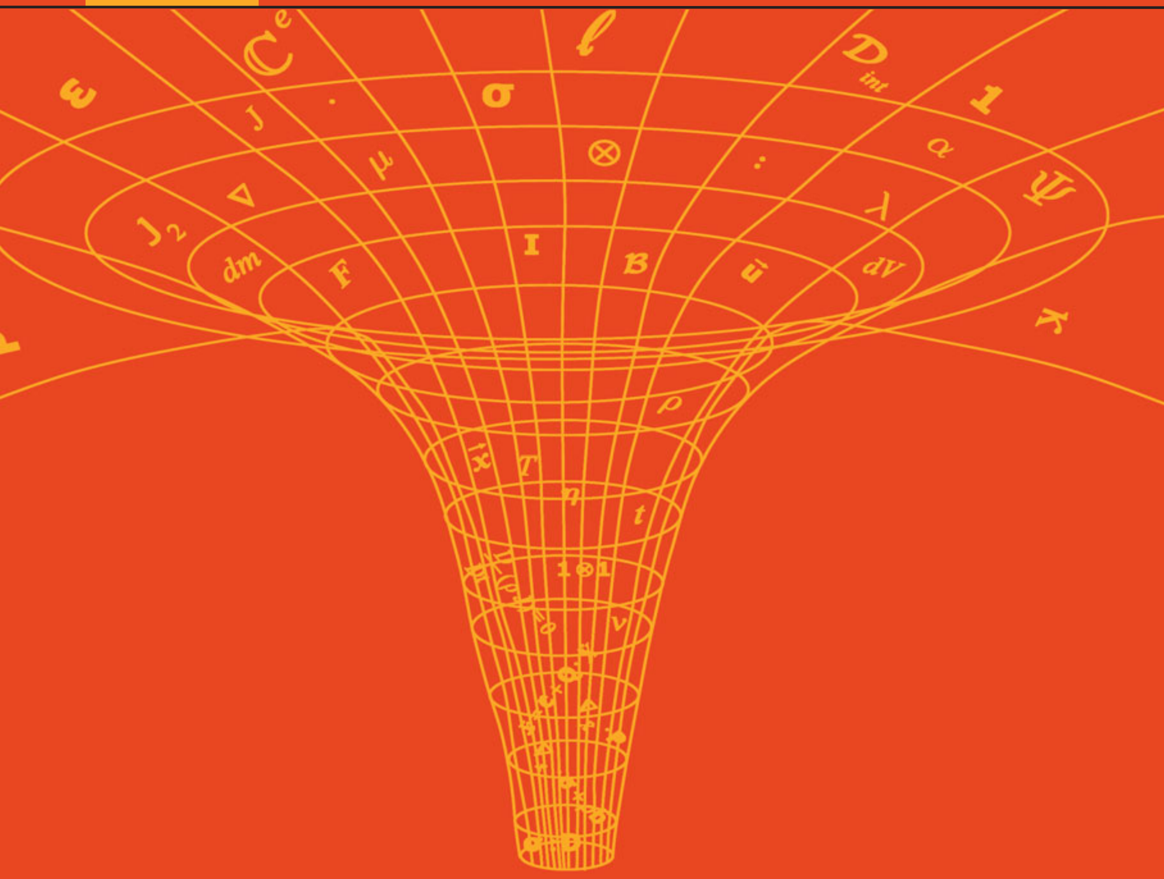
Notes on Continuum Mechanics

Eduardo W.V. Chaves



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Notes on Continuum Mechanics

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Notes on Continuum Mechanics

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Notes on Continuum Mechanics
Eduardo W.V. Chaves

First edition, 2013

To my Parents

Contents

Preface

The Continuum Mechanics is a key subject to several degrees based on physical science, such as: Civil Engineering, Industrial Engineering, Meteorology, Magnetism, Oceanography, Aerodynamics, Hydrodynamics, Marine Engineering, etc.

This book grew out of notes for the course Introduction to Continuum Mechanics of the career of Civil Engineering of the University of Castilla-La Mancha (Spain), and is intended for students who are initiating a university degree based on physical science, and is also intended for PhD students as well researchers.

In order to provide greater clarity for students, this book presents a thorough detail at the time of the demonstration of the equations. At the time of writing the book, the author has had a big concern for trying to unify the existing nomenclature, and to this end has consulted numerous articles and books on the subject. With respect to the notation, the developments of the equations are indiscriminately presented in tensorial, inditial and Voigt notations. Another aspect is that the book is self-contained, so that the concepts used are defined in the text.

Finally, I would like to express my gratitude to: Houzeaux (Guillaume), Vázquez (Mariano), Gallego (Inmaculada), Pulido (Loli), Benítez (José María), Casati (María Jesus), Vélez (Eduardo), Solares (Cristina), Olivares (Miguel Ángel), Escobedo (Fernando), Simarro (Gonzalo), Sanz (Ana), for aid to the revision of the first edition in Spanish. I would also like to thank Toby Wakely for reviewing the English.

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Eduardo W. V. Chaves

Ciudad Real-Spain, October 2012

Abbreviations

IBVP	Initial Boundary Value Problem
BVP	Boundary Value Problem
FEM	Finite Element Method
BEM	Boundary Element Method
FDM	Finite Difference Method

Latin

i.e.	*id est*	that is
et al.	*et alii*	and the others
e.g.	*exempli gratia*	for example
etc.	*et cetera*	and so on
v., vs.	*versus*	versus
viz.	*vidilicet*	namely

Operators and Symbols

$\langle \bullet \rangle = \dfrac{\lvert\bullet\rvert + \bullet}{2}$	Macaulay bracket
$\lVert\bullet\rVert$	Euclidian norm of $\bullet$
$\mathrm{Tr}(\bullet)$	trace of $(\bullet)$
$(\bullet)^T$	transpose of $(\bullet)$
$(\bullet)^{-1}$	inverse of $(\bullet)$
$(\bullet)^{-T}$	inverse of the transpose of $(\bullet)$
$(\bullet)^{sym}$	symmetric part of $(\bullet)$
$(\bullet)^{skew}$	antisymmetric (skew-symmetric) part of $(\bullet)$
$(\bullet)^{sph}$	spherical part of $(\bullet)$
$(\bullet)^{dev}$	deviatoric part of $(\bullet)$
$\lVert\bullet\rVert$	module of $\bullet$
$[[\bullet]]$	jump of $\bullet$
$\cdot$	scalar product
$\det(\bullet) \equiv \lvert\bullet\rvert$	determinant of $(\bullet)$
$\dfrac{D\bullet}{Dt} \equiv \dot{\bullet}$	material time derivative of $(\bullet)$
$\mathrm{cof}(\bullet)$	cofactor of $\bullet$;
$\mathrm{Adj}(\bullet)$	adjugate of $(\bullet)$
$\mathrm{Tr}(\bullet)$	trace of $(\bullet)$
$:$	double scalar product (or double contraction or double dot product)
∇^2	Scalar differential operator
$\otimes$	tensorial product
$\nabla\bullet \equiv \mathrm{grad}(\bullet)$	gradient of $\bullet$
$\nabla\cdot\bullet \equiv \mathrm{div}(\bullet)$	divergence of $\bullet$
$\wedge$	vector product (or cross product)

SI-Units

length	m - metro	energy, work, heat	$J = Nm$ - Joules
mass	kg - kilogram	power	$\frac{J}{s} \equiv W$ watt
time	s - second		
temperature	K - Kelvin	permeability	m^2
velocity	$\frac{m}{s}$	dynamic viscosity	$Pa \times s$
acceleration	$\frac{m}{s^2}$	mass flux	$\frac{kg}{m^2 s}$
energy	$J = Nm$ - Joules	energy flux	$\frac{J}{m^2 s}$
force	N - Newton	thermal conductivity	$\frac{W}{mK}$
pressure, stress	$Pa \equiv \frac{N}{m^2}$ - Pascal	mass density	$\frac{kg}{m^3}$

Prefix	Symbol	10^n	Prefix	Symbol	10^n
pico	p	10^{-12}	kilo	k	10^3
nano	η	10^{-9}	Mega	M	10^6
micro	μ	10^{-6}	Giga	G	10^9
mili	m	10^{-3}	Tera	T	10^{12}
centi	c	10^{-2}			
deci	d	10			

Introduction

1 Mechanics

Broadly speaking, Mechanics is the branch of physics that studies the behavior of a body when it is subjected to forces, (*e.g.* deformation) and how it evolves over time. In general, Mechanics can be classified into:

$$\left.\begin{array}{l}\blacksquare\ \text{Theoretical Mechanics;}\\ \blacksquare\ \text{Applied Mechanics;}\end{array}\right\}\ \text{Continuum Mechanics}$$

- Computational Mechanics.

Theoretical Mechanics establishes the laws that govern a particular physical problem based on fundamental principles.

Applied Mechanics transfers theoretical knowledge to use it in scientific and engineering problems.

Computational Mechanics solves problems by simulation with numerical tools implemented in the computer.

In this book we focus our attention to the Theoretical and Applied Mechanics.

2 What is Continuum Mechanics?

Broadly speaking, Continuum Mechanics is the branch of Mechanics that studies motion (deformation) of a medium that consists of matter subjected to forces. For example, how would a wooden and a concrete beam deform when the same force is applied to them? Another example we can look at is fluids, *e.g.* for a given pressure, how does water (or oil) flow in a pipeline?

2.1 Hypothesis of Continuum Mechanics

As we know, a physical body consists of small molecules (an agglomeration of two or more atoms). Then, by means of sophisticated experiments, we can observe that these "constituents" are not distributed homogeneously, that is, there are gaps (voids) between them. However, within the scope of Continuum Mechanics these phenomenological

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characteristics are ignored. For example, if we are dealing with a fluid in Continuum Mechanics, the properties: mass density, pressure and velocity are assumed to be continuous function. Treating a system of molecules as a continuous medium is valid if we compare the mean free path of molecules (Λ) (average distance particles travel before colliding with each other) with the characteristic physical length scale (ℓ_C). For example, for solids and liquids we have $\Lambda \approx 10^{-7} cm$ and for gases $\Lambda \approx 10^{-6} cm$, Chung (1996). Then the ratio $\frac{\Lambda}{\ell_C}$ is known as the *Knudsen number* (Kn). If this number is much smaller than unity, the domain can be treated as a continuum; otherwise we must use statistical mechanics to obtain the governing equations of the problem whereby we can establish that:

$$Kn = \frac{\Lambda}{\ell} << 1 \quad \Rightarrow \quad \text{macroscopic approach}$$

$$Kn = \frac{\Lambda}{\ell} > 1 \quad \Rightarrow \quad \text{microscopic approach}$$

The fundamental hypothesis in Continuum Mechanics is that the matter of which the medium is made up is continuously distributed and that the variables involved in the problem (*e.g.* velocity, acceleration, pressure, mass density, etc.) are continuous functions.

Then, by means of approximations or additional equations to that initially proposed for the problem, we can characterize a continuum with discontinuous variables associated with the problem, *e.g.* fracture problem and shock waves among others.

2.2 The Continuum

In general, when we apply force to solids they are able to recover their original states when said force is removed. However, this is not the case with fluids, *i.e.* solids and fluids *apparently* act very differently. Therefore, traditionally, continuum mechanics has been divided into two groups: solids and fluids (liquids and gases). As we will see throughout this book, the fundamental equations of Continuum Mechanics are the same for both of these.

For many decades, solid and fluid mechanics have been treated independently from each other. However, nowadays, it is not advisable to work like this. Firstly, it is necessary to simulate more complex materials, *e.g.* materials that have characteristics of solids and fluids simultaneously. These materials, besides presenting elastic properties, (obeying the constitutive law for solids), also exhibit characteristics of fluids due to their viscosity, for example: viscoelastic materials. Secondly, the need to simulate the problem of fluid-solid interaction has improved the relationship between fluids and solids.

Recently, a third branch of continuum mechanics has emerged, which is related to multiphysics problems, characterized by phase change, *e.g.* from solid to liquid phase or vice versa, and which includes mechanical systems that transcend classical mechanical boundary of solids and fluids. Then, traditionally, continuum mechanics can be divided into:

$$\text{The Continuum}\begin{cases}\text{Solids}\\ \text{Fluids}\begin{cases}\text{Liquids}\\ \text{Gases}\end{cases}\\ \text{Multiphysics}\end{cases}$$

3 Scales of Material Studies

According to Willam(2000), materials science can be studied on different scales, (see Figure 1), namely:

- Metric level

  At this level, we include most problems posed in Civil, Mechanical, Aerospace Engineering.
- Millimeter level

  At this level, it may enroll the specimen used to measure the material mechanical properties in the laboratory.
- Micrometer level

  Micro-structural characteristic, such as micro-defects and cement hydration products, are observed at this scale.
- Nanometer level

  At this level, we contemplate atomic and molecular processes.

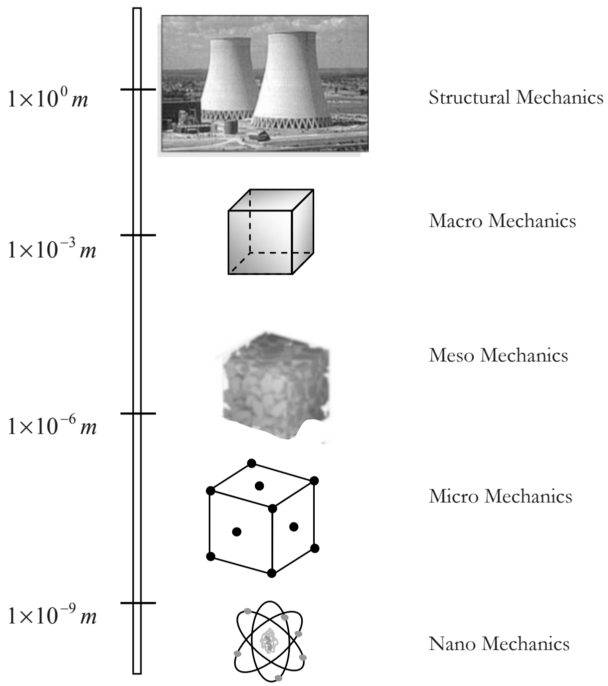

Figure 1: Multiscale in Material Mechanics, Willam(2000).

3.1 Scale Study of Continuum Mechanics

The continuum mechanics is raised at a macroscopic level. That is, the variables of the problem at a macroscopic level are considered as being the average of these variables at a mesoscale level. Let us take, for example, blood, which can be treated in different ways, depending on the scale under consideration. At a $10^{-6}m$ scale, we consider blood flows around a blood cell where the deformation of the cell walls is taken into consideration. Then, at a $10^{-4}m$ scale, we can consider the fluid flow through a set of blood cells, which thus allows us to observe the fluid effects on cells. Next, at a $10^{-3}m$ scale (macroscopic level), we can consider the fluid flow through arteries or veins (ignoring the individual cells) as being a fluid with certain macroscopic properties (*e.g.* velocity, pressure, etc.), (see Figure 2).

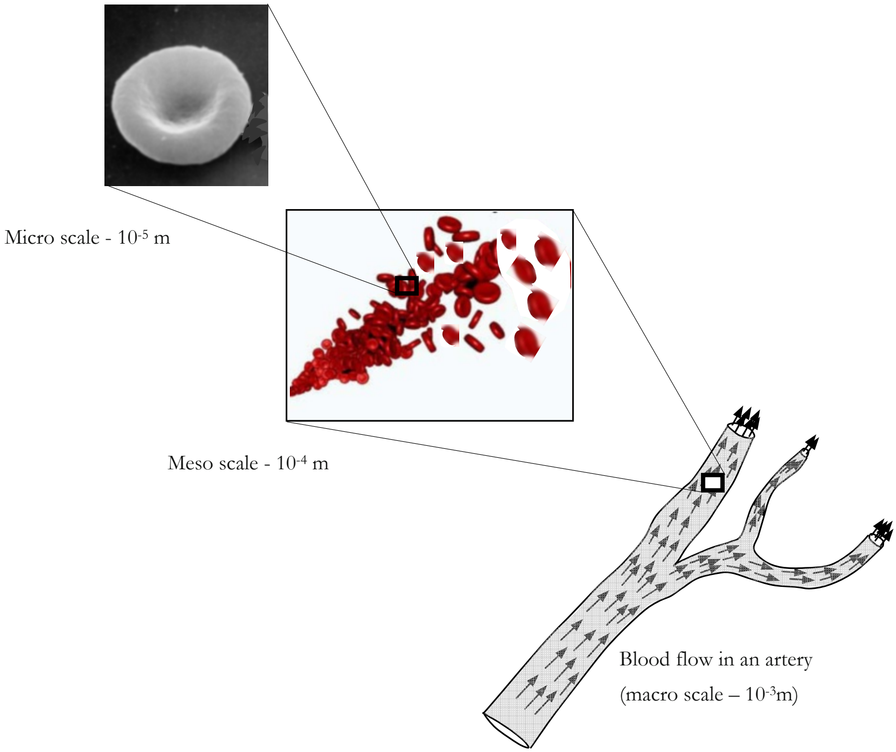

Figure 2: Scale levels in blood.

Another example we can use is a material made up of a mixture of materials such as concrete, which is fundamentally formed by mixing cement, aggregates, and water. At the $10^{-9}m$ scale, we can distinguish the atomic structure of the cement and aggregates. Then, at a $10^{-6}m$ scale it is possible to identify individual cement grains before hydration and grains of calcium silicate and calcium hydroxide can be appreciated, upon hydration. Finally, at the $10^{-3}m$ millimeter scale, we can distinguish individually each of the aggregates and pores (gaps). Note, at this level, the interaction between parts of cement and aggregates is important.

On the $10m$ metric scale and on the $1m$ laboratory scale, the concrete internal structure can be examined to ensure that its properties are identical in all directions and at all its points, which is what characterizes a homogenous and isotropic material.

Another example for understanding in which scale continuum mechanics is raised is by measuring mass density (ρ), which is a macroscopic variable for continuum mechanics.

We can determine the mass density of a cube (with sides a) by dividing its total mass by its volume. So, let us consider a new cube (with sides a') whose volume is less than the first one. In Figure 3(b) we can observe that, depending on the position of the new cube, we can obtain different values for mass density, as different position contain different amounts of matter and voids.

That is, if we can vary the a-dimension from a very small size, we will notice that the mass density value will oscillate, (see Figure 4). However, there will be a a-dimension region in which the mass density value maintains constant. The continuum mechanics is raised into this interval.

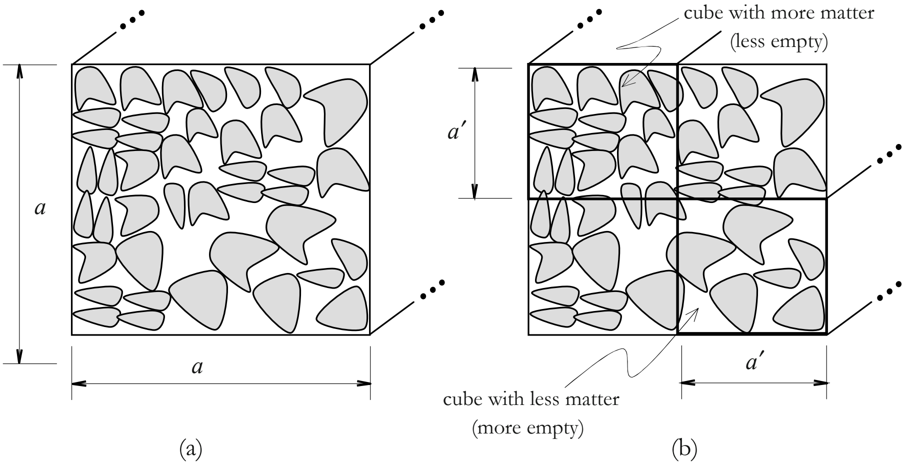

Figure 3: Mass density measurement.

It is possible to extend the continuum mechanics to other scales by adding certain hypothesis, such as the so-called *scale effect*, but this is not a subject covered in this book.

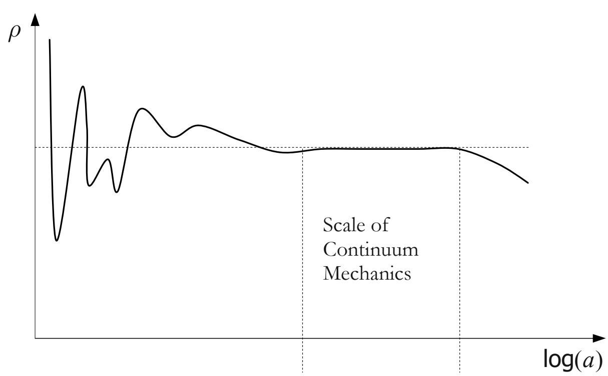

Figure 4: Mass density.

4 The Initial Boundary Value Problem (IBVP)

Continuum mechanics, based on certain principles, attempts to formulate the equations that govern given physical problems by means of partial differential equations. To these we must add the boundary and initial conditions in order to guarantee the uniqueness of the problem. This set of partial differential equations and the boundary initial conditions make up the *Initial Boundary Value Problem* (IBVP), (see Figure 5).

With a static or quasi-static problem the IBVP becomes a Boundary Value Problem (BVP) where the initial conditions are redundant.

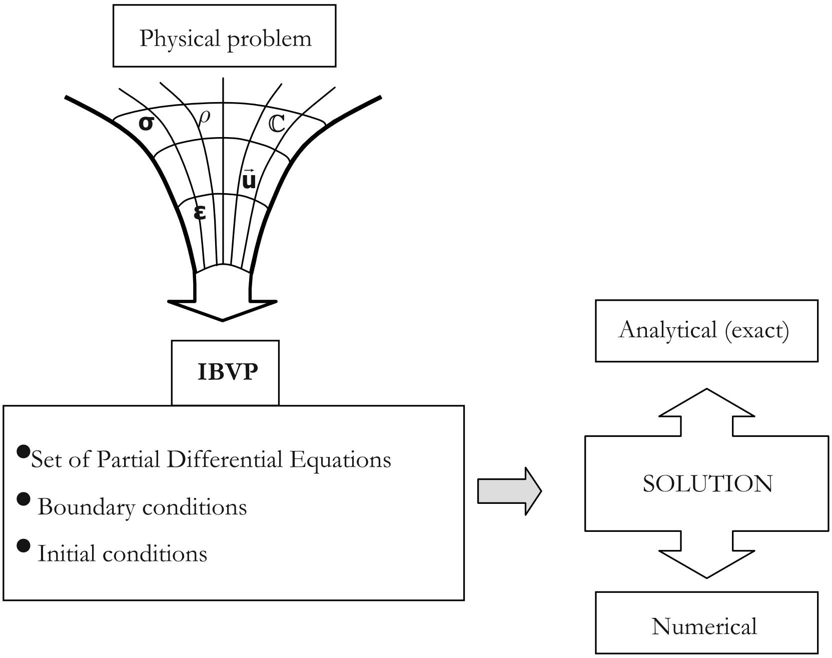

Figure 5: Statement and solution of the problem.

4.1 Solving the IBVP

Once the physical problem is stated, it can be solved and the IBVP solution can be analytical (exact solution), or numerical (approximated solution), (see Figure 5).

In practice, obtaining the analytical solution of the IBVP is very difficult or even impossible because of the problem complexity (*e.g.* due to its geometry, forces, or boundaries), hence we must resort to using IBVP numerical solution. However, obtaining the analytical solution for simple problems is very important since it serves as a reference to indicate the degree of accuracy (precision) of the numerical technique used.

Among the most widely used numerical techniques for the IBVP solution we can list, among others:

- The Finite Differences Method - FDM;
- The Finite Element Method - FEM;

- The Boundary Element Method - BEM;
- The Finite Volume Method - FVM;
- The Meshless Method.

We cannot state that any one of the techniques mentioned above is the best. First we must ask what type of problem we want to solve and, depending on this, one technique or another, or even a combination of different techniques can be used to optimize the solution.

In general, all techniques transform the continuous problem into a discrete system of equations.

The finite difference method (FDM) is based on discretizing the domain by points in which the governing equations are valid. The FDM was the first numerical method to emerge and today it is still in use on problems where stabilization problems occur and is also used to dicretize the time domain.

The finite element method (FEM) is based on discretizing the domain into subdomains called finite elements, in which the governing equations are valid. Moreover, it has proved to be more accurate in solving problems that FDM. Today the FEM technique is the most used and widespread in the solid mechanics field.

Conversely, in the Boundary Element Method (BEM), only the domain boundary is discretized by elements. From a viewpoint of the solution accuracy, the BEM provides more accurate solutions than FEM for elastic problems and it is a better method for working with semi-infinite or infinite problem domains. Nevertheless, the BEM has its downside in nonlinear problems, where we need to dicretize the domain by cells.

Generally, the IBVP contains both spatial variables (displacement, pressure, etc.) and temporary variables (rates of change of spatial variables), so we need to have both *spatial* and *time dicretization* to obtain the numerical solutions. For example, for spatial discretization we can use the FEM and for time discretization we can use another technique, such as the FDM.

4.2 Simplifying the IBVP

There are cases where the problem (IBVP) includes certain features which allow it to be simplified whereby its complexity can be drastically reduced and with which even the analytical problem solution can be obtained. These simplifications will be pointed out and elaborated on in detail throughout this book, but the engineers will have to decide for themselves when these simplifications can be used for a given problem and for this a sound grounding in the general theory is needed.

1

Tensors

1.1 Introduction

As seen previously in the introductory chapter, the goal of continuum mechanics is to establish a set of equations that governs a physical problem from a macroscopic perspective. The physical variables featuring in a problem are represented by tensor fields, in other words, physical phenomena can be shown mathematically by means of tensors whereas tensor fields indicate how tensor values vary in space and time. In these equations one main condition for these physical quantities is they must be independent of the reference system, *i.e.* they must be the same for different observers. However, for matters of convenience, when solving problems, we need to express the tensor in a given coordinate system, hence we have the concept of tensor components, but while tensors are independent of the coordinate system, their components are not and change as the system change.

In this chapter we will learn the language of TENSORS to help us interpret physical phenomena. These tensors can be classified according to the following order:

Zeroth-Order Tensors (Scalars): Among some of the quantities that have magnitude but not direction are *e.g.:* mass density, temperature, and pressure.

First-Order Tensors (Vectors): Quantities that have both magnitude and direction, *e.g.*: velocity, force. The first-order tensor is symbolized with a boldface letter and by an arrow at the top part of the vector, *i.e.:* $\vec{\bullet}$.

Second-Order Tensors: Quantities that have magnitude and two directions, *e.g.* stress and strain. The second-order and higher-order tensors are symbolized with a boldface letter.

In the first part of this chapter we will study several tools to manage tensors (scalars, vectors, second-order tensors, and higher-order tensors) without heeding their dependence

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on space and time. At the end of the chapter we will introduce tensor fields and some field operators which can be used to interpret these fields.

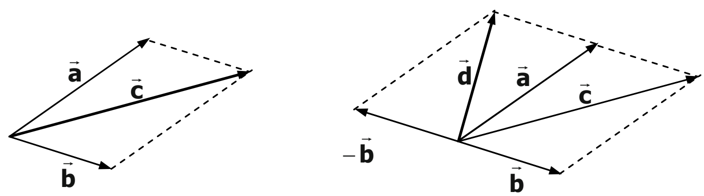
In this textbook we will work indiscriminately with the following notations: tensorial, indicial, and matricial. Additionally, when the tensors are symmetrical, it is also possible to represent their components using the Voigt notation.

1.2 Algebraic Operations with Vectors

There now follows a brief review of vectors, in the Euclidean vector space $(\mathcal{E})$, so that we may become acquainted with the nomenclature used in this textbook.

Addition: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be arbitrary vectors, we can show the sum of adding them, (see Figure 1.1 (a)), with a new vector ($\vec{\mathbf{c}}$) thus defined as:

$$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} = \vec{\mathbf{b}} + \vec{\mathbf{a}} \tag{1.1}$$



a) b)

Figure 1.1: Addition and subtraction of vectors.

Subtraction: The subtraction between two arbitrary vectors ($\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$), (see Figure 1.1 (b)), is given as follows:

$$\vec{\mathbf{d}} = \vec{\mathbf{a}} - \vec{\mathbf{b}} \tag{1.2}$$

Considering three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ the following properties are satisfied:

$$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} = \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}) = \vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} \tag{1.3}$$

Scalar multiplication: Let $\vec{\mathbf{a}}$ be a vector, we can define the scalar multiplication with $\lambda\vec{\mathbf{a}}$. The product of this operation is another vector with the same direction of $\vec{\mathbf{a}}$, and whose length and orientation is defined with the scalar λ as shown in Figure 1.2.

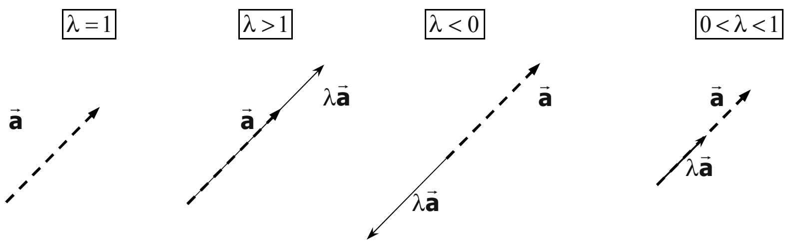


Figure 1.2: Scalar multiplication.

Scalar Product: The *Scalar Product* (also known as the *dot product* or *inner product*) of two vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, denoted by $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$, is defined as follows:

$$\gamma = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\| \cos\theta \tag{1.4}$$

where θ is the angle between the two vectors, (see Figure 1.3(a)), and $\|\bullet\|$ represents the Euclidean norm (or magnitude) of $\bullet$. The result of the operation (1.4) is a scalar. Moreover, we can conclude that $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \vec{\mathbf{b}}\cdot\vec{\mathbf{a}}$. The expression (1.4) is also true when $\vec{\mathbf{a}} = \vec{\mathbf{b}}$, therefore:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{a}}\| \cos\theta \xrightarrow{\theta=0^\circ} \vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{a}}\| \Rightarrow \|\vec{\mathbf{a}}\|^2 = \vec{\mathbf{a}}\cdot\vec{\mathbf{a}} \tag{1.5}$$

Hence, the norm of a vector is $\|\vec{\mathbf{a}}\| = \sqrt{\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}}$.

Unit Vector: A unit vector, associated with the $\vec{\mathbf{a}}$-direction, is shown with a $\hat{\mathbf{a}}$, which has the same direction and orientation of $\vec{\mathbf{a}}$. In this textbook, the hat symbol ($\hat{\bullet}$) denotes a unit vector. Thus, the unit vector, $\hat{\mathbf{a}}$, codirectional with $\vec{\mathbf{a}}$, is defined as:

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \tag{1.6}$$

where $\|\vec{\mathbf{a}}\|$ represents the norm (magnitude) of $\vec{\mathbf{a}}$. If $\hat{\mathbf{a}}$ is the unit vector, then the following must be true:

$$\|\hat{\mathbf{a}}\| = 1 \tag{1.7}$$

Zero Vector (or Null Vector): The zero vector is represented by a:

$$\vec{\mathbf{0}} \tag{1.8}$$

Projection Vector: The projection vector of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$, (see Figure 1.3(b)), is defined as:

$$\overrightarrow{\mathbf{proj}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}} = \|\mathrm{proj}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}\|\hat{\mathbf{b}} \quad \textit{Projection vector of } \vec{\mathbf{a}} \textit{ onto } \vec{\mathbf{b}} \tag{1.9}$$

where $\|\overrightarrow{\mathbf{proj}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}\|$ is the projection of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$, and $\hat{\mathbf{b}}$ is the unit vector associated with the $\vec{\mathbf{b}}$-direction. The magnitude of $\|\overrightarrow{\mathbf{proj}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}\|$ is obtained by means of the scalar product:

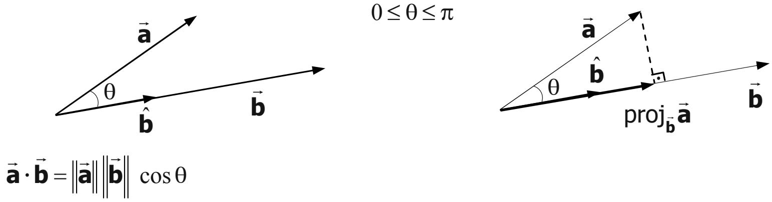
$$\|\overrightarrow{\mathbf{proj}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}\| = \vec{\mathbf{a}}\cdot\hat{\mathbf{b}} \quad \textit{Projection of } \vec{\mathbf{a}} \textit{ onto } \vec{\mathbf{b}} \tag{1.10}$$

So, taking into account the definition of the unit vector, we obtain:

$$\|\overrightarrow{\mathbf{proj}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}\| = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} \tag{1.11}$$

Then, the projection vector, $\overrightarrow{\mathbf{proj}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}}$, can be calculated by:

$$\overrightarrow{\mathbf{proj}}_{\vec{\mathbf{b}}}\vec{\mathbf{a}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|}\hat{\mathbf{b}} = \frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|}\frac{\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|} = \underbrace{\frac{\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}{\|\vec{\mathbf{b}}\|^2}}_{scalar}\vec{\mathbf{b}} \tag{1.12}$$



a) Scalar product b) Projection vector

Figure 1.3: Scalar product and projection vector.

Orthogonality between vectors: Two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal if the *scalar product* between them is zero, *i.e.*:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=0 \tag{1.13}$$

Vector Product (or Cross Product): The *vector product* of two vectors, $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, results in another vector $\vec{\mathbf{c}}$, which is perpendicular to the plane defined by the two input vectors, (see Figure 1.4). The vector product has the following characteristics:

- Representation:

$$\vec{\mathbf{c}}=\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}=-\vec{\mathbf{b}}\wedge\vec{\mathbf{a}} \tag{1.14}$$

- The vector $\vec{\mathbf{c}}$ is orthogonal to the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, thus:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}=\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}=0 \tag{1.15}$$

- The magnitude of $\vec{\mathbf{c}}$ is defined by the formula:

$$\left\|\vec{\mathbf{c}}\right\|=\left\|\vec{\mathbf{a}}\right\|\left\|\vec{\mathbf{b}}\right\|\sin\theta \tag{1.16}$$

where θ measures the smallest angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, (see Figure 1.4).

The magnitude of the vector product $\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}$ is geometrically expressed as the area of the parallelogram defined by the two vectors, (see Figure 1.4):

$$\mathsf{A}=\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| \tag{1.17}$$

Therefore, the triangle area defined by the points OCD, (see Figure 1.4 (a)), is:

$$\mathsf{A}_{\mathsf{T}}=\frac{1}{2}\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| \tag{1.18}$$

If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are linearly dependent, *i.e.* $\vec{\mathbf{a}}=\alpha\vec{\mathbf{b}}$ with α denoting a scalar, the vector product of two linearly dependent vectors becomes a zero vector, $\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}=\alpha\vec{\mathbf{b}}\wedge\vec{\mathbf{b}}=\vec{\mathbf{0}}$.

Scalar Triple Product (or Mixed Product): Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be arbitrary vectors, we can define the *scalar triple product* as:

$$\begin{aligned}\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})&=\vec{\mathbf{b}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}})=\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})=\mathsf{V}\\ \mathsf{V}=-\vec{\mathbf{a}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{b}})&=-\vec{\mathbf{b}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}})=-\vec{\mathbf{c}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}})\end{aligned} \tag{1.19}$$

where the scalar V represents the volume of the parallelepiped defined by $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$, (see Figure 1.5).

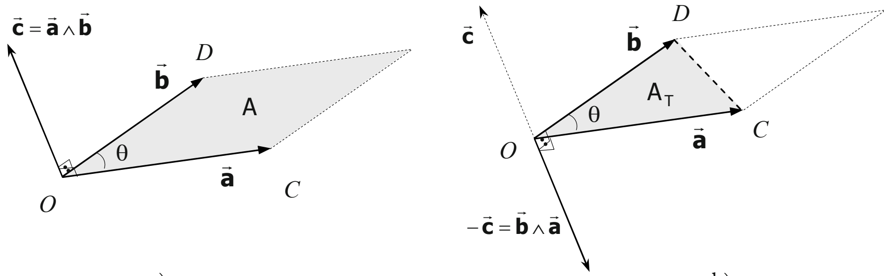


Figure 1.4: Vector product.

If two vectors are linearly dependent then, the scalar triple product is zero, *i.e.*:

$$\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{a}}) = \vec{\mathbf{0}} \tag{1.20}$$

Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, $\vec{\mathbf{d}}$, be vectors and α, β be scalars, the following property is satisfied:

$$(\alpha\, \vec{\mathbf{a}} + \beta\, \vec{\mathbf{b}}) \cdot (\vec{\mathbf{c}} \wedge \vec{\mathbf{d}}) = \alpha\, \vec{\mathbf{a}} \cdot (\vec{\mathbf{c}} \wedge \vec{\mathbf{d}}) + \beta\, \vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \wedge \vec{\mathbf{d}}) \tag{1.21}$$

NOTE: Some authors represent the scalar triple product as, $[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}] \equiv \vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})$, $[\vec{\mathbf{b}}, \vec{\mathbf{c}}, \vec{\mathbf{a}}] \equiv \vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \wedge \vec{\mathbf{a}})$, $[\vec{\mathbf{c}}, \vec{\mathbf{a}}, \vec{\mathbf{b}}] \equiv \vec{\mathbf{c}} \cdot (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$. ■

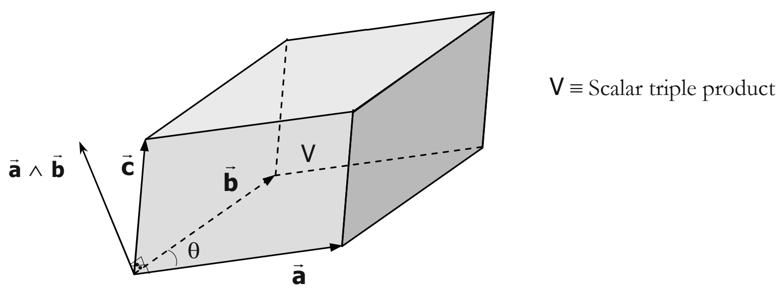


Figure 1.5: Scalar triple product.

Vector Triple Product: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be vectors, we can define the *vector triple product* as $\vec{\mathbf{w}} = \vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}})$. Then, we can demonstrate that the following relationships to be true:

$$\begin{aligned} \vec{\mathbf{w}} = \vec{\mathbf{a}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{c}}) &= -\vec{\mathbf{c}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = \vec{\mathbf{c}} \wedge (\vec{\mathbf{b}} \wedge \vec{\mathbf{a}}) \\ &= (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} \end{aligned} \tag{1.22}$$

whereby it is clear that the result of the vector triple product is another vector $\vec{\mathbf{w}}$, belonging to the plane Π_1 formed by the vectors $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$, (see Figure 1.6).

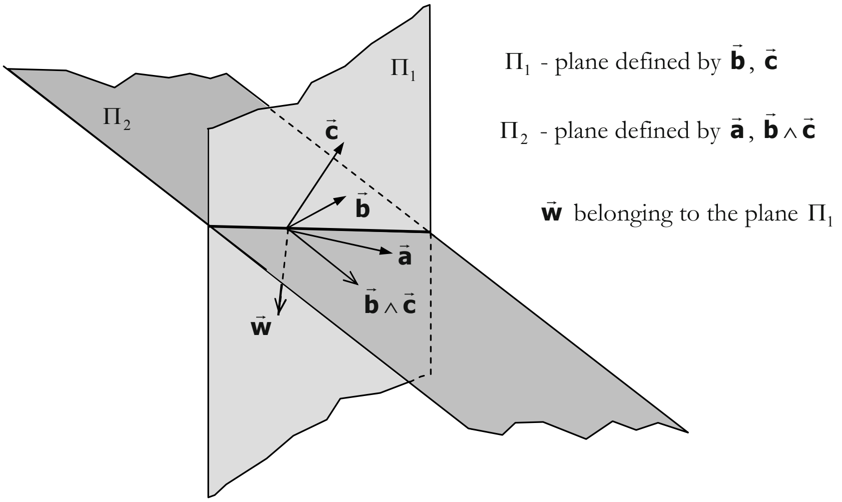


Figure 1.6: Vector triple product.

Problem 1.1: Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be arbitrary vectors. Prove that the following relationship is true:

$$\left(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right) \cdot \left(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right) = \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}\right)\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}\right) - \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)^2$$

Solution:

$$\begin{aligned}
\left(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right) \cdot \left(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right) &= \left\|\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}\right\|^2 \\
&= \left(\left\|\vec{\mathbf{a}}\right\| \left\|\vec{\mathbf{b}}\right\| \sin\theta\right)^2 \\
&= \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 \sin^2\theta \\
&= \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 \left(1 - \cos^2\theta\right) \\
&= \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 - \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 \cos^2\theta \\
&= \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 - \left(\left\|\vec{\mathbf{a}}\right\| \left\|\vec{\mathbf{b}}\right\| \cos\theta\right)^2 \\
&= \left\|\vec{\mathbf{a}}\right\|^2 \left\|\vec{\mathbf{b}}\right\|^2 - \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)^2 \\
&= \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{a}}\right)\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}\right) - \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}\right)^2
\end{aligned}$$

Linear Transformation

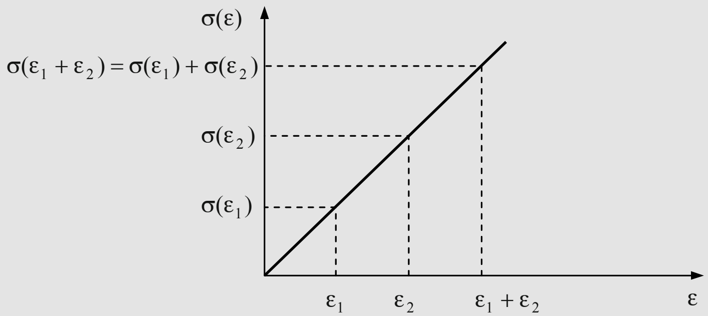
Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be arbitrary vectors, and α be a scalar, we can state F is a linear transformation if the following is true:

- $F(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = F(\vec{\mathbf{u}}) + F(\vec{\mathbf{v}})$
- $F(\alpha\vec{\mathbf{u}}) = \alpha F(\vec{\mathbf{u}})$

Problem 1.2: Given the following functions $\sigma(\varepsilon) = E\varepsilon$ and $\psi(\varepsilon) = \frac{1}{2}E\varepsilon^2$, demonstrate whether these functions show a linear transformation or not.

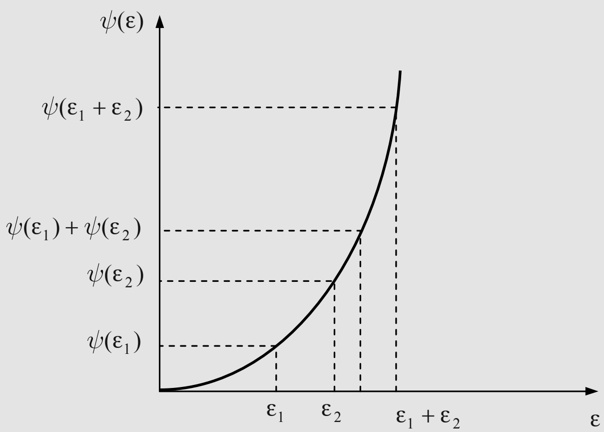
Solution:

$$\sigma(\varepsilon_1 + \varepsilon_2) = E[\varepsilon_1 + \varepsilon_2] = E\varepsilon_1 + E\varepsilon_2 = \sigma(\varepsilon_1) + \sigma(\varepsilon_2) \text{ (linear transformation)}$$



The function $\psi(\varepsilon) = \frac{1}{2}E\varepsilon^2$ does not show a linear transformation because the condition $\psi(\varepsilon_1 + \varepsilon_2) = \psi(\varepsilon_1) + \psi(\varepsilon_2)$ has not been satisfied:

$$\begin{aligned}\psi(\varepsilon_1 + \varepsilon_2) &= \frac{1}{2}E[\varepsilon_1 + \varepsilon_2]^2 = \frac{1}{2}E[\varepsilon_1^2 + 2\varepsilon_1\varepsilon_2 + \varepsilon_2^2] = \frac{1}{2}E\varepsilon_1^2 + \frac{1}{2}E\varepsilon_2^2 + \frac{1}{2}E2\varepsilon_1\varepsilon_2 \\ &= \psi(\varepsilon_1) + \psi(\varepsilon_2) + E\varepsilon_1\varepsilon_2 \neq \psi(\varepsilon_1) + \psi(\varepsilon_2)\end{aligned}$$



1.3 Coordinate Systems

A tensor, which has physical meanings, must be independent of the adopted coordinate system. Sometimes for reasons of convenience, we need to represent a tensor in a specific coordinate system, hence, we have the concept of tensor components, (see Figure 1.7).

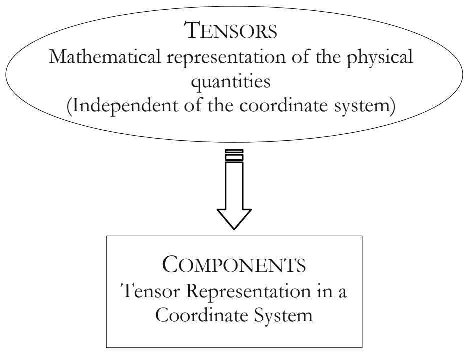


Figure 1.7: Tensor components.

Let $\vec{\mathbf{a}}$ be a first-order tensor (vector) as shown in Figure 1.8 (a), the tensor representation in a general coordinate system, defined as ξ_1, ξ_2, ξ_3, is made up of its components (a_1, a_2, a_3), (see Figure 1.8 (b)). Some examples of coordinate system are: the Cartesian coordinate system; the cylindrical coordinate system; and the spherical coordinate system.

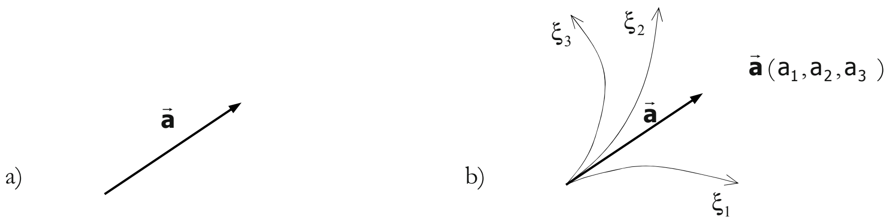


Figure 1.8: Vector representation in a general coordinate system.

1.3.1 Cartesian Coordinate System

The Cartesian coordinate system is defined by three unit vectors: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, denoted by the Cartesian basis, which make up an *orthonormal basis*. The orthonormal basis has the following properties:

1. The vectors that make up this basis are unit vectors:

$$\|\hat{\mathbf{i}}\| = \|\hat{\mathbf{j}}\| = \|\hat{\mathbf{k}}\| = 1 \tag{1.23}$$

or:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \tag{1.24}$$

2. The unit vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) are mutually orthogonal, *i.e.*:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0 \tag{1.25}$$

3. The vector product between the vectors ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) is the following:

$$\hat{\mathbf{i}} \wedge \hat{\mathbf{j}} = \hat{\mathbf{k}} \quad ; \quad \hat{\mathbf{j}} \wedge \hat{\mathbf{k}} = \hat{\mathbf{i}} \quad ; \quad \hat{\mathbf{k}} \wedge \hat{\mathbf{i}} = \hat{\mathbf{j}} \tag{1.26}$$

The direction and orientation of the orthonormal basis can be obtained using the right-hand rule as shown in Figure 1.9.

$$\hat{\mathbf{i}} \wedge \hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{j}} \wedge \hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{k}} \wedge \hat{\mathbf{i}} = \hat{\mathbf{j}} \tag{1.27}$$

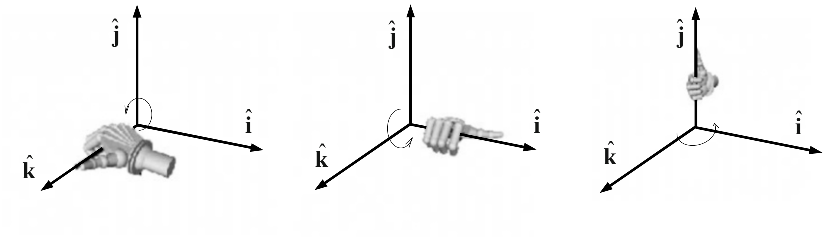


Figure 1.9: The right-hand rule.

1.3.2 Vector Representation in the Cartesian Coordinate System

The vector $\vec{\mathsf{a}}$, (see Figure 1.10), in the Cartesian coordinate system, is represented by its different components (a_x, a_y, a_z) and by the Cartesian bases ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$) as:

$$\vec{\mathsf{a}} = \mathsf{a}_x \hat{\mathbf{i}} + \mathsf{a}_y \hat{\mathbf{j}} + \mathsf{a}_z \hat{\mathbf{k}} \tag{1.28}$$

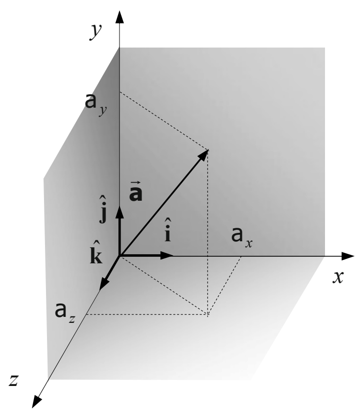


Figure 1.10: Cartesian coordinate system.

Let $\vec{\mathsf{a}}$, $\vec{\mathsf{b}}$, $\vec{\mathsf{c}}$ be arbitrary vectors, we can describe some vector operations in the Cartesian coordinate system, as follows:

- The **scalar product** $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$ becomes a scalar, which is defined in the Cartesian system as:

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})\cdot(b_x\hat{\mathbf{i}}+b_y\hat{\mathbf{j}}+b_z\hat{\mathbf{k}})=(a_xb_x+a_yb_y+a_zb_z) \tag{1.29}$$

Thus, it is true that $\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}=a_xa_x+a_ya_y+a_za_z=a_x^2+a_y^2+a_z^2=\|\vec{\mathbf{a}}\|^2$.

NOTE: The projection of a vector onto a given direction was established in the equation (1.10), thus defining the component concept. For example, if we want to know the vector component along the y-direction, all we need to do is calculate: $\vec{\mathbf{a}}\cdot\hat{\mathbf{j}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})\cdot(\hat{\mathbf{j}})=a_y$.■

- The **norm** of $\vec{\mathbf{a}}$ is:

$$\|\vec{\mathbf{a}}\|=\sqrt{a_x^2+a_y^2+a_z^2} \tag{1.30}$$

- Then, the **unit vector** codirectional with $\vec{\mathbf{a}}$ is:

$$\hat{\mathbf{a}}=\frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|}=\frac{a_x}{\sqrt{a_x^2+a_y^2+a_z^2}}\hat{\mathbf{i}}+\frac{a_y}{\sqrt{a_x^2+a_y^2+a_z^2}}\hat{\mathbf{j}}+\frac{a_z}{\sqrt{a_x^2+a_y^2+a_z^2}}\hat{\mathbf{k}} \tag{1.31}$$

- The **zero vector** is:

$$\vec{\mathbf{0}}=0\,\hat{\mathbf{i}}+0\,\hat{\mathbf{j}}+0\,\hat{\mathbf{k}} \tag{1.32}$$

- **Addition:** The vector sum of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is represented by:

$$\vec{\mathbf{a}}+\vec{\mathbf{b}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})+(b_x\hat{\mathbf{i}}+b_y\hat{\mathbf{j}}+b_z\hat{\mathbf{k}})=(a_x+b_x)\hat{\mathbf{i}}+(a_y+b_y)\hat{\mathbf{j}}+(a_z+b_z)\hat{\mathbf{k}} \tag{1.33}$$

- **Subtraction:** The difference between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is:

$$\vec{\mathbf{a}}-\vec{\mathbf{b}}=(a_x\hat{\mathbf{i}}+a_y\hat{\mathbf{j}}+a_z\hat{\mathbf{k}})-(b_x\hat{\mathbf{i}}+b_y\hat{\mathbf{j}}+b_z\hat{\mathbf{k}})=(a_x-b_x)\hat{\mathbf{i}}+(a_y-b_y)\hat{\mathbf{j}}+(a_z-b_z)\hat{\mathbf{k}} \tag{1.34}$$

- **Scalar multiplication:** The resulting vector defined by $\lambda\vec{\mathbf{a}}$ is:

$$\lambda\vec{\mathbf{a}}=\lambda a_x\hat{\mathbf{i}}+\lambda a_y\hat{\mathbf{j}}+\lambda a_z\hat{\mathbf{k}} \tag{1.35}$$

- The **vector product** $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})$ is evaluated as:

$$\begin{aligned}\vec{\mathbf{c}}=\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}&=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\hat{\mathbf{i}}-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\hat{\mathbf{j}}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\hat{\mathbf{k}}\\ &=(a_yb_z-a_zb_y)\hat{\mathbf{i}}-(a_xb_z-a_zb_x)\hat{\mathbf{j}}+(a_xb_y-a_yb_x)\hat{\mathbf{k}}\end{aligned} \tag{1.36}$$

where the symbol $|\bullet|\equiv\det(\bullet)$ denotes the matrix determinant.

- The **scalar triple product** $[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}]$ is the determinant of the 3 by 3 matrix, defined as:

$$\begin{aligned}
\mathsf{V}(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}) &= \vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\cdot\left(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}\right) = \vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \\
&= a_x\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \\
&= a_x\left(b_y c_z - b_z c_y\right) - a_y\left(b_x c_z - b_z c_x\right) + a_z\left(b_x c_y - b_y c_x\right)
\end{aligned} \tag{1.37}$$

- The **vector triple product** made up of the vectors ($\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$) is obtained, in the Cartesian coordinate system, as:

$$\begin{aligned}
\vec{\mathbf{a}}\wedge\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right) &= \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)\vec{\mathbf{b}} - \left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)\vec{\mathbf{c}} \\
&= \left(\lambda_1 b_x - \lambda_2 c_x\right)\hat{\mathbf{i}} + \left(\lambda_1 b_y - \lambda_2 c_y\right)\hat{\mathbf{j}} + \left(\lambda_1 b_z - \lambda_2 c_z\right)\hat{\mathbf{k}}
\end{aligned} \tag{1.38}$$

where $\lambda_1 = \vec{\mathbf{a}}\cdot\vec{\mathbf{c}} = a_x c_x + a_y c_y + a_z c_z$, and $\lambda_2 = \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z$.

Problem 1.3: Consider the points: $A(1,3,1)$, $B(2,-1,1)$, $C(0,1,3)$ and $D(1,2,4)$, defined in the Cartesian coordinate system.

1) Find the parallelogram area defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$; 2) Find the volume of the parallelepiped defined by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$; 3) Find the projection vector of $\overrightarrow{AB}$ onto $\overrightarrow{BC}$.

Solution:

1) Firstly we calculate the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\vec{\mathbf{a}} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(2\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) - \left(1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = 1\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$$\vec{\mathbf{b}} = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right) - \left(1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = -1\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

With reference to the equation (1.36) we can evaluate the vector product as follows:

$$\vec{\mathbf{a}}\wedge\vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -4 & 0 \\ -1 & -2 & 2 \end{vmatrix} = (-8)\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + (-6)\hat{\mathbf{k}}$$

Then, the parallelogram area can be obtained using definition (1.19), thus:

$$\mathsf{A} = \left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\| = \sqrt{(-8)^2 + (-2)^2 + (-6)^2} = \sqrt{104}$$

2) Next, we can evaluate the vector $\overrightarrow{AD}$ as:

$$\vec{\mathbf{c}} = \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(1\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right) - \left(1\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = 0\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

and using the equation (1.37) we can obtain the volume of the parallelepiped:

$$\begin{aligned}
\mathsf{V}(\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}) = \left\|\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)\right\| &= \left\|\left(0\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right)\cdot\left(-8\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}\right)\right\| \\
&= \left\|0 + 2 - 18\right\| = 16
\end{aligned}$$

3) The $\overrightarrow{BC}$ vector can be calculated as:

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \left(0\hat{\mathbf{i}} + 1\hat{\mathbf{j}} + 3\hat{\mathbf{k}}\right) - \left(2\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 1\hat{\mathbf{k}}\right) = -2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Hence, it is possible to evaluate the projection vector of $\vec{AB}$ onto $\vec{BC}$, (see equation (1.12)), as:

$$\overrightarrow{\mathbf{proj}}_{\vec{BC}}\,\vec{AB} = \frac{\vec{BC}\cdot\vec{AB}}{\underbrace{\vec{BC}\cdot\vec{BC}}_{\|\vec{BC}\|^2}}\vec{BC} = \frac{\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)\cdot\left(1\hat{\mathbf{i}}-4\hat{\mathbf{j}}+0\hat{\mathbf{k}}\right)}{\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)\cdot\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)}\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right)$$

$$= \frac{(-2-8+0)}{(4+4+4)}\left(-2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+2\hat{\mathbf{k}}\right) = \frac{5}{3}\hat{\mathbf{i}} - \frac{5}{3}\hat{\mathbf{j}} - \frac{5}{3}\hat{\mathbf{k}}$$

1.3.3 Einstein Summation Convention (Einstein Notation)

As we saw in equation (1.28) $\vec{\mathbf{a}}$ in the Cartesian coordinate system was defined as:

$$\vec{\mathbf{a}} = \mathsf{a}_x\hat{\mathbf{i}} + \mathsf{a}_y\hat{\mathbf{j}} + \mathsf{a}_z\hat{\mathbf{k}} \tag{1.39}$$

Said expression can be rewritten as:

$$\vec{\mathbf{a}} = \mathsf{a}_1\hat{\mathbf{e}}_1 + \mathsf{a}_2\hat{\mathbf{e}}_2 + \mathsf{a}_3\hat{\mathbf{e}}_3 \tag{1.40}$$

where we have considered that: $\mathsf{a}_1 \equiv \mathsf{a}_x$, $\mathsf{a}_2 \equiv \mathsf{a}_y$, $\mathsf{a}_3 \equiv \mathsf{a}_z$, $\hat{\mathbf{e}}_1 \equiv \hat{\mathbf{i}}$, $\hat{\mathbf{e}}_2 \equiv \hat{\mathbf{j}}$, $\hat{\mathbf{e}}_3 \equiv \hat{\mathbf{k}}$, (see Figure 1.11). In this way we can express equation (1.40) by means of the summation symbol as:

$$\vec{\mathbf{a}} = \mathsf{a}_1\hat{\mathbf{e}}_1 + \mathsf{a}_2\hat{\mathbf{e}}_2 + \mathsf{a}_3\hat{\mathbf{e}}_3 = \sum_{i=1}^{3}\mathsf{a}_i\hat{\mathbf{e}}_i \tag{1.41}$$

Then, we introduce the *summation convention*, according to which the "repeated indices" indicate summation. So, equation (1.41) can be represented as follows:

$$\vec{\mathbf{a}} = \mathsf{a}_1\hat{\mathbf{e}}_1 + \mathsf{a}_2\hat{\mathbf{e}}_2 + \mathsf{a}_3\hat{\mathbf{e}}_3 = \mathsf{a}_i\hat{\mathbf{e}}_i \qquad (i=1,2,3)$$
$$\boxed{\vec{\mathbf{a}} = \mathsf{a}_i\hat{\mathbf{e}}_i \qquad (i=1,2,3)} \tag{1.42}$$

NOTE: The summation notation was introduced by Albert Einstein in 1916, which led to the indicial notation. ■

1.4 Indicial Notation

Using indicial notation, the three axes of the coordinate system are designated by the letter x with a subscript. So, x_i is not a single value but i values, *i.e.* x_1, x_2, x_3 (if $i=1,2,3$) where these values x_1, x_2, x_3 correspond to the axes x, y, z, respectively.

Let $\vec{\mathbf{a}}$ be a vector represented in the Cartesian coordinate system as:

$$\vec{\mathbf{a}} = \mathsf{a}_1\hat{\mathbf{e}}_1 + \mathsf{a}_2\hat{\mathbf{e}}_2 + \mathsf{a}_3\hat{\mathbf{e}}_3 \tag{1.43}$$

where the orthonormal basis is represented by $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$, (see Figure 1.11), and a_1, a_2, a_3 are the vector components. In indicial notation the vector components are represented by a_i. If the range of the subscript is not indicated, we assume that 1,2,3 show these values. Therefore, the vector components are represented as:

$$(\vec{\mathbf{a}})_i = \mathsf{a}_i = \begin{bmatrix} \mathsf{a}_1 \\ \mathsf{a}_2 \\ \mathsf{a}_3 \end{bmatrix} \tag{1.44}$$

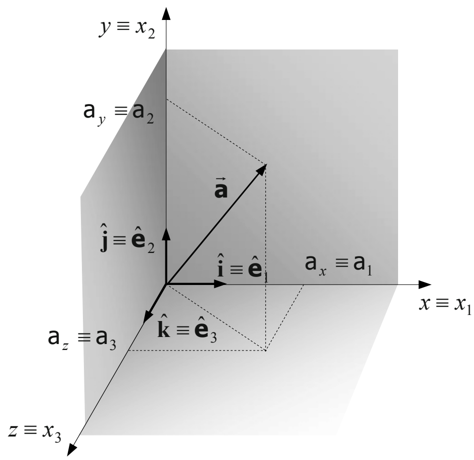


Figure 1.11: Vector representation in the Cartesian coordinate system.

Unit vector components: Let $\vec{\mathbf{a}}$ be a vector, the normalized vector $\hat{\mathbf{a}}$ is defined as:

$$\hat{\mathbf{a}} = \frac{\vec{\mathbf{a}}}{\|\vec{\mathbf{a}}\|} \quad \text{with} \quad \|\hat{\mathbf{a}}\| = 1 \tag{1.45}$$

whose components are:

$$\hat{\mathsf{a}}_i = \frac{\mathsf{a}_i}{\sqrt{\mathsf{a}_1^2 + \mathsf{a}_2^2 + \mathsf{a}_3^2}} = \frac{\mathsf{a}_i}{\sqrt{\mathsf{a}_j \mathsf{a}_j}} = \frac{\mathsf{a}_i}{\sqrt{\mathsf{a}_k \mathsf{a}_k}} \qquad (i,j,k = 1,2,3) \tag{1.46}$$

In light of the previous equation we can emphasize two types of indices:

The *free index (live index)* is that which only appears once in a term of the expression. In the above equation the free index is the (i). The number of the free index indicates the tensor order.

The *dummy index (summation index)* is that which is repeated only twice in a term of the expression, and indicates summation. In the above equation (1.46) the dummy index is the (j), or the (k) index.

OBS.: An index in a term of an expression can only appear once or twice. If it appears more times, then a large error has occurred.

Scalar product: Using definitions (1.4) and (1.29), we can express the scalar product ($\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$) as follows:

$$\gamma=\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}=\|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|\cos\theta=\mathrm{a}_1\mathrm{b}_1+\mathrm{a}_2\mathrm{b}_2+\mathrm{a}_3\mathrm{b}_3=\mathrm{a}_i\mathrm{b}_i=\mathrm{a}_j\mathrm{b}_j \qquad (i,j=1,2,3) \tag{1.47}$$

Problem 1.4: Rewrite the following equations using indicial notation:

1) $a_1x_1x_3+a_2x_2x_3+a_3x_3x_3$

Solution: $a_ix_ix_3$ $\quad(i=1,2,3)$

2) $x_1x_1+x_2x_2$

Solution: x_ix_i $\quad(i=1,2)$

3) $\begin{cases} a_{11}x+a_{12}y+a_{13}z=b_x \\ a_{21}x+a_{22}y+a_{23}z=b_y \\ a_{31}x+a_{32}y+a_{33}z=b_z \end{cases}$

Solution:

$$\begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3 \end{cases} \xrightarrow[\text{index } j]{\text{dummy}} \begin{cases} a_{1j}x_j=b_1 \\ a_{2j}x_j=b_2 \\ a_{3j}x_j=b_3 \end{cases} \xrightarrow[\text{index } i]{\text{free}} \boxed{a_{ij}x_j=b_i}$$

As we can appreciate in this problem, the use of the indicial notation means that the equation becomes very concise. In many cases, if algebraic operation do not use indicial or tensorial notation they become almost impossible to deal with due to the large number of terms involved.

Problem 1.5: Expand the equation: $A_{ij}x_ix_j \qquad (i,j=1,2,3)$

Solution: The indices i,j are dummy indices, and indicate index summation and there is no free index in the expression $A_{ij}x_ix_j$, therefore the result is a scalar. So, we expand first the dummy index i and later the index j to obtain:

$$A_{ij}x_ix_j \xrightarrow{\text{expanding } i} \underbrace{A_{1j}x_1x_j}_{} + \underbrace{A_{2j}x_2x_j}_{} + \underbrace{A_{3j}x_3x_j}_{}$$

$$\text{expanding } j \downarrow \quad \begin{matrix} A_{11}x_1x_1 & A_{21}x_2x_1 & A_{31}x_3x_1 \\ + & + & + \\ A_{12}x_1x_2 & A_{22}x_2x_2 & A_{32}x_3x_2 \\ + & + & + \\ A_{13}x_1x_3 & A_{23}x_2x_3 & A_{33}x_3x_3 \end{matrix}$$

Rearranging the terms we obtain:

$$A_{ij}x_ix_j=A_{11}x_1x_1+A_{12}x_1x_2+A_{13}x_1x_3+A_{21}x_2x_1+A_{22}x_2x_2+ \\ A_{23}x_2x_3+A_{31}x_3x_1+A_{32}x_3x_2+A_{33}x_3x_3$$

1.4.1 Some Operators

1.4.1.1 Kronecker Delta

The *Kronecker delta* δ_{ij} is defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \textit{iff} \quad i = j \\ \\ 0 & \textit{iff} \quad i \neq j \end{cases} \tag{1.48}$$

Also note that the scalar product of the orthonormal basis $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ is equal to 1 if $i = j$ and equal to 0 if $i \neq j$. Hence, $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$ can be expressed in matrix form as:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_{ij} \tag{1.49}$$

An interesting property of the Kronecker delta is shown in the following example. Let V_i be the components of the vector $\vec{V}$, therefore:

$$\delta_{ij} V_i = \delta_{1j} V_1 + \delta_{2j} V_2 + \delta_{3j} V_3 \tag{1.50}$$

As $(j = 1,2,3)$ is a free index, we have three values to be calculated, namely:

$$\left. \begin{aligned} j = 1 \Rightarrow \delta_{ij} V_i = \delta_{11} V_1 + \delta_{21} V_2 + \delta_{31} V_3 = V_1 \\ j = 2 \Rightarrow \delta_{ij} V_i = \delta_{12} V_1 + \delta_{22} V_2 + \delta_{32} V_3 = V_2 \\ j = 3 \Rightarrow \delta_{ij} V_i = \delta_{13} V_1 + \delta_{23} V_2 + \delta_{33} V_3 = V_3 \end{aligned} \right\} \Rightarrow \delta_{ij} V_i = V_j \tag{1.51}$$

That is, in the presence of the Kronecker delta symbol we replace the repeated index as follows:

$$\delta_{ij} V_i = V_j \tag{1.52}$$

For this reason, the Kronecker delta is often called the *substitution operator*.

Other examples using the Kronecker delta are presented below:

$$\delta_{ij} A_{ik} = A_{jk}, \; \delta_{ij} \delta_{ji} = \delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3, \; \delta_{ji} a_{ji} = a_{ii} = a_{11} + a_{22} + a_{33} \tag{1.53}$$

To obtain the components of the vector $\vec{\mathbf{a}}$ in the coordinate system represented by $\hat{\mathbf{e}}_i$, it is sufficient to obtain the scalar product with $\vec{\mathbf{a}}$ and $\hat{\mathbf{e}}_i$, *i.e.* $\vec{\mathbf{a}} \cdot \hat{\mathbf{e}}_i = \mathrm{a}_p \hat{\mathbf{e}}_p \cdot \hat{\mathbf{e}}_i = \mathrm{a}_p \delta_{pi} = \mathrm{a}_i$. With that, it is also possible to represent the vector as:

$$\vec{\mathbf{a}} = \mathrm{a}_i \hat{\mathbf{e}}_i = (\vec{\mathbf{a}} \cdot \hat{\mathbf{e}}_i) \hat{\mathbf{e}}_i \tag{1.54}$$

Problem 1.6: Solve the following equations:

1) $\delta_{ii} \delta_{jj}$

Solution: $\delta_{ii} \delta_{jj} = (\delta_{11} + \delta_{22} + \delta_{33})(\delta_{11} + \delta_{22} + \delta_{33}) = 3 \times 3 = 9$

2) $\delta_{\alpha 1} \delta_{\alpha \gamma} \delta_{\gamma 1}$

Solution: $\delta_{\alpha 1} \delta_{\alpha \gamma} \delta_{\gamma 1} = \delta_{\gamma 1} \delta_{\gamma 1} = \delta_{11} = 1$

NOTE: Note that the following algebraic operation is incorrect $\delta_{\gamma 1} \delta_{\gamma 1} \neq \delta_{\gamma\gamma} = 3 \neq \delta_{11} = 1$, since what must be replaced is the repeated index, not the number ■

1.4.1.2 Permutation Symbol

The *permutation symbol* ϵ_{ijk} (also known as *Levi-Civita symbol* or *alternating symbol)* is defined as:

$$\epsilon_{ijk} = \begin{cases} +1 & if \quad (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \\ -1 & if \quad (i,j,k) \in \{(1,3,2),(3,2,1),(2,1,3)\} \\ 0 & for\ the\ remaining\ cases\ \ i.e.: if\ \ (i=j)\ \ or\ \ (j=k)\ \ or\ \ (i=k) \end{cases} \tag{1.55}$$

NOTE: ϵ_{ijk} are the components of the *Levi-Civita pseudo-tensor*, which will be introduced later on. ■

The values of ϵ_{ijk} can be easily memorized using the mnemonic device shown in Figure 1.12(a), in which if the index values are arranged in a clockwise direction, the value of ϵ_{ijk} is equal to 1, if not it has the value of -1. In the same way we can use this mnemonic device to switch indices, (see Figure 1.12(b)).

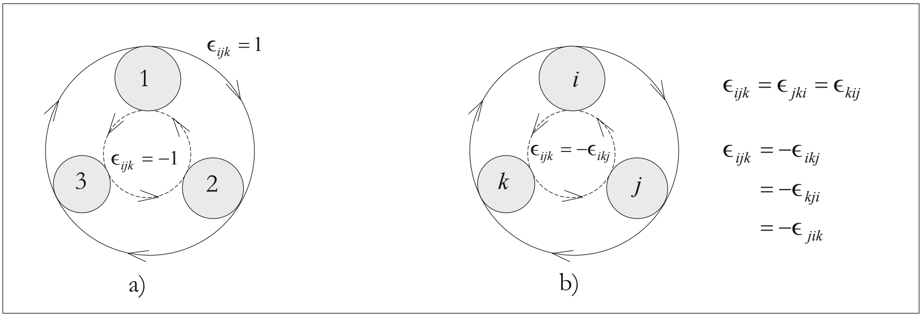


Figure 1.12: Mnemonic device for the permutation symbol.

Another way to express the permutation symbol is by means of its indices:

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i) \tag{1.56}$$

Using both the definition seen in (1.55) and Figure 1.12 (b), it is possible to verify that the following relations are valid:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{jki} = \epsilon_{kij} \\ \epsilon_{ijk} &= -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji} \end{aligned} \tag{1.57}$$

Using the Kronecker delta property, we can state that:

$$\begin{aligned} \epsilon_{ijk} &= \epsilon_{lmn}\delta_{li}\delta_{mj}\delta_{nk} \\ &= \delta_{1i}\delta_{2j}\delta_{3k} - \delta_{1i}\delta_{3j}\delta_{2k} - \delta_{2i}\delta_{1j}\delta_{3k} + \delta_{3i}\delta_{1j}\delta_{2k} + \delta_{2i}\delta_{3j}\delta_{1k} - \delta_{3i}\delta_{2j}\delta_{1k} \\ &= \delta_{1i}\left(\delta_{2j}\delta_{3k} - \delta_{3j}\delta_{2k}\right) - \delta_{1j}\left(\delta_{2i}\delta_{3k} - \delta_{3i}\delta_{2k}\right) + \delta_{1k}\left(\delta_{2i}\delta_{3j} - \delta_{3i}\delta_{2j}\right) \end{aligned} \tag{1.58}$$

The above equation can be represented by means of the following determinant:

$$\epsilon_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \tag{1.59}$$

After which, the term $\epsilon_{ijk}\epsilon_{pqr}$ can be evaluated as follows:

$$\epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \begin{vmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{vmatrix} \tag{1.60}$$

Taking into account that $\det(\mathcal{AB}) = \det(\mathcal{A})\det(\mathcal{B})$, where $\det(\bullet) \equiv |\bullet|$ is the determinant of the matrix $\bullet$, the equation (1.60) can be rewritten as:

$$\epsilon_{ijk}\epsilon_{pqr} = \left| \begin{bmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{bmatrix} \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} \right| \Rightarrow \quad \epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix} \tag{1.61}$$

The term δ_{ip} was obtained by means of the operation $\delta_{1i}\delta_{1p} + \delta_{2i}\delta_{2p} + \delta_{3i}\delta_{3p} = \delta_{mi}\delta_{mp}$ and $\delta_{mi}\delta_{mp} = \delta_{ip}$, the other terms were obtained in a similar fashion.

For the special exception when $r = k$, the equation (1.61) is reduced to:

$$\epsilon_{ijk}\epsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & 3 \end{vmatrix} \quad \Rightarrow \quad \boxed{\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \qquad i,j,k,p,q = 1,2,3} \tag{1.62}$$

Problem 1.7: a) Prove the following is true $\epsilon_{ijk}\epsilon_{pjk} = 2\delta_{ip}$ and $\epsilon_{ijk}\epsilon_{ijk} = 6$. b) Obtain the numerical value of $\epsilon_{ijk}\delta_{2j}\delta_{3k}\delta_{1i}$.

Solution: a) Using the equation in (1.62), *i.e.* $\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$, and by substituting q for j, we obtain:

$$\epsilon_{ijk}\epsilon_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = \delta_{ip}3 - \delta_{ip} = 2\delta_{ip}$$

Based on the above result, it is straight forward to check that:

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} = 6$$

b) $\epsilon_{ijk}\delta_{2j}\delta_{3k}\delta_{1i} = \epsilon_{123} = 1$

- The **vector product** of two vectors ($\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}$) leads to a new vector $\vec{\mathbf{c}}$, defined in (1.36), and the components of $\vec{\mathbf{c}}$, in Cartesian system, are given by:

$$\begin{aligned} \vec{\mathbf{c}} = \vec{\mathbf{a}} \wedge \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \underbrace{(a_2 b_3 - a_3 b_2)}_{c_1}\hat{\mathbf{e}}_1 + \underbrace{(a_3 b_1 - a_1 b_3)}_{c_2}\hat{\mathbf{e}}_2 + \underbrace{(a_1 b_2 - a_2 b_1)}_{c_3}\hat{\mathbf{e}}_3 \end{aligned} \tag{1.63}$$

Using the definition of the permutation symbol ϵ_{ijk}, defined in (1.55), we can express the components of $\vec{\mathbf{c}}$ as follows:

$$\left.\begin{aligned} c_1 &= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2 = \epsilon_{1jk} a_j b_k \\ c_2 &= \epsilon_{231} a_3 b_1 + \epsilon_{213} a_1 b_3 = \epsilon_{2jk} a_j b_k \\ c_3 &= \epsilon_{312} a_1 b_2 + \epsilon_{321} a_2 b_1 = \epsilon_{3jk} a_j b_k \end{aligned}\right\} \Rightarrow c_i = \epsilon_{ijk} a_j b_k \tag{1.64}$$

Then, the vector product ($\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}$) can be represented by means of the permutation symbol as:

$$\begin{aligned} \vec{\mathbf{a}} \wedge \vec{\mathbf{b}} &= \epsilon_{ijk} a_j b_k \hat{\mathbf{e}}_i \\ a_j \hat{\mathbf{e}}_j \wedge b_k \hat{\mathbf{e}}_k &= a_j b_k \epsilon_{ijk} \hat{\mathbf{e}}_i \\ a_j b_k (\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k) &= a_j b_k \epsilon_{ijk} \hat{\mathbf{e}}_i = a_j b_k \epsilon_{jki} \hat{\mathbf{e}}_i \end{aligned} \tag{1.65}$$

Therefore, we can also conclude that the following relationship is valid:

$$\boxed{(\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k) = \epsilon_{ijk}\hat{\mathbf{e}}_i} \tag{1.66}$$

The permutation symbol and the orthonormal basis can be interrelated using the triple scalar product as follows:

$$\hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j = \epsilon_{ijm}\hat{\mathbf{e}}_m \quad \Rightarrow \quad (\hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j)\cdot\hat{\mathbf{e}}_k = \epsilon_{ijm}\hat{\mathbf{e}}_m\cdot\hat{\mathbf{e}}_k = \epsilon_{ijm}\delta_{mk} \Rightarrow (\hat{\mathbf{e}}_i \wedge \hat{\mathbf{e}}_j)\cdot\hat{\mathbf{e}}_k = \epsilon_{ijk} \tag{1.67}$$

- The **triple scalar product** made up of the vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ is expressed by:

$$\lambda = \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = \mathrm{a}_i\hat{\mathbf{e}}_i\cdot(\mathrm{b}_j\hat{\mathbf{e}}_j\wedge\mathrm{c}_k\hat{\mathbf{e}}_k) = \mathrm{a}_i\mathrm{b}_j\mathrm{c}_k\hat{\mathbf{e}}_i\cdot(\hat{\mathbf{e}}_j\wedge\hat{\mathbf{e}}_k) = \epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k \tag{1.68}$$

$$\boxed{\lambda = \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = \epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k \qquad (i,j,k=1,2,3)} \tag{1.69}$$

or

$$\lambda = \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = \vec{\mathbf{b}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}) = \vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \begin{vmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \\ \mathrm{c}_1 & \mathrm{c}_2 & \mathrm{c}_3 \end{vmatrix} \tag{1.70}$$

Starting from the equation (1.69) we can prove the following are true:
$\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = \vec{\mathbf{b}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}) = \vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})$:

$$\begin{aligned}
[\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}] \equiv \vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) &= \epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k \\
&= \epsilon_{jki}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = \vec{\mathbf{b}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{a}}) \equiv [\vec{\mathbf{b}},\vec{\mathbf{c}},\vec{\mathbf{a}}] \\
&= \epsilon_{kij}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = \vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) \equiv [\vec{\mathbf{c}},\vec{\mathbf{a}},\vec{\mathbf{b}}] \\
&= -\epsilon_{ikj}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = -\vec{\mathbf{a}}\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{b}}) \equiv -[\vec{\mathbf{a}},\vec{\mathbf{c}},\vec{\mathbf{b}}] \\
&= -\epsilon_{jik}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = -\vec{\mathbf{b}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{c}}) \equiv -[\vec{\mathbf{b}},\vec{\mathbf{a}},\vec{\mathbf{c}}] \\
&= -\epsilon_{kji}\mathrm{a}_i\mathrm{b}_j\mathrm{c}_k = -\vec{\mathbf{c}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}) \equiv -[\vec{\mathbf{c}},\vec{\mathbf{b}},\vec{\mathbf{a}}]
\end{aligned} \tag{1.71}$$

where we take into account the property of the permutation symbol as given in (1.57).

Problem 1.8: Rewrite the expression $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})$ without using the vector product symbol.

Solution: The vector product $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})$ can be expressed as
$(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \mathrm{a}_j\hat{\mathbf{e}}_j\wedge\mathrm{b}_k\hat{\mathbf{e}}_k = \epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\hat{\mathbf{e}}_i$. Likewise, it is possible to express $(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})$ as
$(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}) = \epsilon_{nlm}\mathrm{c}_l\mathrm{d}_m\hat{\mathbf{e}}_n$, thus:

$$\begin{aligned}
(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}) &= (\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\hat{\mathbf{e}}_i)\cdot(\epsilon_{nlm}\mathrm{c}_l\mathrm{d}_m\hat{\mathbf{e}}_n) = \epsilon_{ijk}\epsilon_{nlm}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_n \\
&= \epsilon_{ijk}\epsilon_{nlm}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m\delta_{in} = \epsilon_{ijk}\epsilon_{ilm}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m
\end{aligned}$$

Taking into account that $\epsilon_{ijk}\epsilon_{ilm} = \epsilon_{jki}\epsilon_{lmi}$ (see equation (1.57)) and by applying the equation (1.62), *i.e.:* $\epsilon_{jki}\epsilon_{lmi} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} = \epsilon_{jki}\epsilon_{ilm}$, we obtain:

$$\epsilon_{ijk}\epsilon_{ilm}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\mathrm{a}_j\mathrm{b}_k\mathrm{c}_l\mathrm{d}_m = \mathrm{a}_l\mathrm{b}_m\mathrm{c}_l\mathrm{d}_m - \mathrm{a}_m\mathrm{b}_l\mathrm{c}_l\mathrm{d}_m$$

Since $\mathrm{a}_l\mathrm{c}_l = (\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})$ and $\mathrm{b}_m\mathrm{d}_m = (\vec{\mathbf{b}}\cdot\vec{\mathbf{d}})$ holds true, we can conclude that:

$$(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}}) = (\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{d}}) - (\vec{\mathbf{a}}\cdot\vec{\mathbf{d}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}})$$

Therefore, it is also valid when $\vec{\mathbf{a}} = \vec{\mathbf{c}}$ and $\vec{\mathbf{b}} = \vec{\mathbf{d}}$, thus:

$$(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})=\left\|\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right\|^2=(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}})-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{a}})=\left\|\vec{\mathbf{a}}\right\|^2\left\|\vec{\mathbf{b}}\right\|^2-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2$$

which is the same equation obtained in **Problem 1.1**.

Problem 1.9: Prove that $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\wedge(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})=\vec{\mathbf{c}}[\vec{\mathbf{d}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})]-\vec{\mathbf{d}}[\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})]$

Solution: Expressing the correct equality term in indicial notation we obtain:

$$\left\{\vec{\mathbf{c}}[\vec{\mathbf{d}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})]-\vec{\mathbf{d}}[\vec{\mathbf{c}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})]\right\}_p=\mathrm{c}_p[\mathrm{d}_i(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)]-\mathrm{d}_p[\mathrm{c}_i(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)]$$

$$\Rightarrow\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_p\mathrm{d}_i-\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k\mathrm{c}_i\mathrm{d}_p\quad\Rightarrow\quad\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k(\mathrm{c}_p\mathrm{d}_i-\mathrm{c}_i\mathrm{d}_p)$$

Using the Kronecker delta the above equation becomes:

$$\Rightarrow\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k(\delta_{pm}\mathrm{c}_m\mathrm{d}_n\delta_{ni}-\delta_{im}\mathrm{c}_m\mathrm{d}_n\delta_{np})\quad\Rightarrow\quad(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)\mathrm{c}_m\mathrm{d}_n(\delta_{pm}\delta_{ni}-\delta_{im}\delta_{np})$$

and by applying the equation $\delta_{pm}\delta_{ni}-\delta_{im}\delta_{np}=\epsilon_{pil}\epsilon_{mnl}$, (see eq. (1.62)), the above equation can be rewritten as follows:

$$\Rightarrow(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)\mathrm{c}_m\mathrm{d}_n(\epsilon_{pil}\epsilon_{mnl})\quad\Rightarrow\quad\epsilon_{pil}[(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)(\epsilon_{mnl}\mathrm{c}_m\mathrm{d}_n)]$$

Since $\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k$ and $\epsilon_{mnl}\mathrm{c}_m\mathrm{d}_n$ represent the components of $(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})$ and $(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})$, respectively, we can conclude that:

$$\epsilon_{pil}[(\epsilon_{ijk}\mathrm{a}_j\mathrm{b}_k)(\epsilon_{mnl}\mathrm{c}_m\mathrm{d}_n)]=[(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\wedge(\vec{\mathbf{c}}\wedge\vec{\mathbf{d}})]_p$$

Problem 1.10: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be linearly independent vectors, and $\vec{\mathbf{v}}$ be a vector, demonstrate that:

$$\vec{\mathbf{v}}=\alpha\vec{\mathbf{a}}+\beta\vec{\mathbf{b}}+\gamma\vec{\mathbf{c}}\neq\vec{\mathbf{0}}$$

where the scalars α, β, γ are given by:

$$\alpha=\frac{\epsilon_{ijk}\mathrm{v}_i\mathrm{b}_j\mathrm{c}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r}\quad;\quad\beta=\frac{\epsilon_{ijk}\mathrm{a}_i\mathrm{v}_j\mathrm{c}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r}\quad;\quad\gamma=\frac{\epsilon_{ijk}\mathrm{a}_i\mathrm{b}_j\mathrm{v}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r}$$

Solution: The scalar product made up of $\vec{\mathbf{v}}$ and $(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})$ becomes:

$$\vec{\mathbf{v}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=\alpha\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})+\beta\underbrace{\vec{\mathbf{b}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})}_{=0}+\gamma\underbrace{\vec{\mathbf{c}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})}_{=0}\quad\Rightarrow\quad\alpha=\frac{\vec{\mathbf{v}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})}{\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})}$$

which is the same as:

$$\alpha=\frac{\begin{vmatrix}\mathrm{v}_1&\mathrm{v}_2&\mathrm{v}_3\\\mathrm{b}_1&\mathrm{b}_2&\mathrm{b}_3\\\mathrm{c}_1&\mathrm{c}_2&\mathrm{c}_3\end{vmatrix}}{\begin{vmatrix}\mathrm{a}_1&\mathrm{a}_2&\mathrm{a}_3\\\mathrm{b}_1&\mathrm{b}_2&\mathrm{b}_3\\\mathrm{c}_1&\mathrm{c}_2&\mathrm{c}_3\end{vmatrix}}=\frac{\begin{vmatrix}\mathrm{v}_1&\mathrm{b}_1&\mathrm{c}_1\\\mathrm{v}_2&\mathrm{b}_2&\mathrm{c}_2\\\mathrm{v}_3&\mathrm{b}_3&\mathrm{c}_3\end{vmatrix}}{\begin{vmatrix}\mathrm{a}_1&\mathrm{b}_1&\mathrm{c}_1\\\mathrm{a}_2&\mathrm{b}_2&\mathrm{c}_2\\\mathrm{a}_3&\mathrm{b}_3&\mathrm{c}_3\end{vmatrix}}=\frac{\epsilon_{ijk}\mathrm{v}_i\mathrm{b}_j\mathrm{c}_k}{\epsilon_{pqr}\mathrm{a}_p\mathrm{b}_q\mathrm{c}_r}$$

One can obtain the parameters β and γ in a similar fashion.

