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Discussion of coupled elastoplasticity and damage constitutive equations for small and finite deformations

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Dedicated to Professor Franz Kollmann on the occasion of his 65th birthday

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Abstract

Three small deformation plasticity models taking into account isotropic damage effects are presented and discussed. The models are formulated in the context of irreversible thermodynamics and the internal state variable theory. They exhibit nonlinear isotropic and nonlinear kinematic hardening. The aim of the paper is first to give a comparative study of the three models with reference to homogeneous and inhomogeneous deformations by using a general damage law. Secondly, and this is the main objective of the paper, we generalize the constitutive models to finite deformations by applying a thermodynamical framework based on the Mandel stress tensor. The responses of the obtained finite deformation models are then discussed for loading processes with homogeneous deformations. © 2000 Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

During the past two decades much research has been concentrated on modelling the progressive material degradation (material damage), which occurs prior to the initiation of a macroscopic crack. To describe such a phenomenon within the

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framework of continuum mechanics several continuum damage models, based on phenomenological considerations, have been developed.

Continuum damage mechanics originates in the early works of Kachanov (1958) and Rabotnov (1968), who considered creep rupture of metals under uniaxial loading. Later, the concepts were extended to model fatigue (Chaboche and Lesne, 1988; Dufailly and Lemaitre, 1995), creep (Hayhurst, 1972; Cocks and Leckie, 1987; Kruch et al., 1991), creep–fatigue interaction (Lemaitre and Plumtree, 1979; Chaboche, 1981) and ductile plastic damage (Lemaitre, 1985; Rousselier, 1987; Lemaitre and Marquis, 1988). Most of these concepts are embedded in the thermodynamics of irreversible processes and the internal state variable theory. To model isotropic damage processes, it suffices to define a scalar damage variable d, representing in some sense the progressive material degradation due to the loading process (Lemaitre and Chaboche, 1990). From a mathematical point of view, $d=0$ characterizes the undamaged or virgin state, whereas $d=1$ corresponds to the rupture of the element into two parts. Isotropic damage formulations are frequently employed in the literature because of their simplicity and efficiency.

In this paper, we focus attention to three plasticity models coupled with damage, which incorporate isotropic and kinematic hardening and were originally formulated in the context of small deformations. The first two models, denoted here as Model A and Model B, were developed by Lemaitre (1992) and Benallal et al. (1988), respectively. The third model is denoted as Model C and can be attributed to Chaboche: Actually, all constitutive equations in this model have been indicated in Chaboche (1981) and recently discussed in more detail in Chaboche (1998), except for the part of the specific free energy function responsible for isotropic hardening, which, even for the undamaged case, differs from corresponding proposals made by Chaboche. Damage is introduced through the concept of effective stress, which relates the stress to the area which effectively supports the load (Rabotnov, 1968), and the hypothesis of strain equivalence, which may be stated as follows (Lemaitre, 1984): "the strain associated with a damaged state under the applied stress is equivalent to the strain associated with its undamaged state under the effective stress". The damage evolution equation, used in this paper, is a generalization of an equation for ductile damage proposed by Dhar et al. (1996), describing nucleation and growth of voids. The three models differ in the definition of the yield function and the laws describing the hardening effects. In particular, a von Mises yield function with kinematic and isotropic hardening is assumed to apply in all cases. However, concerning the yield function, in Model A only the stress is replaced by an effective stress, whereas in Model B the stress as well as the back stress are replaced by effective stresses. In Model C the stress, the back stress and the internal stress describing isotropic hardening are replaced by corresponding effective stresses. Differences in the constitutive behavior of the three models are discussed by calculating the response to loading processes with homogeneous and nonhomogeneous deformations. Some of the model responses have been obtained by using the finite element method.

Next, and this is the main objective of the paper, the constitutive equations of the three models are extended to finite deformations by using the thermodynamically consistent formalism developed in Tsakmakis (1996). A characteristic feature of the theory obtained, is that the back stress tensor describing kinematic hardening is defined to have the same mathematical structure as a Mandel stress tensor. However, the evolution equation governing the response of kinematic hardening is not formulated with respect to the back stress tensor, but with respect to an internal stress tensor, which is related to the back stress tensor. On the other hand, this internal stress tensor has the mathematical structure of a second Piola–Kirchhoff stress tensor relative to the plastic intermediate configuration. In this way, we obtain a very specific form of the hardening rules. Typical properties of the three finite deformation models are discussed by calculating the predicted behavior for uniaxial loadings.

We close this introduction by remarking that our paper does not in the first place aim at a discussion about the relevance of the physical properties described by the constitutive models. The main purpose of the paper is rather to study the generalization of small deformation plasticity models coupled with damage effects to finite deformations by employing a specific thermodynamic approach. To this end three particular models are considered. Clearly, in order to discuss the results we obtain, it is necessary to know the differences in the responses predicted by the models for typical loadings.

2. Preliminaries

First and second-order tensors are denoted by bold-face roman and Greek letters, whereas fourth-order tensors are denoted by bold-face caligraphic letters. In particular, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}$ denote the inner product and the tensor product of the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively.

For second-order tensors $\mathbf{A}$ and $\mathbf{B}$, we write tr $\mathbf{A}$, det $\mathbf{A}$ and $\mathbf{A}^T$ for the trace, the determinant and the transpose of $\mathbf{A}$, while $\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}\left(\mathbf{A}\mathbf{B}^T\right)$ is the inner product between $\mathbf{A}$ and $\mathbf{B}$ and $\|\mathbf{A}\| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$ the Euclidian norm of $\mathbf{A}$. In Cartesian coordinates, $\mathbf{1} = \delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$, represents the second-order identity tensor, where δ_{ij} is the Kronecker delta symbol. Furthermore, we use the notations $\mathbf{A}^D = \mathbf{A} - 1/3(\mathrm{tr}\mathbf{A})\mathbf{1}$ for the deviator of $\mathbf{A}$ and $\mathbf{A}^{T-1} = (\mathbf{A}^{-1})^T$, provided $\mathbf{A}^{-1}$exists.

Let $\boldsymbol{\mathcal{M}} = \mathcal{M}_{ijkl}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l$ be a fourth-order tensor and $\mathbf{A} = A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ a second-order tensor with respect to the orthonormal basis $\{\mathbf{e}_i\}$. Then the following apply:

$$\boldsymbol{\mathcal{M}}^T = \mathcal{M}_{ijkl}\mathbf{e}_k\otimes\mathbf{e}_l\otimes\mathbf{e}_i\otimes\mathbf{e}_j,$$

$$\boldsymbol{\mathcal{M}}[\mathbf{A}] = \mathcal{M}_{ijmn}A_{mn}\mathbf{e}_i\otimes\mathbf{e}_j,$$

$$\mathbf{A}\mathbf{A} = A_{im}A_{mj}\mathbf{e}_i\otimes\mathbf{e}_j.$$

Thus, for second-order tensors $\mathbf{A}$, $\mathbf{B}$,

$$\mathbf{A}\cdot\boldsymbol{\mathcal{M}}[\mathbf{B}] = \mathbf{B}\cdot\boldsymbol{\mathcal{M}}^T[\mathbf{A}].$$

In addition, we write $\mathcal{I}$ for the fourth-order identity tensor

$$\mathcal{I} = \delta_{ik}\delta_{jl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l,$$

which satisfies the property

$$\mathcal{I} = \mathcal{E} + \mathcal{J},$$

with

$$\mathcal{E} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\boldsymbol{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l,$$

$$\mathcal{J} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \boldsymbol{e}_l.$$

Hence, $\mathcal{E}[\mathbf{A}] = 1/2(\mathbf{A} + \mathbf{A}^T)$, $\mathcal{J}[\mathbf{A}] = 1/2(\mathbf{A} - \mathbf{A}^T)$ and $\mathcal{I}[\mathbf{A}] = \mathbf{A}$, for all second-order tensors $\mathbf{A}$. Finally, ( )' denotes the material time derivative of ( ), the variable time being denoted by t.

3. Small deformation damage elastoplasticity

3.1. Thermodynamic framework

We assume an additive decomposition of the linearized Green strain tensor $\tilde{\mathbf{E}}$ into an elastic part $\tilde{\mathbf{E}}_e$ and a plastic part $\tilde{\mathbf{E}}_p$,

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_e + \tilde{\mathbf{E}}_p. \tag{1}$$

Further, it is assumed that the specific free energy function Ψ can be expressed in the form

$$\Psi(t) = \bar{\Psi}(\tilde{\mathbf{E}}_e, \mathbf{Y}, r, d) = \Psi_e(t) + \Psi_p(t), \tag{2}$$

with

$$\Psi_e(t) = \bar{\Psi}_e(\tilde{\mathbf{E}}_e, d) = \frac{1}{2\rho}(1-d)\tilde{\mathbf{E}}_e \cdot \mathcal{C}[\tilde{\mathbf{E}}_e], \tag{3}$$

$$\Psi_p(t) = \Psi_p^{kin}(t) + \Psi_p^{iso}(t). \tag{4}$$

In these equations,

$$\mathcal{C} = 2\mu\mathcal{E} + \lambda(\mathbf{1} \otimes \mathbf{1}) \tag{5}$$

denotes the isotropic fourth-order tensor, with μ, λ being the elasticity constants; ρ is the mass density, $\mathbf{Y}, r\,(r \geqslant 0)$ are second-order tensor and scalar internal strains related to the kinematic and isotropic hardening rule, respectively, and d denotes the isotropic damage variable. We denote by Ψ_{e} and Ψ_{p} the elastic and plastic parts of Ψ, and by $\Psi_{\mathrm{p}}^{\mathrm{kin}}$ and $\Psi_{\mathrm{p}}^{\mathrm{iso}}$ the parts of Ψ_{p} responsible for kinematic and isotropic hardening, respectively. Confining attention to the purely mechanical theory, the second law of thermodynamics in form of the Clausius–Duhem inequality reads

$$\mathbf{T}\cdot\dot{\tilde{\mathbf{E}}} - \rho\dot{\Psi} \geqslant 0 \tag{6}$$

for every admissible process. Here, $\mathbf{T}$ denotes the Cauchy stress tensor. Taking the time derivative of (2), substituting into (6), and making use of (1), we obtain

$$\left(\mathbf{T} - \rho\frac{\partial\bar{\Psi}}{\partial\tilde{\mathbf{E}}_{\mathrm{e}}}\right)\cdot\dot{\tilde{\mathbf{E}}}_{\mathrm{e}} + \mathbf{T}\cdot\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - \rho\frac{\partial\bar{\Psi}}{\partial\mathbf{Y}}\cdot\dot{\mathbf{Y}} - \rho\frac{\partial\bar{\Psi}}{\partial r}\dot{r} - \rho\frac{\partial\bar{\Psi}}{\partial d}\dot{d} \geqslant 0. \tag{7}$$

