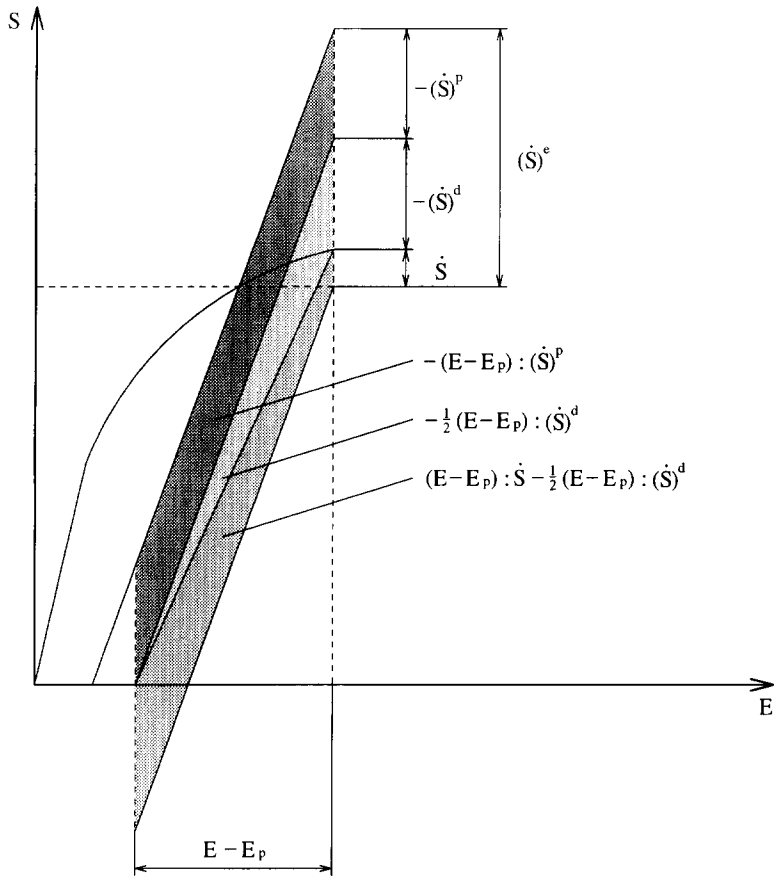

Fig. 2. Schematic representation of the elastic, damage, and plastic *stress* rates in the case of a uniaxial stress–strain curve. The trapezoidal shaded area is the recoverable part of the total rate of work, while the remaining two shaded areas together represent the non-recoverable part of the total rate of work.

and since by eqns (25)–(30)

$$\epsilon_e : \dot{S}_e + \dot{S} : E_p + S : \dot{E}_p = \epsilon : \overset{*}{\tau} + \tau : d_p \,, \tag{142}$$

the expression (140) becomes

$$\dot{\Phi} = \epsilon : \overset{*}{\tau} - \frac{1}{2}\overset{*e}{\mathcal{L}} :: (\epsilon_e \otimes \epsilon_e) + \tau : d_p - \dot{\Upsilon} \,. \tag{143}$$

This can finally be written by using eqns (99)–(103) as

$$\dot{\Phi} = \epsilon : \overset{*}{\tau} - \frac{1}{2}\epsilon_e : (\overset{*}{\tau})^{d} - \epsilon_e : (\overset{*}{\tau})^{p} - \dot{\Upsilon} \,, \tag{144}$$

which is equivalent to eqn (136). The second and third terms on the right-hand side of eqn (144) together represent macroscopically non-recoverable rate of work expanded

on inelastic processes. The remaining part of the total rate of work $\epsilon_e : (\overset{*}{\tau})^e$ (which is identically equal to $\tau : D$), is

$$\epsilon_e : \overset{*}{\tau} - \frac{1}{2}\epsilon_e : (\overset{*}{\tau})^d , \tag{144}$$

and represents the macroscopically recoverable part of the total rate of work.

VIII. DAMAGE POTENTIALS AND EVOLUTION EQUATIONS FOR ELASTIC COMPLIANCE AND STIFFNESS TENSORS

Consider the case when the damage part of the stress and strain rates are defined by eqns (92) and (96). The non-recoverable rate of work expanded on damage processes can then be written as

$$\frac{1}{2}\tau : D^d = \frac{1}{2}\tau : \overset{*e}{\mathcal{M}} : \tau = \overset{*e}{\mathcal{M}} :: \Gamma , \tag{145}$$

where

$$\Gamma = \frac{1}{2}(\tau \otimes \tau) \tag{146}$$

represents a thermodynamic force (affinity) conjugate to elastic compliance $\mathcal{M}$ as a damage tensor. Alternatively, one can express the non-recoverable rate of work expanded on damage processes as

$$-\frac{1}{2}\epsilon_e : (\overset{*}{\tau})^d = -\frac{1}{2}\epsilon_e : \overset{*e}{\mathcal{L}} : \epsilon_e = \overset{*e}{\mathcal{L}} :: G , \tag{147}$$

where

$$G = -\frac{1}{2}(\epsilon_e \otimes \epsilon_e) \tag{148}$$

represents a thermodynamic force (affinity) conjugate to elastic stiffness $\mathcal{L}$ as a damage tensor. Clearly,

$$\overset{*e}{\mathcal{M}} :: \Gamma = \overset{*e}{\mathcal{L}} :: G . \tag{149}$$

The affinities Γ and G have been used in the damage mechanics literature by ORTIZ [1985]; SIMO and JU [1987]; LUBARDA and KRAJCINOVIC [1994b], and others.