Problem 1.11: Prove the relationship given in (1.38) is valid, *i.e.*:

$$\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\vec{\mathbf{b}}-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{c}}\,.$$

Solution: Taking into account that $(\vec{\mathbf{d}})_i=(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})_i=\epsilon_{ijk}\mathrm{b}_j\mathrm{c}_k$ and that $(\vec{\mathbf{a}}\wedge\vec{\mathbf{d}})_q=\epsilon_{qjk}\mathrm{b}_j\mathrm{c}_k$, we obtain:

$$
\begin{aligned}
\left[\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]_q &= \epsilon_{qsi}\mathsf{a}_s(\epsilon_{ijk}\mathsf{b}_j\mathsf{c}_k) = \epsilon_{qsi}\epsilon_{ijk}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k = \epsilon_{qsi}\epsilon_{jki}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k \\
&= (\delta_{qj}\delta_{sk}-\delta_{qk}\delta_{sj})\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k = \delta_{qj}\delta_{sk}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k - \delta_{qk}\delta_{sj}\mathsf{a}_s\mathsf{b}_j\mathsf{c}_k \\
&= \mathsf{a}_k\mathsf{b}_q\mathsf{c}_k - \mathsf{a}_j\mathsf{b}_j\mathsf{c}_q = \mathsf{b}_q(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}) - \mathsf{c}_q(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) \\
&\Rightarrow \left[\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]_q = \left[\vec{\mathbf{b}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}) - \vec{\mathbf{c}}(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\right]_q
\end{aligned}
$$

1.5 Algebraic Operations with Tensors

1.5.1 Dyadic

The *tensor product,* made up of two vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$, becomes a dyad, which is a particular case of a second-order tensor. The dyad is represented by:

$$\vec{\mathbf{u}}\vec{\mathbf{v}} \equiv \vec{\mathbf{u}}\otimes\vec{\mathbf{v}} = \mathbf{A} \tag{1.72}$$

where the operator $\otimes$ denotes the *tensor product*. Then, we define a *dyadic* as a linear combination of dyads. Furthermore, as we will see later, any tensor can be represented by means of a linear combination of dyads, (see Holzapfel (2000)).

The tensor product has the following properties:

1. $$(\vec{\mathbf{u}}\otimes\vec{\mathbf{v}})\cdot\vec{\mathbf{x}} = \vec{\mathbf{u}}(\vec{\mathbf{v}}\cdot\vec{\mathbf{x}}) \equiv \vec{\mathbf{u}}\otimes(\vec{\mathbf{v}}\cdot\vec{\mathbf{x}}) \tag{1.73}$$

2. $$\vec{\mathbf{u}}\otimes(\alpha\vec{\mathbf{v}}+\beta\vec{\mathbf{w}}) = \alpha\vec{\mathbf{u}}\otimes\vec{\mathbf{v}} + \beta\vec{\mathbf{u}}\otimes\vec{\mathbf{w}} \tag{1.74}$$

3. $$\begin{aligned}(\alpha\vec{\mathbf{v}}\otimes\vec{\mathbf{u}}+\beta\vec{\mathbf{w}}\otimes\vec{\mathbf{r}})\cdot\vec{\mathbf{x}} &= \alpha(\vec{\mathbf{v}}\otimes\vec{\mathbf{u}})\cdot\vec{\mathbf{x}} + \beta(\vec{\mathbf{w}}\otimes\vec{\mathbf{r}})\cdot\vec{\mathbf{x}} \\ &= \alpha\left[\vec{\mathbf{v}}\otimes(\vec{\mathbf{u}}\cdot\vec{\mathbf{x}})\right] + \beta\left[\vec{\mathbf{w}}\otimes(\vec{\mathbf{r}}\cdot\vec{\mathbf{x}})\right]\end{aligned} \tag{1.75}$$

where α and β are scalars. By definition, the dyad does not contain the commutative property, *i.e.*, $\vec{\mathbf{u}}\otimes\vec{\mathbf{v}} \neq \vec{\mathbf{v}}\otimes\vec{\mathbf{u}}$.

The equation (1.72) can also be expressed in the Cartesian system as:

$$
\begin{aligned}
\mathbf{A} = \vec{\mathbf{u}}\otimes\vec{\mathbf{v}} &= (\mathsf{u}_i\hat{\mathbf{e}}_i)\otimes(\mathsf{v}_j\hat{\mathbf{e}}_j) \\
&= \mathsf{u}_i\mathsf{v}_j(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j) \qquad (i,j=1,2,3) \\
&= \mathsf{A}_{ij}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)
\end{aligned} \tag{1.76}
$$

$$\boxed{\underbrace{\mathbf{A}}_{Tensor} = \underbrace{\mathsf{A}_{ij}}_{components}\ \underbrace{\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j}_{basis}} \qquad (i,j=1,2,3) \tag{1.77}$$

In this textbook, the components of a second-order tensor can be represented in different ways, namely:

$$
\begin{gathered}
\underbrace{\mathbf{A} = \vec{\mathbf{u}}\otimes\vec{\mathbf{v}}} \\
\downarrow \\
\textit{components} \\
\downarrow \\
(\mathbf{A})_{ij} = (\vec{\mathbf{u}}\otimes\vec{\mathbf{v}})_{ij} = \mathsf{u}_i\mathsf{v}_j = \mathsf{A}_{ij}
\end{gathered} \tag{1.78}
$$

These components are explicitly expressed in matrix form as:

$$(\mathbf{A})_{ij} = \mathsf{A}_{ij} = \mathcal{A} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33} \end{bmatrix} \tag{1.79}$$

It is easy to identify the tensor order by the number of free indices in the tensor components, *i.e.*:

$$\begin{aligned} &\textit{Second-order tensor} && \mathbf{U} = \mathsf{U}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \\ &\textit{Third-order tensor} && \mathbf{T} = \mathsf{T}_{ijk} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \qquad (i,j,k,l = 1,2,3) \\ &\textit{Fourth-order tensor} && \mathbb{I} = \mathbb{I}_{ijkl} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l \end{aligned} \tag{1.80}$$

OBS.: The tensor order is given by the number of free indices in its components.

OBS.: The number of tensor components is given by a^n, where the base a is the maximum value in the index range, and the exponent n is the number of the free index.

Problem 1.12: Define the order of the tensors represented by their Cartesian components: v_i, Φ_{ijk}, F_{ijj}, ε_{ij}, $\mathbb{C}_{ijkl}$, σ_{ij}. Determine the number of components in tensor $\mathbb{C}$.

Solution: The order of the tensor is given by the number of free indices, so it follows that:
First-order tensor (vector): $\vec{v}$, $\vec{\mathbf{F}}$; Second-order tensor: $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$; Third-order tensor: $\boldsymbol{\Phi}$;
Fourth-order tensor: $\mathbb{C}$
The number of tensor components is given by the maximum index range value, *i.e.* $i,j,k,l = 1,2,3$, to the power of the number of free indices which is equal to 4 in the case of $\mathbb{C}_{ijkl}$. Thus, the number of independent components in $\mathbb{C}$ is given by:

$$3^4 = (i=3)\times(j=3)\times(k=3)\times(l=3) = 81$$

The fourth-order tensor $\mathbb{C}_{ijkl}$ has 81 components.

Let $\mathbf{A}$ and $\mathbf{B}$ be second-order tensors, we can then define some algebraic operations including:

- **Addition:** The sum of two tensors of same order is a new tensor defined as follows:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1.81}$$

The components of $\mathbf{C}$ are represented by:

$$(\mathbf{C})_{ij} = (\mathbf{A} + \mathbf{B})_{ij} \qquad \text{or} \qquad \mathsf{C}_{ij} = \mathsf{A}_{ij} + \mathsf{B}_{ij} \tag{1.82}$$

or, in matrix notation as:

$$\mathcal{C} = \mathcal{A} + \mathcal{B} \tag{1.83}$$

- **Multiplication** of a tensor by a scalar: The multiplication of a second-order tensor ($\mathbf{A}$) by a scalar (λ) is defined by a new tensor $\mathbf{D}$, so that:

$$\mathbf{D} = \lambda \mathbf{A} \xrightarrow{\textit{in components}} (\mathbf{D})_{ij} = \lambda (\mathbf{A})_{ij} \tag{1.84}$$

or, in matrix form:

$$\mathcal{A}=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}\longrightarrow \lambda\mathcal{A}=\begin{bmatrix}\lambda A_{11} & \lambda A_{12} & \lambda A_{13}\\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23}\\ \lambda A_{31} & \lambda A_{32} & \lambda A_{33}\end{bmatrix} \tag{1.85}$$

It is also true that:

$$(\lambda \mathbf{A})\cdot\vec{\mathbf{v}}=\lambda(\mathbf{A}\cdot\vec{\mathbf{v}}) \tag{1.86}$$

for any vector $\vec{\mathbf{v}}$.

- **Scalar Product (or Dot Product)**: The scalar product (also known as single contraction) between a second-order tensor $\mathbf{A}$ and a vector $\vec{\mathbf{x}}$ is another vector (first-order tensor) $\vec{\mathbf{y}}$, defined as:

$$\begin{aligned}\vec{\mathbf{y}} &= \mathbf{A}\cdot\vec{\mathbf{x}}\\ &= (A_{jk}\hat{\mathbf{e}}_j\otimes\overbrace{\hat{\mathbf{e}}_k)\cdot(x_l\hat{\mathbf{e}}_l}^{\delta_{kl}})\\ &= A_{jk}x_l\delta_{kl}\hat{\mathbf{e}}_j\\ &= \underbrace{A_{jk}x_k}_{y_j}\hat{\mathbf{e}}_j\\ &= y_j\hat{\mathbf{e}}_j\end{aligned} \tag{1.87}$$

The scalar product between two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is another second-order tensor, that verifies: $\mathbf{A}\cdot\mathbf{B}\neq\mathbf{B}\cdot\mathbf{A}$:

$$\begin{aligned}\mathbf{C}=\mathbf{A}\cdot\mathbf{B} &= (A_{ij}\hat{\mathbf{e}}_i\otimes\overbrace{\hat{\mathbf{e}}_j)\cdot(B_{kl}\hat{\mathbf{e}}_k}^{\delta_{jk}}\otimes\hat{\mathbf{e}}_l)\\ &= A_{ij}B_{kl}\delta_{jk}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l\\ &= \underbrace{A_{ik}B_{kl}}_{\mathcal{AB}}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l\\ &= C_{il}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}\mathbf{D}=\mathbf{B}\cdot\mathbf{A} &= (B_{ij}\hat{\mathbf{e}}_i\otimes\overbrace{\hat{\mathbf{e}}_j)\cdot(A_{kl}\hat{\mathbf{e}}_k}^{\delta_{jk}}\otimes\hat{\mathbf{e}}_l)\\ &= B_{ij}A_{kl}\delta_{jk}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l\\ &= \underbrace{B_{ik}A_{kl}}_{\mathcal{BA}}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l\\ &= D_{il}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l\end{aligned} \tag{1.88}$$

It also satisfies the following properties:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}\quad ;\quad \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})=(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} \tag{1.89}$$

- **The powers of second-order tensors**

The scalar product allows us to define the power of second-order tensors, as seen below:

$$\mathbf{A}^0=\mathbf{1}\quad ;\quad \mathbf{A}^1=\mathbf{A}\quad ;\quad \mathbf{A}^2=\mathbf{A}\cdot\mathbf{A}\quad ;\quad \mathbf{A}^3=\mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A},\ \text{and so on,} \tag{1.90}$$

where $\mathbf{1}$ is the *second-order unit tensor* (also called the *identity tensor*).

- **Double Scalar Product (or Double contraction)**

Consider two dyads, $\mathbf{A}=\vec{\mathbf{c}}\otimes\vec{\mathbf{d}}$ and $\mathbf{B}=\vec{\mathbf{u}}\otimes\vec{\mathbf{v}}$. The double contraction between them is defined in different ways, namely: $\mathbf{A}:\mathbf{B}$ and $\mathbf{A}\cdot\cdot\mathbf{B}$.

Double contraction $(\cdot\cdot)$:

$$(\vec{\mathbf{c}}\otimes\vec{\mathbf{d}})\cdot\cdot(\vec{\mathbf{u}}\otimes\vec{\mathbf{v}})=(\vec{\mathbf{c}}\cdot\vec{\mathbf{v}})(\vec{\mathbf{d}}\cdot\vec{\mathbf{u}}) \tag{1.91}$$

In components

$$
\begin{aligned}
\mathbf{A} \cdot\cdot \mathbf{B} &= (\mathsf{A}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \cdot\cdot (\mathsf{B}_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) \\
&= \mathsf{A}_{ij} \mathsf{B}_{kl} \delta_{jk} \delta_{il} \\
&= \mathsf{A}_{ij} \mathsf{B}_{ji} \\
&= \gamma \qquad (scalar)
\end{aligned} \tag{1.92}
$$

The double contraction $(\cdot\cdot)$ is commutative, *i.e.* $\mathbf{A} \cdot\cdot \mathbf{B} = \mathbf{B} \cdot\cdot \mathbf{A}$.

Double contraction ($:$):

$$
\mathbf{A} : \mathbf{B} = (\vec{\mathbf{c}} \otimes \vec{\mathbf{d}}) : (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) = (\vec{\mathbf{c}} \cdot \vec{\mathbf{u}})(\vec{\mathbf{d}} \cdot \vec{\mathbf{v}}) \tag{1.93}
$$

The double contraction $(:)$ is commutative, so:

$$
\mathbf{B} : \mathbf{A} = (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) : (\vec{\mathbf{c}} \otimes \vec{\mathbf{d}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{c}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{d}}) = (\vec{\mathbf{c}} \cdot \vec{\mathbf{u}})(\vec{\mathbf{d}} \cdot \vec{\mathbf{v}}) = \mathbf{A} : \mathbf{B} \tag{1.94}
$$

The breakdown into its components appears like this:

$$
\begin{aligned}
\mathbf{A} : \mathbf{B} &= (\mathsf{A}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) : (\mathsf{B}_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) \\
&= \mathsf{A}_{ij} \mathsf{B}_{kl} \delta_{ik} \delta_{jl} \\
&= \mathsf{A}_{ij} \mathsf{B}_{ij} \\
&= \lambda \qquad (scalar)
\end{aligned} \tag{1.95}
$$

In general, $\mathbf{A} : \mathbf{B} \neq \mathbf{A} \cdot\cdot \mathbf{B}$, however, they are equal if at least one of them is symmetric, *i.e.* $\mathbf{A}^{sym} : \mathbf{B} = \mathbf{A}^{sym} \cdot\cdot \mathbf{B}$ or $\mathbf{A} : \mathbf{B}^{sym} = \mathbf{A} \cdot\cdot \mathbf{B}^{sym}$, so $\mathbf{A}^{sym} : \mathbf{B}^{sym} = \mathbf{A}^{sym} \cdot\cdot \mathbf{B}^{sym}$.

The double contraction with a third-order tensor ($\mathbf{S}$) and a second-order tensor ($\mathbf{B}$) becomes:

$$
\begin{aligned}
\mathbf{S} : \mathbf{B} &= (\vec{\mathbf{c}} \otimes \vec{\mathbf{d}} \otimes \vec{\mathbf{a}}) : (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{v}})(\vec{\mathbf{d}} \cdot \vec{\mathbf{u}})\vec{\mathbf{c}} \\
\mathbf{B} : \mathbf{S} &= (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) : (\vec{\mathbf{c}} \otimes \vec{\mathbf{d}} \otimes \vec{\mathbf{a}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{c}})(\vec{\mathbf{v}} \cdot \vec{\mathbf{d}})\vec{\mathbf{a}}
\end{aligned} \tag{1.96}
$$

As we can verify the result is a vector. In symbolic notation, the double contraction ($\mathbf{B} : \mathbf{S}$) is represented by:

$$
\mathsf{S}_{ijk} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k : \mathsf{B}_{pq} \hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q = \mathsf{S}_{ijk} \mathsf{B}_{pq} \delta_{jp} \delta_{kq} \hat{\mathbf{e}}_i = \mathsf{S}_{ijk} \mathsf{B}_{jk} \hat{\mathbf{e}}_i \tag{1.97}
$$

The double contraction of a fourth-order tensor ($\mathbb{C}$) with a second-order tensor ($\boldsymbol{\varepsilon}$) is defined as:

$$
\begin{aligned}
\mathbb{C}_{ijkl} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l : \varepsilon_{pq} \hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q = \mathbb{C}_{ijkl} \varepsilon_{pq} \delta_{kp} \delta_{lq} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j &= \mathbb{C}_{ijkl} \varepsilon_{kl} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \\
&= \sigma_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j
\end{aligned} \tag{1.98}
$$

where σ_{ij} are the components of $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$.

Next, we express some properties of the double contraction $(:)$:

$$
\begin{aligned}
&a)\ \mathbf{A}:\mathbf{B}=\mathbf{B}:\mathbf{A}\\
&b)\ \mathbf{A}:(\mathbf{B}+\mathbf{C})=\mathbf{A}:\mathbf{B}+\mathbf{A}:\mathbf{C}\\
&c)\ \lambda(\mathbf{A}:\mathbf{B})=(\lambda\mathbf{A}):\mathbf{B}=\mathbf{A}:(\lambda\mathbf{B})
\end{aligned}
\tag{1.99}
$$

where $\mathbf{A},\mathbf{B},\mathbf{C}$ are second-order tensors, and λ is a scalar.

Via the definition of the double scalar product, it is possible to obtain the components of the second-order tensor $\mathbf{A}$ in the Cartesian system, *i.e.*:

$$
(\mathbf{A})_{ij}=(\mathrm{A}_{kl}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l):(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)=\hat{\mathbf{e}}_i\cdot(\mathrm{A}_{kl}\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l)\cdot\hat{\mathbf{e}}_j=\mathrm{A}_{kl}\delta_{ki}\delta_{lj}=\mathrm{A}_{ij}
\tag{1.100}
$$

If we consider any two vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and an arbitrary second-order tensor, $\mathbf{A}$, we can demonstrate that:

$$
\begin{aligned}
\vec{\mathbf{a}}\cdot\mathbf{A}\cdot\vec{\mathbf{b}}&=\mathrm{a}_p\hat{\mathbf{e}}_p\cdot\mathrm{A}_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\cdot\mathrm{b}_r\hat{\mathbf{e}}_r=\mathrm{a}_p\mathrm{A}_{ij}\mathrm{b}_r\delta_{pi}\delta_{jr}=\mathrm{a}_i\mathrm{A}_{ij}\mathrm{b}_j=\mathrm{A}_{ij}(\mathrm{a}_i\mathrm{b}_j)\\
&=\mathbf{A}:(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})
\end{aligned}
\tag{1.101}
$$

- **Vector product**

The vector product between a second-order tensor $\mathbf{A}$ and a vector $\vec{\mathbf{x}}$ is a second-order tensor given by:

$$
\mathbf{A}\wedge\vec{\mathbf{x}}=(\mathrm{A}_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)\wedge(\mathrm{x}_k\hat{\mathbf{e}}_k)=\epsilon_{ljk}\mathrm{A}_{ij}\mathrm{x}_k\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_l
\tag{1.102}
$$

where we have used the definition (1.67), *i.e.* $\hat{\mathbf{e}}_j\wedge\hat{\mathbf{e}}_k=\epsilon_{ljk}\hat{\mathbf{e}}_l$. In **Problem 1.11**, we have shown that the relation $\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\vec{\mathbf{b}}-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{c}}$ holds, which is also represented by means of dyads as:

$$
\left[\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]_j=(\mathrm{a}_k\mathrm{c}_k)\mathrm{b}_j-(\mathrm{a}_k\mathrm{b}_k)\mathrm{c}_j=(\mathrm{b}_j\mathrm{c}_k-\mathrm{c}_j\mathrm{b}_k)\mathrm{a}_k=\left[(\vec{\mathbf{b}}\otimes\vec{\mathbf{c}}-\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{a}}\right]_j
\tag{1.103}
$$

In the particular case when $\vec{\mathbf{a}}=\vec{\mathbf{c}}$ we obtain:

$$
\begin{aligned}
\left[\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}})\right]_j&=(\mathrm{a}_k\mathrm{a}_k)\mathrm{b}_j-(\mathrm{a}_k\mathrm{b}_k)\mathrm{a}_j=(\mathrm{a}_k\mathrm{a}_k)\mathrm{b}_p\delta_{jp}-(\mathrm{a}_k\mathrm{b}_p\delta_{kp})\mathrm{a}_j\\
&=\left[(\mathrm{a}_k\mathrm{a}_k)\delta_{jp}-(\mathrm{a}_k\delta_{kp})\mathrm{a}_j\right]\mathrm{b}_p=\left[(\mathrm{a}_k\mathrm{a}_k)\delta_{jp}-\mathrm{a}_p\mathrm{a}_j\right]\mathrm{b}_p\\
&=\left\{\left[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\mathbf{1}-\vec{\mathbf{a}}\otimes\vec{\mathbf{a}}\right]\cdot\vec{\mathbf{b}}\right\}_j
\end{aligned}
\tag{1.104}
$$

Thus, the following relationships are valid:

$$
\boxed{
\begin{aligned}
&\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})=(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\vec{\mathbf{b}}-(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{c}}=(\vec{\mathbf{b}}\otimes\vec{\mathbf{c}}-\vec{\mathbf{c}}\otimes\vec{\mathbf{b}})\cdot\vec{\mathbf{a}}\\
&\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}})=\left[(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\mathbf{1}-\vec{\mathbf{a}}\otimes\vec{\mathbf{a}}\right]\cdot\vec{\mathbf{b}}
\end{aligned}}
\tag{1.105}
$$

1.5.1.1 Component Representation of a Second-Order Tensor in the Cartesian Basis

As seen before, a vector which has 3 independent components is represented in a Cartesian space as shown in Figure 1.11. An arbitrary second-order tensor has 9 independent components, so we would need a hyperspace to represent all its components. Afterwards, a device is introduced to represent the second-order tensor components in the Cartesian basis.

An arbitrary second-order tensor $\mathbf{T}$ is represented in the Cartesian basis by:

$$\begin{aligned}\mathbf{T} &= \mathsf{T}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \mathsf{T}_{i1}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{i2}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{i3}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_3 \\ &= \mathsf{T}_{11}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{12}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{13}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 + \\ &+ \mathsf{T}_{21}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{22}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{23}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 + \\ &+ \mathsf{T}_{31}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{32}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{33}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3\end{aligned} \tag{1.106}$$

Next, we calculate the projection of $\mathbf{T}$ onto $\hat{\mathbf{e}}_k$:

$$\mathbf{T}\cdot\hat{\mathbf{e}}_k = \mathsf{T}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k = \mathsf{T}_{ij}\hat{\mathbf{e}}_i\delta_{jk} = \mathsf{T}_{ik}\hat{\mathbf{e}}_i = \mathsf{T}_{1k}\hat{\mathbf{e}}_1 + \mathsf{T}_{2k}\hat{\mathbf{e}}_2 + \mathsf{T}_{3k}\hat{\mathbf{e}}_3 \tag{1.107}$$

thereby defining three vectors, namely:

$$\mathbf{T}\cdot\hat{\mathbf{e}}_k = \mathsf{T}_{ik}\hat{\mathbf{e}}_i \quad \Rightarrow \quad \begin{cases} k=1 \;\Rightarrow\; \mathsf{T}_{i1}\hat{\mathbf{e}}_i = \mathsf{T}_{11}\hat{\mathbf{e}}_1 + \mathsf{T}_{21}\hat{\mathbf{e}}_2 + \mathsf{T}_{31}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)} \\ k=2 \;\Rightarrow\; \mathsf{T}_{i2}\hat{\mathbf{e}}_i = \mathsf{T}_{12}\hat{\mathbf{e}}_1 + \mathsf{T}_{22}\hat{\mathbf{e}}_2 + \mathsf{T}_{32}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_2)} \\ k=3 \;\Rightarrow\; \mathsf{T}_{i3}\hat{\mathbf{e}}_i = \mathsf{T}_{13}\hat{\mathbf{e}}_1 + \mathsf{T}_{23}\hat{\mathbf{e}}_2 + \mathsf{T}_{33}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_3)} \end{cases} \tag{1.108}$$

Graphical representation of these three vectors $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}$, $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_2)}$, $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_3)}$, in the Cartesian basis, is shown in Figure 1.13. Note also that $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}$ is the projection of $\mathbf{T}$ onto $\hat{\mathbf{e}}_1$, $\hat{\mathsf{n}}_i^{(1)} = [1,0,0]$, which can be verified by:

$$(\mathbf{T}\cdot\hat{\mathbf{n}})_i = \begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{21} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{31} & \mathsf{T}_{32} & \mathsf{T}_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{21} \\ \mathsf{T}_{31} \end{bmatrix} = \mathsf{t}_i^{(\hat{\mathbf{e}}_1)} \tag{1.109}$$

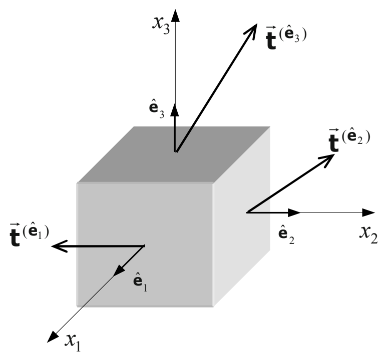


Figure 1.13: The projection of $\mathbf{T}$ in the Cartesian basis.

The same result obtained in (1.109) could have been evaluated by the scalar product of $\mathbf{T}$, given in (1.106), with the basis $\hat{\mathbf{e}}_1$, *i.e.*:

$$\begin{aligned}\mathbf{T}\cdot\hat{\mathbf{e}}_1 &= [\ \mathsf{T}_{11}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{12}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{13}\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_3 + \\ &\quad + \mathsf{T}_{21}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{22}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{23}\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_3 + \\ &\quad + \mathsf{T}_{31}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_1 + \mathsf{T}_{32}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_2 + \mathsf{T}_{33}\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{e}}_3\ ]\cdot\hat{\mathbf{e}}_1 \\ &= \mathsf{T}_{11}\hat{\mathbf{e}}_1 + \mathsf{T}_{21}\hat{\mathbf{e}}_2 + \mathsf{T}_{31}\hat{\mathbf{e}}_3 = \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}\end{aligned} \tag{1.110}$$

where we have used the orthogonality property of the basis, *i.e.* $\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 = 1$, $\hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_1 = 0$, $\hat{\mathbf{e}}_3\cdot\hat{\mathbf{e}}_1 = 0$. Taking into account the components are represented in matrix form, (see Figure 1.14), we can establish that, the diagonal terms (T_{11}, T_{22}, T_{33}) are normal to the plane defined by the unit vectors ($\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$), hence they will be referred to as *normal* *components*. The components displayed tangentially to the plane are called *tangential components*, and correspond to the off-diagonal terms of T_{ij}.

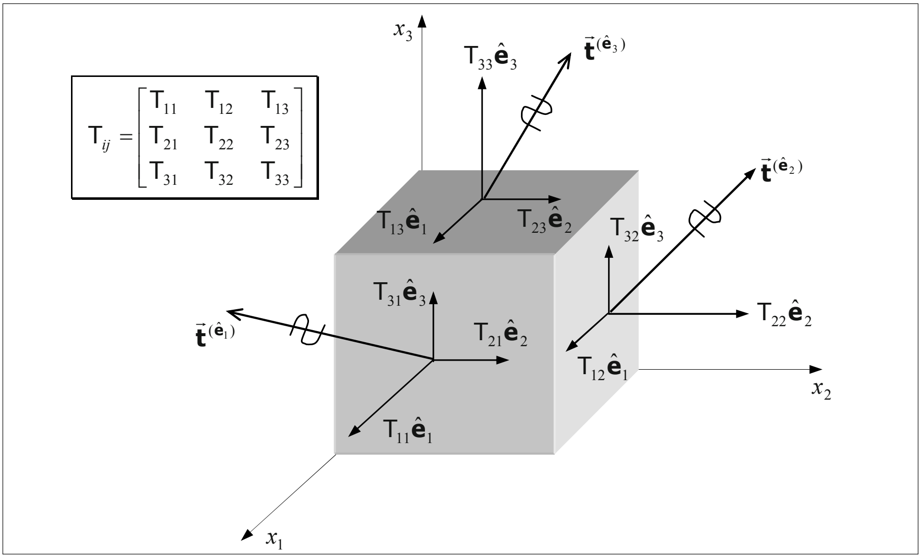


Figure 1.14: Representation of the second-order tensor components in the Cartesian coordinate system.

NOTE: Throughout the textbook, we will use the following notations:

Tensorial notation *Symbolic notation*

$$\mathbf{A}\cdot\mathbf{B} = (\mathsf{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\cdot(\mathsf{B}_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l)$$
$$= \mathsf{A}_{ij}\mathsf{B}_{kl}\delta_{jk}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l) \tag{1.111}$$
$$= \mathsf{A}_{ij}\mathsf{B}_{jl}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l)$$

Cartesian basis

Indicial notation

Note that the index is not repeated more than twice either in symbolic notation or in indicial notation. Also note that the indicial notation is equivalent to the tensor notation only when dealing with scalars, *e.g.* $\mathbf{A}:\mathbf{B} = \mathsf{A}_{ij}\mathsf{B}_{ij} = \lambda$, or $\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = \mathsf{a}_i\mathsf{b}_i$. ■

1.5.2 Properties of Tensors

1.5.2.1 Tensor Transpose

Let $\mathbf{A}$ be a second-order tensor, the *transpose* of $\mathbf{A}$ is defined as:

$$\mathbf{A}^T = \mathsf{A}_{ji}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathsf{A}_{ij}(\hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i) \tag{1.112}$$

If A_{ij} are the components of $\mathbf{A}$, it follows that the components of the transpose are:

$$(\mathbf{A}^T)_{ij} = \mathsf{A}_{ji} \tag{1.113}$$

If $\mathbf{A} = \vec{\mathbf{u}} \otimes \vec{\mathbf{v}}$, the transpose of the dyad $\mathbf{A}$ is given by $\mathbf{A}^T = \vec{\mathbf{v}} \otimes \vec{\mathbf{u}}$:

$$\begin{aligned}
\mathbf{A}^T &= (\vec{\mathbf{u}} \otimes \vec{\mathbf{v}})^T & &= \vec{\mathbf{v}} \otimes \vec{\mathbf{u}} & & \\
&= (\mathsf{u}_i \hat{\mathbf{e}}_i \otimes \mathsf{v}_j \hat{\mathbf{e}}_j)^T & &= \mathsf{v}_j \hat{\mathbf{e}}_j \otimes \mathsf{u}_i \hat{\mathbf{e}}_i & &= \mathsf{v}_i \hat{\mathbf{e}}_i \otimes \mathsf{u}_j \hat{\mathbf{e}}_j \\
&= (\mathsf{u}_i \mathsf{v}_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)^T & &= \mathsf{u}_i \mathsf{v}_j \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i & &= \mathsf{u}_j \mathsf{v}_i \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \\
&= (\mathsf{A}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)^T & &= \mathsf{A}_{ij} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i & &= \mathsf{A}_{ji} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j
\end{aligned} \tag{1.114}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be second-order tensors and α, β be scalars, and the following relationships are valid:

$$(\mathbf{A}^T)^T = \mathbf{A} \quad ; \quad (\alpha\mathbf{B} + \beta\mathbf{A})^T = \alpha\mathbf{B}^T + \beta\mathbf{A}^T \quad ; \quad (\mathbf{B} \cdot \mathbf{A})^T = \mathbf{A}^T \cdot \mathbf{B}^T \tag{1.115}$$

$$\begin{aligned}
\mathbf{A} : \mathbf{B}^T &= (\mathsf{A}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) : (\mathsf{B}_{kl} \hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_k) = \mathsf{A}_{ij} \mathsf{B}_{kl} \delta_{il} \delta_{jk} = \mathsf{A}_{ij} \mathsf{B}_{ji} = \mathbf{A} \cdot\cdot \mathbf{B} \\
\mathbf{A}^T : \mathbf{B} &= (\mathsf{A}_{ij} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i) : (\mathsf{B}_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) = \mathsf{A}_{ij} \mathsf{B}_{kl} \delta_{jk} \delta_{il} = \mathsf{A}_{ij} \mathsf{B}_{ji} = \mathbf{A} \cdot\cdot \mathbf{B}
\end{aligned} \tag{1.116}$$

The transpose of the matrix $\mathcal{A}$ is formed by changing rows for columns and vice versa, *i.e.*:

$$\mathcal{A} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33} \end{bmatrix} \xrightarrow{transpose} \mathcal{A}^T = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{21} & \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{31} & \mathsf{A}_{32} & \mathsf{A}_{33} \end{bmatrix}^T = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{21} & \mathsf{A}_{31} \\ \mathsf{A}_{12} & \mathsf{A}_{22} & \mathsf{A}_{32} \\ \mathsf{A}_{13} & \mathsf{A}_{23} & \mathsf{A}_{33} \end{bmatrix} \tag{1.117}$$

Problem 1.13: Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be arbitrary second-order tensors. Demonstrate that:

$$\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{B}^T \cdot \mathbf{A}) : \mathbf{C} = (\mathbf{A} \cdot \mathbf{C}^T) : \mathbf{B}$$

Solution: Expressing the term $\mathbf{A} : (\mathbf{B} \cdot \mathbf{C})$ in indicial notation we obtain:

$$\begin{aligned}
\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) &= \mathsf{A}_{ij} \hat{\mathbf{e}}_i \otimes \mathbf{e}_j : (\mathsf{B}_{lk} \hat{\mathbf{e}}_l \otimes \mathbf{e}_k \cdot \mathsf{C}_{pq} \hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) \\
&= \mathsf{A}_{ij} \mathsf{B}_{lk} \mathsf{C}_{pq} \hat{\mathbf{e}}_i \otimes \mathbf{e}_j : (\delta_{kp} \hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_q) \\
&= \mathsf{A}_{ij} \mathsf{B}_{lk} \mathsf{C}_{pq} \delta_{kp} \delta_{il} \delta_{jq} = \mathsf{A}_{ij} \mathsf{B}_{ik} \mathsf{C}_{kj}
\end{aligned}$$

Note that, when we are dealing with indicial notation the position of the terms does not matter, *i.e.*:

$$\mathsf{A}_{ij} \mathsf{B}_{ik} \mathsf{C}_{kj} = \mathsf{B}_{ik} \mathsf{A}_{ij} \mathsf{C}_{kj} = \mathsf{A}_{ij} \mathsf{C}_{kj} \mathsf{B}_{ik}$$

We can now observe that the algebraic operation $\mathsf{B}_{ik} \mathsf{A}_{ij}$ is equivalent to the components of the second-order tensor $(\mathbf{B}^T \cdot \mathbf{A})_{kj}$, thus,

$$\mathsf{B}_{ik} \mathsf{A}_{ij} \mathsf{C}_{kj} = (\mathbf{B}^T \cdot \mathbf{A})_{kj} \mathsf{C}_{kj} = (\mathbf{B}^T \cdot \mathbf{A}) : \mathbf{C}.$$

Likewise, we can state that $\mathsf{A}_{ij} \mathsf{C}_{kj} \mathsf{B}_{ik} = (\mathbf{A} \cdot \mathbf{C}^T) : \mathbf{B}$.

Problem 1.14: Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ be vectors and $\mathbf{A}$ be a second-order tensor. Show that the following relationship holds:

$$\vec{\mathbf{u}} \cdot \mathbf{A}^T \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \mathbf{A} \cdot \vec{\mathbf{u}}$$

Solution:

$$\begin{aligned}
\vec{\mathbf{u}} \cdot \mathbf{A}^T \cdot \vec{\mathbf{v}} & & &= \vec{\mathbf{v}} \cdot \mathbf{A} \cdot \vec{\mathbf{u}} \\
\mathsf{u}_i \hat{\mathbf{e}}_i \cdot \mathsf{A}_{jl} \hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_j \cdot \mathsf{v}_k \hat{\mathbf{e}}_k & & &= \mathsf{v}_k \hat{\mathbf{e}}_k \cdot \mathsf{A}_{jl} \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_l \cdot \mathsf{u}_i \hat{\mathbf{e}}_i \\
\mathsf{u}_i \mathsf{A}_{jl} \delta_{il} \mathsf{v}_k \delta_{jk} & & &= \mathsf{v}_k \delta_{kj} \mathsf{A}_{jl} \mathsf{u}_i \delta_{il} \\
\mathsf{u}_l \mathsf{A}_{jl} \mathsf{v}_j & & &= \mathsf{v}_j \mathsf{A}_{jl} \mathsf{u}_l
\end{aligned}$$

1.5.2.2 Symmetry and Antisymmetry

1.5.2.2.1 Symmetric tensor

A second-order tensor $\mathbf{A}$ is symmetric, *i.e.:* $\mathbf{A} \equiv \mathbf{A}^{sym}$, if the tensor is equal to its transpose:

$$\text{if} \quad \mathbf{A} = \mathbf{A}^T \xrightarrow{\text{in components}} \mathsf{A}_{ij} = \mathsf{A}_{ji} \quad \Leftrightarrow \quad \mathbf{A} \text{ is symmetric} \tag{1.118}$$

in matrix form:

$$\mathcal{A} = \mathcal{A}^T \longrightarrow \mathcal{A} \equiv \mathcal{A}^{sym} = \begin{bmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{12} & \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{13} & \mathsf{A}_{23} & \mathsf{A}_{33} \end{bmatrix} \tag{1.119}$$

From the above it is clear that a symmetric second-order tensor has 6 independent components, namely: A_{11}, A_{22}, A_{33}, A_{12}, A_{23}, A_{13}.

According to equation (1.118), a symmetric tensor can be represented by:

$$\begin{aligned} &\mathsf{A}_{ij} = \mathsf{A}_{ji} \\ &\mathsf{A}_{ij} + \mathsf{A}_{ij} = \mathsf{A}_{ij} + \mathsf{A}_{ji} \\ &2\mathsf{A}_{ij} = \mathsf{A}_{ij} + \mathsf{A}_{ji} \\ &\mathsf{A}_{ij} = \frac{1}{2}(\mathsf{A}_{ij} + \mathsf{A}_{ji}) \quad \Rightarrow \quad \mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) \end{aligned} \tag{1.120}$$

A fourth-order tensor $\mathbb{C}$, whose components are $\mathbb{C}_{ijkl}$, may have the following types of symmetries:

Minor symmetry:

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk} = \mathbb{C}_{jilk} \tag{1.121}$$

Major symmetry:

$$\mathbb{C}_{ijkl} = \mathbb{C}_{klij} \tag{1.122}$$

A fourth-order tensor that does not exhibit any kind of symmetry has 81 independent components. If the tensor $\mathbb{C}$ has only minor symmetry, *i.e.* symmetry in $ij = ji(6)$, and symmetry in $kl = lk(6)$, the tensor features 36 independent components. If besides presenting minor symmetry it also provides major symmetry, the tensor features 21 independent components.

1.5.2.2.2 Antisymmetric tensor

A tensor $\mathbf{A}$ is antisymmetric (also called *skew-symmetric tensor* or *skew tensor*), *i.e.:* $\mathbf{A} \equiv \mathbf{A}^{skew}$:

$$\text{if} \quad \mathbf{A} = -\mathbf{A}^T \xrightarrow{\text{in components}} \mathsf{A}_{ij} = -\mathsf{A}_{ji} \quad \Leftrightarrow \quad \mathbf{A} \text{ is antisymmetric} \tag{1.123}$$

which broken down into its components is the same as:

$$\mathcal{A} = -\mathcal{A}^T \longrightarrow \mathcal{A}^{skew} = \begin{bmatrix} 0 & \mathsf{A}_{12} & \mathsf{A}_{13} \\ -\mathsf{A}_{12} & 0 & \mathsf{A}_{23} \\ -\mathsf{A}_{13} & -\mathsf{A}_{23} & 0 \end{bmatrix} \tag{1.124}$$

Therefore, an antisymmetric second-order tensor has 3 independent components, namely: A_{12}, A_{23}, A_{13}.

Under the conditions expressed in (1.123), an antisymmetric tensor can be represented by:

$$\begin{aligned} &\mathsf{A}_{ij} + \mathsf{A}_{ij} = \mathsf{A}_{ij} - \mathsf{A}_{ji} \\ &2\mathsf{A}_{ij} = \mathsf{A}_{ij} - \mathsf{A}_{ji} \\ &\mathsf{A}_{ij} = \frac{1}{2}(\mathsf{A}_{ij} - \mathsf{A}_{ji}) \quad \Rightarrow \quad \mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) \end{aligned} \tag{1.125}$$

Let us consider an antisymmetric second-order tensor denoted by $\mathbf{W}$, then satisfy the above relationship (1.125):

$$\mathsf{W}_{ij} = \frac{1}{2}(\mathsf{W}_{ij} - \mathsf{W}_{ji}) = \frac{1}{2}(\mathsf{W}_{kl}\delta_{ik}\delta_{jl} - \mathsf{W}_{kl}\delta_{jk}\delta_{il}) = \frac{1}{2}\mathsf{W}_{kl}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) \tag{1.126}$$

Using the relation between the Kronecker delta and the permutation symbol given by (1.62), *i.e.* $\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} = -\epsilon_{ijr}\epsilon_{lkr}$, the equation (1.126) is rewritten as:

$$\mathsf{W}_{ij} = -\frac{1}{2}\mathsf{W}_{kl}\epsilon_{ijr}\epsilon_{lkr} \tag{1.127}$$

Expanding the term $\mathsf{W}_{kl}\epsilon_{lkr}$, for the dummy indices (k, l), we can obtain the following nonzero terms:

$$\mathsf{W}_{kl}\epsilon_{lkr} = \mathsf{W}_{12}\epsilon_{21r} + \mathsf{W}_{13}\epsilon_{31r} + \mathsf{W}_{21}\epsilon_{12r} + \mathsf{W}_{23}\epsilon_{32r} + \mathsf{W}_{31}\epsilon_{13r} + \mathsf{W}_{32}\epsilon_{23r} \tag{1.128}$$

thus,

$$\left.\begin{aligned} r=1 \quad &\Rightarrow \quad \mathsf{W}_{kl}\epsilon_{lkr} = -\mathsf{W}_{23} + \mathsf{W}_{32} = -2\mathsf{W}_{23} = 2w_1 \\ r=2 \quad &\Rightarrow \quad \mathsf{W}_{kl}\epsilon_{lkr} = \mathsf{W}_{13} - \mathsf{W}_{31} = 2\mathsf{W}_{13} = 2w_2 \\ r=3 \quad &\Rightarrow \quad \mathsf{W}_{kl}\epsilon_{lkr} = -\mathsf{W}_{12} + \mathsf{W}_{21} = -2\mathsf{W}_{12} = 2w_3 \end{aligned}\right\} \Rightarrow \mathsf{W}_{kl}\epsilon_{lkr} = 2w_r \tag{1.129}$$

In which we assume the following variables have changed:

$$\mathsf{W}_{ij} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ \mathsf{W}_{21} & 0 & \mathsf{W}_{23} \\ \mathsf{W}_{31} & \mathsf{W}_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ -\mathsf{W}_{12} & 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{13} & -\mathsf{W}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{1.130}$$

Hence, we introduce the *axial vector* $\vec{w}$ associated with the antisymmetric tensor, $\mathbf{W}$, as:

$$\vec{w} = w_1\hat{\mathbf{e}}_1 + w_2\hat{\mathbf{e}}_2 + w_3\hat{\mathbf{e}}_3 \tag{1.131}$$

The magnitude of the axial vector $\vec{w}$ is given by:

$$\omega^2 = \|\vec{w}\|^2 = \vec{w}\cdot\vec{w} = w_1^2 + w_2^2 + w_3^2 = \mathsf{W}_{23}^2 + \mathsf{W}_{13}^2 + \mathsf{W}_{12}^2 \tag{1.132}$$

Substituting (1.129) into (1.127) and by considering that $\epsilon_{ijr} = \epsilon_{rij}$ we obtain:

$$\boxed{\mathsf{W}_{ij} = -w_r\epsilon_{rij}} \tag{1.133}$$

Multiplying both sides of the equation (1.133) by ϵ_{kij} we can obtain:

$$\epsilon_{kij}\mathsf{W}_{ij} = -w_r\epsilon_{rij}\epsilon_{kij} = -2w_r\delta_{rk} = -2w_k \tag{1.134}$$

where we have applied the relation $\epsilon_{rij}\epsilon_{kij} = 2\delta_{rk}$, which was evaluated in **Problem 1.7**, thus we can conclude that:

$$\boxed{w_k = -\frac{1}{2}\epsilon_{kij} \mathsf{W}_{ij}} \tag{1.135}$$

Graphical representation of the antisymmetric tensor components and its corresponding axial vector, in the Cartesian system, is shown in Figure 1.15.

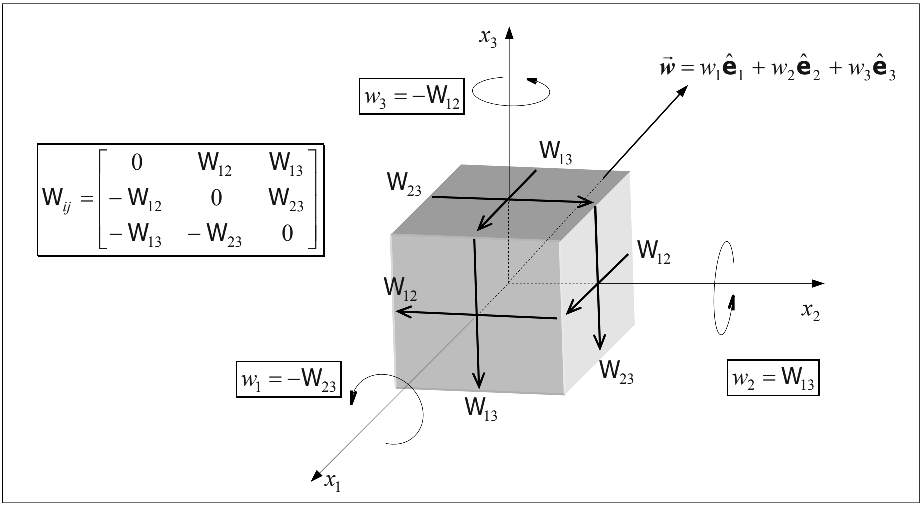


Figure 1.15: Antisymmetric tensor components and the axial vector.

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be arbitrary vectors and $\mathbf{W}$ be an antisymmetric tensor, it follows that:

$$\vec{\mathbf{b}} \cdot \mathbf{W} \cdot \vec{\mathbf{a}} = \vec{\mathbf{a}} \cdot \mathbf{W}^T \cdot \vec{\mathbf{b}} = -\vec{\mathbf{a}} \cdot \mathbf{W} \cdot \vec{\mathbf{b}} \tag{1.136}$$

when $\vec{\mathbf{a}} = \vec{\mathbf{b}}$, it holds that:

$$\vec{\mathbf{a}} \cdot \mathbf{W} \cdot \vec{\mathbf{a}} = \mathbf{W} : (\vec{\mathbf{a}} \otimes \vec{\mathbf{a}}) = 0 \tag{1.137}$$

NOTE: Note that $(\vec{\mathbf{a}} \otimes \vec{\mathbf{a}})$ is a symmetric second-order tensor. Later on we will show that the result of the double contraction between a symmetric tensor and an antisymmetric tensor equals zero. ■

Let us consider an antisymmetric tensor $\mathbf{W}$ and an arbitrary vector $\vec{\mathbf{a}}$. The components of the scalar product $\mathbf{W} \cdot \vec{\mathbf{a}}$ are given by:

$$\begin{aligned} &\mathsf{W}_{ij}\mathsf{a}_j = \mathsf{W}_{i1}\mathsf{a}_1 + \mathsf{W}_{i2}\mathsf{a}_2 + \mathsf{W}_{i3}\mathsf{a}_3 \\ &i=1 \Rightarrow \quad \mathsf{W}_{11}\mathsf{a}_1 + \mathsf{W}_{12}\mathsf{a}_2 + \mathsf{W}_{13}\mathsf{a}_3 \\ &i=2 \Rightarrow \quad \mathsf{W}_{21}\mathsf{a}_1 + \mathsf{W}_{22}\mathsf{a}_2 + \mathsf{W}_{23}\mathsf{a}_3 \\ &i=3 \Rightarrow \quad \mathsf{W}_{31}\mathsf{a}_1 + \mathsf{W}_{32}\mathsf{a}_2 + \mathsf{W}_{33}\mathsf{a}_3 \end{aligned} \tag{1.138}$$

Bearing in mind that the normal components are equal to zero for an antisymmetric tensor, *i.e.*, $\mathsf{W}_{11} = 0$, $\mathsf{W}_{22} = 0$, $\mathsf{W}_{33} = 0$, the scalar product (1.138) becomes:

$$(\mathbf{W} \cdot \vec{\mathbf{a}})_i \Rightarrow \begin{cases} i=1 \Rightarrow & \mathsf{W}_{12}\mathsf{a}_2 + \mathsf{W}_{13}\mathsf{a}_3 \\ i=2 \Rightarrow & \mathsf{W}_{21}\mathsf{a}_1 + \mathsf{W}_{23}\mathsf{a}_3 \\ i=3 \Rightarrow & \mathsf{W}_{31}\mathsf{a}_1 + \mathsf{W}_{32}\mathsf{a}_2 \end{cases} \tag{1.139}$$

The above components are the same as the result of the algebraic operation $\vec{w} \wedge \vec{\mathbf{a}}$:

$$\vec{w} \wedge \vec{\mathbf{a}} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ w_1 & w_2 & w_3 \\ \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 \end{vmatrix}$$
$$= (-w_3\mathrm{a}_2 + w_2\mathrm{a}_3)\hat{\mathbf{e}}_1 + (w_3\mathrm{a}_1 - w_1\mathrm{a}_3)\hat{\mathbf{e}}_2 + (-w_2\mathrm{a}_1 + w_1\mathrm{a}_2)\hat{\mathbf{e}}_3$$
$$= (\mathrm{W}_{12}\mathrm{a}_2 + \mathrm{W}_{13}\mathrm{a}_3)\hat{\mathbf{e}}_1 + (\mathrm{W}_{21}\mathrm{a}_1 + \mathrm{W}_{23}\mathrm{a}_3)\hat{\mathbf{e}}_2 + (\mathrm{W}_{31}\mathrm{a}_1 + \mathrm{W}_{32}\mathrm{a}_2)\hat{\mathbf{e}}_3 \tag{1.140}$$

where $w_1 = -\mathrm{W}_{23} = \mathrm{W}_{32}$, $w_2 = \mathrm{W}_{13} = -\mathrm{W}_{31}$, $w_3 = -\mathrm{W}_{12} = \mathrm{W}_{21}$. Then, given an antisymmetric tensor $\mathbf{W}$ and the axial vector $\vec{w}$ associated with $\mathbf{W}$, it holds that:

$$\boxed{\mathbf{W} \cdot \vec{\mathbf{a}} = \vec{w} \wedge \vec{\mathbf{a}}} \tag{1.141}$$

for any vector $\vec{\mathbf{a}}$. The property (1.141) could easily have been obtained by taking into account the components of $\mathbf{W}$ given by (1.133), *i.e.*:

$$(\mathbf{W} \cdot \vec{\mathbf{a}})_i = \mathrm{W}_{ik}\mathrm{a}_k = -w_j \epsilon_{jik}\mathrm{a}_k = \epsilon_{ijk} w_j \mathrm{a}_k = (\vec{w} \wedge \vec{\mathbf{a}})_i \tag{1.142}$$

The vector $\vec{w}$ can be represented by its magnitude, $\|\vec{w}\| = \omega$, and by the unit vector codirectional with $\vec{w}$, *i.e.* $\vec{w} = \omega\hat{\mathbf{e}}_1^*$. Then, the equation (1.141) can still be expressed as:

$$\mathbf{W} \cdot \vec{\mathbf{a}} = \vec{w} \wedge \vec{\mathbf{a}} = \omega\hat{\mathbf{e}}_1^* \wedge \vec{\mathbf{a}} \tag{1.143}$$

Additionally, we can choose two unit vectors $\hat{\mathbf{e}}_2^*$, $\hat{\mathbf{e}}_3^*$, which make up an orthonormal basis with the unit vector $\hat{\mathbf{e}}_1^*$, (see Figure 1.16), so that:

$$\hat{\mathbf{e}}_1^* = \hat{\mathbf{e}}_2^* \wedge \hat{\mathbf{e}}_3^* \quad ; \quad \hat{\mathbf{e}}_2^* = \hat{\mathbf{e}}_3^* \wedge \hat{\mathbf{e}}_1^* \quad ; \quad \hat{\mathbf{e}}_3^* = \hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_2^* \tag{1.144}$$

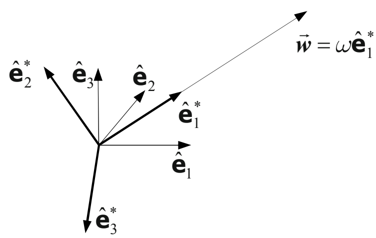


Figure 1.16: Orthonormal basis defined by the axial vector.

By representing the vector $\vec{\mathbf{a}}$ in this new basis, $\vec{\mathbf{a}} = \mathrm{a}_1^*\hat{\mathbf{e}}_1^* + \mathrm{a}_2^*\hat{\mathbf{e}}_2^* + \mathrm{a}_3^*\hat{\mathbf{e}}_3^*$, the relationship shown in (1.143) obtains the form below:

$$\mathbf{W} \cdot \vec{\mathbf{a}} = \omega\hat{\mathbf{e}}_1^* \wedge \vec{\mathbf{a}} = \omega\hat{\mathbf{e}}_1^* \wedge (\mathrm{a}_1^*\hat{\mathbf{e}}_1^* + \mathrm{a}_2^*\hat{\mathbf{e}}_2^* + \mathrm{a}_3^*\hat{\mathbf{e}}_3^*)$$
$$= \omega(\mathrm{a}_1^*\underbrace{\hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_1^*}_{=\hat{\mathbf{0}}} + \mathrm{a}_2^*\underbrace{\hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_2^*}_{=\hat{\mathbf{e}}_3^*} + \mathrm{a}_3^*\underbrace{\hat{\mathbf{e}}_1^* \wedge \hat{\mathbf{e}}_3^*}_{=-\hat{\mathbf{e}}_2^*}) = \omega(\mathrm{a}_2^*\hat{\mathbf{e}}_3^* - \mathrm{a}_3^*\hat{\mathbf{e}}_2^*)$$
$$= \left[\omega(\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^*)\right] \cdot \vec{\mathbf{a}} \tag{1.145}$$

Thus, an antisymmetric tensor can be represented, in the space defined by the axial vector, as follows:

$$\mathbf{W} = \omega(\hat{\mathbf{e}}_3^* \otimes \hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^* \otimes \hat{\mathbf{e}}_3^*) \tag{1.146}$$

Note that by using the antisymmetric tensor representation shown in (1.146), the projections of the tensor $\mathbf{W}$ according to directions $\hat{\mathbf{e}}_1^*$, $\hat{\mathbf{e}}_2^*$ and $\hat{\mathbf{e}}_3^*$ are respectively:

$$\mathbf{W}\cdot\hat{\mathbf{e}}_1^* = \vec{\mathbf{0}} \quad ; \quad \mathbf{W}\cdot\hat{\mathbf{e}}_2^* = \omega\hat{\mathbf{e}}_3^* \quad ; \quad \mathbf{W}\cdot\hat{\mathbf{e}}_3^* = -\omega\hat{\mathbf{e}}_2^* \tag{1.147}$$

We can also verify that:

$$\begin{aligned}\hat{\mathbf{e}}_3^*\cdot\mathbf{W}\cdot\hat{\mathbf{e}}_2^* &= \hat{\mathbf{e}}_3^*\cdot\left[\omega(\hat{\mathbf{e}}_3^*\otimes\hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^*\otimes\hat{\mathbf{e}}_3^*)\right]\cdot\hat{\mathbf{e}}_2^* = \omega \\ \hat{\mathbf{e}}_2^*\cdot\mathbf{W}\cdot\hat{\mathbf{e}}_3^* &= \hat{\mathbf{e}}_2^*\cdot\left[\omega(\hat{\mathbf{e}}_3^*\otimes\hat{\mathbf{e}}_2^* - \hat{\mathbf{e}}_2^*\otimes\hat{\mathbf{e}}_3^*)\right]\cdot\hat{\mathbf{e}}_3^* = -\omega\end{aligned} \tag{1.148}$$

Then, the tensor components of $\mathbf{W}$ in the basis formed by the orthonormal basis $\hat{\mathbf{e}}_1^*$, $\hat{\mathbf{e}}_2^*$, $\hat{\mathbf{e}}_3^*$, are given by:

$$\mathsf{W}_{ij}^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & \omega & 0 \end{bmatrix} \tag{1.149}$$

In Figure 1.17 we can see these components and the axial vector representation. Note that if we take any basis that is formed just by rotation along the $\hat{\mathbf{e}}_1^*$-axis, the components of $\mathbf{W}$ in this new basis will be the same as those provided in (1.149).

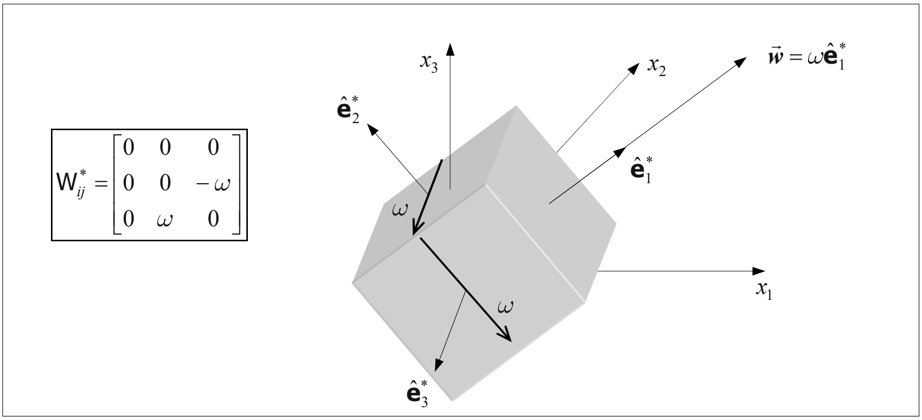


Figure 1.17: Antisymmetric tensor components in the space defined by the axial vector.

1.5.2.2.3 Additive decomposition. Symmetric and antisymmetric part

Any arbitrary second-order tensor $\mathbf{A}$ can be split additively into a symmetric and an antisymmetric part:

$$\mathbf{A} = \underbrace{\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)}_{\mathbf{A}^{sym}} + \underbrace{\frac{1}{2}(\mathbf{A}-\mathbf{A}^T)}_{\mathbf{A}^{skew}} = \mathbf{A}^{sym} + \mathbf{A}^{skew} \tag{1.150}$$

or, into its components:

$$\mathsf{A}_{ij}^{sym} = \frac{1}{2}(\mathsf{A}_{ij}+\mathsf{A}_{ji}) \quad and \quad \mathsf{A}_{ij}^{skew} = \frac{1}{2}(\mathsf{A}_{ij}-\mathsf{A}_{ji}) \tag{1.151}$$

If $\mathbf{A}$ and $\mathbf{B}$ are arbitrary second-order tensors, it holds that:

$$\begin{aligned}(\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A})^{sym} &= \frac{1}{2}\left[(\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A}) + (\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A})^T\right] = \frac{1}{2}\left[\mathbf{A}^T\cdot\mathbf{B}\cdot\mathbf{A} + \mathbf{A}^T\cdot\mathbf{B}^T\cdot\mathbf{A}\right] \\ &= \mathbf{A}^T\cdot\frac{1}{2}\left[\mathbf{B}+\mathbf{B}^T\right]\cdot\mathbf{A} = \mathbf{A}^T\cdot\mathbf{B}^{sym}\cdot\mathbf{A}\end{aligned} \tag{1.152}$$

Problem 1.15: Show that $\boldsymbol{\sigma}:\mathbf{W}=0$ is always true when $\boldsymbol{\sigma}$ is a symmetric second-order tensor and $\mathbf{W}$ is an antisymmetric second-order tensor.

Solution:

$$\boldsymbol{\sigma}:\mathbf{W}=\sigma_{ij}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j):\mathrm{W}_{lk}(\hat{\mathbf{e}}_l\otimes\hat{\mathbf{e}}_k)=\sigma_{ij}\mathrm{W}_{lk}\delta_{il}\delta_{jk}=\sigma_{ij}\mathrm{W}_{ij}\quad\text{(scalar)}$$

Thus,

$$\sigma_{ij}\mathrm{W}_{ij}=\underbrace{\sigma_{1j}\mathrm{W}_{1j}}_{\substack{\sigma_{11}\mathrm{W}_{11}\\+\\\sigma_{12}\mathrm{W}_{12}\\+\\\sigma_{13}\mathrm{W}_{13}}}+\underbrace{\sigma_{2j}\mathrm{W}_{2j}}_{\substack{\sigma_{21}\mathrm{W}_{21}\\+\\\sigma_{22}\mathrm{W}_{22}\\+\\\sigma_{23}\mathrm{W}_{23}}}+\underbrace{\sigma_{3j}\mathrm{W}_{3j}}_{\substack{\sigma_{31}\mathrm{W}_{31}\\+\\\sigma_{32}\mathrm{W}_{32}\\+\\\sigma_{33}\mathrm{W}_{33}}}$$

Taking into account the characteristics of a symmetric and an antisymmetric tensor, *i.e.* $\sigma_{12}=\sigma_{21}$, $\sigma_{31}=\sigma_{13}$, $\sigma_{32}=\sigma_{23}$, and $\mathrm{W}_{11}=\mathrm{W}_{22}=\mathrm{W}_{33}=0$, $\mathrm{W}_{21}=-\mathrm{W}_{12}$, $\mathrm{W}_{31}=-\mathrm{W}_{13}$, $\mathrm{W}_{32}=-\mathrm{W}_{23}$, the equation above becomes:

$$\boldsymbol{\sigma}:\mathbf{W}=0$$

Problem 1.16: Show that a) $\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}$; b) $\mathbf{A}:\mathbf{B}=\mathbf{A}^{sym}:\mathbf{B}^{sym}+\mathbf{A}^{skew}:\mathbf{B}^{skew}$

where $\vec{\mathbf{M}}$ is a vector, and $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{B}$ are arbitrary second-order tensors.

Solution:

a) $\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\left(\mathbf{Q}^{sym}+\mathbf{Q}^{skew}\right)\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}+\vec{\mathbf{M}}\cdot\mathbf{Q}^{skew}\cdot\vec{\mathbf{M}}$

Since the relation $\vec{\mathbf{M}}\cdot\mathbf{Q}^{skew}\cdot\vec{\mathbf{M}}=\mathbf{Q}^{skew}:\underbrace{\left(\vec{\mathbf{M}}\otimes\vec{\mathbf{M}}\right)}_{symmetric\ tensor}=0$ holds, it follows that:

$$\vec{\mathbf{M}}\cdot\mathbf{Q}\cdot\vec{\mathbf{M}}=\vec{\mathbf{M}}\cdot\mathbf{Q}^{sym}\cdot\vec{\mathbf{M}}$$

b)

$$\begin{aligned}\mathbf{A}:\mathbf{B}&=(\mathbf{A}^{sym}+\mathbf{A}^{skew}):(\mathbf{B}^{sym}+\mathbf{B}^{skew})\\&=\mathbf{A}^{sym}:\mathbf{B}^{sym}+\underbrace{\mathbf{A}^{sym}:\mathbf{B}^{skew}}_{=0}+\underbrace{\mathbf{A}^{skew}:\mathbf{B}^{sym}}_{=0}+\mathbf{A}^{skew}:\mathbf{B}^{skew}\\&=\mathbf{A}^{sym}:\mathbf{B}^{sym}+\mathbf{A}^{skew}:\mathbf{B}^{skew}\end{aligned}$$

Then, it is also valid that:

$$\mathbf{A}:\mathbf{B}^{sym}=\mathbf{A}^{sym}:\mathbf{B}^{sym}\quad;\quad\mathbf{A}:\mathbf{B}^{skew}=\mathbf{A}^{skew}:\mathbf{B}^{skew}$$

Problem 1.17: Let $\mathbf{T}$ be an arbitrary second-order tensor, and $\vec{\mathbf{n}}$ be a vector. Check if the relationship $\vec{\mathbf{n}}\cdot\mathbf{T}=\mathbf{T}\cdot\vec{\mathbf{n}}$ is valid.

Solution:

$$\begin{aligned}\vec{\mathbf{n}}\cdot\mathbf{T}&=\mathrm{n}_i\hat{\mathbf{e}}_i\cdot\mathrm{T}_{kl}(\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l)\\&=\mathrm{n}_i\mathrm{T}_{kl}\delta_{ik}\hat{\mathbf{e}}_l\\&=\mathrm{n}_k\mathrm{T}_{kl}\hat{\mathbf{e}}_l\\&=(\mathrm{n}_1\mathrm{T}_{1l}+\mathrm{n}_2\mathrm{T}_{2l}+\mathrm{n}_3\mathrm{T}_{3l})\hat{\mathbf{e}}_l\end{aligned}\qquad\text{and}\qquad\begin{aligned}\mathbf{T}\cdot\vec{\mathbf{n}}&=\mathrm{T}_{lk}(\hat{\mathbf{e}}_l\otimes\hat{\mathbf{e}}_k)\cdot\mathrm{n}_i\hat{\mathbf{e}}_i\\&=\mathrm{n}_i\mathrm{T}_{lk}\delta_{ki}\hat{\mathbf{e}}_l\\&=\mathrm{n}_k\mathrm{T}_{lk}\hat{\mathbf{e}}_l\\&=(\mathrm{n}_1\mathrm{T}_{l1}+\mathrm{n}_2\mathrm{T}_{l2}+\mathrm{n}_3\mathrm{T}_{l3})\hat{\mathbf{e}}_l\end{aligned}$$

With the above we can prove that $\mathrm{n}_k\mathrm{T}_{kl}\neq\mathrm{n}_k\mathrm{T}_{lk}$, then:

$$\vec{\mathbf{n}}\cdot\mathbf{T}\neq\mathbf{T}\cdot\vec{\mathbf{n}}$$

If $\mathbf{T}$ is a symmetric tensor, it follows that the relationship $\vec{\mathbf{n}}\cdot\mathbf{T}^{sym}=\mathbf{T}^{sym}\cdot\vec{\mathbf{n}}$ holds.

Problem 1.18: Obtain the axial vector $\vec{w}$ associated with the antisymmetric tensor $(\vec{x}\otimes\vec{a})^{skew}$.

Solution: Let $\vec{z}$ be an arbitrary vector, it then holds that:

$$(\vec{x}\otimes\vec{a})^{skew}\cdot\vec{z}=\vec{w}\wedge\vec{z}$$

where $\vec{w}$ is the axial vector associated with $(\vec{x}\otimes\vec{a})^{skew}$. Using the definition of an antisymmetric tensor:

$$(\vec{x}\otimes\vec{a})^{skew}=\frac{1}{2}\left[(\vec{x}\otimes\vec{a})-(\vec{x}\otimes\vec{a})^{T}\right]=\frac{1}{2}\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]$$

and by replacing it with $(\vec{x}\otimes\vec{a})^{skew}\cdot\vec{z}=\vec{w}\wedge\vec{z}$, we obtain:

$$\frac{1}{2}\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=\vec{w}\wedge\vec{z}\qquad\Rightarrow\qquad\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=2\vec{w}\wedge\vec{z}$$

By using the equation $\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=\vec{z}\wedge(\vec{x}\wedge\vec{a})$, (see Eq. (1.105)), the above equation becomes:

$$\left[\vec{x}\otimes\vec{a}-\vec{a}\otimes\vec{x}\right]\cdot\vec{z}=\vec{z}\wedge(\vec{x}\wedge\vec{a})=(\vec{a}\wedge\vec{x})\wedge\vec{z}=2\vec{w}\wedge\vec{z}$$

with the above we can conclude that:

$$\vec{w}=\frac{1}{2}(\vec{a}\wedge\vec{x})\ \ \textit{is the axial vector associated with}\ \ (\vec{x}\otimes\vec{a})^{skew}$$

1.5.2.3 Cofactor Tensor. Adjugate of a Tensor

Let $\mathbf{A}$ be a second-order tensor and $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be arbitrary vectors then there is then a unique tensor $\mathrm{cof}(\mathbf{A})$, known as the *cofactor* of $\mathbf{A}$, as we can see below:

$$\mathrm{cof}(\mathbf{A})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})=(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\tag{1.153}$$

We can also define the adjugate of $\mathbf{A}$ as:

$$\mathrm{adj}(\mathbf{A})=[\mathrm{cof}(\mathbf{A})]^{T}\tag{1.154}$$

which satisfies the following condition:

$$[\mathrm{adj}(\mathbf{A})]^{T}=\mathrm{adj}(\mathbf{A}^{T})\tag{1.155}$$

The components of $\mathrm{cof}(\mathbf{A})$ are obtained by expressing the equation (1.153) in terms of its components, *i.e.*:

$$[\mathrm{cof}(\mathbf{A})]_{it}\epsilon_{tpr}\mathrm{a}_{p}\mathrm{b}_{r}=\epsilon_{ijk}\mathrm{A}_{jp}\mathrm{a}_{p}\mathrm{A}_{kr}\mathrm{b}_{r}\qquad\Rightarrow\qquad[\mathrm{cof}(\mathbf{A})]_{it}\epsilon_{tpr}=\epsilon_{ijk}\mathrm{A}_{jp}\mathrm{A}_{kr}\tag{1.156}$$

By multiplying both sides of the equation by ϵ_{qpr} and by also considering that $\epsilon_{tpr}\epsilon_{qpr}=2\delta_{tq}$, we can conclude that:

$$\begin{aligned}&[\mathrm{cof}(\mathbf{A})]_{it}\epsilon_{tpr}=\epsilon_{ijk}\mathrm{A}_{jp}\mathrm{A}_{kr}\qquad\Rightarrow\qquad[\mathrm{cof}(\mathbf{A})]_{it}\underbrace{\epsilon_{tpr}\epsilon_{qpr}}_{=2\delta_{tq}}=\epsilon_{ijk}\epsilon_{qpr}\mathrm{A}_{jp}\mathrm{A}_{kr}\\&\Rightarrow[\mathrm{cof}(\mathbf{A})]_{iq}=\frac{1}{2}\epsilon_{ijk}\epsilon_{qpr}\mathrm{A}_{jp}\mathrm{A}_{kr}\end{aligned}\tag{1.157}$$

1.5.2.4 Tensor Trace

Let's start by defining the trace of the basis $(\hat{\mathbf{e}}_{i}\otimes\hat{\mathbf{e}}_{j})$:

$$\mathrm{Tr}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \tag{1.158}$$

Thus, we can define the trace of a second-order tensor $\mathbf{A}$ as follows:

$$\begin{aligned}\mathrm{Tr}(\mathbf{A}) &= \mathrm{Tr}(\mathrm{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathrm{A}_{ij}\mathrm{Tr}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathrm{A}_{ij}(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = \mathrm{A}_{ij}\delta_{ij} = \mathrm{A}_{ii} \\ &= \mathrm{A}_{11} + \mathrm{A}_{22} + \mathrm{A}_{33}\end{aligned} \tag{1.159}$$

And, the trace of the dyad $(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}})$ can be evaluated as:

$$\begin{aligned}\mathrm{Tr}(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) &= \mathrm{Tr}(\vec{\mathbf{u}} \otimes \vec{\mathbf{v}}) = \mathrm{u}_i\mathrm{v}_j\mathrm{Tr}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathrm{u}_i\mathrm{v}_j(\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j) = \mathrm{u}_i\mathrm{v}_j\delta_{ij} = \mathrm{u}_i\mathrm{v}_i \\ &= \mathrm{u}_1\mathrm{v}_1 + \mathrm{u}_2\mathrm{v}_2 + \mathrm{u}_3\mathrm{v}_3 = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}}\end{aligned} \tag{1.160}$$

NOTE: As we will show later, the tensor trace is an *invariant*, *i.e.* it is independent of the coordinate system. ■

Let $\mathbf{A}$, $\mathbf{B}$ be arbitrary tensors, then:

- The transposed tensor trace is equal to the tensor trace:

$$\mathrm{Tr}\left(\mathbf{A}^T\right) = \mathrm{Tr}(\mathbf{A}) \tag{1.161}$$

- The trace of $(\mathbf{A} + \mathbf{B})$ is the sum of traces:

$$\mathrm{Tr}(\mathbf{A} + \mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B}) \tag{1.162}$$

Proving this is very simple:

$$\begin{aligned}&\mathrm{Tr}(\mathbf{A} + \mathbf{B}) = \mathrm{Tr}(\mathbf{A}) + \mathrm{Tr}(\mathbf{B}) \\ &[(\mathrm{A}_{11} + \mathrm{B}_{11}) + (\mathrm{A}_{22} + \mathrm{B}_{22}) + (\mathrm{A}_{33} + \mathrm{B}_{33})] = (\mathrm{A}_{11} + \mathrm{A}_{22} + \mathrm{A}_{33}) + (\mathrm{B}_{11} + \mathrm{B}_{22} + \mathrm{B}_{33})\end{aligned} \tag{1.163}$$

- The scalar product trace $(\mathbf{A} \cdot \mathbf{B})$ becomes:

$$\begin{aligned}\mathrm{Tr}(\mathbf{A} \cdot \mathbf{B}) &= \mathrm{Tr}\left[(\mathrm{A}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \cdot (\mathrm{B}_{lm}\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_m)\right] \\ &= \mathrm{A}_{ij}\mathrm{B}_{lm}\delta_{jl}\underbrace{\mathrm{Tr}\left[(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_m)\right]}_{\delta_{im}} \\ &= \mathrm{A}_{il}\mathrm{B}_{li} = \mathbf{A} \cdot\cdot \mathbf{B} = \mathrm{Tr}(\mathbf{B} \cdot \mathbf{A})\end{aligned} \tag{1.164}$$

and, the double scalar product (**:**) can be expressed in trace terms as:

$$\begin{aligned}\mathbf{A} : \mathbf{B} &= \mathrm{A}_{ij}\mathrm{B}_{ij} \\ &= \mathrm{A}_{kj}\mathrm{B}_{lj}\delta_{ik}\delta_{il} && = \mathrm{A}_{ik}\mathrm{B}_{il}\delta_{jk}\delta_{jl} \\ &= \underbrace{\mathrm{A}_{kj}\mathrm{B}_{lj}}_{(\mathbf{A}\cdot\mathbf{B}^T)_{kl}}\delta_{kl} && = \underbrace{\mathrm{A}_{ik}\mathrm{B}_{il}}_{(\mathbf{A}^T\cdot\mathbf{B})_{kl}}\delta_{kl} \\ &= \left(\mathbf{A} \cdot \mathbf{B}^T\right)_{kk} && = \left(\mathbf{A}^T \cdot \mathbf{B}\right)_{kk} \\ &= \mathrm{Tr}(\mathbf{A} \cdot \mathbf{B}^T) && = \mathrm{Tr}(\mathbf{A}^T \cdot \mathbf{B})\end{aligned} \tag{1.165}$$

Similarly, it is possible to show that:

$$\mathrm{Tr}(\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C}) = \mathrm{Tr}(\mathbf{B} \cdot \mathbf{C} \cdot \mathbf{A}) = \mathrm{Tr}(\mathbf{C} \cdot \mathbf{A} \cdot \mathbf{B}) = \mathrm{A}_{ij}\mathrm{B}_{jk}\mathrm{C}_{ki} \tag{1.166}$$

$$\begin{aligned}&\mathrm{Tr}(\mathbf{A}) = \mathrm{A}_{ii} \\ &[\mathrm{Tr}(\mathbf{A})]^2 = \mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{A}) = \mathrm{A}_{ii}\mathrm{A}_{jj} \\ &\mathrm{Tr}(\mathbf{A} \cdot \mathbf{A}) = \mathrm{Tr}\left(\mathbf{A}^2\right) = \mathrm{A}_{il}\mathrm{A}_{li}; \quad \mathrm{Tr}(\mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A}) = \mathrm{Tr}\left(\mathbf{A}^3\right) = \mathrm{A}_{ij}\mathrm{A}_{jk}\mathrm{A}_{ki}\end{aligned} \tag{1.167}$$

Problem 1.19: Let $\mathbf{T}$ be a second-order tensor. Show that:

$$\left(\mathbf{T}^m\right)^T=\left(\mathbf{T}^T\right)^m \quad and \quad \mathrm{Tr}\left(\mathbf{T}^T\right)^m=\mathrm{Tr}\left(\mathbf{T}^m\right).$$

Solution:

$$\left(\mathbf{T}^m\right)^T=\left(\mathbf{T}\cdot\mathbf{T}\cdots\mathbf{T}\right)^T=\mathbf{T}^T\cdot\mathbf{T}^T\cdots\mathbf{T}^T=\left(\mathbf{T}^T\right)^m$$

For the second demonstration we can use the trace property $\mathrm{Tr}\left(\mathbf{T}^T\right)=\mathrm{Tr}(\mathbf{T})$, thus:

$$\mathrm{Tr}\left(\mathbf{T}^T\right)^m=\mathrm{Tr}\left(\mathbf{T}^m\right)^T=\mathrm{Tr}\left(\mathbf{T}^m\right)$$

1.5.2.5 Particular Tensors

1.5.2.5.1 Unit Tensors

- The second-order unit tensor, also called the *identity tensor*, is defined as:

$$\mathbf{1}=\delta_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j=\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_i=\boldsymbol{1}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j \tag{1.168}$$

where $\boldsymbol{1}$ is the matrix with the components of tensor $\mathbf{1}$. δ_{ij} is the Kronecker delta symbol defined in (1.48).

- Fourth-order unit tensors can be defined as follows:

$$\mathbb{I}=\mathbf{1}\overline{\otimes}\mathbf{1}=\delta_{ik}\delta_{j\ell}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell=\mathbb{I}_{ijk\ell}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell \tag{1.169}$$

$$\bar{\mathbb{I}}=\mathbf{1}\underline{\otimes}\mathbf{1}=\delta_{i\ell}\delta_{jk}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell=\bar{\mathbb{I}}_{ijk\ell}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell \tag{1.170}$$

$$\bar{\bar{\mathbb{I}}}=\mathbf{1}\otimes\mathbf{1}=\delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell=\bar{\bar{\mathbb{I}}}_{ijk\ell}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell \tag{1.171}$$

Taking into account the fourth-order unit tensors defined above, it holds that:

$$\begin{aligned}\mathbb{I}:\mathbf{A}&=\left(\delta_{ik}\delta_{j\ell}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell\right):\left(\mathsf{A}_{pq}\hat{\mathbf{e}}_p\otimes\hat{\mathbf{e}}_q\right)=\delta_{ik}\delta_{j\ell}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\\&=\delta_{ik}\delta_{j\ell}\mathsf{A}_{k\ell}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)=\mathsf{A}_{ij}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\\&=\mathbf{A}\end{aligned} \tag{1.172}$$

and

$$\begin{aligned}\bar{\mathbb{I}}:\mathbf{A}&=\left(\delta_{i\ell}\delta_{jk}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell\right):\left(\mathsf{A}_{pq}\hat{\mathbf{e}}_p\otimes\hat{\mathbf{e}}_q\right)=\delta_{i\ell}\delta_{jk}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\\&=\delta_{i\ell}\delta_{jk}\mathsf{A}_{k\ell}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)=\mathsf{A}_{ji}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\\&=\mathbf{A}^T\end{aligned} \tag{1.173}$$

and

$$\begin{aligned}\bar{\bar{\mathbb{I}}}:\mathbf{A}&=\left(\delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_\ell\right):\left(\mathsf{A}_{pq}\hat{\mathbf{e}}_p\otimes\hat{\mathbf{e}}_q\right)=\delta_{ij}\delta_{k\ell}\mathsf{A}_{pq}\delta_{kp}\delta_{\ell q}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\\&=\delta_{ij}\delta_{k\ell}\mathsf{A}_{k\ell}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)=\mathsf{A}_{kk}\delta_{ij}\left(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\right)\\&=\mathrm{Tr}(\mathbf{A})\mathbf{1}\end{aligned} \tag{1.174}$$

The symmetric part of the fourth-order unit tensor $\mathbb{I}$ is defined as:

$$\mathbb{I}^{sym}\equiv\mathbf{I}=\frac{1}{2}\left(\mathbf{1}\overline{\otimes}\mathbf{1}+\mathbf{1}\underline{\otimes}\mathbf{1}\right)\xrightarrow{\text{in components}}\mathrm{I}_{ijk\ell}=\frac{1}{2}\left(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\right) \tag{1.175}$$

The property of the tensor product $\overline{\otimes}$ is presented below. Consider a second-order unit tensor, $\mathbf{1}=\delta_{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j$. Then, the tensor product $\overline{\otimes}$ can be defined as:

$$\mathbf{1}\overline{\otimes}\mathbf{1} = (\delta_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\overline{\otimes}(\delta_{k\ell}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell) = \delta_{ij}\delta_{k\ell}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_\ell) \tag{1.176}$$

which is the same as:

$$\mathbb{I} = \mathbf{1}\overline{\otimes}\mathbf{1} = \delta_{ik}\delta_{j\ell}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell) \tag{1.177}$$

and the tensor product $\underline{\otimes}$ is defined as:

$$\mathbf{1}\underline{\otimes}\mathbf{1} = (\delta_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\underline{\otimes}(\delta_{k\ell}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell) = \delta_{ij}\delta_{k\ell}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_\ell \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_j) \tag{1.178}$$

or

$$\bar{\mathbb{I}} = \mathbf{1}\underline{\otimes}\mathbf{1} = \delta_{i\ell}\delta_{jk}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell) \tag{1.179}$$

The antisymmetric part of $\mathbb{I}$ is defined as:

$$\mathbb{I}^{skew} = \frac{1}{2}\left(\mathbf{1}\overline{\otimes}\mathbf{1} - \mathbf{1}\underline{\otimes}\mathbf{1}\right) \xrightarrow{\text{in components}} \mathbb{I}^{skew}_{ijk\ell} = \frac{1}{2}\left(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}\right) \tag{1.180}$$

With a second-order tensor $\mathbf{A}$ and a vector $\vec{\mathbf{b}}$, the following relationships are valid:

$$\begin{aligned}
&\vec{\mathbf{b}} \cdot \mathbf{1} = \vec{\mathbf{b}} \\
&\mathbb{I} : \mathbf{A} = \mathbf{A} \quad ; \quad \mathbb{I}^{sym} : \mathbf{A} = \mathbf{A}^{sym} \\
&\mathbf{A} : \mathbf{1} = \mathrm{Tr}(\mathbf{A}) = \mathsf{A}_{ii} \\
&\mathbf{A}^2 : \mathbf{1} = \mathrm{Tr}(\mathbf{A}^2) = \mathrm{Tr}(\mathbf{A} \cdot \mathbf{A}) = \mathsf{A}_{il}\mathsf{A}_{li} \\
&\mathbf{A}^3 : \mathbf{1} = \mathrm{Tr}(\mathbf{A}^3) = \mathrm{Tr}(\mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A}) = \mathsf{A}_{ij}\mathsf{A}_{jk}\mathsf{A}_{ki}
\end{aligned} \tag{1.181}$$

Problem 1.20: Show that $\mathbf{T} : \mathbf{1} = \mathrm{Tr}(\mathbf{T})$, where $\mathbf{T}$ is an arbitrary second-order tensor.
Solution:

$$\begin{aligned}
\mathbf{T} : \mathbf{1} &= \mathsf{T}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j : \delta_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l \\
&= \mathsf{T}_{ij}\delta_{kl}\delta_{ik}\delta_{jl} \\
&= \mathsf{T}_{ij}\delta_{ij} = \mathsf{T}_{ii} = \mathsf{T}_{jj} \\
&= \mathrm{Tr}(\mathbf{T})
\end{aligned}$$

1.5.2.5.2 Levi-Civita Pseudo-Tensor

The *Levi-Civita Pseudo-Tensor*, also known as the *Permutation Tensor*, is a third-order pseudo-tensor and is denoted by:

$$\boldsymbol{\epsilon} = \epsilon_{ijk}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \tag{1.182}$$

which is not a "true" tensor in the strict meaning of the word, and whose components ϵ_{ijk} were defined in (1.55), the permutation symbol.