In order to ensure that (7) is satisfied for every admissible process, we require

$$\mathbf{T} = \rho\frac{\partial\bar{\Psi}}{\partial\tilde{\mathbf{E}}_{\mathrm{e}}} \tag{8}$$

and

$$\mathbf{T}\cdot\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - \rho\frac{\partial\bar{\Psi}}{\partial\mathbf{Y}}\cdot\dot{\mathbf{Y}} - \rho\frac{\partial\bar{\Psi}}{\partial r}\dot{r} - \rho\frac{\partial\bar{\Psi}}{\partial d}\dot{d} \geqslant 0. \tag{9}$$

Eq. (8), together with (3) leads to

$$\mathbf{T} = (1-d)\mathcal{C}[\tilde{\mathbf{E}}_{\mathrm{e}}]. \tag{10}$$

Note in passing that, in the context of usual plasticity theory, relations (8) and (9) are necessary and sufficient conditions for the validity of (7) in purely elastic processes, i.e. processes with $\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} = \dot{\mathbf{Y}} = \mathbf{0}$ and $\dot{r} = \dot{d} = 0$ However, for an arbitrary loading path, where plastic flow is involved, these relations are only sufficient for inequality (7) to be valid.

Let $\boldsymbol{\xi}, R, \Omega$ be internal stresses, which are thermodynamically conjugate to the internal strains $\mathbf{Y}, r, d$, respectively. That is,

$$\boldsymbol{\xi} := \rho\frac{\partial\bar{\Psi}}{\partial\mathbf{Y}}, \tag{11}$$

$$R := \rho\frac{\partial\bar{\Psi}}{\partial r}, \tag{12}$$

$$\Omega := \rho \frac{\partial \bar{\Psi}}{\partial d}. \tag{13}$$

The present paper is concerned with a so-called stress space formulation of the theory and the stresses $\boldsymbol{\xi}$ and R are supposed to govern the response of kinematic and isotropic hardening, respectively. Usually, $\boldsymbol{\xi}$ is referred to as back stress tensor. Making use of (11)–(13), the so-called intrinsic dissipation inequality (9) can be rewritten as

$$\frac{\mathbf{T}}{1-d} \cdot \dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - \frac{R}{1-d} \dot{r} - \frac{\boldsymbol{\xi}}{1-d} \cdot \dot{\mathbf{Y}} - \frac{\Omega}{1-d} \dot{d} \geqslant 0. \tag{14}$$

Plastic flow is assumed to be described by the associated normality rule

$$\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} = \Lambda \frac{\partial \bar{F}}{\partial \mathbf{T}}, \tag{15}$$

with

$$\Lambda \begin{cases} > 0 & \text{for} \quad F = 0 \; \& \; \dot{F}\Big|_{\tilde{\mathbf{E}}_{\mathrm{p=const.}}} \geqslant 0. \\ = 0 & \text{otherwise} \end{cases} \tag{16}$$

In these relations, F is a scalar valued function, known as yield function ($F = 0$: yield condition), which is assumed to have the form

$$F = \bar{F}(\mathbf{T}, \boldsymbol{\xi}, R, d). \tag{17}$$

Moreover, $F = 0$ & $\dot{F}|_{\tilde{\mathbf{E}}_{\mathrm{p=const.}}} \geqslant 0$ in (16) is known as condition for plastic loading and the multiplier Λ has to be determined from the so-called consistency condition $\dot{F} = 0$.

Eqs. (1)–(17) are common for Models A, B and C, to be presented in the following. Note that in Lemaitre (1992), Benallal et al. (1988) and Chaboche (1998) the evolution equations governing the model responses are derived from two scalar potential functions, namely the free energy function and the dissipation potential function. However, in the present paper, only the free energy function is used. The evolution equations for the internal state variables are then assumed as sufficient conditions for the validity of (9) in every admissible process.

3.2. Model A

This constitutive model, proposed by Lemaitre (1992), can be formulated as follows: It is assumed that the plastic parts of the specific free energy function have the form

$$\Psi_{\mathrm{p}}^{\mathrm{kin}}(t)=\bar{\Psi}_{\mathrm{p}}^{\mathrm{kin}}(\mathbf{Y})=\frac{c}{2\rho}\mathbf{Y}\cdot\mathbf{Y}, \tag{18}$$

$$\Psi_{\mathrm{p}}^{\mathrm{iso}}(t)=\bar{\Psi}_{\mathrm{p}}^{\mathrm{iso}}(r)=\frac{\gamma}{\rho\beta}\left(r+\frac{1}{\beta}e^{-\beta r}\right), \tag{19}$$

where c, γ, β are non negative material parameters. The hardening variables $\boldsymbol{\xi}$ and R, as well as the so-called elastic damage energy release rate $-\Omega_{\mathrm{e}}$, can be deduced from (11)–(13) to be

$$\boldsymbol{\xi}=\rho\frac{\partial\bar{\Psi}_{\mathrm{p}}^{\mathrm{kin}}}{\partial\mathbf{Y}}=c\mathbf{Y}, \tag{20}$$

$$R=\rho\frac{\partial\bar{\Psi}_{\mathrm{p}}^{\mathrm{iso}}}{\partial r}=\frac{\gamma}{\beta}(1-e^{-\beta r}), \tag{21}$$

$$\Omega\equiv\Omega_{\mathrm{e}}=\rho\frac{\partial\bar{\Psi}_{\mathrm{e}}}{\partial d}=-\frac{1}{2}\tilde{\mathbf{E}}_{\mathrm{e}}\cdot\boldsymbol{\mathcal{C}}[\tilde{\mathbf{E}}_{\mathrm{e}}]. \tag{22}$$

A von Mises yield function F with kinematic and isotropic hardening, which takes into account damage effects, is defined by

$$F=\bar{F}(\mathbf{T},\boldsymbol{\xi},R,d)=\sqrt{\frac{3}{2}\left(\frac{\mathbf{T}}{1-d}-\boldsymbol{\xi}\right)^{D}\cdot\left(\frac{\mathbf{T}}{1-d}-\xi\right)^{D}}-R-h, \tag{23}$$

where h is a nonnegative material parameter. If we set

$$k=R+h, \tag{24}$$

Eq. (23) reduces to

$$F=\bar{\bar{F}}(\mathbf{T},\boldsymbol{\xi},k,d)=\sqrt{\frac{3}{2}\left(\frac{\mathbf{T}}{1-d}-\boldsymbol{\xi}\right)^{D}\cdot\left(\frac{\mathbf{T}}{1-d}-\boldsymbol{\xi}\right)^{D}}-k. \tag{25}$$

On assuming, on the one hand $R|_{s=0}=0$, so that $k^{(0)}:=k|_{s=0}=h\geqslant 0$, in view of (24), and on the other hand $r|_{s=0}=0$, we see from (21), that $R\geqslant 0$ and thus $k\geqslant k^{(0)}$. Also, the normality rule (15) implies

$$\dot{\tilde{\mathbf{E}}}_{\mathrm{p}}=\Lambda\frac{\partial\bar{F}}{\partial\mathbf{T}}=\sqrt{\frac{3}{2}}\frac{\Lambda}{1-d}\mathbf{N}=\sqrt{\frac{3}{2}}\dot{s}\mathbf{N},\qquad \mathbf{N}=\frac{\left(\frac{\mathbf{T}}{1-d}-\boldsymbol{\xi}\right)^{D}}{\left\|\left(\frac{\mathbf{T}}{1-d}-\boldsymbol{\xi}\right)^{D}\right\|}, \tag{26}$$

with

$$\dot{s} := \sqrt{\frac{2}{3}\dot{\tilde{\boldsymbol{E}}}_{\mathrm{p}} \cdot \dot{\tilde{\boldsymbol{E}}}_{\mathrm{p}}} = \frac{\Lambda}{1-d}. \tag{27}$$

Substituting (26), (27) and the time derivative of (20) into (14), one gets, with regard to (23) and the yield condition $F=0$,

$$h\dot{s} + R\left(\dot{s} - \frac{\dot{r}}{1-d}\right) + \boldsymbol{\xi}\cdot\left(\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - \frac{\dot{\boldsymbol{\xi}}}{c(1-d)}\right) - \frac{\Omega}{1-d}\dot{d} \geqslant 0. \tag{28}$$

In view of the fact that $(1-d), \dot{s}, h, R, -\Omega \geqslant 0$, sufficient conditions for the validity of (28) in every admissible process may be obtained by setting

$$\dot{s} = \frac{\dot{r}}{1-d}, \tag{29}$$

$$\boldsymbol{\xi}\cdot\left(\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - \frac{\dot{\boldsymbol{\xi}}}{C(1-d)}\right) \geqslant 0, \tag{30}$$

$$\dot{d} \geqslant 0. \tag{31}$$

Note that, in view of (31), damage can only increase or remain constant. Condition (29), together with (21) and the time derivative of (21), yield

$$\dot{R} = (1-d)(\gamma - \beta R)\dot{s}, \tag{32}$$

or, by virtue of (24),

$$\dot{k} = (1-d)(\gamma - \beta(k-h))\dot{s}. \tag{33}$$

On the other hand, a sufficient condition for inequality (30) is given by

$$\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - \frac{\dot{\boldsymbol{\xi}}}{c(1-d)} = \frac{b}{c}\dot{s}\boldsymbol{\xi}, \tag{34}$$

where b is a nonnegative material parameter. Consequently,

$$\dot{\boldsymbol{\xi}} = (1-d)(c\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - b\dot{s}\boldsymbol{\xi}). \tag{35}$$