Suppose that for the considered material degradation processes there is a scalar function $\Omega = \Omega(\Gamma, \mathcal{M}, \rho)$, and a scalar function $\Pi = \Pi(G, \mathcal{L}, \rho)$ such that

$$\overset{*e}{\mathcal{M}} = \dot{\rho}\frac{\partial \Omega}{\partial \Gamma} , \quad \overset{*e}{\mathcal{L}} = \dot{\rho}\frac{\partial \Pi}{\partial G} , \tag{150}$$

where $\dot{\rho}$ is the rate of some monotonically increasing scalar parameter ρ, which can be considered as a measure of the cumulative damage at the considered instant of the deformation process ($\dot{\rho} \geqslant 0$). The function Ω is referred to as a damage potential for the compliance flux $\overset{*e}{\mathcal{M}}$, while the function Π is a damage potential for the stiffness flux

$\overset{*e}{\mathcal{L}}$. The equations appearing in eqn (150) are the evolution equations for the compliance and stiffness tensors.

In view of the principle of material frame indifference (TRUESDELL & NOLL [1965]), the introduced potentials Ω and Π must be isotropic scalar functions of the set of all their respective arguments, i.e. $(\Gamma, \mathcal{M}, \rho)$ and $(G, \mathcal{L}, \rho)$. For example, applied to the potential Ω, this means that the equality

$$\Omega(\mathbf{Q} \otimes \mathbf{Q}\Gamma\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{Q}^{\mathrm{T}},\ \mathbf{Q} \otimes \mathbf{Q}\mathcal{M}\mathbf{Q}^{\mathrm{T}} \otimes \mathbf{Q}^{\mathrm{T}},\ \rho) = \Omega(\Gamma, \mathcal{M}, \rho)\,, \tag{151}$$

holds for any proper orthogonal transformation $\mathbf{Q}$ superimposed on the current configuration. Using relationships (92) and (96), in conjunction with eqn (150), the damage parts of the strain and stress rates are

$$\mathbf{D}^{\mathrm{d}} = \overset{*e}{\mathcal{M}} : \tau = \dot{\rho}\frac{\partial \Omega}{\partial \Gamma} : \tau = \dot{\rho}\frac{\partial \Omega}{\partial \tau} \tag{153}$$

$$(\overset{*}{\tau})^{\mathrm{d}} = \overset{*e}{\mathcal{L}} : \epsilon_{\mathrm{e}} = \dot{\rho}\frac{\partial \Pi}{\partial G} : \epsilon_{\mathrm{e}} = -\dot{\rho}\frac{\partial \Pi}{\partial \epsilon_{\mathrm{e}}}\,. \tag{154}$$

Therefore, the damage potential Ω serves as a potential for the compliance flux $\overset{*e}{\mathcal{M}}$ in the space of affinity Γ, eqn (150a), and as a potential for the damage strain rate $\mathbf{D}^{\mathrm{d}}$ in the space of the stress τ, eqn (153). Similarly, the damage potential Π is a potential for the stiffness flux $\overset{*e}{\mathcal{L}}$ in the space of affinity G, eqn (150b) and a potential for the damage stress rate $(\overset{*}{\tau})^{\mathrm{d}}$ in the space of the elastic strain ϵ_{e}, eqn (154).

For example, if the damage potential for the stiffness flux $\overset{*}{\mathcal{L}}$ is given by

$$\Pi = \frac{1}{2}\epsilon_{\mathrm{e}} : \mathcal{L} : \epsilon_{\mathrm{e}} = -\mathcal{L} :: G\,, \tag{155}$$

the evolution eqn (150b) for the stiffness tensor $\mathcal{L}$ becomes

$$\overset{*e}{\mathcal{L}} = -\dot{\rho}\mathcal{L}\,. \tag{156}$$

This evolution equation corresponds to isotropic damage evolution in the course of which the stiffness tensor Λ is related to the initial stiffness tensor Λ_0 of the undamaged state by

$$\Lambda = (1 - d)\Lambda^0\,, \quad (0 \leqslant d \leqslant 1)\,. \tag{157}$$

In eqn (157), d is a scalar damage variable (KACHANOV [1958]; KRAJCINOVIC [1989]; LEMAITRE [1992]; LUBARDA & KRAJCINOVIC [1994a]), which represents a measure of the accumulated isotropic damage (for example, due to isotropic void growth in a spherically voided material undergoing plastic deformation). The expression defining the rate of monotonically increasing parameter ρ is easily found to be