1.5.2.6 Determinant of a Tensor

The determinant of a tensor is a scalar and is expressed as:

$$\det(\mathbf{A}) \equiv |\mathbf{A}| = \epsilon_{ijk}\mathsf{A}_{1i}\mathsf{A}_{2j}\mathsf{A}_{3k} = \underbrace{\epsilon_{ijk}\mathsf{A}_{i1}\mathsf{A}_{j2}\mathsf{A}_{k3}}_{|\mathbf{A}^T|} \tag{1.183}$$

It is also an invariant (independent of the adopted system). Demonstrating the equation above (1.183) can be done starting directly from the determinant:

$$\begin{aligned}\det(\mathbf{A}) &= |\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33}\end{vmatrix} \\ &= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{21}(A_{12}A_{33} - A_{13}A_{32}) + A_{31}(A_{12}A_{23} - A_{13}A_{22}) \\ &= A_{11}(\epsilon_{1jk}A_{j2}A_{k3}) - A_{21}(-\epsilon_{2jk}A_{j2}A_{k3}) + A_{31}(\epsilon_{3jk}A_{j2}A_{k3}) \\ &= \epsilon_{1jk}A_{11}A_{j2}A_{k3} + \epsilon_{2jk}A_{21}A_{j2}A_{k3} + \epsilon_{3jk}A_{31}A_{j2}A_{k3} \\ &= \epsilon_{ijk}A_{i1}A_{j2}A_{k3}\end{aligned} \tag{1.184}$$

Some determinant properties with second-order tensors are described below:

$$\det(\mathbf{1}) = 1 \tag{1.185}$$

- We can conclude from (1.183) that:

$$\det(\mathbf{A}^T) = \det(\mathbf{A}) \tag{1.186}$$

- We can also show that:

$$\det(\mathbf{A}\cdot\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B}), \quad \det(\alpha\mathbf{A}) = \alpha^3\det(\mathbf{A}) \quad \text{where } \alpha \text{ is a scalar} \tag{1.187}$$

- A tensor $(\mathbf{A})$ is said to be *singular* if $\det(\mathbf{A}) = 0$.
- If you exchange two rows or columns, the determinant sign changes.
- If all elements of a row or column equal zero, the determinant is also zero.
- If you multiply all the elements of a row or column by a constant c (scalar), the determinant is $c|\mathbf{A}|$.
- If two or more rows (or column) are linearly dependent, the determinant is zero.

Problem 1.21: Show that $|\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}A_{rt}A_{jp}A_{kq}$.

Solution:

We start with the following definition:

$$|\mathbf{A}| = \epsilon_{rjk}A_{r1}A_{j2}A_{k3} \quad \Rightarrow \quad |\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}\epsilon_{tpq}A_{r1}A_{j2}A_{k3} \tag{1.188}$$

and also taking into account that the term $\epsilon_{rjk}\epsilon_{tpq}$ can be replaced by (1.61):

$$\begin{aligned}\epsilon_{rjk}\epsilon_{tpq} &= \begin{vmatrix} \delta_{rt} & \delta_{rp} & \delta_{rq} \\ \delta_{jt} & \delta_{jp} & \delta_{jq} \\ \delta_{kt} & \delta_{kp} & \delta_{kq}\end{vmatrix} \\ &= \delta_{rt}\delta_{jp}\delta_{kq} + \delta_{rp}\delta_{jq}\delta_{kt} + \delta_{rq}\delta_{jt}\delta_{kp} - \delta_{rq}\delta_{jp}\delta_{kt} - \delta_{jq}\delta_{kp}\delta_{rt} - \delta_{kq}\delta_{jt}\delta_{rp}\end{aligned} \tag{1.189}$$

Then, by substituting (1.189) into (1.188) we can obtain:

$$\begin{aligned}|\mathbf{A}|\epsilon_{tpq} &= A_{t1}A_{p2}A_{q3} + A_{p1}A_{q2}A_{t3} + A_{q1}A_{t2}A_{p3} - A_{q1}A_{p2}A_{t3} - A_{t1}A_{q2}A_{p3} - A_{p1}A_{t2}A_{q3} \\ &= A_{t1}\left(\epsilon_{1jk}A_{pj}A_{qk}\right) + A_{t2}\left(\epsilon_{2jk}A_{pj}A_{qk}\right) + A_{t3}\left(\epsilon_{3jk}A_{pj}A_{qk}\right) \\ &= \epsilon_{rjk}A_{rt}A_{jp}A_{kq} = \epsilon_{rjk}A_{tr}A_{pj}A_{qk}\end{aligned}$$

Problem 1.22: Show that $|\mathbf{A}| = \frac{1}{6}\epsilon_{rjk}\epsilon_{tpq}A_{rt}A_{jp}A_{kq}$.

Solution:

Starting with the definition $|\mathbf{A}|\epsilon_{tpq} = \epsilon_{rjk}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$, (see **Problem 1.21**), and by multiplying both sides of the equation by ϵ_{tpq}, we obtain:

$$|\mathbf{A}|\epsilon_{tpq}\epsilon_{tpq} = \epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq} \tag{1.190}$$

Using the property defined in expression (1.62), we obtain $\epsilon_{tpq}\epsilon_{tpq} = \delta_{tt}\delta_{pp} - \delta_{tp}\delta_{tp} = \delta_{tt}\delta_{pp} - \delta_{tt} = 6$. Then, the relationship (1.190) becomes:

$$|\mathbf{A}| = \frac{1}{6}\epsilon_{rjk}\epsilon_{tpq}\mathsf{A}_{rt}\mathsf{A}_{jp}\mathsf{A}_{kq}$$

Problem 1.23: Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ be arbitrary vectors and α be a scalar. Show that:

$$\boxed{\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \mu^3 + \mu^2\alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}} \tag{1.191}$$

Solution: The determinant of $\mathbf{A}$ is given by $|\mathbf{A}| = \epsilon_{ijk}\mathsf{A}_{i1}\mathsf{A}_{j2}\mathsf{A}_{k3}$. If we denote by $\mathsf{A}_{ij}=\mu\delta_{ij} + \alpha\mathsf{a}_i\mathsf{b}_j$, thus, $\mathsf{A}_{i1} = \mu\delta_{i1} + \alpha\mathsf{a}_i\mathsf{b}_1$, $\mathsf{A}_{k3} = \mu\delta_{k3} + \alpha\mathsf{a}_k\mathsf{b}_3$, $\mathsf{A}_{k3} = \delta_{k3} + \alpha\mathsf{a}_k\mathsf{b}_3$, then the equation in (1.191) can be rewritten as:

$$\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \epsilon_{ijk}(\mu\delta_{i1} + \alpha\mathsf{a}_i\mathsf{b}_1)(\mu\delta_{j2} + \alpha\mathsf{a}_j\mathsf{b}_2)(\mu\delta_{k3} + \alpha\mathsf{a}_k\mathsf{b}_3) \tag{1.192}$$

By developing the equation (1.192), we obtain:

$$\begin{aligned}\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \epsilon_{ijk}\big[&\mu^3\delta_{i1}\delta_{j2}\delta_{k3} + \mu^2\alpha\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2} + \mu^2\alpha\mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3} + \mu^2\alpha\mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3} + \\ &+ \mu\alpha^2\mathsf{a}_j\mathsf{b}_2\mathsf{a}_k\mathsf{b}_3\delta_{i1} + \mu\alpha^2\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2} + \mu\alpha^2\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3} + \alpha^3\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3\big]\end{aligned}$$

Note that: $\mu^3\epsilon_{ijk}\delta_{i1}\delta_{j2}\delta_{k3} = \mu^3\epsilon_{123} = \mu^3$,

$$\mu^2\alpha(\epsilon_{ijk}\mathsf{a}_k\mathsf{b}_3\delta_{i1}\delta_{j2} + \epsilon_{ijk}\mathsf{a}_j\mathsf{b}_2\delta_{i1}\delta_{k3} + \epsilon_{ijk}\mathsf{a}_i\mathsf{b}_1\delta_{j2}\delta_{k3}) =$$

$$\mu^2\alpha(\epsilon_{12k}\mathsf{a}_k\mathsf{b}_3 + \epsilon_{1j3}\mathsf{a}_j\mathsf{b}_2 + \epsilon_{i23}\mathsf{a}_i\mathsf{b}_1) = \mu^2\alpha(\mathsf{a}_3\mathsf{b}_3 + \mathsf{a}_2\mathsf{b}_2 + \mathsf{a}_1\mathsf{b}_1) = \mu^2\alpha(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})$$

$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2} = \epsilon_{i2k}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3 = \mathsf{a}_1\mathsf{a}_3\mathsf{b}_1\mathsf{b}_3 - \mathsf{a}_3\mathsf{a}_1\mathsf{b}_1\mathsf{b}_3 = 0$$

$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3} = \epsilon_{ij3}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2 = \epsilon_{123}\mathsf{a}_1\mathsf{a}_2\mathsf{b}_1\mathsf{b}_2 - \epsilon_{213}\mathsf{a}_2\mathsf{a}_1\mathsf{b}_1\mathsf{b}_2 = 0$$

$$\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3 = 0$$

Notice that, there was no need to expand the terms $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2}$, $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{b}_1\mathsf{b}_2\delta_{k3}$, and $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3$ to realize that these terms equal zero, since $\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_k\mathsf{b}_1\mathsf{b}_3\delta_{j2} = (\vec{\mathbf{a}}\wedge\vec{\mathbf{a}})_j\mathsf{b}_1\mathsf{b}_3\delta_{j2} = 0$, similarly for other terms.

Taking into account the above considerations we can prove that:

$$\det(\mu\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \mu^3 + \mu^2\alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$$

For the particular case when $\mu = 1$ the above equation becomes:

$$\boxed{\det(\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = 1 + \alpha\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}}$$

Then, it is simple to prove that $\det(\alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = 0$, since

$$\det(\alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = \alpha^3\epsilon_{ijk}\mathsf{a}_i\mathsf{a}_j\mathsf{a}_k\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3 = \alpha^3\mathsf{b}_1\mathsf{b}_2\mathsf{b}_3[\vec{\mathbf{a}}\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{a}})] = 0$$

The following relations are satisfied:

$$\det[\mathbf{1} + \alpha(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) + \beta(\vec{\mathbf{b}}\otimes\vec{\mathbf{a}})] = 1 + \alpha(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) + \beta(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) + \alpha\beta[(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})^2 - (\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}})] \tag{1.193}$$

where α, β are scalars. If $\beta = 0$ we can regain the equation $\det(\mathbf{1} + \alpha\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}) = 1 + \alpha\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}$, (see **Problem 1.23**). If $\alpha = \beta$ we obtain:

$$\det\left(\mathbf{1}+\alpha\,\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}+\alpha\,\vec{\mathbf{b}}\otimes\vec{\mathbf{a}}\right)=1+\alpha\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)+\alpha\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)+\alpha^2\left[\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)^2-\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}\right)\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}}\right)\right]$$
$$=1+\alpha\left[2\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)-\alpha\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)^2\right] \qquad (1.194)$$

where we have used the property $\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)^2-\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{a}}\right)\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{b}}\right)=-\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)^2$, (see **Problem 1.1**).

It is also true that:

$$\det(\alpha\,\mathbf{A}+\beta\,\mathbf{B})=\alpha^3\det(\mathbf{A})+\alpha^2\beta\,\mathrm{Tr}[\mathbf{B}\cdot\mathrm{adj}(\mathbf{A})]+\alpha\beta^2\,\mathrm{Tr}[\mathbf{A}\cdot\mathrm{adj}(\mathbf{B})]+\beta^3\det(\mathbf{B}) \qquad (1.195)$$

Moreover, in the particular case when $\alpha=1$, $\mathbf{A}=\mathbf{1}$, $\mathbf{B}=\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}$, and bearing in mind that $\det\left(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\right)=0$, and $\mathrm{cof}\left(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\right)=\mathbf{0}$, we can conclude that:

$$\det\left(\mathbf{1}+\beta\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\right)=\det(\mathbf{1})+\beta\mathrm{Tr}\left[\vec{\mathbf{a}}\otimes\vec{\mathbf{b}}\cdot\mathbf{1}\right]=1+\beta\mathrm{Tr}\left[a_i b_j\right]=1+\beta\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} \qquad (1.196)$$

which has already been demonstrated in **Problem 1.23**.

We next show that the following property is valid:

$$\boxed{(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]=\det(\mathbf{A})\left[\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}})\right]} \qquad (1.197)$$

To achieve this goal we start with the definition of the scalar triple product given in (1.69), *i.e.* $\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)=\epsilon_{ijk}a_i b_j c_k$, and by multiplying both sides of this equation by the determinant of $\mathbf{A}$ we obtain:

$$\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)|\mathbf{A}|=\epsilon_{ijk}a_i b_j c_k|\mathbf{A}| \qquad (1.198)$$

It was proven in **Problem 1.21** that $|\mathbf{A}|\epsilon_{ijk}=\epsilon_{pqr}A_{pi}A_{qj}A_{rk}$, thus:

$$\vec{\mathbf{a}}\cdot\left(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}\right)|\mathbf{A}|=\epsilon_{ijk}a_i b_j c_k|\mathbf{A}|=\epsilon_{pqr}A_{pi}A_{qj}A_{rk}a_i b_j c_k=\epsilon_{pqr}(A_{pi}a_i)(A_{qj}b_j)(A_{rk}c_k)$$
$$=(\mathbf{A}\cdot\vec{\mathbf{a}})\cdot\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]=\left[(\mathbf{A}\cdot\vec{\mathbf{b}})\wedge(\mathbf{A}\cdot\vec{\mathbf{c}})\right]\cdot(\mathbf{A}\cdot\vec{\mathbf{a}}) \qquad (1.199)$$

1.5.2.7 Inverse of a Tensor

We use the notation $\mathbf{A}^{-1}$ to denote the inverse of $\mathbf{A}$, which is defined as follows:

$$\text{if } |\mathbf{A}|\neq 0 \quad\Leftrightarrow\quad \exists\mathbf{A}^{-1}\ \left|\mathbf{A}\cdot\mathbf{A}^{-1}=\mathbf{A}^{-1}\cdot\mathbf{A}=\mathbf{1}\right. \qquad (1.200)$$

Or in indicial notation:

$$\text{if } |\mathbf{A}|\neq 0 \quad\Leftrightarrow\quad \exists A^{-1}_{ij}\ \left|A_{ik}A^{-1}_{kj}=A^{-1}_{ik}A_{kj}=\delta_{ij}\right. \qquad (1.201)$$

To obtain the inverse of a tensor we start from the definition of the adjugate tensor given in (1.153), *i.e.* $\mathrm{adj}(\mathbf{A}^T)\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})=(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})$. Then by applying the dot product between an arbitrary vector $\vec{\mathbf{d}}$ and this equation we obtain:

$$\left\{[\mathrm{adj}(\mathbf{A})]^T\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\right\}\cdot\vec{\mathbf{d}}=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot\vec{\mathbf{d}}=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot\mathbf{1}\cdot\vec{\mathbf{d}}$$
$$=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot\mathbf{A}\cdot\underbrace{\mathbf{A}^{-1}\cdot\vec{\mathbf{d}}}_{=\vec{\mathbf{c}}} \qquad (1.202)$$

In equation (1.199) we demonstrated that $\vec{\mathbf{c}}\cdot\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)|\mathbf{A}|=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot(\mathbf{A}\cdot\vec{\mathbf{c}})$ thus,

$$\left\{[\mathrm{adj}(\mathbf{A})]^T\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\right\}\cdot\vec{\mathbf{d}}=\left[(\mathbf{A}\cdot\vec{\mathbf{a}})\wedge(\mathbf{A}\cdot\vec{\mathbf{b}})\right]\cdot\mathbf{A}\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{d}}=|\mathbf{A}|\left(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}\right)\cdot\mathbf{A}^{-1}\cdot\vec{\mathbf{d}} \tag{1.203}$$

Denoted by $\vec{\mathbf{p}}=(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})$, the above equation (1.203) can be rearranged as follows:

$$\begin{aligned}&\{[\mathrm{adj}(\mathbf{A})]_{ki}\mathrm{p}_k\}\mathrm{d}_i=|\mathbf{A}|\mathrm{p}_k\mathrm{A}^{-1}_{ki}\mathrm{d}_i\\ \Rightarrow\ &[\mathrm{adj}(\mathbf{A})]_{ki}\mathrm{p}_k\mathrm{d}_i=|\mathbf{A}|\mathrm{A}^{-1}_{ki}\mathrm{p}_k\mathrm{d}_i\\ \Rightarrow\ &[\mathrm{adj}(\mathbf{A})]:\left[(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\otimes\vec{\mathbf{d}}\right]=|\mathbf{A}|\mathbf{A}^{-1}:\left[(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}})\otimes\vec{\mathbf{d}}\right]\end{aligned} \tag{1.204}$$

Thus, we can conclude that:

$$[\mathrm{adj}(\mathbf{A})]=|\mathbf{A}|\mathbf{A}^{-1}\quad\Rightarrow\quad\boxed{\mathbf{A}^{-1}=\frac{1}{|\mathbf{A}|}[\mathrm{adj}(\mathbf{A})]=\frac{1}{|\mathbf{A}|}[\mathrm{cof}(\mathbf{A})]^T} \tag{1.205}$$

- Let $\mathbf{A}$ and $\mathbf{B}$ be invertible tensors, the following properties hold:

$$\begin{aligned}&(\mathbf{A}\cdot\mathbf{B})^{-1}=\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}\\ &\left(\mathbf{A}^{-1}\right)^{-1}=\mathbf{A}\\ &(\beta\mathbf{A})^{-1}=\frac{1}{\beta}\mathbf{A}^{-1}\\ &\det(\mathbf{A}^{-1})=[\det(\mathbf{A})]^{-1}\end{aligned} \tag{1.206}$$

- The following notation will be used to represent the inverse transpose:

$$\mathbf{A}^{-T}\equiv(\mathbf{A}^{-1})^T\equiv(\mathbf{A}^T)^{-1} \tag{1.207}$$

Next, we prove the relation $\mathrm{adj}(\mathbf{A}\cdot\mathbf{B})=\mathrm{adj}(\mathbf{B})\cdot\mathrm{adj}(\mathbf{A})$ holds. To do this, we take the definition of the inverse of a tensor given in (1.205) as a starting point:

$$\begin{aligned}&\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}=\frac{[\mathrm{adj}(\mathbf{B})]}{|\mathbf{B}|}\cdot\frac{[\mathrm{adj}(\mathbf{A})]}{|\mathbf{A}|}\Rightarrow|\mathbf{A}||\mathbf{B}|\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}=[\mathrm{adj}(\mathbf{B})]\cdot[\mathrm{adj}(\mathbf{A})]\\ &\Rightarrow|\mathbf{A}||\mathbf{B}|(\mathbf{A}\cdot\mathbf{B})^{-1}=[\mathrm{adj}(\mathbf{B})]\cdot[\mathrm{adj}(\mathbf{A})]\Rightarrow|\mathbf{A}||\mathbf{B}|\frac{[\mathrm{adj}(\mathbf{A}\cdot\mathbf{B})]}{|\mathbf{A}\cdot\mathbf{B}|}=[\mathrm{adj}(\mathbf{B})]\cdot[\mathrm{adj}(\mathbf{A})]\\ &\Rightarrow\mathrm{adj}(\mathbf{A}\cdot\mathbf{B})=[\mathrm{adj}(\mathbf{B})]\cdot[\mathrm{adj}(\mathbf{A})]\end{aligned} \tag{1.208}$$

where we have used the property $|\mathbf{A}\cdot\mathbf{B}|=|\mathbf{A}||\mathbf{B}|$. Similarly, it is possible to show that $\mathrm{cof}(\mathbf{A}\cdot\mathbf{B})=[\mathrm{cof}(\mathbf{A})]\cdot[\mathrm{cof}(\mathbf{B})]$.

Procedure for obtaining the inverse of the matrix $\mathcal{A}$

1) Obtain the cofactor matrix: $\mathrm{cof}(\mathcal{A})$ as follows:

Consider the matrix $\mathcal{A}$ as:

$$\mathcal{A}=\begin{bmatrix}\mathrm{A}_{11}&\mathrm{A}_{12}&\mathrm{A}_{13}\\ \mathrm{A}_{21}&\mathrm{A}_{22}&\mathrm{A}_{23}\\ \mathrm{A}_{31}&\mathrm{A}_{32}&\mathrm{A}_{33}\end{bmatrix} \tag{1.209}$$

We define the matrix $\mathcal{M}$, where the component $\mathcal{M}_{ij}$ is the determinant of the resulting matrix by removing the i^{th} row and the j^{th} column, *i.e.*:

$$\mathcal{M}=\begin{bmatrix} \begin{vmatrix} \mathsf{A}_{22} & \mathsf{A}_{23} \\ \mathsf{A}_{32} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{21} & \mathsf{A}_{23} \\ \mathsf{A}_{31} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{21} & \mathsf{A}_{22} \\ \mathsf{A}_{31} & \mathsf{A}_{32} \end{vmatrix} \\ \begin{vmatrix} \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{32} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{13} \\ \mathsf{A}_{31} & \mathsf{A}_{33} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{31} & \mathsf{A}_{32} \end{vmatrix} \\ \begin{vmatrix} \mathsf{A}_{12} & \mathsf{A}_{13} \\ \mathsf{A}_{22} & \mathsf{A}_{23} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{13} \\ \mathsf{A}_{21} & \mathsf{A}_{23} \end{vmatrix} & \begin{vmatrix} \mathsf{A}_{11} & \mathsf{A}_{12} \\ \mathsf{A}_{21} & \mathsf{A}_{22} \end{vmatrix} \end{bmatrix} \tag{1.210}$$

Thus, we define the cofactor matrix as:

$$\mathrm{cof}(\mathcal{A})=(-1)^{i+j}\mathcal{M}_{ij} \tag{1.211}$$

2) Obtain the adjugate matrix, $\mathrm{adj}(\mathcal{A})$, as follows:

$$\mathrm{adj}(\mathcal{A})=[\mathrm{cof}(\mathcal{A})]^T \tag{1.212}$$

3) Obtain the inverse matrix:

$$\mathcal{A}^{-1}=\frac{\mathrm{adj}(\mathcal{A})}{|\mathcal{A}|} \tag{1.213}$$

So, the relation $\mathcal{A}[\mathrm{adj}(\mathcal{A})]=|\mathcal{A}|\boldsymbol{1}$ holds, where $\boldsymbol{1}$ is the identity matrix.

Taking into account (1.64), we can express the components of the first, second, and third row of the cofactor matrix, (1.211), as: $\mathsf{M}_{1i}=\epsilon_{ijk}\mathsf{A}_{2j}\mathsf{A}_{3k}$, $\mathsf{M}_{2i}=\epsilon_{ijk}\mathsf{A}_{1j}\mathsf{A}_{3k}$, $\mathsf{M}_{3i}=\epsilon_{ijk}\mathsf{A}_{1j}\mathsf{A}_{2k}$, respectively.

Problem 1.24: Let $\mathbf{A}$ be an arbitrary second-order tensor. Show that there is a nonzero vector $\vec{\mathbf{n}}\neq\vec{\mathbf{0}}$ so that $\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$ if and only if $\det(\mathbf{A})=0$, Chadwick (1976).

Solution: Firstly, we show that, if $\det(\mathbf{A})\equiv|\mathbf{A}|=0\Rightarrow\vec{\mathbf{n}}\neq\vec{\mathbf{0}}$. Secondly, we show that, if $\vec{\mathbf{n}}\neq\vec{\mathbf{0}}\Rightarrow\det(\mathbf{A})\equiv|\mathbf{A}|=0$.

We assume that $\det(\mathbf{A})\equiv|\mathbf{A}|=0$, and we choose an arbitrary basis $\{\vec{\mathbf{f}},\vec{\mathbf{g}},\vec{\mathbf{h}}\}$ (linearly independent). We apply these vectors in the definition seen in (1.197):

$$\vec{\mathbf{f}}\cdot(\vec{\mathbf{g}}\wedge\vec{\mathbf{h}})|\mathbf{A}|=(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot[(\mathbf{A}\cdot\vec{\mathbf{g}})\wedge(\mathbf{A}\cdot\vec{\mathbf{h}})]$$

Due to the fact that $\det(\mathbf{A})\equiv|\mathbf{A}|=0$, the implication is that:

$$(\mathbf{A}\cdot\vec{\mathbf{f}})\cdot[(\mathbf{A}\cdot\vec{\mathbf{g}})\wedge(\mathbf{A}\cdot\vec{\mathbf{h}})]=0$$

Thus, we can conclude that the vectors $(\mathbf{A}\cdot\vec{\mathbf{f}})$, $(\mathbf{A}\cdot\vec{\mathbf{g}})$, $(\mathbf{A}\cdot\vec{\mathbf{h}})$, are linearly dependent. This implies that there are nonzero scalars α, β, γ so that:

$$\alpha(\mathbf{A}\cdot\vec{\mathbf{f}})+\beta(\mathbf{A}\cdot\vec{\mathbf{g}})+\gamma(\mathbf{A}\cdot\vec{\mathbf{h}})=\vec{\mathbf{0}}\Rightarrow\mathbf{A}\cdot(\alpha\vec{\mathbf{f}}+\beta\vec{\mathbf{g}}+\gamma\vec{\mathbf{h}})=\vec{\mathbf{0}}\Rightarrow\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$$

where $\vec{\mathbf{n}}=\alpha\vec{\mathbf{f}}+\beta\vec{\mathbf{g}}+\gamma\vec{\mathbf{h}}\neq\vec{\mathbf{0}}$ since $\{\vec{\mathbf{f}},\vec{\mathbf{g}},\vec{\mathbf{h}}\}$ is linearly independent, (see **Problem 1.10**).

Now we choose two vectors $\vec{\mathbf{k}}$, $\vec{\mathbf{m}}$, which are linearly independent to $\vec{\mathbf{n}}$. We apply definition (1.199) once more:

$$\vec{\mathbf{k}}\cdot(\vec{\mathbf{m}}\wedge\vec{\mathbf{n}})|\mathbf{A}|=(\mathbf{A}\cdot\vec{\mathbf{k}})\cdot[(\mathbf{A}\cdot\vec{\mathbf{m}})\wedge(\mathbf{A}\cdot\vec{\mathbf{n}})]$$

Considering that $\mathbf{A}\cdot\vec{\mathbf{n}}=\vec{\mathbf{0}}$, and $\vec{\mathbf{k}}\cdot(\vec{\mathbf{m}}\wedge\vec{\mathbf{n}})\neq 0$ owing to the fact that $\vec{\mathbf{k}}$, $\vec{\mathbf{m}}$, $\vec{\mathbf{n}}$ are linearly independent, we can conclude that:

$$\underbrace{\vec{\mathbf{k}} \cdot (\vec{\mathbf{m}} \wedge \vec{\mathbf{n}})}_{\neq 0} |\mathbf{A}| = 0 \quad \Rightarrow \quad |\mathbf{A}| = 0$$

1.5.2.8 Orthogonal Tensors

Orthogonal tensors play an important role in continuum mechanics. A second-order tensor ($\mathbf{Q}$) is said to be orthogonal when the transpose ($\mathbf{Q}^T$) is equal to the inverse ($\mathbf{Q}^{-1}$), *i.e.* $\mathbf{Q}^T = \mathbf{Q}^{-1}$. Then, it follows that:

$$\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{1} \quad ; \quad \mathrm{Q}_{ik}\mathrm{Q}_{jk} = \mathrm{Q}_{ki}\mathrm{Q}_{kj} = \delta_{ij} \tag{1.214}$$

A *proper orthogonal tensor* has the following properties:

- The inverse of $\mathbf{Q}$ is equal to the transpose (orthogonality):

$$\mathbf{Q}^{-1} = \mathbf{Q}^T \tag{1.215}$$

- The tensor $\mathbf{Q}$ is a proper, *rotation tensor*, if:

$$\det(\mathbf{Q}) \equiv |\mathbf{Q}| = +1 \tag{1.216}$$

If $|\mathbf{Q}| = -1$, the orthogonal tensor is said to be *improper (a reflection tensor)*.

If $\mathbf{A}$ and $\mathbf{B}$ are orthogonal tensors, the resulting tensor $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$ is also an orthogonal tensor, *i.e.*:

$$\mathbf{C}^{-1} = (\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} = \mathbf{B}^T \cdot \mathbf{A}^T = (\mathbf{A} \cdot \mathbf{B})^T = \mathbf{C}^T \tag{1.217}$$

Consider two arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. An orthogonal transformation applied to these vectors becomes:

$$\tilde{\vec{\mathbf{a}}} = \mathbf{Q} \cdot \vec{\mathbf{a}} \quad ; \quad \tilde{\vec{\mathbf{b}}} = \mathbf{Q} \cdot \vec{\mathbf{b}} \tag{1.218}$$

And the dot product between these new vectors ($\tilde{\vec{\mathbf{a}}}$) and ($\tilde{\vec{\mathbf{b}}}$) is given by:

$$\begin{aligned} \tilde{\vec{\mathbf{a}}} \cdot \tilde{\vec{\mathbf{b}}} &= (\mathbf{Q} \cdot \vec{\mathbf{a}}) \cdot (\mathbf{Q} \cdot \vec{\mathbf{b}}) = \vec{\mathbf{a}} \cdot \underbrace{\mathbf{Q}^T \cdot \mathbf{Q}}_{=\mathbf{1}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \\ \tilde{\mathrm{a}}_i \tilde{\mathrm{b}}_i &= (\mathrm{Q}_{ik}\mathrm{a}_k)(\mathrm{Q}_{ij}\mathrm{b}_j) = \mathrm{a}_k \underbrace{\mathrm{Q}_{ik}\mathrm{Q}_{ij}}_{\delta_{kj}} \mathrm{b}_j = \mathrm{a}_k \mathrm{b}_k \end{aligned} \tag{1.219}$$

So, it is also true when $\tilde{\vec{\mathbf{a}}} = \tilde{\vec{\mathbf{b}}}$, thus $\tilde{\vec{\mathbf{a}}} \cdot \tilde{\vec{\mathbf{a}}} = \left\|\tilde{\vec{\mathbf{a}}}\right\|^2 = \vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = \left\|\vec{\mathbf{a}}\right\|^2$. Therefore, we can conclude that in an orthogonal transformation, the magnitude vectors and the angle between them are maintained, (see Figure 1.18).

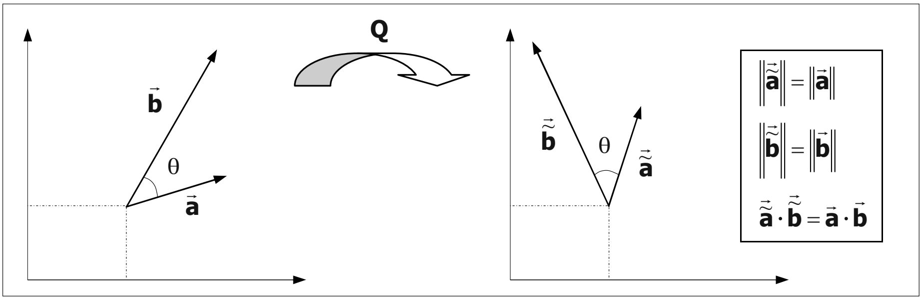


Figure 1.18: Orthogonal transformation applied to vectors.

1.5.2.9 Positive Definite Tensor, Negative Definite Tensor and Semi-Definite Tensors

A tensor is said to be *positive definite* when the following notations hold:

Tensorial notation	*Indicial notation*	*Matrix notation*	
$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} > 0$	$\mathrm{x}_i \mathrm{T}_{ij} \mathrm{x}_j > 0$	$\boldsymbol{x}^T \mathcal{T} \boldsymbol{x} > 0$	(1.220)

for all vectors $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Conversely, a tensor is said to be *negative definite* when these notations hold:

Tensorial notation	*Indicial notation*	*Matrix notation*	
$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} < 0$	$\mathrm{x}_i \mathrm{T}_{ij} \mathrm{x}_j < 0$	$\boldsymbol{x}^T \mathcal{T} \boldsymbol{x} < 0$	(1.221)

for all vectors $\vec{\mathbf{x}} \neq \vec{\mathbf{0}}$.

A tensor is said to be *semi-positive definite* if $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \geq 0$ for all vectors $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Similarly, we define a *semi-negative definite tensor* when the following holds: $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \leq 0$.

If $\alpha = \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \mathbf{T} : (\hat{\mathbf{x}} \otimes \hat{\mathbf{x}}) = \mathrm{T}_{ij} \mathrm{x}_i \mathrm{x}_j$, then the derivative of α with respect to $\vec{\mathbf{x}}$ is given by:

$$\frac{\partial \alpha}{\partial \mathrm{x}_k} = \mathrm{T}_{ij} \frac{\partial \mathrm{x}_i}{\partial \mathrm{x}_k} \mathrm{x}_j + \mathrm{T}_{ij} \mathrm{x}_i \frac{\partial \mathrm{x}_j}{\partial \mathrm{x}_k} = \mathrm{T}_{ij} \delta_{ik} \mathrm{x}_j + \mathrm{T}_{ij} \mathrm{x}_i \delta_{jk} = \mathrm{T}_{kj} \mathrm{x}_j + \mathrm{T}_{ik} \mathrm{x}_i = \left(\mathrm{T}_{ki} + \mathrm{T}_{ik} \right) \mathrm{x}_i \tag{1.222}$$

Thus, we can conclude that:

$$\frac{\partial \alpha}{\partial \hat{\mathbf{x}}} = 2 \mathbf{T}^{sym} \cdot \hat{\mathbf{x}} \quad \Rightarrow \quad \frac{\partial^2 \alpha}{\partial \hat{\mathbf{x}} \otimes \partial \hat{\mathbf{x}}} = 2 \mathbf{T}^{sym} \tag{1.223}$$

Remember that it is also true that $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \hat{\mathbf{x}} \cdot \mathbf{T}^{sym} \cdot \hat{\mathbf{x}}$, therefore if the symmetric part of a tensor is positive definite the tensor is too.

NOTE: As we will demonstrate later, the eigenvalues must be positive for $\mathbf{T}$ to be positive definite. The proof is in the subsection "Spectral Representation of Tensors". ■

Problem 1.25: Let $\boldsymbol{F}$ be an arbitrary second-order tensor. Show that the resulting tensors $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are *symmetric tensors* and *semi-positive definite tensors*. Also check in what condition are $\boldsymbol{C}$ and $\boldsymbol{b}$ *positive definite tensors*.
Solution: Symmetry:

$$\boldsymbol{C}^T = (\boldsymbol{F}^T \cdot \boldsymbol{F})^T = \boldsymbol{F}^T \cdot (\boldsymbol{F}^T)^T = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{C}$$
$$\boldsymbol{b}^T = (\boldsymbol{F} \cdot \boldsymbol{F}^T)^T = (\boldsymbol{F}^T)^T \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{F}^T = \boldsymbol{b}$$

Thus, we have shown that $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are symmetric tensors.

To prove that the tensors $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are semi-positive definite tensors, we start with the definition of a semi-positive definite tensor, *i.e.*, a tensor $\mathbf{A}$ is semi-positive definite if $\hat{\mathbf{x}} \cdot \mathbf{A} \cdot \hat{\mathbf{x}} \geq 0$ holds, for all $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Thus:

$$\begin{aligned} \hat{\mathbf{x}} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F}) \cdot \hat{\mathbf{x}} &= \boldsymbol{F} \cdot \hat{\mathbf{x}} \cdot \boldsymbol{F} \cdot \hat{\mathbf{x}} \\ &= (\boldsymbol{F} \cdot \hat{\mathbf{x}}) \cdot (\boldsymbol{F} \cdot \hat{\mathbf{x}}) \\ &= \left\| \boldsymbol{F} \cdot \hat{\mathbf{x}} \right\|^2 \geq 0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \hat{\mathbf{x}} \cdot (\boldsymbol{F} \cdot \boldsymbol{F}^T) \cdot \hat{\mathbf{x}} &= \hat{\mathbf{x}} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \hat{\mathbf{x}} \\ &= (\boldsymbol{F}^T \cdot \hat{\mathbf{x}}) \cdot (\boldsymbol{F}^T \cdot \hat{\mathbf{x}}) \\ &= \left\| \boldsymbol{F}^T \cdot \hat{\mathbf{x}} \right\|^2 \geq 0 \end{aligned}$$

Or in indicial notation:

$$\begin{aligned} \mathrm{x}_i C_{ij} \mathrm{x}_j &= \mathrm{x}_i (F_{ki} F_{kj}) \mathrm{x}_j \\ &= (F_{ki} \mathrm{x}_i)(F_{kj} \mathrm{x}_j) \\ &= \left\| F_{ki} \mathrm{x}_i \right\|^2 \geq 0 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \mathrm{x}_i b_{ij} \mathrm{x}_j &= \mathrm{x}_i (F_{ik} F_{jk}) \mathrm{x}_j \\ &= (F_{ik} \mathrm{x}_i)(F_{jk} \mathrm{x}_j) \\ &= \left\| F_{ik} \mathrm{x}_i \right\|^2 \geq 0 \end{aligned}$$

Thus, we proved that $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are semi-positive definite tensors. Note that $\hat{\mathbf{x}} \cdot \boldsymbol{C} \cdot \hat{\mathbf{x}} = \left\| \boldsymbol{F} \cdot \hat{\mathbf{x}} \right\|^2$ equals zero, when $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$, if $\boldsymbol{F} \cdot \hat{\mathbf{x}} = \vec{\mathbf{0}}$. Furthermore, by definition $\boldsymbol{F} \cdot \hat{\mathbf{x}} = \vec{\mathbf{0}}$ with $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$ if and only if $\det(\boldsymbol{F}) = 0$, (see **Problem 1.24**). Then, the tensors $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ are positive definite if and only if $\det(\boldsymbol{F}) \neq 0$.

1.5.2.10 Additive Decomposition of Tensors

Given two arbitrary tensors $\mathbf{S}$, $\mathbf{T} \neq \mathbf{0}$, and a scalar α, we can represent $\mathbf{S}$ by means of the following additive decomposition of tensors:

$$\mathbf{S} = \alpha \mathbf{T} + \mathbf{U} \qquad where \qquad \mathbf{U} = \mathbf{S} - \alpha \mathbf{T} \tag{1.224}$$

Note that, depending on the value of α, we have an infinite number of possibilities for representing $\mathbf{S}$. But, if $\mathrm{Tr}(\mathbf{T} \cdot \mathbf{U}^T) = \mathrm{Tr}(\mathbf{U} \cdot \mathbf{T}^T) = 0$, the additive decomposition is unique. From the relationship in (1.224), we can evaluate the value of α as follows:

$$\mathbf{S} \cdot \mathbf{T}^T = \alpha \mathbf{T} \cdot \mathbf{T}^T + \mathbf{U} \cdot \mathbf{T}^T \Rightarrow \mathrm{Tr}(\mathbf{S} \cdot \mathbf{T}^T) = \alpha \mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T) + \underbrace{\mathrm{Tr}(\mathbf{U} \cdot \mathbf{T}^T)}_{=0} = \alpha \mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T) \tag{1.225}$$

$$\Rightarrow \qquad \alpha = \frac{\mathrm{Tr}(\mathbf{S} \cdot \mathbf{T}^T)}{\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T)} \tag{1.226}$$

For example, let us suppose that $\mathbf{T} = \mathbf{1}$. In this case α is evaluated as follows:

$$\alpha = \frac{\mathrm{Tr}(\mathbf{S} \cdot \mathbf{T}^T)}{\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}^T)} = \frac{\mathrm{Tr}(\mathbf{S} \cdot \mathbf{1})}{\mathrm{Tr}(\mathbf{1} \cdot \mathbf{1})} = \frac{\mathrm{Tr}(\mathbf{S})}{\mathrm{Tr}(\mathbf{1})} = \frac{\mathrm{Tr}(\mathbf{S})}{3} \tag{1.227}$$

We can then define $\mathbf{U}$ as:

$$\mathbf{U} = \mathbf{S} - \alpha \mathbf{T} = \mathbf{S} - \frac{\mathrm{Tr}(\mathbf{S})}{3} \mathbf{1} \equiv \mathbf{S}^{dev} \tag{1.228}$$

Thus:

$$\mathbf{S} = \frac{\mathrm{Tr}(\mathbf{S})}{3} \mathbf{1} + \mathbf{S}^{dev} = \mathbf{S}^{sph} + \mathbf{S}^{dev} \tag{1.229}$$

NOTE: $\mathbf{S}^{sph} = \frac{\text{Tr}(\mathbf{S})}{3}\mathbf{1}$ is the *spherical* part of the tensor $\mathbf{S}$, and $\mathbf{S}^{dev} = \mathbf{S} - \frac{\text{Tr}(\mathbf{S})}{3}\mathbf{1}$ is known as a *deviatoric tensor*. ■

Now suppose that $\mathbf{T}$ is given by $\mathbf{T} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^T)$ then α can be evaluated as follows:

$$\alpha = \frac{\text{Tr}(\mathbf{S} \cdot \mathbf{T}^T)}{\text{Tr}(\mathbf{T} \cdot \mathbf{T}^T)} = \frac{\frac{1}{2}\text{Tr}\left[\mathbf{S} \cdot (\mathbf{S} + \mathbf{S}^T)^T\right]}{\frac{1}{4}\text{Tr}\left[(\mathbf{S} + \mathbf{S}^T) \cdot (\mathbf{S} + \mathbf{S}^T)^T\right]} = 1 \tag{1.230}$$

We can then define $\mathbf{U}$ as $\mathbf{U} = \mathbf{S} - \alpha\mathbf{T} = \mathbf{S} - \frac{1}{2}(\mathbf{S} + \mathbf{S}^T) = \frac{1}{2}(\mathbf{S} - \mathbf{S}^T)$. Then we obtain $\mathbf{S}$ represented by the additive decomposition as follows:

$$\mathbf{S} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^T) + \frac{1}{2}(\mathbf{S} - \mathbf{S}^T) = \mathbf{S}^{sym} + \mathbf{S}^{skew} \tag{1.231}$$

which is the same as the equation obtained in (1.150) in which we split the tensor into symmetric and antisymmetric parts.

Problem 1.26: Find a fourth-order tensor $\mathbb{P}$ so that $\mathbb{P} : \mathbf{A} = \mathbf{A}^{dev}$, where $\mathbf{A}$ is a second-order tensor.
Solution: Taking into account the additive decomposition into spherical and deviatoric parts, we obtain:

$$\mathbf{A} = \mathbf{A}^{sph} + \mathbf{A}^{dev} = \frac{\text{Tr}(\mathbf{A})}{3}\mathbf{1} + \mathbf{A}^{dev} \quad \Rightarrow \quad \mathbf{A}^{dev} = \mathbf{A} - \frac{\text{Tr}(\mathbf{A})}{3}\mathbf{1}$$

Referring to the definition of fourth-order unit tensors seen in (1.172), and (1.174), where the relations $\bar{\bar{\mathbb{I}}} : \mathbf{A} = \text{Tr}(\mathbf{A})\mathbf{1}$ and $\mathbb{I} : \mathbf{A} = \mathbf{A}$ hold, we can now state:

$$\mathbf{A}^{dev} = \mathbf{A} - \frac{\text{Tr}(\mathbf{A})}{3}\mathbf{1} = \mathbb{I} : \mathbf{A} - \frac{1}{3}\bar{\bar{\mathbb{I}}} : \mathbf{A} = \left(\mathbb{I} - \frac{1}{3}\bar{\bar{\mathbb{I}}}\right) : \mathbf{A} = \left(\mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}\right) : \mathbf{A}$$

Therefore, we can conclude that:

$$\boxed{\mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}}$$

The tensor $\mathbb{P}$ is known as a *fourth-order projection tensor*, Holzapfel(2000).

1.5.3 Transformation Law of the Tensor Components

The tensor components depend on the coordinate system, so, if the coordinate system is changed due to a rotation so do the tensor components. The tensor components between these coordinate systems are interrelated to each other by the component transformation law, which is defined below, (see Figure 1.19).

Consider a Cartesian coordinate system (x_1, x_2, x_3) formed by the orthogonal basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$, (see Figure 1.20). In this system, an arbitrary vector $\vec{\mathbf{v}}$ is represented by its components as follows:

$$\vec{\mathbf{v}} = \mathrm{v}_i \hat{\mathbf{e}}_i = \mathrm{v}_1 \hat{\mathbf{e}}_1 + \mathrm{v}_2 \hat{\mathbf{e}}_2 + \mathrm{v}_3 \hat{\mathbf{e}}_3 \tag{1.232}$$

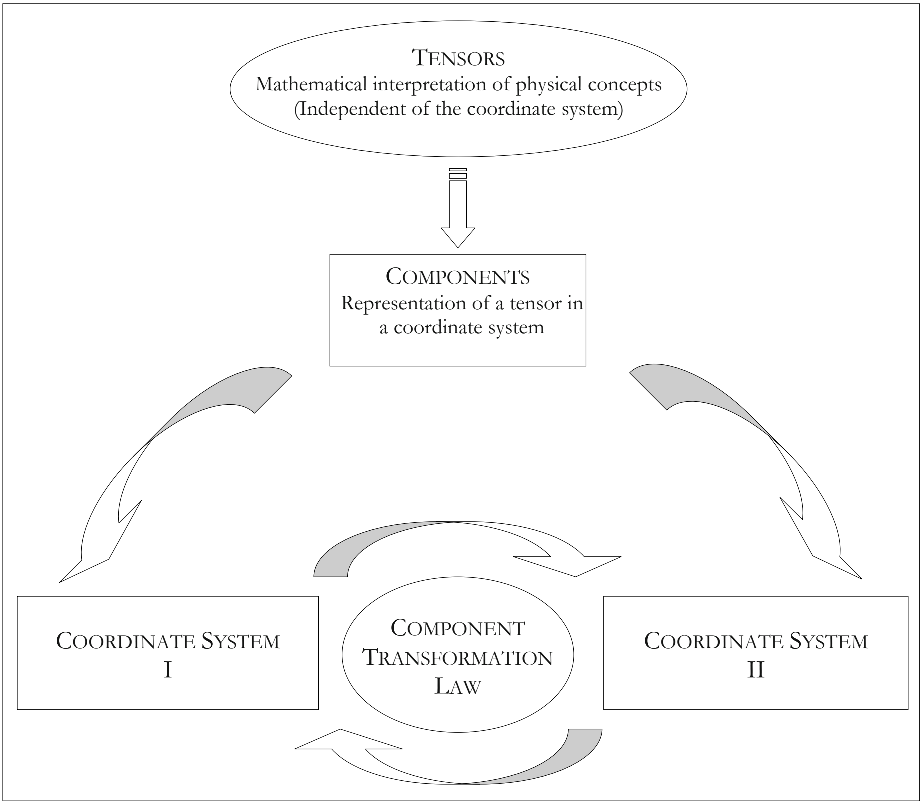


Figure 1.19: Transformation law of the tensor components.

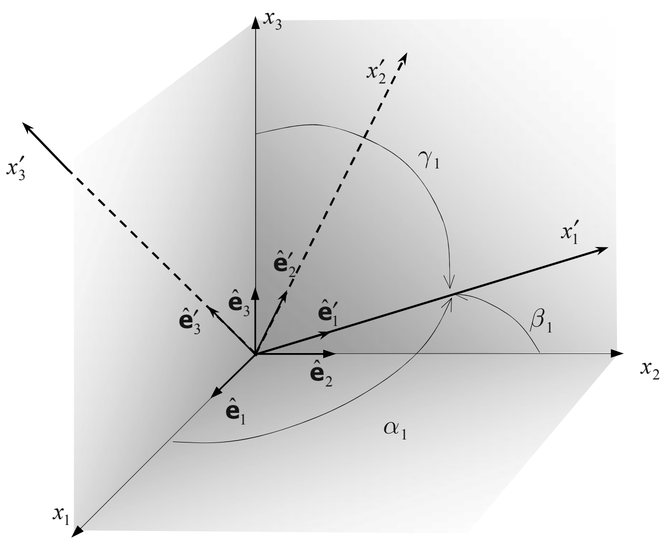


Figure 1.20: Rotation of the Cartesian system.

The components, v_i, are represented in matrix form as:

$$(\vec{\mathbf{v}})_i = \mathrm{v}_i = \boldsymbol{v} = \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{bmatrix} \tag{1.233}$$

Now consider a new coordinate system (x'_1, x'_2, x'_3) represented by the orthogonal basis $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$, (see Figure 1.20). In this new system, the vector $\vec{\mathbf{v}}$ is represented by $\mathrm{v}'_j \hat{\mathbf{e}}'_j$. As mentioned before, a tensor is independent of the adopted system, so:

$$\vec{\mathbf{v}} = \mathrm{v}'_k \hat{\mathbf{e}}'_k = \mathrm{v}_j \hat{\mathbf{e}}_j \tag{1.234}$$

To obtain the components of a tensor in a given system one need only make the dot product between the tensor and the system basis:

$$\begin{aligned} \mathrm{v}'_k \hat{\mathbf{e}}'_k \cdot \hat{\mathbf{e}}'_i &= (\mathrm{v}_j \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}'_i \\ \mathrm{v}'_k \delta_{ki} &= (\mathrm{v}_j \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}'_i \\ \mathrm{v}'_i &= (\mathrm{v}_1 \hat{\mathbf{e}}_1 + \mathrm{v}_2 \hat{\mathbf{e}}_2 + \mathrm{v}_3 \hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}'_i \end{aligned} \tag{1.235}$$

Or in matrix form:

$$\begin{bmatrix} \mathrm{v}'_1 \\ \mathrm{v}'_2 \\ \mathrm{v}'_3 \end{bmatrix} = \begin{bmatrix} (\mathrm{v}_1 \hat{\mathbf{e}}_1 + \mathrm{v}_2 \hat{\mathbf{e}}_2 + \mathrm{v}_3 \hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}'_1 \\ (\mathrm{v}_1 \hat{\mathbf{e}}_1 + \mathrm{v}_2 \hat{\mathbf{e}}_2 + \mathrm{v}_3 \hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}'_2 \\ (\mathrm{v}_1 \hat{\mathbf{e}}_1 + \mathrm{v}_2 \hat{\mathbf{e}}_2 + \mathrm{v}_3 \hat{\mathbf{e}}_3) \cdot \hat{\mathbf{e}}'_3 \end{bmatrix} \tag{1.236}$$

After restructuring, the previous equation looks like:

$$\begin{bmatrix} \mathrm{v}'_1 \\ \mathrm{v}'_2 \\ \mathrm{v}'_3 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}'_1 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_1 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}'_2 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_2 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}'_3 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_3 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}'_3 \end{bmatrix} \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \\ \mathrm{v}_3 \end{bmatrix} \quad \therefore \quad a_{ij} = \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}'_i = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j \tag{1.237}$$

Or in indicial notation:

$$\mathrm{v}'_i = a_{ij} \mathrm{v}_j \tag{1.238}$$

where we have introduced the transformation matrix $\mathcal{A} \equiv a_{ij}$ as:

$$\begin{aligned} \mathcal{A} \equiv a_{ij} &= \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}'_1 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_1 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}'_1 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}'_2 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_2 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}'_2 \\ \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}'_3 & \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}'_3 & \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}'_3 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}'_3 \cdot \hat{\mathbf{e}}_3 \end{bmatrix} \\ a_{ij} \equiv \mathcal{A} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{aligned} \tag{1.239}$$

The matrix ($\mathcal{A}$) is not symmetric, *i.e.* $\mathcal{A} \neq \mathcal{A}^T$. With reference to the scalar product $\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j = \|\hat{\mathbf{e}}'_i\| \|\hat{\mathbf{e}}_j\| \cos(x'_i, x_j) = \cos(x'_i, x_j)$, (see equation (1.4)), the relationship in (1.237) is expressed by means of the direction cosines as:

$$\underbrace{\begin{bmatrix} \mathsf{v}'_1 \\ \mathsf{v}'_2 \\ \mathsf{v}'_3 \end{bmatrix}}_{\boldsymbol{v}'} = \underbrace{\begin{bmatrix} \cos(x'_1, x_1) & \cos(x'_1, x_2) & \cos(x'_1, x_3) \\ \cos(x'_2, x_1) & \cos(x'_2, x_2) & \cos(x'_2, x_3) \\ \cos(x'_3, x_1) & \cos(x'_3, x_2) & \cos(x'_3, x_3) \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} \mathsf{v}_1 \\ \mathsf{v}_2 \\ \mathsf{v}_3 \end{bmatrix}}_{\boldsymbol{v}} \qquad \boxed{\boldsymbol{v}' = \mathcal{A}\boldsymbol{v}} \tag{1.240}$$

The direction cosines of a vector are those of the angles between the vector and the three coordinate axes. According to Figure 1.20 we can verify that $\cos\alpha_1 = \cos(x'_1, x_1)$, $\cos\beta_1 = \cos(x'_1, x_2)$ and $\cos\gamma_1 = \cos(x'_1, x_3)$.

In the equation (1.235) we have projected the vector onto $\hat{\mathbf{e}}'_i$. Now, we can project the vector onto $\hat{\mathbf{e}}_i$:

$$\begin{aligned} \mathsf{v}_k \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i &= \mathsf{v}'_j \hat{\mathbf{e}}'_j \cdot \hat{\mathbf{e}}_i \\ \mathsf{v}_k \delta_{ki} &= \mathsf{v}'_j a_{ji} \\ \mathsf{v}_i &= \mathsf{v}'_j a_{ji} \\ \boldsymbol{v} &= \mathcal{A}^T \boldsymbol{v}' \end{aligned} \tag{1.241}$$

Therefore, it is also true that:

$$\boxed{\hat{\mathbf{e}}_i = a_{ji} \hat{\mathbf{e}}'_j} \tag{1.242}$$

The inverse relationship of equation (1.240) is obtained as follows:

$$\mathcal{A}^{-1}\boldsymbol{v}' = \mathcal{A}^{-1}\mathcal{A}\boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{v} = \mathcal{A}^{-1}\boldsymbol{v}' \tag{1.243}$$

and by comparing the equations (1.243) with (1.241) we can conclude that the matrix $\mathcal{A}$ is an orthogonal matrix, *i.e.*:

$$\mathcal{A}^{-1} = \mathcal{A}^T \quad \Rightarrow \quad \mathcal{A}^T \mathcal{A} = \mathbf{1} \xrightarrow[\text{notation}]{\text{Indicial}} a_{ki} a_{kj} = \delta_{ij} \tag{1.244}$$

<u>Second-order tensor</u>

Consider a coordinate system formed by the orthogonal basis $\hat{\mathbf{e}}_i$ then, how the basis changes from the $\hat{\mathbf{e}}_i$ system to a new one represented by the orthogonal basis $\hat{\mathbf{e}}'_i$. This is illustrated in transformation law as $\hat{\mathbf{e}}_k = a_{ik}\hat{\mathbf{e}}'_i$, which allow us to represent a second-order tensor $\mathbf{T}$ as follows:

$$\begin{aligned} \mathbf{T} &= \mathsf{T}_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l \\ &= \mathsf{T}_{kl} a_{ik} \hat{\mathbf{e}}'_i \otimes a_{jl} \hat{\mathbf{e}}'_j \\ &= \mathsf{T}_{kl} a_{ik} a_{jl} \, \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \\ &= \mathsf{T}'_{ij} \, \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \end{aligned} \tag{1.245}$$

Then, the transformation law of the components between systems for a second-order tensor is given by:

$$\mathsf{T}'_{ij} = \mathsf{T}_{kl} a_{ik} a_{jl} = a_{ik} \mathsf{T}_{kl} a_{jl} \xrightarrow{\text{Matrix form}} \mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T \tag{1.246}$$

<u>Third-order tensor</u>

A third-order tensor ($\mathbf{S}$) can be shown in two systems represented by orthogonal bases $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$ as follows:

$$\begin{aligned}\mathbf{S} &= \mathsf{S}_{lmn}\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_m \otimes \hat{\mathbf{e}}_n \\ &= \mathsf{S}_{lmn} a_{il}\hat{\mathbf{e}}'_i \otimes a_{jm}\hat{\mathbf{e}}'_j \otimes a_{kn}\hat{\mathbf{e}}'_k \\ &= \mathsf{S}_{lmn} a_{il} a_{jm} a_{kn}\ \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \otimes \hat{\mathbf{e}}'_k \\ &= \mathsf{S}'_{ijk}\ \hat{\mathbf{e}}'_i \otimes \hat{\mathbf{e}}'_j \otimes \hat{\mathbf{e}}'_k\end{aligned} \tag{1.247}$$

In conclusion the components of the third-order tensor in the new basis $(\hat{\mathbf{e}}'_i)$ are:

$$\mathsf{S}'_{ijk} = \mathsf{S}_{lmn} a_{il} a_{jm} a_{kn} \tag{1.248}$$

The following table summarizes the transformation law of the components according to the tensor rank:

rank	from $(x_1, x_2, x_3) \xrightarrow{to} (x'_1, x'_2, x'_3)$	from $(x'_1, x'_2, x'_3) \xrightarrow{to} (x_1, x_2, x_3)$
0 (scalar)	$\lambda' = \lambda$	$\lambda = \lambda'$
1 (vector)	$S'_i = a_{ij} S_j$	$S_i = a_{ji} S'_j$
2	$S'_{ij} = a_{ik} a_{jl} S_{kl}$	$S_{ij} = a_{ki} a_{lj} S'_{kl}$
3	$S'_{ijk} = a_{il} a_{jm} a_{kn} S_{lmn}$	$S_{ijk} = a_{li} a_{mj} a_{nk} S'_{lmn}$
4	$S'_{ijkl} = a_{im} a_{jn} a_{kp} a_{lq} S_{mnpq}$	$S_{ijkl} = a_{mi} a_{nj} a_{pk} a_{ql} S'_{mnpq}$

(1.249)

Problem 1.27: Obtain the components of $\mathbf{T}'$, given by the transformation:

$$\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T$$

where the components of $\mathbf{T}$ and $\mathbf{A}$ are shown, respectively, as T_{ij} and a_{ij}. Afterwards, given that a_{ij} are the components of the transformation matrix, represent graphically the components of the tensors $\mathbf{T}$ and $\mathbf{T}'$ on both systems.

Solution: The expression $\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^T$ in symbolic notation is given by:

$$\begin{aligned}\mathsf{T}'_{ab}(\hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b) &= a_{rs}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_s) \cdot \mathsf{T}_{pq}(\hat{\mathbf{e}}_p \otimes \hat{\mathbf{e}}_q) \cdot a_{kl}(\hat{\mathbf{e}}_l \otimes \hat{\mathbf{e}}_k) \\ &= a_{rs}\mathsf{T}_{pq} a_{kl}\delta_{sp}\delta_{ql}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k) \\ &= a_{rp}\mathsf{T}_{pq} a_{kq}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k)\end{aligned}$$

To obtain the components of $\mathbf{T}'$ one only need make the double scalar product with the basis $(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)$, the result of which is:

$$\begin{aligned}&\mathsf{T}'_{ab}(\hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b) : (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = a_{rp}\mathsf{T}_{pq} a_{kq}(\hat{\mathbf{e}}_r \otimes \hat{\mathbf{e}}_k) : (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \\ &\mathsf{T}'_{ab}\delta_{ai}\delta_{bj} = a_{rp}\mathsf{T}_{pq} a_{kq}\delta_{ri}\delta_{kj} \\ &\mathsf{T}'_{ij} = a_{ip}\mathsf{T}_{pq} a_{jq}\end{aligned}$$

The above eqaution is shown in matrix notation as:

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T \xrightarrow{inverse} \mathcal{T} = \mathcal{A}^{-1}\,\mathcal{T}'\,\mathcal{A}^{-T}$$

Since $\mathcal{A}$ is an orthogonal matrix, it holds that $\mathcal{A}^T = \mathcal{A}^{-1}$. Thus, $\mathcal{T} = \mathcal{A}^T\,\mathcal{T}'\,\mathcal{A}$. The graphical representation of the tensor components in both systems can be seen in Figure 1.21.

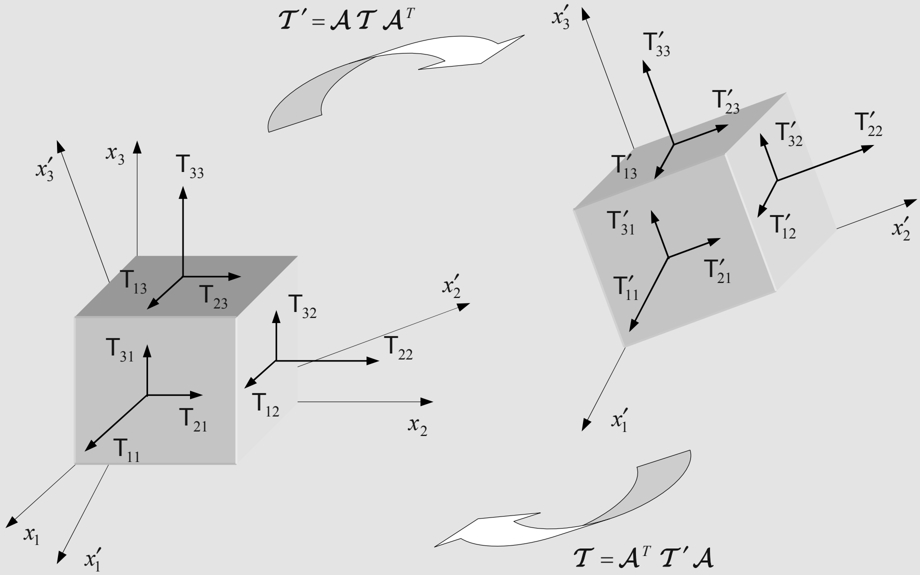


Figure 1.21: Transformation law of the second-order tensor components.

Problem 1.28: Let $\mathbf{T}$ be a symmetric second-order tensor and $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ be scalars, where:

$$\boxed{I_{\mathbf{T}} = \mathrm{Tr}(\mathbf{T}) = \mathsf{T}_{ii} \quad ; \quad II_{\mathbf{T}} = \frac{1}{2}\left\{I_{\mathbf{T}}^2 - \mathrm{Tr}(\mathbf{T}^2)\right\} \quad ; \quad III_{\mathbf{T}} = \det(\mathbf{T})}$$

Show that $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ are invariant with a change of basis.

Solution:

a) Taking into account the transformation law for the second-order tensor components given in (1.249), *i.e.* $\mathsf{T}'_{ij} = a_{ik}a_{jl}\mathsf{T}_{kl}$ or in matrix form $\mathcal{T}' = \mathcal{A}\mathcal{T}\mathcal{A}^T$. Then, T'_{ii} is:

$$\mathsf{T}'_{ii} = a_{ik}a_{il}\mathsf{T}_{kl} = \delta_{kl}\mathsf{T}_{kl} = \mathsf{T}_{kk} = I_{\mathbf{T}}$$

Hence we have proved that $I_{\mathbf{T}}$ is independent of the adopted system.

b) To prove that $II_{\mathbf{T}}$ is an invariant, one only need show that $\mathrm{Tr}(\mathbf{T}^2)$ is one also, since $I_{\mathbf{T}}^2$ is already an invariant.

$$\begin{aligned}
\mathrm{Tr}(\mathbf{T}'^2) = \mathrm{Tr}(\mathbf{T}'\cdot\mathbf{T}') = \mathbf{T}':\mathbf{T}' = \mathsf{T}'_{ij}\mathsf{T}'_{ij} &= (a_{ik}a_{jl}\mathsf{T}_{kl})(a_{ip}a_{jq}\mathsf{T}_{pq}) \\
&= \underbrace{a_{ik}a_{ip}}_{\delta_{kp}}\underbrace{a_{jl}a_{jq}}_{\delta_{lq}}\mathsf{T}_{kl}\mathsf{T}_{pq} \\
&= \mathsf{T}_{pl}\mathsf{T}_{pl} \\
&= \mathbf{T}:\mathbf{T} = \mathrm{Tr}(\mathbf{T}\cdot\mathbf{T}) = \mathrm{Tr}(\mathbf{T}^2)
\end{aligned}$$

c)

$$\det(\mathbf{T}') = \det(\mathcal{T}') = \det(\mathcal{A}\mathcal{T}\mathcal{A}^T) = \underbrace{\det(\mathcal{A})}_{=1}\det(\mathcal{T})\underbrace{\det(\mathcal{A}^T)}_{=1} = \det(\mathcal{T})$$

Consider now four sets of coordinate systems, represented by (x_1, x_2, x_3), (x'_1, x'_2, x'_3), (x''_1, x''_2, x''_3) and (x'''_1, x'''_2, x'''_3), (see Figure 1.22), and consider also the following transformation matrices:

$\mathcal{A}$: Transformation matrix from (x_1, x_2, x_3) to (x'_1, x'_2, x'_3);

$\mathcal{B}$: Transformation matrix from (x'_1, x'_2, x'_3) to (x''_1, x''_2, x''_3);

$\mathcal{C}$: Transformation matrix from (x''_1, x''_2, x''_3) to (x'''_1, x'''_2, x'''_3).

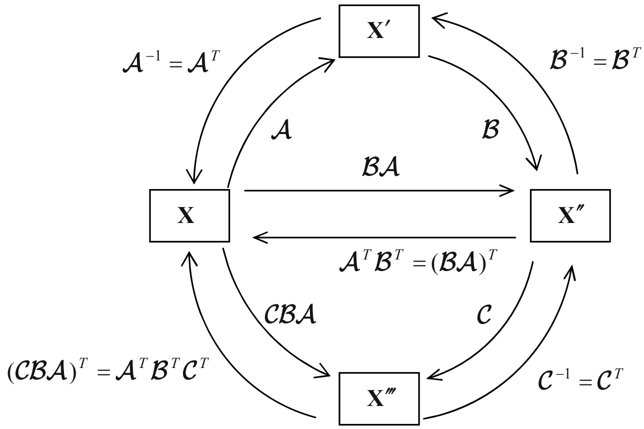


Figure 1.22: Transformations matrices between several systems.

If we consider a $\boldsymbol{v}$ column matrix made up of components of $\vec{\mathbf{v}}$ in the coordinate system (x_1, x_2, x_3), the components of this vector in the system (x'_1, x'_2, x'_3) are given by:

$$\boldsymbol{v}' = \mathcal{A}\boldsymbol{v} \tag{1.250}$$

and the inverse transformation of relation (1.250) is:

$$\boldsymbol{v} = \mathcal{A}^T \boldsymbol{v}' \tag{1.251}$$

Now, starting with the system (x'_1, x'_2, x'_3), the components of the vector in the system (x''_1, x''_2, x''_3) are given by:

$$\boldsymbol{v}'' = \mathcal{B}\boldsymbol{v}' \tag{1.252}$$

and the inverse transformation is:

$$\boldsymbol{v}' = \mathcal{B}^T \boldsymbol{v}'' \tag{1.253}$$

By substituting the equation (1.250) into (1.252) we obtain:

$$\boldsymbol{v}'' = \mathcal{B}\mathcal{A}\boldsymbol{v} \tag{1.254}$$

The resulting matrix $\mathcal{B}\mathcal{A}$ is also an orthogonal matrix, and shows the transformation matrix from (x_1, x_2, x_3) to (x''_1, x''_2, x''_3), (see Figure 1.22). The inverse form of (1.254) is evaluated by substituting (1.253) into (1.251), the result of which is:

$$\boldsymbol{v} = \mathcal{A}^T \mathcal{B}^T \boldsymbol{v}'' \tag{1.255}$$

This equation could have been obtained by using equation (1.254), *i.e.*:

$$(\mathcal{B}\mathcal{A})^{-1}\boldsymbol{v}'' = (\mathcal{B}\mathcal{A})^{-1}(\mathcal{B}\mathcal{A})\boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{v} = (\mathcal{B}\mathcal{A})^{-1}\boldsymbol{v}'' = \mathcal{A}^{-1}\mathcal{B}^{-1}\boldsymbol{v}'' = \mathcal{A}^T\mathcal{B}^T\boldsymbol{v}'' \tag{1.256}$$

Then, it is easy to find the components of the vector in the coordinate system (x'''_1, x'''_2, x'''_3), (see Figure 1.22):

$$\boldsymbol{v}''' = \mathcal{C}\mathcal{B}\mathcal{A}\boldsymbol{v} \quad \xrightarrow{\text{inverse form}} \quad \boldsymbol{v} = \mathcal{A}^T\mathcal{B}^T\mathcal{C}^T\boldsymbol{v}''' \tag{1.257}$$

1.5.3.1 Component Transformation Law in Two Dimensions (2D)

Now, consider two sets of coordinate systems, shown in Figure 1.23.

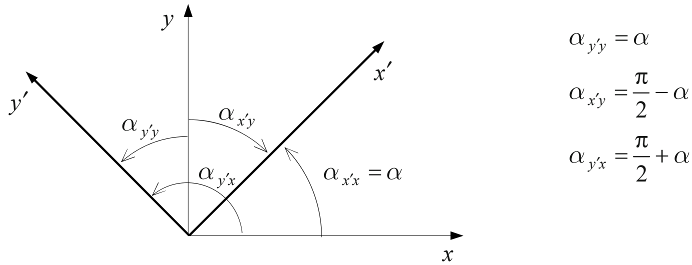


Figure 1.23: Transformation of a coordinate system in 2D.

The transformation matrix from $(x - y)$ to $(x' - y')$ is given by direction cosines, (see Figure 1.23), as:

$$\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha_{x'x}) & \cos(\alpha_{x'y}) & 0 \\ \cos(\alpha_{y'x}) & \cos(\alpha_{y'y}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.258}$$

By using trigonometric identities we can deduce that:

$$\begin{aligned} &\alpha_{x'x} = \alpha_{y'y} \Rightarrow \cos(\alpha_{x'x}) = \cos(\alpha_{y'y}) = \cos(\alpha)\,, \ \cos(\alpha_{x'y}) = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha)\,, \\ &\cos(\alpha_{y'x}) = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha) \end{aligned} \tag{1.259}$$

Thus, the transformation matrix in 2D is dependent on a single parameter, α, *i.e.*:

$$\mathcal{A} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{1.260}$$

Another way to prove (1.260) is by considering the vector position of the point P in both systems, (see Figure 1.24).

Moreover, in view of Figure 1.24, said coordinates are interrelated as shown below:

$$\begin{aligned} &\begin{cases} x'_P = x_P \cos(\alpha) + y_P \cos(\beta) \\ y'_P = -x_P \cos(\beta) + y_P \cos\left(\frac{\pi}{2} - \beta\right) \end{cases} \Rightarrow \begin{cases} x'_P = x_P \cos(\alpha) + y_P \cos\left(\frac{\pi}{2} - \alpha\right) \\ y'_P = -x_P \cos\left(\frac{\pi}{2} - \alpha\right) + y_P \cos(\alpha) \end{cases} \\ &\Rightarrow \begin{cases} x'_P = x_P \cos(\alpha) + y_P \sin(\alpha) \\ y'_P = -x_P \sin(\alpha) + y_P \cos(\alpha) \end{cases} \end{aligned} \tag{1.261}$$

Or in matrix form:

$$\begin{bmatrix} x'_P \\ y'_P \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x_P \\ y_P \end{bmatrix} \xrightarrow[\text{transformation}]{\text{Inverse}} \begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}^{-1} \begin{bmatrix} x'_P \\ y'_P \end{bmatrix} \tag{1.262}$$

Since $\mathcal{A}^{-1} = \mathcal{A}^T$, it is true that:

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x'_P \\ y'_P \end{bmatrix} \tag{1.263}$$

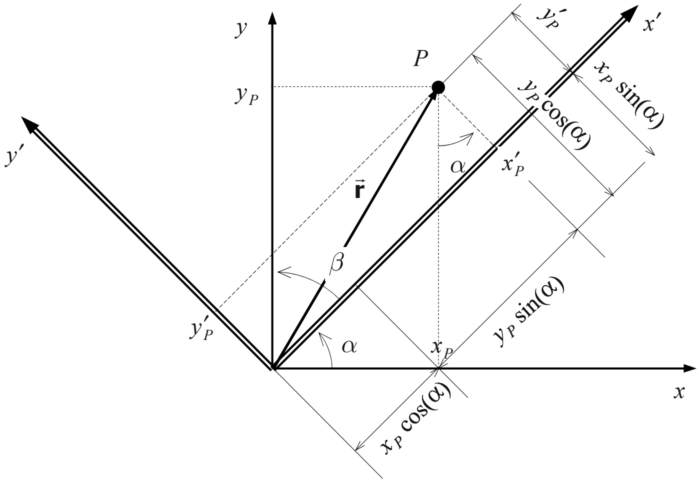


Figure 1.24: Transformation of a coordinate system in 2D.

Problem 1.29: Find the transformation matrix between the systems: x, y, z and x''', y''', z'''. These systems are represented in Figure 1.25.

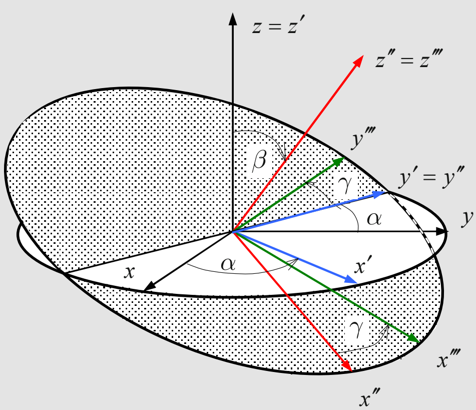
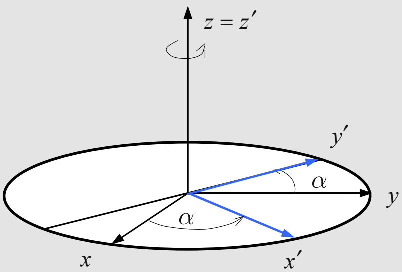


Figure 1.25: Rotation.

Solution: The coordinate system x''', y''', z''' can be obtained by different combinations of rotations as follows:

- Rotation along the z-axis

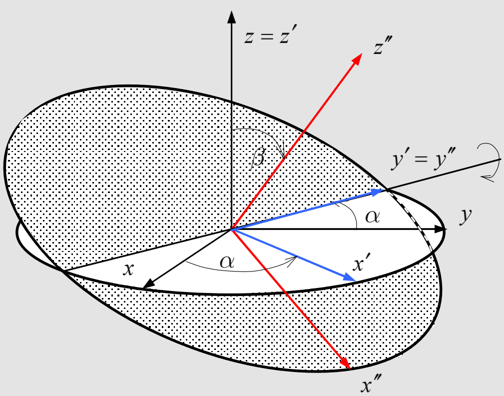


from x, y, z to x', y', z'

$$\mathcal{A} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $0 \le \alpha \le 360^{\circ}$

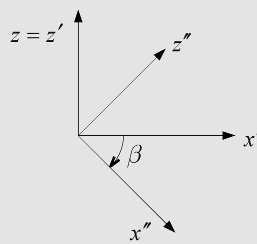
- Rotation along the y'-axis



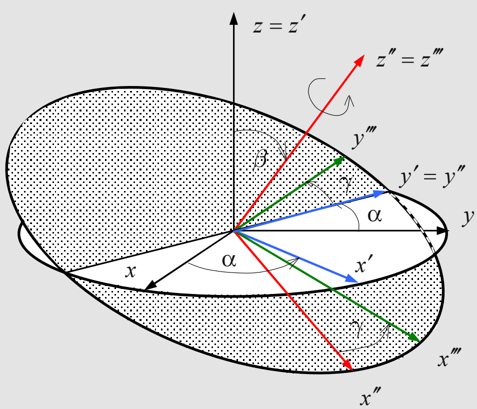
from x', y', z' to x'', y'', z''

$$\mathcal{B} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

with $0 \le \beta \le 180^{\circ}$



- Rotation along the z''-axis



from x'', y'', z'' to x''', y''', z'''

$$\mathcal{C} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $0 \le \gamma \le 360^{\circ}$

The transformation matrix from (x, y, z) to (x''', y''', z'''), (see Figure 1.22), is given by:

$$\mathcal{D} = \mathcal{CBA}$$

After multiplying the matrices, we obtain:

$$\mathcal{D} = \begin{bmatrix} (\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma) & (\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma) & -\sin\beta\cos\gamma \\ (-\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma) & (-\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma) & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{bmatrix}$$

The angles α, β, γ are known as *Euler angles* and were introduced by Leonhard Euler to describe the orientation of a rigid body motion.

Problem 1.30: Let $\mathbf{T}$ be a second-order tensor whose components in the Cartesian system (x_1, x_2, x_3) are given by:

$$(\mathbf{T})_{ij} = \mathsf{T}_{ij} = \mathcal{T} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given that the transformation matrix between two systems, (x_1, x_2, x_3)- (x'_1, x'_2, x'_3), is:

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Obtain the tensor components T_{ij} in the new coordinate system (x'_1, x'_2, x'_3).

Solution: As defined in equation (1.249), the transformation law for second-order tensor components is:

$$\mathsf{T}'_{ij} = a_{ik} a_{jl} \mathsf{T}_{kl}$$

To enable the previous calculation to be carried out in matrix form we use:

$$\mathsf{T}'_{ij} = \underbrace{[a_{ik}]\,[\mathsf{T}_{kl}]}\;[a_{lj}]^T$$

Thus

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T$$

$$\mathcal{T}' = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

On carrying out the operation of the previous matrices we now have:

$$\mathcal{T}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

NOTE: As we can verify in the above example, the components of the tensor $\mathbf{T}$, in the new basis, have one particular feature, *i.e.* the off-diagonal terms are equal to zero. The question now is: Given an arbitrary tensor $\mathbf{T}$, is there a transformation which results in the off-diagonal terms being zero? This type of problem is called the *eigenvalue and eigenvector* problem. ■

1.5.4 Eigenvalue and Eigenvector Problem

As we have seen, the scalar product between a second-order tensor $\mathbf{T}$ and a vector (or unit vector $\hat{\mathbf{n}}'$) leads to a vector. In other words, projecting a second-order tensor onto a certain direction results in a vector that does not necessarily have the same direction as $\hat{\mathbf{n}}'$, (see Figure 1.26(a)).

The aim of the eigenvalue and eigenvector problem is to find a direction $\hat{\mathbf{n}}$, in such a way that the resulting vector, $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \mathbf{T}\cdot\hat{\mathbf{n}}$, coincides with it, (see Figure 1.26 (b)).

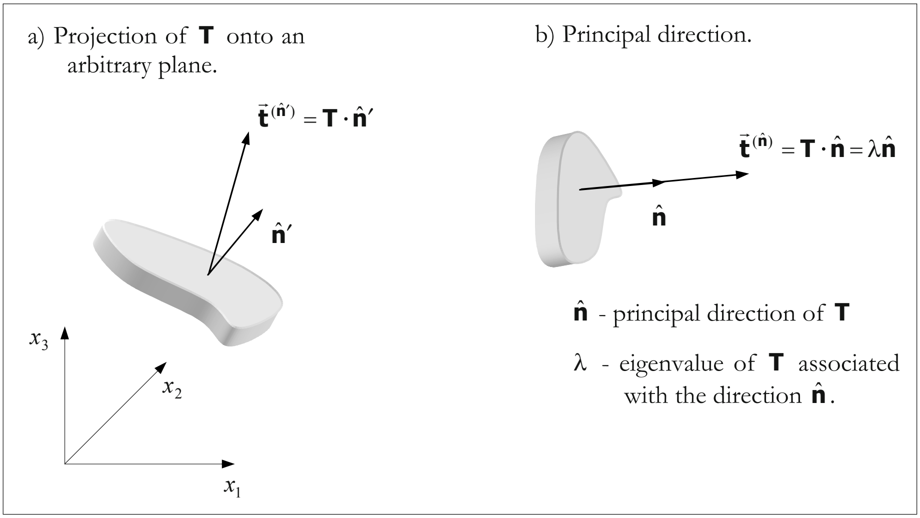


Figure 1.26: Projecting a tensor onto a direction.

Let $\mathbf{T}$ be a second-order tensor. A vector $\hat{\mathbf{n}}$ is said to be *eigenvector* of $\mathbf{T}$ if there is a scalar λ, called the *eigenvalue*, so that:

$$\mathbf{T}\cdot\hat{\mathbf{n}} = \lambda\hat{\mathbf{n}} \tag{1.264}$$

The equation (1.264) can be rearranged in indicial notation as:

$$\begin{aligned}
&\mathsf{T}_{ij}\hat{\mathsf{n}}_j = \lambda\hat{\mathsf{n}}_i \\
&\Rightarrow \mathsf{T}_{ij}\hat{\mathsf{n}}_j - \lambda\hat{\mathsf{n}}_i = \mathsf{0}_i \\
&\Rightarrow (\mathsf{T}_{ij} - \lambda\delta_{ij})\hat{\mathsf{n}}_j = \mathsf{0}_i \quad \xrightarrow[\textit{notation}]{\textit{Tensorial}} \quad (\mathbf{T} - \lambda\mathbf{1})\cdot\hat{\mathbf{n}} = \vec{\mathbf{0}}
\end{aligned} \tag{1.265}$$

The previous set of homogeneous equations only have nontrivial solution, *i.e.* $\hat{\mathbf{n}} \neq \vec{\mathbf{0}}$, if and only if:

$$\det(\mathbf{T} - \lambda\mathbf{1}) = 0 \quad ; \quad \left|\mathsf{T}_{ij} - \lambda\delta_{ij}\right| = 0 \tag{1.266}$$

The determinant (1.266) is called the *characteristic determinant* of the tensor $\mathbf{T}$, explicitly given by:

$$\begin{vmatrix} \mathsf{T}_{11}-\lambda & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{21} & \mathsf{T}_{22}-\lambda & \mathsf{T}_{23} \\ \mathsf{T}_{31} & \mathsf{T}_{32} & \mathsf{T}_{33}-\lambda \end{vmatrix} = 0 \tag{1.267}$$

Developing this determinant, we obtain the *characteristic polynomial*, which is shown by a cubic equation in λ:

$$\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0 \tag{1.268}$$

where $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$ are the *principal invariants* of $\mathbf{T}$, and are defined in components terms as:

$$\begin{aligned}
I_{\mathbf{T}} &= \mathrm{Tr}(\mathbf{T}) = \mathsf{T}_{ii} \\
II_{\mathbf{T}} &= \frac{1}{2}\left[(\mathrm{Tr}\mathbf{T})^2 - \mathrm{Tr}(\mathbf{T}^2)\right] \\
&= \frac{1}{2}\left\{ \mathrm{Tr}(\mathsf{T}_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)\mathrm{Tr}(\mathsf{T}_{kl}\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) - \mathrm{Tr}\left[(\mathsf{T}_{ij}\hat{\mathbf{e}}_i \otimes \overbrace{\hat{\mathbf{e}}_j)\cdot(\mathsf{T}_{kl}\hat{\mathbf{e}}_k}^{\delta_{jk}} \otimes \hat{\mathbf{e}}_l)\right] \right\} \\
&= \frac{1}{2}\left\{ \mathsf{T}_{ij}\delta_{ij}\mathsf{T}_{kl}\delta_{kl} - \mathsf{T}_{ij}\mathsf{T}_{kl}\delta_{jk}\mathrm{Tr}\left[(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l)\right] \right\} \\
&= \frac{1}{2}\left\{ \mathsf{T}_{ii}\mathsf{T}_{kk} - \mathsf{T}_{ij}\mathsf{T}_{kl}\delta_{jk}\delta_{il} \right\} \\
&= \frac{1}{2}\left\{ \mathsf{T}_{ii}\mathsf{T}_{kk} - \mathsf{T}_{ij}\mathsf{T}_{ji} \right\} = \mathsf{M}_{ii} = \mathrm{Tr}\left[\mathrm{cof}(\mathbf{T})\right] \\
III_{\mathbf{T}} &= \det(\mathbf{T}) = \left|\mathsf{T}_{ij}\right| = \epsilon_{ijk}\mathsf{T}_{i1}\mathsf{T}_{j2}\mathsf{T}_{k3}
\end{aligned} \tag{1.269}$$

where M_{ii} is the matrix trace defined in equation (1.210), $\mathsf{M}_{ii} = \mathsf{M}_{11} + \mathsf{M}_{22} + \mathsf{M}_{33}$. More explicitly the invariants are given by:

$$\begin{aligned}
I_{\mathbf{T}} &= \mathsf{T}_{11} + \mathsf{T}_{22} + \mathsf{T}_{33} \\
II_{\mathbf{T}} &= \begin{vmatrix} \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{32} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{13} \\ \mathsf{T}_{31} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} \\ \mathsf{T}_{21} & \mathsf{T}_{22} \end{vmatrix} \\
&= \mathsf{T}_{22}\mathsf{T}_{33} - \mathsf{T}_{23}\mathsf{T}_{32} + \mathsf{T}_{11}\mathsf{T}_{33} - \mathsf{T}_{13}\mathsf{T}_{31} + \mathsf{T}_{11}\mathsf{T}_{22} - \mathsf{T}_{12}\mathsf{T}_{21} \\
III_{\mathbf{T}} &= \mathsf{T}_{11}(\mathsf{T}_{22}\mathsf{T}_{33} - \mathsf{T}_{32}\mathsf{T}_{23}) - \mathsf{T}_{12}(\mathsf{T}_{21}\mathsf{T}_{33} - \mathsf{T}_{31}\mathsf{T}_{23}) + \mathsf{T}_{13}(\mathsf{T}_{21}\mathsf{T}_{32} - \mathsf{T}_{31}\mathsf{T}_{22})
\end{aligned} \tag{1.270}$$

If $\mathbf{T}$ is a symmetric tensor, the principal invariants are summarized as follows:

$$\begin{aligned}
I_{\mathbf{T}} &= \mathsf{T}_{11} + \mathsf{T}_{22} + \mathsf{T}_{33} \\
II_{\mathbf{T}} &= \mathsf{T}_{11}\mathsf{T}_{22} + \mathsf{T}_{11}\mathsf{T}_{33} + \mathsf{T}_{22}\mathsf{T}_{33} - \mathsf{T}_{12}^2 - \mathsf{T}_{13}^2 - \mathsf{T}_{23}^2 \\
III_{\mathbf{T}} &= \mathsf{T}_{11}\mathsf{T}_{22}\mathsf{T}_{33} + \mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} + \mathsf{T}_{13}\mathsf{T}_{12}\mathsf{T}_{23} - \mathsf{T}_{12}^2\mathsf{T}_{33} - \mathsf{T}_{23}^2\mathsf{T}_{11} - \mathsf{T}_{13}^2\mathsf{T}_{22}
\end{aligned} \tag{1.271}$$

The eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, are found by solving the characteristic polynomial (1.268). Once the eigenvalues are evaluated, the eigenvectors are found by applying equation (1.265), *i.e.* $(\mathsf{T}_{ij} - \lambda_1\delta_{ij})\hat{\mathsf{n}}_j^{(1)} = \mathsf{0}_i$, $(\mathsf{T}_{ij} - \lambda_2\delta_{ij})\hat{\mathsf{n}}_j^{(2)} = \mathsf{0}_i$, $(\mathsf{T}_{ij} - \lambda_3\delta_{ij})\hat{\mathsf{n}}_j^{(3)} = \mathsf{0}_i$. These eigenvectors constitute a new space denoted as the *principal space*.