3.3. Model B

This model, which has been introduced by Benallal et al. (1988), is similar to Model A, apart from the definition of the yield function and the evolution equation describing kinematic hardening. This means, that Eqs. (18), (19), and therefore also (20)–(22), are assumed to apply. But now, the yield function is defined by

$$F = \bar{F}(\mathbf{T}, \boldsymbol{\xi}, R, d) = \sqrt{\frac{3}{2}\left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D} \cdot \left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D}} - R - h, \tag{36}$$

$$= \bar{\bar{F}}(\mathbf{T}, \boldsymbol{\xi}, k, d) = \sqrt{\frac{3}{2}\left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D} \cdot \left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D}} - k, \tag{37}$$

$$k = R + h. \tag{38}$$

As in Section 3.2, h denotes a nonnegative material parameter. Again, the flow rule (15) implies

$$\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} = \Lambda \frac{\partial \bar{F}}{\partial \mathbf{T}} = \sqrt{\frac{3}{2}} \frac{\Lambda}{1-d} \mathbf{N} = \sqrt{\frac{3}{2}} \dot{s} \mathbf{N}, \qquad \mathbf{N} = \frac{(\mathbf{T}-\boldsymbol{\xi})^{D}}{\left\| (\mathbf{T}-\boldsymbol{\xi})^{D} \right\|}, \tag{39}$$

with

$$\dot{s} := \sqrt{\frac{2}{3} \dot{\tilde{\mathbf{E}}}_{\mathrm{p}} \cdot \dot{\tilde{\mathbf{E}}}_{\mathrm{p}}} = \frac{\Lambda}{1-d}. \tag{40}$$

Taking into account these relations as well as (20) and the yield condition $F=0$, the dissipation inequality (14) becomes

$$h\dot{s} + R\left(\dot{s} - \frac{\dot{r}}{1-d}\right) + \boldsymbol{\xi} \cdot \left(\frac{\dot{\tilde{\mathbf{E}}}_{\mathrm{p}}}{1-d} - \frac{\dot{\boldsymbol{\xi}}}{c(1-d)}\right) - \frac{\Omega}{1-d} \dot{d} \geqslant 0. \tag{41}$$

It can readily seen that relation (29) and the inequalities

$$\boldsymbol{\xi} \cdot \left(\frac{\dot{\tilde{\mathbf{E}}}_{\mathrm{p}}}{1-d} - \frac{\dot{\boldsymbol{\xi}}}{c(1-d)}\right) \geqslant 0, \tag{42}$$

$$\dot{d} \geqslant 0, \tag{43}$$

are sufficient conditions for (41) to be satisfied for all admissible processes. Clearly, as in Model A, the isotropic hardening rule (33) holds:

$$\dot{k} = (1-d)(\gamma - \beta(k-h))\dot{s}. \tag{44}$$

In analogy to (34), a sufficient condition

$$\frac{\dot{\tilde{\mathbf{E}}}_{\mathrm{p}}}{1-d} - \frac{\dot{\boldsymbol{\xi}}}{c(1-d)} = \frac{b}{c} \dot{s} \boldsymbol{\xi} \tag{45}$$

can be chosen for (42). Here, b is a nonnegative material parameter. Hence,

$$\dot{\boldsymbol{\xi}} = c\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - (1-d)b\dot{s}\boldsymbol{\xi}. \tag{46}$$

It is of interest to remark that, compared to Model A, also the back stress $\boldsymbol{\xi}$ in the yield function is replaced by the effective back stress $\boldsymbol{\xi}/(1-d)$, but the part of the specific free energy function $\Psi_{\mathrm{p}}^{\mathrm{kin}}$ is not assumed to be affected by the damage variable d.

3.4. Model C

If we consistently replace in the constitutive theory all variables of stress type by corresponding effective stresses, then we obtain a constitutive model, which can be attributed to Chaboche. Indeed, Chaboche (1981, 1998) has proposed to replace in the constitutive theory $\mathbf{T}, \boldsymbol{\xi}$ and R by $\mathbf{T}/(1-d), \boldsymbol{\xi}/(1-d)$ and $R/(1-d)$ (cf. also related remarks in Ju, 1989). However, as remarked below, the form of $\Psi_{\mathrm{p}}^{\mathrm{iso}}$ assumed here differs from corresponding proposals made by Chaboche.

In order to replace in the constitutive theory the internal stresses responsible for kinematic and isotropic hardening by corresponding effective stresses, we suppose the functions $\Psi_{\mathrm{p}}^{\mathrm{kin}}$ and $\Psi_{\mathrm{p}}^{\mathrm{iso}}$to be affected by the damage variable d in the form

$$\Psi_{\mathrm{p}}^{\mathrm{kin}}(t) = \bar{\Psi}_{\mathrm{p}}^{\mathrm{kin}}(\mathbf{Y}, d) = \frac{(1-d)}{2\rho} c\mathbf{Y}\cdot\mathbf{Y}, \tag{47}$$

$$\Psi_{\mathrm{p}}^{\mathrm{iso}}(t) = \bar{\Psi}_{\mathrm{p}}^{\mathrm{iso}}(r, d) = \frac{(1-d)}{2\rho}\left(\gamma r^2 + 2R^{(0)} r\right), \tag{48}$$

with $c, \gamma, R^{(0)} := R|_{s=0}$ being nonnegative material parameters. It follows that

$$\boldsymbol{\xi} = \rho \frac{\partial \bar{\Psi}_{\mathrm{p}}^{\mathrm{kin}}}{\partial \mathbf{Y}} = (1-d)c\mathbf{Y}, \tag{49}$$

$$R = \rho \frac{\partial \bar{\Psi}_{\mathrm{p}}^{\mathrm{iso}}}{\partial r} = (1-d)\left(\gamma r + R^{(0)}\right), \tag{50}$$

$$\Omega \equiv \Omega_e + \Omega_p = \rho \frac{\partial \bar{\Psi}}{\partial d} = -\frac{1}{2}\left(\tilde{\mathbf{E}}_{\mathrm{e}}\cdot\boldsymbol{\mathcal{C}}\left[\tilde{\mathbf{E}}_{\mathrm{e}}\right] + \gamma r^2 + 2R^{(0)} r + c\mathbf{Y}\cdot\mathbf{Y}\right). \tag{51}$$

The yield function is now defined by

$$F = \bar{F}(\mathbf{T}, \boldsymbol{\xi}, R, d) = \sqrt{\frac{3}{2}\left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D}\cdot\left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D}} - \frac{R}{1-d} - h, \tag{52}$$

with $h \geqslant 0$. Note that h is a material parameter only, and that R is the variable governing the response of isotropic hardening. Hence, only R has to be replaced by an effective stress variable. If we introduce k in the form

$$k := R + (1-d)h, \tag{53}$$

then Eq. (52) can be expressed in terms of k:

$$F = \bar{\bar{F}}(\mathbf{T}, \boldsymbol{\xi}, k, d) = \sqrt{\frac{3}{2}\left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D} \cdot \left(\frac{\mathbf{T}-\boldsymbol{\xi}}{1-d}\right)^{D}} - \frac{k}{1-d}. \tag{54}$$

Moreover, the normality rule (15) furnishes

$$\dot{\bar{\mathbf{E}}}_{\mathrm{p}} = \Lambda \frac{\partial \bar{F}}{\partial \mathbf{T}} = \sqrt{\frac{3}{2}} \frac{\Lambda}{1-d} \mathbf{N} = \sqrt{\frac{3}{2}} \dot{s} \mathbf{N}, \qquad \mathbf{N} = \frac{(\mathbf{T}-\boldsymbol{\xi})^{D}}{\left\|(\mathbf{T}-\boldsymbol{\xi})^{D}\right\|}, \tag{55}$$

with

$$\dot{s} := \sqrt{\frac{2}{3} \dot{\bar{\mathbf{E}}}_{\mathrm{p}} \cdot \dot{\bar{\mathbf{E}}}_{\mathrm{p}}} = \frac{\Lambda}{1-d}. \tag{56}$$

Taking into account these relations as well as Eqs. (49), (52) and the yield condition $F = 0$, the dissipation inequality (14) can be rewritten as

$$h\dot{s} + \frac{R}{1-d}(\dot{s} - \dot{r} + \frac{\boldsymbol{\xi}}{1-d} \cdot \left(\dot{\bar{\mathbf{E}}}_{\mathrm{p}} - \frac{1}{c}\left(\frac{\boldsymbol{\xi}}{1-d}\right)^{\cdot} \right) - \frac{\Omega}{1-d}\dot{d} \geqslant 0. \tag{57}$$

On assuming $r \geqslant 0$, it follows from (50) and (51) that $R \geqslant 0$ and $-\Omega \geqslant 0$. Hence,

$$\dot{s} \geqslant \dot{r}, \tag{58}$$

$$\frac{\boldsymbol{\xi}}{1-d} \cdot \left(\dot{\bar{\mathbf{E}}}_{\mathrm{p}} - \frac{1}{c}\left(\frac{\boldsymbol{\xi}}{1-d}\right)^{\cdot} \right) \geqslant 0, \tag{59}$$

$$\dot{d} \geqslant 0 \tag{60}$$

are sufficient conditions for the validity of (57) in every admissible process. It is obvious, that

$$\dot{r} = \left(1 - \frac{\beta}{\gamma}\left(\frac{R}{1-d} - \left.\frac{R}{1-d}\right|_{s=0}\right)\right)\dot{s} = \left(1 - \frac{\beta}{\gamma}\left(\frac{R}{1-d} - R^{(0)}\right)\right)\dot{s} \tag{61}$$

satisfies inequality (58) in a sufficient way, provided that $\beta \geqslant 0$ and $d|_{s=0} = 0$. Taking the time derivative of (50), along with (61), we obtain

$$\left(\frac{R}{1-d}\right)^{\cdot} = \left(\gamma - \beta\left(\frac{R}{1-d} - R^{(0)}\right)\right)\dot{s}, \tag{62}$$

or by virtue of (53),

$$\left(\frac{k}{1-d}\right)^{\cdot} = \left(\gamma - \beta\left(\frac{k}{1-d} - k^{(0)}\right)\right)\dot{s}. \tag{63}$$

Here,

$$k^{(0)} := k|_{s=0} \equiv \left.\frac{k}{1-d}\right|_{s=0} = R^{(0)} + h. \tag{64}$$

Futhermore,

$$\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - \frac{1}{c}\left(\frac{\boldsymbol{\xi}}{1-d}\right)^{\cdot} = \frac{b}{c}\dot{s}\left(\frac{\boldsymbol{\xi}}{1-d}\right), \tag{65}$$

with $b \geqslant 0$, ensures the satisfaction of inequality (59) and results to the following evolution equation for the back stress tensor $\boldsymbol{\xi}$:

$$\left(\frac{\boldsymbol{\xi}}{1-d}\right)^{\cdot} = c\dot{\tilde{\mathbf{E}}}_{\mathrm{p}} - b\dot{s}\left(\frac{\boldsymbol{\xi}}{1-d}\right). \tag{66}$$

It is remarked, that, to our knowledge, $\Psi_{\mathrm{p}}^{\mathrm{iso}}$ in (48) differs from corresponding proposals made by Chaboche in various papers, even if damage is absent (see e.g. Chaboche, 1993a,b, 1996). As shown in Tsakmakis (1998), for the undamaged case, the assumption (48), together with the hardening law (61) (respectively (62)) are sufficiently flexible to allow well modelling of the energy stored in the material. For more details about this problem see Chaboche (1993a,b 1996), Tsakmakis (1998) and the literature cited thereby. However, since the purpose of the present paper is to discuss the structure of constitutive models in a purely mechanical context, we will concentrate in the following to the evolution equations (33), (44) and (63) governing the response of the variable k responsible for isotropic hardening.