$$\dot{\rho} = \frac{\dot{d}}{1 - d}\,. \tag{158}$$

Integrating eqn (158) it follows that $\rho = -\ln(1 - d)$. Therefore, as damage evolves and the damage variable d increases from zero to its maximum value which is equal to

one, the parameter ρ increases from zero to infinity. For further discussion of various damage measures, the reader is referred to JANSON and HULT [1977]; and KRAJCINOVIC *et al.* [1993]. In a similar manner, by taking the damage potential for the compliance flux $\overset{*e}{\mathcal{M}}$ to be

$$\Omega = \frac{1}{2}\tau : \mathcal{M} : \tau = \mathcal{M} :: \Gamma \,, \tag{159}$$

from eqn (150a) it follows that

$$\overset{*e}{\mathcal{M}} = \dot{\rho}\mathcal{M} \,. \tag{160}$$

In view of eqns (156) and (160), the relationship (149) leads to the identity

$$\mathcal{M} :: \Gamma = -\mathcal{L} :: G \,, \tag{161}$$

which clearly holds since both sides of eqn (161) are equal to $\frac{1}{2}\tau : \epsilon_e$. An alternative representation of the damage potential, analogous to that used by ORTIZ [1985], is $\Omega = (\Gamma :: \Gamma)^{1/2}$, which has the gradient $\partial\Omega/\partial\Gamma$ proportional to Γ. ORTIZ [1985] analyzed an important issue of modeling different tensile and compressive responses of brittle materials by employing positive and negative projections of stress and strain tensors. Using this method it is possible to distinguish between two basic, tensile and compressive, damage evolution modes. The reader is also referred to YAZDANI and SCHREYER [1990].

If the Lagrange strain and its conjugate Piola–Kirchhoff stress tensors are used to cast the formulation of the analysis, the thermodynamic force conjugate to the compliance tensor $\mathbf{M}_0$ is

$$\Gamma_0 = \frac{1}{2}(S \otimes S) \,. \tag{162}$$

The thermodynamic force conjugate to the stiffness tensor Λ_0 is

$$G_0 = -\frac{1}{2}(E - E_p) \otimes (E - E_p) \,. \tag{163}$$

The evolution equations for the compliance and stiffness tensors $\mathbf{M}_0$ and Λ_0 are in terms of dual damage potentials $\Omega_0 = \Omega_0(G_0, \mathbf{M}_0, \rho_0)$ and $\Pi_0 = \Pi_0(\Gamma_0, \Lambda_0, \rho_0)$, given by

$$\dot{\mathbf{M}}_0 = \dot{\rho}_0 \frac{\partial \Omega_0}{\partial \Gamma_0} \,, \quad \dot{\Lambda}_0 = \dot{\rho}_0 \frac{\partial \Pi_0}{\partial G_0} \,, \tag{164}$$

where ρ_0 is the corresponding measure of the cumulative damage. The damage parts of the rate of the Lagrangian strain and the Piola–Kirchhoff stress tensors are

$$(\dot{E})^d = \dot{\mathbf{M}}_0 : S = \dot{\rho}_0 \frac{\partial \Omega_0}{\partial \Gamma_0} : S = \dot{\rho}_0 \frac{\partial \Omega_0}{\partial S} \tag{165}$$

$$(\dot{S})^{\mathrm{d}} = \dot{\Lambda}_0 : (E - E_{\mathrm{p}}) = \dot{\rho}_0 \frac{\partial \Pi_0}{\partial G_0} : (E - E_{\mathrm{p}}) = -\dot{\rho}_0 \frac{\partial \Pi_0}{\partial E} \,. \tag{166}$$

The introduced affinities Γ_0 and G_0 are related to the affinities Γ and G by

$$\Gamma_0 = F^{-1} \otimes F^{-1} \, \Gamma \, F^{-\mathrm{T}} \otimes F^{-\mathrm{T}} \,, \quad G_0 = F^{\mathrm{T}} \otimes F^{\mathrm{T}} G F \otimes F \,. \tag{167}$$

VIII.1. *Damage surface*

To define the onset or continuation of damage, a concept of the damage surface is customarily introduced (DOUGILL [1975,1983]; DRAGON & MRÓZ [1979]; HOLCOMB [1981]; HOLCOMB & COSTIN [1986]; ORTIZ [1985]; SIMO & JU [1987]; ASHBY & SAMMIS [1990]; LUBARDA & KRAJCINOVIC [1994b], etc.), in analogy to the yield surface of the plasticity theory. This can be accomplished by using either of the introduced affinities conjugate to the corresponding damage tensors. The selection is governed primarily by either analytical or computational advantages associated with a particular choice. Consider the following damage surface introduced in the space of the affinity G_0

$$\pi_0(G_0, \Lambda_0, \rho_0) = 0 \,. \tag{168}$$

By definition, the function π_0 is non-negative for all possible states of damage. The evolution of damage is possible only when the equality (168) is satisfied. In the course of a loading process, during which the damage gradually evolves, the affinity G_0 remains on the expanding (or moving) damage surface. Consequently, the consistency condition $\dot{\pi} = 0$ holds, which gives

$$\frac{\partial \pi_0}{\partial G_0} :: \dot{G}_0 + \frac{\partial \pi_0}{\partial \Lambda_0} :: \dot{\Lambda}_0 + \frac{\partial \pi_0}{\partial \rho_0} \dot{\rho}_0 = 0 \,. \tag{169}$$