If $\mathbf{T}$ is a symmetric tensor, the principal space is defined by an orthonormal basis and all eigenvalues are real numbers. If the three eigenvalues are different, $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the three principal directions are unique. If two of them are equal, *e.g.* $\lambda_1 = \lambda_2 \neq \lambda_3$, we can state that the principal direction, $\hat{\mathbf{n}}^{(3)}$, associated with the eigenvalue λ_3, is unique, and, any direction defined in the plane normal to $\hat{\mathbf{n}}^{(3)}$ is a principal direction, and othorgonality is the only constraint to determining $\hat{\mathbf{n}}^{(1)}$ and $\hat{\mathbf{n}}^{(2)}$. If $\lambda_1 = \lambda_2 = \lambda_3$, any direction is principal. A tensor that has three equal eigenvalues is called a *Spherical Tensor*, (see Appendix A-The Tensor ellipsoid).

The $\mathbf{T}$-components in the principal space are only made up of normal components, *i.e.*:

$$\mathsf{T}'_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \tag{1.272}$$

Therefore, the principal invariants can also be evaluated by:

$$I_{\mathbf{T}} = \mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3 \, , \; II_{\mathbf{T}} = \mathsf{T}_1\mathsf{T}_2 + \mathsf{T}_2\mathsf{T}_3 + \mathsf{T}_1\mathsf{T}_3 \, , \; III_{\mathbf{T}} = \mathsf{T}_1\mathsf{T}_2\mathsf{T}_3 \tag{1.273}$$

whose values must match the values obtained in (1.270), since they are invariant with a change of basis.

If $\mathbf{T}$ is a spherical tensor, *i.e.* $\mathsf{T}_1 = \mathsf{T}_2 = \mathsf{T}_3 = \mathsf{T}$, it holds that $I_{\mathbf{T}}^2 = 3\,II_{\mathbf{T}}$, $III_{\mathbf{T}} = \mathsf{T}^3$.

Let $\mathbf{W}$ be an antisymmetric tensor. The principal invariants of $\mathbf{W}$ are given by:

$$\begin{aligned}
I_{\mathbf{W}} &= \mathrm{Tr}(\mathbf{W}) = 0 \\
II_{\mathbf{W}} &= \frac{1}{2}\left[(\mathrm{Tr}\mathbf{W})^2 - \mathrm{Tr}(\mathbf{W}^2)\right] = \frac{-\mathrm{Tr}(\mathbf{W}^2)}{2} \\
&= \begin{vmatrix} 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{23} & 0 \end{vmatrix} + \begin{vmatrix} 0 & \mathsf{W}_{13} \\ -\mathsf{W}_{13} & 0 \end{vmatrix} + \begin{vmatrix} 0 & \mathsf{W}_{12} \\ -\mathsf{W}_{12} & 0 \end{vmatrix} \\
&= \mathsf{W}_{23}\mathsf{W}_{23} + \mathsf{W}_{13}\mathsf{W}_{13} + \mathsf{W}_{12}\mathsf{W}_{12} \\
&= \omega^2 \\
III_{\mathbf{W}} &= 0
\end{aligned} \tag{1.274}$$

where $\omega^2 = \|\vec{w}\|^2 = \vec{w} \cdot \vec{w} = \mathsf{W}_{23}^2 + \mathsf{W}_{13}^2 + \mathsf{W}_{12}^2$ as defined in (1.132). Then, the characteristic equation for an antisymmetric tensor is reduced to:

$$\lambda^3 - \lambda^2 I_{\mathbf{W}} + \lambda II_{\mathbf{W}} - III_{\mathbf{W}} = 0 \quad \Rightarrow \quad \lambda^3 + \omega^2\lambda = 0 \quad \Rightarrow \quad \lambda\left(\lambda^2 + \omega^2\right) = 0 \tag{1.275}$$

In this case, one eigenvalue is real and equal to zero and the others are imaginary roots:

$$\lambda^2 + \omega^2 = 0 \quad \Rightarrow \quad \lambda^2 = -\omega^2 = 0 \quad \Rightarrow \quad \lambda_{(1,2)} = \pm\omega\sqrt{-1} = \pm\omega\, i \tag{1.276}$$

1.5.4.1 The Orthogonality of the Eigenvectors

Consider a symmetric second-order tensor $\mathbf{T}$. By the definition of eigenvalues, given in (1.264), if λ_1, λ_2, λ_3 are the eigenvalues of $\mathbf{T}$, then it follows that:

$$\mathbf{T} \cdot \hat{\mathbf{n}}^{(1)} = \lambda_1 \hat{\mathbf{n}}^{(1)} \quad ; \quad \mathbf{T} \cdot \hat{\mathbf{n}}^{(2)} = \lambda_2 \hat{\mathbf{n}}^{(2)} \qquad \mathbf{T} \cdot \hat{\mathbf{n}}^{(3)} = \lambda_3 \hat{\mathbf{n}}^{(3)} \tag{1.277}$$

Applying the dot product between $\hat{\mathbf{n}}^{(2)}$ and $\mathbf{T} \cdot \hat{\mathbf{n}}^{(1)} = \lambda_1 \hat{\mathbf{n}}^{(1)}$, and the dot product between $\hat{\mathbf{n}}^{(1)}$ and $\mathbf{T} \cdot \hat{\mathbf{n}}^{(2)} = \lambda_2 \hat{\mathbf{n}}^{(2)}$ we obtain:

$$\begin{aligned}
\hat{\mathbf{n}}^{(2)} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}^{(1)} &= \lambda_1 \hat{\mathbf{n}}^{(2)} \cdot \hat{\mathbf{n}}^{(1)} \\
\hat{\mathbf{n}}^{(1)} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}^{(2)} &= \lambda_2 \hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(2)}
\end{aligned} \tag{1.278}$$

Since $\mathbf{T}$ is symmetric, it holds that $\hat{\mathbf{n}}^{(2)} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}^{(1)} = \hat{\mathbf{n}}^{(1)} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}^{(2)}$, so:

$$\lambda_1 \hat{\mathbf{n}}^{(2)} \cdot \hat{\mathbf{n}}^{(1)} = \lambda_2 \hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(2)} = \lambda_2 \hat{\mathbf{n}}^{(2)} \cdot \hat{\mathbf{n}}^{(1)} \tag{1.279}$$

$$\Rightarrow (\lambda_1 - \lambda_2)\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(2)} = 0 \tag{1.280}$$

To satisfy the equation (1.280), with $\lambda_1 \neq \lambda_2 \neq 0$, the following must be true:

$$\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(2)} = 0 \tag{1.281}$$

Similarly, it is possible to show that $\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(3)} = 0$ and $\hat{\mathbf{n}}^{(2)} \cdot \hat{\mathbf{n}}^{(3)} = 0$ and then we can conclude that the eigenvectors are mutually orthogonal, and constitute an orthogonal basis, (see Figure 1.27), where the transformation matrix between systems is:

$$\mathcal{A} = \begin{bmatrix} \hat{\mathbf{n}}^{(1)} \\ \hat{\mathbf{n}}^{(2)} \\ \hat{\mathbf{n}}^{(3)} \end{bmatrix} = \begin{bmatrix} \hat{n}_1^{(1)} & \hat{n}_2^{(1)} & \hat{n}_3^{(1)} \\ \hat{n}_1^{(2)} & \hat{n}_2^{(2)} & \hat{n}_3^{(2)} \\ \hat{n}_1^{(3)} & \hat{n}_2^{(3)} & \hat{n}_3^{(3)} \end{bmatrix} \tag{1.282}$$

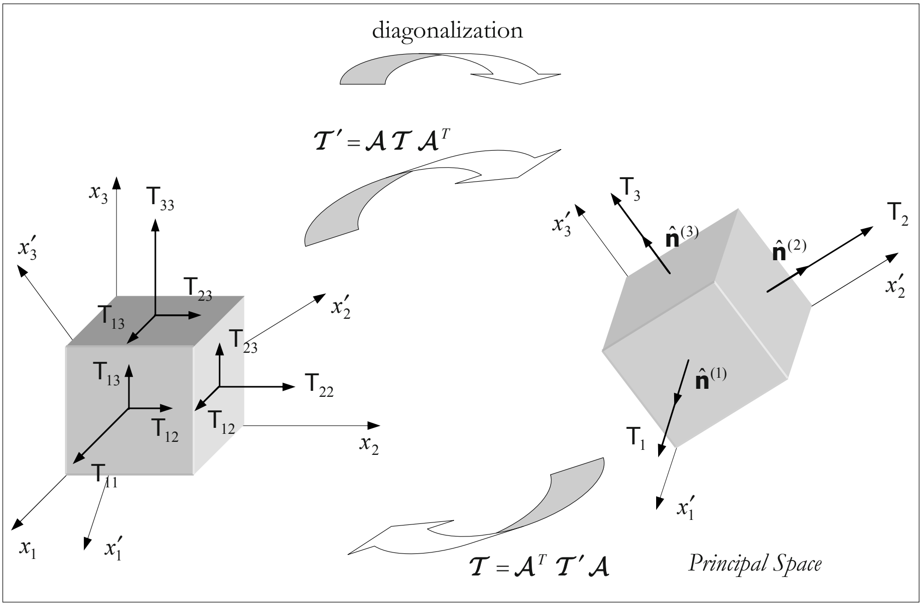


Figure 1.27: Diagonalization.

Problem 1.31: Show that the following relations are invariants:

$$C_1^2 + C_2^2 + C_3^2 \quad ; \quad C_1^3 + C_2^3 + C_3^3 \quad ; \quad C_1^4 + C_2^4 + C_3^4$$

where C_1, C_2, C_3 are the eigenvalues of the second-order tensor $\boldsymbol{C}$.

Solution: Any combination of invariants is also an invariant, so, on this basis, we can try to express the above expressions in terms of their principal invariants.

$$I_{\boldsymbol{C}}^2 = (C_1 + C_2 + C_3)^2 = C_1^2 + C_2^2 + C_3^2 + 2\underbrace{(C_1C_2 + C_1C_3 + C_2C_3)}_{II_{\boldsymbol{C}}} \Rightarrow C_1^2 + C_2^2 + C_3^2 = I_{\boldsymbol{C}}^2 - 2\,II_{\boldsymbol{C}}$$

So, we have proved that $C_1^2 + C_2^2 + C_3^2$ is an invariant. Similarly, we can obtain:

$$C_1^3 + C_2^3 + C_3^3 = I_{\boldsymbol{C}}^3 - 3\,II_{\boldsymbol{C}} I_{\boldsymbol{C}} + 3\,III_{\boldsymbol{C}}$$
$$C_1^4 + C_2^4 + C_3^4 = I_{\boldsymbol{C}}^4 - 4\,II_{\boldsymbol{C}} I_{\boldsymbol{C}}^2 + 4\,III_{\boldsymbol{C}} I_{\boldsymbol{C}} + 2\,II_{\boldsymbol{C}}^2$$

Problem 1.32: Let $\mathbf{Q}$ be a proper orthogonal tensor, and $\mathbf{E}$ be an arbitrary second-order tensor. Show that the eigenvalues of $\mathbf{E}$ do not change with the following orthogonal transformation:

$$\mathbf{E}^* = \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T$$

Solution: We can prove this as follows:

$$\begin{aligned}
0 &= \det(\mathbf{E}^* - \lambda \mathbf{1}) \\
&= \det(\mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T - \lambda \mathbf{1}) \\
&= \det(\mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^T - \mathbf{Q} \cdot \lambda \mathbf{1} \cdot \mathbf{Q}^T) \\
&= \det[\mathbf{Q} \cdot (\mathbf{E} - \lambda \mathbf{1}) \cdot \mathbf{Q}^T] \\
&= \underbrace{\det(\mathbf{Q})}_{1} \det(\mathbf{E} - \lambda \mathbf{1}) \underbrace{\det(\mathbf{Q}^T)}_{1} \\
&= \det(\mathbf{E} - \lambda \mathbf{1})
\end{aligned}
\qquad
\begin{aligned}
0 &= \det(\mathsf{E}^*_{ij} - \lambda \delta_{ij}) \\
&= \det(\mathsf{Q}_{ik} \mathsf{E}_{kp} \mathsf{Q}_{jp} - \lambda \delta_{ij}) \\
&= \det(\mathsf{Q}_{ik} \mathsf{E}_{kp} \mathsf{Q}_{jp} - \lambda \mathsf{Q}_{ik} \mathsf{Q}_{jp} \delta_{kp}) \\
&= \det[\mathsf{Q}_{ik} (\mathsf{E}_{kp} - \lambda \delta_{kp}) \mathsf{Q}_{jp}] \\
&= \det(\mathsf{Q}_{ik}) \det(\mathsf{E}_{kp} - \lambda \delta_{kp}) \det(\mathsf{Q}_{jp}) \\
&= \det(\mathsf{E}_{kp} - \lambda \delta_{kp})
\end{aligned}$$

Thus, we have proved that $\mathbf{E}$ and $\mathbf{E}^*$ have the same eigenvalues.

1.5.4.2 Solution of the Cubic Equation

Let $\mathbf{T}$ be a symmetric second-order tensor. The roots of the characteristic equation ($\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0$) are all real numbers, and are expressed as:

$$\begin{aligned}
\lambda_1 &= 2S\left[\cos\left(\frac{\alpha}{3}\right)\right] + \frac{I_{\mathbf{T}}}{3} \\
\lambda_2 &= 2S\left[\cos\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right)\right] + \frac{I_{\mathbf{T}}}{3} \\
\lambda_3 &= 2S\left[\cos\left(\frac{\alpha}{3} + \frac{4\pi}{3}\right)\right] + \frac{I_{\mathbf{T}}}{3}
\end{aligned} \tag{1.283}$$

where

$$R = \frac{I_{\mathbf{T}}^2 - 3 II_{\mathbf{T}}}{3}; \quad S = \sqrt{\frac{R}{3}}; \quad Q = \frac{I_{\mathbf{T}} II_{\mathbf{T}}}{3} - III_{\mathbf{T}} - \frac{2I_{\mathbf{T}}^3}{27}; \quad T = \sqrt{\frac{R^3}{27}}; \quad \alpha = \arccos\left(-\frac{Q}{2T}\right)$$

where α is in radians. (1.284)

By restructuring the solution (1.283) in matrix form, we obtain:

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \underbrace{\frac{I_{\mathbf{T}}}{3}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Spherical part}} + \underbrace{2S\begin{bmatrix} \cos\left(\frac{\alpha}{3}\right) & 0 & 0 \\ 0 & \cos\left(\frac{\alpha}{3} + \frac{2\pi}{3}\right) & 0 \\ 0 & 0 & \cos\left(\frac{\alpha}{3} + \frac{4\pi}{3}\right) \end{bmatrix}}_{\text{Deviatoric part}} \tag{1.285}$$

where we clearly distinguish the spherical and the deviatoric part of the tensor in the principal space. Note that, if $\mathbf{T}$ is a spherical tensor the following relationship holds $I_{\mathbf{T}}^2 = 3 II_{\mathbf{T}}$, then $S = 0$.

Problem 1.33: Find the principal values and directions of the second-order tensor $\mathbf{T}$, where the Cartesian components of $\mathbf{T}$ are:

$$(\mathbf{T})_{ij} = \mathsf{T}_{ij} = \mathcal{T} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: We need to find nontrivial solutions for $(\mathsf{T}_{ij} - \lambda\delta_{ij})\hat{\mathsf{n}}_j = 0_i$, which are constrained by $\hat{\mathsf{n}}_j\hat{\mathsf{n}}_j = 1$ (unit vector). As we have seen, the nontrivial solution requires that:

$$\left|\mathsf{T}_{ij} - \lambda\delta_{ij}\right| = 0$$

Explicitly, the above equation is:

$$\begin{vmatrix} \mathsf{T}_{11}-\lambda & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{21} & \mathsf{T}_{22}-\lambda & \mathsf{T}_{23} \\ \mathsf{T}_{31} & \mathsf{T}_{32} & \mathsf{T}_{33}-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Developing the above determinant, we can obtain the cubic equation:

$$(1-\lambda)\left[(3-\lambda)^2 - 1\right] = 0$$
$$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

We could have obtained the characteristic equation directly in terms of invariants:

$$I_{\mathbf{T}} = \mathrm{Tr}(\mathsf{T}_{ij}) = \mathsf{T}_{ii} = \mathsf{T}_{11} + \mathsf{T}_{22} + \mathsf{T}_{33} = 7$$

$$II_{\mathbf{T}} = \frac{1}{2}\left(\mathsf{T}_{ii}\mathsf{T}_{jj} - \mathsf{T}_{ij}\mathsf{T}_{ij}\right) = \begin{vmatrix} \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{32} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{13} \\ \mathsf{T}_{31} & \mathsf{T}_{33} \end{vmatrix} + \begin{vmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} \\ \mathsf{T}_{21} & \mathsf{T}_{22} \end{vmatrix} = 14$$

$$III_{\mathbf{T}} = \left|\mathsf{T}_{ij}\right| = \epsilon_{ijk}\mathsf{T}_{i1}\mathsf{T}_{j2}\mathsf{T}_{k3} = 8$$

Then, using the equation in (1.268), the characteristic equation is:

$$\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0 \quad \rightarrow \quad \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$$

On solving the cubic equation we obtain three real roots, namely:

$$\lambda_1 = 1; \quad \lambda_2 = 2; \quad \lambda_3 = 4$$

We can also verify that:

$$I_{\mathbf{T}} = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 4 = 7 \checkmark$$
$$II_{\mathbf{T}} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1\times 2 + 2\times 4 + 4\times 1 = 14 \checkmark$$
$$III_{\mathbf{T}} = \lambda_1\lambda_2\lambda_3 = 8 \checkmark$$

Thus, we can see that the invariants are the same as those evaluated previously.

<u>Principal directions:</u>

Each eigenvalue, λ_i, is associated with a corresponding eigenvector, $\hat{\mathbf{n}}^{(i)}$. We can use the equation in (1.265), *i.e.* $(\mathsf{T}_{ij} - \lambda\delta_{ij})\hat{\mathsf{n}}_j = \mathbf{0}_i$, to obtain the principal directions.

- $\lambda_1 = 1$

$$\begin{bmatrix} 3-\lambda_1 & -1 & 0 \\ -1 & 3-\lambda_1 & 0 \\ 0 & 0 & 1-\lambda_1 \end{bmatrix}\begin{bmatrix} \mathsf{n}_1 \\ \mathsf{n}_2 \\ \mathsf{n}_3 \end{bmatrix} = \begin{bmatrix} 3-1 & -1 & 0 \\ -1 & 3-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix}\begin{bmatrix} \mathsf{n}_1 \\ \mathsf{n}_2 \\ \mathsf{n}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

These become the following system of equations:

$$\begin{cases} 2n_1 - n_2 = 0 \\ -n_1 + 2n_2 = 0 \\ 0n_3 = 0 \end{cases} \Rightarrow n_1 = n_2 = 0$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1$$

Then we can conclude that: $\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = [0 \quad 0 \quad \pm 1]$.

NOTE: This solution could have been directly determined by the specific features of the $\mathcal{T}$ matrix. As the terms $T_{13} = T_{23} = T_{31} = T_{32} = 0$ imply that $T_{33} = 1$ is already a principal value, then, consequently, the original direction is a principal direction. ■

$$\lambda_2 = 2$$

$$\begin{bmatrix} 3-\lambda_2 & -1 & 0 \\ -1 & 3-\lambda_2 & 0 \\ 0 & 0 & 1-\lambda_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-2 & -1 & 0 \\ -1 & 3-2 & 0 \\ 0 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} n_1 - n_2 = 0 \Rightarrow n_1 = n_2 \\ -n_1 + n_2 = 0 \\ -n_3 = 0 \end{cases}$$

The first two equations are linearly dependent, after which we need an additional equation:

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_1^2 = 1 \Rightarrow n_1 = \pm\sqrt{\frac{1}{2}}$$

Thus:

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \left[\pm\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

$$\lambda_3 = 4$$

$$\begin{bmatrix} 3-\lambda_3 & -1 & 0 \\ -1 & 3-\lambda_3 & 0 \\ 0 & 0 & 1-\lambda_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 3-4 & -1 & 0 \\ -1 & 3-4 & 0 \\ 0 & 0 & 1-4 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -n_1 - n_2 = 0 \\ -n_1 - n_2 = 0 \\ -3n_3 = 0 \end{cases} \Rightarrow n_1 = -n_2$$

$$n_i n_i = n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow 2n_2^2 = 1 \Rightarrow n_2 = \pm\sqrt{\frac{1}{2}}$$

Then:

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \left[\mp\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

Afterwards, we summarize the eigenvalues and eigenvectors of $\mathbf{T}$:

$$\lambda_1 = 1 \quad \Rightarrow \quad \hat{n}_i^{(1)} = [0 \quad 0 \quad \pm 1]$$

$$\lambda_2 = 2 \quad \Rightarrow \quad \hat{n}_i^{(2)} = \left[\pm\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

$$\lambda_3 = 4 \quad \Rightarrow \quad \hat{n}_i^{(3)} = \left[\mp\sqrt{\frac{1}{2}} \quad \pm\sqrt{\frac{1}{2}} \quad 0 \right]$$

NOTE: The tensor components of this problem are the same as those used in **Problem 1.30**. Additionally, we can verify that the eigenvectors make up the transformation matrix, $\mathcal{A}$, between the original system, (x_1, x_2, x_3), and the principal space, (x'_1, x'_2, x'_3), (see **Problem 1.30**). ■

1.5.5 Spectral Representation of Tensors

Based on the solution of the equation in (1.268), if $\mathbf{T}$ is a symmetric second-order tensor there are three real eigenvalues: T_1, T_2, T_3 each of which is associated with an eigenvector, *i.e.*:

$$\begin{aligned} \mathsf{T}_1 &\Rightarrow \hat{\mathsf{n}}_i^{(1)} = \left[\hat{\mathsf{n}}_1^{(1)} \quad \hat{\mathsf{n}}_2^{(1)} \quad \hat{\mathsf{n}}_3^{(1)}\right] \\ \mathsf{T}_2 &\Rightarrow \hat{\mathsf{n}}_i^{(2)} = \left[\hat{\mathsf{n}}_1^{(2)} \quad \hat{\mathsf{n}}_2^{(2)} \quad \hat{\mathsf{n}}_3^{(2)}\right] \\ \mathsf{T}_3 &\Rightarrow \hat{\mathsf{n}}_i^{(3)} = \left[\hat{\mathsf{n}}_1^{(3)} \quad \hat{\mathsf{n}}_2^{(3)} \quad \hat{\mathsf{n}}_3^{(3)}\right] \end{aligned} \tag{1.286}$$

The principal space is formed by the orthogonal basis $\hat{\mathbf{n}}^{(1)}, \hat{\mathbf{n}}^{(2)}, \hat{\mathbf{n}}^{(3)}$, and the tensor components are represented by their eigenvalues as:

$$\mathsf{T}'_{ij} = \mathcal{T}' = \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \tag{1.287}$$

With reference to the fact that eigenvectors form a transformation matrix, $\mathcal{A}$, so that:

$$\mathcal{T}' = \mathcal{A}\,\mathcal{T}\,\mathcal{A}^T \tag{1.288}$$

Since $\mathcal{A}^{-1} = \mathcal{A}^T$, the inverse form is:

$$\mathcal{T} = \mathcal{A}^T\,\mathcal{T}'\,\mathcal{A} \tag{1.289}$$

where

$$\mathcal{A} = \begin{bmatrix} \hat{\mathbf{n}}^{(1)} \\ \hat{\mathbf{n}}^{(2)} \\ \hat{\mathbf{n}}^{(3)} \end{bmatrix} = \begin{bmatrix} \hat{\mathsf{n}}_1^{(1)} & \hat{\mathsf{n}}_2^{(1)} & \hat{\mathsf{n}}_3^{(1)} \\ \hat{\mathsf{n}}_1^{(2)} & \hat{\mathsf{n}}_2^{(2)} & \hat{\mathsf{n}}_3^{(2)} \\ \hat{\mathsf{n}}_1^{(3)} & \hat{\mathsf{n}}_2^{(3)} & \hat{\mathsf{n}}_3^{(3)} \end{bmatrix} \tag{1.290}$$

Explicitly, the relation in (1.289) is given by:

$$\begin{aligned} \begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33} \end{bmatrix} &= \begin{bmatrix} \hat{\mathsf{n}}_1^{(1)} & \hat{\mathsf{n}}_1^{(2)} & \hat{\mathsf{n}}_1^{(3)} \\ \hat{\mathsf{n}}_2^{(1)} & \hat{\mathsf{n}}_2^{(2)} & \hat{\mathsf{n}}_2^{(3)} \\ \hat{\mathsf{n}}_3^{(1)} & \hat{\mathsf{n}}_3^{(2)} & \hat{\mathsf{n}}_3^{(3)} \end{bmatrix} \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \begin{bmatrix} \hat{\mathsf{n}}_1^{(1)} & \hat{\mathsf{n}}_2^{(1)} & \hat{\mathsf{n}}_3^{(1)} \\ \hat{\mathsf{n}}_1^{(2)} & \hat{\mathsf{n}}_2^{(2)} & \hat{\mathsf{n}}_3^{(2)} \\ \hat{\mathsf{n}}_1^{(3)} & \hat{\mathsf{n}}_2^{(3)} & \hat{\mathsf{n}}_3^{(3)} \end{bmatrix} \\ &= \mathcal{A}^T \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \mathcal{A} \\ &= \mathsf{T}_1 \mathcal{A}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathsf{T}_2 \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathsf{T}_3 \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{A} \end{aligned} \tag{1.291}$$

Whereas:

$$
\begin{aligned}
\mathcal{A}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} &= \begin{bmatrix} \hat{n}_1^{(1)}\hat{n}_1^{(1)} & \hat{n}_1^{(1)}\hat{n}_2^{(1)} & \hat{n}_1^{(1)}\hat{n}_3^{(1)} \\ \hat{n}_2^{(1)}\hat{n}_1^{(1)} & \hat{n}_2^{(1)}\hat{n}_2^{(1)} & \hat{n}_2^{(1)}\hat{n}_3^{(1)} \\ \hat{n}_3^{(1)}\hat{n}_1^{(1)} & \hat{n}_3^{(1)}\hat{n}_2^{(1)} & \hat{n}_3^{(1)}\hat{n}_3^{(1)} \end{bmatrix} = \hat{n}_i^{(1)}\hat{n}_j^{(1)} \\
\mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} &= \begin{bmatrix} \hat{n}_1^{(2)}\hat{n}_1^{(2)} & \hat{n}_1^{(2)}\hat{n}_2^{(2)} & \hat{n}_1^{(2)}\hat{n}_3^{(2)} \\ \hat{n}_2^{(2)}\hat{n}_1^{(2)} & \hat{n}_2^{(2)}\hat{n}_2^{(2)} & \hat{n}_2^{(2)}\hat{n}_3^{(2)} \\ \hat{n}_3^{(2)}\hat{n}_1^{(2)} & \hat{n}_3^{(2)}\hat{n}_2^{(2)} & \hat{n}_3^{(2)}\hat{n}_3^{(2)} \end{bmatrix} = \hat{n}_i^{(2)}\hat{n}_j^{(2)} \\
\mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathcal{A} &= \begin{bmatrix} \hat{n}_1^{(3)}\hat{n}_1^{(3)} & \hat{n}_1^{(3)}\hat{n}_2^{(3)} & \hat{n}_1^{(3)}\hat{n}_3^{(3)} \\ \hat{n}_2^{(3)}\hat{n}_1^{(3)} & \hat{n}_2^{(3)}\hat{n}_2^{(3)} & \hat{n}_2^{(3)}\hat{n}_3^{(3)} \\ \hat{n}_3^{(3)}\hat{n}_1^{(3)} & \hat{n}_3^{(3)}\hat{n}_2^{(3)} & \hat{n}_3^{(3)}\hat{n}_3^{(3)} \end{bmatrix} = \hat{n}_i^{(3)}\hat{n}_j^{(3)}
\end{aligned}
\tag{1.292}
$$

Then, it is possible to represent the components of a second-order tensor in function of their eigenvalues and eigenvectors (spectral representation) as:

$$
\mathsf{T}_{ij} = \mathsf{T}_1\, \hat{n}_i^{(1)}\hat{n}_j^{(1)} + \mathsf{T}_2 \hat{n}_i^{(2)}\hat{n}_j^{(2)} + \mathsf{T}_3\, \hat{n}_i^{(3)}\hat{n}_j^{(3)} \tag{1.293}
$$

As we can see, the tensor is represented as a linear combination of dyads and the above representation in tensorial notation becomes:

$$
\mathbf{T} = \mathsf{T}_1\, \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \mathsf{T}_2\, \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \mathsf{T}_3\, \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \tag{1.294}
$$

or:

$$
\boxed{\mathbf{T} = \sum_{a=1}^{3} \mathsf{T}_a\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}} \quad \textit{Spectral representation of a second-order tensor} \tag{1.295}
$$

which is the *spectral representation* of the tensor. Note that, in the above equation we have to resort to the summation symbol, because the dummy index appears thrice in the expression.

NOTE: The spectral representation in (1.295) could easily have been obtained from the definition of the second-order unit tensor, given in (1.168), *i.e.* $\mathbf{1} = \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i$, which can also be represented by means of the summation symbol as $\mathbf{1} = \sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$. Then, it follows that:

$$
\mathbf{T} = \mathbf{T} \cdot \mathbf{1} = \mathbf{T} \cdot \left(\sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) = \sum_{a=1}^{3} \mathbf{T} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \mathsf{T}_a\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.296}
$$

where we have used the definition of eigenvalue and eigenvector $\mathbf{T} \cdot \hat{\mathbf{n}}^{(a)} = \mathsf{T}_a\, \hat{\mathbf{n}}^{(a)}$. ■

We now consider the orthogonal tensor $\mathbf{R}$. The orthogonal transformation applied to the unit vector $\hat{\mathbf{N}}$ leads to the unit vector $\hat{\mathbf{n}}$, *i.e.* $\hat{\mathbf{n}} = \mathbf{R} \cdot \hat{\mathbf{N}}$. Therefore, it is also possible to represent the orthogonal tensor $\mathbf{R}$ as follows:

$$
\mathbf{R} = \mathbf{R} \cdot \mathbf{1} = \mathbf{R} \cdot \left(\sum_{a=1}^{3} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right) = \sum_{a=1}^{3} \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \tag{1.297}
$$

The spectral representation is very useful for making algebraic operations with tensors. For example, tensor power in the principal space can be expressed as:

$$\left(\mathbf{T}^n\right)_{ij} = \begin{bmatrix} \mathsf{T}_1^n & 0 & 0 \\ 0 & \mathsf{T}_2^n & 0 \\ 0 & 0 & \mathsf{T}_3^n \end{bmatrix} \tag{1.298}$$

So, the spectral representation of $\mathbf{T}^n$ is given by:

$$\mathbf{T}^n = \sum_{a=1}^{3} \mathsf{T}_a^n \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.299}$$

Now, if we need the square root of the tensor, $\sqrt{\mathbf{T}}$, this can easily be obtained from the spectral representation as:

$$\sqrt{\mathbf{T}} = \sum_{a=1}^{3} \sqrt{\mathsf{T}_a} \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.300}$$

Next, we can show that a positive definite tensor has positive eigenvalues. For this purpose, we can consider a semi-positive definite tensor, $\mathbf{T}$, by which the condition $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \geq 0$ holds for all $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. Replacing the tensor by its spectral representation, we obtain:

$$\begin{aligned} & \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} \geq 0 \\ & \Rightarrow \hat{\mathbf{x}} \cdot \left(\sum_{a=1}^{3} \mathsf{T}_a \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) \cdot \hat{\mathbf{x}} \geq 0 \\ & \Rightarrow \sum_{a=1}^{3} \mathsf{T}_a \,\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \hat{\mathbf{x}} \geq 0 \end{aligned} \tag{1.301}$$

Note that the result of the operation $(\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)})$ is a scalar, thus:

$$\begin{aligned} & \sum_{a=1}^{3} \mathsf{T}_a \,\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \hat{\mathbf{x}} \geq 0 \quad \Rightarrow \quad \sum_{a=1}^{3} \mathsf{T}_a \underbrace{\left[(\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(a)})^2\right]}_{>0} \geq 0 \\ & \Rightarrow \mathsf{T}_1 \,(\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(1)})^2 + \mathsf{T}_2 \,(\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(2)})^2 + \mathsf{T}_3 \,(\hat{\mathbf{x}} \cdot \hat{\mathbf{n}}^{(3)})^2 \geq 0 \end{aligned} \tag{1.302}$$

The above expression must hold for all $\hat{\mathbf{x}} \neq \vec{\mathbf{0}}$. If we take $\hat{\mathbf{x}} = \hat{\mathbf{n}}^{(1)}$, the above equation is reduced to $\mathsf{T}_1 \,(\hat{\mathbf{n}}^{(1)} \cdot \hat{\mathbf{n}}^{(1)})^2 = \mathsf{T}_1 \geq 0$. The same is true for T_2 and T_3. Thus, we have demonstrated that if a tensor is semi-positive definite, its eigenvalues are greater than or equal to zero, *i.e.* $\mathsf{T}_1 \geq 0$, $\mathsf{T}_2 \geq 0$, $\mathsf{T}_3 \geq 0$. Therefore we can conclude that a tensor is positive definite, *i.e.* $\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} > 0$, if and only if its eigenvalues are positive and nonzero, *i.e.* $\mathsf{T}_1 > 0$, $\mathsf{T}_2 > 0$, $\mathsf{T}_3 > 0$. Consequently, the positive definite tensor trace is greater than zero. If the positive definite tensor trace is zero, this implies that the tensor is the zero tensor.

The spectral representation of the fourth-order unit tensor, $\mathbb{I}$, can be obtained starting from the definition in (1.169), *i.e.*:

$$\mathbb{I} = \delta_{ik}\delta_{j\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \sum_{a=1}^{3}\sum_{b=1}^{3} \hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b \otimes \hat{\mathbf{e}}_a \otimes \hat{\mathbf{e}}_b \tag{1.303}$$

As $\mathbb{I}$ is an isotropic tensor, (see 1.5.8 Isotropic and Anisotropic tensors), then the representation in (1.303) is also valid in any orthonormal basis, $\hat{\mathbf{n}}^{(a)}$, so:

$$\mathbb{I} = \sum_{a=1}^{3}\sum_{b=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \tag{1.304}$$

Similarly, we obtain the spectral representation for $\bar{\mathbb{I}}$ and $\bar{\bar{\mathbb{I}}}$ as:

$$\bar{\mathbb{I}} = \delta_{i\ell}\delta_{jk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_i \tag{1.305}$$

$$\bar{\mathbb{I}} = \sum_{a,b=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.306}$$

and

$$\bar{\bar{\mathbb{I}}} = \delta_{ij}\delta_{k\ell}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_\ell = \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k \tag{1.307}$$

$$\bar{\bar{\mathbb{I}}} = \sum_{a,b=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} \tag{1.308}$$

Problem 1.34: Let $\boldsymbol{\omega}$ be an antisymmetric second-order tensor and $\mathbf{V}$ be a positive definite symmetric tensor whose spectral representation is given by:

$$\mathbf{V} = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$$

Show that the antisymmetric tensor $\boldsymbol{\omega}$ can be represented by:

$$\boldsymbol{\omega} = \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \omega_{ab} \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Demonstrate also that:

$$\boldsymbol{\omega}\cdot\mathbf{V} - \mathbf{V}\cdot\boldsymbol{\omega} = \sum_{\substack{a,b=1 \\ a\neq b}}^{3} \omega_{ab}(\lambda_b - \lambda_a)\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Solution:
It is true that

$$\boldsymbol{\omega}\cdot\mathbf{1} = \boldsymbol{\omega}\cdot\left(\sum_{a=1}^{3}\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}\right) = \sum_{a=1}^{3}\boldsymbol{\omega}\cdot\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3}\left(\vec{w}\wedge\hat{\mathbf{n}}^{(a)}\right)\otimes\hat{\mathbf{n}}^{(a)}$$
$$= \sum_{a,b=1}^{3} w_b\left(\hat{\mathbf{n}}^{(b)}\wedge\hat{\mathbf{n}}^{(a)}\right)\otimes\hat{\mathbf{n}}^{(a)}$$

where we have applied an antisymmetric tensor property $\boldsymbol{\omega}\cdot\hat{\mathbf{n}} = \vec{w}\wedge\hat{\mathbf{n}}$, where $\vec{w}$ is the axial vector associated with $\boldsymbol{\omega}$. Expanding the above equation, we obtain:

$$\begin{aligned}
\boldsymbol{\omega} &= w_b\left(\hat{\mathbf{n}}^{(b)}\wedge\hat{\mathbf{n}}^{(1)}\right)\otimes\hat{\mathbf{n}}^{(1)} + w_b\left(\hat{\mathbf{n}}^{(b)}\wedge\hat{\mathbf{n}}^{(2)}\right)\otimes\hat{\mathbf{n}}^{(2)} + w_b\left(\hat{\mathbf{n}}^{(b)}\wedge\hat{\mathbf{n}}^{(3)}\right)\otimes\hat{\mathbf{n}}^{(3)} = \\
&= w_1\left(\hat{\mathbf{n}}^{(1)}\wedge\hat{\mathbf{n}}^{(1)}\right)\otimes\hat{\mathbf{n}}^{(1)} + w_2\left(\hat{\mathbf{n}}^{(2)}\wedge\hat{\mathbf{n}}^{(1)}\right)\otimes\hat{\mathbf{n}}^{(1)} + w_3\left(\hat{\mathbf{n}}^{(3)}\wedge\hat{\mathbf{n}}^{(1)}\right)\otimes\hat{\mathbf{n}}^{(1)} + \\
&+ w_1\left(\hat{\mathbf{n}}^{(1)}\wedge\hat{\mathbf{n}}^{(2)}\right)\otimes\hat{\mathbf{n}}^{(2)} + w_2\left(\hat{\mathbf{n}}^{(2)}\wedge\hat{\mathbf{n}}^{(2)}\right)\otimes\hat{\mathbf{n}}^{(2)} + w_3\left(\hat{\mathbf{n}}^{(3)}\wedge\hat{\mathbf{n}}^{(2)}\right)\otimes\hat{\mathbf{n}}^{(2)} + \\
&+ w_1\left(\hat{\mathbf{n}}^{(1)}\wedge\hat{\mathbf{n}}^{(3)}\right)\otimes\hat{\mathbf{n}}^{(3)} + w_2\left(\hat{\mathbf{n}}^{(2)}\wedge\hat{\mathbf{n}}^{(3)}\right)\otimes\hat{\mathbf{n}}^{(3)} + w_3\left(\hat{\mathbf{n}}^{(3)}\wedge\hat{\mathbf{n}}^{(3)}\right)\otimes\hat{\mathbf{n}}^{(3)}
\end{aligned}$$

On simplifying the above expression we obtain:

$$\begin{aligned}
\boldsymbol{\omega} = &-w_2\left(\hat{\mathbf{n}}^{(3)}\right)\otimes\hat{\mathbf{n}}^{(1)} + w_3\left(\hat{\mathbf{n}}^{(2)}\right)\otimes\hat{\mathbf{n}}^{(1)} + \\
&+ w_1\left(\hat{\mathbf{n}}^{(3)}\right)\otimes\hat{\mathbf{n}}^{(2)} - w_3\left(\hat{\mathbf{n}}^{(1)}\right)\otimes\hat{\mathbf{n}}^{(2)} + \\
&- w_1\left(\hat{\mathbf{n}}^{(2)}\right)\otimes\hat{\mathbf{n}}^{(3)} + w_2\left(\hat{\mathbf{n}}^{(1)}\right)\otimes\hat{\mathbf{n}}^{(3)}
\end{aligned}$$

Taking into account that $w_1 = -\omega_{23} = \omega_{32}$, $w_2 = \omega_{13} = -\omega_{31}$, $w_3 = -\omega_{12} = \omega_{21}$, the above equation becomes:

$$\boldsymbol{\omega} = \omega_{31}\,\hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(1)} + \omega_{21}\,\hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(1)} + \\ + \omega_{32}\,\hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(2)} + \omega_{12}\,\hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(2)} + \\ + \omega_{23}\,\hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(3)} + \omega_{13}\,\hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(3)}$$

which is the same as:

$$\boldsymbol{\omega} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

The terms $\boldsymbol{\omega}\cdot\mathbf{V}$ and $\mathbf{V}\cdot\boldsymbol{\omega}$ can be expressed as follows:

$$\boldsymbol{\omega}\cdot\mathbf{V} = \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right)\cdot\left(\sum_{b=1}^{3} \lambda_b\,\hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)}\right)$$

$$= \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\cdot\hat{\mathbf{n}}^{(b)} \otimes \hat{\mathbf{n}}^{(b)} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

and

$$\mathbf{V}\cdot\boldsymbol{\omega} = \left(\sum_{a=1}^{3} \lambda_a\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right) = \sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_a \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Then,

$$\boldsymbol{\omega}\cdot\mathbf{V} - \mathbf{V}\cdot\boldsymbol{\omega} = \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_b \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right) - \left(\sum_{\substack{a,b=1\\a\neq b}}^{3} \lambda_a \omega_{ab}\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}\right)$$

$$= \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}(\lambda_b - \lambda_a)\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

Similarly, it is possible to show that:

$$\boldsymbol{\omega}\cdot\mathbf{V}^2 - \mathbf{V}^2\cdot\boldsymbol{\omega} = \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}(\lambda_b^2 - \lambda_a^2)\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}$$

1.5.6 Cayley-Hamilton Theorem

The Cayley-Hamilton theorem states that any tensor, $\mathbf{T}$, satisfies its own characteristic equation, *i.e.* if the eigenvalues of $\mathbf{T}$ satisfy the equation $\lambda^3 - \lambda^2 I_{\mathbf{T}} + \lambda II_{\mathbf{T}} - III_{\mathbf{T}} = 0$, so does the tensor $\mathbf{T}$:

$$\mathbf{T}^3 - \mathbf{T}^2 I_{\mathbf{T}} + \mathbf{T} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0} \tag{1.309}$$

One of the applications of the Cayley-Hamilton theorem is to express the power of tensor, $\mathbf{T}^n$, as a combination of $\mathbf{T}^{n-1}$, $\mathbf{T}^{n-2}$, $\mathbf{T}^{n-3}$. For example, $\mathbf{T}^4$ is obtained as:

$$\mathbf{T}^3\cdot\mathbf{T} - \mathbf{T}^2\cdot\mathbf{T} I_{\mathbf{T}} + \mathbf{T}\cdot\mathbf{T} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1}\cdot\mathbf{T} = \mathbf{0} \quad\Rightarrow\quad \mathbf{T}^4 = \mathbf{T}^3 I_{\mathbf{T}} - \mathbf{T}^2 II_{\mathbf{T}} + III_{\mathbf{T}}\mathbf{T} \tag{1.310}$$

Using the Cayley-Hamilton theorem, it is possible to express the third invariant as a function of traces. According to the Cayley-Hamilton theorem, the expression

$\mathbf{T}^3 - I_{\mathbf{T}}\mathbf{T}^2 + II_{\mathbf{T}}\mathbf{T} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0}$ remains valid. Additionally, by applying the double scalar product with the second-order unit tensor, $\mathbf{1}$, we obtain:

$$\mathbf{T}^3 : \mathbf{1} - I_{\mathbf{T}}\mathbf{T}^2 : \mathbf{1} + II_{\mathbf{T}}\mathbf{T} : \mathbf{1} - III_{\mathbf{T}}\mathbf{1} : \mathbf{1} = \mathbf{0} : \mathbf{1} \tag{1.311}$$

Taking into consideration $\mathbf{T}^3 : \mathbf{1} = \mathrm{Tr}(\mathbf{T}^3)$, $\mathbf{T}^2 : \mathbf{1} = \mathrm{Tr}(\mathbf{T}^2)$, $\mathbf{T} : \mathbf{1} = \mathrm{Tr}(\mathbf{T})$, $\mathbf{1} : \mathbf{1} = \mathrm{Tr}(\mathbf{1}) = 3$, $\mathbf{0} : \mathbf{1} = \mathrm{Tr}(\mathbf{0}) = 0$ in the equation (1.311) we obtain:

$$\begin{aligned} &\mathrm{Tr}(\mathbf{T}^3) - I_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}^2) + II_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}) - III_{\mathbf{T}}\underbrace{\mathrm{Tr}(\mathbf{1})}_{=3} = 0 \\ &\Rightarrow III_{\mathbf{T}} = \frac{1}{3}\left[\mathrm{Tr}(\mathbf{T}^3) - I_{\mathbf{T}}\mathrm{Tr}(\mathbf{T}^2) + II_{\mathbf{T}}\mathrm{Tr}(\mathbf{T})\right] \end{aligned} \tag{1.312}$$

Replacing the values of the invariants, $I_{\mathbf{T}}$, $II_{\mathbf{T}}$, given by equation (1.269), we obtain:

$$III_{\mathbf{T}} = \frac{1}{3}\left\{\mathrm{Tr}(\mathbf{T}^3) - \frac{3}{2}\mathrm{Tr}(\mathbf{T}^2)\mathrm{Tr}(\mathbf{T}) + \frac{1}{2}\left[\mathrm{Tr}(\mathbf{T})\right]^3\right\} \tag{1.313}$$

or in indicial notation

$$III_{\mathbf{T}} = \frac{1}{3}\left\{\mathrm{T}_{ij}\mathrm{T}_{jk}\mathrm{T}_{ki} - \frac{3}{2}\mathrm{T}_{ij}\mathrm{T}_{ji}\mathrm{T}_{kk} + \frac{1}{2}\mathrm{T}_{ii}\mathrm{T}_{jj}\mathrm{T}_{kk}\right\} \tag{1.314}$$

Problem 1.35: Based on the Cayley-Hamilton theorem, find the inverse of a tensor $\mathbf{T}$ in terms of tensor power.
Solution: The Cayley-Hamilton theorem states that:

$$\mathbf{T}^3 - \mathbf{T}^2 I_{\mathbf{T}} + \mathbf{T} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0}$$

Carrying out the dot product between the previous equation and the tensor $\mathbf{T}^{-1}$, we obtain:

$$\begin{aligned} &\mathbf{T}^3 \cdot \mathbf{T}^{-1} - \mathbf{T}^2 \cdot \mathbf{T}^{-1} I_{\mathbf{T}} + \mathbf{T} \cdot \mathbf{T}^{-1} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} \cdot \mathbf{T}^{-1} = \mathbf{0} \cdot \mathbf{T}^{-1} \\ &\mathbf{T}^2 - \mathbf{T} I_{\mathbf{T}} + \mathbf{1} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{T}^{-1} = \mathbf{0} \\ &\Rightarrow \mathbf{T}^{-1} = \frac{1}{III_{\mathbf{T}}}\left(\mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + II_{\mathbf{T}}\mathbf{1}\right) \end{aligned}$$

The Cayley-Hamilton theorem also applies to square matrices of order n. Let $\mathcal{A}_{n\times n}$ be a square n by n matrix. The characteristic determinant is given by:

$$\left|\lambda \mathbf{1}_{n\times n} - \mathcal{A}\right| = 0 \tag{1.315}$$

where $\mathbf{1}_{n\times n}$ is the identity n by n matrix. Developing the determinant (1.315) we ontain:

$$\lambda^n - I_1\lambda^{n-1} + I_2\lambda^{n-2} - \cdots (-1)^n I_n = 0 \tag{1.316}$$

where $I_1, I_2, \cdots, I_n$ are the invariants of $\mathcal{A}$. In the particular case when $n = 3$, the invariants are the same obtained for a second-order tensor, *i.e.*: $I_1 = I_{\mathbf{A}}$, $I_2 = II_{\mathbf{A}}$, $I_3 = III_{\mathbf{A}}$. Applying the Cayley-Hamilton theorem it is true that:

$$\mathcal{A}^n - I_1\mathcal{A}^{n-1} + I_2\mathcal{A}^{n-2} - \cdots + (-1)^n I_n \mathbf{1} = \mathbf{0} \tag{1.317}$$

By means of the relationship (1.317), we can obtain the inverse of the matrix $\mathcal{A}_{n\times n}$ by multiplying all the terms by the inverse, $\mathcal{A}^{-1}$, *i.e.*:

$$\begin{aligned}&\mathcal{A}^n\mathcal{A}^{-1} - I_1\mathcal{A}^{n-1}\mathcal{A}^{-1} + I_2\mathcal{A}^{n-2}\mathcal{A}^{-1} - \cdots + (-1)^n I_n \mathbf{1}\mathcal{A}^{-1} = \mathbf{0}\\ &\Rightarrow \mathcal{A}^{n-1} - I_1\mathcal{A}^{n-2} + I_2\mathcal{A}^{n-3} - \cdots (-1)^{n-1} I_{n-1}\mathbf{1} + (-1)^n I_n\mathcal{A}^{-1} = \mathbf{0}\end{aligned} \tag{1.318}$$

then

$$\mathcal{A}^{-1} = \frac{(-1)^{n-1}}{I_n}\left(\mathcal{A}^{n-1} - I_1\mathcal{A}^{n-2} + I_2\mathcal{A}^{n-3} - \cdots (-1)^{n-1} I_{n-1}\mathbf{1}\right) \tag{1.319}$$

I_n is the determinant of $\mathcal{A}_{n\times n}$. Then, the inverse exists if $I_n = \det(\mathcal{A}) \neq 0$.

Problem 1.36: Check the Cayley-Hamilton theorem by using a second-order tensor whose Cartesian components are given by:

$$\mathcal{T} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:
The Cayley-Hamilton theorem states that:

$$\mathcal{T}^3 - \mathcal{T}^2 I_{\mathbf{T}} + \mathcal{T} II_{\mathbf{T}} - III_{\mathbf{T}}\mathbf{1} = \mathbf{0}$$

where $I_{\mathbf{T}} = 5 + 2 + 1 = 8$, $II_{\mathbf{T}} = 10 + 2 + 5 = 17$, $III_{\mathbf{T}} = 10$, and

$$\mathcal{T}^3 = \begin{bmatrix} 5^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 125 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad \mathcal{T}^2 = \begin{bmatrix} 5^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By applying the Cayley-Hamilton theorem, we can verify that it is true:

$$\begin{bmatrix} 125 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 8\begin{bmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 17\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1.5.7 Norms of Tensors

The magnitude (module) of a tensor, also known as the *Frobenius norm*, is given below:

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_i v_i} \quad \text{(vector)} \tag{1.320}$$

$$\|\mathbf{T}\| = \sqrt{\mathbf{T}:\mathbf{T}} = \sqrt{\mathsf{T}_{ij}\mathsf{T}_{ij}} \quad \text{(second-order tensor)} \tag{1.321}$$

$$\|\mathbf{A}\| = \sqrt{\mathbf{A}:\cdot\mathbf{A}} = \sqrt{\mathsf{A}_{ijk}\mathsf{A}_{ijk}} \quad \text{(third-order tensor)} \tag{1.322}$$

$$\|\mathbb{C}\| = \sqrt{\mathbb{C}::\mathbb{C}} = \sqrt{\mathbb{C}_{ijkl}\mathbb{C}_{ijkl}} \quad \text{(fourth-order tensor)} \tag{1.323}$$

Interpreting the Frobenius norm of $\mathbf{T}$ is done by considering the principal space of $\mathbf{T}$ where $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3$ are the eigenvalues of $\mathbf{T}$. In this space, it follows that:

$$\|\mathbf{T}\| = \sqrt{\mathbf{T}:\mathbf{T}} = \sqrt{\mathsf{T}_{ij}\mathsf{T}_{ij}} = \sqrt{\mathsf{T}_1^2 + \mathsf{T}_2^2 + \mathsf{T}_3^2} = \sqrt{I_{\mathbf{T}}^2 - 2\,II_{\mathbf{T}}} \tag{1.324}$$

As we can verify $\|\mathbf{T}\|$ is an invariant, and $\|\mathbf{T}\|$ represents a measurement of distance as shown in Figure 1.28.

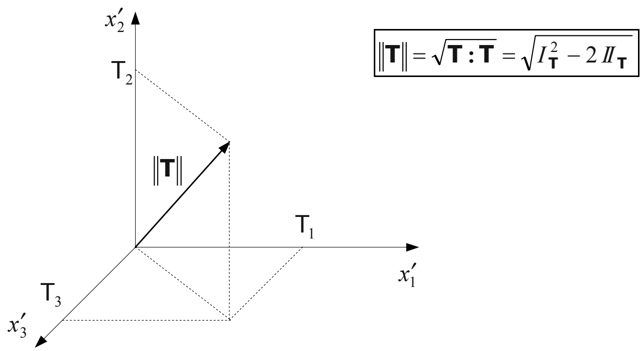


Figure 1.28: Norm of a second-order tensor.

1.5.8 Isotropic and Anisotropic Tensor

A tensor is called *isotropic* when its components are the same in any coordinate system, otherwise the tensor is said to be *anisotropic*.

Let $\mathcal{T}$ and $\mathcal{T}'$ represent the tensor components $\mathbf{T}$ in the systems $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_i'$, respectively, so, the tensor is isotropic if $\mathcal{T} = \mathcal{T}'$ on any arbitrary basis.

Isotropic first-order tensor

Let $\vec{\mathbf{v}}$ be a vector that is represented by its components, $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$, in the coordinate system x_1, x_2, x_3. The representation of these components in a new coordinate system, x_1', x_2', x_3', are given by $\mathrm{v}_1', \mathrm{v}_2', \mathrm{v}_3'$, so the transformation law for these components is:

$$\vec{\mathbf{v}} = \mathrm{v}_i \hat{\mathbf{e}}_i = \mathrm{v}_j' \hat{\mathbf{e}}_j' \quad \Rightarrow \quad \mathrm{v}_i' = a_{ij} \mathrm{v}_j \tag{1.325}$$

By definition, $\vec{\mathbf{v}}$ is an isotropic tensor if it holds that $\mathrm{v}_i = \mathrm{v}_i'$, and this is only possible if $\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_j'$, *i.e.* there is no change of system, or if the tensor is the zero vector, *i.e.* $\mathrm{v}_i = \mathrm{v}_i' = 0_i$. Then, the unique isotropic first-order tensor is the zero vector $\vec{\mathbf{0}}$.

Isotropic second-order tensor

An example of a second-order isotropic tensor is the unit tensor, $\mathbf{1}$, whose components are represented by δ_{kl} (Kronecker delta). In the demonstration, we use the transformation law for a second-order tensor components, obtained in (1.248), thus:

$$\delta_{ij}' = a_{ik} a_{jl} \delta_{kl} = \underbrace{a_{ik} a_{jk}}_{\mathcal{A}\mathcal{A}^T = \mathbf{1}} = \delta_{ij} \tag{1.326}$$

An immediate observation of the isotropy of unit tensor $\mathbf{1}$ is that any spherical tensor ($\alpha\mathbf{1}$) is also an isotropic tensor. So, if a second-order tensor is isotropic it is spherical and vice versa.

Isotropic third-order tensor

An example of a third-order isotropic tensor is the Levi-Civita pseudo-tensor, defined in (1.182), which is not a "real" tensor in the strict meaning of the word. With reference to the transformation law for the third-order tensor components, (see equation (1.248)), we can conclude that:

$$\epsilon'_{ijk}=a_{il}a_{jm}a_{kn}\epsilon_{lmn}=\underbrace{|\mathcal{A}|}_{1}\epsilon_{ijk}=\epsilon_{ijk}\quad\text{(see **Problem 1.21**)}\tag{1.327}$$

Isotropic fourth-order tensor

With reference to the transformation law for fourth-order tensor components, (see equation (1.249)), it is possible to demonstrate that the following tensors are isotropic:

$$\bar{\bar{\mathbb{I}}}_{ijkl}=\delta_{ij}\delta_{kl}\quad;\quad\mathbb{I}_{ijkl}=\delta_{ik}\delta_{jl}\quad;\quad\bar{\mathbb{I}}_{ijkl}=\delta_{il}\delta_{jk}\tag{1.328}$$

Therefore, any fourth-order isotropic tensor can be represented by a linear combination of the three tensors given in (1.328), *e.g.*:

$$\begin{aligned}\mathbb{D}&=a_0\bar{\bar{\mathbb{I}}}+a_1\mathbb{I}+a_2\bar{\mathbb{I}}\\ \mathbb{D}&=a_0\mathbf{1}\otimes\mathbf{1}+a_1\mathbf{1}\overline{\otimes}\mathbf{1}+a_2\mathbf{1}\underline{\otimes}\mathbf{1}\\ \mathbb{D}_{ijkl}&=a_0\delta_{ij}\delta_{kl}+a_1\delta_{ik}\delta_{jl}+a_2\delta_{il}\delta_{jk}\end{aligned}\tag{1.329}$$

Problem 1.37: Let $\mathbb{C}$ be a fourth-order tensor, whose components are given by:

$$\mathbb{C}_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

where λ, μ are constant real numbers. Show that $\mathbb{C}$ is an isotropic tensor.

Solution:

Applying the transformation law for fourth-order tensor components:

$$\mathbb{C}'_{ijkl}=a_{im}a_{jn}a_{kp}a_{lq}\mathbb{C}_{mnpq}$$

and by replacing the relation $\mathbb{C}_{mnpq}=\lambda\delta_{mn}\delta_{pq}+\mu(\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})$ in the above equation, we obtain:

$$\begin{aligned}\mathbb{C}'_{ijkl}&=a_{im}a_{jn}a_{kp}a_{lq}[\lambda\delta_{mn}\delta_{pq}+\mu(\delta_{mp}\delta_{nq}+\delta_{mq}\delta_{np})]\\&=\lambda a_{im}a_{jn}a_{kp}a_{lq}\delta_{mn}\delta_{pq}+\mu(a_{im}a_{jn}a_{kp}a_{lq}\delta_{mp}\delta_{nq}+a_{im}a_{jn}a_{kp}a_{lq}\delta_{mq}\delta_{np})\\&=\lambda a_{in}a_{jn}a_{kq}a_{lq}+\mu(a_{ip}a_{jq}a_{kp}a_{lq}+a_{iq}a_{jn}a_{kn}a_{lq})\\&=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\\&=\mathbb{C}_{ijkl}\end{aligned}$$

which is proof that $\mathbb{C}$ is an isotropic tensor.

1.5.9 Coaxial Tensors

Two arbitrary second-order tensors, $\mathbf{T}$ and $\mathbf{S}$, are coaxial tensors if they have the same eigenvectors. It is easy to show that if two tensors are coaxial, this means the dot product between them is commutative, and vice versa, *i.e.*:

$$\text{if}\quad\mathbf{T}\cdot\mathbf{S}=\mathbf{S}\cdot\mathbf{T}\quad\Leftrightarrow\quad\mathbf{S},\mathbf{T}\text{ are coaxial}\tag{1.330}$$

If $\mathbf{T}$ and $\mathbf{S}$ are coaxial as well as symmetric tensors, the spectral representations of these tensors are given by:

$$\mathbf{T}=\sum_{a=1}^{3}\mathsf{T}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\quad;\quad\mathbf{S}=\sum_{a=1}^{3}\mathsf{S}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\tag{1.331}$$

An immediate result of (1.330) is that the tensor $\mathbf{S}$ and its inverse $\mathbf{S}^{-1}$ are coaxial tensors:

$$\mathbf{S}^{-1}\cdot\mathbf{S}=\mathbf{S}\cdot\mathbf{S}^{-1}=\mathbf{1}$$

$$\mathbf{S}=\sum_{a=1}^{3}\mathsf{S}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\qquad;\qquad \mathbf{S}^{-1}=\sum_{a=1}^{3}\frac{1}{\mathsf{S}_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)} \tag{1.332}$$

where S_a, $\dfrac{1}{\mathsf{S}_a}$, are the eigenvalues of $\mathbf{S}$ and $\mathbf{S}^{-1}$, respectively.

If $\mathbf{S}$ and $\mathbf{T}$ are coaxial symmetric tensors, the resulting tensor $(\mathbf{S}\cdot\mathbf{T})$ becomes another symmetric tensor. To prove this we start from the definition of coaxial tensors:

$$\mathbf{T}\cdot\mathbf{S}=\mathbf{S}\cdot\mathbf{T}\quad\Rightarrow\quad \mathbf{T}\cdot\mathbf{S}-\mathbf{S}\cdot\mathbf{T}=\mathbf{0}\quad\Rightarrow\quad \mathbf{T}\cdot\mathbf{S}-(\mathbf{T}\cdot\mathbf{S})^T=\mathbf{0}\Rightarrow 2(\mathbf{T}\cdot\mathbf{S})^{skew}=\mathbf{0} \tag{1.333}$$

Then, if the antisymmetric part of a tensor is a zero tensor, it follows that this tensor is symmetric:

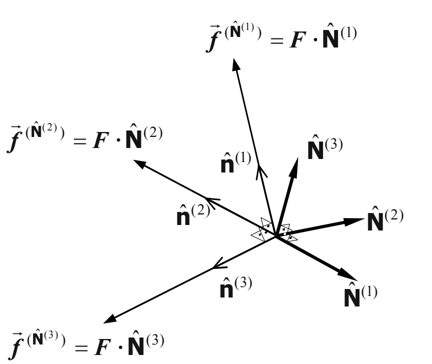
$$(\mathbf{T}\cdot\mathbf{S})^{skew}=\mathbf{0}\quad\Rightarrow\quad(\mathbf{T}\cdot\mathbf{S})\equiv(\mathbf{T}\cdot\mathbf{S})^{sym} \tag{1.334}$$

1.5.10 Polar Decomposition

Let $\boldsymbol{F}$ be an arbitrary nonsingular second-order tensor, *i.e.* $(\det(\boldsymbol{F})\neq 0\Rightarrow\exists\boldsymbol{F}^{-1})$. Additionally, as previously seen, it satisfies the condition $\boldsymbol{F}\cdot\hat{\mathbf{N}}=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}})}\right\|\hat{\mathbf{n}}=\lambda_{(\hat{\mathbf{n}})}\hat{\mathbf{n}}\neq\vec{\mathbf{0}}$, since $\det(\boldsymbol{F})\neq 0$. After that, given an orthonormal basis $\hat{\mathbf{N}}^{(a)}$, we can obtain:

$$\begin{aligned}
&\boldsymbol{F}^{-1}\cdot\boldsymbol{F}=\mathbf{1}=\sum_{a=1}^{3}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\\
&\Rightarrow\boldsymbol{F}=\boldsymbol{F}\cdot\mathbf{1}=\boldsymbol{F}\cdot\sum_{a=1}^{3}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}=\sum_{a=1}^{3}\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\\
&\Rightarrow\boldsymbol{F}=\sum_{a=1}^{3}\lambda_a\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}
\end{aligned}\tag{1.335}$$

NOTE: The representation of $\boldsymbol{F}$, given in (1.335), is not the spectral representation of $\boldsymbol{F}$ in the strict sense of the word, *i.e.*, λ_a are not eigenvalues of $\boldsymbol{F}$, and neither $\hat{\mathbf{n}}^{(a)}$ nor $\hat{\mathbf{N}}^{(a)}$ are eigenvectors of $\boldsymbol{F}$. ■



$$\boxed{\begin{aligned}
\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(1)}&=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(1)})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(1)})}\right\|\hat{\mathbf{n}}^{(1)}=\lambda_1\hat{\mathbf{n}}^{(1)}\\
\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(2)}&=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(2)})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(2)})}\right\|\hat{\mathbf{n}}^{(2)}=\lambda_2\hat{\mathbf{n}}^{(2)}\\
\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(3)}&=\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(3)})}=\left\|\vec{\boldsymbol{f}}^{(\hat{\mathbf{N}}^{(3)})}\right\|\hat{\mathbf{n}}^{(3)}=\lambda_3\hat{\mathbf{n}}^{(3)}
\end{aligned}}$$

Figure 1.29: Projecting $\boldsymbol{F}$ onto $\hat{\mathbf{N}}^{(a)}$.

Note that for the arbitrary orthonormal basis $\hat{\mathbf{N}}^{(a)}$, the new basis $\hat{\mathbf{n}}^{(a)}$ will not necessarily be orthonormal. We seek to find a basis $\hat{\mathbf{N}}^{(a)}$ so that the new basis $\hat{\mathbf{n}}^{(a)}$ is orthonormal, (see Figure 1.29), *i.e.* $\vec{f}^{(\hat{\mathbf{N}}^{(1)})} \cdot \vec{f}^{(\hat{\mathbf{N}}^{(2)})} = 0$, $\vec{f}^{(\hat{\mathbf{N}}^{(2)})} \cdot \vec{f}^{(\hat{\mathbf{N}}^{(3)})} = 0$, $\vec{f}^{(\hat{\mathbf{N}}^{(3)})} \cdot \vec{f}^{(\hat{\mathbf{N}}^{(1)})} = 0$. Then we look for a space in accordance with the following orthogonal transformation $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)}$, which ensures $\hat{\mathbf{n}}^{(a)}$ orthonormality since an orthogonal transformation changes neither angles between vectors nor their magnitudes.

Now, consider that there is a transformation from $\hat{\mathbf{N}}^{(a)}$ to $\hat{\mathbf{n}}^{(a)}$, which is given by the following orthogonal transformation $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)}$, then we can state that:

$$\begin{gathered} \boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \lambda_a \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \mathbf{R} \cdot \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \mathbf{R} \cdot \mathbf{U} \\ \boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} \quad \Rightarrow \quad \mathbf{U} = \mathbf{R}^T \cdot \boldsymbol{F} \end{gathered} \tag{1.336}$$

where we have defined the tensor $\mathbf{U} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$. Note that $\mathbf{U}$ is a symmetric tensor, *i.e.* $\mathbf{U} = \mathbf{U}^T$. This condition is easily verified by the fact that $\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$ is also symmetric. Now considering that $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \Rightarrow \hat{\mathbf{N}}^{(a)} = \mathbf{R}^T \cdot \hat{\mathbf{n}}^{(a)} = \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R}$, we obtain:

$$\begin{gathered} \boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R} = \mathbf{V} \cdot \mathbf{R} \\ \boldsymbol{F} = \mathbf{V} \cdot \mathbf{R} \quad \Rightarrow \quad \mathbf{V} = \boldsymbol{F} \cdot \mathbf{R}^T \end{gathered} \tag{1.337}$$

where we have defined the symmetric second-order tensor $\mathbf{V} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$. By comparing the spectral representation of $\mathbf{U}$ with $\mathbf{V}$, we can conclude that they have the same eigenvalues but different eigenvectors, and they are related by $\hat{\mathbf{n}}^{(a)} = \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)}$.

With reference to the above considerations, we can define the polar decomposition:

$$\boxed{\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}} \quad \textit{Polar Decomposition} \tag{1.338}$$

Carrying out the dot product between $\boldsymbol{F}^T$ and $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U}$, we obtain:

$$\underbrace{\boldsymbol{F}^T \cdot \boldsymbol{F}}_{\boldsymbol{C}} = \boldsymbol{F}^T \cdot \mathbf{R} \cdot \mathbf{U} = (\mathbf{R}^T \cdot \boldsymbol{F})^T \cdot \mathbf{U} = \mathbf{U}^T \cdot \mathbf{U} = \mathbf{U}^2 \quad \Rightarrow \quad \mathbf{U} = \pm\sqrt{\boldsymbol{F}^T \cdot \boldsymbol{F}} = \pm\sqrt{\boldsymbol{C}} \tag{1.339}$$

Moreover, by carrying out the dot product between $\boldsymbol{F} = \mathbf{V} \cdot \mathbf{R}$ and $\boldsymbol{F}^T$, we obtain:

$$\underbrace{\boldsymbol{F} \cdot \boldsymbol{F}^T}_{\boldsymbol{b}} = \mathbf{V} \cdot \mathbf{R} \cdot \boldsymbol{F}^T = \mathbf{V} \cdot (\boldsymbol{F} \cdot \mathbf{R}^T)^T = \mathbf{V} \cdot \mathbf{V}^T = \mathbf{V}^2 \quad \Rightarrow \quad \mathbf{V} = \pm\sqrt{\boldsymbol{F} \cdot \boldsymbol{F}^T} = \pm\sqrt{\boldsymbol{b}} \tag{1.340}$$

Since $\det(\boldsymbol{F}) \neq 0$, the tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ are positive definite symmetric tensors, (see **Problem 1.25**), which implies that the eigenvalues of $\boldsymbol{C}$ and $\boldsymbol{b}$ are all real and positive. However, up to now, $\det(\boldsymbol{F}) \neq 0$ is the only restriction imposed on the tensor $\boldsymbol{F}$. Therefore, we have the following possibilities:

- If $\det(\boldsymbol{F}) > 0$

In this scenario, we have $\det(\boldsymbol{F}) = \det(\mathbf{R})\det(\mathbf{U}) = \det(\mathbf{V})\det(\mathbf{R}) > 0$, which results in the following cases:

$$\begin{cases}\mathbf{R} - \text{Proper ortogonal tensor} \\ \mathbf{U}, \mathbf{V} - \text{Positive definite tensors}\end{cases} \text{or} \begin{cases}\mathbf{R} - \text{Improper orthogonal tensor} \\ \mathbf{U}, \mathbf{V} - \text{Negative definite tensors}\end{cases}$$

- If $\det(\boldsymbol{F}) < 0$

In this situation, we have $\det(\boldsymbol{F}) = \det(\mathbf{R})\det(\mathbf{U}) = \det(\mathbf{V})\det(\mathbf{R}) < 0$, which give us the following cases:

$$\begin{cases}\mathbf{R} - \text{Proper orthogonal tensor} \\ \mathbf{U}, \mathbf{V} - \text{Negative definite tensors}\end{cases} \text{or} \begin{cases}\mathbf{R} - \text{Improper orthogonal tensor} \\ \mathbf{U}, \mathbf{V} - \text{Positive definite tensors}\end{cases}$$

NOTE: In Chapter 2 we will work with some special tensors where $\boldsymbol{F}$ is a nonsingular tensor, $\det(\boldsymbol{F}) \neq 0$, and $\det(\boldsymbol{F}) > 0$. $\mathbf{U}$ and $\mathbf{V}$ are positive definite tensors and $\mathbf{R}$ is a rotation tensor, *i.e.* a proper orthogonal tensor. ■

1.5.11 Partial Derivative with Tensors

The first derivative of a tensor with respect to itself is defined as:

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}} \equiv \mathbf{A}_{,\mathbf{A}} = \frac{\partial \mathsf{A}_{ij}}{\partial \mathsf{A}_{kl}}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) = \delta_{ik}\delta_{jl}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l) = \mathbb{I} \tag{1.341}$$

The derivative of a tensor trace with respect to a tensor:

$$\frac{\partial[\mathrm{Tr}(\mathbf{A})]}{\partial \mathbf{A}} \equiv [\mathrm{Tr}(\mathbf{A})]_{,\mathbf{A}} = \frac{\partial \mathsf{A}_{kk}}{\partial \mathsf{A}_{ij}}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \delta_{ki}\delta_{kj}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \delta_{ij}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \mathbf{1} \tag{1.342}$$

The derivative of the tensor trace squared with respect to the tensor is given by:

$$\frac{\partial[\mathrm{Tr}(\mathbf{A})]^2}{\partial \mathbf{A}} = 2\mathrm{Tr}(\mathbf{A})\frac{\partial[\mathrm{Tr}(\mathbf{A})]}{\partial \mathbf{A}} = 2\mathrm{Tr}(\mathbf{A})\mathbf{1} \tag{1.343}$$

And, the derivative of the trace of the tensor squared with respect to tensor is given by:

$$\begin{aligned}\frac{\partial[\mathrm{Tr}(\mathbf{A}^2)]}{\partial \mathbf{A}} &= \frac{\partial(\mathsf{A}_{sr}\mathsf{A}_{rs})}{\partial \mathsf{A}_{ij}}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = \left[\mathsf{A}_{rs}\frac{\partial(\mathsf{A}_{sr})}{\partial \mathsf{A}_{ij}} + \mathsf{A}_{sr}\frac{\partial(\mathsf{A}_{rs})}{\partial \mathsf{A}_{ij}}\right](\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \\ &= [\mathsf{A}_{rs}\delta_{si}\delta_{rj} + \mathsf{A}_{sr}\delta_{ri}\delta_{sj}](\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = [\mathsf{A}_{ji} + \mathsf{A}_{ji}](\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) \\ &= 2\mathsf{A}_{ji}(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j) = 2\mathbf{A}^T\end{aligned} \tag{1.344}$$

We leave the reader with the following demonstration:

$$\frac{\partial[\mathrm{Tr}(\mathbf{A}^3)]}{\partial \mathbf{A}} = 3(\mathbf{A}^2)^T \tag{1.345}$$

Then, if we are considering a symmetric second-order tensor, $\mathbf{C}$, it is true that $\frac{\partial[\mathrm{Tr}(\mathbf{C})]}{\partial \mathbf{C}} = \mathbf{1}$, $\frac{\partial[\mathrm{Tr}(\mathbf{C})]^2}{\partial \mathbf{C}} = 2\mathrm{Tr}(\mathbf{C})\mathbf{1}$, $\frac{\partial[\mathrm{Tr}(\mathbf{C}^2)]}{\partial \mathbf{C}} = 2\mathbf{C}^T = 2\mathbf{C}$, $\frac{\partial[\mathrm{Tr}(\mathbf{C}^3)]}{\partial \mathbf{C}} = 3(\mathbf{C}^2)^T = 3\mathbf{C}^2$.

Moreover, we can say that the derivative of the Frobenius norm of $\mathbf{C}$ is given by:

$$\frac{\partial\|\mathbf{C}\|}{\partial\mathbf{C}}=\frac{\partial\left(\sqrt{\mathbf{C}:\mathbf{C}}\right)}{\partial\mathbf{C}}=\frac{\partial\left(\sqrt{\mathrm{Tr}(\mathbf{C}\cdot\mathbf{C}^T)}\right)}{\partial\mathbf{C}}=\frac{\partial\left(\sqrt{\mathrm{Tr}(\mathbf{C}^2)}\right)}{\partial\mathbf{C}}=\frac{1}{2}\left[\mathrm{Tr}(\mathbf{C}^2)\right]^{\frac{-1}{2}}\left[\mathrm{Tr}(\mathbf{C}^2)\right]_{,\mathbf{C}}$$
$$=\frac{1}{2}\left[\mathrm{Tr}(\mathbf{C}^2)\right]^{\frac{-1}{2}}2\mathbf{C} \tag{1.346}$$

or:

$$\frac{\partial\|\mathbf{C}\|}{\partial\mathbf{C}}=\frac{\mathbf{C}}{\sqrt{\mathrm{Tr}(\mathbf{C}^2)}}=\frac{\mathbf{C}}{\|\mathbf{C}\|} \tag{1.347}$$

Another interesting derivative is presented below:

$$\frac{\partial\left(\mathrm{n}_i\mathrm{C}_{ij}\mathrm{n}_j\right)}{\partial\mathrm{n}_k}=\frac{\partial\mathrm{n}_i}{\partial\mathrm{n}_k}\mathrm{C}_{ij}\mathrm{n}_j+\mathrm{n}_i\mathrm{C}_{ij}\frac{\partial\mathrm{n}_j}{\partial\mathrm{n}_k}=\delta_{ik}\mathrm{C}_{ij}\mathrm{n}_j+\mathrm{n}_i\mathrm{C}_{ij}\delta_{jk}=\mathrm{C}_{kj}\mathrm{n}_j+\mathrm{n}_i\mathrm{C}_{ik}$$
$$=\mathrm{C}_{kj}\mathrm{n}_j+\mathrm{C}_{jk}\mathrm{n}_j=(\mathrm{C}_{kj}+\mathrm{C}_{jk})\mathrm{n}_j=2\mathrm{C}_{kj}^{sym}\mathrm{n}_j=2\mathrm{C}_{kj}\mathrm{n}_j \tag{1.348}$$

where we have assumed that $\mathbf{C}$ is symmetric, *i.e.*, $\mathrm{C}_{kj}=\mathrm{C}_{jk}$.

Let $\mathbf{C}$ be a symmetric second-order tensor. The partial derivative of $\mathbf{C}^{-1}$ with respect to the tensor $\mathbf{C}$ is obtained by using the following relationship:

$$\frac{\partial\mathbf{1}}{\partial\mathbf{C}}=\frac{\partial\left(\mathbf{C}^{-1}\cdot\mathbf{C}\right)}{\partial\mathbf{C}}=\mathbb{O} \tag{1.349}$$

where $\mathbb{O}$ is the fourth-order zero tensor and the above equation in indicial notation becomes:

$$\frac{\partial\left(\mathrm{C}_{iq}^{-1}\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}=\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{qj}+\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}=\mathbb{O}_{ikjl}$$
$$\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{qj}=-\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}\quad\Rightarrow\quad\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{qj}\mathrm{C}_{jr}^{-1}=-\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{jr}^{-1}$$
$$\Rightarrow\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\delta_{qr}=-\mathrm{C}_{iq}^{-1}\frac{\partial\left(\mathrm{C}_{qj}\right)}{\partial\mathrm{C}_{kl}}\mathrm{C}_{jr}^{-1} \tag{1.350}$$

whereas $\mathrm{C}_{qj}=\frac{1}{2}\left(\mathrm{C}_{qj}+\mathrm{C}_{jq}\right)$, so we can conclude that:

$$\frac{\partial\left(\mathrm{C}_{iq}^{-1}\right)}{\partial\mathrm{C}_{kl}}\delta_{qr}=-\mathrm{C}_{iq}^{-1}\frac{1}{2}\left[\frac{\partial\left(\mathrm{C}_{qj}+\mathrm{C}_{jq}\right)}{\partial\mathrm{C}_{kl}}\right]\mathrm{C}_{jr}^{-1}$$
$$\frac{\partial\left(\mathrm{C}_{ir}^{-1}\right)}{\partial\mathrm{C}_{kl}}=-\mathrm{C}_{iq}^{-1}\frac{1}{2}\left[\delta_{qk}\delta_{jl}+\delta_{jk}\delta_{ql}\right]\mathrm{C}_{jr}^{-1}=-\frac{1}{2}\left[\mathrm{C}_{iq}^{-1}\delta_{qk}\delta_{jl}\mathrm{C}_{jr}^{-1}+\mathrm{C}_{iq}^{-1}\delta_{jk}\delta_{ql}\mathrm{C}_{jr}^{-1}\right]$$
$$\frac{\partial\left(\mathrm{C}_{ir}^{-1}\right)}{\partial\mathrm{C}_{kl}}=-\frac{1}{2}\left[\mathrm{C}_{ik}^{-1}\mathrm{C}_{lr}^{-1}+\mathrm{C}_{il}^{-1}\mathrm{C}_{kr}^{-1}\right] \tag{1.351}$$

Or in tensorial notation:

$$\frac{\partial\mathbf{C}^{-1}}{\partial\mathbf{C}}=-\frac{1}{2}\left[\mathbf{C}^{-1}\overline{\otimes}\mathbf{C}^{-1}+\mathbf{C}^{-1}\underline{\otimes}\mathbf{C}^{-1}\right] \tag{1.352}$$

NOTE: Note that, if we had not replaced the symmetric part of C_{qj} in (1.351), we would have found that $\dfrac{\partial(\mathsf{C}^{-1}_{iq})}{\partial \mathsf{C}_{kl}}\delta_{qr} = -\mathsf{C}^{-1}_{iq}\dfrac{\partial(\mathsf{C}_{qj})}{\partial \mathsf{C}_{kl}}\mathsf{C}^{-1}_{jr} = -\mathsf{C}^{-1}_{iq}\delta_{qk}\delta_{jl}\mathsf{C}^{-1}_{jr} = -\mathsf{C}^{-1}_{ik}\mathsf{C}^{-1}_{lr}$, which is a non-symmetric tensor. ■

1.5.11.1 Partial Derivative of Invariants

Let $\mathbf{T}$ be a second-order tensor. The partial derivative of $I_{\mathbf{T}}$ with respect to $\mathbf{T}$, (see equation (1.342)), is:

$$\frac{\partial[I_{\mathbf{T}}]}{\partial \mathbf{T}} = \frac{\partial[\mathrm{Tr}(\mathbf{T})]}{\partial \mathbf{T}} = [\mathrm{Tr}(\mathbf{T})]_{,\mathbf{T}} = \mathbf{1} \tag{1.353}$$

The partial derivative of $II_{\mathbf{T}}$ with respect to $\mathbf{T}$, (see equation (1.342)), is:

$$\begin{aligned}\frac{\partial[II_{\mathbf{T}}]}{\partial \mathbf{T}} = \frac{\partial}{\partial \mathbf{T}}\left\{\frac{1}{2}\left[[\mathrm{Tr}(\mathbf{T})]^2 - \mathrm{Tr}(\mathbf{T}^2)\right]\right\} &= \frac{1}{2}\left[\frac{\partial[\mathrm{Tr}(\mathbf{T})]^2}{\partial \mathbf{T}} - \frac{\partial[\mathrm{Tr}(\mathbf{T}^2)]}{\partial \mathbf{T}}\right] \\ &= \frac{1}{2}\left[2(\mathrm{Tr}\mathbf{T})\mathbf{1} - 2\mathbf{T}^T\right] \\ &= \mathrm{Tr}(\mathbf{T})\mathbf{1} - \mathbf{T}^T\end{aligned} \tag{1.354}$$

Next, we apply the Cayley-Hamilton theorem so as to represent $\mathbf{T}$ as:

$$\begin{aligned}&\mathbf{T}^3 : \mathbf{T}^{-2} - I_{\mathbf{T}}\mathbf{T}^2 : \mathbf{T}^{-2} + II_{\mathbf{T}}\mathbf{T} : \mathbf{T}^{-2} - III_{\mathbf{T}}\mathbf{1} : \mathbf{T}^{-2} = \mathbf{0} \\ &\mathbf{T} - I_{\mathbf{T}}\mathbf{1} + II_{\mathbf{T}}\mathbf{T}^{-1} - III_{\mathbf{T}}\mathbf{T}^{-2} = \mathbf{0} \\ &\Rightarrow \mathbf{T} = I_{\mathbf{T}}\mathbf{1} - II_{\mathbf{T}}\mathbf{T}^{-1} + III_{\mathbf{T}}\mathbf{T}^{-2}\end{aligned} \tag{1.355}$$

By substituting (1.355) into the equation in (1.354), we obtain:

$$\frac{\partial[II_{\mathbf{T}}]}{\partial \mathbf{T}} = \mathrm{Tr}(\mathbf{T})\mathbf{1} - \mathbf{T}^T = \mathrm{Tr}(\mathbf{T})\mathbf{1} - \left(I_{\mathbf{T}}\mathbf{1} - II_{\mathbf{T}}\mathbf{T}^{-1} + III_{\mathbf{T}}\mathbf{T}^{-2}\right)^T = \left(II_{\mathbf{T}}\mathbf{T}^{-1} - III_{\mathbf{T}}\mathbf{T}^{-2}\right)^T \tag{1.356}$$

To find the partial derivative of the third invariant, we can start with the definition given in (1.313), so:

$$\begin{aligned}\frac{\partial[III_{\mathbf{T}}]}{\partial \mathbf{T}} &= \frac{\partial}{\partial \mathbf{T}}\left\{\frac{1}{3}\mathrm{Tr}(\mathbf{T}^3) - \frac{1}{2}\mathrm{Tr}(\mathbf{T}^2)\mathrm{Tr}(\mathbf{T}) + \frac{1}{6}[\mathrm{Tr}(\mathbf{T})]^3\right\} \\ &= \frac{1}{3}3(\mathbf{T}^2)^T - \frac{1}{2}\frac{\partial[\mathrm{Tr}(\mathbf{T}^2)]}{\partial \mathbf{T}}\mathrm{Tr}(\mathbf{T}) - \frac{1}{2}\mathrm{Tr}(\mathbf{T}^2)\frac{\partial[\mathrm{Tr}(\mathbf{T})]}{\partial \mathbf{T}} + \frac{3}{6}[\mathrm{Tr}(\mathbf{T})]^2\mathbf{1} \\ &= (\mathbf{T}^2)^T - \mathrm{Tr}(\mathbf{T})\mathbf{T}^T - \frac{1}{2}\mathrm{Tr}(\mathbf{T}^2)\mathbf{1} + \frac{1}{2}[\mathrm{Tr}(\mathbf{T})]^2\mathbf{1} \\ &= (\mathbf{T}^2)^T - \mathrm{Tr}(\mathbf{T})\mathbf{T}^T + \frac{1}{2}\left[[\mathrm{Tr}(\mathbf{T})]^2 - \mathrm{Tr}(\mathbf{T}^2)\right]\mathbf{1} \\ &= (\mathbf{T}^2)^T - I_{\mathbf{T}}\mathbf{T}^T + II_{\mathbf{T}}\mathbf{1}\end{aligned} \tag{1.357}$$

Once again using the Cayley-Hamilton theorem we obtain:

$$\begin{aligned}&\mathbf{T}^3 \cdot \mathbf{T}^{-1} - I_{\mathbf{T}}\mathbf{T}^2 \cdot \mathbf{T}^{-1} + II_{\mathbf{T}}\mathbf{T} \cdot \mathbf{T}^{-1} - III_{\mathbf{T}}\mathbf{1} \cdot \mathbf{T}^{-1} = \mathbf{0} \\ &\mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + II_{\mathbf{T}}\mathbf{1} - III_{\mathbf{T}}\mathbf{T}^{-1} = \mathbf{0} \\ &\Rightarrow III_{\mathbf{T}}\mathbf{T}^{-1} = \mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + II_{\mathbf{T}}\mathbf{1}\end{aligned} \tag{1.358}$$

and the transpose:

$$\left(III_{\mathbf{T}}\mathbf{T}^{-1}\right)^T = \left(\mathbf{T}^2 - I_{\mathbf{T}}\mathbf{T} + II_{\mathbf{T}}\mathbf{1}\right)^T = \left(\mathbf{T}^2\right)^T - I_{\mathbf{T}}\mathbf{T}^T + II_{\mathbf{T}}\mathbf{1} \tag{1.359}$$

By comparing (1.357) with (1.359) we find another way to express the derivative of $III_{\mathbf{T}}$ with respect to $\mathbf{T}$, *i.e.*:

$$\frac{\partial[III_{\mathbf{T}}]}{\partial\mathbf{T}} = \left(III_{\mathbf{T}}\mathbf{T}^{-1}\right)^T = III_{\mathbf{T}}\mathbf{T}^{-T} \tag{1.360}$$

1.5.11.2 Time Derivative of Tensors

Let us assume a second-order tensor depends on the time, t, *i.e.* $\mathbf{T} = \mathbf{T}(t)$. Then, we define the first time derivative and the second time derivative of the tensor $\mathbf{T}$, respectively, as:

$$\frac{D}{Dt}\mathbf{T} = \dot{\mathbf{T}} \qquad ; \qquad \frac{D^2}{Dt^2}\mathbf{T} = \ddot{\mathbf{T}} \tag{1.361}$$

The time derivative of a tensor determinant is defined as:

$$\frac{D}{Dt}[\det(\mathbf{T})] = \frac{D\mathrm{T}_{ij}}{Dt}\operatorname{cof}\left(\mathrm{T}_{ij}\right) \tag{1.362}$$

where $\operatorname{cof}\left(\mathrm{T}_{ij}\right)$ is the cofactor of T_{ij} and defined as $\left[\operatorname{cof}\left(\mathrm{T}_{ij}\right)\right]^T = \det(\mathbf{T})\left(\mathbf{T}^{-1}\right)_{ij}$.