3.5. Characterization of damage evolution

A general evolution equation describing ductile damage is given by

$$\dot{d} = a_0\dot{s} + (a_1 + a_2 d)\frac{(-\Omega)^n}{(1-d)^q}\dot{s} \geqslant 0, \tag{67}$$

with a_0, a_1, a_2, n, q being nonnegative material parameters. It is well known that ductile fracture occurs mainly due to void nucleation, growth and finally coalescence into a crack. By analogy with Dhar et al. (1996), the first term in (67), which does not depend on Ω, can be interpreted to represent the evolution of damage due to void nucleation, while the other two terms, which depend on Ω, describe the evolution of damage due to void growth. It was outlined in Dhar et al. (1996), that this is in agreement with experimental results by Leroy et al. (1989), who studied the nucleation and growth of voids in spheroidized carbon-steels during tensile loading. Earlier investigations by McClintock (1968) and Rice and Tracey (1969), who considered isolated cylindrical and spherical voids in a non-hardening material, showed a very strong dependency of void growth on the stress triaxiality, which is given by the ratio hydrostatic stress to von Mises equivalent stress. Here, the influence of the triaxiality ratio on the damage evolution is taken into account by Ω. We recall, that $\Omega \equiv \Omega_e$ for Model A and Model B, whereas $\Omega \neq \Omega_e$ for Model C. In order to have a common basis when discussing the three models, we replace Ω by Ω_e in (67), so that

$$\dot{d} = a_0 \dot{s} + (a_1 + a_2 d) \frac{(-\Omega_e)^n}{(1-d)^q} \dot{s}, \tag{68}$$

with

$$\Omega_e = -\frac{1}{(1-d)^2} \left\{ \frac{1}{4\mu} \mathbf{T}^D \cdot \mathbf{T}^D + \frac{1}{6(2\mu + 3\lambda)} (\mathrm{tr}\mathbf{T})^2 \right\}, \tag{69}$$

by virtue of (3), (5) and (10).

Eq. (68) contains the following damage evolution models:

- $n = 1, q = 0$ (Dhar et al., 1996),
- $n = 1, q = 0, a_0 = 0, a_1 = 0$ (Tai and Yang, 1986),
- $a_0 = 0, a_2 = 0$ (Lemaitre, 1987a,b).

For the case of Model C and (homogeneous) strain controlled uniaxial loading, the effect of the parameters a_0, a_1, a_2, n, q on the strain–stress-response and the damage evolution is illustrated in Figs. 1–5. In these figures, $\sigma = T_{11}$ and $e = (l - l_0)/l_0$, where l_0, l are the lengths of the tensile specimen at the beginning of deformation and at the actual state, respectively. In all cases the material parameters $\mu = 38461.5$ MPa, $\lambda = 57692.3$ MPa, $h = k_0 = 150$ MPa, $\beta = 100$, $\gamma = 8000$ MPa, $b = 100$ and $c = 5000$ Mpa are chosen to apply (for more details see also Lämmer, 1998).

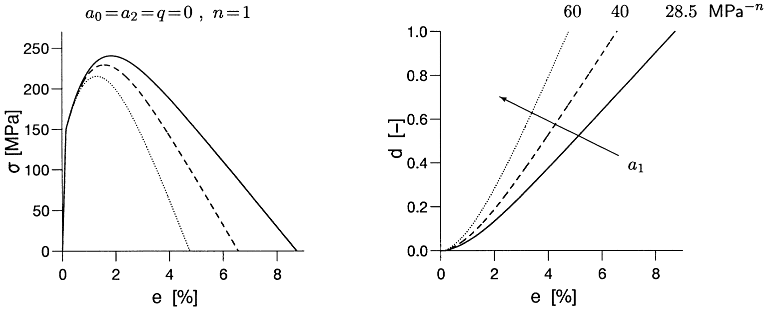

Fig. 1. Effect of the parameter a_1.

The comparative study of the three models, given in Section 3.6 relies upon the special case $a_0 = a_2 = q = 0$ and $n = 1$. This results to the damage law

$$\dot{d} = -a_1 \Omega_e \dot{s}, \tag{70}$$

which has been previously postulated by Lemaitre (1992).

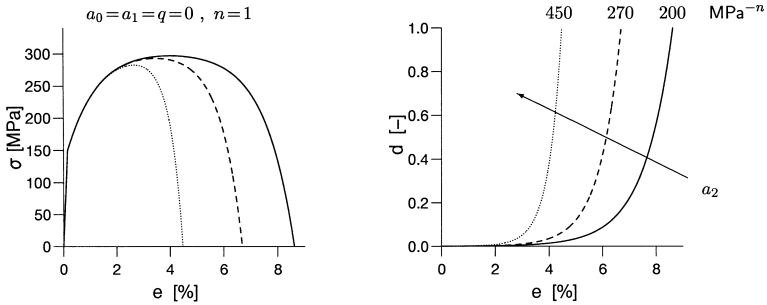

Fig. 2. Effect of the parameter a_2.

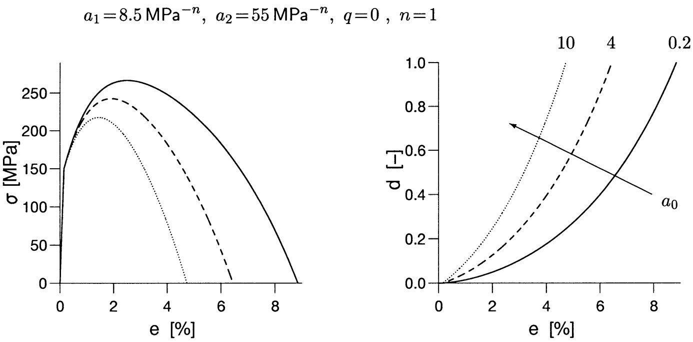

Fig. 3. Effect of the parameter a_0.

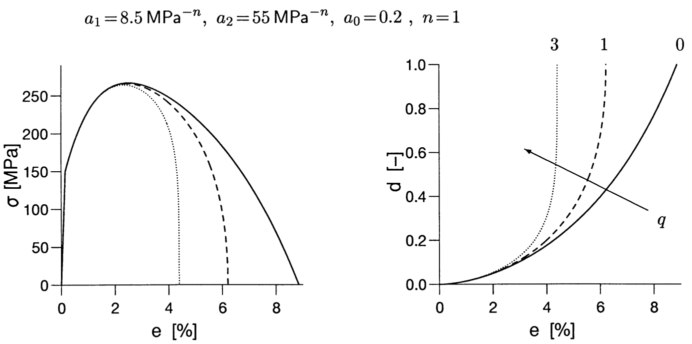

Fig. 4. Effect of the parameter q.

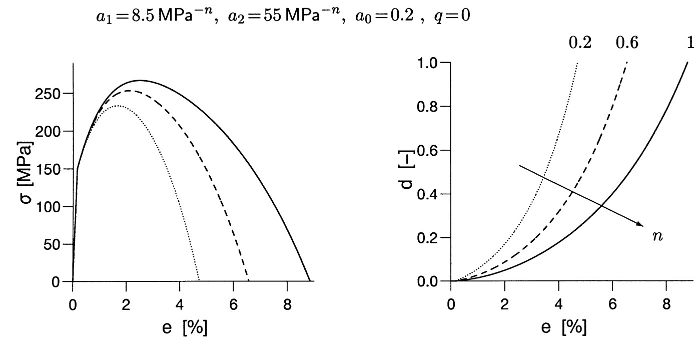

Fig. 5. Effect of the parameter n.

3.6. Comparative study

The three models, presented above, differ in the definition of the specific free energy function, the yield function as well as the evolution equations governing the response of isotropic and kinematic hardening. It is perhaps of interest to note, that in the absence of damage the three models reduce to the same plasticity law discussed in various papers by Chaboche (e.g. Chaboche and Rousselier, 1983a,b; Chaboche, 1986).

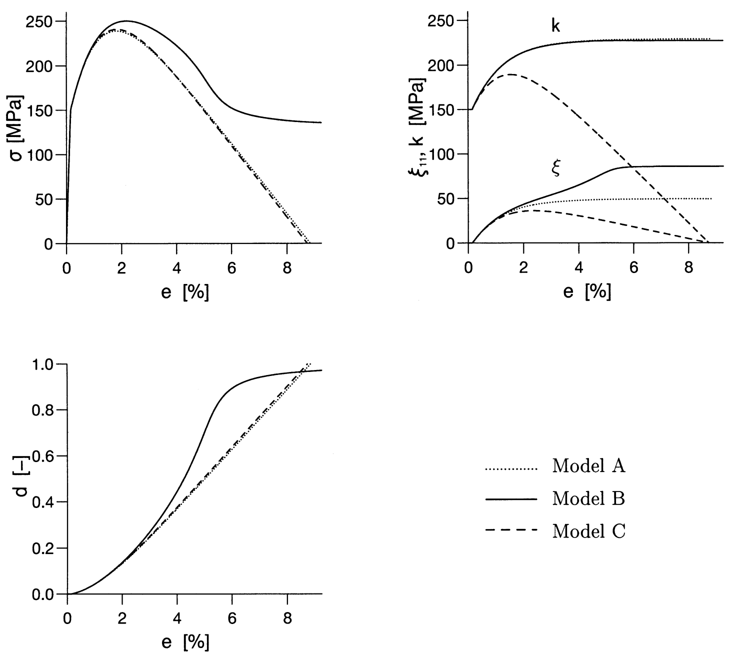

Fig. 6. Comparison of the model responses for uniaxial tension (small deformation formulation); assigned material parameters: $\mu=38461.5$ MPa, $\lambda=57692.3$ MPa, $h=k_0=150$ MPa, $\beta=100$, $\gamma=8000$ MPa, $b=100$, $c=5000$ MPa, $a_0=0$,$a_1=28.5$/MPa, $a_2=0$, $n=1$, $q=0$.