During unloading from the current damage surface, $\dot{\pi} < 0$ and $\dot{\rho} = 0$; hence $(\partial \pi_0 / \partial G_0) :: \dot{G}_0 < 0$. If $\dot{\pi} = 0$ and $\dot{\rho} = 0$, the rate of affinity defines a loading during which the state moves along the fixed damage surface (neutral loading); hence from eqn (169) $(\partial \pi_0 / \partial G_0) :: \dot{G}_0 = 0$. The remaining possibility, defined by the rate of affinity $\dot{G}_0$ satisfying the inequality

$$\frac{\partial \pi_0}{\partial G_0} :: \dot{G}_0 > 0 \tag{170}$$

constitutes the damage loading condition. The rate of cumulative damage parameter is, in this case, determined by substituting the evolution eqn (164b) for the stiffness flux $\dot{\Lambda}_0$ into the consistency condition (169). Hence,

$$\dot{\rho}_0 = \frac{1}{h_0} \left(\frac{\partial \pi_0}{\partial G_0} :: \dot{G}_0 \right) = \frac{1}{h_0} \frac{\partial \pi_0}{\partial E} : (\dot{E} - \dot{E}_{\mathrm{p}}) \,, \tag{171}$$

where

$$h_0 = -\left(\frac{\partial \pi_0}{\partial \Lambda_0} :: \frac{\partial \Pi_0}{\partial G_0} + \frac{\partial \pi_0}{\partial \rho_0} \right) . \tag{172}$$

The stiffness flux is determined by substituting eqn (171) into the evolution eqn (164b)

$$\dot{\Lambda}_0 = \frac{1}{h_0}\left(\frac{\partial \Pi_0}{\partial G_0} \otimes \frac{\partial \pi_0}{\partial G_0}\right) :: \dot{G}_0 \,. \tag{173}$$

The damage part of the rate of the Piola–Kirchoff stress eqn (166) can be expressed as

$$(\dot{S})^{\mathrm{d}} = -\frac{1}{h_0}\left(\frac{\partial \Pi_0}{\partial E} \otimes \frac{\partial \pi_0}{\partial E}\right) : (\dot{E} - \dot{E}_{\mathrm{p}}) \,. \tag{174}$$

If the damage potential and damage surface functions are identical ($\Pi_0 = \pi_0$), the introduced damage rules are referred to as being associative. In a general case $\Pi_0 \neq \pi_0$, the damage rules are nonassociative.

When the damage surface is introduced in the space of the affinity Γ_0 by equation

$$\omega_0(\Gamma_0, \mathrm{M}_0, \rho_0) = 0 \,, \tag{175}$$

it follows that the rate of the cumulative damage parameter is

$$\dot{\rho}_0 = \frac{1}{h_0}\left(\frac{\partial \omega_0}{\partial \Gamma_0} :: \dot{\Gamma}_0\right) = \frac{1}{h_0}\left(\frac{\partial \omega_0}{\partial S} : \dot{S}\right) , \tag{176}$$

where

$$h_0 = -\left(\frac{\partial \omega_0}{\partial \mathrm{M}_0} :: \frac{\partial \Omega_0}{\partial \Gamma_0} + \frac{\partial \omega_0}{\partial \rho_0}\right) . \tag{177}$$

The compliance flux is derived by substituting eqn (176) into the evolution eqn (164a),

$$\dot{\mathrm{M}}_0 = \frac{1}{h_0}\left(\frac{\partial \Omega_0}{\partial \Gamma_0} \otimes \frac{\partial \omega_0}{\partial \Gamma_0}\right) :: \dot{\Gamma}_0 \,. \tag{178}$$

The damage part of the rate of the Lagrangian strain eqn (165) can be expressed as

$$(\dot{E})^{\mathrm{d}} = \frac{1}{h_0}\left(\frac{\partial \Omega_0}{\partial S} \otimes \frac{\partial \omega_0}{\partial S}\right) : \dot{S} \,. \tag{179}$$

Note that the rate of the Piola–Kirchhoff stress in eqns (176) and (179) can be expressed in terms of the convected rate of the Kirchhoff stress by using the identity

$$\frac{\partial \omega_0}{\partial S} : \dot{S} \equiv \frac{\partial \omega_0}{\partial \tau} : \overset{*}{\tau} \,. \tag{180}$$