Problem 1.38: Consider that $J = [\det(\boldsymbol{b})]^{\frac{1}{2}} = (III_{\boldsymbol{b}})^{\frac{1}{2}}$, where $\boldsymbol{b}$ is a symmetric second-order tensor, *i.e.* $\boldsymbol{b} = \boldsymbol{b}^T$. Obtain the partial derivatives of J and $\ln(J)$ with respect to $\boldsymbol{b}$.

Solution:

$$\Rightarrow \quad \frac{\partial J}{\partial \boldsymbol{b}} = \frac{\partial\left[(III_{\boldsymbol{b}})^{\frac{1}{2}}\right]}{\partial \boldsymbol{b}}$$

$$= \frac{1}{2}(III_{\boldsymbol{b}})^{\frac{-1}{2}}\frac{\partial III_{\boldsymbol{b}}}{\partial \boldsymbol{b}} = \frac{1}{2}(III_{\boldsymbol{b}})^{\frac{-1}{2}}\,III_{\boldsymbol{b}}\boldsymbol{b}^{-T}$$

$$= \frac{1}{2}(III_{\boldsymbol{b}})^{\frac{1}{2}}\boldsymbol{b}^{-1} = \frac{1}{2}J\,\boldsymbol{b}^{-1}$$

$$\Rightarrow \quad \frac{\partial[\ln(J)]}{\partial \boldsymbol{b}} = \frac{\partial\left[\ln\left(III_{\boldsymbol{b}}^{\frac{1}{2}}\right)\right]}{\partial \boldsymbol{b}} = \frac{1}{2\,III_{\boldsymbol{b}}}\frac{\partial III_{\boldsymbol{b}}}{\partial \boldsymbol{b}} = \frac{1}{2}\boldsymbol{b}^{-1}$$

1.5.12 Spherical and Deviatoric Tensors

Any tensor can be decomposed into a spherical and a deviatoric part, so, for a given second-order tensor $\mathbf{T}$, this decomposition is represented by:

$$\mathbf{T} = \mathbf{T}^{sph} + \mathbf{T}^{dev} = \frac{\mathrm{Tr}(\mathbf{T})}{3}\mathbf{1} + \mathbf{T}^{dev} = \frac{I_{\mathbf{T}}}{3}\mathbf{1} + \mathbf{T}^{dev} = \mathrm{T}_m\mathbf{1} + \mathbf{T}^{dev} \tag{1.363}$$

The deviatoric part of the tensor $\mathbf{T}$ is defined as:

$$\mathbf{T}^{dev} = \mathbf{T} - \frac{\mathrm{Tr}(\mathbf{T})}{3}\mathbf{1} = \mathbf{T} - \mathrm{T}_m \mathbf{1} \tag{1.364}$$

For the following operations, we consider that $\mathbf{T}$ is a symmetric tensor, $\mathbf{T} = \mathbf{T}^T$, then under this condition the deviatoric tensor components, T_{ij}^{dev}, become:

$$\begin{aligned} \mathrm{T}_{ij}^{dev} &= \begin{bmatrix} \mathrm{T}_{11}^{dev} & \mathrm{T}_{12}^{dev} & \mathrm{T}_{13}^{dev} \\ \mathrm{T}_{12}^{dev} & \mathrm{T}_{22}^{dev} & \mathrm{T}_{23}^{dev} \\ \mathrm{T}_{13}^{dev} & \mathrm{T}_{23}^{dev} & \mathrm{T}_{33}^{dev} \end{bmatrix} = \begin{bmatrix} \mathrm{T}_{11} - \mathrm{T}_m & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{12} & \mathrm{T}_{22} - \mathrm{T}_m & \mathrm{T}_{23} \\ \mathrm{T}_{13} & \mathrm{T}_{23} & \mathrm{T}_{33} - \mathrm{T}_m \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3}(2\mathrm{T}_{11} - \mathrm{T}_{22} - \mathrm{T}_{33}) & \mathrm{T}_{12} & \mathrm{T}_{13} \\ \mathrm{T}_{12} & \frac{1}{3}(2\mathrm{T}_{22} - \mathrm{T}_{11} - \mathrm{T}_{33}) & \mathrm{T}_{23} \\ \mathrm{T}_{13} & \mathrm{T}_{23} & \frac{1}{3}(2\mathrm{T}_{33} - \mathrm{T}_{11} - \mathrm{T}_{22}) \end{bmatrix} \end{aligned} \tag{1.365}$$

Graphical representations of the Cartesian components of the spherical and deviatoric parts are shown in Figure 1.30.

In the following subsections we obtain the deviatoric tensor invariants in terms of the principal invariants of $\mathbf{T}$.

1.5.12.1 First Invariant of the Deviatoric Tensor

$$I_{\mathbf{T}^{dev}} = \mathrm{Tr}(\mathbf{T}^{dev}) = \mathrm{Tr}\left[\mathbf{T} - \frac{\mathrm{Tr}(\mathbf{T})}{3}\mathbf{1}\right] = \mathrm{Tr}(\mathbf{T}) - \frac{\mathrm{Tr}(\mathbf{T})}{3}\underbrace{\mathrm{Tr}(\mathbf{1})}_{\delta_{ii}=3} = 0 \tag{1.366}$$

Thus, we can conclude that the trace of any deviatoric tensor is equal to zero.

1.5.12.2 Second Invariant of the Deviatoric Tensor

For simplicity we can use the principal space to obtain the second and third invariant of the deviatoric tensor. In the principal space the components of $\mathbf{T}$ are given by:

$$\mathrm{T}_{ij} = \begin{bmatrix} \mathrm{T}_1 & 0 & 0 \\ 0 & \mathrm{T}_2 & 0 \\ 0 & 0 & \mathrm{T}_3 \end{bmatrix} \tag{1.367}$$

The principal invariants of $\mathbf{T}$: $I_{\mathbf{T}} = \mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3$, $I\!I_{\mathbf{T}} = \mathrm{T}_1\mathrm{T}_2 + \mathrm{T}_2\mathrm{T}_3 + \mathrm{T}_3\mathrm{T}_1$, $I\!I\!I_{\mathbf{T}} = \mathrm{T}_1\mathrm{T}_2\mathrm{T}_3$.

The deviatoric components, $\mathbf{T}^{dev} = \mathbf{T} - \mathrm{T}_m\mathbf{1}$, in the principal space are:

$$\mathrm{T}_{ij}^{dev} = \begin{bmatrix} \mathrm{T}_1 - \mathrm{T}_m & 0 & 0 \\ 0 & \mathrm{T}_2 - \mathrm{T}_m & 0 \\ 0 & 0 & \mathrm{T}_3 - \mathrm{T}_m \end{bmatrix} \tag{1.368}$$

So, the second invariant of deviatoric tensor $\mathbf{T}^{dev}$ is evaluated as follows:

$$\begin{aligned} I\!I_{\mathbf{T}^{dev}} &= (\mathrm{T}_1 - \mathrm{T}_m)(\mathrm{T}_2 - \mathrm{T}_m) + (\mathrm{T}_1 - \mathrm{T}_m)(\mathrm{T}_3 - \mathrm{T}_m) + (\mathrm{T}_2 - \mathrm{T}_m)(\mathrm{T}_3 - \mathrm{T}_m) \\ &= (\mathrm{T}_1\mathrm{T}_2 + \mathrm{T}_1\mathrm{T}_3 + \mathrm{T}_2\mathrm{T}_3) - 2\mathrm{T}_m(\mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3) + 3\mathrm{T}_m^2 \\ &= I\!I_{\mathbf{T}} - \frac{2I_{\mathbf{T}}}{3}(I_{\mathbf{T}}) + \frac{I_{\mathbf{T}}^2}{3} \\ &= \frac{1}{3}\left(3\,I\!I_{\mathbf{T}} - I_{\mathbf{T}}^2\right) \end{aligned} \tag{1.369}$$

We could also have obtained the above result, by directly starting from the definition of the second invariant of a tensor given in (1.269), *i.e.*:

$$
\begin{aligned}
I\!I_{\mathbf{T}^{dev}} &= \frac{1}{2}\left\{ (I_{\mathbf{T}^{dev}})^2 - \mathrm{Tr}\left[(\mathbf{T}^{dev})^2 \right] \right\} = -\frac{1}{2}\left\{ \mathrm{Tr}\left[(\mathbf{T}^{dev})^2 \right] \right\} \\
&= \frac{1}{2}\left\{ -\mathrm{Tr}\left[(\mathbf{T} - \mathrm{T}_m \mathbf{1})^2 \right] \right\} \\
&= \frac{1}{2}\left\{ -\mathrm{Tr}\left[(\mathbf{T}^2 - 2\mathrm{T}_m \mathbf{T}\cdot\mathbf{1} + \mathrm{T}_m^2 \mathbf{1}) \right] \right\} \\
&= \frac{1}{2}\left[-\mathrm{Tr}(\mathbf{T}^2) + 2\mathrm{T}_m \mathrm{Tr}(\mathbf{T}) - \mathrm{T}_m^2 \mathrm{Tr}(\mathbf{1}) \right] \\
&= \frac{1}{2}\left[-\mathrm{Tr}(\mathbf{T}^2) + 2\frac{I_{\mathbf{T}}}{3} I_{\mathbf{T}} - \frac{I_{\mathbf{T}}^2}{9} 3 \right] \\
&= \frac{1}{2}\left[-\mathrm{Tr}(\mathbf{T}^2) + \frac{I_{\mathbf{T}}^2}{3} \right]
\end{aligned}
\tag{1.370}
$$

Observing that $\mathrm{Tr}(\mathbf{T}^2) = \mathrm{T}_1^2 + \mathrm{T}_2^2 + \mathrm{T}_3^2 = I_{\mathbf{T}}^2 - 2 I\!I_{\mathbf{T}}$, (see **Problem 1.31**), the equation (1.370) becomes:

$$
I\!I_{\mathbf{T}^{dev}} = \frac{1}{2}\left[-I_{\mathbf{T}}^2 + 2 I\!I_{\mathbf{T}} + \frac{I_{\mathbf{T}}^2}{3} \right] = \frac{1}{2}\left[2 I\!I_{\mathbf{T}} - \frac{2 I_{\mathbf{T}}^2}{3} \right] = \frac{1}{3}\left(3 I\!I_{\mathbf{T}} - I_{\mathbf{T}}^2 \right) \tag{1.371}
$$

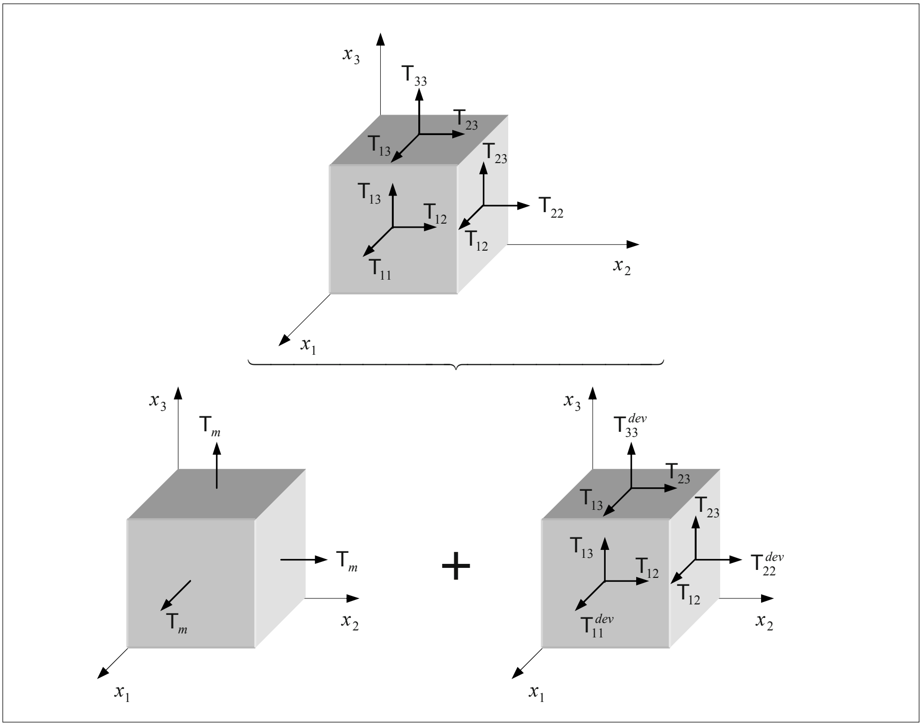


Figure 1.30: Spherical and deviatoric part.

Another equation for $\mathit{II}_{\mathbf{T}^{dev}}$ is presented in terms of deviatoric tensor components. To calculate this, we can apply the equation (1.370):

$$\mathit{II}_{\mathbf{T}^{dev}} = -\frac{1}{2}\mathrm{Tr}\left[(\mathbf{T}^{dev})^2\right] = -\frac{1}{2}\mathrm{Tr}\left[(\mathbf{T}^{dev}\cdot\mathbf{T}^{dev})\right] = -\frac{1}{2}\mathbf{T}^{dev} \cdot\cdot\, \mathbf{T}^{dev} = -\frac{1}{2}\mathsf{T}_{ij}^{dev}\mathsf{T}_{ji}^{dev} \tag{1.372}$$

Expanding the previous equation we obtain:

$$\mathit{II}_{\mathbf{T}^{dev}} = -\frac{1}{2}\left[(\mathsf{T}_{11}^{dev})^2 + (\mathsf{T}_{22}^{dev})^2 + (\mathsf{T}_{33}^{dev})^2 + 2(\mathsf{T}_{12}^{dev})^2 + 2(\mathsf{T}_{13}^{dev})^2 + 2(\mathsf{T}_{23}^{dev})^2\right] \tag{1.373}$$

Additionally, in the space of the principal directions we obtain:

$$\mathit{II}_{\mathbf{T}^{dev}} = -\frac{1}{2}\mathsf{T}_{ij}^{dev}\mathsf{T}_{ji}^{dev} = -\frac{1}{2}\left[(\mathsf{T}_{1}^{dev})^2 + (\mathsf{T}_{2}^{dev})^2 + (\mathsf{T}_{3}^{dev})^2\right] \tag{1.374}$$

Another way to express the second invariant is shown below:

$$\begin{aligned}\mathit{II}_{\mathbf{T}^{dev}} &= \begin{vmatrix}\mathsf{T}_{22}^{dev} & \mathsf{T}_{23}^{dev}\\ \mathsf{T}_{23}^{dev} & \mathsf{T}_{33}^{dev}\end{vmatrix} + \begin{vmatrix}\mathsf{T}_{11}^{dev} & \mathsf{T}_{13}^{dev}\\ \mathsf{T}_{13}^{dev} & \mathsf{T}_{33}^{dev}\end{vmatrix} + \begin{vmatrix}\mathsf{T}_{11}^{dev} & \mathsf{T}_{12}^{dev}\\ \mathsf{T}_{12}^{dev} & \mathsf{T}_{22}^{dev}\end{vmatrix}\\ &= -\frac{1}{2}\left[-2\mathsf{T}_{22}^{dev}\mathsf{T}_{33}^{dev} - 2\mathsf{T}_{11}^{dev}\mathsf{T}_{33}^{dev} - 2\mathsf{T}_{11}^{dev}\mathsf{T}_{22}^{dev}\right] - (\mathsf{T}_{12}^{dev})^2 - (\mathsf{T}_{23}^{dev})^2 - (\mathsf{T}_{13}^{dev})^2\end{aligned} \tag{1.375}$$

or

$$\begin{aligned}\mathit{II}_{\mathbf{T}^{dev}} = -\frac{1}{2}\Big[&\left(\mathsf{T}_{22}^{dev}\right)^2 - 2\mathsf{T}_{22}^{dev}\mathsf{T}_{33}^{dev} + \left(\mathsf{T}_{33}^{dev}\right)^2 + \left(\mathsf{T}_{11}^{dev}\right)^2 - 2\mathsf{T}_{11}^{dev}\mathsf{T}_{33}^{dev} + \left(\mathsf{T}_{33}^{dev}\right)^2 + \\ &\left(\mathsf{T}_{11}^{dev}\right)^2 - 2\mathsf{T}_{11}^{dev}\mathsf{T}_{22}^{dev} + \left(\mathsf{T}_{22}^{dev}\right)^2\Big] + \left(\mathsf{T}_{11}^{dev}\right)^2 + \left(\mathsf{T}_{22}^{dev}\right)^2 + \left(\mathsf{T}_{33}^{dev}\right)^2\\ &- (\mathsf{T}_{12}^{dev})^2 - (\mathsf{T}_{23}^{dev})^2 - (\mathsf{T}_{13}^{dev})^2\end{aligned} \tag{1.376}$$

Note that, from equation (1.373), we can state that:

$$(\mathsf{T}_{11}^{dev})^2 + (\mathsf{T}_{22}^{dev})^2 + (\mathsf{T}_{33}^{dev})^2 = -2\,\mathit{II}_{\mathbf{T}^{dev}} - 2(\mathsf{T}_{12}^{dev})^2 - 2(\mathsf{T}_{13}^{dev})^2 - 2(\mathsf{T}_{23}^{dev})^2 \tag{1.377}$$

Substituting (1.377) into (1.376), we find:

$$\mathit{II}_{\mathbf{T}^{dev}} = -\frac{1}{6}\left[(\mathsf{T}_{22}^{dev} - \mathsf{T}_{33}^{dev})^2 + (\mathsf{T}_{11}^{dev} - \mathsf{T}_{33}^{dev})^2 + (\mathsf{T}_{11}^{dev} - \mathsf{T}_{22}^{dev})^2\right] - (\mathsf{T}_{12}^{dev})^2 - (\mathsf{T}_{23}^{dev})^2 - (\mathsf{T}_{13}^{dev})^2 \tag{1.378}$$

Moreover, if we consider the principal space we obtain:

$$\mathit{II}_{\mathbf{T}^{dev}} = -\frac{1}{6}\left[(\mathsf{T}_{2}^{dev} - \mathsf{T}_{3}^{dev})^2 + (\mathsf{T}_{1}^{dev} - \mathsf{T}_{3}^{dev})^2 + (\mathsf{T}_{1}^{dev} - \mathsf{T}_{2}^{dev})^2\right] \tag{1.379}$$

1.5.12.3 Third Invariant of Deviatoric Tensor

The third invariant of the deviatoric tensor is given by:

$$\begin{aligned}\mathit{III}_{\mathbf{T}^{dev}} &= (\mathsf{T}_1 - \mathsf{T}_m)(\mathsf{T}_2 - \mathsf{T}_m)(\mathsf{T}_3 - \mathsf{T}_m)\\ &= \mathsf{T}_1\mathsf{T}_2\mathsf{T}_3 - \mathsf{T}_m(\mathsf{T}_1\mathsf{T}_2 + \mathsf{T}_1\mathsf{T}_3 + \mathsf{T}_2\mathsf{T}_3) + \mathsf{T}_m^2(\mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3) - \mathsf{T}_m^3\\ &= \mathit{III}_{\mathbf{T}} - \frac{I_{\mathbf{T}}}{3}\mathit{II}_{\mathbf{T}} + \frac{I_{\mathbf{T}}^2}{9}I_{\mathbf{T}} - \frac{I_{\mathbf{T}}^3}{27}\\ &= \mathit{III}_{\mathbf{T}} - \frac{I_{\mathbf{T}}\mathit{II}_{\mathbf{T}}}{3} + \frac{2I_{\mathbf{T}}^3}{27}\\ &= \frac{1}{27}\left(2I_{\mathbf{T}}^3 - 9I_{\mathbf{T}}\mathit{II}_{\mathbf{T}} + 27\,\mathit{III}_{\mathbf{T}}\right)\end{aligned} \tag{1.380}$$

Another way of expressing the third invariant is:

$$III_{\mathbf{T}^{dev}} = \mathsf{T}_1^{dev}\mathsf{T}_2^{dev}\mathsf{T}_3^{dev} = \frac{1}{3}\mathsf{T}_{ij}^{dev}\mathsf{T}_{jk}^{dev}\mathsf{T}_{ki}^{dev} \tag{1.381}$$

Problem 1.39: Let $\boldsymbol{\sigma}$ be a symmetric second-order tensor, and $\mathbf{s} \equiv \boldsymbol{\sigma}^{dev}$ be a deviatoric tensor. Prove that $\mathbf{s} : \dfrac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \mathbf{s}$. Also show that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors.

Solution: First, we make use of the definition of a deviatoric tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma}^{sph} + \mathbf{s} = \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} + \mathbf{s} \quad \Rightarrow \quad \mathbf{s} = \boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}.$$

Afterwards we calculate:

$$\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \frac{\partial\left[\boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}\right]}{\partial \boldsymbol{\sigma}} = \frac{\partial[\boldsymbol{\sigma}]}{\partial \boldsymbol{\sigma}} - \frac{1}{3}\frac{\partial[I_{\boldsymbol{\sigma}}]}{\partial \boldsymbol{\sigma}}\mathbf{1}$$

which in indicial notation is:

$$\frac{\partial \mathsf{s}_{ij}}{\partial \sigma_{kl}} = \frac{\partial \sigma_{ij}}{\partial \sigma_{kl}} - \frac{1}{3}\frac{\partial[I_{\boldsymbol{\sigma}}]}{\partial \sigma_{kl}}\delta_{ij} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}$$

Therefore

$$\begin{aligned}\mathsf{s}_{ij}\frac{\partial \mathsf{s}_{ij}}{\partial \sigma_{kl}} &= \mathsf{s}_{ij}\left(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{kl}\delta_{ij}\right) = \mathsf{s}_{ij}\delta_{ik}\delta_{jl} - \frac{1}{3}\mathsf{s}_{ij}\delta_{kl}\delta_{ij} = \mathsf{s}_{kl} - \frac{1}{3}\delta_{kl}\underbrace{\mathsf{s}_{ii}}_{=0} \\ &= \mathsf{s}_{kl}\end{aligned}$$

$$\mathbf{s} : \frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \mathbf{s}$$

To show that two tensors are coaxial, we must prove that $\boldsymbol{\sigma}^{dev} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{dev}$:

$$\begin{aligned}\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{dev} &= \boldsymbol{\sigma} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{sph}) = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}^{sph} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} \\ &= \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1} \cdot \boldsymbol{\sigma} \\ &= \left(\boldsymbol{\sigma} - \frac{I_{\boldsymbol{\sigma}}}{3}\mathbf{1}\right) \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{\sigma}\end{aligned}$$

Therefore, we have shown that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors. In other words, they have the same principal directions.

1.6 The Tensor-Valued Tensor Function

Tensor-valued tensor function can be of the types: scalar, vector, or higher-order tensors. As examples of scalar-valued tensor functions we can list:

$$\begin{aligned} \Psi &= \Psi(\mathbf{T}) = \det(\mathbf{T}) \\ \Psi &= \Psi(\mathbf{T},\mathbf{S}) = \mathbf{T}:\mathbf{S} \end{aligned} \tag{1.382}$$

where $\mathbf{T}$ and $\mathbf{S}$ are second-order tensors. Additionally, as an example of a second-order-valued tensor function we have:

$$\Pi = \Pi(\mathbf{T}) = \alpha\mathbf{1} + \beta\mathbf{T} \tag{1.383}$$

where α and β are scalars.

1.6.1 The Tensor Series

The function $f(x)$ can be approximated by the Taylor series as $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(a)}{\partial x^n}(x-a)^n$, where $n!$ denotes the factorial of n, and $f(a)$ is the value of the function at the application point $x = a$. We can extrapolate that definition for use on tensors. For example, let us suppose we have a scalar-valued tensor function ψ in terms of a second-order tensor, $\boldsymbol{E}$, then we can approximate $\psi(\boldsymbol{E})$ as:

$$\begin{aligned} \psi(\boldsymbol{E}) &\approx \frac{1}{0!}\psi(\boldsymbol{E}_0) + \frac{1}{1!}\frac{\partial\psi(\boldsymbol{E}_0)}{\partial E_{ij}}(E_{ij} - E_{0ij}) + \frac{1}{2!}\frac{\partial^2\psi(\boldsymbol{E}_0)}{\partial E_{ij}\partial E_{kl}}(E_{ij} - E_{ij0})(E_{kl} - E_{kl0}) + \cdots \\ &\approx \psi_0 + \frac{\partial\psi(\boldsymbol{E}_0)}{\partial\boldsymbol{E}} : (\boldsymbol{E} - \boldsymbol{E}_0) + \frac{1}{2}(\boldsymbol{E} - \boldsymbol{E}_0) : \frac{\partial^2\psi(\boldsymbol{E}_0)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}} : (\boldsymbol{E} - \boldsymbol{E}_0) + \cdots \end{aligned} \tag{1.384}$$

A second-order-valued tensor function, $\mathbf{S}(\boldsymbol{E})$, can be approximated as:

$$\begin{aligned} \mathbf{S}(\boldsymbol{E}) &\approx \frac{1}{0!}\mathbf{S}(\boldsymbol{E}_0) + \frac{1}{1!}\frac{\partial\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}} : (\boldsymbol{E} - \boldsymbol{E}_0) + \frac{1}{2!}(\boldsymbol{E} - \boldsymbol{E}_0) : \frac{\partial^2\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}} : (\boldsymbol{E} - \boldsymbol{E}_0) + \cdots \\ &\approx \mathbf{S}_0 + \frac{\partial\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}} : (\boldsymbol{E} - \boldsymbol{E}_0) + \frac{1}{2}(\boldsymbol{E} - \boldsymbol{E}_0) : \frac{\partial^2\mathbf{S}(\boldsymbol{E}_0)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}} : (\boldsymbol{E} - \boldsymbol{E}_0) + \cdots \end{aligned} \tag{1.385}$$

Other tensor algebraic expressions can be represented by series, *e.g.*:

$$\begin{aligned} \exp^{\mathbf{S}} &= \mathbf{1} + \mathbf{S} + \frac{1}{2!}\mathbf{S}^2 + \frac{1}{3!}\mathbf{S}^3 + \cdots \\ \ln(\mathbf{1} + \mathbf{S}) &= \mathbf{S} - \frac{1}{2}\mathbf{S}^2 + \frac{1}{3}\mathbf{S}^3 - \cdots \\ \sin(\mathbf{S}) &= \mathbf{S} - \frac{1}{3!}\mathbf{S}^3 + \frac{1}{5!}\mathbf{S}^5 - \cdots \end{aligned} \tag{1.386}$$

With reference to the spectral representation of a symmetric second-order tensor, $\mathbf{S}$, it is also true that:

$$\exp^{\mathbf{S}} = \sum_{a=1}^{3}\left(1+\lambda_a + \frac{\lambda_a^2}{2!} + \frac{\lambda_a^3}{3!} + \cdots\right)\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \exp^{\lambda_a} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}$$
$$\ln(\mathbf{1}+\mathbf{S}) = \sum_{a=1}^{3}\left(\lambda_a - \frac{1}{2}\lambda_a^2 + \frac{1}{3}\lambda_a^3 + \cdots\right)\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \ln(1+\lambda_a)\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{1.387}$$

where λ_a and $\hat{\mathbf{n}}^{(a)}$ are the eigenvalues and eigenvectors, respectively, of the tensor $\mathbf{S}$.

1.6.2 The Tensor-Valued Isotropic Tensor Function

A second-order-valued tensor function, $\Pi = \Pi(\mathbf{T})$, is isotropic if after an orthogonal transformation the following condition is satisfied:

$$\Pi^*(\mathbf{T}) = \mathbf{Q}\cdot\Pi(\mathbf{T})\cdot\mathbf{Q}^T = \underbrace{\Pi\left(\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T\right)}_{\Pi\left(\mathbf{T}^*\right)} \tag{1.388}$$

We can show that $\Pi(\mathbf{T})$ has the same principal directions of $\mathbf{T}$, *i.e.* $\Pi(\mathbf{T})$ and $\mathbf{T}$ are coaxial tensors. To demonstrate this we can regard the components of $\mathbf{T}$ in the principal space as:

$$(\mathbf{T})_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.389}$$

Then the tensor function is given in terms of the principal values of $\mathbf{T}$: $\Pi = \Pi(\lambda_1, \lambda_2, \lambda_3)$ and, the transformation of $\mathbf{T}$ is given by:

$$\mathbf{T}^* = \mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T \tag{1.390}$$

Likewise, for the tensor function Π:

$$\Pi^*(\mathbf{T}) = \mathbf{Q}\cdot\Pi(\mathbf{T})\cdot\mathbf{Q}^T \tag{1.391}$$

If we take as the orthogonal tensor components:

$$(\mathbf{Q})_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1.392}$$

After having done the calculation for the matrices (1.391), we obtain:

$$\Pi^* = \begin{bmatrix} \Pi_{11} & -\Pi_{12} & -\Pi_{13} \\ -\Pi_{12} & \Pi_{22} & -\Pi_{23} \\ -\Pi_{13} & -\Pi_{23} & \Pi_{33} \end{bmatrix} = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{12} & \Pi_{22} & \Pi_{23} \\ \Pi_{13} & \Pi_{23} & \Pi_{33} \end{bmatrix} = \Pi$$
$$\Rightarrow \Pi^* = \begin{bmatrix} \Pi_{11} & 0 & 0 \\ 0 & \Pi_{22} & 0 \\ 0 & 0 & \Pi_{33} \end{bmatrix} \tag{1.393}$$

To satisfy that $\Pi^* = \Pi$ (isotropy), we conclude that $\Pi_{12} = \Pi_{13} = \Pi_{23} = 0$. Therefore, $\Pi(\mathbf{T})$ and $\mathbf{T}$ have the same principal directions.

Once again we observe, a tensor function $\Pi(\mathbf{T})$. This tensor function is isotropic if and only if it can be represented by the following linear transformation, Truesdell & Noll (1965):

$$\Pi = \Pi(\mathbf{T}) = \Phi_0 \mathbf{1} + \Phi_1 \mathbf{T} + \Phi_2 \mathbf{T}^2 \tag{1.394}$$

where Φ_0, Φ_1, Φ_2 are functions of the tensor $\mathbf{T}$ invariants or functions of the $\mathbf{T}$ eigenvalues.

A brief demonstration follows. We can now consider the spectral representations of $\mathbf{T}$ and Π, respectively:

$$\mathbf{T} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \lambda_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \lambda_2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \lambda_3 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \tag{1.395}$$

$$\Pi = \sum_{a=1}^{3} \omega_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \omega_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \omega_2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \omega_3 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \tag{1.396}$$

Note that $\mathbf{T}$ and Π have the same principal directions $\hat{\mathbf{n}}^{(i)}$. Then, we can put the following set of equations together:

$$\begin{cases} \mathbf{1} = \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \\ \mathbf{T} = \lambda_1 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \lambda_2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \lambda_3 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \\ \mathbf{T}^2 = \lambda_1^2 \hat{\mathbf{n}}^{(1)} \otimes \hat{\mathbf{n}}^{(1)} + \lambda_3^2 \hat{\mathbf{n}}^{(2)} \otimes \hat{\mathbf{n}}^{(2)} + \lambda_3^2 \hat{\mathbf{n}}^{(3)} \otimes \hat{\mathbf{n}}^{(3)} \end{cases} \tag{1.397}$$

Solving the set above, we obtain $\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \equiv \mathbf{M}^{(a)}$ as a function of the tensor $\mathbf{T}$, and we obtain:

$$\begin{aligned} \mathbf{M}^{(1)} &= \frac{\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)}\mathbf{1} - \frac{(\lambda_2+\lambda_3)}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)}\mathbf{T} + \frac{\mathbf{T}^2}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)} \\ \mathbf{M}^{(2)} &= \frac{\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\mathbf{1} - \frac{(\lambda_1+\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\mathbf{T} + \frac{\mathbf{T}^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \\ \mathbf{M}^{(3)} &= \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\mathbf{1} - \frac{(\lambda_1+\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\mathbf{T} + \frac{\mathbf{T}^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \end{aligned} \tag{1.398}$$

It is evident that, if we substitute the values of $\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \equiv \mathbf{M}^{(a)}$ in equation (1.395) we obtain: $\mathbf{T} = \mathbf{T}$. Now, if we substitute the values of $\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \equiv \mathbf{M}^{(a)}$ in equation (1.396), we obtain:

$$\Pi = \Pi(\mathbf{T}) = \Phi_0 \mathbf{1} + \Phi_1 \mathbf{T} + \Phi_2 \mathbf{T}^2 \tag{1.399}$$

where the coefficients Φ_0, Φ_1, and Φ_2 are functions of the eigenvalues of $\mathbf{T}$, $(\lambda_1 \neq \lambda_2 \neq \lambda_3)$, and given by:

$$\begin{aligned} \Phi_0 &= \frac{\omega_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)} + \frac{\omega_2\lambda_1\lambda_3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\omega_3\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \\ \Phi_1 &= -\frac{\omega_1(\lambda_2+\lambda_3)}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)} - \frac{\omega_2(\lambda_1+\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} - \frac{\omega_3(\lambda_1+\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \\ \Phi_2 &= \frac{\omega_1}{(\lambda_1-\lambda_3)(\lambda_1-\lambda_2)} + \frac{\omega_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\omega_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \end{aligned} \tag{1.400}$$

We can now show that if a tensor function $\Pi(\mathbf{T})$ is given in (1.399), this tensor function is isotropic:

$$\begin{aligned}\Pi^*(\mathbf{T}) &= \mathbf{Q}\cdot\Pi(\mathbf{T})\cdot\mathbf{Q}^T = \mathbf{Q}\cdot\left(\Phi_0\mathbf{1}+\Phi_1\mathbf{T}+\Phi_2\mathbf{T}^2\right)\cdot\mathbf{Q}^T \\ &= \Phi_0\mathbf{Q}\cdot\mathbf{1}\cdot\mathbf{Q}^T + \Phi_1\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}^T + \Phi_2\mathbf{Q}\cdot\mathbf{T}^2\cdot\mathbf{Q}^T = \Phi_0\mathbf{1}+\Phi_1\mathbf{T}^* + \Phi_2\mathbf{T}^{*2} \\ &= \Pi(\mathbf{T}^*)\end{aligned} \tag{1.401}$$

1.6.3 The Derivative of the Tensor-Valued Tensor Function

Firstly, we refer to a scalar-valued tensor function:

$$\Pi = \Pi(\mathbf{A}) \tag{1.402}$$

The partial derivative of $\Pi(\mathbf{A})$ with respect to $\mathbf{A}$ is defined as:

$$\frac{\partial\Pi}{\partial\mathbf{A}} \equiv \Pi_{,\mathbf{A}} = \frac{\partial\Pi}{\partial\mathrm{A}_{ij}}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j) \tag{1.403}$$

where the comma denotes a partial derivative.

Then the second derivative of $\Pi(\mathbf{A})$ becomes a fourth-order tensor:

$$\frac{\partial^2\Pi}{\partial\mathbf{A}\otimes\partial\mathbf{A}} = \Pi_{,\mathbf{AA}} = \frac{\partial^2\Pi}{\partial\mathrm{A}_{ij}\partial\mathrm{A}_{kl}}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l) = \mathbb{D}_{ijkl}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_k\otimes\hat{\mathbf{e}}_l) \tag{1.404}$$

Let $\boldsymbol{C}$ and $\boldsymbol{b}$ be positive definite symmetric second-order tensors defined as:

$$\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F} \quad ; \quad \boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{F}^T \tag{1.405}$$

where $\boldsymbol{F}$ is an arbitrary second-order tensor with the restriction $\det(\boldsymbol{F})>0$ imposed on it. We must also bear in mind that there is a scalar-valued isotropic tensor function, $\Psi = \Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})$, expressed in terms of the principal invariants of $\boldsymbol{C}$, where $I_{\boldsymbol{C}} = I_{\boldsymbol{b}}$, $II_{\boldsymbol{C}} = II_{\boldsymbol{b}}$, $III_{\boldsymbol{C}} = III_{\boldsymbol{b}}$. Next, we can find the partial derivative of Ψ with respect to $\boldsymbol{C}$, and with respect to $\boldsymbol{b}$. We must also verify that the following relation holds:

$$\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T = \Psi_{,\boldsymbol{b}}\cdot\boldsymbol{b} \tag{1.406}$$

By applying the chain rule for derivative we obtain:

$$\Psi_{,\boldsymbol{C}} = \frac{\partial\Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})}{\partial\boldsymbol{C}} = \frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\frac{\partial I_{\boldsymbol{C}}}{\partial\boldsymbol{C}} + \frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}\frac{\partial II_{\boldsymbol{C}}}{\partial\boldsymbol{C}} + \frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}\frac{\partial III_{\boldsymbol{C}}}{\partial\boldsymbol{C}} \tag{1.407}$$

Considering the partial derivatives of the principal invariants, we can state that:

$$\begin{aligned}&\frac{\partial I_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = \mathbf{1} \\ &\frac{\partial II_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}^T = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C} = II_{\boldsymbol{C}}\boldsymbol{C}^{-1} - III_{\boldsymbol{C}}\boldsymbol{C}^{-2} \\ &\frac{\partial III_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = III_{\boldsymbol{C}}\boldsymbol{C}^{-T} = III_{\boldsymbol{C}}\boldsymbol{C}^{-1} = \boldsymbol{C}^2 - I_{\boldsymbol{C}}\boldsymbol{C} + II_{\boldsymbol{C}}\mathbf{1}\end{aligned} \tag{1.408}$$

Now, by substituting the following values $\frac{\partial I_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = \mathbf{1}$, $\frac{\partial II_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}$ and $\frac{\partial III_{\boldsymbol{C}}}{\partial\boldsymbol{C}} = III_{\boldsymbol{C}}\boldsymbol{C}^{-1}$ into the equation in (1.407), we obtain:

$$\Psi_{,\boldsymbol{C}} = \frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\mathbf{1} + \frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}(I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}) + \frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}\,III_{\boldsymbol{C}}\boldsymbol{C}^{-1} \tag{1.409}$$

$$\boxed{\Psi_{,\boldsymbol{C}} = \left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C} + \left(\frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}} \tag{1.410}$$

Another way to express the relation (1.410) is by substituting $\frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \mathbf{1}$, $\frac{\partial I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}$ and $\frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \boldsymbol{C}^2 - I_{\boldsymbol{C}}\boldsymbol{C} + I\!I_{\boldsymbol{C}}\mathbf{1}$ into the equation in (1.407), thus:

$$\boxed{\Psi_{,\boldsymbol{C}} = \left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}} + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{C} + \left(\frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C}^2} \tag{1.411}$$

If we now consider the values $\frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \mathbf{1}$, $\frac{\partial I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1} - I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-2}$, $\frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1}$, in the equation in (1.407), we obtain:

$$\boxed{\Psi_{,\boldsymbol{C}} = \left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\right)\mathbf{1} + \left(\frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} I\!I_{\boldsymbol{C}} + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1} - \left(\frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-2}} \tag{1.412}$$

If we now observe both $I_{\boldsymbol{C}} = I_{\boldsymbol{b}}$, $I\!I_{\boldsymbol{C}} = I\!I_{\boldsymbol{b}}$, $I\!I\!I_{\boldsymbol{C}} = I\!I\!I_{\boldsymbol{b}}$, and the equation in (1.410), we can draw the conclusion that:

$$\boxed{\Psi_{,\boldsymbol{b}} = \left(\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}} + \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{b}}} I_{\boldsymbol{b}}\right)\mathbf{1} - \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{b}}}\boldsymbol{b} + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{b}}} I\!I\!I_{\boldsymbol{b}}\boldsymbol{b}^{-1}} \tag{1.413}$$

Using the equation in (1.410), the equation $\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T$ becomes:

$$\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T = \left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{F}\cdot\mathbf{1}\cdot\boldsymbol{F}^T - \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}}\boldsymbol{F}\cdot\boldsymbol{C}\cdot\boldsymbol{F}^T + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\boldsymbol{F}\cdot\boldsymbol{C}^{-1}\cdot\boldsymbol{F}^T \tag{1.414}$$

Then, if we observe that:

$$\begin{aligned}
&\Rightarrow \boldsymbol{F}\cdot\mathbf{1}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{b}\\
&\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F}\\
&\Rightarrow \boldsymbol{F}\cdot\boldsymbol{C}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{b}\cdot\boldsymbol{b} = \boldsymbol{b}^2
\end{aligned} \tag{1.415}$$

$$\begin{aligned}
&\boldsymbol{C}^{-1} = \boldsymbol{F}^{-1}\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{F}\\
&\Rightarrow \boldsymbol{F}\cdot\boldsymbol{C}^{-1}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{b}^{-1}\cdot\boldsymbol{b}
\end{aligned} \tag{1.416}$$

The equation (1.414) can be rewritten as:

$$\begin{aligned}
\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T &= \left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{b} - \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}}\boldsymbol{b}^2 + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\boldsymbol{b}^{-1}\cdot\boldsymbol{b}\\
&= \left[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{C}}}\boldsymbol{b} + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\boldsymbol{b}^{-1}\right]\cdot\boldsymbol{b}
\end{aligned} \tag{1.417}$$

In light of the equation in (1.413) and (1.417), we can draw the conclusion that:

$$\begin{aligned}
\boldsymbol{F}\cdot\Psi_{,\boldsymbol{C}}\cdot\boldsymbol{F}^T &= \left[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}} + \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{b}}} I_{\boldsymbol{b}}\right)\mathbf{1} - \frac{\partial\Psi}{\partial I\!I_{\boldsymbol{b}}}\boldsymbol{b} + \frac{\partial\Psi}{\partial I\!I\!I_{\boldsymbol{b}}} I\!I\!I_{\boldsymbol{b}}\boldsymbol{b}^{-1}\right]\cdot\boldsymbol{b}\\
&= \Psi_{,\boldsymbol{b}}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\Psi_{,\boldsymbol{b}}
\end{aligned} \tag{1.418}$$

which indicates that $\Psi_{,\boldsymbol{b}}$ and $\boldsymbol{b}$ are coaxial tensors.

Once again, we can observe $\boldsymbol{C}$ given by the equation in (1.405). Next, we can evaluate the derivative of the scalar-valued tensor function, $\Psi = \Psi(\boldsymbol{C})$, with respect to the tensor $\boldsymbol{F}$:

$$\Psi_{,\boldsymbol{F}} = \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{F}} = \frac{\partial \Psi}{\partial \boldsymbol{C}} : \frac{\partial \boldsymbol{C}}{\partial \boldsymbol{F}} \xrightarrow{\text{indicial notation}} (\Psi_{,\boldsymbol{F}})_{kl} = \frac{\partial \Psi}{\partial C_{ij}} \frac{\partial C_{ij}}{\partial F_{kl}} \tag{1.419}$$

The derivative of tensor $\boldsymbol{C}$ with respect to $\boldsymbol{F}$ is evaluated as follows:

$$\begin{aligned} \frac{\partial C_{ij}}{\partial F_{kl}} &= \frac{\partial (F_{qi} F_{qj})}{\partial F_{kl}} \\ &= \frac{\partial (F_{qi})}{\partial F_{kl}} F_{qj} + F_{qi} \frac{\partial (F_{qj})}{\partial F_{kl}} \\ &= \delta_{qk} \delta_{il} F_{qj} + \delta_{qk} \delta_{jl} F_{qi} \\ &= \delta_{il} F_{kj} + \delta_{jl} F_{ki} \end{aligned} \tag{1.420}$$

Then, by substituting (1.420) into (1.419), we obtain:

$$\begin{aligned} (\Psi_{,\boldsymbol{F}})_{kl} &= \frac{\partial \Psi}{\partial C_{ij}} (\delta_{il} F_{kj} + \delta_{jl} F_{ki}) \\ &= F_{kj} \frac{\partial \Psi}{\partial C_{lj}} + F_{ki} \frac{\partial \Psi}{\partial C_{il}} \end{aligned} \tag{1.421}$$

Due to the symmetry of $\boldsymbol{C}$, *i.e.* $C_{lj} = C_{jl}$, we can draw the conclusion that:

$$(\Psi_{,\boldsymbol{F}})_{kl} = 2 \frac{\partial \Psi}{\partial C_{lj}} F_{kj} = 2 \frac{\partial \Psi}{\partial C_{jl}} F_{kj} \quad \Rightarrow \quad \boxed{\Psi_{,\boldsymbol{F}} = 2 \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = 2 \boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}}} \tag{1.422}$$

Now, suppose that $\boldsymbol{C}$ is given by the equation $\boldsymbol{C} = \mathbf{U}^T \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{U} = \mathbf{U}^2$, where $\mathbf{U}$ is a symmetric second-order tensor. To find $\Psi(\boldsymbol{C})_{,\mathbf{U}}$ we can use the same equation as in (1.422), *i.e.*:

$$\Psi_{,\mathbf{U}} = 2 \Psi_{,\boldsymbol{C}} \cdot \mathbf{U} = 2 \mathbf{U} \cdot \Psi_{,\boldsymbol{C}} \tag{1.423}$$

Therefore, we can draw the conclusion that $\Psi_{,\boldsymbol{C}}$ and $\mathbf{U}$ are coaxial tensors.

Let $\mathbf{A}$ be a symmetric second-order tensor, and $\Psi = \Psi(\mathbf{A})$ be a scalar-valued tensor function. The following relationships hold:

$$\begin{aligned} \Psi_{,\boldsymbol{b}} &= 2 \boldsymbol{b} \cdot \Psi_{,\mathbf{A}} \quad \text{for} \quad \mathbf{A} = \boldsymbol{b}^T \cdot \boldsymbol{b} \\ \Psi_{,\boldsymbol{b}} &= 2 \Psi_{,\mathbf{A}} \cdot \boldsymbol{b} \quad \text{for} \quad \mathbf{A} = \boldsymbol{b} \cdot \boldsymbol{b}^T \\ \Psi_{,\boldsymbol{b}} &= 2 \boldsymbol{b} \cdot \Psi_{,\mathbf{A}} = 2 \Psi_{,\mathbf{A}} \cdot \boldsymbol{b} \\ &= \boldsymbol{b} \cdot \Psi_{,\mathbf{A}} + \Psi_{,\mathbf{A}} \cdot \boldsymbol{b} \qquad \text{for} \quad \mathbf{A} = \boldsymbol{b} \cdot \boldsymbol{b} \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{b}^T \end{aligned} \tag{1.424}$$

1.7 The Voigt Notation

When dealing with symmetric tensors, it may be advantageous to just work with the independent components. For example, a symmetric second-order tensor has 6 independent components, so, it is possible to represent these components by a column matrix as follows:

$$\mathsf{T}_{ij}=\begin{bmatrix}\mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13}\\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23}\\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33}\end{bmatrix}\xrightarrow{Voigt}\{\mathcal{T}\}=\begin{bmatrix}\mathsf{T}_{11}\\ \mathsf{T}_{22}\\ \mathsf{T}_{33}\\ \mathsf{T}_{12}\\ \mathsf{T}_{23}\\ \mathsf{T}_{13}\end{bmatrix} \tag{1.425}$$

This representation is called the *Voigt Notation*. It is also possible to represent a second-order tensor as:

$$\mathsf{E}_{ij}=\begin{bmatrix}\mathsf{E}_{11} & \mathsf{E}_{12} & \mathsf{E}_{13}\\ \mathsf{E}_{12} & \mathsf{E}_{22} & \mathsf{E}_{23}\\ \mathsf{E}_{13} & \mathsf{E}_{23} & \mathsf{E}_{33}\end{bmatrix}\xrightarrow{Voigt}\{\mathcal{E}\}=\begin{bmatrix}\mathsf{E}_{11}\\ \mathsf{E}_{22}\\ \mathsf{E}_{33}\\ 2\mathsf{E}_{12}\\ 2\mathsf{E}_{23}\\ 2\mathsf{E}_{13}\end{bmatrix} \tag{1.426}$$

As we have seen before, a fourth-order tensor, $\mathbb{C}$, that presents minor symmetry, *i.e.* $\mathbb{C}_{ijkl}=\mathbb{C}_{jikl}=\mathbb{C}_{ijlk}=\mathbb{C}_{jilk}$, has $6\times 6=36$ independent components. Note that, due to the symmetry of (ij) we have 6 independent components, and due to the symmetry of (kl) we have 6 independent components. In Voigt Notation we can represent these components in a 6-by-6 matrix as:

$$[\mathcal{C}]=\begin{bmatrix}\mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & \mathbb{C}_{1123} & \mathbb{C}_{1113}\\ \mathbb{C}_{2211} & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \mathbb{C}_{2212} & \mathbb{C}_{2223} & \mathbb{C}_{2213}\\ \mathbb{C}_{3311} & \mathbb{C}_{3322} & \mathbb{C}_{3333} & \mathbb{C}_{3312} & \mathbb{C}_{3323} & \mathbb{C}_{3313}\\ \mathbb{C}_{1211} & \mathbb{C}_{1222} & \mathbb{C}_{1233} & \mathbb{C}_{1212} & \mathbb{C}_{1223} & \mathbb{C}_{1213}\\ \mathbb{C}_{2311} & \mathbb{C}_{2322} & \mathbb{C}_{2333} & \mathbb{C}_{2312} & \mathbb{C}_{2323} & \mathbb{C}_{2313}\\ \mathbb{C}_{1311} & \mathbb{C}_{1322} & \mathbb{C}_{1333} & \mathbb{C}_{1312} & \mathbb{C}_{1323} & \mathbb{C}_{1313}\end{bmatrix} \tag{1.427}$$

In addition to minor symmetry the tensor also has major symmetry, *i.e.* $\mathbb{C}_{ijkl}=\mathbb{C}_{klij}$, and the number of independent components have reduced to 21. One can easily memorize the order of the components in the matrix $[\mathcal{C}]$ if we consider the order of the second-order tensor in Voigt Notation, *i.e.*:

$$\begin{bmatrix}(11)\\(22)\\(33)\\(12)\\(23)\\(13)\end{bmatrix}\begin{bmatrix}(11) & (22) & (33) & (12) & (23) & (13)\end{bmatrix} \tag{1.428}$$

1.7.1 The Unit Tensors in Voigt Notation

The second-order unit tensor is represented in the Voigt notation as:

$$\delta_{ij} \equiv \boldsymbol{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{Voigt} \{\delta\} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{1.429}$$

In the subsection *1.5.2.5.1 Unit Tensors* we have defined three fourth-order unit tensors, namely, $\mathbb{I}_{ijk\ell} = \delta_{ik}\delta_{j\ell}$, $\bar{\mathbb{I}}_{ijk\ell} = \delta_{i\ell}\delta_{jk}$ and $\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell}$, among which only $\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell}$ is a symmetric tensor. The representation of $\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell}$ in Voigt notation can be evaluated by observing how a symmetric fourth-order tensor is represented in (1.427), thus:

$$\bar{\bar{\mathbb{I}}}_{ijk\ell} = \delta_{ij}\delta_{k\ell} \xrightarrow{Voigt} [\bar{\bar{\mathcal{I}}}] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.430}$$

where $\bar{\bar{\mathbb{I}}}_{1111} = \delta_{11}\delta_{11} = 1$, $\bar{\bar{\mathbb{I}}}_{1122} = \delta_{11}\delta_{22} = 1$, and so on.

The components of a fourth-order unit tensor, $\mathbb{I}^{sym}$, are represented by $\mathbf{I}_{ijk\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$, which in Voigt notation becomes:

$$\mathbf{I}_{ijk\ell} \xrightarrow{Voigt} [\mathcal{I}] = \begin{bmatrix} \mathbf{I}_{1111} & \mathbf{I}_{1122} & \mathbf{I}_{1133} & \mathbf{I}_{1112} & \mathbf{I}_{1123} & \mathbf{I}_{1113} \\ \mathbf{I}_{2211} & \mathbf{I}_{2222} & \mathbf{I}_{2233} & \mathbf{I}_{2212} & \mathbf{I}_{2223} & \mathbf{I}_{2213} \\ \mathbf{I}_{3311} & \mathbf{I}_{3322} & \mathbf{I}_{3333} & \mathbf{I}_{3312} & \mathbf{I}_{3323} & \mathbf{I}_{3313} \\ \mathbf{I}_{1211} & \mathbf{I}_{1222} & \mathbf{I}_{1233} & \mathbf{I}_{1212} & \mathbf{I}_{1223} & \mathbf{I}_{1213} \\ \mathbf{I}_{2311} & \mathbf{I}_{2322} & \mathbf{I}_{2333} & \mathbf{I}_{2312} & \mathbf{I}_{2323} & \mathbf{I}_{2313} \\ \mathbf{I}_{1311} & \mathbf{I}_{1322} & \mathbf{I}_{1333} & \mathbf{I}_{1312} & \mathbf{I}_{1323} & \mathbf{I}_{1313} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{1.431}$$

and the inverse of the equation in (1.431) becomes:

$$[\mathcal{I}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{1.432}$$

1.7.2 The Scalar Product in Voigt Notation

The dot product between a symmetric second-order tensor, $\mathbf{T}$, and a vector $\vec{\mathbf{n}}$, is given by $\vec{\mathbf{b}} = \mathbf{T} \cdot \vec{\mathbf{n}}$ where the components of $\vec{\mathbf{b}}$ can be evaluated as follows:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \Rightarrow \begin{cases} b_1 = T_{11}n_1 + T_{12}n_2 + T_{13}n_3 \\ b_2 = T_{12}n_1 + T_{22}n_2 + T_{23}n_3 \\ b_3 = T_{13}n_1 + T_{23}n_2 + T_{33}n_3 \end{cases} \tag{1.433}$$

By observing how a second-order tensor is presented in Voigt notation, as in (1.425), the scalar product (1.433) can be represented in the Voigt notation as:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{bmatrix}}_{[\overline{\mathcal{N}}]^T} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{13} \end{bmatrix} \quad \Rightarrow \quad \{\boldsymbol{b}\} = [\overline{\mathcal{N}}]^T \{\mathcal{T}\} \tag{1.434}$$

1.7.3 The Component Transformation Law in Voigt Notation

The component transformation law for a second-order tensor is defined as:

$$T'_{ij} = T_{kl} a_{ik} a_{jl} \tag{1.435}$$

or in matrix form:

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{12} & T'_{22} & T'_{23} \\ T'_{13} & T'_{23} & T'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T \tag{1.436}$$

By multiplying the matrices and by rearranging the result in Voigt notation we obtain:

$$\{\mathcal{T}'\} = [\mathcal{M}]\{\mathcal{T}\} \tag{1.437}$$

where:

$$\{\mathcal{T}'\} = \begin{bmatrix} T'_{11} \\ T'_{22} \\ T'_{33} \\ T'_{12} \\ T'_{23} \\ T'_{13} \end{bmatrix} \quad ; \quad \{\mathcal{T}\} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{13} \end{bmatrix} \tag{1.438}$$

and $[\mathcal{M}]$ is the transformation matrix for the second-order tensor components in Voigt Notation. The matrix $[\mathcal{M}]$ is given by:

$$[\mathcal{M}] = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{11}a_{12} & 2a_{12}a_{13} & 2a_{11}a_{13} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{21}a_{22} & 2a_{22}a_{23} & 2a_{21}a_{23} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{31}a_{32} & 2a_{32}a_{33} & 2a_{31}a_{33} \\ a_{21}a_{11} & a_{22}a_{12} & a_{13}a_{23} & (a_{11}a_{22} + a_{12}a_{21}) & (a_{13}a_{22} + a_{12}a_{23}) & (a_{13}a_{21} + a_{11}a_{23}) \\ a_{31}a_{21} & a_{32}a_{22} & a_{33}a_{23} & (a_{31}a_{22} + a_{32}a_{21}) & (a_{33}a_{22} + a_{32}a_{23}) & (a_{33}a_{21} + a_{31}a_{23}) \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & (a_{31}a_{12} + a_{32}a_{11}) & (a_{33}a_{12} + a_{32}a_{13}) & (a_{33}a_{11} + a_{31}a_{13}) \end{bmatrix} \tag{1.439}$$

If the representation of tensor components is shown in (1.426), equation (1.436) in Voigt Notation becomes:

$$\{\mathcal{E}'\} = [\mathcal{N}]\{\mathcal{E}\} \tag{1.440}$$

where

$$[\mathcal{N}] = \begin{bmatrix} a_{11}{}^2 & a_{12}{}^2 & a_{13}{}^2 & a_{11}a_{12} & a_{12}a_{13} & a_{11}a_{13} \\ a_{21}{}^2 & a_{22}{}^2 & a_{23}{}^2 & a_{21}a_{22} & a_{22}a_{23} & a_{21}a_{23} \\ a_{31}{}^2 & a_{32}{}^2 & a_{33}{}^2 & a_{31}a_{32} & a_{32}a_{33} & a_{31}a_{33} \\ 2a_{21}a_{11} & 2a_{22}a_{12} & 2a_{13}a_{23} & (a_{11}a_{22}+a_{12}a_{21}) & (a_{13}a_{22}+a_{12}a_{23}) & (a_{13}a_{21}+a_{11}a_{23}) \\ 2a_{31}a_{21} & 2a_{32}a_{22} & 2a_{33}a_{23} & (a_{31}a_{22}+a_{32}a_{21}) & (a_{33}a_{22}+a_{32}a_{23}) & (a_{33}a_{21}+a_{31}a_{23}) \\ 2a_{31}a_{11} & 2a_{32}a_{12} & 2a_{33}a_{13} & (a_{31}a_{12}+a_{32}a_{11}) & (a_{33}a_{12}+a_{32}a_{13}) & (a_{33}a_{11}+a_{31}a_{13}) \end{bmatrix} \tag{1.441}$$

The matrices (1.439) and (1.441) are not orthogonal matrices, *i.e.* $[\mathcal{M}]^{-1} \neq [\mathcal{M}]^T$ and $[\mathcal{N}]^{-1} \neq [\mathcal{N}]^T$. However, it is possible to show that $[\mathcal{M}]^{-1} = [\mathcal{N}]^T$.

1.7.4 Spectral Representation in Voigt Notation

Regarding the spectral representation of a symmetric tensor $\mathbf{T}$:

$$\mathbf{T} = \sum_{a=1}^{3} \mathsf{T}_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \xrightarrow{\text{Matricial form}} \mathcal{T} = \mathcal{A}^T \mathcal{T}' \mathcal{A} \tag{1.442}$$

where $\mathcal{A}$ is the transformation matrix between the original set and the principal space, made up of the eigenvectors $\hat{\mathbf{n}}^{(a)}$. The above equation can be rewritten in terms of components as follows:

$$\begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33} \end{bmatrix} = \mathcal{A}^T \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathcal{A} + \mathcal{A}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} \mathcal{A} \tag{1.443}$$

or

$$\begin{aligned} \begin{bmatrix} \mathsf{T}_{11} & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \mathsf{T}_{22} & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \mathsf{T}_{33} \end{bmatrix} &= \mathsf{T}_1 \begin{bmatrix} a_{11}^2 & a_{11}a_{12} & a_{11}a_{13} \\ a_{11}a_{12} & a_{12}^2 & a_{12}a_{13} \\ a_{11}a_{13} & a_{12}a_{13} & a_{13}^2 \end{bmatrix} + \mathsf{T}_2 \begin{bmatrix} a_{21}^2 & a_{21}a_{22} & a_{21}a_{23} \\ a_{21}a_{22} & a_{22}^2 & a_{22}a_{23} \\ a_{21}a_{23} & a_{22}a_{23} & a_{23}^2 \end{bmatrix} \\ &+ \mathsf{T}_3 \begin{bmatrix} a_{31}^2 & a_{31}a_{32} & a_{31}a_{33} \\ a_{31}a_{32} & a_{32}^2 & a_{32}a_{33} \\ a_{31}a_{33} & a_{32}a_{33} & a_{33}^2 \end{bmatrix} \end{aligned} \tag{1.444}$$

By regarding how second-order tensors are presented in Voigt Notation as in (1.438), the spectral representation of a second-order tensor in Voigt notation becomes:

$$\{\mathcal{T}\} = \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{33} \\ \mathsf{T}_{12} \\ \mathsf{T}_{23} \\ \mathsf{T}_{13} \end{bmatrix} = \mathsf{T}_1 \begin{bmatrix} a_{11}^2 \\ a_{12}^2 \\ a_{13}^2 \\ a_{11}a_{12} \\ a_{12}a_{13} \\ a_{11}a_{13} \end{bmatrix} + \mathsf{T}_2 \begin{bmatrix} a_{21}^2 \\ a_{22}^2 \\ a_{23}^2 \\ a_{21}a_{22} \\ a_{22}a_{23} \\ a_{21}a_{23} \end{bmatrix} + \mathsf{T}_3 \begin{bmatrix} a_{31}^2 \\ a_{32}^2 \\ a_{33}^2 \\ a_{31}a_{32} \\ a_{32}a_{33} \\ a_{31}a_{33} \end{bmatrix} \tag{1.445}$$

1.7.5 Deviatoric Tensor Components in Voigt Notation

Observing the components of the deviatoric tensor:

$$\mathsf{T}_{ij}^{dev} = \begin{bmatrix} \frac{1}{3}(2\mathsf{T}_{11} - \mathsf{T}_{22} - \mathsf{T}_{33}) & \mathsf{T}_{12} & \mathsf{T}_{13} \\ \mathsf{T}_{12} & \frac{1}{3}(2\mathsf{T}_{22} - \mathsf{T}_{11} - \mathsf{T}_{33}) & \mathsf{T}_{23} \\ \mathsf{T}_{13} & \mathsf{T}_{23} & \frac{1}{3}(2\mathsf{T}_{33} - \mathsf{T}_{11} - \mathsf{T}_{22}) \end{bmatrix} \tag{1.446}$$

T_{ij}^{dev} in Voigt notation is given by:

$$\begin{bmatrix} \mathsf{T}_{11}^{dev} \\ \mathsf{T}_{22}^{dev} \\ \mathsf{T}_{33}^{dev} \\ \mathsf{T}_{12}^{dev} \\ \mathsf{T}_{23}^{dev} \\ \mathsf{T}_{13}^{dev} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{33} \\ \mathsf{T}_{12} \\ \mathsf{T}_{23} \\ \mathsf{T}_{13} \end{bmatrix} \tag{1.447}$$

Problem 1.40: Let $\mathbf{T}(\vec{x},t)$ be a symmetric second-order tensor, which is expressed in terms of the position $(\vec{x})$ and time (t). Also, bear in mind that the tensor components, along direction x_3, are equal to zero, *i.e.* $\mathsf{T}_{13} = \mathsf{T}_{23} = \mathsf{T}_{33} = 0$.

NOTE: In the next section we will define $\mathbf{T}(\vec{x},t)$ as a field tensor, *i.e.* the value of $\mathbf{T}$ depends on position and time. As we will see later, if the tensor is independent of any one direction at all points $(\vec{x})$, *e.g.* if $\mathbf{T}(\vec{x},t)$ is independent of the x_3-direction, (see Figure 1.31), the problem becomes a two-dimensional problem (plane state) so that the problem is greatly simplified. ■

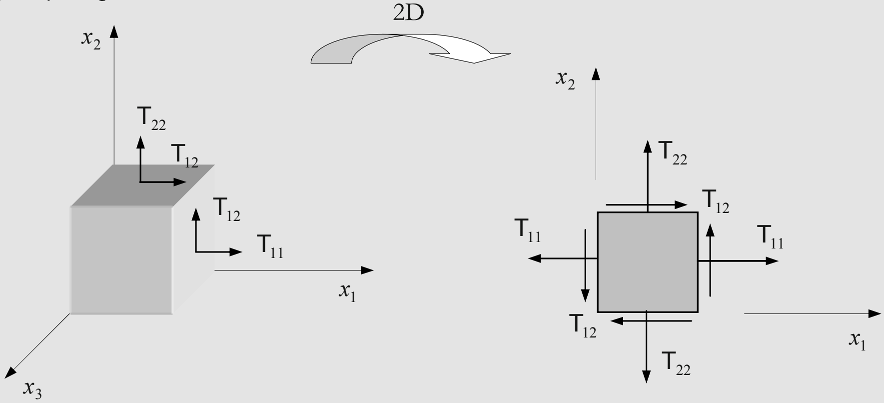


Figure 1.31: A two-dimensional problem (2D).

a) Obtain T'_{11}, T'_{22}, T'_{12} in the new reference system $(x'_1 - x'_2)$ defined in Figure 1.32.
b) Obtain the value of θ so that θ corresponds to the principal direction of $\mathbf{T}$, and also find an equation for the principal values of $\mathbf{T}$.
c) Evaluate the values of T'_{ij}, $(i,j=1,2)$, when $\mathsf{T}_{11}=1$, $\mathsf{T}_{22}=2$, $\mathsf{T}_{12}=-4$ and $\theta=45^{\circ}$. Also, obtain the principal values and principal directions.
d) Draw a graph that shows the relationship between θ and components T'_{11}, T'_{22} and T'_{12}, and in which the angle varies from 0° to 360°.
Hint: Use the Voigt Notation, and express the results in terms of 2θ.

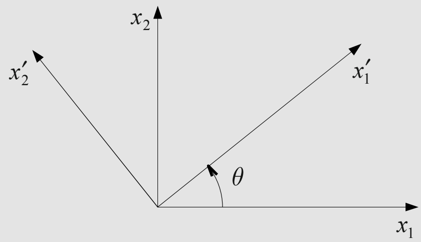


Figure 1.32: A two-dimensional problem (2D).

Solution:

a) Here we can apply the transformation law obtained in (1.437), which after removing rows and columns associated with the x_3-direction becomes:

$$\begin{bmatrix} \mathsf{T}'_{11} \\ \mathsf{T}'_{22} \\ \mathsf{T}'_{12} \end{bmatrix} = \begin{bmatrix} a_{11}{}^2 & a_{12}{}^2 & 2a_{11}a_{12} \\ a_{21}{}^2 & a_{22}{}^2 & 2a_{21}a_{22} \\ a_{21}a_{11} & a_{22}a_{12} & a_{11}a_{22} + a_{12}a_{21} \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix} \tag{1.448}$$

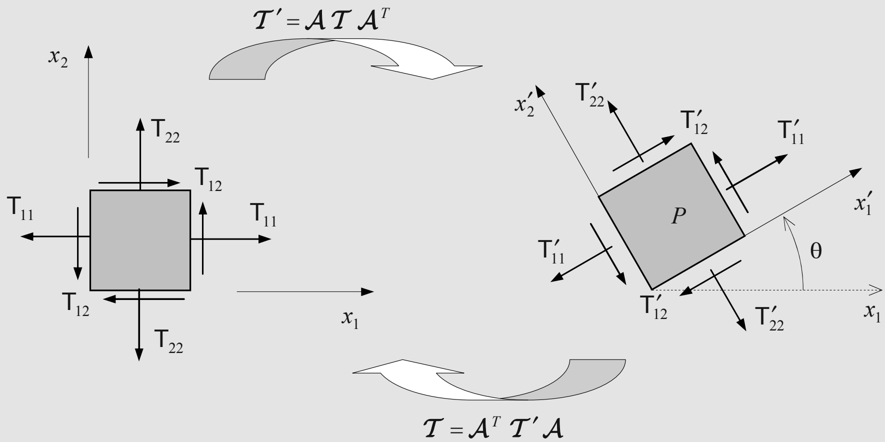


Figure 1.33: Transformation law for (2D) tensor components.

The transformation matrix, a_{ij}, in the plane, can be evaluated in terms of a single parameter, θ:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.449}$$

By substituting the matrix components a_{ij} given in (1.449) into (1.448) we obtain:

$$\begin{bmatrix} \mathsf{T}'_{11} \\ \mathsf{T}'_{22} \\ \mathsf{T}'_{12} \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{bmatrix} \mathsf{T}_{11} \\ \mathsf{T}_{22} \\ \mathsf{T}_{12} \end{bmatrix} \tag{1.450}$$

Making use of the following trigonometric identities, $2\cos\theta\sin\theta = \sin 2\theta$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, (1.450) becomes:

$$\begin{bmatrix} T'_{11} \\ T'_{22} \\ T'_{12} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{1+\cos 2\theta}{2}\right) & \left(\dfrac{1-\cos 2\theta}{2}\right) & \sin 2\theta \\ \left(\dfrac{1-\cos 2\theta}{2}\right) & \left(\dfrac{1+\cos 2\theta}{2}\right) & -\sin 2\theta \\ \left(-\dfrac{\sin 2\theta}{2}\right) & \left(\dfrac{\sin 2\theta}{2}\right) & \cos 2\theta \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix}$$

Explicitly, the above components are given by:

$$\begin{cases} T'_{11} = \left(\dfrac{1+\cos 2\theta}{2}\right) T_{11} + \left(\dfrac{1-\cos 2\theta}{2}\right) T_{22} + T_{12}\sin 2\theta \\ T'_{22} = \left(\dfrac{1-\cos 2\theta}{2}\right) T_{11} + \left(\dfrac{1+\cos 2\theta}{2}\right) T_{22} - T_{12}\sin 2\theta \\ T'_{12} = \left(-\dfrac{\sin 2\theta}{2}\right) T_{11} + \left(\dfrac{\sin 2\theta}{2}\right) T_{22} + T_{12}\cos 2\theta \end{cases}$$

Reordering the previous equation, we obtain:

$$\boxed{\begin{cases} T'_{11} = \left(\dfrac{T_{11}+T_{22}}{2}\right) + \left(\dfrac{T_{11}-T_{22}}{2}\right)\cos 2\theta + T_{12}\sin 2\theta \\ T'_{22} = \left(\dfrac{T_{11}+T_{22}}{2}\right) - \left(\dfrac{T_{11}-T_{22}}{2}\right)\cos 2\theta - T_{12}\sin 2\theta \\ T'_{12} = -\left(\dfrac{T_{11}-T_{22}}{2}\right)\sin 2\theta + T_{12}\cos 2\theta \end{cases}} \tag{1.451}$$

b) Recalling that the principal directions are characterized by the lack of any tangential components, *i.e.* $T_{ij} = 0$ if $i \neq j$, in order to find the principal directions in the plane, we let $T'_{12} = 0$, hence:

$$T'_{12} = -\left(\frac{T_{11}-T_{22}}{2}\right)\sin 2\theta + T_{12}\cos 2\theta = 0 \Rightarrow \left(\frac{T_{11}-T_{22}}{2}\right)\sin 2\theta = T_{12}\cos 2\theta$$

$$\Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = \frac{2T_{12}}{T_{11}-T_{22}} \Rightarrow \mathrm{tg}(2\theta) = \frac{2T_{12}}{T_{11}-T_{22}}$$

Then, the angle corresponding to the principal direction is:

$$\boxed{\theta = \frac{1}{2}\mathrm{arctg}\left(\frac{2T_{12}}{T_{11}-T_{22}}\right)} \tag{1.452}$$

To find the principal values (eigenvalues) we must solve the following characteristic equation:

$$\begin{vmatrix} T_{11}-T & T_{12} \\ T_{12} & T_{22}-T \end{vmatrix} = 0 \quad \Rightarrow \quad T^2 - T(T_{11}+T_{22}) + (T_{11}T_{22} - T_{12}^2) = 0$$

And by evaluating the quadratic equation we obtain:

$$\begin{aligned} T_{(1,2)} &= \frac{-[-(T_{11}+T_{22})] \pm \sqrt{[-(T_{11}+T_{22})]^2 - 4(1)(T_{11}T_{22} - T_{12}^2)}}{2(1)} \\ &= \frac{T_{11}+T_{22}}{2} \pm \sqrt{\frac{[(T_{11}+T_{22})]^2 - 4(T_{11}T_{22} - T_{12}^2)}{4}} \end{aligned}$$

By rearranging the above equation we obtain the principal values for the two-dimensional case as:

$$\boxed{\mathsf{T}_{(1,2)} = \frac{\mathsf{T}_{11} + \mathsf{T}_{22}}{2} \pm \sqrt{\left(\frac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)^2 + \mathsf{T}_{12}^2}} \tag{1.453}$$

c) We directly apply equation (1.451) to evaluate the values of the components T'_{ij}, $(i,j=1,2)$, where $\mathsf{T}_{11}=1$, $\mathsf{T}_{22}=2$, $\mathsf{T}_{12}=-4$ and $\theta=45^\circ$, *i.e.*:

$$\begin{cases} \mathsf{T}'_{11} = \left(\dfrac{1+2}{2}\right) + \left(\dfrac{1-2}{2}\right)\cos 90^\circ - 4\sin 90^\circ = -2.5 \\ \mathsf{T}'_{22} = \left(\dfrac{1+2}{2}\right) - \left(\dfrac{1-2}{2}\right)\cos 90^\circ + 4\sin 90^\circ = 5.5 \\ \mathsf{T}'_{12} = -\left(\dfrac{1-2}{2}\right)\sin 90^\circ - 4\cos 90^\circ = 0.5 \end{cases}$$

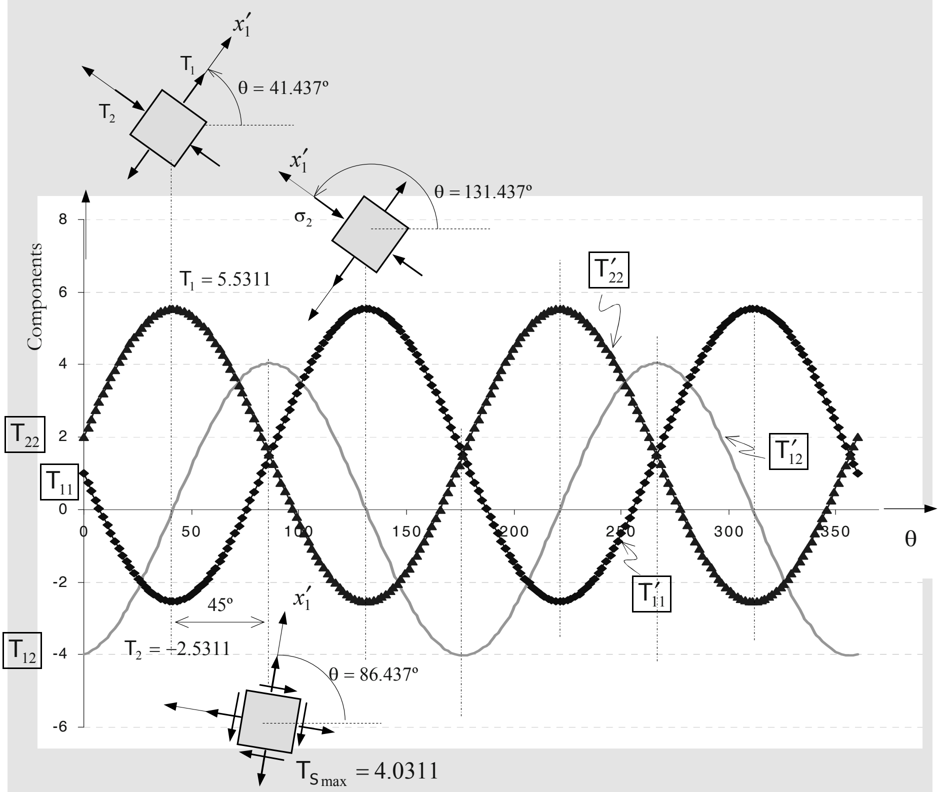
And the angle corresponding to the principal direction is:

$$\theta = \frac{1}{2}\operatorname{arctg}\left(\frac{2\mathsf{T}_{12}}{\mathsf{T}_{11} - \mathsf{T}_{22}}\right) = \frac{2\times(-4)}{1-2} \quad \Rightarrow \quad (\theta = 41.4375^\circ)$$

The principal values of $\mathbf{T}(\vec{x},t)$ can be evaluated as follows:

$$\mathsf{T}_{(1,2)} = \frac{\mathsf{T}_{11} + \mathsf{T}_{22}}{2} \pm \sqrt{\left(\frac{\mathsf{T}_{11} - \mathsf{T}_{22}}{2}\right)^2 + \mathsf{T}_{12}^2} \quad \Rightarrow \quad \begin{cases} \mathsf{T}_1 = 5.5311 \\ \mathsf{T}_2 = -2.5311 \end{cases}$$

d) By referring to equation in (1.451) and by varying θ from 0° to 360°, we can obtain different values of T'_{11}, T'_{22}, T'_{12}, which are illustrated in the following graph:



1.8 Tensor Fields

A *tensor field* indicates how the tensor, $\mathbf{T}(\vec{x},t)$, varies in space ($\vec{x}$) and time (t). In this section, we regard the tensor field as a differentiable function of position and time. For more information about it, we need to define some operators, *e.g. gradient*, *divergence*, *curl*, which we can use as indicators of how these fields vary in space.

A tensor field which is independent of time is called a stationary or steady-state tensor field, *i.e.* $\mathbf{T} = \mathbf{T}(\vec{x})$. However, if the field is only dependent on t then it is said to be *homogeneous* or *uniform*. That is, $\mathbf{T}(t)$ has the same value at every $\vec{x}$ position.

Tensor fields can be classified according to their order as: scalar, vector, second-order tensor fields, etc. As an example of a scalar field we can quote temperature $T(\vec{x},t)$ and in Figure 1.34(a) we can see temperature distribution over time $t = t_1$. Then, as an example of a vector field we can quote velocity $\vec{v}(\vec{x},t)$ and Figure 1.34(b) shows velocity distribution, in which each point is associated with a vector $\vec{v}$ over time $t = t_1$.

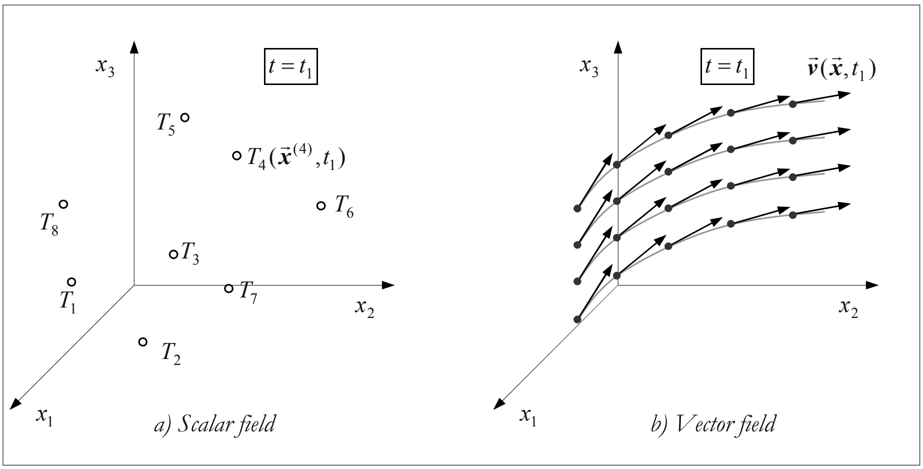


Figure 1.34: Examples of tensor fields.