In order to investigate characteristic properties of the three models, we first consider the model responses due to (homogeneous) strain controlled uniaxial loading. Fig. 6 illustrates the behaviour of the stresses $\sigma = T_{11}, \xi = \xi_{11}, k$ and the damage variable d as a function of the nominal strain e, defined in Section 3.5. The constitutive behaviours, displayed in Fig. 6, are calculated by using the same material parameters for all three models. These parameters are given in Fig. 6 as well. When comparing the results, one should concentrate on the $(e-\sigma)^-$ and $(e–d)^-$ diagrams only, since the physical meaning of the internal stresses ξ and k is not unique within each model.

It can be seen that Model A and Model C predict almost identical $(e-\sigma)^-$ and $(e–d)^-$ responses, which essentially differ from the corresponding responses due to Model B. It should be noted, that just before submitting this paper, a similiar uniaxial comparative study by Chaboche (1998), but without isotropic hardening and without reference to Model B, has come to our knowledge. In every case, from such uniaxial results it should not be concluded, that Model A and Model C exhibit almost identical behaviour in every deformation process. To demonstrate this, we consider a notched circular cylinder tensile specimen, subjected to prescribed displacement along the upper boundaries. The predicted responses have been calculated by employing the ABAQUS finite element code. This code provides a user subroutine, in which the three models have been implemented. The finite element mesh, the imposed loading and the material parameters used, are shown in Fig. 7. Because of various symmetry conditions, only a quarter of the specimen has been discretized with 111 eight-node axisymmetric solid elements.

Fig. 8 illustrates for the three models the radial distribution of the damage variable d and the plastic arc length s, with r being the radius in the plane through the notch root. The results are referred to the overall resulting strain $e^* = (L - L_0)/L_0$, which is a measure for the global deformation of the inhornogeneously deformed specimen. It can be seen, that for $e_0^* = 0.2\%$, all three models predict identical radial distributions. However, significant quantitative differences between the responses predicted by all three models may be recognized for $e_1^* = 0.46\%$. Thus, the three models generally exhibit different constitutive behaviour. Further remarks and comments on the model properties are postponed until Section 4.6.

Next, we proceed to generalize the three models to take into account finite deformations. To this end, we make use of the formalism developed in Tsakmakis (1996).

4. Generalization to finite deformations

4.1. Kinematics, strain and stress measures

Let $\mathbf{F}$ ($\det\mathbf{F} > 0$) be the deformation gradient tensor and

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p \tag{71}$$

the multiplicative decomposition of $\mathbf{F}$ into elastic and plastic parts. It is assumed that $\det \mathbf{F}_e > 0$ and thus $\det \mathbf{F}_p > 0$. The transformation $\mathbf{F}_p$ introduces a so-called plastic intermediate configuration denoted by $\hat{R}_t$. In the reference and current configuration

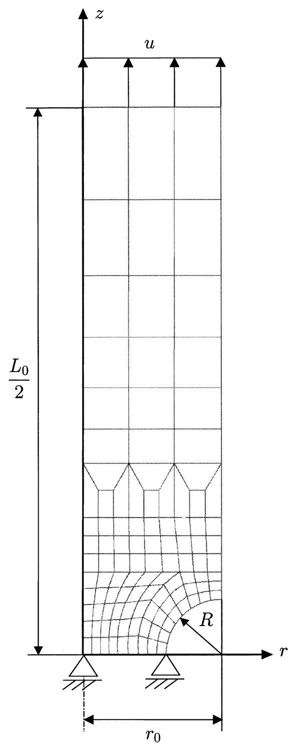

Fig. 7. Circular notched specimen: $L_0=8$ mm, $r_0=1$ mm, $R=0.4$ mm. Assigned material parameters: $\mu=76923.1$ MPa, $\lambda=115384.6$ MPa, $h=k_0=400$ MPa, $\beta=10$, $\gamma=1500$ MPa, $b=10$, $c=1500$ MPa, $a_0=0$, $a_1=4.2$ MPa^{-1}, $a_2=0$ Mpa^{-1}, $n=1$, $q=0$.

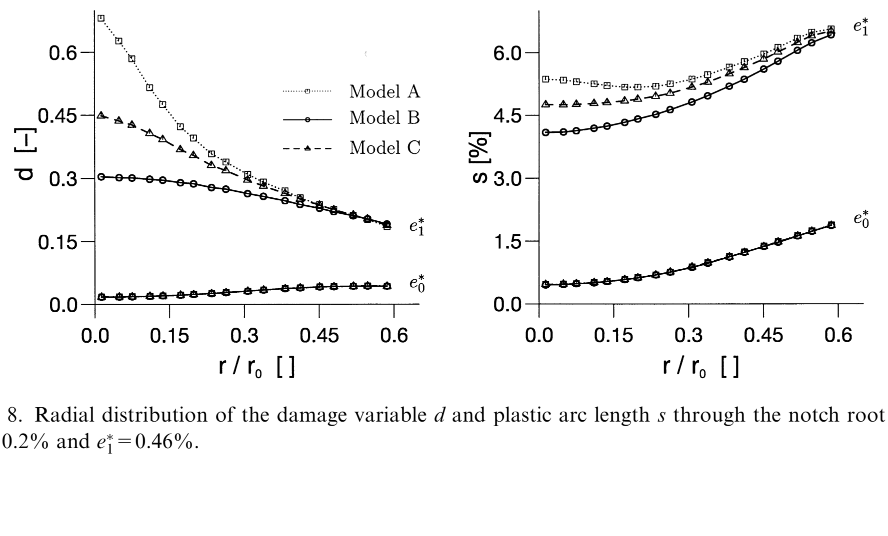

Fig. 8. Radial distribution of the damage variable d and plastic arc length s through the notch root for $e_0^*=0.2\%$ and $e_1^*=0.46\%$.

the material body occupies the regions R_R and R_t in the three-dimensional Euclidian point space E, respectively. For later reference, we introduce the relations

$$\hat{\mathbf{L}}_{\mathrm{p}} = \dot{\mathbf{F}}_{\mathrm{p}}\mathbf{F}_{\mathrm{p}}^{-1}, \qquad \hat{\mathbf{D}}_{\mathrm{p}} = \frac{1}{2}\left(\hat{\mathbf{L}}_{\mathrm{p}} + \hat{\mathbf{L}}_{\mathrm{p}}^{T}\right). \tag{72}$$

Plastic incompressibility, which is assumed to apply, is expressed by det $\mathbf{F}_p = 1$ implying tr $\hat{\mathbf{L}}_{\mathrm{p}} \equiv \mathrm{tr}\hat{\mathbf{D}}_{\mathrm{p}} = 0$. With respect to $\hat{R}_t$, we introduce the strain tensors $\hat{\Gamma}$, $\hat{\Gamma}_{\mathrm{e}}$, $\hat{\Gamma}_{\mathrm{p}}$

$$\hat{\mathbf{\Gamma}} = \hat{\mathbf{\Gamma}}_{\mathrm{e}} + \hat{\mathbf{\Gamma}}_{\mathrm{p}}, \tag{73}$$

$$\hat{\mathbf{\Gamma}}_{\mathrm{e}} = \frac{1}{2}\left(\mathbf{F}_{\mathrm{e}}^{T}\mathbf{F}_{\mathrm{e}} - \mathbf{1}\right), \tag{74}$$

$$\hat{\mathbf{\Gamma}}_{\mathrm{p}} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}_{\mathrm{p}}^{T-1}\mathbf{F}_{\mathrm{p}}^{-1}\right), \tag{75}$$

and the Oldroyd derivatives

$$(\ )^{\triangle} = (\ )^{\cdot} + \hat{\mathbf{L}}_{\mathrm{p}}^{T}(\ ) + (\ )\hat{\mathbf{L}}_{\mathrm{p}}, \tag{76}$$

$$(\ )^{\nabla} = (\ )^{\cdot} - \hat{\mathbf{L}}_{\mathrm{p}}(\ ) - (\ )\hat{\mathbf{L}}_{\mathrm{p}}^{T}. \tag{77}$$

It is easy to verify, that

$$\overset{\triangle}{\hat{\mathbf{\Gamma}}}_{\mathrm{p}} = \dot{\hat{\mathbf{\Gamma}}}_{\mathrm{p}} + \hat{\mathbf{L}}_{\mathrm{p}}^{T}\hat{\mathbf{\Gamma}}_{\mathrm{p}} + \hat{\mathbf{\Gamma}}_{\mathrm{p}}\hat{\mathbf{L}}_{\mathrm{p}} = \hat{\mathbf{D}}_{\mathrm{p}}. \tag{78}$$

Further, we denote by $\mathbf{T}$ the Cauchy stress tensor, by $\mathbf{S}$ the weighted Cauchy stress tensor,

$$\mathbf{S} = (\det \mathbf{F})\mathbf{T}, \tag{79}$$

and by

$$\hat{\mathbf{T}} = \mathbf{F}_{\mathrm{e}}^{-1}\mathbf{S}\mathbf{F}_{\mathrm{e}}^{T-1} \tag{80}$$

the second Piola–Kirchhoff stress tensor relative to $\hat{R}_t$. For further details on kinematics we refer to Haupt and Tsakmakis (1989, 1996) and Tsakmakis (1996).