IX. CONSTITUTIVE EQUATIONS FOR INELASTIC STRESS AND STRAIN RATES

To complete the constitutive analysis of damage-elastoplasticity, a constitutive equation for the inelastic stress or strain rate has to be constructed. If the constitutive

structure for the inelastic stress rate is sought, the inelastic potential and the inelastic yield surface are introduced in the appropriate strain space. If the constitutive structure for the inelastic strain rate is considered, the inelastic potential and the inelastic yield surface are introduced in the appropriate stress space. A duality of the two formulations was recently discussed in the context of elastoplasticity by LUBARDA [1994b]. The formulation of elastoplasticity theory employing the yield surfaces in stress and strain spaces have been thoroughly studied in the past by HILL [1959,1968,1978]; ILYUSHIN [1961]; HILL and RICE [1973]; NAGHDI and TRAPP [1975]; CASEY and NAGHDI [1984,1988,1992]; YODER and IWAN [1981], and others. In this section both formulations are used, i.e. the constitutive equations are developed for the inelastic parts of the stress and strain rates using the inelastic potentials and the yield surfaces in the strain and stress space, respectively. It should, however, be pointed out that the constitutive equations for the total inelastic parts of the stress and strain rates are constructed. This is in variance with most other approaches in damage mechanics literature, where the constitutive equations for the damage and plastic parts of the stress or strain rates are sought separately. It seems to be more convenient and physically appealing to construct the constitutive equations for the total inelastic stress or strain rates, and determine the purely plastic part by subtracting the damage part from the total inelastic stress or strain rate. Indeed, if the material behavior is such that the Ilyushin postulate can be adopted (the net work during arbitrary closed strain cycle being positive, provided that inelastic deformation occurred during the cycle), it follows that the total inelastic stress and strain rates are normal to the respective yield surfaces in strain and stress spaces (ILYUSHIN [1961]; HILL [1968,1978]; PALGEN & DRUCKER [1983]).

IX.1. *Inelastic strain rate*

The constitutive structure for the inelastic part of the rate of the Lagrangian strain tensor $(\dot{E})^{\mathrm{i}}$ is in this subsection derived by introducing the inelastic potential

$$\mathcal{F}_0 = \mathcal{F}(S, \mathbf{M}_0, \rho_0, \mathbf{A}_\mathrm{p}), \tag{181}$$

such that

$$(\dot{E})^{\mathrm{i}} = \dot{\gamma}_0 \frac{\partial \mathcal{F}_0}{\partial S}. \tag{182}$$

The conditions for the existence of inelastic potentials for a class of solids exhibiting time dependent and independent inelasticity as a consequence of specific structural rearrangements on the microscale (such as metals deforming plastically through dislocation motion), has been studied by RICE [1971]. Among other results, it is shown that the normality structure in macroscopic laws arises when each local microstructural rearrangement proceeds at a rate governed by its associated thermodynamic force. For example, in metal plasticity the slip rate at a particular slip system is governed by the resolved shear stress on that system.

In eqn (182), $\dot{\gamma}$ is the inelastic loading parameter, while $\mathbf{A}_\mathrm{p}$ in eqn (181) represents suitable plastic internal variables, selected in accordance with considered material model and the pertinent deformation modes. For example, $\mathbf{A}_\mathrm{p}$ can include the plastic

strain tensor, a selected scalar hardening parameter, etc. In the absence of the time-dependent (viscous) processes, the evolution equation for these variables is assumed to be

$$\dot{\mathrm{A}}_{\mathrm{p}} = \mathcal{A}_{\mathrm{p}} : (\dot{E})^{\mathrm{p}} , \tag{183}$$

where $\mathcal{A}_{\mathrm{p}}$ stands for an appropriate tensor function corresponding to a particular plastic internal variable.

To define the onset and evolution of inelastic deformation, the inelastic yield surface in the Piola–Kirchhoff stress space is introduced by the equation

$$f_0(S, \mathrm{M}_0, \rho_0, \mathrm{A}_{\mathrm{p}}) = 0 . \tag{184}$$

The points inside this yield surface correspond to possible states of unloading, and the points outside correspond to the states of increasing inelastic loading. In the context of the plasticity theory without damage the yield surface introduced in the Piola–Kirchhoff stress space was used in the constitutive formulation by GREEN and NAGHDI [1965]; MANDEL [1973]; LUBLINER [1968]; CASEY and NAGHDI [1984,1988,1992], and others. Although the formulation of analysis appears to be straightforward, a care must be taken to select appropriately the arguments needed in the structure of the yield surface, in order to reproduce the results of classical plasticity formulation, such as those corresponding to isotropic or kinematic hardening with the Mises type yield condition in the Cauchy stress space. Some of these issues are discussed by VOYIADJIS and KIOUSIS [1987]; CASEY and NAGHDI [1992]; and LUBARDA [1994b].