Scalar Field

$$\phi = \phi(\vec{x},t) \tag{1.454}$$

Vector Field

Tensorial notation $\quad \vec{\mathbf{v}} = \vec{\mathbf{v}}(\vec{x},t)$

Indicial notation $\quad \mathsf{v}_i = \mathsf{v}_i(\vec{x},t)$ (1.455)

Second-Order Tensor Field

Tensorial notation $\quad \mathbf{T} = \mathbf{T}(\vec{x},t)$

Indicial notation $\quad \mathsf{T}_{ij} = \mathsf{T}_{ij}(\vec{x},t)$ (1.456)

1.8.1 Scalar Fields

The next analysis is carry out with reference to a stationary scalar field, *i.e.* $\phi = \phi(\vec{\boldsymbol{x}})$, with continuous values of $\partial\phi / \partial x_1$, $\partial\phi / \partial x_2$ and $\partial\phi / \partial x_3$. Then, observe that the value of the scalar function at point $(\vec{\boldsymbol{x}})$ is $\phi(\vec{\boldsymbol{x}})$, and if we observe a second point located at $(\vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}})$, the *total derivative* (*differential*) of the function ϕ is defined as:

$$\begin{gathered} \phi(\vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}) - \phi(\vec{\boldsymbol{x}}) \equiv d\phi \\ \phi(x_1 + dx_1,\ x_2 + dx_2,\ x_3 + dx_3) - \phi(x_1,\ x_2,\ x_3) \equiv d\phi \end{gathered} \tag{1.457}$$

For any continuous function $\phi(x_1, x_2, x_3)$, $d\phi$ is linearly related to dx_1, dx_2, dx_3. This linear relationship can be evaluated by the chain rule of differentiation as:

$$\begin{gathered} d\phi = \frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2 + \frac{\partial\phi}{\partial x_3} dx_3 \\ d\phi = \phi_{,i}\, dx_i \end{gathered} \tag{1.458}$$

The differentiation of the components of a tensor, with respect to coordinates x_i, is expressed by the differential operator:

$$\frac{\partial \bullet}{\partial x_i} \equiv \bullet_{,i} \tag{1.459}$$

1.8.2 Gradient

The gradient of a scalar field

The gradient $\nabla_{\vec{x}}\phi$ or $\mathsf{grad}\phi$ is defined as:

$$\nabla_{\vec{x}}\phi \longrightarrow d\phi = \nabla_{\vec{x}}\phi \cdot d\vec{\boldsymbol{x}} \tag{1.460}$$

where the operator $\nabla_{\vec{x}}$ is known as the *Nabla symbol*. Expressing the equation (1.460) in the Cartesian basis we obtain:

$$\begin{aligned} &\frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2 + \frac{\partial\phi}{\partial x_3} dx_3 = \\ &\quad = \left[(\nabla_{\vec{x}}\phi)_{x_1} \hat{\mathbf{e}}_1 + (\nabla_{\vec{x}}\phi)_{x_2} \hat{\mathbf{e}}_2 + (\nabla_{\vec{x}}\phi)_{x_3} \hat{\mathbf{e}}_3\right] \cdot \left[(dx_1)\hat{\mathbf{e}}_1 + (dx_2)\hat{\mathbf{e}}_2 + (dx_3)\hat{\mathbf{e}}_3\right] \end{aligned} \tag{1.461}$$

Evaluating the above scalar product we find:

$$\frac{\partial\phi}{\partial x_1} dx_1 + \frac{\partial\phi}{\partial x_2} dx_2 + \frac{\partial\phi}{\partial x_3} dx_3 = (\nabla_{\vec{x}}\phi)_{x_1} dx_1 + (\nabla_{\vec{x}}\phi)_{x_2} dx_2 + (\nabla_{\vec{x}}\phi)_{x_3} dx_3 \tag{1.462}$$

Therefore, we can draw the conclusion that the $\nabla_{\vec{x}}\phi$ components in the Cartesian basis are:

$$(\nabla_{\vec{x}}\phi)_1 \equiv \frac{\partial\phi}{\partial x_1} \quad ; \quad (\nabla_{\vec{x}}\phi)_2 \equiv \frac{\partial\phi}{\partial x_2} \quad ; \quad (\nabla_{\vec{x}}\phi)_3 \equiv \frac{\partial\phi}{\partial x_3} \tag{1.463}$$

Hence, the gradient in terms of components is defined as such:

$$\nabla_{\vec{x}}\phi = \frac{\partial\phi}{\partial x_1} \hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial x_2} \hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial x_3} \hat{\mathbf{e}}_3 \tag{1.464}$$

The Nabla symbol $\nabla_{\vec{x}}$ is defined as:

$$\boxed{\nabla_{\vec{x}} = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i \equiv \partial_{,i}\,\hat{\mathbf{e}}_i} \quad \textit{Nabla symbol} \tag{1.465}$$

The geometric meaning of $\nabla_{\vec{x}}\phi$

- The direction of $\nabla_{\vec{x}}\phi$ is normal to the equiscalar surface, *i.e.* it is perpendicular to the isosurface $\phi = const$. The direction of $\nabla_{\vec{x}}\phi$ points to the direction where ϕ is increasing the most, (see Figure 1.35).
- The magnitude of $\nabla_{\vec{x}}\phi$ is the rate of change of ϕ, *i.e.* the gradient of ϕ.

The normal vector to this surface is obtained as follows:

$$\hat{\mathbf{n}} = \frac{\nabla_{\vec{x}}\phi}{\|\nabla_{\vec{x}}\phi\|} \tag{1.466}$$

The surface $\phi = const$, called the surface level, or isosurface or equiscalar surface, is the surface formed by points which all have the same value of ϕ, so, if we move along the level surface the values of the function do not change.

The gradient of a vector field $\vec{\mathbf{v}}(\vec{x})$:

$$\mathrm{grad}(\vec{\mathbf{v}}) \equiv \nabla_{\vec{x}}\vec{\mathbf{v}} \tag{1.467}$$

Using the definition of $\nabla_{\vec{x}}$, given in (1.465), the gradient of the vector field becomes:

$$\nabla_{\vec{x}}\vec{\mathbf{v}} = \frac{\partial(\mathrm{v}_i\hat{\mathbf{e}}_i)}{\partial x_j} \otimes \hat{\mathbf{e}}_j = (\mathrm{v}_i\hat{\mathbf{e}}_i)_{,j} \otimes \hat{\mathbf{e}}_j = \mathrm{v}_{i,j}\,\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \tag{1.468}$$

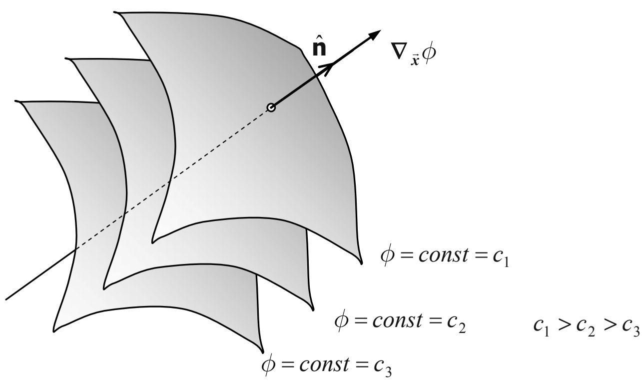


Figure 1.35: Gradient of ϕ.

Therefore, we can define the gradient of a tensor field $(\bullet(\vec{x},t))$ in the Cartesian basis as:

$$\boxed{\nabla_{\vec{x}}(\bullet) = \frac{\partial(\bullet)}{\partial x_j} \otimes \hat{\mathbf{e}}_j} \quad \textit{Gradient of a tensor field in the Cartesian basis} \tag{1.469}$$

As noted, the gradient of a vector field becomes a second-order tensor field, whose components are:

$$\mathrm{v}_{i,j} \equiv \frac{\partial \mathrm{v}_i}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{v}_1}{\partial x_1} & \frac{\partial \mathrm{v}_1}{\partial x_2} & \frac{\partial \mathrm{v}_1}{\partial x_3} \\ \frac{\partial \mathrm{v}_2}{\partial x_1} & \frac{\partial \mathrm{v}_2}{\partial x_2} & \frac{\partial \mathrm{v}_2}{\partial x_3} \\ \frac{\partial \mathrm{v}_3}{\partial x_1} & \frac{\partial \mathrm{v}_3}{\partial x_2} & \frac{\partial \mathrm{v}_3}{\partial x_3} \end{bmatrix} \tag{1.470}$$

The gradient of a second-order tensor field $\mathbf{T}(\vec{x})$:

$$\nabla_{\vec{x}} \mathbf{T} = \frac{\partial (T_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)}{\partial x_k} \otimes \hat{\mathbf{e}}_k = T_{ij,k} \; \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \tag{1.471}$$

and its components are represented by:

$$(\nabla_{\vec{x}} \mathbf{T})_{ijk} \equiv T_{ij,k} \tag{1.472}$$

Problem 1.41: Find the gradient of the function $f(x_1, x_2) = \cos(x_1) + \exp^{x_1 x_2}$ at the point $(x_1 = 0, x_2 = 1)$.

Solution: By definition, the gradient of a scalar function is given by:

$$\nabla_{\vec{x}} f = \frac{\partial f}{\partial x_1} \; \hat{\mathbf{e}}_1 + \frac{\partial f}{\partial x_2} \; \hat{\mathbf{e}}_2$$

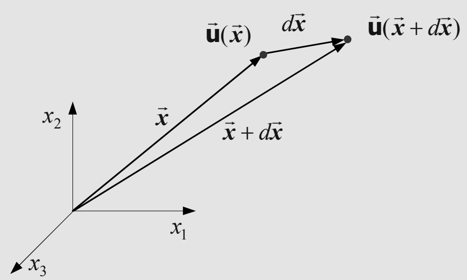
where: $\frac{\partial f}{\partial x_1} = -\sin(x_1) + x_2 \exp^{x_1 x_2}$; $\frac{\partial f}{\partial x_2} = x_1 \exp^{x_1 x_2}$

$$\nabla_{\vec{x}} f(x_1, x_2) = \left[-\sin(x_1) + x_2 \exp^{x_1 x_2} \right] \hat{\mathbf{e}}_1 + \left[x_1 \exp^{x_1 x_2} \right] \hat{\mathbf{e}}_2 \Rightarrow \nabla_{\vec{x}} f(0,1) = [1] \; \hat{\mathbf{e}}_1 + [0] \; \hat{\mathbf{e}}_2 = 2 \hat{\mathbf{e}}_1$$

Problem 1.42: Let $\vec{\mathbf{u}}(\vec{x})$ be a stationary vector field. a) Obtain the components of the differential $d\vec{\mathbf{u}}$. b) Now, consider that $\vec{\mathbf{u}}(\vec{x})$ represents a displacement field, and is independent of x_3. With these conditions, graphically illustrate the displacement field in the differential area element $dx_1 dx_2$.

Solution: According to the differential and gradient definitions, it holds that:

$$\boxed{\begin{aligned} d\vec{\mathbf{u}} &\equiv \vec{\mathbf{u}}(\vec{x} + d\vec{x}) - \vec{\mathbf{u}}(\vec{x}) \\ d\vec{\mathbf{u}} &= \nabla_{\vec{x}} \vec{\mathbf{u}} \cdot d\vec{x} \end{aligned}}$$



Thus, the components are defined as:

$$d\mathrm{u}_i = \frac{\partial \mathrm{u}_i}{\partial x_j} dx_j \quad \Rightarrow \quad \begin{bmatrix} d\mathrm{u}_1 \\ d\mathrm{u}_2 \\ d\mathrm{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial x_1} & \frac{\partial \mathrm{u}_1}{\partial x_2} & \frac{\partial \mathrm{u}_1}{\partial x_3} \\ \frac{\partial \mathrm{u}_2}{\partial x_1} & \frac{\partial \mathrm{u}_2}{\partial x_2} & \frac{\partial \mathrm{u}_2}{\partial x_3} \\ \frac{\partial \mathrm{u}_3}{\partial x_1} & \frac{\partial \mathrm{u}_3}{\partial x_2} & \frac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

or:

$$\begin{cases} d\mathrm{u}_1 = \dfrac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_1}{\partial x_2} dx_2 + \dfrac{\partial \mathrm{u}_1}{\partial x_3} dx_3 \\ d\mathrm{u}_2 = \dfrac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_2}{\partial x_2} dx_2 + \dfrac{\partial \mathrm{u}_2}{\partial x_3} dx_3 \\ d\mathrm{u}_3 = \dfrac{\partial \mathrm{u}_3}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_3}{\partial x_2} dx_2 + \dfrac{\partial \mathrm{u}_3}{\partial x_3} dx_3 \end{cases}$$

with

$$\begin{cases} d\mathrm{u}_1 = \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_1(x_1, x_2, x_3) \\ d\mathrm{u}_2 = \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_2(x_1, x_2, x_3) \\ d\mathrm{u}_3 = \mathrm{u}_3(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - \mathrm{u}_3(x_1, x_2, x_3) \end{cases}$$

As the field is independent of x_3, the displacement field in the differential area element is defined as:

$$\begin{cases} d\mathrm{u}_1 = \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2) - \mathrm{u}_1(x_1, x_2) = \dfrac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_1}{\partial x_2} dx_2 \\ d\mathrm{u}_2 = \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2) - \mathrm{u}_2(x_1, x_2) = \dfrac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_2}{\partial x_2} dx_2 \end{cases}$$

or:

$$\begin{cases} \mathrm{u}_1(x_1 + dx_1, x_2 + dx_2) = \mathrm{u}_1(x_1, x_2) + \dfrac{\partial \mathrm{u}_1}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_1}{\partial x_2} dx_2 \\ \mathrm{u}_2(x_1 + dx_1, x_2 + dx_2) = \mathrm{u}_2(x_1, x_2) + \dfrac{\partial \mathrm{u}_2}{\partial x_1} dx_1 + \dfrac{\partial \mathrm{u}_2}{\partial x_2} dx_2 \end{cases}$$

Note that the above equation is equivalent to the Taylor series expansion taking into account only up to linear terms. The representation of the displacement field in the differential area element is shown in Figure 1.36.

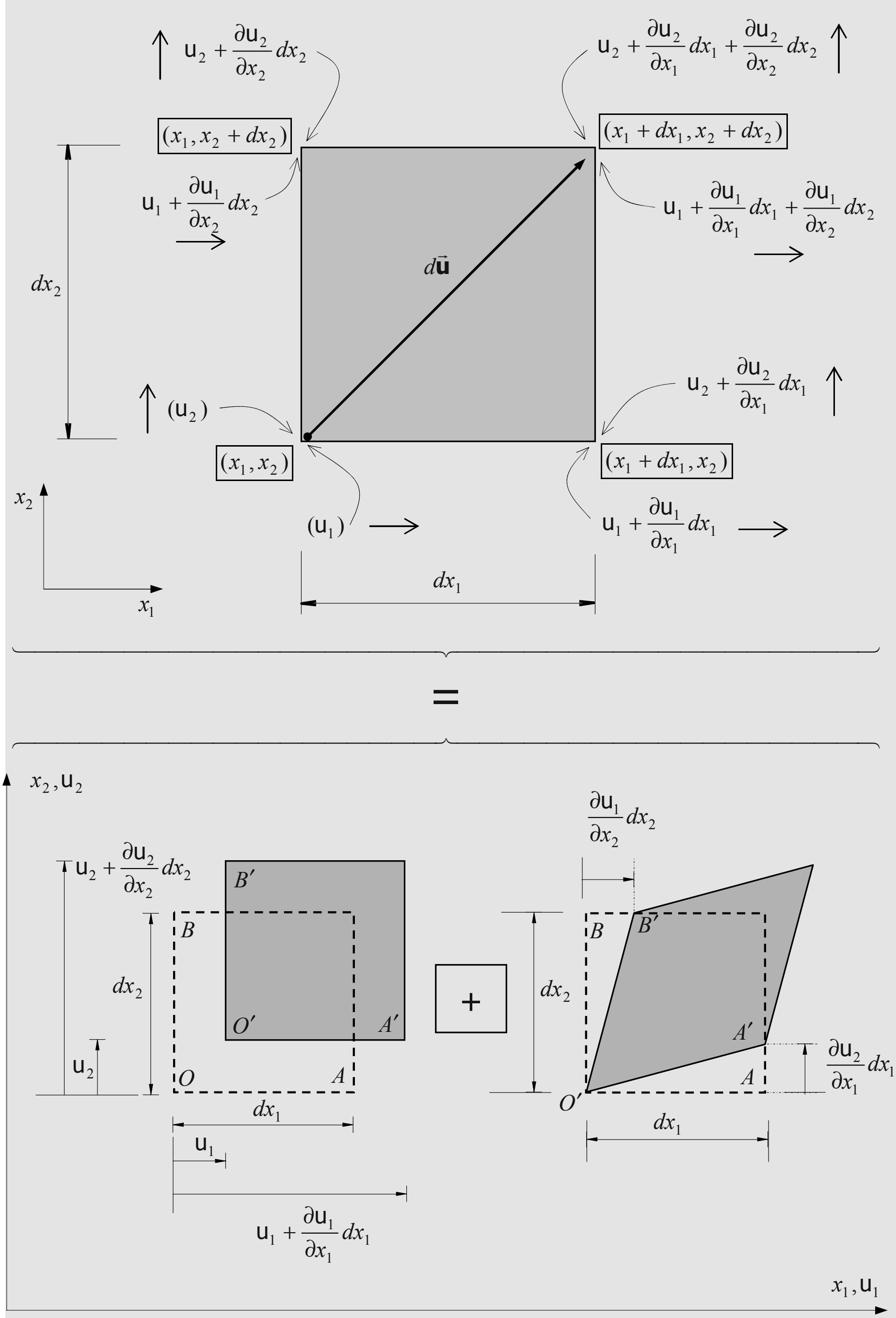


Figure 1.36: Displacement field in the differential area element.

1.8.3 Divergence

The **divergence of a vector field**, $\vec{\mathbf{v}}(\vec{x})$, is denoted as follows:

$$\operatorname{div}(\vec{\mathbf{v}}) \equiv \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} \tag{1.473}$$

which by definition is:

$$\operatorname{div}(\vec{\mathbf{v}}) \equiv \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} = \nabla_{\vec{x}} \vec{\mathbf{v}} : \mathbf{1} = \operatorname{Tr}(\nabla_{\vec{x}} \vec{\mathbf{v}}) \tag{1.474}$$

Then:

$$\begin{aligned} \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} = \nabla_{\vec{x}} \vec{\mathbf{v}} : \mathbf{1} &= \left[\mathrm{v}_{i,j}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right] : \left[\delta_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l\right] = \mathrm{v}_{i,j} \delta_{kl} \delta_{ik} \delta_{jl} = \mathrm{v}_{k,k} \\ &= \frac{\partial \mathrm{v}_1}{\partial x_1} + \frac{\partial \mathrm{v}_2}{\partial x_2} + \frac{\partial \mathrm{v}_3}{\partial x_3} \end{aligned} \tag{1.475}$$

or

$$\begin{aligned} \nabla_{\vec{x}} \cdot \vec{\mathbf{v}} = \nabla_{\vec{x}} \vec{\mathbf{v}} : \mathbf{1} &= \left[\mathrm{v}_{i,j}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right] : \left[\delta_{kl} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l\right] \\ &= \left[\mathrm{v}_{i,j} \delta_{kl} \delta_{lj} \hat{\mathbf{e}}_i\right] \cdot \hat{\mathbf{e}}_k \\ &= \left[\mathrm{v}_{i,k} \hat{\mathbf{e}}_i\right] \cdot \hat{\mathbf{e}}_k \\ &= \frac{\partial \left[\mathrm{v}_i \hat{\mathbf{e}}_i\right]}{\partial x_k} \cdot \hat{\mathbf{e}}_k \end{aligned} \tag{1.476}$$

Which we can use to insert the following operator into the Cartesian basis:

$$\boxed{\nabla_{\vec{x}} \cdot (\bullet) = \frac{\partial (\bullet)}{\partial x_k} \cdot \hat{\mathbf{e}}_k} \quad \textit{Divergence of } (\bullet) \textit{ in Cartesian basis} \tag{1.477}$$

We can also verify that, when divergence is applied to a tensor field its rank decreases by one order.

Divergence of a second-order tensor field $\mathbf{T}(\vec{x})$

The divergence of a second-order tensor field $\mathbf{T}$ is denoted by $\nabla_{\vec{x}} \cdot \mathbf{T} = \nabla_{\vec{x}} \mathbf{T} : \mathbf{1}$, which becomes a vector:

$$\begin{aligned} \nabla_{\vec{x}} \cdot \mathbf{T} \equiv \operatorname{div}\mathbf{T} &= \frac{\partial (T_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)}{\partial x_k} \cdot \hat{\mathbf{e}}_k \\ &= \frac{\partial T_{ij}}{\partial x_k} \delta_{jk} \hat{\mathbf{e}}_i \\ &= \mathrm{T}_{ik,k} \hat{\mathbf{e}}_i \end{aligned} \tag{1.478}$$

NOTE: In this text book, when dealing with *gradient* or *divergence* of a tensor field, *e.g.* $\nabla_{\vec{x}} \vec{\mathbf{v}}$ (the gradient of the vector field), $\nabla_{\vec{x}} \mathbf{T}$ (the gradient of a second-order tensor field), $\nabla_{\vec{x}} \cdot \mathbf{T}$ (divergence of a second-order tensor field), this does not indicate that we are making a tensor operation between a vector and a tensor, *i.e.* $\nabla_{\vec{x}} \vec{\mathbf{v}} \neq (\vec{\nabla}_{\vec{x}}) \otimes (\vec{\mathbf{v}})$, $\nabla_{\vec{x}} \mathbf{T} \neq (\vec{\nabla}_{\vec{x}}) \otimes (\mathbf{T})$ and $\nabla_{\vec{x}} \cdot \mathbf{T} \neq (\vec{\nabla}_{\vec{x}}) \cdot (\mathbf{T})$ and so on. In this textbook, $\nabla_{\vec{x}}$ is an operator which must be applied to the entire tensor field, so, the tensor must be inside the operator, (see equations (1.477) and (1.469)). Nevertheless, it is possible to relate $\nabla_{\vec{x}} \vec{\mathbf{v}}$, $\nabla_{\vec{x}} \mathbf{T}$ or $\nabla_{\vec{x}} \cdot \mathbf{T}$ to tensor operations between tensors, and it is easy to show that:

$$
\begin{aligned}
&\nabla_{\vec{x}}\vec{\mathbf{v}} = (\vec{\mathbf{v}})\otimes(\vec{\nabla}_{\vec{x}}) \\
&\nabla_{\vec{x}}\mathbf{T} = (\mathbf{T})\otimes(\vec{\nabla}_{\vec{x}}) \\
&\nabla_{\vec{x}}\cdot\mathbf{T} = (\mathbf{T})\cdot(\vec{\nabla}_{\vec{x}}) = (\vec{\nabla}_{\vec{x}})\cdot(\mathbf{T}^T) \quad \blacksquare
\end{aligned}
\tag{1.479}
$$

Once the Nabla symbol is defined we introduce the *Laplacian operator* ∇^2 as:

$$
\begin{aligned}
&\nabla_{\vec{x}}^{\ 2} = \nabla_{\vec{x}}\cdot\nabla_{\vec{x}} = \left(\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\hat{\mathbf{e}}_j\right)\cdot\hat{\mathbf{e}}_i\right) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\delta_{ij} = \frac{\partial^2}{\partial x_i\partial x_i} \\
&\nabla_{\vec{x}}^{\ 2} \equiv \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2} = \partial,_k\,\partial,_k = \partial,_{kk}
\end{aligned}
\tag{1.480}
$$

Then, the vector Laplacian of a vector field, $\vec{\mathbf{v}}(\vec{x})$, is given by:

$$
\nabla_{\vec{x}}^{\ 2}\vec{\mathbf{v}} = \nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\vec{\mathbf{v}}) \xrightarrow{\text{components}} \left[\nabla_{\vec{x}}^{\ 2}\vec{\mathbf{v}}\right]_i = \left[\nabla_{\vec{x}}\cdot(\nabla_{\vec{x}}\vec{\mathbf{v}})\right]_i = \mathrm{v}_{i,kk}
\tag{1.481}
$$

Problem 1.43: Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be vectors. Show that the following identity $\nabla_{\vec{x}}\cdot(\vec{\mathbf{a}}+\vec{\mathbf{b}}) = \nabla_{\vec{x}}\cdot\vec{\mathbf{a}}+\nabla_{\vec{x}}\cdot\vec{\mathbf{b}}$ holds.

Solution:

Observing that $\vec{\mathbf{a}} = \mathrm{a}_j\hat{\mathbf{e}}_j$, $\vec{\mathbf{b}} = \mathrm{b}_k\hat{\mathbf{e}}_k$, $\nabla_{\vec{x}} = \hat{\mathbf{e}}_i\dfrac{\partial}{\partial x_i}$, we can express $\nabla_{\vec{x}}\cdot(\vec{\mathbf{a}}+\vec{\mathbf{b}})$ as:

$$
\frac{\partial(\mathrm{a}_j\hat{\mathbf{e}}_j+\mathrm{b}_k\hat{\mathbf{e}}_k)}{\partial x_i}\cdot\hat{\mathbf{e}}_i = \frac{\partial\mathrm{a}_j}{\partial x_i}\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_i+\frac{\partial\mathrm{b}_k}{\partial x_i}\hat{\mathbf{e}}_k\cdot\hat{\mathbf{e}}_i = \frac{\partial\mathrm{a}_i}{\partial x_i}+\frac{\partial\mathrm{b}_i}{\partial x_i} = \nabla_{\vec{x}}\cdot\vec{\mathbf{a}}+\nabla_{\vec{x}}\cdot\vec{\mathbf{b}}
$$

Working directly with indicial notation we obtain:

$$
\nabla_{\vec{x}}\cdot(\vec{\mathbf{a}}+\vec{\mathbf{b}}) = (\mathrm{a}_i+\mathrm{b}_i),_i = \mathrm{a}_i,_i+\mathrm{b}_i,_i = \nabla_{\vec{x}}\cdot\vec{\mathbf{a}}+\nabla_{\vec{x}}\cdot\vec{\mathbf{b}}
$$

Problem 1.44: Find the components of $(\nabla_{\vec{x}}\vec{\mathbf{a}})\cdot\vec{\mathbf{b}}$.

Solution: Bearing in mind that $\vec{\mathbf{a}} = \mathrm{a}_j\hat{\mathbf{e}}_j$, $\vec{\mathbf{b}} = \mathrm{b}_k\hat{\mathbf{e}}_k$, $\nabla_{\vec{x}} = \hat{\mathbf{e}}_i\dfrac{\partial}{\partial x_i}$ $(i=1,2,3)$, the following is true:

$$
(\nabla_{\vec{x}}\vec{\mathbf{a}})\cdot\vec{\mathbf{b}} = \left(\frac{\partial(\mathrm{a}_j\hat{\mathbf{e}}_j)}{\partial x_i}\otimes\hat{\mathbf{e}}_i\right)\cdot(\mathrm{b}_k\hat{\mathbf{e}}_k) = \left(\frac{\partial\mathrm{a}_j}{\partial x_i}\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_i\right)\cdot(\mathrm{b}_k\hat{\mathbf{e}}_k) = \mathrm{b}_k\delta_{ik}\frac{\partial\mathrm{a}_j}{\partial x_i}\hat{\mathbf{e}}_j = \mathrm{b}_k\frac{\partial\mathrm{a}_j}{\partial x_k}\hat{\mathbf{e}}_j
$$

Expanding the dummy index k, we obtain:

$$
\mathrm{b}_k\frac{\partial\mathrm{a}_j}{\partial x_k} = \mathrm{b}_1\frac{\partial\mathrm{a}_j}{\partial x_1}+\mathrm{b}_2\frac{\partial\mathrm{a}_j}{\partial x_2}+\mathrm{b}_3\frac{\partial\mathrm{a}_j}{\partial x_3}
$$

Thus,

$$
\begin{aligned}
j=1 \quad &\Rightarrow \quad \mathrm{b}_1\frac{\partial\mathrm{a}_1}{\partial x_1}+\mathrm{b}_2\frac{\partial\mathrm{a}_1}{\partial x_2}+\mathrm{b}_3\frac{\partial\mathrm{a}_1}{\partial x_3} \\
j=2 \quad &\Rightarrow \quad \mathrm{b}_1\frac{\partial\mathrm{a}_2}{\partial x_1}+\mathrm{b}_2\frac{\partial\mathrm{a}_2}{\partial x_2}+\mathrm{b}_3\frac{\partial\mathrm{a}_2}{\partial x_3} \\
j=3 \quad &\Rightarrow \quad \mathrm{b}_1\frac{\partial\mathrm{a}_3}{\partial x_1}+\mathrm{b}_2\frac{\partial\mathrm{a}_3}{\partial x_2}+\mathrm{b}_3\frac{\partial\mathrm{a}_3}{\partial x_3}
\end{aligned}
$$

Problem 1.45: Prove that the following relationship is valid:

$$\nabla_{\vec{x}} \cdot \left(\frac{\vec{\mathbf{q}}}{T} \right) = \frac{1}{T} \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} - \frac{1}{T^2} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T$$

where $\vec{\mathbf{q}}(\vec{x}, t)$ is an arbitrary vector field, and $T(\vec{x}, t)$ is a scalar field.

Solution:

$$\nabla_{\vec{x}} \cdot \left(\frac{\vec{\mathbf{q}}}{T} \right) = \frac{\partial}{\partial x_i} \left(\frac{\mathrm{q}_i}{T} \right) \equiv \left(\frac{\mathrm{q}_i}{T} \right)_{,i} = \frac{1}{T} \mathrm{q}_{i,i} - \frac{1}{T^2} \mathrm{q}_i T_{,i}$$

$$= \frac{1}{T} \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} - \frac{1}{T^2} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \quad \text{(scalar)}$$

1.8.4 The Curl

The curl of a vector field

The curl (or rotor) of a vector field, $\vec{\mathbf{v}}(\vec{x})$ is denoted by $\mathrm{curl}(\vec{\mathbf{v}}) \equiv \mathrm{rot}(\vec{\mathbf{v}}) \equiv \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$, and is defined in the Cartesian basis as:

$$\boxed{\vec{\nabla}_{\vec{x}} \wedge (\bullet) = \frac{\partial}{\partial x_j} \hat{\mathbf{e}}_j \wedge (\bullet)} \quad \textit{The curl (rotor) of a tensor field in the Cartesian basis} \tag{1.482}$$

Note that the curl is already a tensor operator between two vectors. Using the definition of the vector product we obtain the curl of a vector field as:

$$\mathrm{rot}(\vec{\mathbf{v}}) = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} = \frac{\partial}{\partial x_j} \hat{\mathbf{e}}_j \wedge (\mathrm{v}_k \hat{\mathbf{e}}_k) = \frac{\partial \mathrm{v}_k}{\partial x_j} \hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k = \frac{\partial \mathrm{v}_k}{\partial x_j} \epsilon_{ijk} \hat{\mathbf{e}}_i = \epsilon_{ijk} \mathrm{v}_{k,j} \hat{\mathbf{e}}_i \tag{1.483}$$

where ϵ_{ijk} is the permutation symbol defined in (1.55). Moreover, we have applied the definition $\hat{\mathbf{e}}_j \wedge \hat{\mathbf{e}}_k = \epsilon_{ijk} \hat{\mathbf{e}}_i$ and we can also note that:

$$\begin{aligned} \mathrm{rot}(\vec{\mathbf{v}}) = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} &= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{vmatrix} = \epsilon_{ijk} \mathrm{v}_{k,j} \hat{\mathbf{e}}_i \\ &= \left(\frac{\partial \mathrm{v}_3}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial \mathrm{v}_1}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial \mathrm{v}_2}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_2} \right) \hat{\mathbf{e}}_3 \end{aligned} \tag{1.484}$$

We can verify that the antisymmetric part of a vector field gradient, which is illustrated by $(\nabla_{\vec{x}} \vec{\mathbf{v}})^{skew} \equiv \mathbf{W}$, has as components:

$$\begin{aligned} \left[(\nabla_{\vec{x}} \vec{\mathbf{v}})^{skew} \right]_{ij} \equiv \mathrm{v}^{skew}_{i,j} &= \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial \mathrm{v}_1}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_1} \right) & \frac{1}{2}\left(\frac{\partial \mathrm{v}_1}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_1} \right) \\ \frac{1}{2}\left(\frac{\partial \mathrm{v}_2}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_2} \right) & 0 & \frac{1}{2}\left(\frac{\partial \mathrm{v}_2}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_2} \right) \\ \frac{1}{2}\left(\frac{\partial \mathrm{v}_3}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_3} \right) & \frac{1}{2}\left(\frac{\partial \mathrm{v}_3}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_3} \right) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \mathrm{W}_{12} & \mathrm{W}_{13} \\ \mathrm{W}_{21} & 0 & \mathrm{W}_{23} \\ \mathrm{W}_{31} & \mathrm{W}_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{W}_{12} & \mathrm{W}_{13} \\ -\mathrm{W}_{12} & 0 & \mathrm{W}_{23} \\ -\mathrm{W}_{13} & -\mathrm{W}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \end{aligned} \tag{1.485}$$

where w_1, w_2, w_3 are the components of the axial vector $\vec{w}$ associated with $\mathbf{W}$, (see subsection: 1.5.2.2.2. Antisymmetric Tensor).

With reference to the definition of the curl in (1.484) and the relationship in (1.485), we can conclude that:

$$\begin{aligned}\mathrm{rot}(\vec{\mathbf{v}}) \equiv \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}} &= \left(\frac{\partial \mathrm{v}_3}{\partial x_2} - \frac{\partial \mathrm{v}_2}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial \mathrm{v}_1}{\partial x_3} - \frac{\partial \mathrm{v}_3}{\partial x_1}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial \mathrm{v}_2}{\partial x_1} - \frac{\partial \mathrm{v}_1}{\partial x_2}\right)\hat{\mathbf{e}}_3 \\ &= 2\mathrm{W}_{32}\hat{\mathbf{e}}_1 + 2\mathrm{W}_{13}\hat{\mathbf{e}}_2 + 2\mathrm{W}_{21}\hat{\mathbf{e}}_3 \\ &= 2\left(w_1\hat{\mathbf{e}}_1 + w_2\hat{\mathbf{e}}_2 + w_3\hat{\mathbf{e}}_3\right) \\ &= 2\vec{w}\end{aligned} \tag{1.486}$$

And, if we use the identity in (1.141), we obtain:

$$\mathbf{W} \cdot \vec{\mathbf{v}} = \vec{w} \wedge \vec{\mathbf{v}} = \frac{1}{2}\left(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}\right) \wedge \vec{\mathbf{v}} \tag{1.487}$$

It could be interesting to note that the equation in (1.486) can be obtained by means of **Problem 1.18**, in which we showed that $\frac{1}{2}(\vec{a} \wedge \vec{x})$ is the axial vector associated with the antisymmetric tensor $(\vec{x} \otimes \vec{a})^{skew}$. Therefore, the axial vector associated with the antisymmetric tensor $\mathbf{W} = (\nabla_{\vec{x}}\vec{\mathbf{v}})^{skew} = \left[(\vec{\mathbf{v}}) \otimes (\vec{\nabla})\right]^{skew}$ is the vector $\frac{1}{2}\left(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}\right)$.

As we can see, the curl describes the rotational tendency of the vector field.

Summary

•	Divergence $\mathrm{div}(\bullet) \equiv \nabla_{\vec{x}} \cdot \bullet$	Gradient $\mathrm{grad}(\bullet) \equiv \nabla_{\vec{x}}\, \bullet$	Curl $\mathrm{rot}(\bullet) \equiv \vec{\nabla}_{\vec{x}} \wedge \bullet$
Scalar		vector	
Vector	Scalar	Second-order tensor	Vector
Second-order tensor	Vector	Third-order tensor	Second-order tensor

We can now present some equations:

- $\mathrm{rot}(\lambda\vec{\mathbf{a}}) = \vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}}) = \lambda(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) + (\nabla_{\vec{x}}\lambda \wedge \vec{\mathbf{a}})$

  The result of the algebraic operation $\vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}})$ is a vector, whose components are given by:

$$\begin{aligned}\left[\vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}})\right]_i &= \epsilon_{ijk}(\lambda \mathrm{a}_k)_{,j} \\ &= \epsilon_{ijk}(\lambda_{,j}\mathrm{a}_k + \lambda \mathrm{a}_{k,j}) \\ &= \epsilon_{ijk}\lambda \mathrm{a}_{k,j} + \epsilon_{ijk}\lambda_{,j}\mathrm{a}_k \\ &= \lambda(\nabla_{\vec{x}} \wedge \vec{\mathbf{a}})_i + \epsilon_{ijk}(\nabla_{\vec{x}}\lambda)_j \mathrm{a}_k \\ &= \lambda(\nabla_{\vec{x}} \wedge \vec{\mathbf{a}})_i + (\nabla_{\vec{x}}\lambda \wedge \vec{\mathbf{a}})_i\end{aligned} \tag{1.488}$$

We can use the above equation to check that the relationship $\mathrm{rot}(\lambda\vec{\mathbf{a}}) = \vec{\nabla}_{\vec{x}} \wedge (\lambda\vec{\mathbf{a}}) = \lambda(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) + (\nabla_{\vec{x}}\lambda \wedge \vec{\mathbf{a}})$ holds.

- $$\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}}) = (\nabla_{\vec{x}} \cdot \vec{\mathbf{b}})\vec{\mathbf{a}} - (\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\vec{\mathbf{b}} + (\nabla_{\vec{x}}\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}} - (\nabla_{\vec{x}}\vec{\mathbf{b}}) \cdot \vec{\mathbf{a}} \tag{1.489}$$

The components of the vector product $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})$ are given by $(\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})_k = \epsilon_{kij} \mathrm{a}_i \mathrm{b}_j$, thus:

$$\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})\right]_l = \epsilon_{lpk} (\epsilon_{kij} \mathrm{a}_i \mathrm{b}_j)_{,p} = \epsilon_{kij} \epsilon_{lpk} (\mathrm{a}_{i,p} \mathrm{b}_j + \mathrm{a}_i \mathrm{b}_{j,p}) \tag{1.490}$$

Regarding that $\epsilon_{kij} = \epsilon_{ijk}$ and $\epsilon_{ijk}\epsilon_{lpk} = \delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl}$, the above equation becomes:

$$\begin{aligned}
\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\mathbf{a}} \wedge \vec{\mathbf{b}})\right]_l &= \epsilon_{kij} \epsilon_{lpk} (\mathrm{a}_{i,p} \mathrm{b}_j + \mathrm{a}_i \mathrm{b}_{j,p}) \quad = (\delta_{il}\delta_{jp} - \delta_{ip}\delta_{jl})(\mathrm{a}_{i,p} \mathrm{b}_j + \mathrm{a}_i \mathrm{b}_{j,p}) \\
&= \delta_{il}\delta_{jp} \mathrm{a}_{i,p} \mathrm{b}_j - \delta_{ip}\delta_{jl} \mathrm{a}_{i,p} \mathrm{b}_j + \delta_{il}\delta_{jp} \mathrm{a}_i \mathrm{b}_{j,p} - \delta_{ip}\delta_{jl} \mathrm{a}_i \mathrm{b}_{j,p} \\
&= \mathrm{a}_{l,p} \mathrm{b}_p - \mathrm{a}_{p,p} \mathrm{b}_l + \mathrm{a}_l \mathrm{b}_{p,p} - \mathrm{a}_p \mathrm{b}_{l,p}
\end{aligned} \tag{1.491}$$

We can also verify that $\left[(\nabla_{\vec{x}}\vec{\mathbf{a}}) \cdot \vec{\mathbf{b}}\right]_l = \mathrm{a}_{l,p} \mathrm{b}_p$, $\left[(\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\vec{\mathbf{b}}\right]_l = \mathrm{a}_{p,p} \mathrm{b}_l$, $\left[(\nabla_{\vec{x}} \cdot \vec{\mathbf{b}})\vec{\mathbf{a}}\right]_l = \mathrm{a}_l \mathrm{b}_{p,p}$, $\left[(\nabla_{\vec{x}}\vec{\mathbf{b}}) \cdot \vec{\mathbf{a}}\right]_l = \mathrm{a}_p \mathrm{b}_{l,p}$.

- $$\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}}) = \nabla_{\vec{x}} (\nabla_{\vec{x}} \cdot \vec{\mathbf{a}}) - \nabla_{\vec{x}}^2 \vec{\mathbf{a}} \tag{1.492}$$

The components of $(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})$ are given by $(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})_i = \underbrace{\epsilon_{ijk} \mathrm{a}_{k,j}}_{\mathrm{c}_i}$, thus:

$$\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})\right]_q = \epsilon_{qli} \mathrm{c}_{i,l} = \epsilon_{qli} (\epsilon_{ijk} \mathrm{a}_{k,j})_{,l} = \epsilon_{qli} \epsilon_{ijk} \mathrm{a}_{k,jl} \tag{1.493}$$

Once again considering that $\epsilon_{qli}\epsilon_{ijk} = \epsilon_{qli}\epsilon_{jki} = \delta_{qj}\delta_{lk} - \delta_{qk}\delta_{lj}$, the above equation becomes:

$$\begin{aligned}
\left[\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{a}})\right]_q &= \epsilon_{qli} \epsilon_{ijk} \mathrm{a}_{k,jl} \quad = (\delta_{qj}\delta_{lk} - \delta_{qk}\delta_{lj}) \mathrm{a}_{k,jl} = \delta_{qj}\delta_{lk} \mathrm{a}_{k,jl} - \delta_{qk}\delta_{lj} \mathrm{a}_{k,jl} \\
&= \mathrm{a}_{k,kq} - \mathrm{a}_{q,ll}
\end{aligned} \tag{1.494}$$

where $\left[\nabla_{\vec{x}} (\nabla_{\vec{x}} \cdot \vec{\mathbf{a}})\right]_q = \mathrm{a}_{k,kq}$ and $\left[\nabla_{\vec{x}}^2 \vec{\mathbf{a}}\right]_q = \mathrm{a}_{q,ll}$.

- $$\nabla_{\vec{x}} \cdot (\psi \nabla_{\vec{x}} \phi) = \psi \nabla_{\vec{x}}^2 \phi + (\nabla_{\vec{x}} \psi) \cdot (\nabla_{\vec{x}} \phi) \tag{1.495}$$

$$\nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi) = (\phi \psi_{,i})_{,i} = \phi \psi_{,ii} + \phi_{,i} \psi_{,i} = \phi \nabla_{\vec{x}}^2 \psi + (\nabla_{\vec{x}} \phi) \cdot (\nabla_{\vec{x}} \psi) \tag{1.496}$$

where ϕ and ψ are scalar fields. Other interesting equations derived from the above are:

$$\begin{aligned}
\nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi) &= \phi \nabla_{\vec{x}}^2 \psi + (\nabla_{\vec{x}} \phi) \cdot (\nabla_{\vec{x}} \psi) \\
\nabla_{\vec{x}} \cdot (\psi \nabla_{\vec{x}} \phi) &= \psi \nabla_{\vec{x}}^2 \phi + (\nabla_{\vec{x}} \psi) \cdot (\nabla_{\vec{x}} \phi)
\end{aligned} \tag{1.497}$$

After subtracting the above two identities we obtain:

$$\begin{aligned}
&\nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi) - \nabla_{\vec{x}} \cdot (\psi \nabla_{\vec{x}} \phi) = \phi \nabla_{\vec{x}}^2 \psi - \psi \nabla_{\vec{x}}^2 \phi \\
&\Rightarrow \nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi - \psi \nabla_{\vec{x}} \phi) = \phi \nabla_{\vec{x}}^2 \psi - \psi \nabla_{\vec{x}}^2 \phi
\end{aligned} \tag{1.498}$$

1.8.5 The Conservative Field

A vector field, $\vec{\mathbf{b}}(\vec{x}, t)$, is said to be conservative if there exists a differentiable scalar field, $\phi(\vec{x}, t)$, so that:

$$\vec{\mathbf{b}} = \nabla_{\vec{x}} \phi \tag{1.499}$$

If the function ϕ satisfies the relation (1.499), then ϕ is a *potential function* of $\vec{\mathbf{b}}(\vec{x},t)$.

A necessary but insufficient condition for $\vec{\mathbf{b}}(\vec{x},t)$ to be conservative is that $\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}} = \vec{\mathbf{0}}$. In other words, given a conservative field, the curl $\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}}$ equals zero. However, if the curl of a vector field equals zero, this does not necessarily mean that the field is conservative.

Problem 1.46: Let ϕ be a scalar field, and $\vec{\mathbf{u}}$ be a vector field. a) Show that $\nabla_{\vec{x}} \cdot \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) = 0$ and $\vec{\nabla}_{\vec{x}} \wedge \left(\nabla_{\vec{x}}\phi\right) = \vec{\mathbf{0}}$.
b) Show that $\vec{\nabla}_{\vec{x}} \wedge \left[\left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v}\ \right] = (\nabla_{\vec{x}} \cdot \vec{v})(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) + \left[\nabla_{\vec{x}}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right] \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})$;
c) Referring $\vec{\boldsymbol{\omega}} = \nabla_{\vec{x}} \wedge \vec{v}$, show that $\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}^2 \vec{v}) = \nabla_{\vec{x}}^2 (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = \nabla_{\vec{x}}^2 \vec{\boldsymbol{\omega}}$.

Solution:
Regarding that: $\vec{\nabla}_{\vec{x}} \wedge \vec{v} = \epsilon_{ijk} v_{k,j} \hat{\mathbf{e}}_i$

$$\nabla_{\vec{x}} \cdot \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) = \frac{\partial}{\partial x_l}\left(\epsilon_{ijk} v_{k,j} \hat{\mathbf{e}}_i\right) \cdot \hat{\mathbf{e}}_l = \epsilon_{ijk} \frac{\partial}{\partial x_l}\left(v_{k,j}\right)\delta_{il} = \epsilon_{ijk} \frac{\partial}{\partial x_i}\left(v_{k,j}\right) = \epsilon_{ijk} v_{k,ji}$$

The second derivative of $\vec{v}$ is symmetrical with ij, *i.e.* $v_{k,ji} = v_{k,ij}$, while ϵ_{ijk} is antisymmetric with ij, *i.e.*, $\epsilon_{ijk} = -\epsilon_{jik}$, thus:

$$\epsilon_{ijk} v_{k,ji} = \epsilon_{ij1} v_{1,ji} + \epsilon_{ij2} v_{2,ji} + \epsilon_{ij3} v_{3,ji} = 0$$

We can observe that $\epsilon_{ij1} v_{1,ji}$ equals the double scalar product by using a symmetric and an antisymmetric tensor, so $\epsilon_{ij1} v_{1,ji} = 0$.
Likewise, we can show that:

$$\vec{\nabla}_{\vec{x}} \wedge \left(\nabla_{\vec{x}}\phi\right) = \epsilon_{ijk}\phi_{,kj}\ \hat{\mathbf{e}}_i = 0_i \hat{\mathbf{e}}_i = \vec{\mathbf{0}}$$

b) Denoting by $\vec{\boldsymbol{\omega}} = \vec{\nabla}_{\vec{x}} \wedge \vec{v}$ we obtain:

$$\vec{\nabla}_{\vec{x}} \wedge \left[\left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v}\ \right] = \vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v})$$

Observing the equation in (1.489), it holds that:

$$\vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v}) = (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} - (\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}})\vec{v} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot \vec{\boldsymbol{\omega}}$$

Note that $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}} = \nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) = 0$. Then, we can draw the conclusion that:

$$\begin{aligned}\vec{\nabla}_{\vec{x}} \wedge (\vec{\boldsymbol{\omega}} \wedge \vec{v}) &= (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} + (\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}) \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot \vec{\boldsymbol{\omega}} \\ &= (\nabla_{\vec{x}} \cdot \vec{v})(\vec{\nabla}_{\vec{x}} \wedge \vec{v}) + \left[\nabla_{\vec{x}}(\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right] \cdot \vec{v} - (\nabla_{\vec{x}}\vec{v}) \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\end{aligned}$$

c) Observing the equation in (1.492) we obtain:

$$\begin{aligned}\nabla_{\vec{x}}^2 \vec{v} &= \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{v}) \\ &= \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) - \vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}}\end{aligned}$$

Applying the curl to the above equation we obtain:

$$\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}^2 \vec{v}) = \underbrace{\vec{\nabla}_{\vec{x}} \wedge \left[\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v})\right]}_{=\vec{\mathbf{0}}} - \vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}})$$

Referring once again to the equation in (1.492) to express the term $\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}})$:

$$\begin{aligned}\vec{\nabla}_{\vec{x}} \wedge (\nabla_{\vec{x}}^2 \vec{v}) &= -\vec{\nabla}_{\vec{x}} \wedge (\vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{\omega}}) = -\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}}) + \nabla_{\vec{x}}^2 \vec{\boldsymbol{\omega}} = -\nabla_{\vec{x}}\underbrace{\left[\nabla_{\vec{x}} \cdot (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\right]}_{=0} + \nabla_{\vec{x}}^2 \vec{\boldsymbol{\omega}} \\ &= \nabla_{\vec{x}}^2 (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\end{aligned}$$

1.9 Theorems Involving Integrals

1.9.1 Integration by Parts

Integration by parts states that:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx \tag{1.500}$$

where $v'(x) = \frac{dv}{dx}$, and the functions $u(x)$, $v(x)$ are differentiable in $a \le x \le b$.

1.9.2 The Divergence Theorem

Given a domain $\mathcal{B}$ with a volume V, and bounded by the surface S, (see Figure 1.37), the divergence theorem, also called the *Gauss' theorem*, applied to the vector field states that:

$$\begin{aligned} &\int_V \nabla_{\vec{x}} \cdot \vec{\mathbf{v}}\; dV = \int_S \vec{\mathbf{v}} \cdot \hat{\mathbf{n}}\, dS = \int_S \vec{\mathbf{v}} \cdot d\vec{\boldsymbol{S}} \\ &\int_V \mathsf{v}_{i,i}\; dV = \int_S \mathsf{v}_i \hat{\mathsf{n}}_i\; dS = \int_S \mathsf{v}_i\; dS_i \end{aligned} \tag{1.501}$$

where $\hat{\mathbf{n}}$ is the outward unit normal to surface S.

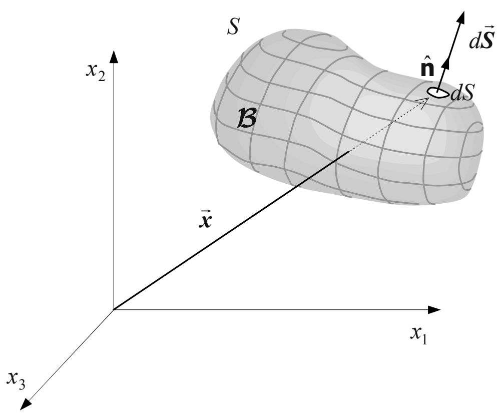


Figure 1.37.

Let $\mathbf{T}$ be a second-order tensor field defined in the domain $\mathcal{B}$. The divergence theorem applied to this field is defined as:

$$\int_V \nabla_{\vec{x}} \cdot \mathbf{T}\; dV = \int_S \mathbf{T} \cdot \hat{\mathbf{n}}\, dS = \int_S \mathbf{T} \cdot d\vec{\boldsymbol{S}} \quad \Bigg| \quad \int_V \mathsf{T}_{ij,j}\; dV = \int_S \mathsf{T}_{ij} \hat{\mathsf{n}}_j\; dS = \int_S \mathsf{T}_{ij}\; dS_j \tag{1.502}$$

By using the divergence theorem we can also demonstrate that:

$$\begin{aligned}\int_V (x_k)_{,j}\, dV &= \int_V (\delta_{ik} x_i)_{,j}\, dV &&= \int_S \delta_{ik} x_i \hat{\mathsf{n}}_j\, dS \\ &= \int_V \left[\delta_{ik,j} x_i + \delta_{ik} x_{i,j}\right] dV &&= \int_S x_k \hat{\mathsf{n}}_j\, dS \\ &= \int_V x_{k,j}\, dV &&= \int_S x_k \hat{\mathsf{n}}_j\, dS\end{aligned} \tag{1.503}$$

in which we have assumed that $\delta_{ik,j} = 0_{ikj}$. Additionally, by observing that $x_{k,j} = \delta_{kj}$, we can obtain:

$$\delta_{kj} \int_V dV = \int_S x_k \hat{\mathsf{n}}_j\, dS \quad \Rightarrow \quad V\delta_{kj} = \int_S x_k \hat{\mathsf{n}}_j\, dS$$
$$V\mathbf{1} = \int_S \vec{\boldsymbol{x}} \otimes \hat{\mathbf{n}}\, dS \tag{1.504}$$

Given a second-order tensor $\boldsymbol{\sigma}$ defined in the domain $\mathcal{B}$, the following is valid:

$$\begin{aligned}\int_V (x_i \sigma_{jk})_{,k}\, dV &= \int_V (x_i \sigma_{jk})_{,k}\, dV = \int_S x_i \sigma_{jk} \hat{\mathsf{n}}_k\, dS \\ &= \int_V \left[x_{i,k} \sigma_{jk} + x_i \sigma_{jk,k}\right] dV = \int_S x_i \sigma_{jk} \hat{\mathsf{n}}_k\, dS \\ &= \int_V \left[\delta_{ik} \sigma_{jk} + x_i \sigma_{jk,k}\right] dV = \int_S x_i \sigma_{jk} \hat{\mathsf{n}}_k\, dS\end{aligned} \tag{1.505}$$

Hence proving that:

$$\int_V x_i \sigma_{jk,k}\, dV = \int_S x_i \sigma_{jk} \hat{\mathsf{n}}_k\, dS - \int_V \sigma_{ji}\, dV$$
$$\int_V \vec{\boldsymbol{x}} \otimes \nabla_{\vec{x}} \cdot \boldsymbol{\sigma}\, dV = \int_S \vec{\boldsymbol{x}} \otimes (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})\, dS - \int_V \boldsymbol{\sigma}^T\, dV \tag{1.506}$$

or

$$\int_V \vec{\boldsymbol{x}} \otimes \nabla_{\vec{x}} \cdot \boldsymbol{\sigma}\, dV = \int_S \left(\vec{\boldsymbol{x}} \otimes \boldsymbol{\sigma}\right) \cdot d\vec{\boldsymbol{S}} - \int_V \boldsymbol{\sigma}^T\, dV \tag{1.507}$$

Likewise, one can prove that:

$$\int_V (\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}) \otimes \vec{\boldsymbol{x}}\, dV = \int_S (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \otimes \vec{\boldsymbol{x}}\, dS - \int_V \boldsymbol{\sigma}\, dV \tag{1.508}$$

Problem 1.47: Let Ω be a domain bounded by Γ as shown in Figure 1.38. Further consider that $\mathbf{m}$ is a second-order tensor field and ω is a scalar field. Show that the following relationship holds:

$$\int_\Omega \left[\mathbf{m} : \nabla_{\vec{x}} (\nabla_{\vec{x}} \omega)\right] d\Omega = \int_\Gamma \left[(\nabla_{\vec{x}} \omega) \cdot \mathbf{m}\right] \cdot \hat{\mathbf{n}}\, d\Gamma - \int_\Omega \left[(\nabla_{\vec{x}} \cdot \mathbf{m}) \cdot \nabla_{\vec{x}} \omega\right] d\Omega$$

$$\int_{\Omega}\left[\mathsf{m}_{ij}\omega_{,ij}\right]d\Omega=\int_{\Gamma}(\omega_{,i}\,\mathsf{m}_{ij})\hat{\mathsf{n}}_j\;d\Gamma-\int_{\Omega}\left[\mathsf{m}_{ij,j}\omega_{,i}\right]d\Omega$$

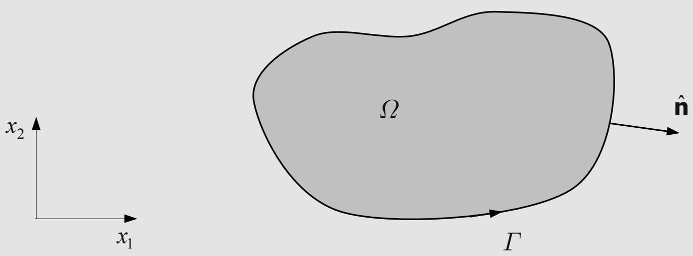


Figure 1.38

Solution: We could directly apply the definition of integration by parts to demonstrate the above relationship. But, here we will start with the definition of the divergence theorem. That is, given a tensor field $\vec{\mathbf{v}}$, it is true that:

$$\int_{\Omega}\nabla_{\vec{x}}\cdot\vec{\mathbf{v}}\;d\Omega=\int_{\Gamma}\vec{\mathbf{v}}\cdot\hat{\mathbf{n}}\,d\Gamma\xrightarrow{indicial}\int_{\Omega}\mathsf{v}_{j,j}\;\;d\Omega=\int_{\Gamma}\mathsf{v}_j\hat{\mathsf{n}}_j\;\;d\Gamma$$

Observing that the tensor $\vec{\mathbf{v}}$ is the result of the algebraic operation $\vec{\mathbf{v}}=\nabla_{\vec{x}}\omega\cdot\mathbf{m}$ and the equivalent in indicial notation to $\mathsf{v}_j=\omega_{,i}\,\mathsf{m}_{ij}$, and by substituting it in the above equation we obtain:

$$\begin{aligned}\int_{\Omega}\mathsf{v}_{j,j}\;d\Omega=\int_{\Gamma}\mathsf{v}_j\hat{\mathsf{n}}_j\;d\Gamma\quad&\Rightarrow\int_{\Omega}\left[\omega_{,i}\,\mathsf{m}_{ij}\right]_{,j}\;d\Omega=\int_{\Gamma}\omega_{,i}\,\mathsf{m}_{ij}\hat{\mathsf{n}}_j\;\;d\Gamma\\&\Rightarrow\int_{\Omega}\left[\omega_{,ij}\,\mathsf{m}_{ij}+\omega_{,i}\,\mathsf{m}_{ij,j}\right]d\Omega=\int_{\Gamma}\omega_{,i}\,\mathsf{m}_{ij}\hat{\mathsf{n}}_j\;\;d\Gamma\\&\Rightarrow\int_{\Omega}\left[\omega_{,ij}\,\mathsf{m}_{ij}\right]d\Omega=\int_{\Gamma}\omega_{,i}\,\mathsf{m}_{ij}\hat{\mathsf{n}}_j\;\;d\Gamma-\int_{\Omega}\left[\omega_{,i}\,\mathsf{m}_{ij,j}\right]d\Omega\end{aligned}$$

The above equation in tensorial notation becomes:

$$\int_{\Omega}\left[\mathbf{m}:\nabla_{\vec{x}}(\nabla_{\vec{x}}\omega)\right]d\Omega=\int_{\Gamma}\left[(\nabla_{\vec{x}}\omega)\cdot\mathbf{m}\right]\cdot\hat{\mathbf{n}}d\Gamma-\int_{\Omega}\left[\nabla_{\vec{x}}\omega\cdot(\nabla_{\vec{x}}\cdot\mathbf{m})\right]d\Omega$$

NOTE: Consider now the domain defined by the volume V, which is bounded by the surface S with the outward unit normal to the surface $\hat{\mathbf{n}}$. If $\vec{N}$ is a vector field and T is a scalar field, it is also true that:

$$\int_V N_i T_{,ij}\;dV=\int_S N_i T_{,i}\,\hat{\mathsf{n}}_j dS-\int_V N_{i,j}T_{,i}\;dV$$

$$\Rightarrow\int_V\vec{N}\cdot\nabla_{\vec{x}}(\nabla_{\vec{x}}T)dV=\int_S(\nabla_{\vec{x}}T\cdot\vec{N})\otimes\hat{\mathbf{n}}dS-\int_V\nabla_{\vec{x}}T\cdot\nabla_{\vec{x}}\vec{N}\;dV$$

where we have directly applied the definition of integration by parts.

1.9.3 Independence of Path

A curve which connects two points A and B is denoted by the path from A to B, (see Figure 1.39). We can then establish the condition by which a line integral is independent of path, (see Figure 1.39).

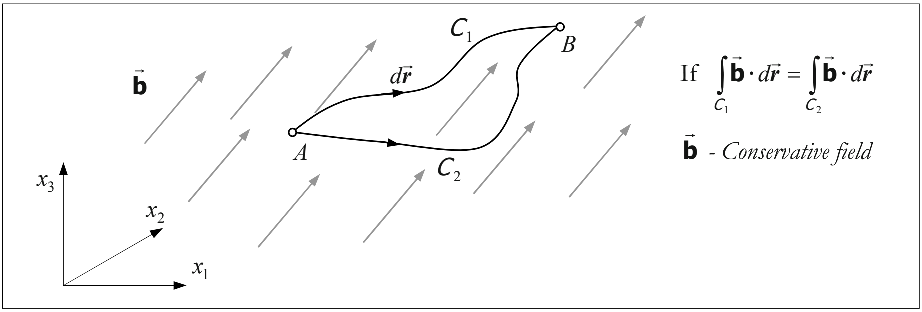


Figure 1.39: Path independence.

Let $\vec{\mathbf{b}}(\vec{x})$ be a continuous vector fields, then the integral $\int_C \vec{\mathbf{b}} \cdot d\vec{r}$ is *independent of the path* if and only if $\vec{\mathbf{b}}$ is a *conservative field*. This means that there is a scalar field ϕ so that $\vec{\mathbf{b}} = \nabla_{\vec{x}}\phi$. Regarding the above, we can draw the conclusion that:

$$\begin{gathered}\int_A^B \vec{\mathbf{b}} \cdot d\vec{r} = \int_A^B \nabla_{\vec{x}}\phi \cdot d\vec{r} \\ \int_A^B (\mathrm{b}_1\hat{\mathbf{e}}_1 + \mathrm{b}_2\hat{\mathbf{e}}_2 + \mathrm{b}_3\hat{\mathbf{e}}_3) \cdot d\vec{r} = \int_A^B \left(\frac{\partial\phi}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial\phi}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial\phi}{\partial x_3}\hat{\mathbf{e}}_3 \right) \cdot d\vec{r}\end{gathered} \tag{1.509}$$

Thus

$$\mathrm{b}_1 = \frac{\partial\phi}{\partial x_1} \quad ; \quad \mathrm{b}_2 = \frac{\partial\phi}{\partial x_2} \quad ; \quad \mathrm{b}_3 = \frac{\partial\phi}{\partial x_3} \tag{1.510}$$

As the field is conservative, the curl of $\vec{\mathbf{b}}$ is the zero vector:

$$\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{b}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 \end{vmatrix} = 0_i \tag{1.511}$$

We can therefore conclude that:

$$\begin{cases} \dfrac{\partial \mathrm{b}_3}{\partial x_2} - \dfrac{\partial \mathrm{b}_2}{\partial x_3} = 0 \\ \dfrac{\partial \mathrm{b}_1}{\partial x_3} - \dfrac{\partial \mathrm{b}_3}{\partial x_1} = 0 \\ \dfrac{\partial \mathrm{b}_2}{\partial x_1} - \dfrac{\partial \mathrm{b}_1}{\partial x_2} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial \mathrm{b}_3}{\partial x_2} = \dfrac{\partial \mathrm{b}_2}{\partial x_3} \\ \dfrac{\partial \mathrm{b}_1}{\partial x_3} = \dfrac{\partial \mathrm{b}_3}{\partial x_1} \\ \dfrac{\partial \mathrm{b}_2}{\partial x_1} = \dfrac{\partial \mathrm{b}_1}{\partial x_2} \end{cases} \tag{1.512}$$

Therefore, if the above condition is not satisfied, the field is not conservative.

1.9.4 The Kelvin-Stokes' Theorem

Let S be a regular surface, (see Figure 1.40), and $\vec{\mathbf{F}}(\vec{x},t)$ be a vector field. According to the Kelvin-Stokes' Theorem:

$$\boxed{\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{\Gamma}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot d\vec{\boldsymbol{S}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{n}} dS} \tag{1.513}$$

If $\hat{\mathbf{p}}$ denotes the unit vector tangent to the boundary Γ, the Stokes' theorem becomes:

$$\oint_{\Gamma} \vec{\mathbf{F}} \cdot \hat{\mathbf{p}} d\Gamma = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot d\vec{\boldsymbol{S}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{n}} dS \tag{1.514}$$

With reference to the vector representation in the Cartesian basis: $\vec{\mathbf{F}} = \mathsf{F}_1 \hat{\mathbf{e}}_1 + \mathsf{F}_2 \hat{\mathbf{e}}_2 + \mathsf{F}_3 \hat{\mathbf{e}}_3$, $d\vec{\boldsymbol{S}} = dS_1 \hat{\mathbf{e}}_1 + dS_2 \hat{\mathbf{e}}_2 + dS_3 \hat{\mathbf{e}}_3$, $d\vec{\boldsymbol{\Gamma}} = dx_1 \hat{\mathbf{e}}_1 + dx_2 \hat{\mathbf{e}}_2 + dx_3 \hat{\mathbf{e}}_3$, the components of the curl of $\vec{\mathbf{F}}$ are given by:

$$\left(\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}\right)_i = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \mathsf{F}_1 & \mathsf{F}_2 & \mathsf{F}_3 \end{vmatrix} = \left(\frac{\partial \mathsf{F}_3}{\partial x_2} - \frac{\partial \mathsf{F}_2}{\partial x_3} \right) \hat{\mathbf{e}}_1 + \left(\frac{\partial \mathsf{F}_1}{\partial x_3} - \frac{\partial \mathsf{F}_3}{\partial x_1} \right) \hat{\mathbf{e}}_2 + \left(\frac{\partial \mathsf{F}_2}{\partial x_1} - \frac{\partial \mathsf{F}_1}{\partial x_2} \right) \hat{\mathbf{e}}_3 \tag{1.515}$$

Next, the Stokes' theorem expressed in terms of components becomes:

$$\oint_{\Gamma} \mathsf{F}_1 dx_1 + \mathsf{F}_2 dx_2 + \mathsf{F}_3 dx_3 = \int_{\Omega} \left(\frac{\partial \mathsf{F}_3}{\partial x_2} - \frac{\partial \mathsf{F}_2}{\partial x_3} \right) dS_1 + \left(\frac{\partial \mathsf{F}_1}{\partial x_3} - \frac{\partial \mathsf{F}_3}{\partial x_1} \right) dS_2 + \left(\frac{\partial \mathsf{F}_2}{\partial x_1} - \frac{\partial \mathsf{F}_1}{\partial x_2} \right) dS_3 \tag{1.516}$$

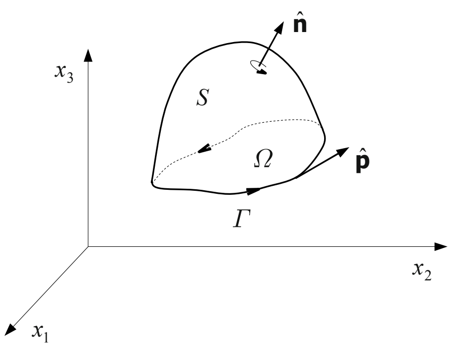


Figure 1.40: Stokes' theorem.

In the special case when the surface S coincides with the plane Ω, (see Figure 1.41), the equation (1.516) remains valid. Then, if the domain Ω coincides with the plane $x_1 - x_2$, the equation (1.513) becomes:

$$\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{\Gamma}} = \int_{\Omega} (\vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{F}}) \cdot \hat{\mathbf{e}}_3 dS \tag{1.517}$$

which is known as the Stokes' theorem in the plane or *Green's theorem*, which is expressed in terms of components as:

$$\oint_{\Gamma} \mathsf{F}_1 dx_1 + \mathsf{F}_2 dx_2 = \int_{\Omega} \left(\frac{\partial \mathsf{F}_2}{\partial x_1} - \frac{\partial \mathsf{F}_1}{\partial x_2} \right) dS_3 \tag{1.518}$$

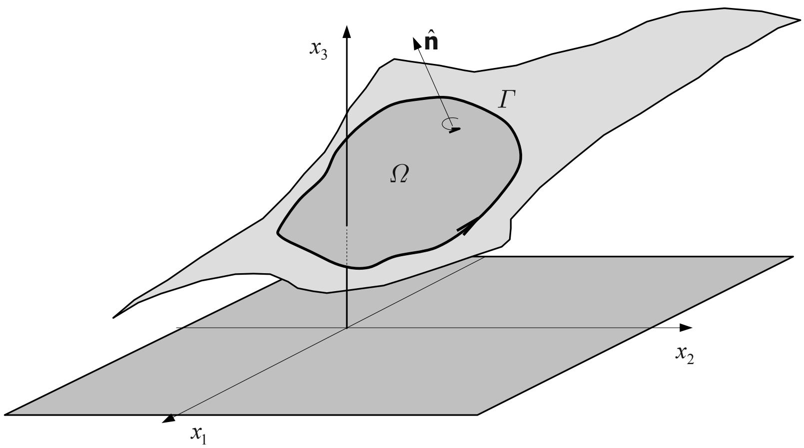


Figure 1.41.

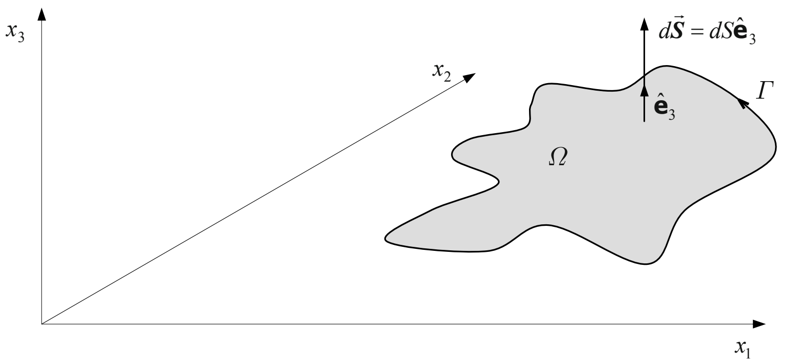


Figure 1.42: Green's theorem.

1.9.5 Green's Identities

Let $\vec{\mathbf{F}}$ be a vector field, and by applying the divergence theorem we obtain:

$$\int_V \nabla_{\vec{x}} \cdot \vec{\mathbf{F}} \; dV = \int_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}} \, dS \tag{1.519}$$

With reference to the equations (1.496) and (1.498), *i.e.*:

$$\nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi) = \phi \nabla_{\vec{x}}^{\,2} \psi + (\nabla_{\vec{x}} \phi) \cdot (\nabla_{\vec{x}} \psi) \tag{1.520}$$

$$\nabla_{\vec{x}} \cdot (\phi \nabla_{\vec{x}} \psi - \psi \nabla_{\vec{x}} \phi) = \phi \nabla_{\vec{x}}^{\,2} \psi - \psi \nabla_{\vec{x}}^{\,2} \phi \tag{1.521}$$

and also regarding that $\vec{\mathbf{F}} = \phi \nabla_{\vec{x}} \psi$, and by substituting (1.520) into (1.519) we obtain:

$$\int_V \phi\nabla_{\vec{x}}^2\psi + (\nabla_{\vec{x}}\phi)\cdot(\nabla_{\vec{x}}\psi)\; dV = \int_S \phi\nabla_{\vec{x}}\psi\cdot\hat{\mathbf{n}}\, dS$$
$$\Rightarrow \int_V (\nabla_{\vec{x}}\phi)\cdot(\nabla_{\vec{x}}\psi)\; dV = \int_S \phi\nabla_{\vec{x}}\psi\cdot\hat{\mathbf{n}}\, dS - \int_V \phi\nabla_{\vec{x}}^2\psi\; dV \tag{1.522}$$

which is known as *Green's first identity*.

Now, if we substituting (1.521) into (1.519) we obtain:

$$\int_V \phi\nabla_{\vec{x}}^2\psi - \psi\nabla_{\vec{x}}^2\phi\; dV = \int_S (\phi\nabla_{\vec{x}}\psi - \psi\nabla_{\vec{x}}\phi)\cdot\hat{\mathbf{n}}\, dS \tag{1.523}$$

which is known as *Green's second identity*.

Problem 1.48: Let $\vec{\mathbf{b}}$ be a vector field, which is defined as $\vec{\mathbf{b}} = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$. Show that:

$$\int_S \lambda \mathrm{b}_i \hat{\mathrm{n}}_i\; dS = \int_V \lambda_{,i}\, \mathrm{b}_i\; dV$$

where $\lambda = \lambda(\vec{x})$.

Solution: The Cartesian components of $\vec{\mathbf{b}} = \vec{\nabla}_{\vec{x}} \wedge \vec{\mathbf{v}}$ are $\mathrm{b}_i = \epsilon_{ijk}\mathrm{v}_{k,j}$ and by substituting them in the above surface integral we obtain:

$$\int_S \lambda \mathrm{b}_i \hat{\mathrm{n}}_i\; dS = \int_S \lambda\epsilon_{ijk}\mathrm{v}_{k,j}\hat{\mathrm{n}}_i\; dS$$

Applying the divergence theorem we obtain:

$$\begin{aligned}\int_S \lambda \mathrm{b}_i \hat{\mathrm{n}}_i\; dS = \int_S \lambda\epsilon_{ijk}\mathrm{v}_{k,j}\hat{\mathrm{n}}_i\; dS &= \int_V (\epsilon_{ijk}\lambda\mathrm{v}_{k,j})_{,i}\; dV \\ &= \int_V (\epsilon_{ijk}\lambda_{,i}\,\mathrm{v}_{k,j} + \epsilon_{ijk}\lambda\mathrm{v}_{k,ji})\, dV \\ &= \int_V (\lambda_{,i}\,\underbrace{\epsilon_{ijk}\mathrm{v}_{k,j}}_{\mathrm{b}_i} + \lambda\underbrace{\epsilon_{ijk}\mathrm{v}_{k,ji}}_{0})\, dV = \int_V \lambda_{,i}\,\mathrm{b}_i\; dV\end{aligned}$$

Appendix A

Graphical Representation of a Second-Order Tensor

A.1 Projecting a Second-Order Tensor onto a Particular Direction

A.1.1 Normal and Tangential Components

Projecting a second-order tensor ($\mathbf{T}$) onto the $\hat{\mathbf{n}}$-direction results in $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \mathbf{T}\cdot\hat{\mathbf{n}}$ whose vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ can be split into:

$$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \vec{\mathbf{T}}_N + \vec{\mathbf{T}}_S \tag{A.1}$$

where $\vec{\mathbf{T}}_N$ is the normal vector, and $\vec{\mathbf{T}}_S$ is the tangential vector, (see Figure A.1).

If we then bear in mind that $\hat{\mathbf{n}}$ and $\hat{\mathbf{s}}$ are unit vectors according to the directions $\vec{\mathbf{T}}_N$ and $\vec{\mathbf{T}}_S$, respectively, the vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ can also be expressed as:

$$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \mathsf{T}_N\hat{\mathbf{n}} + \mathsf{T}_S\hat{\mathbf{s}} \tag{A.2}$$

where T_N and T_S are the magnitudes of $\vec{\mathbf{T}}_N$ and $\vec{\mathbf{T}}_S$, respectively.

The vector $\vec{\mathbf{T}}_N$ can also be evaluated as follows:

$$\begin{aligned}
\vec{\mathbf{T}}_N &= \mathsf{T}_N\,\hat{\mathbf{n}} \equiv \mathsf{T}_N \otimes \hat{\mathbf{n}} \\
&= (\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} = (\hat{\mathbf{n}}\cdot\vec{\mathbf{t}}^{(\hat{\mathbf{n}})})\,\hat{\mathbf{n}} \\
&= \underbrace{\left[\hat{\mathbf{n}}\cdot\mathbf{T}\cdot\hat{\mathbf{n}}\right]}_{\mathsf{T}_N}\hat{\mathbf{n}}
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
\mathsf{T}_{N\,i} &= \mathsf{T}_N\,\hat{\mathsf{n}}_i \\
&= (\mathsf{t}_k^{(\hat{\mathbf{n}})}\hat{\mathsf{n}}_k)\,\hat{\mathsf{n}}_i = (\hat{\mathsf{n}}_k\mathsf{t}_k^{(\hat{\mathbf{n}})})\,\hat{\mathsf{n}}_i \\
&= \underbrace{\left[\hat{\mathsf{n}}_k\mathsf{T}_{kj}\hat{\mathsf{n}}_j\right]}_{\mathsf{T}_N}\hat{\mathsf{n}}_i
\end{aligned} \tag{A.3}$$

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_3,

Thus:

$$\mathsf{T}_N = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}} = \hat{\mathsf{n}}_k \mathsf{T}_{kj} \hat{\mathsf{n}}_j \tag{A.4}$$

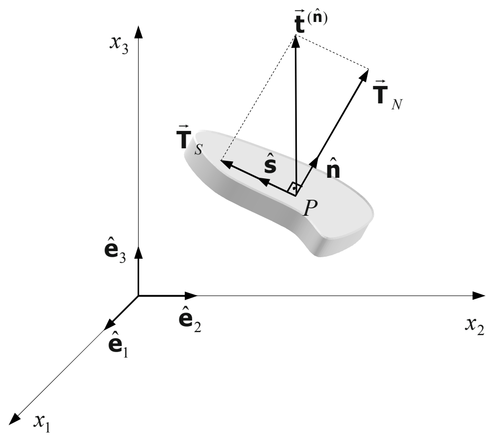


Figure A.1: Normal and tangential vectors.

As we saw in Chapter 1, $\mathbf{T}$ is a positive definite tensor if $\mathsf{T}_N = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}} > 0$ for all $\hat{\mathbf{n}} \neq \vec{\mathbf{0}}$. It is also true that $\mathsf{T}_N = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \mathbf{T}^{sym} \cdot \hat{\mathbf{n}}$, thus, if the symmetric part of a tensor is a positive definite tensor then the tensor will be also.

The vector $\vec{\mathbf{T}}_S$ can also be expressed in terms of T_S and $\hat{\mathbf{s}}$ as:

$$\begin{aligned} \vec{\mathbf{T}}_S &= \mathsf{T}_S \, \hat{\mathbf{s}} \\ &= (\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{s}})\hat{\mathbf{s}} = (\hat{\mathbf{s}} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})})\hat{\mathbf{s}} \\ &= \underbrace{\left[\hat{\mathbf{s}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}\right]}_{\mathsf{T}_S}\hat{\mathbf{s}} \end{aligned} \qquad \begin{aligned} \vec{\mathsf{T}}_{S\,i} &= \mathsf{T}_S \, \hat{\mathsf{s}}_i \\ &= (\mathsf{t}_j^{(\hat{\mathbf{n}})} \hat{\mathsf{s}}_j)\hat{\mathsf{s}}_i \\ &= \underbrace{\left[\mathsf{T}_{jk} \hat{\mathsf{n}}_k \hat{\mathsf{s}}_j\right]}_{\mathsf{T}_S}\hat{\mathsf{s}}_i \end{aligned} \tag{A.5}$$

Then, another way to evaluate the tangential vector can be by means of the equation in (A.1):

$$\vec{\mathbf{T}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \vec{\mathbf{T}}_N = \mathbf{T} \cdot \hat{\mathbf{n}} - \left[\mathbf{T} : (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})\right]\hat{\mathbf{n}} \tag{A.6}$$

Note, the magnitude of $\vec{\mathbf{T}}_S$ can also be obtained by means of the Pythagorean Theorem, *i.e.:*

$$\left\| \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \right\|^2 = \left\| \vec{\mathbf{T}}_N \right\|^2 + \left\| \vec{\mathbf{T}}_S \right\|^2 \quad \Rightarrow \quad \mathsf{T}_S^2 = \mathsf{t}_i^{(\hat{\mathbf{n}})} \mathsf{t}_i^{(\hat{\mathbf{n}})} - \mathsf{T}_N^2 \tag{A.7}$$

where $\mathsf{t}_i^{(\hat{\mathbf{n}})} \mathsf{t}_i^{(\hat{\mathbf{n}})} = \mathsf{T}_{ij} \mathsf{T}_{ik} \hat{\mathsf{n}}_j \hat{\mathsf{n}}_k$.

The question now is: on what plane is the maximum normal and tangential component? The answer to this problem is related to the maximum and minimum values of a function, which will be discussed in the next subsection.

Problem A.1: Show that $\vec{\mathbf{T}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot (\mathbf{1} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$, where $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ is vector resulting from projecting the second-order tensor $\mathbf{T}$ onto the $\hat{\mathbf{n}}$-direction, and $\vec{\mathbf{T}}_S$ is the tangential vector.

Solution 1: If we consider that $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \vec{\mathbf{T}}_N + \vec{\mathbf{T}}_S$ and (A.3) we can state that:

$$\vec{\mathbf{T}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \left[\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}}\right] \hat{\mathbf{n}} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot (\mathbf{1} - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$$

Solution 2: We can also solve the problem just using the components of the equation (A.6), $\vec{\mathbf{T}}_S = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - [\boldsymbol{\sigma} : (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}})]\hat{\mathbf{n}}$, *i.e.*:

$$\mathsf{T}_{S\,i} = \mathsf{t}_i^{(\hat{\mathbf{n}})} - [(\hat{\mathsf{n}}_k \hat{\mathsf{n}}_l \mathsf{T}_{kl})]\hat{\mathsf{n}}_i = \mathsf{t}_i^{(\hat{\mathbf{n}})} - \hat{\mathsf{n}}_i \hat{\mathsf{n}}_k \mathsf{t}_k^{(\hat{\mathbf{n}})} = \mathsf{t}_k^{(\hat{\mathbf{n}})} \delta_{ik} - \hat{\mathsf{n}}_i \hat{\mathsf{n}}_k \mathsf{t}_k^{(\mathbf{n})} = \mathsf{t}_k^{(\hat{\mathbf{n}})} (\delta_{ik} - \hat{\mathsf{n}}_i \hat{\mathsf{n}}_k)$$

A.1.2 The Maximum and Minimum Normal Components

As we have seen previously the normal component is given by $\mathsf{T}_N = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}$ with the constraint $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$ (the unit vector). The maximum and minimum values of T_N with this constraint can be evaluated by means of the Lagrange multiplier method which consists in constructing a function such as:

$$\mathcal{L}(\hat{\mathbf{n}}, \mu) = \mathsf{T}_N - \mu(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} - 1) = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}} - \mu(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} - 1) \tag{A.8}$$

where μ is known as the Lagrange multiplier. Then, the derivative of the function $\mathcal{L}(\hat{\mathbf{n}})$ with respect to $\hat{\mathbf{n}}$ and μ, yields the following set of equations:

$$\begin{aligned} \frac{\partial \mathcal{L}(\hat{\mathbf{n}}, \mu)}{\partial \hat{\mathbf{n}}} &= 2\mathbf{T}^{sym} \cdot \hat{\mathbf{n}} - 2\mu\hat{\mathbf{n}} = \vec{\mathbf{0}} \quad \Rightarrow \quad (\mathbf{T}^{sym} - \mu\mathbf{1}) \cdot \hat{\mathbf{n}} = \vec{\mathbf{0}} \\ \frac{\partial \mathcal{L}(\hat{\mathbf{n}}, \mu)}{\partial \mu} &= \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} - 1 = 0 \quad \Rightarrow \quad \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1 \end{aligned} \tag{A.9}$$

The first set can only be solved if and only if $\det(\mathbf{T}^{sym} - \mu\mathbf{1}) = 0$, which is the eigenvalue problem of the symmetric part of $\mathbf{T}$. That is, the maximum and minimum of T_N correspond to the eigenvalues of $\mathbf{T}^{sym}$. Now, if we consider that T_1^{sym}, T_2^{sym}, T_3^{sym}, are the eigenvalues of $\mathbf{T}^{sym}$, we can then restructure these such that:

$$\mathsf{T}_{\mathrm{I}}^{sym} > \mathsf{T}_{\mathrm{II}}^{sym} > \mathsf{T}_{\mathrm{III}}^{sym} \tag{A.10}$$

Then, the maximum value of T_N is defined by $\mathsf{T}_{\mathrm{I}}^{sym}$, and the minimum value by $\mathsf{T}_{\mathrm{III}}^{sym}$.

OBS.: When the nomenclature $\mathsf{T}_{\mathrm{I}}, \mathsf{T}_{\mathrm{II}}, \mathsf{T}_{\mathrm{III}}$ are used to represent the eigenvalues, $\mathsf{T}_{\mathrm{I}} > \mathsf{T}_{\mathrm{II}} > \mathsf{T}_{\mathrm{III}}$ is implicit.

NOTE: As expected, the extreme values of T_N relate to the symmetric part $\mathbf{T}^{sym}$ because the antisymmetric part plays no role in the normal component, *i.e.* $\mathsf{T}_N = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \mathbf{T}^{sym} \cdot \hat{\mathbf{n}}$.

A.1.3 The Maximum and Minimum Tangential Component

In this section we will consider a symmetric second-order tensor, *i.e.* $\mathbf{T}=\mathbf{T}^T$. For the sake of convenience, we will work here in the principal space of $\mathbf{T}$ where the tensor components are just represented by the normal components, (see Figure A.2).

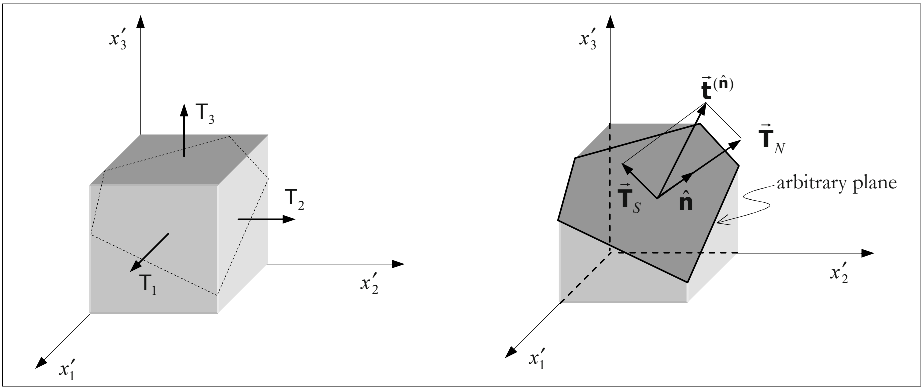


Figure A.2: Principal space.

Then, the normal component T_N for the $\hat{\mathbf{n}}$-direction can be obtained by means of the equation in (A.4) as follows:

$$\mathsf{T}_N = \mathsf{t}_i^{(\hat{\mathbf{n}})}\hat{\mathsf{n}}_i = \mathsf{T}_{ij}\hat{\mathsf{n}}_j\hat{\mathsf{n}}_i = \mathsf{T}_1\hat{\mathsf{n}}_1^2 + \mathsf{T}_2\hat{\mathsf{n}}_2^2 + \mathsf{T}_3\hat{\mathsf{n}}_3^2 \tag{A.11}$$

Note that, on the particular plane $\hat{\mathsf{n}}_i = [1,0,0] \Rightarrow \mathsf{T}_N = \mathsf{T}_1$.