4.2. Thermodynamic framework

Following Tsakmakis (1996), we primarily formulate the constitutive equations relative to the plastic intermediate configuration. In analogy to (2)–(4), we choose a specific free energy function of the form

$$\Psi(t) = \hat{\Psi}\left(\hat{\mathbf{\Gamma}}_{\mathrm{e}}, \hat{\mathbf{Y}}, r, d\right) = \Psi_{\mathrm{e}}(t) + \Psi_{\mathrm{p}}(t), \tag{81}$$

with

$$\Psi_{\mathrm{e}}(t) = \hat{\Psi}_{\mathrm{e}}\left(\hat{\mathbf{\Gamma}}_{\mathrm{e}}, d\right) = \frac{1}{2\rho_{\mathrm{R}}}(1-d)\hat{\mathbf{\Gamma}}_{\mathrm{e}} \cdot \mathcal{C}\left[\hat{\mathbf{\Gamma}}_{\mathrm{e}}\right], \tag{82}$$

$$\Psi_{\mathrm{p}}(t) = \Psi_{\mathrm{p}}^{\mathrm{kin}}(t) + \Psi_{\mathrm{p}}^{\mathrm{iso}}(t). \tag{83}$$

As in Section 3, r is the scalar valued internal strain responsible for isotropic hardening and $d\,(0 \leqslant d \leqslant 1)$ the scalar valued variable describing isotropic damage. Kinematic hardening is elaborated by means of the second-order strain tensor $\hat{\mathbf{Y}}$. This strain measure is defined to have the mathematical structure of a tensor relative to the plastic intermediate configuration, like the tensors $\hat{\mathbf{\Gamma}}$, $\hat{\mathbf{\Gamma}}_{\mathrm{e}}$ or $\hat{\mathbf{\Gamma}}_{\mathrm{p}}$.

In view of isothermal deformations with uniform temperature distribution assumed in this paper, the Clausius–Duhem inequality reads as follows:

$$\mathbf{S} \cdot \mathbf{D} - \rho_{\mathrm{R}} \dot{\Psi} = \hat{\mathbf{T}} \cdot \overset{\triangle}{\hat{\Gamma}} - \rho \mathrm{R} \dot{\Psi} \geqslant 0. \tag{84}$$

Here, ρ_{R} is the mass density in the reference configuration. In the following, elastic isotropy is assumed to apply. This means, that $\hat{\Psi}$ is an isotropic tensor function of $\hat{\mathbf{\Gamma}}_{\mathrm{e}}$. After some rearrangement of terms, where use has been made of (81), (73), (76), $(72)_2$, (78) and the property $\frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{\Gamma}}_{\mathrm{e}}} \hat{\mathbf{\Gamma}}_{\mathrm{e}} = \hat{\mathbf{\Gamma}}_{\mathrm{e}} \frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{\Gamma}}_{\mathrm{e}}}$, resulting from the elastic isotropy assumed, inequality (84) becomes

$$\left(\hat{\mathbf{T}} - \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{\Gamma}}_{\mathrm{e}}}\right) \cdot \overset{\triangle}{\hat{\mathbf{\Gamma}}}_{\mathrm{e}} + \left(\hat{\mathbf{T}} + 2\rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{\Gamma}}_{\mathrm{e}}} \hat{\mathbf{\Gamma}}_{\mathrm{e}}\right) \cdot \hat{\mathbf{D}}_{\mathrm{p}} - \rho_{\mathrm{R}} \left(\frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{Y}}} \cdot \dot{\hat{\mathbf{Y}}} + \frac{\partial \hat{\Psi}}{\partial r} \dot{r} + \frac{\partial \hat{\Psi}}{\partial d} \dot{d}\right) \geqslant 0. \tag{85}$$

In the case where plastic flow is involved,

$$\hat{\mathbf{T}} = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{\Gamma}}_{\mathrm{e}}}, \tag{86}$$

$$\left(\hat{\mathbf{T}} + 2\rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{\Gamma}}_{\mathrm{e}}} \hat{\mathbf{\Gamma}}_{\mathrm{e}}\right) \cdot \hat{\mathbf{D}}_{\mathrm{p}} - \rho_{\mathrm{R}} \left(\frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{Y}}} \cdot \dot{\hat{\mathbf{Y}}} + \frac{\partial \hat{\Psi}}{\partial r} \dot{r} + \frac{\partial \hat{\Psi}}{\partial d} \dot{d}\right) \geqslant 0 \tag{87}$$

are sufficient conditions for the validity of (85) in every admissible process. Thus, by inserting (82) in (86),

$$\hat{\mathbf{T}} = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}_{\mathrm{e}}}{\partial \hat{\mathbf{\Gamma}}_{\mathrm{e}}} = (1-d)\mathcal{C}\left[\hat{\mathbf{\Gamma}}_{\mathrm{e}}\right]. \tag{88}$$

With regard to the definitions

$$\hat{\mathbf{Z}} := \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial \hat{\mathbf{Y}}}, \tag{89}$$

$$R := \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial r}, \tag{90}$$

$$\Omega := \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial d}, \tag{91}$$

the intrinsic dissipation inequality (87) can be rewritten as

$$\frac{\hat{\mathbf{P}}}{1-d} \cdot \hat{\mathbf{D}}_{\mathrm{p}} - \frac{\hat{\mathbf{Z}}}{1-d} \cdot \dot{\hat{\mathbf{Y}}} - \frac{R}{1-d} \dot{r} - \frac{\Omega}{1-d} \dot{d} \geqslant 0, \tag{92}$$

where $\hat{\mathbf{P}}$ denotes the so-called Mandel stress tensor, defined by

$$\hat{\mathbf{P}} := \left(\mathbf{1} + 2\hat{\mathbf{\Gamma}}_{\mathrm{e}}\right)\hat{\mathbf{T}}. \tag{93}$$

The tensor $\hat{\mathbf{Z}}$, introduced by (89), is a stress tensor relative to the plastic intermediate configuration with similar mathematical structure as the stress tensor $\hat{\mathbf{T}}$. In absence of damage ($d=0$) and isotropic hardening ($r\equiv 0$), (92) reduces to

$$\hat{\mathbf{P}} \cdot \hat{\mathbf{D}}_{\mathrm{p}} - \hat{\mathbf{Z}} \cdot \dot{\hat{\mathbf{Y}}} \geqslant 0. \tag{94}$$

Now, we briefly sketch the manner of proceeding in Tsakmakis (1996) (see also Diegele et al., 1995) for satisfying (94). In order to explain the main steps, it suffices to concentrate on plastic incompressibility. Especially, it is assumed that $\Psi_{\mathrm{p}} = \frac{c}{2\rho_R} \hat{\mathbf{Y}} \cdot \hat{\mathbf{Y}}$, with c being a nonnegative material parameter, so that $\hat{\mathbf{Z}} = c\hat{\mathbf{Y}}$. Further, we assume the existence of a von Mises yield function in the space of Mandel stress tensors ($h\equiv$const.)

$$F = \hat{F}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}\right) = \sqrt{\frac{3}{2}\left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right)^{D} \cdot \left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right)^{D}} - h, \tag{95}$$

where $\hat{\boldsymbol{\xi}}$ is the back stress tensor. This tensor is postulated to have the same mathematical structure as $\hat{\mathbf{P}}$ and is defined by

$$\hat{\boldsymbol{\xi}} := \left(\mathbf{1} + 2\hat{\mathbf{Y}}\right)\hat{\mathbf{Z}}. \tag{96}$$

This way, inequality (94) becomes

$$\begin{aligned} &\left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right) \cdot \hat{\mathbf{D}}_{\mathrm{p}} + \left(\mathbf{1} + \frac{2}{c}\hat{\mathbf{Z}}\right)\hat{\mathbf{Z}} \cdot \hat{\mathbf{D}}_{\mathrm{p}} - \frac{1}{c}\hat{\mathbf{Z}} \cdot \dot{\hat{\mathbf{Z}}} \\ &\quad = \left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right) \cdot \hat{\mathbf{D}}_{\mathrm{p}} + \hat{\mathbf{Z}} \cdot \hat{\mathbf{D}}_{\mathrm{p}} + \frac{1}{c}\hat{\mathbf{Z}} \cdot \left(\hat{\mathbf{L}}_{\mathrm{p}}\hat{\mathbf{Z}} + \hat{\mathbf{Z}}\hat{\mathbf{L}}_{\mathrm{p}}^{T}\right) - \frac{1}{c}\hat{\mathbf{Z}} \cdot \dot{\hat{\mathbf{Z}}} \\ &\quad = \left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right) \cdot \hat{\mathbf{D}}_{\mathrm{p}} + \hat{\mathbf{Z}} \cdot \left(\hat{\mathbf{D}}_{\mathrm{p}} - \frac{1}{c}\overset{\nabla}{\hat{\mathbf{Z}}}\right) \geqslant 0, \end{aligned} \tag{97}$$

where use has been made of $(72)_2$, (77). Provided a normality rule of the form $\hat{\mathbf{D}}_p = \Lambda \frac{\partial \hat{F}}{\partial \mathbf{P}}$ holds, inequality (97) may be satisfied by requiring $\hat{\mathbf{Z}} \cdot (\hat{\mathbf{D}}_p - \frac{1}{c} \overset{\nabla}{\hat{\mathbf{Z}}}) \geqslant 0$. Clearly, a sufficient condition for this inequality is $\hat{\mathbf{D}}_p - \frac{1}{c} \overset{\nabla}{\hat{\mathbf{Z}}} = \frac{b}{c} \dot{s} \hat{\mathbf{Z}}$ with $b =$ const $\geqslant 0$ and $\dot{s} = \sqrt{\frac{2}{3} \hat{\mathbf{D}}_p \cdot \hat{\mathbf{D}}_p}$. Hence,

$$\overset{\nabla}{\hat{\mathbf{Z}}} = c \hat{\mathbf{D}}_p - b \dot{s} \hat{\mathbf{Z}}. \tag{98}$$

Proceeding to extend this approach to take into account isotropic hardening and isotropic damage effects, we assume the existence of a yield function in the space of Mandel stress tensors

$$F = \hat{F}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}, R, d\right), \tag{99}$$

which is defined to be an appropriately modified von Mises yield function. While we define the back stress tensor $\hat{\boldsymbol{\xi}}$ as in Eq. (96), we assume that $\tilde{\boldsymbol{E}}_p$ must be replaced by $\hat{\mathbf{D}}_p$ and that an associated normality rule, formulated relative to the plastic intermediate configuration, holds:

$$\hat{\mathbf{D}}_p \equiv \overset{\triangle}{\hat{\mathbf{\Gamma}}}_p = \Lambda \frac{\partial \hat{F}}{\partial \hat{\mathbf{P}}} \tag{100}$$

with

$$\Lambda \begin{cases} > 0 & \text{for} \quad F = 0 \ \& \ \dot{F}\Big|_{\mathbf{F}_{p=\text{const}}} \geqslant 0. \\ = 0 & \text{otherwise} \end{cases} \tag{101}$$

With regard to these preliminaries the three models may be generalized to finite deformations as follows.