The consistency condition for continuing inelastic deformation is obtained by differentiating eqn (184)

$$\frac{\partial f_0}{\partial S} : \dot{S} + \frac{\partial f_0}{\partial \mathrm{M}_0} :: \dot{\mathrm{M}}_0 + \frac{\partial f_0}{\partial \rho_0}\dot{\rho}_0 + \frac{\partial f_0}{\partial \mathrm{A}_{\mathrm{p}}} : \dot{\mathrm{A}}_{\mathrm{p}} = 0 . \tag{185}$$

Since neither damage nor plastic processes take place when the stress point moves inside or tangentially to the yield surface, the inelastic loading condition is

$$\frac{\partial f_0}{\partial S} : \dot{S} > 0 . \tag{186}$$

To derive the expression for the loading parameter $\dot{\gamma}_0$, the rate of plastic variables eqn (183) is expressed by using eqns (165) and (182) as

$$\dot{\mathrm{A}}_{\mathrm{p}} = \mathcal{A}_{\mathrm{p}} : (\dot{E})^{\mathrm{p}} = \mathcal{A}_{\mathrm{p}} : [(\dot{E})^{\mathrm{i}} - (\dot{E})^{\mathrm{d}}] = \mathcal{A}_{\mathrm{p}} : (\dot{\gamma}_0 \frac{\partial \mathcal{F}_0}{\partial S} - \dot{\rho}_0 \frac{\partial \Omega_0}{\partial S}) . \tag{187}$$

In view of eqn (164a), eqn (187) and the expression for the rate of the cumulative damage parameter (176), the consistency condition (185) now becomes

$$\frac{\partial f_0}{\partial S} : \dot{S} + \frac{\alpha_0}{h_0}\frac{\partial \omega_0}{\partial S} : \dot{S} - \beta_0 \dot{\gamma}_0 = 0 . \tag{188}$$

The parameter h_0 in eqn (188) is defined by eqn (177), while the parameters α_0 and β_0 are defined by

$$\alpha_0 = \frac{\partial f_0}{\partial \mathbf{M}_0} :: \frac{\partial \Omega_0}{\partial \Gamma_0} + \frac{\partial f_0}{\partial \rho_0} - \frac{\partial f_0}{\partial \mathbf{A}_p} : \mathcal{A}_p : \frac{\partial \Omega_0}{\partial S}, \quad \beta_0 = -\frac{\partial f_0}{\partial \mathbf{A}_p} : \mathcal{A}_p : \frac{\partial \mathcal{F}_0}{\partial S}. \tag{189}$$

The loading parameter follows by solving eqn (188) for $\dot{\gamma}_0$, which gives

$$\dot{\gamma}_0 = \frac{1}{H_0}\left(\frac{\partial f_0}{\partial S} + \kappa_0 \frac{\partial \omega_0}{\partial S}\right) : \dot{S}, \tag{190}$$

where $H_0 = \beta_0$ and $\kappa_0 = \alpha_0/h_0$.
Consequently, the inelastic part of the strain rate eqn (182) is governed by the constitutive equation

$$(\dot{E})^i = \frac{1}{H_0}\left[\frac{\partial \mathcal{F}_0}{\partial S} \otimes \left(\frac{\partial f_0}{\partial S} + \kappa_0 \frac{\partial \omega_0}{\partial S}\right)\right] : \dot{S}. \tag{191}$$

The constitutive equation for the overall damage-elastoplasticity according to eqn (72) is obtained by adding to eqn (191) the elastic part of the stress rate defined by eqn (73). Hence,

$$\dot{E} = \left\{\mathbf{M}_0 + \frac{1}{H_0}\left[\frac{\partial \mathcal{F}_0}{\partial S} \otimes \left(\frac{\partial f_0}{\partial S} + \kappa_0 \frac{\partial \omega_0}{\partial S}\right)\right]\right\} : \dot{S}. \tag{192}$$

Expression (192) can be alternatively written, in view of the relationships (12), (14), (69b) and (180), as

$$\mathbf{D} = \left\{\mathcal{M} + \frac{1}{H_0}\left[\frac{\partial \mathcal{F}_0}{\partial \tau} \otimes \left(\frac{\partial f_0}{\partial \tau} + \kappa_0 \frac{\partial \omega_0}{\partial \tau}\right)\right]\right\} : \overset{*}{\tau}. \tag{193}$$

If a current inelastic deformation involves only damage and not plastic processes, the loading parameter $\dot{\gamma}_0$ becomes equal to the rate of the cumulative damage parameter $\dot{\rho}_0$. On the other hand, when current inelastic deformation involves plasticity without damage, the loading parameter (190) reduces to $\dot{\gamma}_0 = (\partial f_0/\partial S) : \dot{S}/H_0$. Hence, in this case eqn (192) becomes

$$\dot{E} = \left[\mathbf{M}_0 + \frac{1}{H_0}\left(\frac{\partial \mathcal{F}_0}{\partial S} \otimes \frac{\partial f_0}{\partial S}\right)\right] : \dot{S}. \tag{194}$$

Since in general $\mathcal{F}_0 \neq f_0$, the instantaneous elastoplastic compliance tensor in eqn (194) does not possess the self-adjoint, reciprocal symmetry. The corresponding plastic flow rule is said to be nonassociative, and is commonly employed in the analysis of pressure sensitive dilatant materials with internal friction, such as brittle rock masses, granular and other geomaterials (RUDNICKI & RICE [1975]; NEMAT-NASSER [1983]).