The tangential component can then be evaluated as follows:

$$\mathsf{T}_S^2 = \left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\right\|^2 - \mathsf{T}_N^2 = \mathsf{t}_i^{(\hat{\mathbf{n}})}\mathsf{t}_i^{(\hat{\mathbf{n}})} - \mathsf{T}_N^2 = \mathsf{T}_{ij}\mathsf{T}_{ik}\hat{\mathsf{n}}_j\hat{\mathsf{n}}_k - \mathsf{T}_N^2 \tag{A.12}$$

Next, by combining the equations (A.11) and (A.12) we find:

$$\mathsf{T}_S^2 = \mathsf{T}_1^2\hat{\mathsf{n}}_1^2 + \mathsf{T}_2^2\hat{\mathsf{n}}_2^2 + \mathsf{T}_3^2\hat{\mathsf{n}}_3^2 - \left(\mathsf{T}_1\hat{\mathsf{n}}_1^2 + \mathsf{T}_2\hat{\mathsf{n}}_2^2 + \mathsf{T}_3\hat{\mathsf{n}}_3^2\right)^2 \tag{A.13}$$

If we now ask what values of $\hat{\mathsf{n}}_i$ maximize the function T_S^2, this problem is equivalent to find extreme values of the function:

$$F(\hat{\mathbf{n}}) = \mathsf{T}_S^2 - \mu\left(\hat{\mathsf{n}}_i\hat{\mathsf{n}}_i - 1\right) \tag{A.14}$$

where μ is the Lagrange multiplier, with the constraint $\hat{\mathsf{n}}_i\hat{\mathsf{n}}_i = 1$. Then:

$$\frac{\partial F(\hat{\mathbf{n}})}{\partial \mathsf{n}_j} = 0_j \qquad ; \qquad \frac{\partial F(\hat{\mathbf{n}})}{\partial \mu} = 0 \tag{A.15}$$

and by solving the above set of equations we obtain:

$$\begin{aligned}
\hat{\mathsf{n}}_1\left\{\mathsf{T}_1^2 - 2\mathsf{T}_1\left(\mathsf{T}_1\hat{\mathsf{n}}_1^2 + \mathsf{T}_2\hat{\mathsf{n}}_2^2 + \mathsf{T}_3\hat{\mathsf{n}}_3^2\right) + \mu\right\} &= 0\\
\hat{\mathsf{n}}_2\left\{\mathsf{T}_2^2 - 2\mathsf{T}_2\left(\mathsf{T}_1\hat{\mathsf{n}}_1^2 + \mathsf{T}_2\hat{\mathsf{n}}_2^2 + \mathsf{T}_3\hat{\mathsf{n}}_3^2\right) + \mu\right\} &= 0\\
\hat{\mathsf{n}}_3\left\{\mathsf{T}_3^2 - 2\mathsf{T}_3\left(\mathsf{T}_1\hat{\mathsf{n}}_1^2 + \mathsf{T}_2\hat{\mathsf{n}}_2^2 + \mathsf{T}_3\hat{\mathsf{n}}_3^2\right) + \mu\right\} &= 0
\end{aligned} \tag{A.16}$$

with the constraint $\hat{\mathsf{n}}_i\hat{\mathsf{n}}_i = 1$. Then, the analytical solution of the previous set of equations results in the following possible solutions:

solutions	$\hat{n}_1$	$\hat{n}_2$	$\hat{n}_3$	T_S
(1)	$\hat{n}_1^{(1)} = \pm 1$	$\hat{n}_2^{(1)} = 0$	$\hat{n}_3^{(1)} = 0$	$T_S = 0$
(2)	$\hat{n}_1^{(2)} = 0$	$\hat{n}_2^{(2)} = \pm 1$	$\hat{n}_3^{(2)} = 0$	$T_S = 0$
(3)	$\hat{n}_1^{(3)} = 0$	$\hat{n}_1^{(3)} = 0$	$\hat{n}_1^{(3)} = \pm 1$	$T_S = 0$
(4)	0	$\pm \frac{1}{\sqrt{2}}$	$\pm \frac{1}{\sqrt{2}}$	$T_S = \pm \frac{T_2 - T_3}{2}$
(5)	$\pm \frac{1}{\sqrt{2}}$	0	$\pm \frac{1}{\sqrt{2}}$	$T_S = \pm \frac{T_1 - T_3}{2}$
(6)	$\pm \frac{1}{\sqrt{2}}$	$\pm \frac{1}{\sqrt{2}}$	0	$T_S = \pm \frac{T_1 - T_2}{2}$

(A.17)

where the values of T_S were obtained by substituting the values of $\hat{n}_i$ into (A.13).

The first three sets of solutions give us the minimum values of T_S, which correspond precisely to the principal directions.

For solutions (4), (5) and (6) the planes are outlined as shown in Figure A.3, Figure A.4 and Figure A.5, respectively.

Then, by ordering the eigenvalues (the principal values T_1, T_2, T_3) such that $T_I > T_{II} > T_{III}$ we can find the absolute maximum tangential component:

$$T_{S\,\max} = \frac{T_I - T_{III}}{2} \tag{A.18}$$

which corresponds to solution (5) of (A.17).

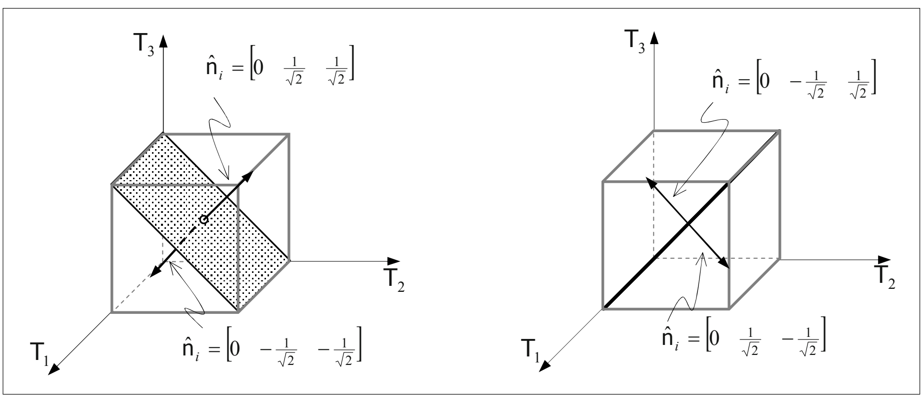


Figure A.3: The maximum relative for T_S, solution (4).

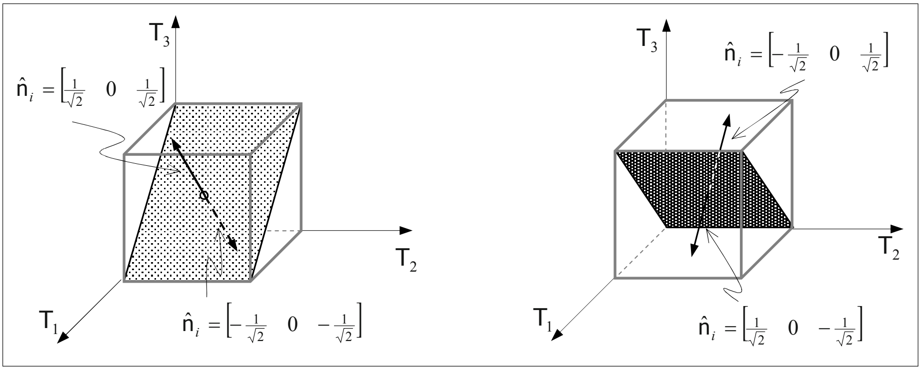


Figure A.4: The maximum relative for T_S, solution (5).

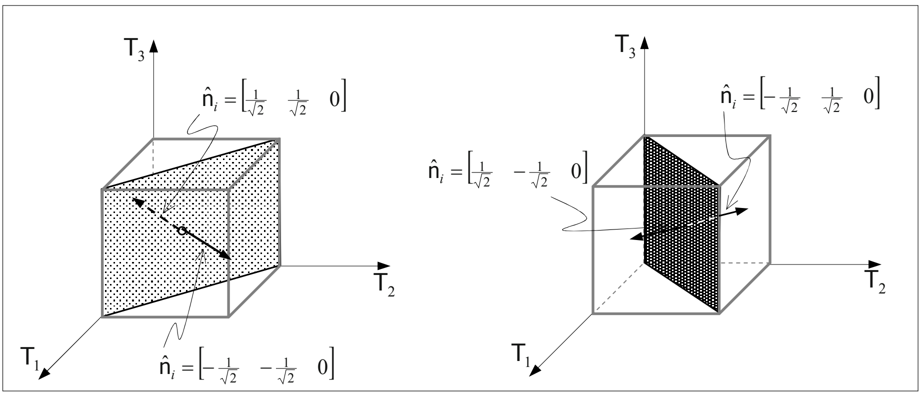


Figure A.5: The maximum relative for T_S, solution (6).

A.2 Graphical Representation of an Arbitrary Second-Order Tensor

If the second-order tensor Cartesian components are known, it is possible to evaluate the normal and tangential components (T_N, T_S) on any plane defined by the normal $\hat{\mathbf{n}}$, with the constraint $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = \hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$. Our goal in this section is to draw a graph in which the abscissa is represented by the normal components (T_N) and the ordinate represents the tangential component (T_S). This procedure can be carried out either numerically or analytically.

Firstly, we will adopt a numerical procedure, *i.e.* we will randomly evaluate different values for the normal $\hat{\mathbf{n}}$, and then we will obtain the corresponding values of (T_N, T_S), whose values will be plotted on the graph $T_N \times T_S$. In this way we will obtain all feasible values of the vector resulting from projecting the tensor onto a direction. Similarly, we will also construct a graph which corresponds to the symmetric part, *i.e.* $(T_N^{(sym)} = T_N) \times T_S^{(sym)}$.

The first example, (see Figure A.6), is a non-symmetric tensor and it is noteworthy that it is positive definite since $\mathsf{T}_N = \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}} > 0$ for all $\hat{\mathbf{n}} \neq \vec{\mathbf{0}}$. We will also verify that it has three real eigenvalues, which correspond to $\mathsf{T}_S = 0$. Note, the maximum and minimum values for the T_N coincide with the eigenvalues of the symmetric part of $\mathbf{T}$.

Then, for the tangential vector, we can carry out the following decomposition:

$$\vec{\mathbf{T}}_S = \underbrace{[\hat{\mathbf{s}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]}_{\mathsf{T}_S}\hat{\mathbf{s}} = [\hat{\mathbf{s}} \cdot \mathbf{T}^{sym} \cdot \hat{\mathbf{n}} + \hat{\mathbf{s}} \cdot \mathbf{T}^{skew} \cdot \hat{\mathbf{n}}]\hat{\mathbf{s}} \tag{A.19}$$

When $\hat{\mathbf{n}}$ is one of the principal directions of the symmetric part, we have:

$$\vec{\mathbf{T}}_S = [\hat{\mathbf{s}} \cdot \mathbf{T}^{sym} \cdot \hat{\mathbf{n}} + \hat{\mathbf{s}} \cdot \mathbf{T}^{skew} \cdot \hat{\mathbf{n}}]\hat{\mathbf{s}} = [\hat{\mathbf{s}} \cdot \lambda\hat{\mathbf{n}} + \hat{\mathbf{s}} \cdot \mathbf{T}^{skew} \cdot \hat{\mathbf{n}}]\hat{\mathbf{s}} = [\hat{\mathbf{s}} \cdot \mathbf{T}^{skew} \cdot \hat{\mathbf{n}}]\hat{\mathbf{s}} \tag{A.20}$$

where we have considered that $\hat{\mathbf{s}} \cdot \hat{\mathbf{n}} = 0$, since the unit vectors $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ are orthogonal to each other. So, the tangential component can be obtained as:

$$\mathsf{T}_S = \hat{\mathbf{s}} \cdot \mathbf{T}^{skew} \cdot \hat{\mathbf{n}} = \hat{\mathbf{s}} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew} = \pm\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew}\right\| \hat{\mathbf{s}} \cdot \hat{\mathbf{s}} = \pm\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew}\right\| \tag{A.21}$$

We must emphasize that this procedure is only valid when $\hat{\mathbf{n}}$ is a principal direction of $\mathbf{T}^{sym}$, but not on an arbitrary plane.

For example, for the eigenvalue $\mathsf{T}_I^{sym} = 10.55$, which is associated with the eigenvector $\hat{\mathsf{n}}_j^{(1)} = [-0.45229371; -0.561517458; -0.692913086]$, we have:

$$\mathsf{T}_S = \pm\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew}\right\| = \pm\left\| \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}}_{\mathsf{T}_{ij}^{anti}} \underbrace{\begin{bmatrix} -0.45229371 \\ -0.561517458 \\ -0.692913086 \end{bmatrix}}_{\hat{\mathsf{n}}_j^{(1)}} \right\| = \pm\left\| \begin{bmatrix} 0.1313956 \\ 1.8381199 \\ -1.5753286 \end{bmatrix} \right\| = \pm 2.424378 \tag{A.22}$$

for the eigenvalue $\mathsf{T}_{II}^{sym} = 3.61$, we find:

$$\mathsf{T}_S = \pm\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew}\right\| = \pm\left\| \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0.88542667 \\ 0.18949182 \\ 0.42439659 \end{bmatrix} \right\| = \pm\left\| \begin{bmatrix} -0.234905 \\ 0.036633496 \\ -0.50644304 \end{bmatrix} \right\| = \pm 0.55947 \tag{A.23}$$

and, for the eigenvalue $\mathsf{T}_{III}^{sym} = 0.84$, we obtain:

$$\mathsf{T}_S = \pm\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew}\right\| = \pm\left\| \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0.107004733 \\ -0.80547563 \\ 0.582888489 \end{bmatrix} \right\| = \pm\left\| \begin{bmatrix} -1.3883641 \\ -1.2727817 \\ -1.5039465 \end{bmatrix} \right\| = \pm 2.41 \tag{A.24}$$

Note that the maximum and minimum normal components of $\mathbf{T}$ corresponds to the eigenvalues of $\mathbf{T}^{sym}$, *i.e.*, $\mathsf{T}_{N_{\max}} = \mathsf{T}_I^{sym} = 10.55$ and $\mathsf{T}_{N_{\min}} = \mathsf{T}_{III}^{sym} = 0.84$. With respect to the symmetric part, the maximum tangential component is equal to the radius of the circle formed by $\mathsf{T}_I^{sym} = 10.55$ and $\mathsf{T}_{III}^{sym} = 0.84$, (see Figure A.6), which is $\mathsf{T}_{S\ \max}^{sym} = \dfrac{10.55 - 0.84}{2} = 4.86$.

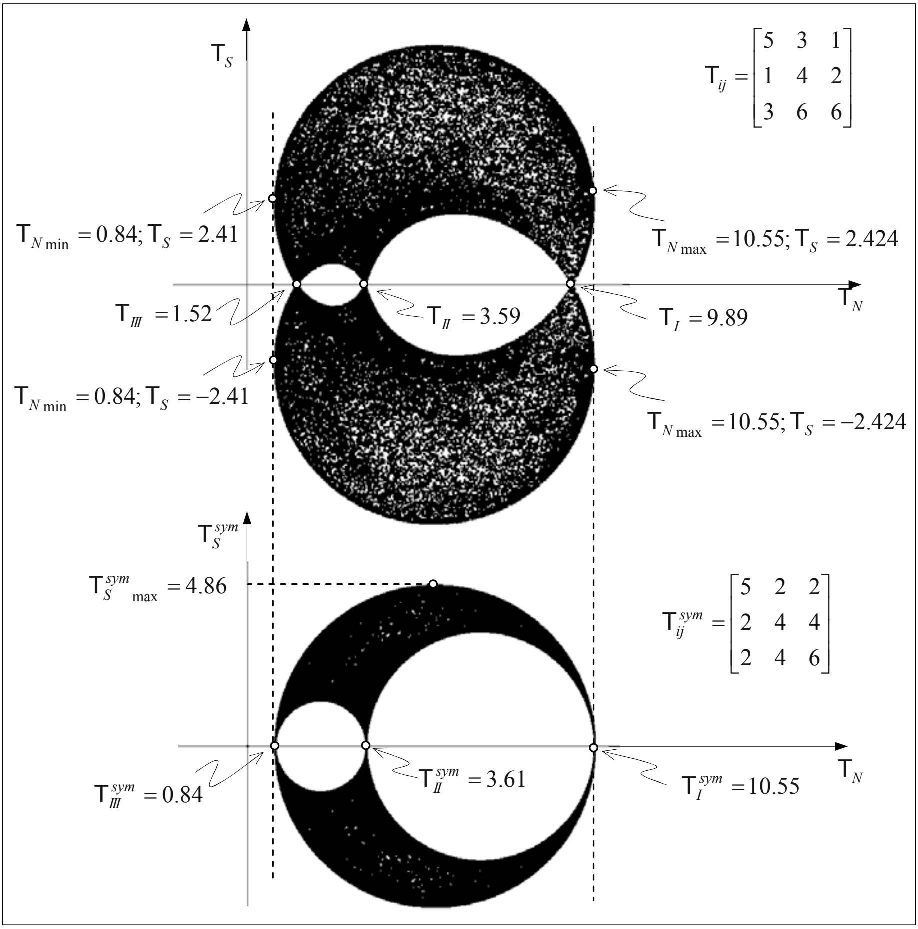


Figure A.6: Graphical representation of a positive definite tensor.

The second example is a symmetric tensor, which has two equal eigenvalues. We can now verify that the possible values for (T_N, T_S) are limited to the circumference radius $R = \frac{T_{\mathrm{I}} - T_{\mathrm{III}}}{2} = 2.5$ and centered at the point ($T_N = \frac{T_{\mathrm{I}} + T_{\mathrm{III}}}{2} = 1.5$, $T_S = 0$), (see Figure A.7). Intuitively, this leads us to believe that the graphical representation of a spherical tensor reduces to a single point.

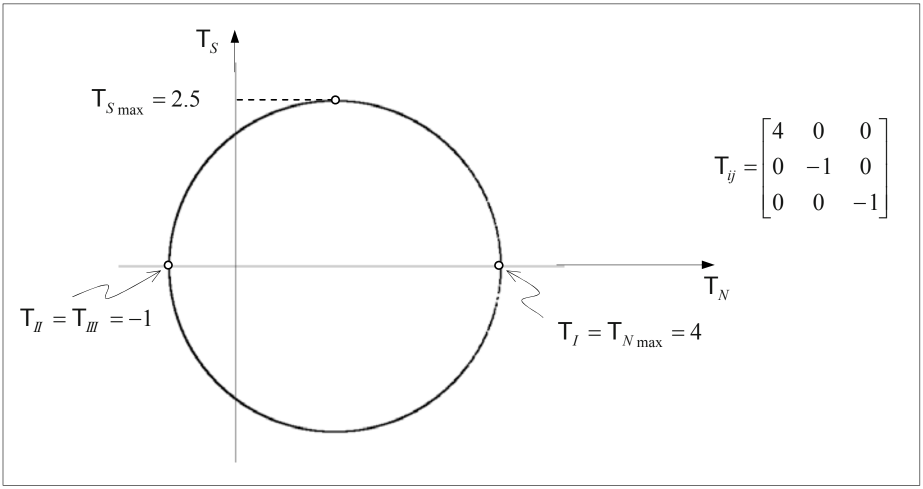


Figure A.7: Graphical representation of a symmetric tensor with two equal eigenvalues.

The third example is a non-symmetric tensor, which has one real eigenvalue that is equal to $\mathsf{T}_1 = -0.964$, (see Figure A.8).

In the principal direction of the symmetric part of $\mathbf{T}$, we have the following values for the tangential component:

For the eigenvalue $\mathsf{T}_I^{sym} = 9.894$, we have:

$$\mathsf{T}_S = \pm\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew}\right\| = \pm\left\|\begin{bmatrix} 0 & 5 & 1.5 \\ -5 & 0 & -3 \\ -1.5 & 3 & 0 \end{bmatrix}\begin{bmatrix} -0.707427855 \\ -0.514420622 \\ -0.484682632 \end{bmatrix}\right\| = \pm\left\|\begin{bmatrix} -3.299127 \\ 4.991187 \\ -0.482120 \end{bmatrix}\right\| = \pm 6.0 \qquad \text{(A.25)}$$

and the eigenvalue $\mathsf{T}_{III}^{sym} = -2.02$, we have:

$$\mathsf{T}_S = \pm\left\|\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}_{skew}\right\| = \pm\left\|\begin{bmatrix} 0 & 5 & 1.5 \\ -5 & 0 & -3 \\ -1.5 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0.0575152387 \\ -0.72538138475 \\ 0.685940116897 \end{bmatrix}\right\| = \pm\left\|\begin{bmatrix} -2.5979967 \\ -2.3453965 \\ -2.2624170 \end{bmatrix}\right\| = \pm 4.1676 \qquad \text{(A.26)}$$

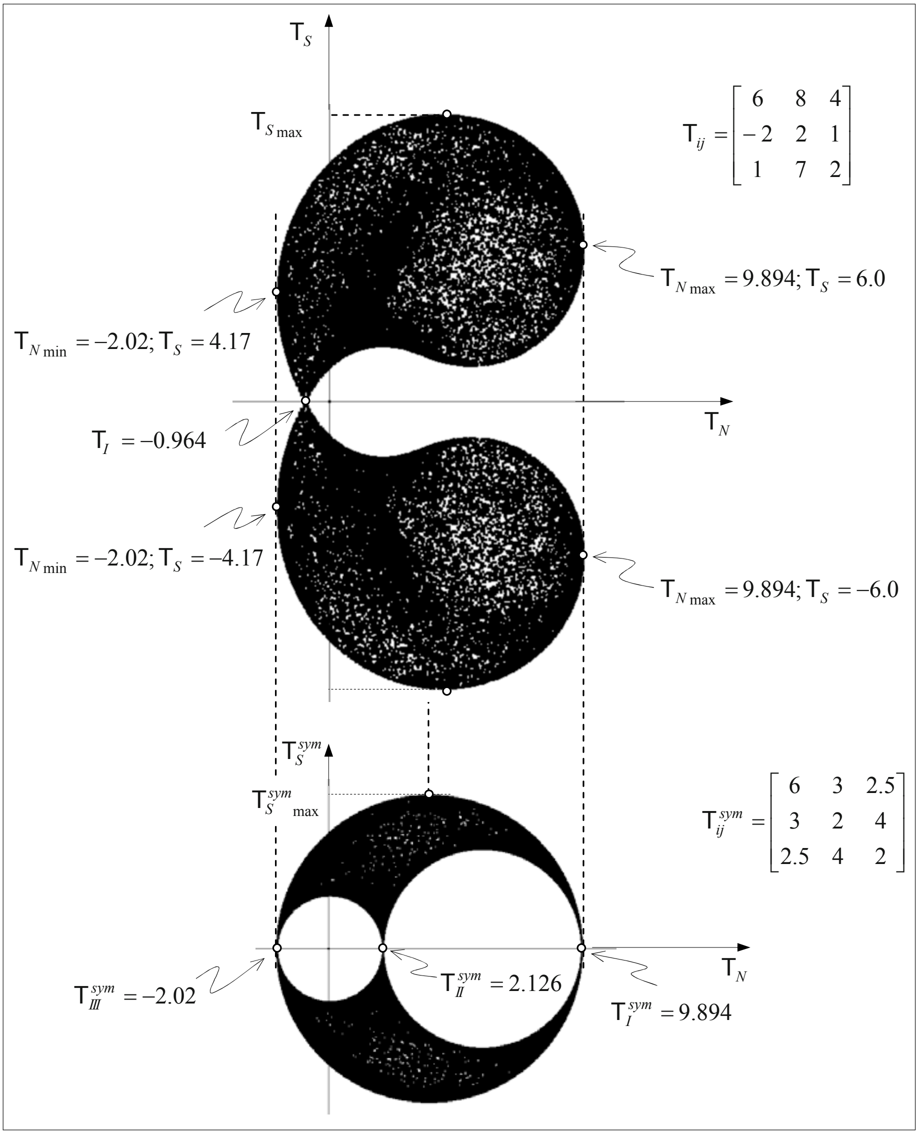


Figure A.8: Graphical representation of a non-symmetric tensor with only one real eigenvalue.

A.2.1 Graphical Representation of a Symmetric Second-Order Tensor (Mohr's Circle)

In this subsection we will focus our attention on the graphical representation of a symmetric second-order tensor, which is denoted by the *Mohr's Circle*.

Once again, we will work in the principal space, and we will assume that the eigenvalues of $\mathbf{T}$ are ordered such that $T_I > T_{II} > T_{III}$. We will start from the equation in (A.7), *i.e.:*

$$\mathsf{T}_S^2 + \mathsf{T}_N^2 = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \left\| \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \right\|^2 \tag{A.27}$$

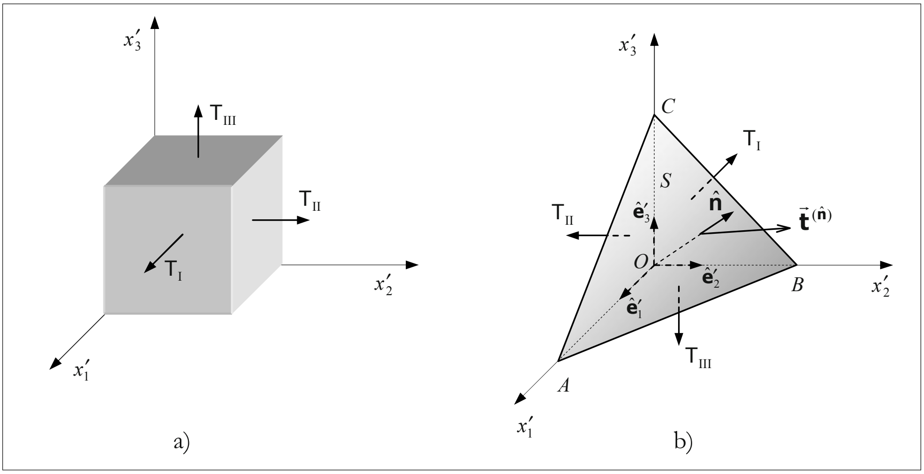


Figure A.9: An arbitrary plane in the principal space.

The components of the vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$, in an arbitrary plane, were obtained in the Eq. (A.1). Then the components of $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \mathbf{T} \cdot \hat{\mathbf{n}}$ in the principal space, (see Figure A.9(b)), are given by:

$$\mathsf{t}_1^{(\hat{\mathbf{n}})} = \mathsf{T}_I \hat{\mathsf{n}}_1 \quad ; \quad \mathsf{t}_2^{(\hat{\mathbf{n}})} = \mathsf{T}_{II} \hat{\mathsf{n}}_2 \quad ; \quad \mathsf{t}_3^{(\hat{\mathbf{n}})} = \mathsf{T}_{III} \hat{\mathsf{n}}_3 \tag{A.28}$$

Next, the dot product $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ is evaluated as follows:

$$\begin{aligned} \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} &= \mathsf{t}_i^{(\hat{\mathbf{n}})} \hat{\mathbf{e}}_i' \cdot \mathsf{t}_j^{(\hat{\mathbf{n}})} \hat{\mathbf{e}}_j' = \mathsf{t}_i^{(\hat{\mathbf{n}})} \mathsf{t}_j^{(\hat{\mathbf{n}})} \delta_{ij} = \mathsf{t}_i^{(\hat{\mathbf{n}})} \mathsf{t}_i^{(\hat{\mathbf{n}})} \\ &= \mathsf{t}_1^{(\hat{\mathbf{n}})} \mathsf{t}_1^{(\hat{\mathbf{n}})} + \mathsf{t}_2^{(\hat{\mathbf{n}})} \mathsf{t}_2^{(\hat{\mathbf{n}})} + \mathsf{t}_3^{(\hat{\mathbf{n}})} \mathsf{t}_3^{(\hat{\mathbf{n}})} \\ &= \mathsf{T}_I^2 \hat{\mathsf{n}}_1^2 + \mathsf{T}_{II}^2 \hat{\mathsf{n}}_2^2 + \mathsf{T}_{III}^2 \hat{\mathsf{n}}_3^2 \end{aligned} \tag{A.29}$$

and by combining the equations (A.27) and (A.29), we obtain:

$$\mathsf{T}_S^2 + \mathsf{T}_N^2 = \mathsf{T}_I^2 \hat{\mathsf{n}}_1^2 + \mathsf{T}_{II}^2 \hat{\mathsf{n}}_2^2 + \mathsf{T}_{III}^2 \hat{\mathsf{n}}_3^2 \tag{A.30}$$

Then, the normal component T_N, in the principal space, can be expressed as:

$$\mathsf{T}_N = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = \mathsf{T}_{ij} \hat{\mathsf{n}}_j \hat{\mathsf{n}}_i = \mathsf{T}_I \hat{\mathsf{n}}_1^2 + \mathsf{T}_{II} \hat{\mathsf{n}}_2^2 + \mathsf{T}_{III} \hat{\mathsf{n}}_3^2 \tag{A.31}$$

where we have used the equation in (A.4).

Let us consider the constraint $\hat{\mathsf{n}}_i \hat{\mathsf{n}}_i = 1 \Rightarrow \hat{\mathsf{n}}_1^2 = 1 - \hat{\mathsf{n}}_2^2 - \hat{\mathsf{n}}_3^2$ which if we substitute into (A.31) give us the value of $\hat{\mathsf{n}}_2^2$:

$$\mathsf{T}_N = \mathsf{T}_I (1 - \hat{\mathsf{n}}_2^2 - \hat{\mathsf{n}}_3^2) + \mathsf{T}_{II} \hat{\mathsf{n}}_2^2 + \mathsf{T}_{III} \hat{\mathsf{n}}_3^2 \quad \Rightarrow \quad \hat{\mathsf{n}}_2^2 = \frac{\mathsf{T}_N - \mathsf{T}_{III} \hat{\mathsf{n}}_3^2 + \mathsf{T}_I \hat{\mathsf{n}}_3^2 - \mathsf{T}_I}{(\mathsf{T}_{II} - \mathsf{T}_I)} \tag{A.32}$$

Then, substituting $(\hat{\mathsf{n}}_1^2 = 1 - \hat{\mathsf{n}}_2^2 - \hat{\mathsf{n}}_3^2)$ into the equation (A.30) yields:

$$\begin{aligned} \mathsf{T}_S^2 + \mathsf{T}_N^2 &= \mathsf{T}_I^2 \hat{\mathsf{n}}_1^2 + \mathsf{T}_{II}^2 \hat{\mathsf{n}}_2^2 + \mathsf{T}_{III}^2 \hat{\mathsf{n}}_3^2 \\ &= \mathsf{T}_I^2 (1 - \hat{\mathsf{n}}_2^2 - \hat{\mathsf{n}}_3^2) + \mathsf{T}_{II}^2 \hat{\mathsf{n}}_2^2 + \mathsf{T}_{III}^2 \hat{\mathsf{n}}_3^2 \end{aligned} \tag{A.33}$$

and substituting $\hat{\mathsf{n}}_2^2$, obtained in (A.32), into the above equation results in:

$$\mathsf{T}_S^2+\mathsf{T}_N^2=\left[(\mathsf{T}_{III}-\mathsf{T}_I)(\mathsf{T}_{III}-\mathsf{T}_{II})\right]\hat{\mathsf{n}}_3^2+\mathsf{T}_{II}\mathsf{T}_N+\mathsf{T}_I\,\mathsf{T}_N-\mathsf{T}_{II}\mathsf{T}_{III} \tag{A.34}$$

Next, if we evaluate the term $\hat{\mathsf{n}}_3^2$ we obtain:

$$\hat{\mathsf{n}}_3^2=\frac{(\mathsf{T}_N-\mathsf{T}_I)(\mathsf{T}_N-\mathsf{T}_{II})+\mathsf{T}_S^2}{(\mathsf{T}_{III}-\mathsf{T}_I)(\mathsf{T}_{III}-\mathsf{T}_{II})} \tag{A.35}$$

and we can find $\hat{\mathsf{n}}_1^2$ and $\hat{\mathsf{n}}_2^2$ in a similar fashion. Finally, we can summarize the results as follows:

$$\begin{aligned}
\hat{\mathsf{n}}_1^2&=\frac{(\mathsf{T}_N-\mathsf{T}_{II})(\mathsf{T}_N-\mathsf{T}_{III})+\mathsf{T}_S^2}{(\mathsf{T}_I-\mathsf{T}_{II})(\mathsf{T}_I-\mathsf{T}_{III})}\geq 0 &&\text{(a)}\\
\hat{\mathsf{n}}_2^2&=\frac{(\mathsf{T}_N-\mathsf{T}_{III})(\mathsf{T}_N-\mathsf{T}_I)+\mathsf{T}_S^2}{(\mathsf{T}_{II}-\mathsf{T}_{III})(\mathsf{T}_{II}-\mathsf{T}_I)}\geq 0 &&\text{(b)}\\
\hat{\mathsf{n}}_3^2&=\frac{(\mathsf{T}_N-\mathsf{T}_I)(\mathsf{T}_N-\mathsf{T}_{II})+\mathsf{T}_S^2}{(\mathsf{T}_{III}-\mathsf{T}_I)(\mathsf{T}_{III}-\mathsf{T}_{II})}\geq 0 &&\text{(c)}
\end{aligned} \tag{A.36}$$

Then, if we consider that $\mathsf{T}_I>\mathsf{T}_{II}>\mathsf{T}_{III}$ we can verify that the equations (A.36) (a) and (c) have positive denominators; so, their nominators must also be positive. However, the equation in (A.36) (b) has negative denominators, so its nominator must be negative too. *i.e.*:

$$\left.\begin{aligned}
\hat{\mathsf{n}}_1^2&=\frac{\left[(\mathsf{T}_N-\mathsf{T}_{II})(\mathsf{T}_N-\mathsf{T}_{III})+\mathsf{T}_S^2\right]\geq 0}{\left[(\mathsf{T}_I-\mathsf{T}_{II})(\mathsf{T}_I-\mathsf{T}_{III})\right]>0}\geq 0\\
\hat{\mathsf{n}}_2^2&=\frac{\left[(\mathsf{T}_N-\mathsf{T}_{III})(\mathsf{T}_N-\mathsf{T}_I)+\mathsf{T}_S^2\right]\leq 0}{\left[(\mathsf{T}_{II}-\mathsf{T}_{III})(\mathsf{T}_{II}-\mathsf{T}_I)\right]<0}\geq 0\\
\hat{\mathsf{n}}_3^2&=\frac{\left[(\mathsf{T}_N-\mathsf{T}_I)(\mathsf{T}_N-\mathsf{T}_{II})+\mathsf{T}_S^2\right]\geq 0}{\left[(\mathsf{T}_{III}-\mathsf{T}_I)(\mathsf{T}_{III}-\mathsf{T}_{II})\right]>0}\geq 0
\end{aligned}\right\}\Rightarrow\begin{cases}
(\mathsf{T}_N-\mathsf{T}_{II})(\mathsf{T}_N-\mathsf{T}_{III})+\mathsf{T}_S^2\geq 0\\
(\mathsf{T}_N-\mathsf{T}_{III})(\mathsf{T}_N-\mathsf{T}_I)+\mathsf{T}_S^2\leq 0\\
(\mathsf{T}_N-\mathsf{T}_I)(\mathsf{T}_N-\mathsf{T}_{II})+\mathsf{T}_S^2\geq 0
\end{cases} \tag{A.37}$$

Then after some algebraic manipulations, the previous inequalities in (A.37) become:

$$\begin{aligned}
\mathsf{T}_S^2+\left[\mathsf{T}_N-\tfrac{1}{2}(\mathsf{T}_{II}+\mathsf{T}_{III})\right]^2&\geq\left[\tfrac{1}{2}(\mathsf{T}_{II}-\mathsf{T}_{III})\right]^2\\
\mathsf{T}_S^2+\left[\mathsf{T}_N-\tfrac{1}{2}(\mathsf{T}_I+\mathsf{T}_{III})\right]^2&\leq\left[\tfrac{1}{2}(\mathsf{T}_I-\mathsf{T}_{III})\right]^2\\
\mathsf{T}_S^2+\left[\mathsf{T}_N-\tfrac{1}{2}(\mathsf{T}_I+\mathsf{T}_{II})\right]^2&\geq\left[\tfrac{1}{2}(\mathsf{T}_I-\mathsf{T}_{II})\right]^2
\end{aligned} \tag{A.38}$$

The above shows equations of circles. The first circle with the center $\frac{1}{2}(\mathsf{T}_{II}+\mathsf{T}_{III})$ and radius $\frac{1}{2}(\mathsf{T}_{II}-\mathsf{T}_{III})$, indicates that the feasible points for the pair $(\mathsf{T}_N;\mathsf{T}_S)$ are outside the circle C_1 , including the circumference, (see Figure A.10). The second circle with the center $\frac{1}{2}(\mathsf{T}_I+\mathsf{T}_{III})$ and radius $\frac{1}{2}(\mathsf{T}_I-\mathsf{T}_{III})$, indicates that the feasible values for the pair $(\mathsf{T}_N;\mathsf{T}_S)$ are inside the circle C_2 , including the circumference. The third inequality indicates that the feasible values for the pair $(\mathsf{T}_N;\mathsf{T}_S)$ are outside the circle C_3 , which is defined by, the radius $\frac{1}{2}(\mathsf{T}_I-\mathsf{T}_{II})$ and center $\frac{1}{2}(\mathsf{T}_I+\mathsf{T}_{II})$. Then, taking into account the three equations simultaneously, the feasible region is formed by the pair $(\mathsf{T}_N;\mathsf{T}_S)$, defined in the gray area which includes the circumferences C_1, C_2 and C_3, (see Figure A.10).

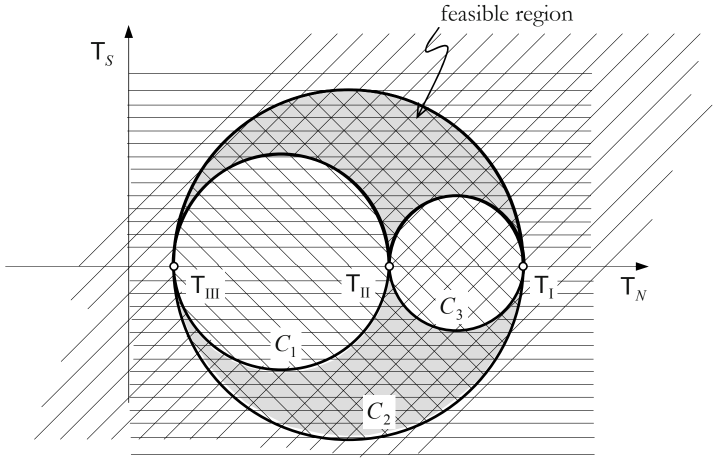


Figure A.10: Mohr's circle – feasible region.

In the Mohr's circle, (see Figure A.11), we can identify the maximum values of $\mathsf{T}_{S\,\max}$, which were also obtained in the set of solutions (A.17).

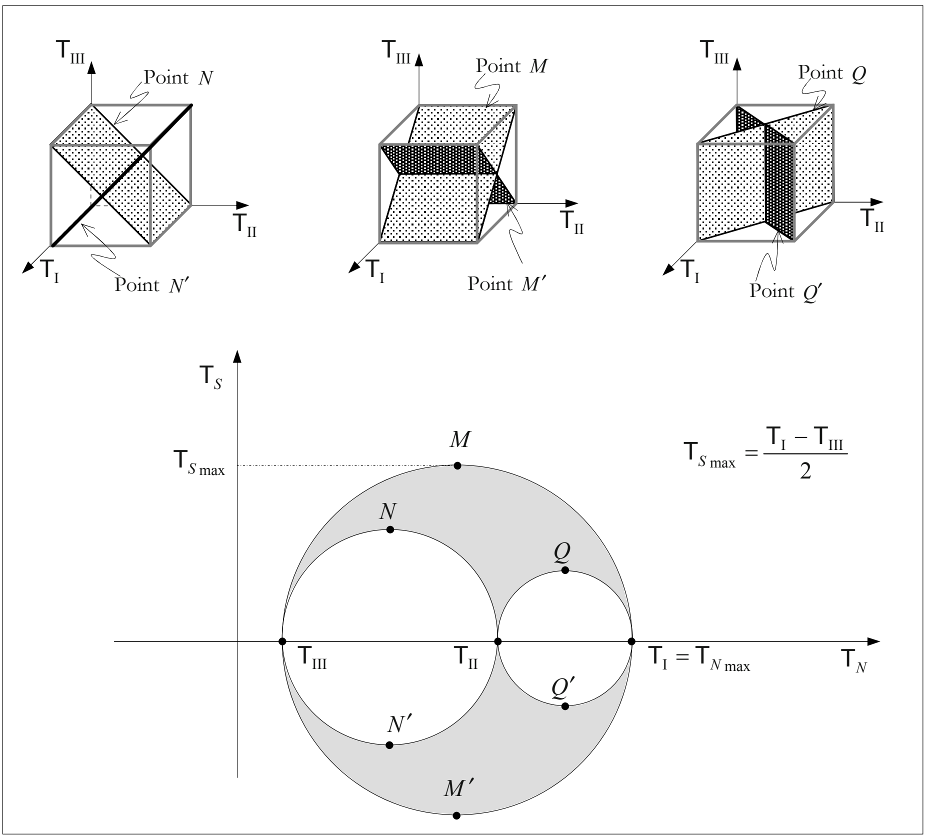


Figure A.11: Mohr's circle.

A.3 The Tensor Ellipsoid

Let us consider a symmetric second-order tensor $\mathbf{T}$ which is represented in the principal space by its eigenvalues (T_1, T_2, T_3) and where the following is fulfilled:

$$\mathbf{T}\cdot\hat{\mathbf{n}}=\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \xrightarrow{components} \begin{cases} t_1^{(\hat{\mathbf{n}})}=T_1\hat{n}_1 \\ t_2^{(\hat{\mathbf{n}})}=T_2\hat{n}_2 \\ t_3^{(\hat{\mathbf{n}})}=T_3\hat{n}_3 \end{cases} \Rightarrow \begin{cases} \hat{n}_1=\dfrac{t_1^{(\hat{\mathbf{n}})}}{T_1} \\ \hat{n}_2=\dfrac{t_2^{(\hat{\mathbf{n}})}}{T_2} \\ \hat{n}_3=\dfrac{t_3^{(\hat{\mathbf{n}})}}{T_3} \end{cases} \tag{A.39}$$

Our goal now is to define the surface, in the principal space, that describes the vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ for all possible values of $\hat{\mathbf{n}}$. Then, if we consider the constraint of $\hat{\mathbf{n}}$, *i.e.* $\hat{n}_1^2+\hat{n}_2^2+\hat{n}_3^2=1$, and substitute the n_i equation given by (A.39) we obtain:

$$\frac{t_1^{(\hat{\mathbf{n}})^2}}{T_1^2}+\frac{t_2^{(\hat{\mathbf{n}})^2}}{T_2^2}+\frac{t_3^{(\hat{\mathbf{n}})^2}}{T_3^2}=1 \tag{A.40}$$

which represents an ellipsoid in the principal space of $\mathbf{T}$, (see Figure A.12). This surface describes the feasible values for t_1, t_2 and t_3. When two eigenvalues are equal, the surface becomes an ellipsoid of revolution, whereas when the three eigenvalues are equal, the surface is a sphere, so, tensors that exhibit this property are called *Spherical Tensors*, and any direction for these is a principal direction.

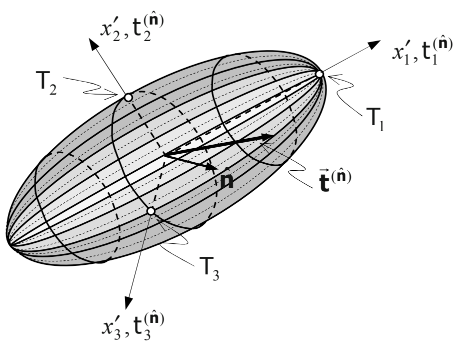


Figure A.12: The tensor ellipsoid.

A.4 Graphical Representation of the Spherical and Deviatoric Parts

A.4.1 The Octahedral Vector

Firstly, we define an *octahedral plane*, also called the *deviatoric plane*, which is that in which the normal is at an equal angle with each principal axis. In Figure A.13, the plane formed by the points ABC is an octahedral plane whose normal is defined by $\hat{\mathbf{n}}$ and can easily be evaluated by using the definition of the unit vector $\hat{n}_i \hat{n}_i = \hat{n}_1^2 + \hat{n}_2^2 + \hat{n}_3^2 = 1$. Then, by considering that $\hat{n}_1 = \hat{n}_2 = \hat{n}_3$ we obtain $3\hat{n}_1^2 = 1$, after which the plane with the normal $\hat{n}_i = [\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}]$, in the principal space, is said to be an octahedral plane. So, the vector that comes out of projecting a second-order tensor onto the octahedral plane is denoted by the *octahedral vector* $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ which can be decomposed into normal and tangential components, (see Figure A.13), so defining the *normal octahedral vector*, $\vec{\mathbf{T}}_N^{oct}$, and the *tangential octahedral vector*, $\vec{\mathbf{T}}_S^{oct}$.

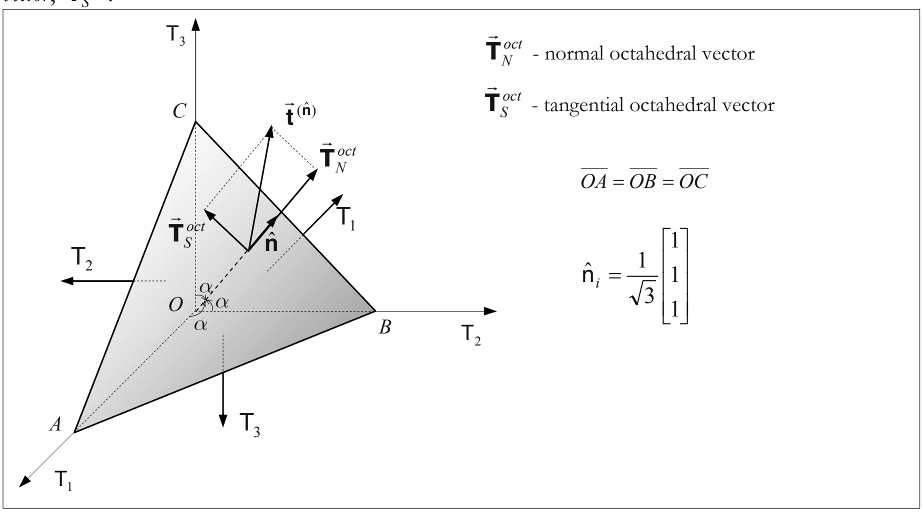


Figure A.13: The octahedral plane and octahedral vector (principal space).

The octahedral vector can be expressed in terms of components in the form:

$$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \frac{T_1}{\sqrt{3}} \hat{\mathbf{e}}'_1 + \frac{T_2}{\sqrt{3}} \hat{\mathbf{e}}'_2 + \frac{T_3}{\sqrt{3}} \hat{\mathbf{e}}'_3 \tag{A.41}$$

Then, the magnitude of $\vec{\mathbf{T}}_N^{oct}$ can be evaluated by projecting $\vec{\mathbf{T}}_N^{oct}$ onto $\hat{\mathbf{n}}$:

$$\begin{aligned} T_N^{oct} &= \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \hat{\mathbf{n}} = \left(\frac{T_1}{\sqrt{3}} \hat{\mathbf{e}}'_1 + \frac{T_2}{\sqrt{3}} \hat{\mathbf{e}}'_2 + \frac{T_3}{\sqrt{3}} \hat{\mathbf{e}}'_3 \right) \cdot \left(\frac{\hat{\mathbf{e}}'_1}{\sqrt{3}} + \frac{\hat{\mathbf{e}}'_2}{\sqrt{3}} + \frac{\hat{\mathbf{e}}'_3}{\sqrt{3}} \right) \\ &= \frac{1}{3}(T_1 + T_2 + T_3) = \frac{1}{3} T_{ii} = \frac{I_{\mathbf{T}}}{3} = T_m \end{aligned} \tag{A.42}$$

where T_N^{oct} is called the *octahedral normal component*. Then, the magnitude of $\vec{\mathbf{T}}_S^{oct}$, called the *octahedral tangential component*, is obtained as follows:

$$\mathsf{T}_S^{oct\,2} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \cdot \vec{\mathbf{t}}^{(\hat{\mathbf{n}})} - \mathsf{T}_N^{oct\,2} = \frac{1}{3}\left[\mathsf{T}_1^2 + \mathsf{T}_2^2 + \mathsf{T}_3^2\right] - \frac{1}{9}\left[\mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3\right]^2 = \frac{1}{9}\left(2I_{\mathbf{T}}^2 - 6\,II_{\mathbf{T}}\right) \tag{A.43}$$

Note, the above equation can also be expressed as:

$$\begin{aligned}\mathsf{T}_S^{oct} &= \frac{1}{3}\sqrt{(\mathsf{T}_1 - \mathsf{T}_2)^2 + (\mathsf{T}_2 - \mathsf{T}_3)^2 + (\mathsf{T}_3 - \mathsf{T}_1)^2} \\ &= \frac{1}{3}\sqrt{(\mathsf{T}_{11} - \mathsf{T}_{22})^2 + (\mathsf{T}_{22} - \mathsf{T}_{33})^2 + (\mathsf{T}_{33} - \mathsf{T}_{11})^2 + 6\left(\mathsf{T}_{12}^2 + \mathsf{T}_{23}^2 + \mathsf{T}_{13}^2\right)} \\ &= \sqrt{\frac{-2}{3}\,II_{\mathbf{T}^{dev}}}\end{aligned} \tag{A.44}$$

or in terms of the principal values of the deviatoric tensor $\mathbf{T}^{dev}$:

$$\mathsf{T}_S^{oct} = \sqrt{\frac{\left(\mathsf{T}_1^{dev}\right)^2 + \left(\mathsf{T}_2^{dev}\right)^2 + \left(\mathsf{T}_3^{dev}\right)^2}{3}} \tag{A.45}$$

Then, in summary, we have:

$$\boxed{\mathsf{T}_N^{oct} = \frac{I_{\mathbf{T}}}{3} = \mathsf{T}_m} \quad \textit{Octahedral normal component} \tag{A.46}$$

$$\boxed{\mathsf{T}_S^{oct} = \frac{1}{3}\sqrt{2I_{\mathbf{T}}^2 - 6\,II_{\mathbf{T}}} = \sqrt{\frac{-2}{3}\,II_{\mathbf{T}^{dev}}} = \sqrt{\frac{\left(\mathsf{T}_1^{dev}\right)^2 + \left(\mathsf{T}_2^{dev}\right)^2 + \left(\mathsf{T}_3^{dev}\right)^2}{3}}} \quad \textit{Octahedral tangential component} \tag{A.47}$$

Notice that the octahedral normal and tangential components are the same for the 8 octahedral planes, (see Figure A.14).

Then, let us consider the principal space once again, here represented by the orthonormal basis $(\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{e}}_3')$, (see Figure A.15). Now, we can plot the coordinates $(\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3)$, which are denoted by the P in Figure A.15.

Note, any plane normal to the straight line $\overline{OA}$ is an octahedral (deviatoric) plane and the specific plane passing through the origin is denoted by Π. Finally, the straight line $\overline{OA}$ is called the *spherical axis* (or *hydrostatic axis*).

Let us now consider a deviatoric plane passing through the point P, which is denoted by Π', (see Figure A.15). Then, we will define three vectors, namely, $\vec{OP}$, $\vec{OA}$ and $\vec{AP}$.

Next, the vector $\vec{OP}$ can be expressed in terms of the principal values of $\mathbf{T}$ as:

$$\vec{OP} = \mathsf{T}_1\hat{\mathbf{e}}_1' + \mathsf{T}_2\hat{\mathbf{e}}_2' + \mathsf{T}_3\hat{\mathbf{e}}_3' \tag{A.48}$$

and according to Figure A.15, the magnitude of $\vec{OA}$ is:

$$\begin{aligned}\|\vec{OA}\| = p = \vec{OP}\cdot\hat{\mathbf{n}} &= \left(\mathsf{T}_1\hat{\mathbf{e}}_1' + \mathsf{T}_2\hat{\mathbf{e}}_2' + \mathsf{T}_3\hat{\mathbf{e}}_3'\right)\cdot\left(\frac{\hat{\mathbf{e}}_1'}{\sqrt{3}} + \frac{\hat{\mathbf{e}}_2'}{\sqrt{3}} + \frac{\hat{\mathbf{e}}_3'}{\sqrt{3}}\right) = \frac{\mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3}{\sqrt{3}} \\ &= \frac{3\mathsf{T}_m}{\sqrt{3}} = \sqrt{3}\,\mathsf{T}_m = \sqrt{3}\,\mathsf{T}_N^{oct}\end{aligned} \tag{A.49}$$

$$\boxed{p = \sqrt{3}\,\mathsf{T}_m = \sqrt{3}\,\mathsf{T}_N^{oct}} \tag{A.50}$$

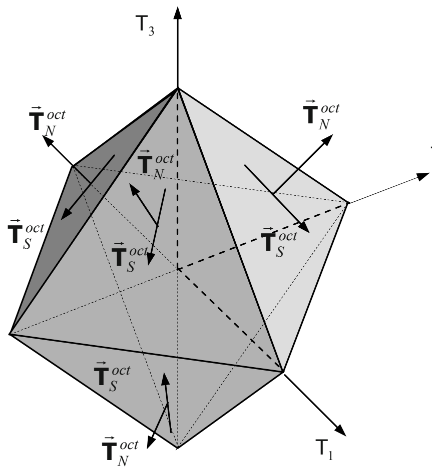


Figure A.14: Vectors on the octahedral planes.

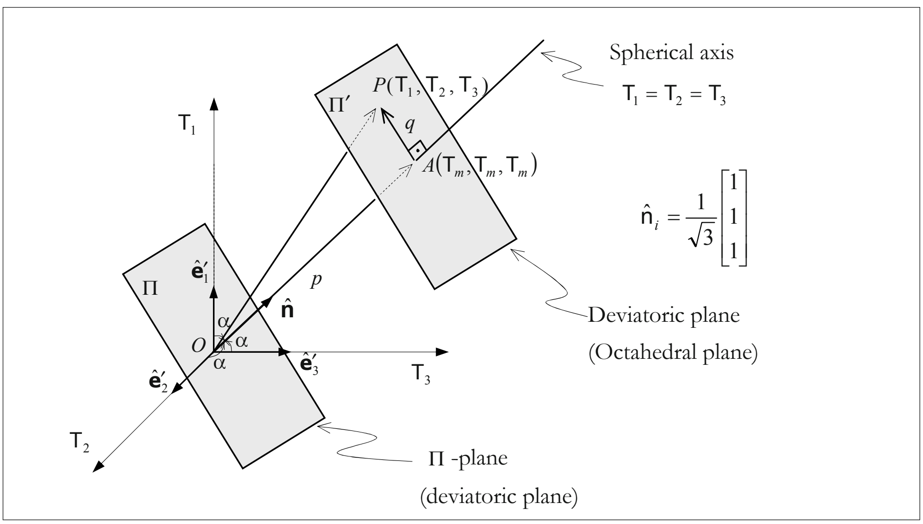


Figure A.15: Principal space.

Therefore, the vector $\vec{OA}$ is defined as:

$$\vec{OA} = \|\vec{OA}\|\,\hat{\mathbf{n}} = \sqrt{3}\mathsf{T}_m\left(\frac{\hat{\mathbf{e}}'_1}{\sqrt{3}} + \frac{\hat{\mathbf{e}}'_2}{\sqrt{3}} + \frac{\hat{\mathbf{e}}'_3}{\sqrt{3}}\right) = \mathsf{T}_m\hat{\mathbf{e}}'_1 + \mathsf{T}_m\hat{\mathbf{e}}'_2 + \mathsf{T}_m\hat{\mathbf{e}}'_3 \tag{A.51}$$

Then, the coordinates of point A are $(\mathsf{T}_m, \mathsf{T}_m, \mathsf{T}_m)$ and once the vectors $\vec{OP}$ and $\vec{OA}$ have been defined, the vector $\vec{AP}$ can be evaluated by adding the following vectors, (see Figure A.15):

$$\vec{AP} = \vec{OP} - \vec{OA} \tag{A.52}$$

Then, taking into account (A.48) and (A.51), the above equation becomes:

$$\begin{aligned}\vec{AP} &= \mathsf{T}_1\hat{\mathbf{e}}_1' + \mathsf{T}_2\hat{\mathbf{e}}_2' + \mathsf{T}_3\hat{\mathbf{e}}_3' - \left(\mathsf{T}_m\hat{\mathbf{e}}_1' + \mathsf{T}_m\hat{\mathbf{e}}_2' + \mathsf{T}_m\hat{\mathbf{e}}_3'\right) \\ &= (\mathsf{T}_1 - \mathsf{T}_m)\hat{\mathbf{e}}_1' + (\mathsf{T}_2 - \mathsf{T}_m)\hat{\mathbf{e}}_2' + (\mathsf{T}_3 - \mathsf{T}_m)\hat{\mathbf{e}}_3' \\ &= \mathsf{T}_1^{dev}\hat{\mathbf{e}}_1' + \mathsf{T}_2^{dev}\hat{\mathbf{e}}_2' + \mathsf{T}_3^{dev}\hat{\mathbf{e}}_3'\end{aligned} \tag{A.53}$$

and using the definition $\mathsf{T}_{ij}^{dev} = \mathsf{T}_{ij} - \mathsf{T}_m \delta_{ij}$, the above can also be expressed as:

$$\vec{AP} = \mathsf{T}_1^{dev}\hat{\mathbf{e}}_1' + \mathsf{T}_2^{dev}\hat{\mathbf{e}}_2' + \mathsf{T}_3^{dev}\hat{\mathbf{e}}_3' \tag{A.54}$$

Notice that, the components of $\vec{AP}$ represent the principal values of the deviatoric part T_{ij}^{dev}. The magnitude of $\vec{AP}$ is then given by:

$$q = \left\|\vec{AP}\right\| = \sqrt{\left(\mathsf{T}_1^{dev}\right)^2 + \left(\mathsf{T}_2^{dev}\right)^2 + \left(\mathsf{T}_3^{dev}\right)^2} \tag{A.55}$$

Taking into account the expression of $I\!I_{\mathbf{T}^{dev}}$, given by $2\,I\!I_{\mathbf{T}^{dev}} = -(\mathsf{T}_1^{dev})^2 - (\mathsf{T}_2^{dev})^2 - (\mathsf{T}_3^{dev})^2$, the above equation becomes:

$$\boxed{q = \sqrt{-2\,I\!I_{\mathbf{T}^{dev}}} = \sqrt{3}\,\mathsf{T}_S^{oct}} \tag{A.56}$$

Note, we could also have obtained the $\vec{AP}$ module by using Pythagoras' theorem:

$$\left\|\vec{AP}\right\|^2 = \left\|\vec{OP}\right\|^2 - \left\|\vec{OA}\right\|^2 = \mathsf{T}_1^2 + \mathsf{T}_2^2 + \mathsf{T}_3^2 - \frac{1}{3}\left(\mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3\right)^2 \tag{A.57}$$

$$\Rightarrow \left\|\vec{AP}\right\| = \sqrt{\frac{2}{3}\left(\mathsf{T}_1^2 + \mathsf{T}_2^2 + \mathsf{T}_3^2 - \mathsf{T}_1\mathsf{T}_2 - \mathsf{T}_2\mathsf{T}_3 - \mathsf{T}_3\mathsf{T}_1\right)} = q \tag{A.58}$$

where q indicates how faraway is the tensor state is from the spherical state, (see Figure A.15).

Note, in some cases working in the Π-plane could be advantageous and for this reason we define some parameters on it.

We will next analyze the projection of the principal space onto the Π-plane, (see Figure A.16).

Then, to find the components of the unit vector $\hat{\mathbf{e}}_1'' = a_1\hat{\mathbf{e}}_1' + a_2\hat{\mathbf{e}}_2' + a_3\hat{\mathbf{e}}_3'$ described in Figure A.16, we will consider the principal space as shown in Figure A.17, where $\cos\beta = \sin\alpha = \frac{\sqrt{2}}{\sqrt{3}} = a_1$, $a_2 = a_3$ holds. If we also consider that the hydrostatic axis is orthogonal to the deviatoric plane, we obtain:

$$\begin{aligned}&\hat{\mathbf{e}}_1''\cdot\hat{\mathbf{n}} = \left(a_1\hat{\mathbf{e}}_1' + a_2\hat{\mathbf{e}}_2' + a_3\hat{\mathbf{e}}_3'\right)\cdot\frac{1}{\sqrt{3}}\left(\hat{\mathbf{e}}_1' + \hat{\mathbf{e}}_2' + \hat{\mathbf{e}}_3'\right) = 0 \Rightarrow \frac{1}{\sqrt{3}}\left(a_1 + a_2 + a_3\right) = 0 \\ &\Rightarrow a_2 + a_3 = -a_1 = -\frac{\sqrt{2}}{\sqrt{3}} = -\frac{2}{\sqrt{6}}\end{aligned} \tag{A.59}$$

In addition to this we have:

$$a_2 = a_3 = -\frac{1}{2}a_1 = -\frac{\sqrt{2}}{2\sqrt{3}} = -\frac{1}{\sqrt{6}} \tag{A.60}$$

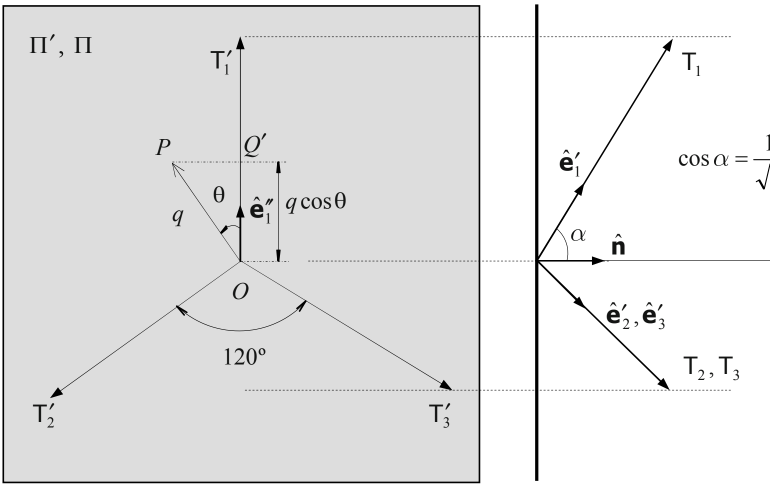


Figure A.16: Projection of the principal space onto the Π -plane.

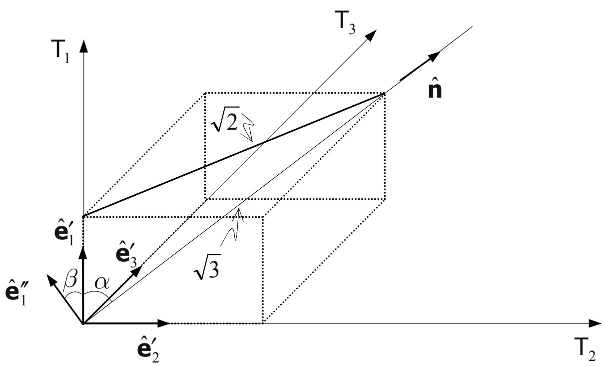


Figure A.17: Principal space.

Thus:

$$\hat{\mathbf{e}}_1'' = \frac{1}{\sqrt{6}}(2\hat{\mathbf{e}}_1' - \hat{\mathbf{e}}_2' - \hat{\mathbf{e}}_3') \tag{A.61}$$

The, the projection of $\overrightarrow{OP}$ onto $\hat{\mathbf{e}}_1''$ is evaluated as follows:

$$\begin{aligned}\overrightarrow{OP}\cdot\hat{\mathbf{e}}_1'' &= (\mathsf{T}_1^{dev}\hat{\mathbf{e}}_1' - \mathsf{T}_2^{dev}\hat{\mathbf{e}}_2' - \mathsf{T}_3^{dev}\hat{\mathbf{e}}_3')\cdot\frac{1}{\sqrt{6}}(2\hat{\mathbf{e}}_1' - \hat{\mathbf{e}}_2' - \hat{\mathbf{e}}_3') \\ &= \frac{1}{\sqrt{6}}(2\mathsf{T}_1^{dev} - \mathsf{T}_2^{dev} - \mathsf{T}_3^{dev}) = q\cos\theta\end{aligned} \tag{A.62}$$

Then, if we consider that $\mathsf{T}_1^{dev} + \mathsf{T}_2^{dev} + \mathsf{T}_3^{dev} = 0 \Rightarrow -\mathsf{T}_1^{dev} = \mathsf{T}_2^{dev} + \mathsf{T}_3^{dev}$, the above equation becomes:

$$q\cos\theta = \overrightarrow{OP}\cdot\hat{\mathbf{e}}_1'' = \frac{1}{\sqrt{6}}(2\mathsf{T}_1^{dev} + \mathsf{T}_1^{dev}) = \frac{3}{\sqrt{6}}\mathsf{T}_1^{dev} = \sqrt{\frac{3}{2}}\ \mathsf{T}_1^{dev} \tag{A.63}$$

and by considering that $q = \sqrt{-2\,I\!I_{\mathbf{T}^{dev}}}$, we obtain:

$$\begin{aligned} q\cos\theta = \sqrt{\frac{3}{2}}\ \mathsf{T}_1^{dev} &\Rightarrow \sqrt{-2\,I\!I_{\mathbf{T}^{dev}}}\cos\theta = \sqrt{\frac{3}{2}}\ \mathsf{T}_1^{dev} \\ \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}\frac{\mathsf{T}_1^{dev}}{\sqrt{-I\!I_{\mathbf{T}^{dev}}}} \quad &\Rightarrow \quad \mathsf{T}_1^{dev} = \frac{2}{\sqrt{3}}\sqrt{-I\!I_{\mathbf{T}^{dev}}}\cos\theta \end{aligned} \tag{A.64}$$

Likewise, we can find T_2^{dev} and T_3^{dev}. Then, we can express the principal values, $\mathsf{T}_{ij} = \mathsf{T}_m\delta_{ij} + \mathsf{T}_{ij}^{dev}$, as follows:

$$\begin{aligned} \begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} &= \begin{bmatrix} \mathsf{T}_m & 0 & 0 \\ 0 & \mathsf{T}_m & 0 \\ 0 & 0 & \mathsf{T}_m \end{bmatrix} + \begin{bmatrix} \mathsf{T}_1^{dev} & 0 & 0 \\ 0 & \mathsf{T}_2^{dev} & 0 \\ 0 & 0 & \mathsf{T}_3^{dev} \end{bmatrix} \\ &= \begin{bmatrix} \mathsf{T}_m & 0 & 0 \\ 0 & \mathsf{T}_m & 0 \\ 0 & 0 & \mathsf{T}_m \end{bmatrix} + \frac{2}{\sqrt{3}}\sqrt{-I\!I_{\mathbf{T}^{dev}}} \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos(\theta - \frac{2\pi}{3}) & 0 \\ 0 & 0 & \cos(\theta + \frac{2\pi}{3}) \end{bmatrix} \end{aligned} \tag{A.65}$$

with $0 \le \theta \le \pi/3$. Note, the tensor state can also be expressed in terms of (p, q, θ):

$$\begin{bmatrix} \mathsf{T}_1 & 0 & 0 \\ 0 & \mathsf{T}_2 & 0 \\ 0 & 0 & \mathsf{T}_3 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}}_{\substack{Spherical \\ part}} + \underbrace{\sqrt{\frac{2}{3}}\,q \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos(\theta - \frac{2\pi}{3}) & 0 \\ 0 & 0 & \cos(\theta + \frac{2\pi}{3}) \end{bmatrix}}_{\substack{Deviatoric \\ part}} \tag{A.66}$$

Then, substituting the $\cos\theta$, given by the equation in (A.64), into the trigonometric relationship $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, yields:

$$\cos 3\theta = 4\left(\frac{\sqrt{3}}{2}\frac{\mathsf{T}_1^{dev}}{\sqrt{-I\!I_{\mathbf{T}^{dev}}}}\right)^3 - 3\left(\frac{\sqrt{3}}{2}\frac{\mathsf{T}_1^{dev}}{\sqrt{-I\!I_{\mathbf{T}^{dev}}}}\right) = \frac{3\sqrt{3}}{2\sqrt{-I\!I_{\mathbf{T}^{dev}}}^{\,3}}\left[\left(\mathsf{T}_1^{dev}\right)^3 + \mathsf{T}_1^{dev}\, I\!I_{\mathbf{T}^{dev}}\right] \tag{A.67}$$

and if we take into account that $I\!I_{\mathbf{T}^{dev}} = (\mathsf{T}_1^{dev}\mathsf{T}_2^{dev} + \mathsf{T}_2^{dev}\mathsf{T}_3^{dev} + \mathsf{T}_1^{dev}\mathsf{T}_3^{dev})$, the above equation becomes:

$$\begin{aligned} \cos 3\theta &= 4\left(\frac{\sqrt{3}}{2}\frac{\mathsf{T}_1^{dev}}{\sqrt{-I\!I_{\mathbf{T}}}}\right)^3 - 3\left(\frac{\sqrt{3}}{2}\frac{\mathsf{T}_1^{dev}}{\sqrt{-I\!I_{\mathbf{T}}}}\right) \\ \cos 3\theta &= \frac{3\sqrt{3}}{2\sqrt{-I\!I_{\mathbf{T}^{dev}}}^{\,3}}\left[\left(\mathsf{T}_1^{dev}\right)^3 + \left(\mathsf{T}_1^{dev}\right)^2\underbrace{\left(\mathsf{T}_2^{dev} + \mathsf{T}_3^{dev}\right)}_{-\mathsf{T}_1^{dev}} + \underbrace{\mathsf{T}_1^{dev}\mathsf{T}_2^{dev}\mathsf{T}_3^{dev}}_{\mathsf{J}_3\ I\!I\!I_{\mathbf{T}^{dev}}}\right] \end{aligned} \tag{A.68}$$

$$\cos 3\theta = \frac{3\sqrt{3}\, I\!I\!I_{\mathbf{T}^{dev}}}{2\sqrt{-I\!I_{\mathbf{T}^{dev}}}^{\,3}} \tag{A.69}$$

Note, the terms $I\!I_{\mathbf{T}^{dev}}$, $I\!I\!I_{\mathbf{T}^{dev}}$ are invariants, hence $\cos 3\theta$ is an invariant too.

2

Continuum Kinematics

2.1 Introduction

A material body (continuum) in motion, starting from an initial state $(t=0)\equiv t_0$, will have different configurations over time, (see Figure 2.1). In this Chapter we will study the *description of motion* also called *Kinematics*, thus establishing the equations of motion that allow us to characterize how the continuum evolves and how continuum properties, *e.g.* displacement, velocity, acceleration, mass density, temperature, etc., change over time. To do this, we will consider the *initial configuration*, also known as the *reference* or *undeformed configuration*, characterized by material body $\mathcal{B}_0$ at time $t=0$, and we will also consider the generic configuration $\mathcal{B}_t$ at time t called the *current configuration*, also known as the *actual* or *deformed configuration*.

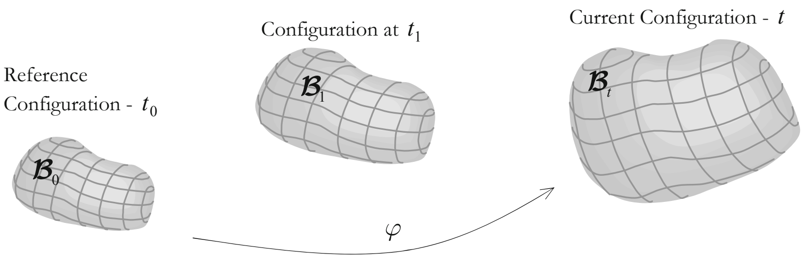


Figure 2.1: Motion of a material body.

We will start by describing the motion of a single particle in the continuum. Then, we will study how the relative distances between particles change during this motion and afterwards we will define some deformation and strain tensors, but before we can carry out these objectives we will define some continuum properties.

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_4,

2.2 The Continuous Medium

Any continuous medium (continuum) is assigned a positive scalar quantity called *mass*. It is assumed that the mass is continuously distributed in the continuum, without there being any discontinuities. A continuum is said to be *homogeneous* if its properties are the same throughout the continuum.

Now, let us consider a sphere centered at point P in the initial configuration, (see Figure 2.2). The volume and mass of this sphere are denoted by ΔV_0 and Δm, respectively. Then, we can define the *mass density* of the particle in the initial configuration as:

$$\rho_0(\vec{\boldsymbol{X}}) = \lim_{\Delta V_0 \to 0} \frac{\Delta m}{\Delta V_0} = \frac{dm}{dV_0} \qquad \left[\frac{kg}{m^3}\right] \tag{2.1}$$

Starting from this concept we can define a particle as a dimensionless element that has physical properties such as mass density, velocity, temperature, etc.

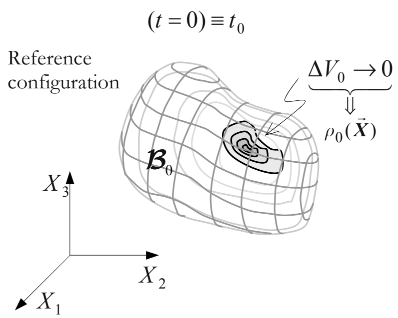


Figure 2.2: Mass density in the initial configuration.

We can define a continuum medium as a set of particles arranged in an area without discontinuities, in which there is a one-to-one correspondence (*i.e.* bijection) between possible configurations. Then we can define some terminologies that are used throughout this chapter, *e.g.*:

Particle (material point): a small volume element that has certain physical properties, *e.g.* mass density (ρ), velocity ($\vec{\boldsymbol{v}}$), temperature (T), etc.

Points: A place in space, position;

Particle trajectory (or path line): The locus of the points occupied by a single particle during motion, (see Figure 2.3).

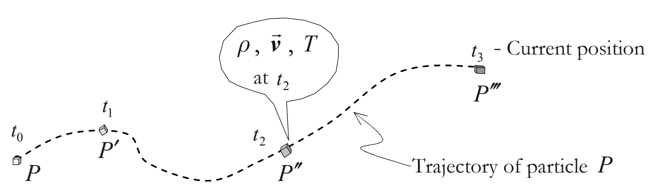


Figure 2.3: Particle trajectory.

2.2.1 Kinds of Motion

Motion of a continuous medium, also denoted by deformation, is characterized by the following types:

Rigid Body Motion: Characterized by maintaining the original shape of the body after motion, *i.e.* it is characterized by preserving the distance between particles. The rigid body motion can be classified by: *translation* and/or *rotation*.

Motion with Deformation: Characterized by changes of distance between particles.

In general, motion is characterized by deformation and rigid body motion simultaneously.

2.2.1.1 Rigid Body Motion

As we have seen previously, in rigid body motion the distances between particles remain unchanged. Then we can establish an equation that governs this motion. To do this let us consider a Cartesian system $OX_1X_2X_3$ which is attached to the body, so, the position vector of any particle with respect to this system remains unchanged during motion. We can also adopt a second Cartesian system $ox_1x_2x_3$, which is represented by the orthonormal basis ($\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$), (see Figure 2.4).

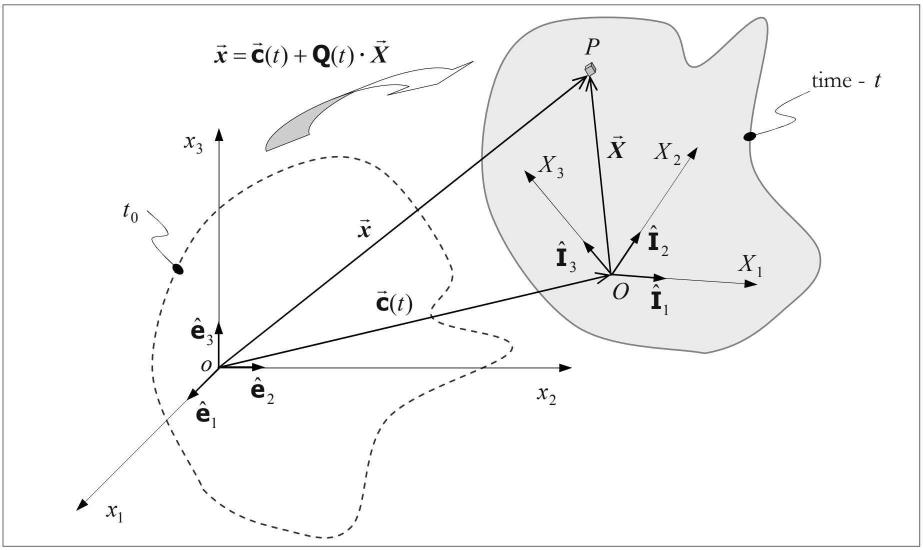


Figure 2.4: Rigid body motion.

If $\vec{X}$ and $\vec{x}$ are the position vectors of material point P with respect to the systems $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{I}}_i$, respectively, (see Figure 2.4), the following relationship is satisfied:

$$\vec{x} = \vec{\mathbf{c}} + \vec{X} \tag{2.2}$$

where $\vec{\mathbf{c}}(t)$ is a time-dependent vector that describes the translation motion of the system $\hat{\mathbf{I}}_i$. The above equation (2.2) in symbolic notation can be represented as:

$$x_p \hat{\mathbf{e}}_p = \mathrm{c}_k \hat{\mathbf{e}}_k + X_j \hat{\mathbf{I}}_j \tag{2.3}$$

The components of (2.3) in the system $ox_1x_2x_3$ are obtained by means of the dot product with respect to the basis $\hat{\mathbf{e}}_i$, *i.e.*:

$$
\begin{aligned}
x_p \hat{\mathbf{e}}_p \cdot \hat{\mathbf{e}}_i &= \mathrm{c}_k \hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}_i + X_j \hat{\mathbf{I}}_j \cdot \hat{\mathbf{e}}_i \\
x_p \delta_{pi} &= \mathrm{c}_k \delta_{ik} + X_j a_{ji} \\
x_i &= \mathrm{c}_i + X_j a_{ji}
\end{aligned}
\tag{2.4}
$$

in where $\hat{\mathbf{I}}_j \cdot \hat{\mathbf{e}}_i = a_{ji}$ is the transformation matrix from the system $\hat{\mathbf{I}}_i$ to the system $\hat{\mathbf{e}}_i$, and where it holds that $a_{ik} a_{kj} = \delta_{ij}$, *i.e.* a_{ji} is an orthogonal matrix. Note also that the equation (2.4) holds for any adopted system. If we consider that $\mathrm{Q}_{ij} = a_{ji}$, where Q_{ij} are orthogonal tensor components, we can sum up the equation in (2.4) as:

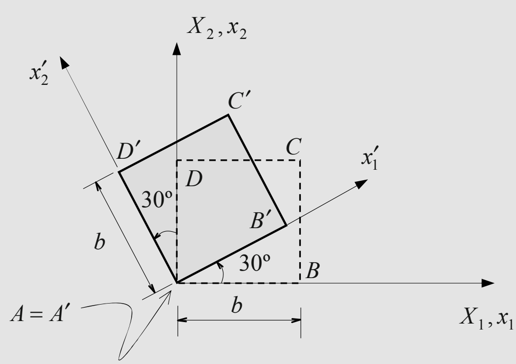
$$\boxed{\vec{x} = \vec{\mathbf{c}} + \mathbf{Q} \cdot \vec{X}} \quad \textit{Rigid body motion equations} \tag{2.5}$$

which describes rigid body motion.

NOTE: The transformation law of components and orthogonal transformation are closely interrelated although they have completely different meanings. ■

Problem 2.1: A continuum is defined by a square with sides b, subjected to rigid body motion which is defined by rotating the continuum counterclockwise by an angle of 30° to the origin. Find the equations of motion. Also obtain the new position of particle D.

Hint: Consider the systems $\vec{x}$ and $\vec{X}$ to be superimposed.



Solution: We apply the rigid body motion equations $\vec{x} = \vec{\mathbf{c}} + \mathbf{Q} \cdot \vec{X} = \mathbf{Q} \cdot \vec{X}$, to $\vec{\mathbf{c}} = \vec{\mathbf{0}}$. The components of $\mathbf{Q}$ are the same as the components of the transformation matrix from the $\vec{x}'$-system to the $\vec{x}$-system, *i.e.*:

$$\mathrm{Q}_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, the continuum particles are governed by the equations of motion:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

A particle which initially was at point D ($X_1 = 0, X_2 = b, X_3 = 0$) moves into the following position:

$$\begin{Bmatrix} x_1^D \\ x_2^D \\ x_3^D \end{Bmatrix} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ b \\ 0 \end{Bmatrix} = \begin{Bmatrix} -b\sin 30^\circ \\ b\cos 30^\circ \\ 0 \end{Bmatrix}$$

In the above example we have adopted the system, $\vec{X}$, fixed in space and time. When we are establishing the equations of motion for a continuum subjected to deformation, we can adopt a system fixed in space and time which is called the *material system*. There is also a second system defined by $\vec{x}$ and is called the *spatial system*.

In general, throughout this chapter we will adopt $\vec{\mathbf{c}} = \vec{\mathbf{0}}$, (see Figure 2.5), or to put it another way, the spatial and the material axes will be superimposed as shown in Figure 2.5.

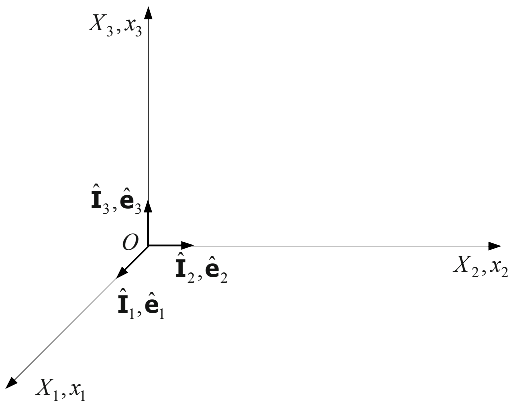


Figure 2.5: Spatial and material axes superimposed.

2.2.2 Types of Configurations

We define two types of configurations adopted in this chapter, (see Figure 2.6), namely:

- *The Reference configuration* or the *initial configuration*: the configuration at the instant of time $(t=0) \equiv t_0$, is considered to be the undeformed configuration in which the particle P is identified by the position vector $\vec{X}^P$.
- *The Current configuration* or *deformed configuration*: the configuration at the instant of time t.

As we have seen before, a continuum is defined as a set of particles arranged in an area without discontinuities, in which there is a one-to-one correspondence (*i.e.* bijection) between possible configurations. So, if motion is characterized by the bijective function, φ, this ensures the existence of the inverse function φ^{-1}, (see Figure 2.6).

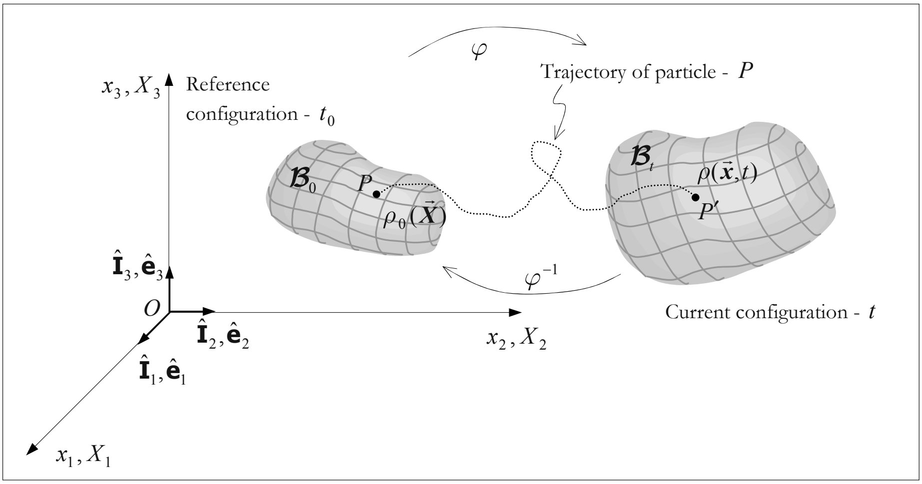


Figure 2.6: Initial and current configurations.

2.2.2.1 Mass Density

As with the definition of mass density in the reference configuration given in (2.1), we define mass density in the current configuration, (see Figure 2.7), as:

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV} \qquad \left[\frac{kg}{m^3}\right] \tag{2.6}$$

Mass density is a scalar field, which is a function of position and time, *i.e.* $\rho = \rho(\vec{x}, t)$.