4.3. Model A

In analogy to Section 3.2, the plastic parts of the specific free energy function are chosen in the form

$$\Psi_p^{\text{kin}}(t) = \Psi_p^{\text{kin}}\left(\hat{\mathbf{Y}}\right) = \frac{c}{2\rho_R} \hat{\mathbf{Y}} \cdot \hat{\mathbf{Y}}, \tag{102}$$

$$\Psi_p^{\text{iso}}(t) = \hat{\Psi}_p^{\text{iso}}(r) = \frac{\gamma}{\rho_R \beta} \left(r + \frac{1}{\beta} e^{-\beta r} \right), \tag{103}$$

where c, γ, β are nonnegative material parameters. Consequently, the potential relations (89)–(91) lead to

$$\hat{\mathbf{Z}} = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}_{\mathrm{p}}^{\mathrm{kin}}}{\partial \hat{\mathbf{Y}}} = c\hat{\mathbf{Y}}, \tag{104}$$

$$k = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}_{\mathrm{p}}^{\mathrm{iso}}}{\partial r} = \frac{\gamma}{\beta}\left(1 - e^{-\beta r}\right), \tag{105}$$

$$\Omega \equiv \Omega_{\mathrm{e}} = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}_{\mathrm{e}}}{\partial d} = -\frac{1}{2}\hat{\mathbf{\Gamma}}_{\mathrm{e}} \cdot \mathcal{C}\left[\hat{\mathbf{\Gamma}}_{\mathrm{e}}\right]. \tag{106}$$

Furthermore, the yield function, formulated in terms of Mandel stresses, is defined by ($h \geqslant 0$)

$$F = \hat{F}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}, R, d\right) = \sqrt{\frac{3}{2}\left(\frac{\hat{\mathbf{P}}}{1-d} - \hat{\boldsymbol{\xi}}\right)^{D} \cdot \left(\frac{\hat{\mathbf{P}}}{1-d} - \hat{\boldsymbol{\xi}}\right)^{D}} - R - h, \tag{107}$$

$$= \hat{\hat{F}}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}, k, d\right) = \sqrt{\frac{3}{2}\left(\frac{\hat{\mathbf{P}}}{1-d} - \hat{\boldsymbol{\xi}}\right)^{D} \cdot \left(\frac{\hat{\mathbf{P}}}{1-d} - \hat{\boldsymbol{\xi}}\right)^{D}} - k, \tag{108}$$

$$k := R + h, \tag{109}$$

so that the flow rule (100) reads as follows:

$$\hat{\mathbf{D}}_{\mathrm{p}} = \sqrt{\frac{3}{2}}\frac{\Lambda}{1-d}\hat{\mathbf{N}} = \sqrt{\frac{3}{2}}\dot{s}\hat{\mathbf{N}}, \qquad \hat{\mathbf{N}} = \frac{\left(\frac{\hat{\mathbf{P}}}{1-d} - \hat{\boldsymbol{\xi}}\right)^{D}}{\left\|\left(\frac{\hat{\mathbf{P}}}{1-d} - \hat{\boldsymbol{\xi}}\right)^{D}\right\|}, \tag{110}$$

$$\dot{s} := \sqrt{\frac{2}{3}\hat{\mathbf{D}}_{\mathrm{p}} \cdot \hat{\mathbf{D}}_{\mathrm{p}}} = \frac{\Lambda}{1-d}\,. \tag{111}$$

Substituting (110), (111) and the time derivative of (104) into (92), we obtain after some algebraic manipulations, together with (77), $(72)_2$, (96), (107) and the yield condition $F=0$,

$$h\dot{s} + R\left(\dot{s} - \frac{\dot{r}}{1-d}\right) + \hat{\mathbf{Z}} \cdot \left\{\left(\mathbf{1} - \frac{2d}{1-d}\hat{\mathbf{Y}}\right)\hat{\mathbf{D}}_{\mathrm{p}} - \frac{\overset{\triangledown}{\hat{\mathbf{Z}}}}{c(1-d)}\right\} - \frac{\Omega}{1-d}\dot{d} \geqslant 0. \tag{112}$$

By dealing with the terms responsible for isotropic hardening and damage effects as in Section 3.2, we get the relations (31), (33):

$$\dot{d} \geqslant 0, \tag{113}$$

$$\dot{k} = (1-d)(\gamma - \beta(k-h))\dot{s}. \tag{114}$$

Therefore, it suffices to take care that the term in (112), which is responsible for kinematic hardening, is nonnegative:

$$\hat{\mathbf{Z}} \cdot \left\{ \left(\mathbf{1} - \frac{2d}{1-d} \hat{\mathbf{Y}} \right) \hat{\mathbf{D}}_{\mathrm{p}} - \frac{\overset{\triangledown}{\hat{\mathbf{Z}}}}{c(1-d)} \right\} \geqslant 0. \tag{115}$$

As in Section 3.2, we set the term in brace positive proportional to $\hat{\mathbf{Z}}$, in order to satisfy this inequality:

$$\left(\mathbf{1} - \frac{2d}{1-d} \hat{\mathbf{Y}} \right) \hat{\mathbf{D}}_{\mathrm{p}} - \frac{\overset{\triangledown}{\hat{\mathbf{Z}}}}{c(1-d)} = \frac{b}{c} \dot{s} \hat{\mathbf{Z}}, \tag{116}$$

or

$$\overset{\triangledown}{\hat{\mathbf{Z}}} = (1-d) \left\{ c \left(\mathbf{1} - \frac{2d}{c(1-d)} \hat{\mathbf{Z}} \right) \hat{\mathbf{D}}_{\mathrm{p}} - b \dot{s} \hat{\mathbf{Z}} \right\}, \tag{117}$$

where b is a nonnegative material parameter and use has been made of (104).

4.4. Model B

As mentioned in Section 3.3, this model differs from Model A in the definition of the yield function and the kinematic hardening rule. By introducing a yield function

$$F = \hat{F}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}, R, d \right) = \sqrt{\frac{3}{2} \left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d} \right)^{D} \cdot \left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d} \right)^{D}} - R - h, \tag{118}$$

$$= \hat{\hat{F}}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}, k, d \right) = \sqrt{\frac{3}{2} \left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d} \right)^{D} \cdot \left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d} \right)^{D}} - k, \tag{119}$$

$$k := R + h, \tag{120}$$

the normality rule (100) takes the form

$$\hat{\mathbf{D}}_{\mathrm{p}} = \sqrt{\frac{3}{2}}\frac{\Lambda}{1-d}\hat{\mathbf{N}} = \sqrt{\frac{3}{2}}\dot{s}\hat{\mathbf{N}}, \qquad \hat{\mathbf{N}} = \frac{\left(\hat{\mathbf{P}} - \boldsymbol{\xi}\right)^{D}}{\left\| \left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right)^{D} \right\|}, \tag{121}$$

$$\dot{s} := \sqrt{\frac{2}{3}\hat{\mathbf{D}}_{\mathrm{p}} \cdot \hat{\mathbf{D}}_{\mathrm{p}}} = \frac{\Lambda}{1-d}. \tag{122}$$

Making use of (104), (96), (77), $(72)_2$, together with (121), (122), (118) and the yield condition $F=0$, the dissipation inequality (92) can be rewritten as

$$h\dot{s} + R\left(\dot{s} - \frac{\dot{r}}{1-d}\right) + \hat{\mathbf{Z}} \cdot \left\{ \frac{\hat{\mathbf{D}}_{\mathrm{p}}}{1-d} - \frac{\overset{\triangledown}{\hat{\mathbf{Z}}}}{c(1-d)} \right\} - \frac{\Omega}{1-d}\dot{d} \geqslant 0. \tag{123}$$

By treating the terms in (123), responsible for isotropic hardening and damage effects, as in Section 3.2 or 3.3 (see also Section 4.3), we get the relations

$$\dot{d} \geqslant 0, \tag{124}$$

$$\dot{k} = (1-d)(\gamma - \beta(k-h))\dot{s}, \tag{125}$$

and inequality (123) is satisfied as soon as

$$\hat{\mathbf{Z}} \cdot \left\{ \frac{\hat{\mathbf{D}}_{\mathrm{p}}}{1-d} - \frac{\overset{\triangledown}{\hat{\mathbf{Z}}}}{c(1-d)} \right\} \geqslant 0. \tag{126}$$

Evidently,

$$\frac{\hat{\mathbf{D}}_{\mathrm{p}}}{1-d} - \frac{\overset{\triangledown}{\hat{\mathbf{Z}}}}{c(1-d)} = \frac{b}{c}\dot{s}\hat{\mathbf{Z}} \qquad (b \geqslant 0), \tag{127}$$

or

$$\overset{\triangledown}{\hat{\mathbf{Z}}} = c\hat{\mathbf{D}}_{\mathrm{p}} - (1-d)b\dot{s}\hat{\mathbf{Z}} \tag{128}$$

ensures the validity of (126).