IX.2. *Inelastic stress rate*

A formulation dual to the one presented in the previous subsection is obtained by introducing the inelastic potential and the yield surface in the Lagrangian strain space,

and constructing the constitutive equation for the inelastic part of the Piola–Kirchhoff stress tensor. Although the analysis appears to be somewhat more involved, the advantage of this approach is that a specific or explicit representation of the yield surface in the Lagrangian strain space may be simpler in form and easier to use than that of the yield surface in the Piola–Kirchhoff stress space (LUBARDA [1994b]). Therefore, let the inelastic potential be defined as

$$\mathcal{G}_0 = \mathcal{G}_0(E - E_p, \Lambda_0, \rho_0, \mathbf{B}_p) \,, \tag{195}$$

such that the inelastic part of the rate of the Piola–Kirchhoff stress tensor $(\dot{S})^i$ is

$$(\dot{S})^i = -\dot{\gamma}_0 \frac{\partial \mathcal{G}_0}{\partial E} \,. \tag{196}$$

In eqn (196), $\dot{\gamma}_0$ again denotes the inelastic loading parameter, while $\mathbf{B}_p$ in eqn (195), analogous to $\mathbf{A}_p$ in eqn (181), represents plastic internal variables, selected in accordance with the considered material model and the pertinent deformation modes. For example, $\mathbf{B}_p$ can include the plastic strain tensor, a selected scalar hardening parameter, etc. In the absence of the time-dependent (viscous) processes, the evolution equation for these variables is assumed to be

$$\dot{\mathbf{B}}_p = -\mathcal{B}_p : (\dot{S})^p \,, \tag{197}$$

where $\mathcal{B}_p$ stands for the appropriate tensor function corresponding to a particular plastic internal variable introduced in the analysis. For example, if $\mathbf{B}_p$ is the plastic strain E_p the tensor $\mathcal{B}_p$ is by eqn (110) the fourth-order compliance tensor $\mathbf{M}_0$.

To define the onset and evolution of inelastic deformation, the inelastic yield surface in the Lagrangian strain is next introduced by the equation

$$g_0(E - E_p, \Lambda_0, \rho_0, \mathbf{B}_p) = 0 \,. \tag{198}$$

The points inside this yield surface correspond to possible states of unloading, and the points outside correspond to the states of increasing inelastic loading. The consistency condition is, therefore,

$$\frac{\partial g_0}{\partial E} : (\dot{E} - \dot{E}_p) + \frac{\partial g_0}{\partial \Lambda_0} :: \dot{\Lambda}_0 + \frac{\partial g_0}{\partial \rho_0} \dot{\rho}_0 + \frac{\partial g_0}{\partial \mathbf{B}_p} : \dot{\mathbf{B}}_p = 0 \,. \tag{199}$$

The inelastic loading condition is

$$\frac{\partial g_0}{\partial E} : \dot{E} > 0 \,, \tag{200}$$

since neither damage nor plastic processes take place when the strain point moves inside or tangentially to the yield surface. To derive the expression for the loading parameter $\dot{\gamma}_0$, the rate of plastic variables is expressed by using eqns (166) and (196) as

$$\dot{\mathbf{B}}_{\mathrm{p}} = -\mathcal{B}_{\mathrm{p}} : (\dot{S})^{\mathrm{p}} = -\mathcal{B}_{\mathrm{p}} : [(\dot{S})^{\mathrm{i}} - (\dot{S})^{\mathrm{d}}] = \mathcal{B}_{\mathrm{p}} : (\dot{\gamma}_0 \frac{\partial \mathcal{G}_0}{\partial E} - \dot{\rho}_0 \frac{\partial \Pi_0}{\partial E}) \,. \tag{201}$$

In view of eqn (164b), eqn (201) and the expression for the rate of the cumulative damage parameter (171), the consistency condition (199) becomes

$$\frac{\partial g_0}{\partial E} : (\dot{E} - \dot{E}_{\mathrm{p}}) + \frac{\alpha_0}{h_0} [\frac{\partial \pi_0}{\partial E} : (\dot{E} - \dot{E}_{\mathrm{p}})] - \beta_0 \dot{\gamma}_0 = 0 \,. \tag{202}$$

In eqn (202), h_0 is defined by eqn (172), while the parameters α_0 and β_0 are defined by

$$\alpha_0 = \frac{\partial g_0}{\partial \Lambda_0} :: \frac{\partial \Pi_0}{\partial G_0} + \frac{\partial g_0}{\partial \rho_0} - \frac{\partial g_0}{\partial \mathbf{B}_{\mathrm{p}}} : \mathcal{B}_{\mathrm{p}} : \frac{\partial \Pi_0}{\partial E} \,, \quad \beta_0 = -\frac{\partial g_0}{\partial \mathbf{B}_{\mathrm{p}}} : \mathcal{B}_{\mathrm{p}} : \frac{\partial \mathcal{G}_0}{\partial E} \,. \tag{203}$$

Furthermore,

$$\dot{E}_{\mathrm{p}} = (\dot{E})^{\mathrm{p}} = -\mathbf{M}_0 : (\dot{S})^{\mathrm{p}} = -\mathbf{M}_0 : [(\dot{S})^{\mathrm{i}} - (\dot{S})^{\mathrm{d}}] = \mathbf{M}_0 : (\dot{\gamma}_0 \frac{\partial \mathcal{G}_0}{\partial E} - \dot{\rho}_0 \frac{\partial \Pi_0}{\partial E}) \,, \tag{204}$$

which, by using the expression (171), gives

$$\dot{\rho}_0 = \frac{1}{h_0} [\frac{\partial \pi_0}{\partial E} : (\dot{E} - \dot{E}_{\mathrm{p}})] = \frac{1}{h_0 - a_0} [(\frac{\partial \pi_0}{\partial E} : \dot{E}) - b_0 \dot{\gamma}_0] \,, \tag{205}$$

and

$$\dot{E}_{\mathrm{p}} = -\mathbf{M}_0 : [\frac{1}{h_0 - a_0} (\frac{\partial \pi_0}{\partial E} : \dot{E}) \frac{\partial \Pi_0}{\partial E} - \dot{\gamma}_0 (\frac{\partial \mathcal{G}_0}{\partial E} + \frac{b_0}{h_0 - a_0} \frac{\partial \Pi_0}{\partial E})] \,. \tag{206}$$

In eqns (205) and (206) the parameters a_0 and b_0 are defined by

$$a_0 = \frac{\partial \pi_0}{\partial E} : \mathbf{M}_0 : \frac{\partial \Pi_0}{\partial E} \,, \quad b_0 = \frac{\partial \pi_0}{\partial E} : \mathbf{M}_0 : \frac{\partial \mathcal{G}_0}{\partial E} \,. \tag{207}$$

Substituting eqns (205) and (206) into the consistency condition (202) and solving for $\dot{\gamma}_0$, the loading parameter can be expressed as

$$\dot{\gamma}_0 = \frac{1}{H_0} (\frac{\partial g_0}{\partial E} + \kappa_0 \frac{\partial \pi_0}{\partial E}) : \dot{E} \,, \tag{208}$$

where $H_0 = b_0(\kappa_0 + \eta_0)$, and

$$\kappa_0 = \frac{\alpha_0 + d_0}{h_0 - a_0} \,, \qquad \eta_0 = \frac{\beta_0 + c_0}{b_0} \,. \tag{209}$$

The parameters c_0 and d_0 in eqn (209) are defined by formulas analogous to eqn (207), i.e.

$$c_0 = \frac{\partial g_0}{\partial E} : \mathbf{M}_0 : \frac{\partial \mathcal{G}_0}{\partial E} , \qquad d_0 = \frac{\partial g_0}{\partial E} : \mathbf{M}_0 : \frac{\partial \Pi_0}{\partial E} . \tag{210}$$

Consequently, the inelastic part of the stress rate eqn (196) is governed by the constitutive equation

$$(\dot{S})^{\mathrm{i}} = -\frac{1}{H_0}[\frac{\partial \mathcal{G}_0}{\partial E} \otimes (\frac{\partial g_0}{\partial E} + \kappa_0 \frac{\partial \pi_0}{\partial E})] : \dot{E} . \tag{211}$$

Finally, the constitutive structure for the overall damage-elastoplasticity is according to eqn (76) obtained by adding to eqn (211) the elastic part of the stress rate defined by eqn (77), which gives

$$\dot{S} = \{\Lambda_0 - \frac{1}{H_0}[\frac{\partial \mathcal{G}_0}{\partial E} \otimes (\frac{\partial g_0}{\partial E} + \kappa_0 \frac{\partial \pi_0}{\partial E})]\} : \dot{E} . \tag{212}$$

In the case when the current inelastic deformation involves only damage and not plastic processes, the loading parameter $\dot{\gamma}_0$ becomes equal to the rate of the cumulative damage parameter $\dot{\rho}_0$. On the other hand, when the current inelastic deformation involves plasticity without damage, the loading parameter (208) reduces to

$$\dot{\gamma}_0 = \frac{1}{H_0}(\frac{\partial g_0}{\partial E} : \dot{E}) , \tag{213}$$

where the parameter H_0 is equal to $b_0\eta_0$.

Returning to the general case when the inelastic deformation involves both damage and plastic processes, once the loading parameter expression (208) is available, the rate of the cumulative damage parameter can be calculated by substituting eqn (208) into eqn (205). This gives

$$\dot{\rho}_0 = \frac{b_0}{H_0(h_0 - a_0)} (\eta_0 \frac{\partial \pi_0}{\partial E} - \frac{\partial g_0}{\partial E}) : \dot{E} . \tag{214}$$

If no plastic processes take place, eqn (214) can be shown to reduce to $(\partial \pi_0 / \partial E) : \dot{E}/h_0$.

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