4.5. Model C

Keeping in mind the formulation in Section 3.4 the plastic parts of the specific free energy function are chosen as follows:

$$\Psi_{\mathrm{p}}^{\mathrm{iso}}(t) = \hat{\Psi}_{\mathrm{p}}^{\mathrm{kin}}\left(\hat{\mathbf{Y}}, d\right) = \frac{(1-d)}{2\rho_{\mathrm{R}}} c\hat{\mathbf{Y}}\cdot\hat{\mathbf{Y}}, \tag{129}$$

$$\Psi_{\mathrm{p}}^{\mathrm{iso}}(t) = \hat{\Psi}_{\mathrm{p}}^{\mathrm{iso}}(r, d) = \frac{(1-d)}{2\rho_{\mathrm{R}}}\left(\gamma r^2 + 2R^{(0)} r\right). \tag{130}$$

These, together with (82) imply

$$\hat{\mathbf{Z}} = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}_{\mathrm{p}}^{\mathrm{kin}}}{\partial \hat{\mathbf{Y}}} = (1-d)c\hat{\mathbf{Y}}, \tag{131}$$

$$k = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}_{\mathrm{p}}^{\mathrm{iso}}}{\partial r} = (1-d)\left(\gamma r + R^{(0)}\right), \tag{132}$$

$$\Omega \equiv \Omega_{\mathrm{e}} + \Omega_{\mathrm{p}} = \rho_{\mathrm{R}} \frac{\partial \hat{\Psi}}{\partial d} = -\frac{1}{2}\left(\hat{\boldsymbol{\Gamma}}_{\mathrm{e}}\cdot\boldsymbol{\mathcal{C}}\left[\hat{\boldsymbol{\Gamma}}_{\mathrm{e}}\right] + \gamma r^2 + 2R^{(0)} r + c\hat{\mathbf{Y}}\cdot\hat{\mathbf{Y}}\right). \tag{133}$$

As in Section 3.4, the material parameters $c, \gamma, R^{(0)} := R|_{s=0}$ are nonnegative. In analogy to (52), we consider a yield function

$$F = \hat{F}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}, R, d\right) = \sqrt{\frac{3}{2}\left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d}\right)^D \cdot \left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d}\right)^D} - \frac{R}{1-d} - h, \tag{134}$$

$$= \hat{\hat{F}}\left(\hat{\mathbf{P}}, \hat{\boldsymbol{\xi}}, k, d\right) = \sqrt{\frac{3}{2}\left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d}\right)^D \cdot \left(\frac{\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}}{1-d}\right)^D} - \frac{k}{1-d}, \tag{135}$$

$$k := R + (1-d)h, \tag{136}$$

which combined with (100) yield the flow rule

$$\hat{\mathbf{D}}_{\mathrm{p}} = \sqrt{\frac{3}{2}}\frac{\Lambda}{1-d}\hat{\mathbf{N}} = \sqrt{\frac{3}{2}}\dot{s}\hat{\mathbf{N}}, \qquad \hat{\mathbf{N}} = \frac{\left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right)^D}{\left\|\left(\hat{\mathbf{P}} - \hat{\boldsymbol{\xi}}\right)^D\right\|}, \tag{137}$$

$$\dot{s} := \sqrt{\frac{2}{3}\hat{\mathbf{D}}_{\mathrm{p}}\cdot\hat{\mathbf{D}}_{\mathrm{p}}} = \frac{\Lambda}{1-d}. \tag{138}$$

After some rearrangement of terms, using (131), (96), (77), $(72)_2$, (137), (138), (134) and the yield condition $F=0$, the dissipation inequality (92) becomes

$$h\dot{s}+\frac{R}{1-d}(\dot{s}-\dot{r})+\left(\frac{\hat{\mathbf{Z}}}{1-d}\right)\cdot\left\{\hat{\mathbf{D}}_{\mathrm{p}}-\frac{1}{c}\left(\frac{\hat{\mathbf{Z}}}{1-d}\right)^{\nabla}\right\}-\frac{\Omega}{1-d}\dot{d}\geqslant 0. \tag{139}$$

By means of steps parallel to those in Section 3.4, we see that

$$\left(\frac{k}{1-d}\right)^{\cdot}=\left(\gamma-\beta\left(\frac{k}{1-d}-k^{(0)}\right)\right)\dot{s}, \tag{140}$$

$$\frac{\hat{\mathbf{Z}}}{1-d}\cdot\left\{\hat{\mathbf{D}}_{\mathrm{p}}-\frac{1}{c}\left(\frac{\hat{\mathbf{Z}}}{1-d}\right)^{\nabla}\right\}\geqslant 0, \tag{141}$$

$$\dot{d}\geqslant 0, \tag{142}$$

are sufficient conditions for inequality (139). Obviously, (141) can be satisfied by setting the term enclosed in braces positive proportional to $\hat{\mathbf{Z}}/(1-d)$:

$$\hat{\mathbf{D}}_{\mathrm{p}}-\frac{1}{c}\left(\frac{\hat{\mathbf{Z}}}{1-d}\right)^{\nabla}=\frac{b}{c}\dot{s}\left(\frac{\hat{\mathbf{Z}}}{1-d}\right) \qquad (b\geqslant 0), \tag{143}$$

or

$$\left(\frac{\hat{\mathbf{Z}}}{1-d}\right)^{\nabla}=c\hat{\mathbf{D}}_{\mathrm{p}}-b\dot{s}\left(\frac{\hat{\mathbf{Z}}}{1-d}\right). \tag{144}$$

4.6. Discussion of the model responses—concluding remarks

For all models, the damage evolution equation is identical to that for small deformations (cf. Eq (70))

$$\dot{d}=-a_1\Omega_{\mathrm{e}}\dot{s}. \tag{145}$$

But now Ω_{e} reads

$$\Omega_{\mathrm{e}}=-\frac{1}{(1-d)^2}\left\{\frac{1}{4\mu}\hat{\mathbf{T}}^{D}\cdot\hat{\mathbf{T}}^{D}+\frac{1}{6(2\mu+3\lambda)}\left(\mathrm{tr}\hat{\mathbf{T}}\right)^{2}\right\}. \tag{146}$$

Fig. 9 displays characteristic features of the models for (homogeneous) strain controlled uniaxial tension loadings. In these diagrams, σ_{W} denotes the uniaxial component of the weighted Cauchy stress tensor $\mathbf{S}, \sigma_W \equiv S_{11}$, while $\hat{\xi}_{11}$ is the uniaxial stress component of the internal stress $\hat{\boldsymbol{\xi}}$. Further, $\varepsilon = \ln(l/l_0)$ denotes the logarithmic strain, where l_0, l are the lengths of the tensile specimen at the beginning of the deformation and at the actual state, respectively.

Comparing Fig. 9 with Fig. 6, we see that for the three models, the responses for uniaxial tension remain essentially the same for both the small and the finite deformation formulation. As outlined in the introduction, the main objective of this paper is to discuss the extension of the three small deformation models to finite deformations in a thermodynamically consistent way. Generally, the extension of constitutive equations from small to finite deformations is not unique. Typically, it depends on the choice of the variables and associated time derivatives used to formulate the theory. In the present work, use has been made of the thermodynamical approach previously developed by Tsakmakis (1996). A characteristic feature of this approach is that the constitutive theory is first formulated relative to the plastic intermediate configuration by means of the Mandel stress tensor. Also, the back stress tensor $\hat{\boldsymbol{\xi}}$ is defined to obey the mathematical structure of a Mandel stress tensor. However, the evolution equation governing the response of kinematic hardening has not been formulated with respect to $\hat{\boldsymbol{\xi}}$, but with respect to an internal stress tensor $\hat{\mathbf{Z}}$, which is defined to have the mathematical structure of a second Piola–Kirchhoff stress tensor relative to the plastic intermediate configuration.

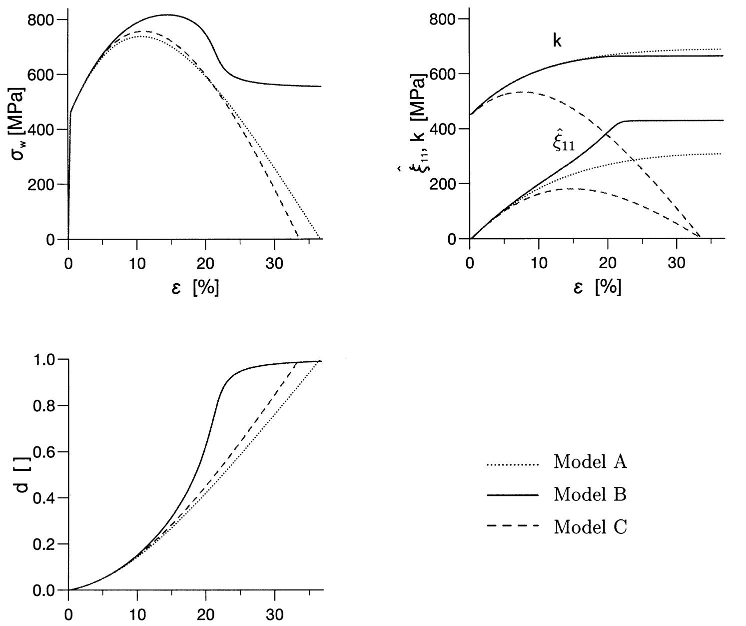

Fig. 9. Comparison of the model responses for uniaxial tension (finite deformation formulation); assigned material parameters: $\mu = 76\,923$ MPa, $\lambda = 115\,384.6$ MPa, $h = k_0 = 450$ MPa, $\beta = 10$, $\gamma = 2700$ MPa, $b = 10$, $c = 2500$ MPa, $a_0 = 0$, $a_1 = 1.25$/MPa, $a_2 = 0$, $n = 1$, $q = 0$.

It must be emphasized that a discussion of the physical properties described by the models was not an aim of the present paper. However, it seems that the constitutive behavior predicted by Model B should be interpreted (at least for metallic materials) to be not physically plausible for $d \rightarrow 1$. (To our knowledge no such constitutive behavior has been experimentally observed.) Also, we remark that, in what concerns Model B and Model C, the evolution equations for kinematic hardening (128) and (144) have the same structure as Eqs. (46) and (66), respectively. On the other hand, because of the term $(\mathbf{1} - \frac{2d}{c(1-d)}\hat{\mathbf{Z}})$, the hardening law (117) for Model A has formally not the same structure as Eq. (35). Consequently, the adopted thermodynamic formalism (respectively, the variables used in this formalism) could be interpreted to be not the appropriate one for generalizing Model A to finite deformations. Hence, from a formal point of view, according to our approach, Model C and its finite deformation version should be preferred.

Finally, it should be mentioned that for the purposes of our paper, it was sufficient to discuss the finite deformation versions of the models by using homogeneous deformations only. However, it is of interest to compare the responses predicted by the small deformation model versions with those predicted by the finite deformation ones for inhomogeneous deformations. Such a comparison for a simplified version of Model C is given in Diegele et al. (2000) and Lämmer (1998) for the cases without and with coupled damage effects, respectively.

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