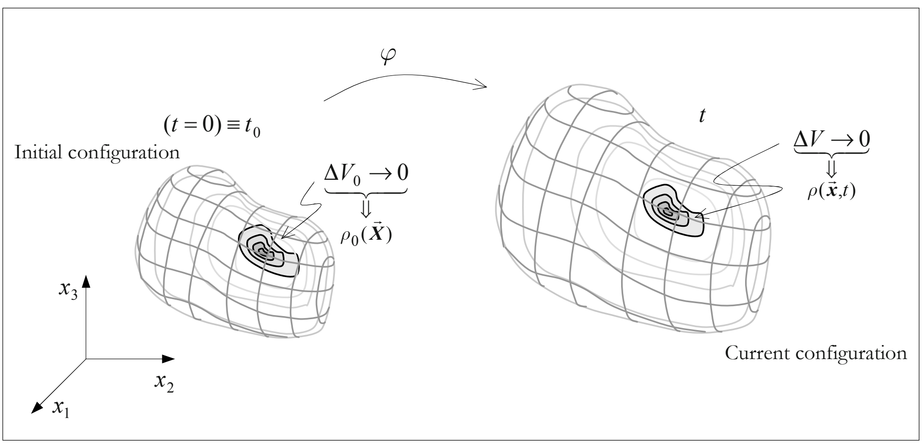


Figure 2.7: Mass density.

2.3 Description of Motion

2.3.1 Material and Spatial Coordinates

Let us consider the material body $\mathcal{B}_0$ in the initial configuration, (see Figure 2.8). At an arbitrary time (t), the material body occupies a new position in space, $\mathcal{B}_t$. We now focus our attention on a particle of the continuum, which is denoted by the particle P.

NOTE: With regard to nomenclature, the particles are identified by labels. These labels are the positions they occupied in the reference configuration. For instance, the particle which occupied the point $P(X_1, X_2, X_3)$ in the reference configuration will be denoted by particle P, (see Figure 2.8). ■

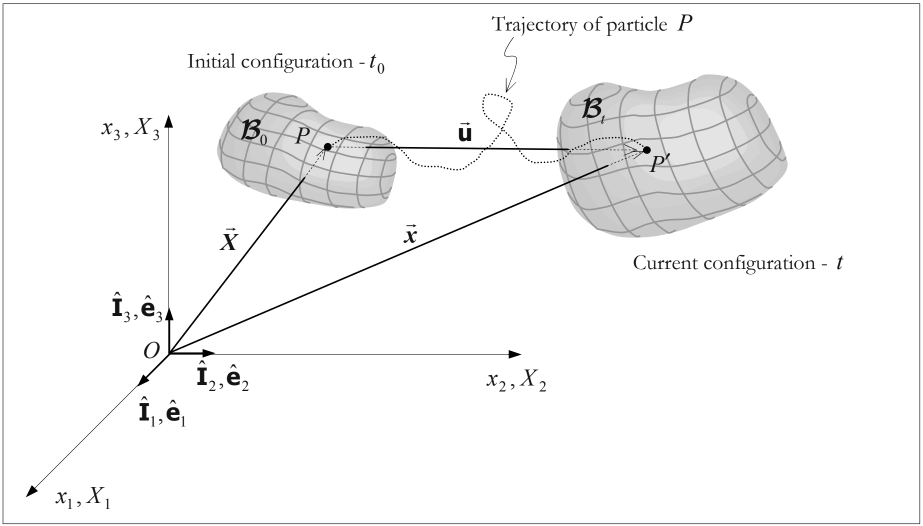


Figure 2.8: Initial and current configurations.

The position of a particle is characterized by the *position vector*. The position vector of particle P in the reference configuration, $t = 0 \equiv t_0$, is given by:

$$\vec{X} = X_1\hat{\mathbf{e}}_1 + X_2\hat{\mathbf{e}}_2 + X_3\hat{\mathbf{e}}_3 \tag{2.7}$$

which thus defines the *material coordinate:*

$$X_i = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{2.8}$$

In the current configuration (deformed configuration) particle P occupies the position P', and the position vector is given by:

$$\vec{x} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \tag{2.9}$$

which defines the *spatial coordinate:*

$$x_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2.10}$$

2.3.2 The Displacement Vector

By definition, the displacement vector ($\vec{\mathbf{u}}$) of a particle is the difference between the position vector in the current configuration ($\vec{x}$) and the position vector in the reference configuration ($\vec{X}$), (see Figure 2.8), *i.e.*:

$$\boxed{\vec{\mathbf{u}} = \vec{x} - \vec{X}} \qquad \Bigg| \qquad \mathrm{u}_i = x_i - X_i \qquad [m] \tag{2.11}$$

2.3.3 The Velocity Vector

The velocity of a particle is defined by the rate of change of the position vector, *i.e.*:

$$\vec{V} = \frac{d\vec{x}}{dt} \equiv \dot{\vec{x}} = \frac{d\left(\vec{\mathbf{u}} + \vec{X}\right)}{dt} = \frac{d\vec{\mathbf{u}}}{dt} + \underbrace{\frac{d\vec{X}}{dt}}_{=\vec{\mathbf{0}}} = \frac{d\vec{\mathbf{u}}}{dt} \equiv \dot{\vec{\mathbf{u}}} \qquad \left[\frac{m}{s}\right] \tag{2.12}$$

2.3.4 The Acceleration Vector

The acceleration of a particle is the rate of change of velocity, *i.e.*:

$$\vec{A} = \frac{d\vec{V}}{dt} \equiv \dot{\vec{V}} = \frac{d^2\vec{x}}{dt^2} \equiv \ddot{\vec{x}} = \ddot{\vec{\mathbf{u}}} \qquad \left[\frac{m}{s^2}\right] \tag{2.13}$$

2.3.5 Lagrangian and Eulerian Descriptions

Continuum properties, *e.g.*: mass density, temperature, velocity, acceleration, etc., are intrinsic in particles (material points), and such properties may change over time. As mentioned before, continuum motion is characterized by the bijective function (φ), and the inverse function (φ^{-1}). This ensures that we can correlate continuum properties between the current and reference configurations. In other words, the study of motion can be carried out either in the current or reference configuration.

2.3.5.1 Lagrangian Description of Motion

The particle in motion can be described in terms of material coordinates ($\vec{X}$) and time as:

$$\vec{x} = \vec{x}\left(\vec{X}, t\right) \qquad \Bigg| \qquad x_i = x_i\left(X_1, X_2, X_3, t\right) = x_i\left(\vec{X}, t\right) \tag{2.14}$$

The equations of motion (2.14) are called the *Lagrangian* or *Material Description of the motion*. The above parametric equation gives us the current position $\vec{x}$, at time t, of a particle that occupied position $\vec{X}$ in the reference configuration, at time t_0. The equation in (2.14), applied to particle P, provides us with the unique path line (trajectory) of the particle, (see Figure 2.9).

2.3.5.2 Eulerian Description of Motion

Particle motion can also be described in terms of spatial coordinates ($\vec{x}$) and time as:

$$\vec{X} = \vec{X}(\vec{x}, t) \qquad \Big| \qquad X_i = X_i(x_1, x_2, x_3, t) = X_i(\vec{x}, t) \tag{2.15}$$

The above equation give us the original position $\vec{X}$, at time t_0, of a particle which at the present time (t) has the coordinates (x_1, x_2, x_3), (see Figure 2.9).

The necessary and sufficient condition for there to be an inverse is:

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} \neq 0 \qquad \left[\frac{m^3}{m^3} \right] \tag{2.16}$$

where J is called the *Jacobian determinant*, also knows as the volume ration.

> **OBS.:** *The Axiom of Impenetrability:* Two particles can not occupy the same place at the same time. As discussed later, this condition is ensured when the Jacobian determinant is positive, *i.e.* $J > 0$.

In view of Figure 2.9 we can observe that:

$$\vec{x}^P(\vec{X}, t = 0) = \vec{X}^P \tag{2.17}$$

and

$$\underbrace{\begin{aligned} \vec{x}(\vec{X}^P, t_0) &= \vec{x}^P \\ \vec{x}(\vec{X}^P, t_1) &= \vec{x}^{P'} \\ \vec{x}(\vec{X}^P, t_2) &= \vec{x}^{P''} \end{aligned}}_{Trajectory\ of\ particle\ P} \qquad \underbrace{\begin{aligned} \vec{X}(\vec{x}^P, t_0) &= \vec{X}^P \\ \vec{X}(\vec{x}^{S'}, t_1) &= \vec{X}^S \\ \vec{X}(\vec{x}^{Q''}, t_2) &= \vec{X}^Q \end{aligned}}_{\substack{Particles\ at\ point\ P, \\ at\ different\ time}} \tag{2.18}$$

2.3.5.3 Lagrangian and Eulerian Variables

Any physical quantity ($\mathcal{Z}$) assigned to the continuum ($\mathcal{B}$) can be expressed as a Lagrangian ($\mathbf{Z}(\vec{X}, t)$) or Eulerian ($\mathbf{z}(\vec{x}, t)$) description, and are related in the following ways:

$$\mathbf{Z}(\vec{X}, t) = \mathbf{Z}(\vec{X}(\vec{x}, t), t) = \mathbf{z}(\vec{x}, t) \quad ; \quad \mathbf{z}(\vec{x}, t) = \mathbf{z}(\vec{x}(\vec{X}, t), t) = \mathbf{Z}(\vec{X}, t) \tag{2.19}$$

NOTE: Some authors try to differentiate a Lagrangian from Eulerian variable by using upper and lower-case letters, respectively. As a general rule we have not adopted this convention in this publication. In this textbook when we are dealing with a Lagrangian variable we will indicate explicitly by its arguments, *i.e.* $\vec{V} \equiv \vec{v}(\vec{X}, t)$. And if we are dealing with an Eulerian variable it will be indicated as follows $\vec{v}(\vec{x}, t)$. ■

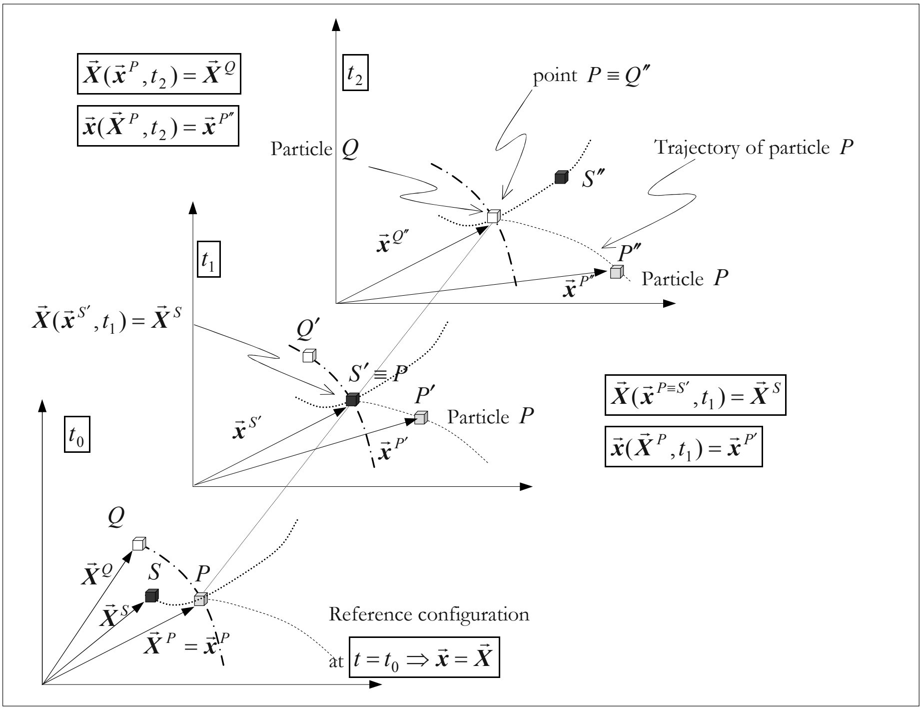


Figure 2.9: Lagrangian and Eulerian description.

Problem 2.2: Consider the following equations of motion in the Lagrangian description:

$$\begin{cases} x_1(\vec{X},t) = X_2 t^2 + X_1 \\ x_2(\vec{X},t) = X_3 t + X_2 \\ x_3(\vec{X},t) = X_3 \end{cases} \xrightarrow{\text{Matrix form}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & t^2 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \tag{2.20}$$

Is the motion above possible? If so, find the displacement, velocity and acceleration fields in Lagrangian and Eulerian descriptions. Consider a particle P that at time $t = 0$ was at the point defined by the triple equation $X_1 = 2, X_2 = 1, X_3 = 3$. Find the velocity of P at time $t = 1s$ and $t = 2s$.

Solution: Motion is possible if $J \neq 0$, thus

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} 1 & t^2 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

The displacement vector field is given by the definition in (2.11), $\vec{\mathbf{u}} = \vec{x} - \vec{X}$. Using the equations of motion (2.20) we obtain:

$$\begin{cases} \mathsf{u}_1(\vec{X},t) = x_1(\vec{X},t) - X_1 = [X_2 t^2 + X_1] - X_1 = X_2 t^2 \\ \mathsf{u}_2(\vec{X},t) = x_2(\vec{X},t) - X_2 = [X_3 t + X_2] - X_2 = X_3 t \\ \mathsf{u}_3(\vec{X},t) = x_3(\vec{X},t) - X_3 = [X_3] - X_3 = 0 \end{cases} \tag{2.21}$$

which are the components of the displacement vector in the Lagrangian description. Here, velocity and acceleration can be evaluated as follows:

$$\begin{cases} V_1 \equiv v_1(\vec{X},t) = \dfrac{d\mathsf{u}_1(\vec{X},t)}{dt} = \dfrac{d}{dt}\left(X_2 t^2\right) = 2X_2 t \\ V_2 \equiv v_2(\vec{X},t) = \dfrac{d\mathsf{u}_2(\vec{X},t)}{dt} = \dfrac{d}{dt}\left(X_3 t\right) = X_3 \\ V_3 \equiv v_3(\vec{X},t) = \dfrac{d\mathsf{u}_3(\vec{X},t)}{dt} = \dfrac{d}{dt}\left(X_2 t\right) = 0 \end{cases} \quad ; \quad \begin{cases} A_1 \equiv a_1(\vec{X},t) = \dfrac{dV_1}{dt} = 2X_2 \\ A_2 \equiv a_2(\vec{X},t) = \dfrac{dV_2}{dt} = 0 \\ A_3 \equiv a_3(\vec{X},t) = \dfrac{dV_3}{dt} = 0 \end{cases} \tag{2.22}$$

The inverse form of (2.20) provides us the equations of motion in the Eulerian description:

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} 1 & -t^2 & t^3 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \Rightarrow \begin{cases} X_1(\vec{x},t) = x_1 - t^2 x_2 + t^3 x_3 \\ X_2(\vec{x},t) = x_2 - t x_3 \\ X_3(\vec{x},t) = x_3 \end{cases} \tag{2.23}$$

Then, the displacement, velocity and acceleration fields in Eulerian description can be evaluated by substituting equation (2.23) into the equations (2.21) and (2.22), *i.e.*:

$$\begin{cases} \mathsf{u}_1\left(\vec{X}(\vec{x},t),t\right) = X_2(\vec{x},t)t^2 = (x_2 - tx_3)t^2 = \mathsf{u}_1(\vec{x},t) \\ \mathsf{u}_2\left(\vec{X}(\vec{x},t),t\right) = X_3(\vec{x},t)t = x_3 t = \mathsf{u}_2(\vec{x},t) \\ \mathsf{u}_3\left(\vec{X}(\vec{x},t),t\right) = \mathsf{u}_3(\vec{x},t) = 0 \end{cases} \tag{2.24}$$

$$\begin{cases} V_1\left(\vec{X}(\vec{x},t),t\right) = 2X_2(\vec{x},t)t = 2(x_2 - tx_3)t = v_1(\vec{x},t) \\ V_2\left(\vec{X}(\vec{x},t),t\right) = X_3(\vec{x},t) = x_3 = v_2(\vec{x},t) \\ V_3\left(\vec{X}(\vec{x},t),t\right) = v_3(\vec{x},t) = 0 \end{cases} \tag{2.25}$$

$$\begin{cases} A_1\left(\vec{X}(\vec{x},t),t\right) = 2X_2(\vec{x},t) = 2(x_2 - tx_3) = a_1(\vec{x},t) \\ A_2\left(\vec{X}(\vec{x},t),t\right) = a_2(\vec{x},t) = 0 \\ A_3\left(\vec{X}(\vec{x},t),t\right) = a_3(\vec{x},t) = 0 \end{cases} \tag{2.26}$$

Taking into account the Lagrangian description of velocity given in (2.22), the velocity of particle $P(X_1 = 2, X_2 = 1, X_3 = 3)$ at time $t = 1s$ is given by:

$$v_1(\vec{X},t) = 2X_2 t = 2m/s \quad ; \quad v_2(\vec{X},t) = X_3 = 3m/s \quad ; \quad v_3(\vec{X},t) = 0$$

We can also observe that at time $t = 1s$ the particle P occupies the position:

$$x_1 = X_2 t^2 + X_1 = 3 \quad ; \quad x_2 = X_3 t + X_2 = 4 \quad ; \quad x_3 = X_3 = 3$$

So, the velocity of the particle P can also be evaluated by (2.25) as:

$$\begin{cases} v_1(\vec{x},t) = 2(x_2 - tx_3)t = 2(4 + 1\times 3)\times 1 = 2m/s \\ v_2(\vec{x},t) = x_3 = 3m/s \\ v_3(\vec{x},t) = 0 \end{cases}$$

Note that, the velocities obtained via the Lagrangian or Eulerian description are the same, since velocity is an intrinsic property of the particle.

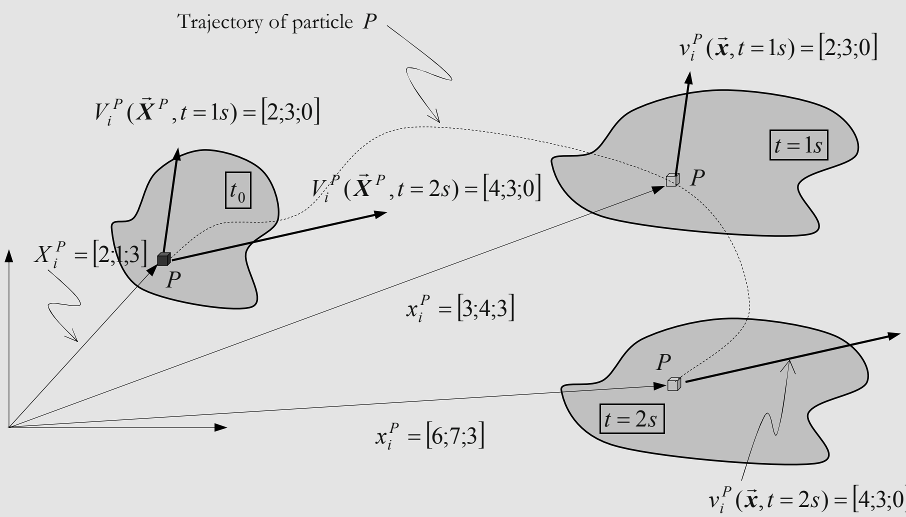
We can also provide the velocity of the particle P at time $t = 2s$:

$$\begin{cases} V_1 \equiv v_1\left(\vec{X},t\right) = 2X_2 t = 2\times 2\times 1 = 4m/s \\ V_2 \equiv v_2\left(\vec{X},t\right) = X_3 = 3m/s \\ V_3 \equiv v_3\left(\vec{X},t\right) = 0 \end{cases}$$

At time $t = 2s$ the new position of P is:

$$\begin{cases} x_1(\vec{X},t) = X_2 t^2 + X_1 = 6 \\ x_2(\vec{X},t) = X_3 t + X_2 = 7 \\ x_3(\vec{X},t) = X_3 = 3 \end{cases}$$

As we can verify the Lagrangian description of motion $\vec{x}(\vec{X},t)$ describes the trajectory of P.



2.4 The Material Time Derivative

The rate of change of a physical quantity, *e.g.*: velocity, temperature, mass density, etc., is called the *material time derivative,* and is denoted by $\frac{D}{Dt}$. For example, let us consider an observer who is travelling with particle P and is recording how temperature changes over time, (see Figure 2.10); this rate of change of temperature is denoted by the material time derivative of temperature. By this example, it seemed reasonable to conclude that the material time derivative of a property depends on whether the property is a Lagrangian or Eulerian variable.

- If the property is in Lagrangian description we have:

$$\theta = \theta(X_1, X_2, X_3, t) \tag{2.27}$$

In this case, the material time derivative is expressed in the form:

$$\boxed{\dot{\theta}(\vec{X},t) \equiv \frac{D\theta(\vec{X},t)}{Dt} = \frac{d\theta(\vec{X},t)}{dt}} \tag{2.28}$$

When the property is described in terms of material coordinates, this implication is that the property represented is connected to the same particle during motion.

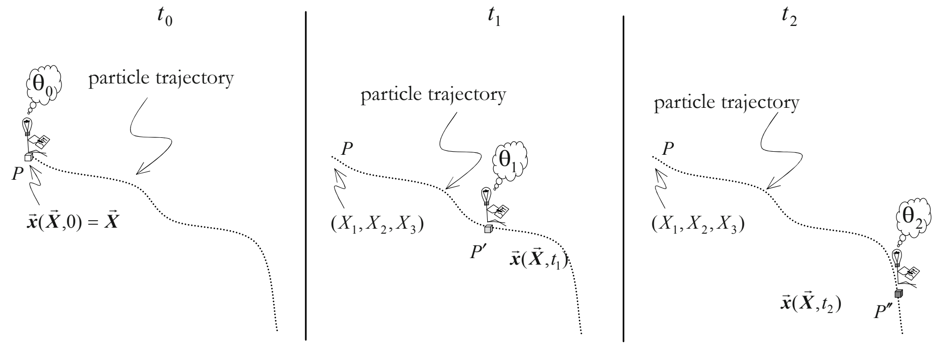


Figure 2.10: The rate of change of temperature – Lagrangian description.

- If the property is in Eulerian description we have:

$$\theta = \theta(x_1, x_2, x_3, t) \tag{2.29}$$

In this description the observer is not traveling with the particle, but fixed at one point (x_1, x_2, x_3) watching the particles passing. According to Figure 2.11, equation (2.29), at time t_1, provides us with the property of particle Q, which takes the point (x_1, x_2, x_3). Later, t_2, in equation (2.29) give us the property of another particle, *e.g.* R, and at time t_3, the equation (2.29) give us the value of the property of particle P, (see Figure 2.11).

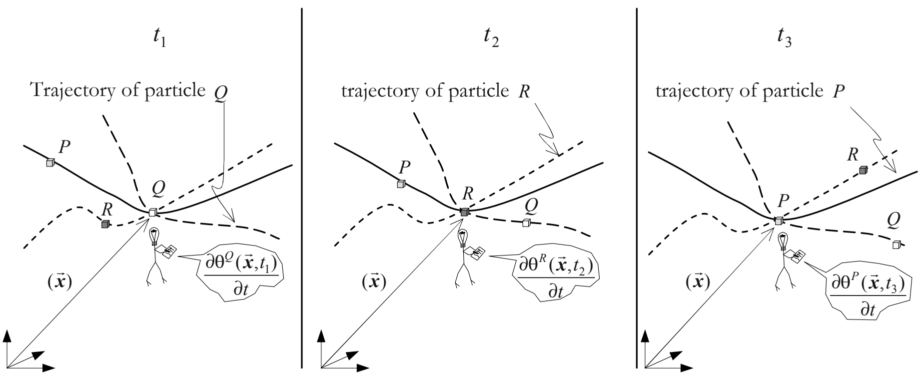


Figure 2.11: The rate of change of temperature – Eulerian description.

It must be emphasized that the material time derivative is related to the derivative with respect to time of an intrinsic property of the particle, *i.e.* it is related to the same particle. But an observer fixed at a spatial point (x_1, x_2, x_3) only has information on the *local rate of change*. In order to be fully informed, we need to know how the property of this particle changes along its path line- this additional information is known as the *convective rate of change*, which is related to mass transport. Afterwards, to obtain the material time derivative of an Eulerian property $\theta = \theta(\vec{x}, t)$ we must take into consideration:

- *The local rate of change*;
- *The convective rate of change*

So,

$$\boxed{\begin{aligned}\dot{\theta}(\vec{x},t) \equiv \frac{D\theta(\vec{x},t)}{Dt} &= \underbrace{\frac{\partial\theta(\vec{x},t)}{\partial t}}_{\substack{\text{local}\\ \text{rate of change}}} + \underbrace{\frac{\partial\theta(\vec{x},t)}{\partial x_k}\frac{\partial x_k(\vec{X},t)}{\partial t}}_{\substack{\text{Convective}\\ \text{rate of change}}} \\ &= \frac{\partial\theta(\vec{x},t)}{\partial t} + \frac{\partial\theta(\vec{x},t)}{\partial x_k} v_k(\vec{X},t)\end{aligned}} \tag{2.30}$$

where $\vec{v}(\vec{X},t) \equiv \dot{\vec{x}}(\vec{X},t)$ is the particle velocity, which can also be expressed in Eulerian description by substituting the equations of motion, *i.e.* $\vec{v}(\vec{X}(\vec{x},t),t) = \vec{v}(\vec{x},t)$.

Then, we can define the material time derivative operator for an Eulerian property, $\bullet(\vec{x},t)$, as:

$$\boxed{\frac{D\blacksquare(\vec{x},t)}{Dt} = \frac{\partial\blacksquare(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\blacksquare(\vec{x},t)\cdot\vec{v}(\vec{x},t)} \quad \textit{Material time derivative for an Eulerian variable} \tag{2.31}$$

or in indicial notation as:

$$\frac{D\blacksquare(\vec{x},t)}{Dt} = \frac{\partial\blacksquare(\vec{x},t)}{\partial t} + \frac{\partial\blacksquare(\vec{x},t)}{\partial x_k} v_k \tag{2.32}$$

2.4.1 Velocity and Acceleration in Eulerian Description

We have defined the velocity of particle P as:

$$\vec{V}^P(\vec{X},t) = \frac{D}{Dt}\vec{x}(\vec{X},t) = \dot{\vec{x}}^P = \left(\frac{d\vec{x}}{dt}\right)_{\vec{x}=\vec{x}^P} = \frac{d(\vec{\mathbf{u}}+\vec{X})}{dt} = \frac{d\vec{\mathbf{u}}(\vec{X},t)}{dt} \tag{2.33}$$

which is the Lagrangian velocity. To obtain the Eulerian velocity, $\vec{v}(\vec{x},t)$, we have to substituting the inverse equation of motion, *i.e.* $\vec{V}^P(\vec{X},t) = \vec{V}^P(\vec{X}(\vec{x},t),t) = \vec{v}^P(\vec{x},t)$.

The Lagrangian acceleration was obtained as follows:

$$\vec{A}^P(\vec{X},t) = \dot{\vec{V}}^P = \ddot{\vec{x}}^P \equiv \frac{D^2}{Dt^2}\vec{x}(\vec{X},t) \tag{2.34}$$

The Eulerian acceleration can be evaluated either by substituting the inverse equation of motion into the equation in (2.34) or via the definition of the material time derivative for an Eulerian property, *i.e.*:

$$\begin{gathered} a_i^P(\vec{x},t) = \frac{D}{Dt}v_i(\vec{x},t) = \frac{\partial v_i(\vec{x},t)}{\partial t} + \frac{\partial v_i(\vec{x},t)}{\partial x_k}\frac{\partial x_k}{\partial t} = \frac{\partial v_i(\vec{x},t)}{\partial t} + \underbrace{\frac{\partial v_i(\vec{x},t)}{\partial x_k} v_k(\vec{x},t)}_{\substack{\textit{convective}\\ \textit{acceleration}}} \\ \boxed{\vec{a}(\vec{x},t) \equiv \frac{D\vec{v}(\vec{x},t)}{Dt} = \frac{\partial\vec{v}(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\vec{v}\cdot\vec{v}(\vec{x},t)} \end{gathered} \tag{2.35}$$

The Eulerian acceleration in matrix form can be evaluated as follows:

$$a_i(\vec{\boldsymbol{x}},t)=\begin{Bmatrix}a_1\\a_2\\a_3\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial v_1}{\partial t}\\\dfrac{\partial v_2}{\partial t}\\\dfrac{\partial v_3}{\partial t}\end{Bmatrix}+\begin{bmatrix}\dfrac{\partial v_1}{\partial x_1}&\dfrac{\partial v_1}{\partial x_2}&\dfrac{\partial v_1}{\partial x_3}\\\dfrac{\partial v_2}{\partial x_1}&\dfrac{\partial v_2}{\partial x_2}&\dfrac{\partial v_2}{\partial x_3}\\\dfrac{\partial v_3}{\partial x_1}&\dfrac{\partial v_3}{\partial x_2}&\dfrac{\partial v_3}{\partial x_3}\end{bmatrix}\begin{Bmatrix}v_1\\v_2\\v_3\end{Bmatrix} \tag{2.36}$$

Returning to **Problem 2.2**, the Eulerian velocity field was obtained as:

$$v_1(\vec{\boldsymbol{x}},t)=2(x_2-tx_3)t \quad ; \quad v_2(\vec{\boldsymbol{x}},t)=x_3 \quad ; \quad v_3(\vec{\boldsymbol{x}},t)=0 \tag{2.37}$$

Explicitly, the Eulerian acceleration can also be evaluated by using the definition in (2.35):

$$\begin{aligned}a_i^P(\vec{\boldsymbol{x}},t)&=\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial t}+\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_k}v_k(\vec{\boldsymbol{x}},t)\\&=\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial t}+\left[\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_1}v_1(\vec{\boldsymbol{x}},t)+\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_2}v_2(\vec{\boldsymbol{x}},t)+\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_3}v_3(\vec{\boldsymbol{x}},t)\right]\end{aligned} \tag{2.38}$$

Thus, the components a_i are given by:

$$\begin{aligned}a_1^P(\vec{\boldsymbol{x}},t)&=\frac{\partial v_1(\vec{\boldsymbol{x}},t)}{\partial t}+\left[\frac{\partial v_1(\vec{\boldsymbol{x}},t)}{\partial x_1}v_1(\vec{\boldsymbol{x}},t)+\frac{\partial v_1(\vec{\boldsymbol{x}},t)}{\partial x_2}v_2(\vec{\boldsymbol{x}},t)+\frac{\partial v_1(\vec{\boldsymbol{x}},t)}{\partial x_3}v_3(\vec{\boldsymbol{x}},t)\right]\\&=[2x_2-4x_3t]+[0+2x_3t-0]\\&=2(x_2-x_3t)\\a_2^P(\vec{\boldsymbol{x}},t)&=\frac{\partial v_2(\vec{\boldsymbol{x}},t)}{\partial t}+\left[\frac{\partial v_2(\vec{\boldsymbol{x}},t)}{\partial x_1}v_1(\vec{\boldsymbol{x}},t)+\frac{\partial v_2(\vec{\boldsymbol{x}},t)}{\partial x_2}v_2(\vec{\boldsymbol{x}},t)+\frac{\partial v_2(\vec{\boldsymbol{x}},t)}{\partial x_3}v_3(\vec{\boldsymbol{x}},t)\right]\\&=0\\a_3^P(\vec{\boldsymbol{x}},t)&=\frac{\partial v_3(\vec{\boldsymbol{x}},t)}{\partial t}+\left[\frac{\partial v_3(\vec{\boldsymbol{x}},t)}{\partial x_1}v_1(\vec{\boldsymbol{x}},t)+\frac{\partial v_3(\vec{\boldsymbol{x}},t)}{\partial x_2}v_2(\vec{\boldsymbol{x}},t)+\frac{\partial v_3(\vec{\boldsymbol{x}},t)}{\partial x_3}v_3(\vec{\boldsymbol{x}},t)\right]\\&=0\end{aligned} \tag{2.39}$$

whose components are the same as those obtained in (2.26).

2.4.2 Stationary Fields

A field $\phi(\vec{\boldsymbol{x}},t)$ is said to be stationary if the local rate of change does not vary over time:

$$\frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial t}=\boldsymbol{0} \quad \Rightarrow \quad \phi=\phi(\vec{\boldsymbol{x}}) \quad \textit{Steady state (stationary) field} \tag{2.40}$$

For example, let us consider a stationary (steady state) velocity field as shown in Figure 2.12. Then, as we can verify, the field representation for any time, *e.g.* t_1 and t_2, does not change. However, that does not mean that the velocities of the particles do not change over time. In light of Figure 2.12, we can now focus our attention on the fixed spatial point $\vec{\boldsymbol{x}}^*$. At time t_1 the particle Q is passing through point $\vec{\boldsymbol{x}}^*$ with velocity $\vec{\boldsymbol{v}}^*$. Let us also consider another particle P, which is passing through another point with velocity $\vec{\boldsymbol{v}}^P(t_1)\neq\vec{\boldsymbol{v}}^*$. At time t_2 the particle P is now passing through the point $\vec{\boldsymbol{x}}^*$. It follows that if we are dealing with a steady state velocity field, then the velocity of particle P at $\vec{\boldsymbol{x}}^*$ must be $\vec{\boldsymbol{v}}^*$, *i.e.* $\vec{\boldsymbol{v}}^P(t_2)=\vec{\boldsymbol{v}}^*$. We can easily contrast this with the material time derivative of velocity, which is always associated with the same particle, *i.e.*:

$$\frac{D\vec{v}(\vec{x},t)}{Dt} \equiv \vec{a}(\vec{x},t) = \underbrace{\frac{\partial \vec{v}(\vec{x},t)}{\partial t}}_{=\vec{\mathbf{0}}\,(\text{Stationay})} + \nabla_{\vec{x}}\vec{v} \cdot \vec{v}(\vec{x}) = \nabla_{\vec{x}}\vec{v} \cdot \vec{v}(\vec{x}) = \vec{a}(\vec{x}) \tag{2.41}$$

The rate of change of velocity (acceleration) will be zero if the velocity field is stationary $\left(\frac{\partial \vec{v}(\vec{x},t)}{\partial t} = \vec{\mathbf{0}}\right)$ and homogeneous ($\nabla_{\vec{x}}\vec{v} = \mathbf{0}$).

We can also verify that, although spatial velocity is independent of time, that does not mean material velocity is also, since:

$$\vec{v}(\vec{x}) = \vec{v}(\vec{x}(\vec{X},t)) = \vec{v}(\vec{X},t) \tag{2.42}$$

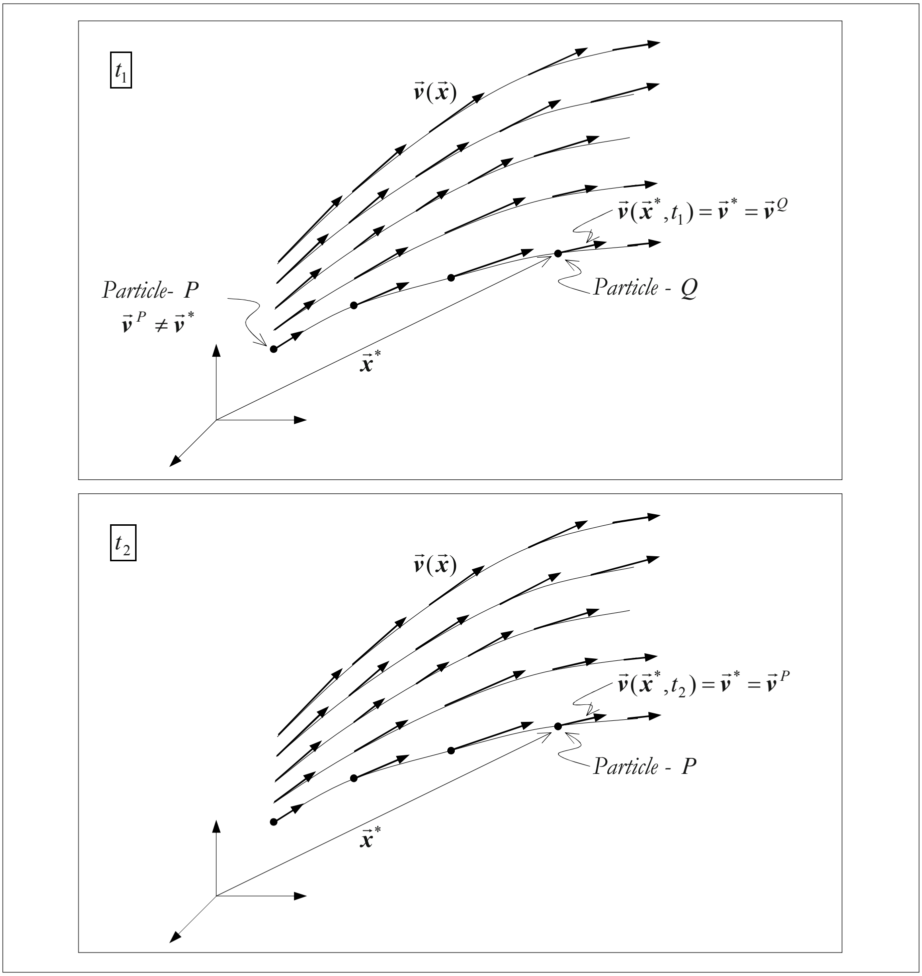


Figure 2.12: Steady velocity field.

NOTE: As we can verify, there are two ways of analyzing the continuum. Either we can follow the particles and see how their properties change over time or we can focus our attention on a fixed spatial region and check how the continuum properties change over time. While the first option is most often used in solid mechanics, the second one is widespread in the field of fluid mechanics. ■

2.4.3 Streamlines

Given a spatial velocity field at time t, we can define a *streamline* to the curve in which the tangent at each point has the same direction as the velocity. In general, the streamline and the trajectory do not coincide, but in steady state motion they do.

Problem 2.3: The acceleration vector field is described by:

$$\vec{a}(\vec{x},t)=\frac{D\vec{v}}{Dt}=\frac{\partial\vec{v}}{\partial t}+\nabla_{\vec{x}}\vec{v}\cdot\vec{v}$$

Show that acceleration can also be written as:

$$\frac{D\vec{v}}{Dt}=\frac{\partial\vec{v}}{\partial t}+\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})=\frac{\partial\vec{v}}{\partial t}+\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{v}\wedge\mathrm{rot}\,\vec{v}=\frac{\partial\vec{v}}{\partial t}+\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)+\mathrm{rot}\,\vec{v}\wedge\vec{v}$$

Solution:

To prove the above relationship one need only demonstrate that:

$$\nabla_{\vec{x}}\vec{v}\cdot\vec{v}=\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})$$

Expressing the terms on the right of the equation in symbolic notation we obtain:

$$\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})=\frac{1}{2}\left[\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}(v_jv_j)\right]-(v_i\hat{\mathbf{e}}_i)\wedge\left[\frac{\partial}{\partial x_r}\hat{\mathbf{e}}_r\wedge(v_s\hat{\mathbf{e}}_s)\right]$$

Using the definition of the permutation symbol (see Chapter 1) we can express the vector product as:

$$\begin{aligned}\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})&=\frac{1}{2}\left[\hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}(v_jv_j)\right]-(v_i\hat{\mathbf{e}}_i)\wedge\epsilon_{rst}\frac{\partial v_s}{\partial x_r}\hat{\mathbf{e}}_t\\&=\frac{1}{2}\left[\hat{\mathbf{e}}_i2v_j\frac{\partial v_j}{\partial x_i}\right]-\epsilon_{rst}\epsilon_{itk}v_i\frac{\partial v_s}{\partial x_r}\hat{\mathbf{e}}_k\end{aligned}$$

where we have used the equation $\hat{\mathbf{e}}_i\wedge\hat{\mathbf{e}}_t=\epsilon_{itk}\hat{\mathbf{e}}_k$. In Chapter 1 we also proved that $\epsilon_{rst}\epsilon_{itk}=\epsilon_{rst}\epsilon_{kit}=\delta_{rk}\delta_{si}-\delta_{ri}\delta_{sk}$, then:

$$\begin{aligned}\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})&=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-(\delta_{rk}\delta_{si}-\delta_{ri}\delta_{sk})v_i\frac{\partial v_s}{\partial x_r}\hat{\mathbf{e}}_k\\&=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-\left(\delta_{rk}\delta_{si}v_i\frac{\partial v_s}{\partial x_r}-\delta_{ri}\delta_{sk}v_i\frac{\partial v_s}{\partial x_r}\right)\hat{\mathbf{e}}_k\\&=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-\left(v_s\frac{\partial v_s}{\partial x_k}-v_i\frac{\partial v_k}{\partial x_i}\right)\hat{\mathbf{e}}_k\end{aligned}$$

$$\begin{aligned}\nabla_{\vec{x}}\left(\frac{v^2}{2}\right)-\vec{v}\wedge(\vec{\nabla}_{\vec{x}}\wedge\vec{v})&=v_j\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-v_s\frac{\partial v_s}{\partial x_k}\hat{\mathbf{e}}_k+v_i\frac{\partial v_k}{\partial x_i}\hat{\mathbf{e}}_k\\&=\delta_{sj}v_s\frac{\partial v_j}{\partial x_i}\hat{\mathbf{e}}_i-v_s\frac{\partial v_s}{\partial x_k}\delta_{ik}\hat{\mathbf{e}}_i+v_i\frac{\partial v_k}{\partial x_i}\hat{\mathbf{e}}_k\\&=v_s\frac{\partial v_s}{\partial x_i}\hat{\mathbf{e}}_i-v_s\frac{\partial v_s}{\partial x_i}\hat{\mathbf{e}}_i+v_i\frac{\partial v_k}{\partial x_i}\hat{\mathbf{e}}_k\\&=\frac{\partial v_k\hat{\mathbf{e}}_k}{\partial x_i}v_i=\frac{\partial(\vec{v})}{\partial x_i}v_i\\&=\nabla_{\vec{x}}\vec{v}\cdot\vec{v}\end{aligned}$$

Problem 2.4: Consider the equations of motion $\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)$ and the temperature field $T(\vec{\boldsymbol{x}},t)$ given by:

$$\begin{cases} x_1 = X_1(1+t) \\ x_2 = X_2(1+t) \\ x_3 = X_3 \end{cases} \quad ; \quad T(\vec{\boldsymbol{x}}) = x_1^2 + x_2^2$$

Find the rate of change of temperature for the particle P at time $t = 1s$ given that particle P was at point $(X_1 = 3, X_2 = 1, X_3 = 0)$ at time $t = 0$.

Solution 1:

In this first solution we first obtain the material time derivative of the Lagrangian temperature, so, we have to obtain the temperature in Lagrangian description $T(\vec{\boldsymbol{X}},t)$ (Lagrangian temperature):

$$T(\vec{\boldsymbol{x}}) = x_1^2 + x_2^2$$
$$\downarrow$$
$$\substack{\textit{By substituting} \\ \textit{the equation of motion}}$$
$$\downarrow$$
$$T(\vec{\boldsymbol{X}},t) = X_1^2(1+t)^2 + X_2^2(1+t)^2$$

The material time derivative of the Lagrangian temperature is given by:

$$\dot{T}(\vec{\boldsymbol{X}},t) \equiv \frac{DT}{Dt} = \frac{dT(\vec{\boldsymbol{X}},t)}{dt} = 2X_1^2(1+t) + 2X_2^2(1+t)$$

By substituting $t = 1s$, $(X_1 = 3, X_2 = 1, X_3 = 0)$, into the above equation we obtain:

$$\Rightarrow \dot{T}(\vec{\boldsymbol{X}},t) = 2X_1^2(1+t) + 2X_2^2(1+t) = 2(3)^2(1+1) + 2(1)^2(1+1) = 40$$

Solution 2:

In this second solution we directly use the definition of material time derivative of the Eulerian variable, (see Eq. (2.30)). Then $\dot{T}(\vec{\boldsymbol{x}},t) = \dfrac{DT}{Dt} = \dfrac{\partial T(\vec{\boldsymbol{x}})}{\partial t} + \dfrac{\partial T(\vec{\boldsymbol{x}})}{\partial x_k} v_k(\vec{\boldsymbol{x}},t)$.

From the equations of motion we obtain:

$$\begin{cases} x_1 = X_1(1+t) \\ x_2 = X_2(1+t) \\ x_3 = X_3 \end{cases} \xrightarrow{\textit{velocity}} \begin{cases} v_1(\vec{\boldsymbol{X}},t) = X_1 \\ v_2(\vec{\boldsymbol{X}},t) = X_2 \\ v_3(\vec{\boldsymbol{X}},t) = 0 \end{cases}$$

The equations of motion in Eulerian description are given by:

$$\begin{cases} x_1 = X_1(1+t) \\ x_2 = X_2(1+t) \\ x_3 = X_3 \end{cases} \xrightarrow{\textit{inverse of the motion}} \begin{cases} X_1 = \dfrac{x_1}{(1+t)} \\ X_2 = \dfrac{x_2}{(1+t)} \\ X_3 = x_3 \end{cases}$$

So, it is possible to obtain the Eulerian velocity as follows:

$$\begin{cases} V_1(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = X_1(\vec{\boldsymbol{x}},t) = \dfrac{x_1}{(1+t)} = v_1(\vec{\boldsymbol{x}},t) \\ V_2 = (\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = X_2(\vec{\boldsymbol{x}},t) = \dfrac{x_2}{(1+t)} = v_2(\vec{\boldsymbol{x}},t) \\ V_3 = v_3(\vec{\boldsymbol{x}},t) = 0 \end{cases}$$

Afterwards, the material time derivative of the Eulerian temperature, $T(\vec{\boldsymbol{x}},t)$, is given by:

$$\Rightarrow \frac{DT(\vec{\boldsymbol{x}},t)}{Dt} \equiv \dot{T}(\vec{\boldsymbol{x}},t) = \underbrace{\frac{\partial T(\vec{\boldsymbol{x}})}{\partial t}}_{=0\,(\text{Stationary field})} + \left[\frac{\partial T}{\partial x_1} v_1 + \frac{\partial T}{\partial x_2} v_2 + \frac{\partial T}{\partial x_3} v_3 \right]$$

$$\Rightarrow \dot{T}(\vec{\boldsymbol{x}},t) = 2x_1 \frac{x_1}{1+t} + 2x_2 \frac{x_2}{1+t} + 0 \quad \Rightarrow \quad \dot{T}(\vec{\boldsymbol{x}},t) = \frac{2x_1^2}{1+t} + \frac{2x_2^2}{1+t} = \frac{2}{1+t}(x_1^2 + x_2^2)$$

The position of particle P at time $t = 1s$ is evaluated as follows:

$$\begin{cases} x_1 = X_1(1+t) = 3(1+1) = 6 \\ x_2 = X_2(1+t) = 1(1+1) = 2 \\ x_3 = X_3 = 0 \end{cases}$$

Then, by substituting the spatial coordinates in the expression of the material time derivative of temperature we obtain:

$$\dot{T}(\vec{\boldsymbol{x}},t) = \dot{T}(x_1 = 6, x_2 = 2, x_3 = 0, t = 1) = \frac{2}{1+t}(x_1^2 + x_2^2) = \frac{2}{1+1}(6^2 + 2^2) = 40$$

Alternatively, the expression $\dot{T}(\vec{\boldsymbol{x}},t)$ could also have been obtained as:

$$\dot{T}(\vec{\boldsymbol{X}},t) = 2X_1^2(1+t) + 2X_2^2(1+t)$$

$$\dot{T}(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t) = 2[X_1(\vec{\boldsymbol{x}},t)]^2(1+t) + 2[X_2(\vec{\boldsymbol{x}},t)]^2(1+t) = 2\left[\frac{x_1}{(1+t)}\right]^2(1+t) + 2\left[\frac{x_2}{(1+t)}\right]^2(1+t)$$

$$= \frac{2}{(1+t)}(x_1^2 + x_2^2) = \dot{T}(\vec{\boldsymbol{x}},t)$$

2.5 The Deformation Gradient

2.5.1 Introduction

In the previous section we studied the description of a particle in motion without looking at how the relative motion between particles changed. In this section we analyze how distances between particles change during motion after which we define some deformation and strain tensors. To do this, let us consider two neighboring particles in the reference configuration, which are denoted by P and Q, (see Figure 2.13).

2.5.2 Stretch and Unit Extension

Let $d\vec{\boldsymbol{X}}$ be a vector joining two points P and Q in the reference configuration, defining a line element. The unit vector associated with the $d\vec{\boldsymbol{X}}$-direction is represented by $\hat{\boldsymbol{M}}$. After motion, particles P and Q occupy new positions P' and Q', respectively. In this new configuration (current configuration), the vector joining the points P' and Q' is represented by $d\vec{\boldsymbol{x}}$, which is associated with the unit vector $\hat{\boldsymbol{m}}$, (see Figure 2.13), and the magnitudes of $d\vec{\boldsymbol{X}}$ and $d\vec{\boldsymbol{x}}$ are denoted, respectively, by:

$$\| \overrightarrow{PQ} \| = \| d\vec{\boldsymbol{X}} \| = dS \quad ; \quad \| \overrightarrow{P'Q'} \| = \| d\vec{\boldsymbol{x}} \| = ds \tag{2.43}$$

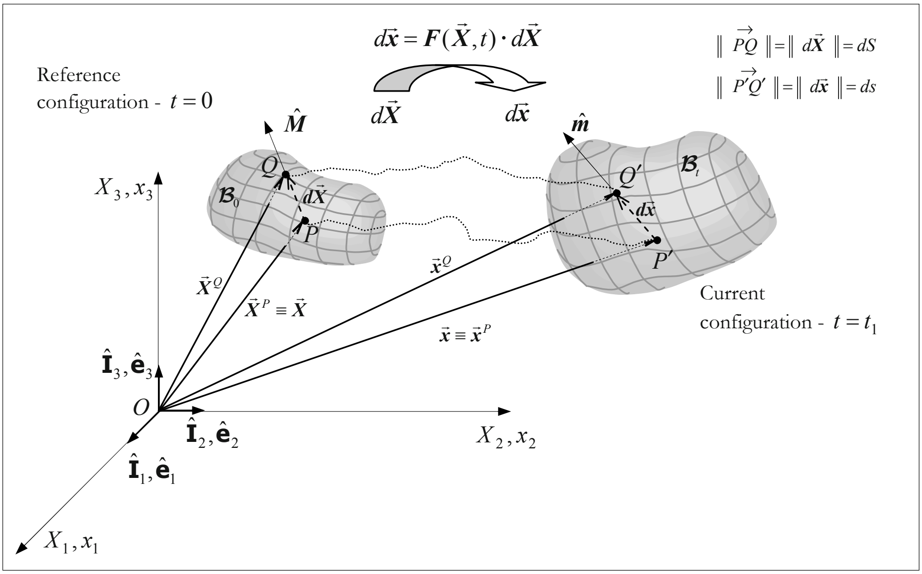


Figure 2.13: Continuum deformation.

Next we can define some parameters related to the magnitudes of line elements:

- The *Stretch (or Stretch ratio)*, $\lambda_{\hat{m}}$, associated with the $\hat{m}$-direction is given by:

$$\boxed{\lambda_{\hat{m}} = \frac{\|d\vec{x}\|}{\|d\vec{X}\|} = \frac{ds}{dS} \qquad where \qquad \lambda_{\hat{m}} > 0} \quad \text{Stretch} \tag{2.44}$$

The possible values of $\lambda_{\hat{m}}$ are in the range: $\underbrace{0}_{ds\to 0} < \lambda_{\hat{m}} < \underbrace{\infty}_{ds\to\infty}$. And due to the axiom of impenetrability $ds \neq 0$, the stretch has to be nonzero, $\lambda_{\hat{m}} \neq 0$, if this is not so, the implication is that two particles are occupying the same place at the same time which has no physical meaning. Then, we can draw the conclusion that:

$\lambda_{\hat{m}} = 1$ - There is no elongation;

$0 < \lambda_{\hat{m}} < 1$ - There is a shortening of the line element $\overrightarrow{PQ}$;

$\lambda_{\hat{m}} > 1$ - There is an increase in distance between particles (elongation).

- The *Unit Extension*, $\varepsilon_{\hat{m}}$, is defined as:

$$\boxed{\varepsilon_{\hat{m}} = \frac{\|d\vec{x}\| - \|d\vec{X}\|}{\|d\vec{X}\|} = \frac{ds - dS}{dS}} \quad \text{Unit extension} \tag{2.45}$$

The possible values of the unit extension are within the range of $-1 < \varepsilon_{\hat{m}} < \infty$. The stretch is related to the unit extension by:

$$\varepsilon_{\hat{m}} = \frac{ds - dS}{dS} = \frac{ds}{dS} - 1 = \lambda_{\hat{m}} - 1 \qquad \Leftrightarrow \qquad ds = (\varepsilon_{\hat{m}} + 1) dS = \lambda_{\hat{m}} dS \tag{2.46}$$

2.5.3 The Material and Spatial Deformation Gradient

Our goal now is to find the relationship between the line elements $d\vec{\boldsymbol{X}}$ and $d\vec{\boldsymbol{x}}$. By considering the material description of motion $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)$, and by applying vector addition, (see Figure 2.13), we obtain:

$$\vec{\boldsymbol{X}}^Q = \vec{\boldsymbol{X}}^P + d\vec{\boldsymbol{X}} \quad ; \quad d\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^Q(\vec{\boldsymbol{X}}^Q,t) - \vec{\boldsymbol{x}}^P(\vec{\boldsymbol{X}}^P,t) \tag{2.47}$$

If we observe that $\vec{\boldsymbol{x}}^Q(\vec{\boldsymbol{X}}^Q,t) = \vec{\boldsymbol{x}}^P(\vec{\boldsymbol{X}}^P + d\vec{\boldsymbol{X}},t) = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}} + d\vec{\boldsymbol{X}},t)$, the vector field $d\vec{\boldsymbol{x}}$ in the current configuration becomes:

$$\begin{aligned}
d\vec{\boldsymbol{x}} &= \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}} + d\vec{\boldsymbol{X}},t) - \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t) \\
d\vec{\boldsymbol{x}} &= \left[x_i(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3, t) - x_i(X_1, X_2, X_3, t)\right]\hat{\mathbf{e}}_i
\end{aligned} \tag{2.48}$$

Then by applying Taylor series to represent the function in (2.48) we obtain:

$$d\vec{\boldsymbol{x}} = \left[\frac{\partial x_i}{\partial X_1}dX_1 + \frac{\partial x_i}{\partial X_2}dX_2 + \frac{\partial x_i}{\partial X_3}dX_3\right]\hat{\mathbf{e}}_i + O(\left\|d\vec{\boldsymbol{X}}\right\|^2) \Rightarrow d\vec{\boldsymbol{x}} = \frac{\partial x_i}{\partial X_j}dX_j\,\hat{\mathbf{e}}_i + O(\left\|d\vec{\boldsymbol{X}}\right\|^2) \tag{2.49}$$

Since points P and Q are close together, higher order terms can be discarded, which leads to:

$$d\vec{\boldsymbol{x}} = \frac{\partial x_i}{\partial X_k}dX_k\,\hat{\mathbf{e}}_i = F_{ik}dX_k\,\hat{\mathbf{e}}_i \tag{2.50}$$

or expressed in compact form:

$$\boxed{d\vec{\boldsymbol{x}} = \boldsymbol{F}\cdot d\vec{\boldsymbol{X}}} \tag{2.51}$$

where $\boldsymbol{F}$ is a two-point tensor and is known as the *material deformation gradient* or simply the *deformation gradient*. The relation in (2.51) is a linear transformation so $\boldsymbol{F}$ relates $d\vec{\boldsymbol{X}}$ (undeformed configuration) to $d\vec{\boldsymbol{x}}$ (deformed configuration), (see Figure 2.13). The equation in (2.51) could have been obtained by directly starting from the gradient definition, (see Chapter 1-Tensors). That is, if $\phi = \phi(\vec{\boldsymbol{x}},t)$ is a scalar field, the total derivative ($d\phi$) is given by the equation: $d\phi(\vec{\boldsymbol{x}},t) = \nabla\phi\cdot d\vec{\boldsymbol{x}} = \dfrac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial\vec{\boldsymbol{x}}}\cdot d\vec{\boldsymbol{x}}$. Then, if we have the vector field $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)$, the total derivative (differential) is:

$$d\vec{\boldsymbol{x}} = \frac{\partial\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)}{\partial\vec{\boldsymbol{X}}}\cdot d\vec{\boldsymbol{X}} = \nabla_{\vec{X}}\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)\cdot d\vec{\boldsymbol{X}} = \boldsymbol{F}\cdot d\vec{\boldsymbol{X}} \tag{2.52}$$

The components of $d\vec{\boldsymbol{x}}$ in Cartesian system can be evaluated by means of the scalar product (dot product):

$$(d\vec{\boldsymbol{x}})_j = d\vec{\boldsymbol{x}}\cdot\hat{\mathbf{e}}_j = F_{ik}dX_k\underbrace{\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j}_{\delta_{ij}} = F_{jk}dX_k \tag{2.53}$$

The tensor $\boldsymbol{F}$ can also be expressed as:

$$\boldsymbol{F} = \frac{\partial\vec{\boldsymbol{x}}}{\partial X_1}\hat{\mathbf{e}}_1 + \frac{\partial\vec{\boldsymbol{x}}}{\partial X_2}\hat{\mathbf{e}}_2 + \frac{\partial\vec{\boldsymbol{x}}}{\partial X_3}\hat{\mathbf{e}}_3 = \frac{\partial x_i}{\partial X_j}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j = x_{i,J}\,\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j \tag{2.54}$$

whose components can be expressed in matrix form as:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = x_{i,J} = \mathcal{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{2.55}$$

NOTE: Sometimes, for purposes of clarification, we use subscript in capital letters to differentiate material coordinates, *i.e.* $\bullet_{i,J} \equiv \frac{\partial \bullet_i}{\partial X_j} \neq \bullet_{i,j} \equiv \frac{\partial \bullet_i}{\partial x_j}$. ■

NOTE: In this publication, we denote the material gradient by $\mathrm{Grad}(\bullet) \equiv \nabla_{\vec{X}}(\bullet) = \frac{\partial(\bullet)}{\partial X_i} \otimes \hat{\mathbf{e}}_i$ and the spatial gradient by $\mathrm{grad}(\bullet) \equiv \nabla_{\vec{x}}(\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes \hat{\mathbf{e}}_i$. ■

We can also find the inverse transformation of the equation in (2.51) *i.e.*:

$$\boxed{d\vec{\boldsymbol{X}} = \boldsymbol{F}^{-1} \cdot d\vec{\boldsymbol{x}}} \tag{2.56}$$

where $\boldsymbol{F}^{-1}$ is the *spatial deformation gradient*, which is defined as:

$$\boldsymbol{F}^{-1} = \nabla_{\vec{x}} \vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t) \qquad\qquad F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} = X_{I,j} \tag{2.57}$$

Explicitly, the components of $\boldsymbol{F}^{-1}$ are given by:

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} = \mathcal{F}^{-1} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} \tag{2.58}$$

We can now show, (see Chapter 1), that the following relationships are valid:

$$\left(\boldsymbol{F}^{-1}\right)_{il} = \frac{[\mathrm{cof}(\boldsymbol{F})]_{li}}{J} = \frac{1}{2J} \epsilon_{lmn} \epsilon_{ijk} F_{mj} F_{nk} = \frac{1}{2J} \epsilon_{lmn} \epsilon_{IJK} x_{m,J} x_{n,K} \tag{2.59}$$

$$J = \det(\boldsymbol{F}) = \frac{1}{6} \epsilon_{lmn} \epsilon_{IJK} F_{li} F_{mj} F_{nk} = \frac{1}{6} \epsilon_{lmn} \epsilon_{IJK} x_{l,I} x_{m,J} x_{n,K} \tag{2.60}$$

The derivative of the equation (2.60) with respect to $\boldsymbol{F}$ becomes:

$$\frac{dJ}{dF_{pq}} = \frac{dJ}{dx_{p,Q}} = \frac{1}{6} \epsilon_{lmn} \epsilon_{IJK} \left[\frac{\partial x_{l,I}}{\partial x_{p,Q}} x_{m,J} x_{n,K} + x_{l,I} \frac{\partial x_{m,J}}{\partial x_{p,Q}} x_{n,K} + x_{l,I} x_{m,J} \frac{\partial x_{n,K}}{\partial x_{p,Q}} \right] \tag{2.61}$$

which becomes:

$$
\begin{aligned}
\frac{dJ}{dx_{p,Q}} &= \frac{1}{6}\epsilon_{lmn}\epsilon_{IJK}\left[\delta_{lp}\delta_{IQ}x_{m,J}x_{n,K} + \delta_{mp}\delta_{JQ}x_{l,I}x_{n,K} + \delta_{np}\delta_{KQ}x_{l,I}x_{m,J}\right] \\
&= \frac{1}{6}\left[\epsilon_{pmn}\epsilon_{QJK}x_{m,J}x_{n,K} + \epsilon_{lpn}\epsilon_{IQK}x_{l,I}x_{n,K} + \epsilon_{lmp}\epsilon_{IJQ}x_{l,I}x_{m,J}\right] \\
&= \frac{1}{6}\left[\epsilon_{pmn}\epsilon_{QJK}x_{m,J}x_{n,K} + \epsilon_{pnl}\epsilon_{QKI}x_{n,K}x_{l,I} + \epsilon_{plm}\epsilon_{QIJ}x_{l,I}x_{m,J}\right] \\
&= \frac{1}{2}\epsilon_{pmn}\epsilon_{QJK}x_{m,J}x_{n,K} = \frac{1}{2}\epsilon_{pmn}\epsilon_{qjk}F_{mj}F_{nk}
\end{aligned}
\tag{2.62}
$$

Referring to the definitions for cofactors and inverting tensors, the following relationship holds:

$$
\begin{aligned}
\frac{dJ}{dF_{pq}} = \frac{dJ}{dx_{p,Q}} &= \frac{1}{2}\epsilon_{pmn}\epsilon_{qjk}F_{mj}F_{nk} \\
&= [\mathrm{cof}(\boldsymbol{F})]_{pq} = |\boldsymbol{F}|F^{-1}_{qp} = JX_{Q,p}
\end{aligned}
\tag{2.63}
$$

We could have obtained the above equation by means of the definition for third invariant derivatives in respect to tensors, (see Chapter 1), *i.e.*:

$$
\frac{\partial[\, III_{\boldsymbol{F}}\,]}{\partial \boldsymbol{F}} = \frac{\partial[\det(\boldsymbol{F})]}{\partial \boldsymbol{F}} = III_{\boldsymbol{F}}\,\boldsymbol{F}^{-T} = J\boldsymbol{F}^{-T}
\tag{2.64}
$$

In comparison with (2.63) it is also true that:

$$
J^{-1}x_{q,P} = [\mathrm{cof}(\boldsymbol{F}^{-1})]_{pq} = \frac{1}{2}\epsilon_{pmn}\epsilon_{qjk}F^{-1}_{mj}F^{-1}_{nk}
\tag{2.65}
$$

The derivative of the equation (2.59) with respect to $\vec{\boldsymbol{X}}$ becomes:

$$
\frac{\partial\left(JX_{Q,p}\right)}{\partial X_Q} \equiv (JF^{-1}_{qp})_{,q} = 0_p \quad ; \quad \nabla_{\vec{X}}\cdot\left(J\boldsymbol{F}^{-T}\right) = \vec{\mathbf{0}}
\tag{2.66}
$$

or

$$
\frac{\partial\left(J^{-1}x_{q,p}\right)}{\partial x_q} \equiv (J^{-1}F_{qp})_{,q} = 0_p \quad ; \quad \nabla_{\vec{x}}\cdot\left(J^{-1}\boldsymbol{F}^{T}\right) = \vec{\mathbf{0}}
\tag{2.67}
$$

The following provides proof of the above. From (2.63) we obtain

$$
\begin{aligned}
\left(JX_{q,p}\right)_{,q} &= \left[\frac{1}{2}\epsilon_{pmn}\epsilon_{qjk}F_{mj}F_{nk}\right]_{,q} = \frac{1}{2}\epsilon_{pmn}\epsilon_{qjk}\left[F_{mj,q}F_{nk} + F_{mj}F_{nk,q}\right] \\
&= \frac{1}{2}\epsilon_{pmn}\epsilon_{qjk}\left(x_{m,jq}x_{n,k} + x_{m,j}x_{n,kq}\right) = 0_p
\end{aligned}
\tag{2.68}
$$

Note that, in kq the tensor $\epsilon_{qjk} = \epsilon_{jkq} = -\epsilon_{jqk}$ is antisymmetric while the tensor $x_{n,kq}$ is symmetric. Therefore $\epsilon_{qjk}x_{n,kq} = 0_{jn}$. We can obtain the same result for $\epsilon_{qjk}x_{m,jq} = 0_{km}$. Likewise, it is possible to demonstrate that $\left(J^{-1}x_{q,p}\right)_{,q} = 0_p$.

Using the above definitions, it can be shown that, if $\vec{\mathbf{u}}(\vec{x},t)$ and $\boldsymbol{\sigma}(\vec{x},t)$ represent a vector and a second-order tensor field, respectively, they satisfy the following relationships:

$$
\begin{aligned}
\nabla_{\vec{X}}\cdot\vec{\mathbf{u}}(\vec{X},t) &= J\nabla_{\vec{x}}\cdot\left[J^{-1}\boldsymbol{F}\cdot\vec{\mathbf{u}}(\vec{x},t)\right] \\
\nabla_{\vec{X}}\cdot\boldsymbol{\sigma}(\vec{X},t) &= J\nabla_{\vec{x}}\cdot\left[J^{-1}\boldsymbol{F}\cdot\boldsymbol{\sigma}(\vec{x},t)\right]
\end{aligned}
\tag{2.69}
$$

To prove this we can use indicial notation:

$$J\nabla_{\vec{x}} \cdot \left[J^{-1}\boldsymbol{F} \cdot \vec{\mathbf{u}}(\vec{x},t)\right] \xrightarrow{indicial} J\left(J^{-1}F_{ij}\mathrm{u}_j\right)_{,i} = J\left[\underbrace{\left(J^{-1}F_{ij}\right)_{,i}}_{=0_j}\mathrm{u}_j + \left(J^{-1}F_{ij}\right)\mathrm{u}_{j,i}\right]$$

$$J\left(J^{-1}F_{ij}\mathrm{u}_j\right)_{,i} = F_{ij}\mathrm{u}_{j,i} = \frac{\partial x_i}{\partial X_j}\frac{\partial \mathrm{u}_j(\vec{x},t)}{\partial x_i} = \frac{\partial \mathrm{u}_j(\vec{x},t)}{\partial x_i}\frac{\partial x_i}{\partial X_j} = \frac{\partial \mathrm{u}_j(\vec{X},t)}{\partial X_j} = \nabla_{\vec{X}} \cdot \vec{\mathbf{u}}(\vec{X},t) \tag{2.70}$$

Likewise:

$$J\nabla_{\vec{x}} \cdot \left[J^{-1}\boldsymbol{F} \cdot \boldsymbol{\sigma}(\vec{x},t)\right] \xrightarrow{indicial} J\left(J^{-1}F_{ik}\sigma_{kj}\right)_{,i} = J\left[\underbrace{\left(J^{-1}F_{ik}\right)_{,i}}_{=0_k}\sigma_{kj} + \left(J^{-1}F_{ik}\right)\sigma_{kj,i}\right]$$

$$J\left(J^{-1}F_{ik}\right)\sigma_{kj,i} = F_{ik}\sigma_{kj,i} = \frac{\partial x_i}{\partial X_k}\frac{\partial \sigma_{kj}(\vec{x},t)}{\partial x_i} = \frac{\partial \sigma_{kj}(\vec{x},t)}{\partial x_i}\frac{\partial x_i}{\partial X_j} = \frac{\partial \sigma_{kj}(\vec{X},t)}{\partial X_j} \tag{2.71}$$

$$\Rightarrow J\nabla_{\vec{x}} \cdot \left[J^{-1}\boldsymbol{F} \cdot \boldsymbol{\sigma}(\vec{x},t)\right] = \nabla_{\vec{X}} \cdot \boldsymbol{\sigma}(\vec{X},t)$$

Problem 2.5: Let $\phi(\vec{X},t)$ be a scalar field in Lagrangian (material) description. Find the relationship between the material gradient of $\phi(\vec{X},t)$, *i.e.* $\nabla_{\vec{X}}\phi(\vec{X},t)$, and the spatial gradient of $\phi(\vec{x},t)$, *i.e.* $\nabla_{\vec{x}}\phi(\vec{x},t)$.

Solution: Remember that a Lagrangian variable $\phi(\vec{X},t)$ can be expressed in the Eulerian (current) configuration by means of the equations of motion, *i.e.*:

$$\phi(\vec{X},t) = \phi(\vec{X}(\vec{x},t),t) = \phi(\vec{x},t) .$$

Then, from the scalar gradient definition we obtain:

$$\nabla_{\vec{X}}\phi(\vec{X},t) = \frac{\partial\phi(\vec{X},t)}{\partial\vec{X}} = \frac{\partial\phi(\vec{X}(\vec{x},t),t)}{\partial\vec{x}} \cdot \frac{\partial\vec{x}}{\partial\vec{X}} = \frac{\partial\phi(\vec{x},t)}{\partial\vec{x}} \cdot \boldsymbol{F} = \nabla_{\vec{x}}\phi(\vec{x},t) \cdot \boldsymbol{F}$$

In addition we have the inverse form:

$$\nabla_{\vec{x}}\phi(\vec{x},t) = \frac{\partial\phi(\vec{x},t)}{\partial\vec{x}} = \frac{\partial\phi(\vec{x}(\vec{X},t),t)}{\partial\vec{X}} \cdot \frac{\partial\vec{X}}{\partial\vec{x}} = \frac{\partial\phi(\vec{X},t)}{\partial\vec{X}} \cdot \boldsymbol{F}^{-1} = \nabla_{\vec{X}}\phi(\vec{X},t) \cdot \boldsymbol{F}^{-1}$$

2.5.4 Displacement Gradient Tensors (Material and Spatial)

Let $\vec{\mathbf{u}}$ be a displacement field, (see equation (2.11)). The displacement components in Lagrangian (material) and Eulerian (spatial) descriptions are, respectively:

$$\mathrm{u}_i(\vec{X},t) = x_i(\vec{X},t) - X_i \qquad \text{and} \qquad \mathrm{u}_i(\vec{x},t) = x_i - X_i(\vec{x},t) \tag{2.72}$$

Taking the partial derivative of $\mathrm{u}_i(\vec{X},t)$ with respect to the material coordinates $\vec{X}$, we obtain:

Indicial notation	Tensorial notation
$\dfrac{\partial \mathrm{u}_i(\vec{X},t)}{\partial X_j} = \dfrac{\partial x_i(\vec{X},t)}{\partial X_j} - \dfrac{\partial X_i}{\partial X_j}$	
$\mathrm{u}_{i,J}(\vec{X},t) = F_{ij} - \delta_{ij}$	$\nabla_{\vec{X}}\vec{\mathbf{u}}(\vec{X},t) \equiv \boldsymbol{J}(\vec{X},t) = \boldsymbol{F} - \mathbf{1}$

(2.73)

where $\boldsymbol{J}$ is known as the *material displacement gradient tensor*.

Taking now the partial derivative of equation (2.72) with respect to spatial coordinates $\vec{x}$, we obtain:

Indicial notation	Tensorial notation	
$\dfrac{\partial \mathrm{u}_i(\vec{x},t)}{\partial x_j} = \dfrac{\partial x_i}{\partial x_j} - \dfrac{\partial X_i(\vec{x},t)}{\partial x_j}$ $\mathrm{u}_{i,j}(\vec{x},t) = \delta_{ij} - F^{-1}_{ij}$	$\nabla_{\vec{x}} \vec{\mathbf{u}}(\vec{x},t) \equiv \boldsymbol{j}(\vec{x},t) = \mathbf{1} - \boldsymbol{F}^{-1}$	(2.74)

where $\boldsymbol{j}$ is known as the *spatial displacement gradient tensor*.

The components of $\boldsymbol{J}$ and $\boldsymbol{j}$ can be represented, respectively, as:

$$\boldsymbol{J}_{ij} = \frac{\partial \mathrm{u}_i(\vec{X},t)}{\partial X_j} = \begin{bmatrix} \dfrac{\partial \mathrm{u}_1}{\partial X_1} & \dfrac{\partial \mathrm{u}_1}{\partial X_2} & \dfrac{\partial \mathrm{u}_1}{\partial X_3} \\ \dfrac{\partial \mathrm{u}_2}{\partial X_1} & \dfrac{\partial \mathrm{u}_2}{\partial X_2} & \dfrac{\partial \mathrm{u}_2}{\partial X_3} \\ \dfrac{\partial \mathrm{u}_3}{\partial X_1} & \dfrac{\partial \mathrm{u}_3}{\partial X_2} & \dfrac{\partial \mathrm{u}_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} - 1 & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} - 1 & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} - 1 \end{bmatrix} \tag{2.75}$$

$$\boldsymbol{j}_{ij} = \frac{\partial \mathrm{u}_i(\vec{x},t)}{\partial x_j} = \begin{bmatrix} \dfrac{\partial \mathrm{u}_1}{\partial x_1} & \dfrac{\partial \mathrm{u}_1}{\partial x_2} & \dfrac{\partial \mathrm{u}_1}{\partial x_3} \\ \dfrac{\partial \mathrm{u}_2}{\partial x_1} & \dfrac{\partial \mathrm{u}_2}{\partial x_2} & \dfrac{\partial \mathrm{u}_2}{\partial x_3} \\ \dfrac{\partial \mathrm{u}_3}{\partial x_1} & \dfrac{\partial \mathrm{u}_3}{\partial x_2} & \dfrac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{\partial X_1}{\partial x_1} & -\dfrac{\partial X_1}{\partial x_2} & -\dfrac{\partial X_1}{\partial x_3} \\ -\dfrac{\partial X_2}{\partial x_1} & 1 - \dfrac{\partial X_2}{\partial x_2} & -\dfrac{\partial X_2}{\partial x_3} \\ -\dfrac{\partial X_3}{\partial x_1} & -\dfrac{\partial X_3}{\partial x_2} & 1 - \dfrac{\partial X_3}{\partial x_3} \end{bmatrix} \tag{2.76}$$

By referring to (2.74) we obtain:

$$\boldsymbol{j}(\vec{x},t) = \mathbf{1} - \boldsymbol{F}^{-1} \quad \Rightarrow \quad \boldsymbol{j}(\vec{x},t) \cdot \boldsymbol{F} = \left(\mathbf{1} - \boldsymbol{F}^{-1}\right) \cdot \boldsymbol{F} = \boldsymbol{F} - \mathbf{1} \tag{2.77}$$

and by comparing the above equation with that in (2.73) we can draw the conclusion that $\boldsymbol{J}(\vec{X},t)$ and $\boldsymbol{j}(\vec{x},t)$ are interrelated by:

$$\boldsymbol{J}(\vec{X},t) = \boldsymbol{j}(\vec{x},t) \cdot \boldsymbol{F} \quad ; \quad \nabla_{\vec{X}} \vec{\mathbf{u}}(\vec{X},t) = \nabla_{\vec{x}} \vec{\mathbf{u}}(\vec{x},t) \cdot \boldsymbol{F} \tag{2.78}$$

It is interesting to compare the above with the outcome of **Problem 2.5**.

Problem 2.6: Consider a continuum in which the displacement field is described by the following equations:

$$\begin{cases} \mathrm{u}_1 = 2X_1^2 + X_1 X_2 \\ \mathrm{u}_2 = X_2^2 \\ \mathrm{u}_3 = 0 \end{cases}$$

By definition, a *material curve* is always formed by the same particles. Let $\overline{OP}$ and $\overline{OT}$ be material lines in the reference configuration, where $O(X_1 = 0, X_2 = 0, X_3 = 0)$, $P(X_1 = 1, X_2 = 1, X_3 = 0)$ and $T(X_1 = 1, X_2 = 0, X_3 = 0)$. Find the material curves in the current configuration. Also find the deformation gradient.

Solution:

a) The equations of motion can be obtained by means of the displacement field, (see Eq. (2.72)), *i.e.*:

$$\mathrm{u}_i = x_i - X_i$$

$$\begin{cases} x_1 = \mathrm{u}_1 + X_1 \\ x_2 = \mathrm{u}_2 + X_2 \\ x_3 = \mathrm{u}_3 + X_3 \end{cases} \xrightarrow[\text{the values of } \mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3]{\text{substituting}} \begin{cases} x_1 = X_1 + 2X_1^2 + X_1 X_2 \\ x_2 = X_2 + X_2^2 \\ x_3 = X_3 \end{cases}$$

Then, to obtain the material curve, one need only substitute the material coordinates with the particles belonging to the line $\overline{OP}$ in the equations of motion, (see Figure 2.14). Notice that the material curve $\overline{OP}$ in the current configuration is no longer a straight line, but the line $\overline{OT}$ is still a straight line in the current configuration (see Figure 2.15).
The components of the deformation gradient, (see Eq. (2.55)), can be obtained as follows:

$$F_{jk} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} (1+4X_1+X_2) & X_1 & 0 \\ 0 & 1+2X_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

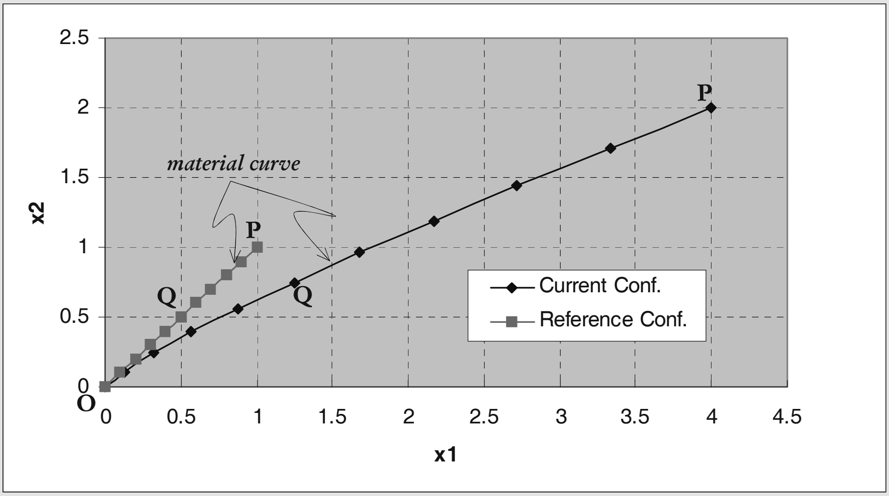


Figure 2.14: Deformation of the material curve $\overline{OP}$.

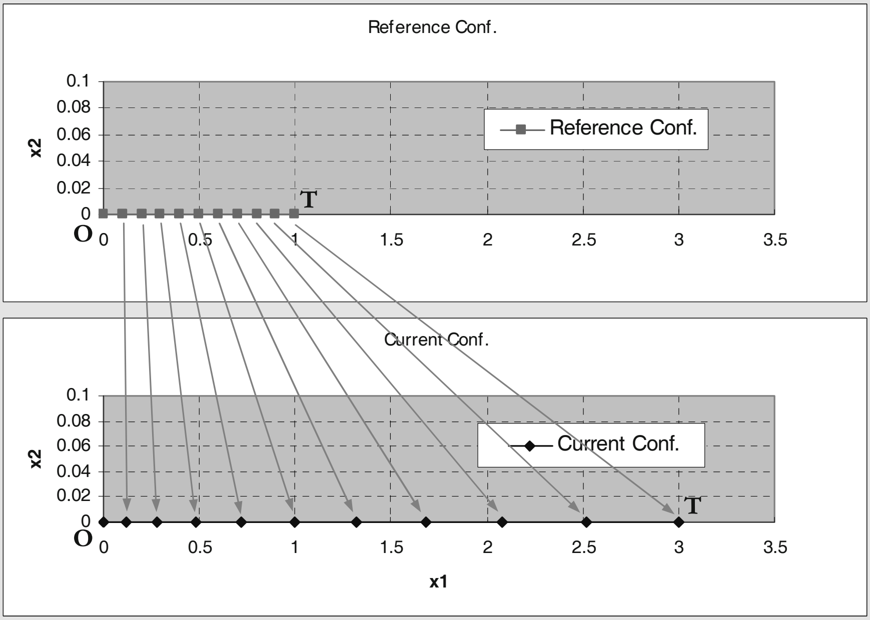


Figure 2.15: Deformation of the material curve $\overline{OT}$.

2.5.5 Material Time Derivative of the Deformation Gradient. Material Time Derivative of the Jacobian Determinant

2.5.5.1 Material Time Derivative of $\boldsymbol{F}$. The Spatial Velocity Gradient

The material time derivative of $\boldsymbol{F}$ is given by

$$
\begin{aligned}
\frac{D}{Dt}F_{ij} \equiv \dot{F}_{ij} = \frac{\partial}{\partial t}\frac{\partial x_i(\vec{\boldsymbol{X}},t)}{\partial X_j} &= \frac{\partial}{\partial X_j}\underbrace{\frac{\partial x_i(\vec{\boldsymbol{X}},t)}{\partial t}}_{\dot{x}_i} &&= \frac{\partial v_i(\vec{\boldsymbol{X}},t)}{\partial X_j} \\
&= \dot{x}_{i,J} &&= v_{I,J}
\end{aligned}
\tag{2.79}
$$

Expressing velocity in Eulerian coordinates, *i.e.* $v_i(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t)$, and by using the chain rule of the derivatives we obtain:

$$
\begin{aligned}
\dot{F}_{ij} &= \frac{\partial v_i(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}},t),t)}{\partial X_j} &&= \frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_k}\frac{\partial x_k(\vec{\boldsymbol{X}},t)}{\partial X_j} \\
&= v_{I,J} &&= v_{i,k}x_{k,J} \\
& &&= v_{i,k}\frac{\partial x_k}{\partial X_j} = \ell_{ik}\frac{\partial x_k}{\partial X_j} = \ell_{ik}F_{kj}
\end{aligned}
\tag{2.80}
$$

The above equation is represented in tensor notation as:

$$
\boxed{\dot{\boldsymbol{F}} = \boldsymbol{\ell}\cdot\boldsymbol{F}}
\tag{2.81}
$$

where $\boldsymbol{\ell}$ is the *spatial velocity gradient*, and is defined as:

$$\boxed{\boldsymbol{\ell}(\vec{\boldsymbol{x}},t) = \nabla_{\vec{x}}\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}} \quad \textit{Spatial velocity gradient} \quad \left[\frac{m}{m\,s}\right] \tag{2.82}$$

Problem 2.7: Let $d\vec{\boldsymbol{x}}$ be a differential line element in the current configuration. Find the material time derivative of $d\vec{\boldsymbol{x}}$.
Solution:

$$\frac{D}{Dt} d\vec{\boldsymbol{x}} = \frac{D}{Dt}(\boldsymbol{F} \cdot d\vec{\boldsymbol{X}}) = \frac{D}{Dt}(\boldsymbol{F}) \cdot d\vec{\boldsymbol{X}} + \boldsymbol{F} \cdot \underbrace{\frac{D}{Dt}(d\vec{\boldsymbol{X}})}_{\vec{\boldsymbol{0}}} = \boldsymbol{\ell} \cdot \underbrace{\boldsymbol{F} \cdot d\vec{\boldsymbol{X}}}_{d\vec{x}}$$

$$= \boldsymbol{\ell} \cdot d\vec{\boldsymbol{x}} \equiv \nabla_{\vec{x}}\vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{x}}$$

And, whose components are represented by:

$$\left(\frac{D}{Dt} d\vec{\boldsymbol{x}}\right)_i = v_{i,k}\, dx_k = \frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial x_k} dx_k$$

2.5.5.2 Rate-of-Deformation and Spin Tensors

The spatial velocity gradient $\boldsymbol{\ell}$ can be decomposed into a symmetric and an antisymmetric part, *i.e.*:

$$\boldsymbol{\ell} = \boldsymbol{\ell}^{sym} + \boldsymbol{\ell}^{skew} = \frac{1}{2}(\boldsymbol{\ell} + \boldsymbol{\ell}^T) + \frac{1}{2}(\boldsymbol{\ell} - \boldsymbol{\ell}^T) = \mathbf{D} + \mathbf{W} \quad \left[\frac{m}{m\,s}\right] \tag{2.83}$$

Whereby, we can define the following tensors:

$\boldsymbol{\ell}^{sym} \equiv \mathbf{D}(\vec{\boldsymbol{x}},t)$ - the *rate-of-deformation tensor;*

$\boldsymbol{\ell}^{skew} \equiv \mathbf{W}(\vec{\boldsymbol{x}},t)$ - the *spin, rate-of-rotation tensor* or *vorticity tensor.*

The components of $\mathbf{D}$ and $\mathbf{W}$, respectively, are:

$$\mathsf{D}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad ; \quad \mathsf{W}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{2.84}$$

The spin tensor has three independent components and can be represented as:

$$\mathsf{W}_{ij} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ \mathsf{W}_{21} & 0 & \mathsf{W}_{23} \\ \mathsf{W}_{31} & \mathsf{W}_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{W}_{12} & \mathsf{W}_{13} \\ -\mathsf{W}_{12} & 0 & \mathsf{W}_{23} \\ -\mathsf{W}_{13} & -\mathsf{W}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{2.85}$$

where w_i are the axial vector components associated with the antisymmetric tensor $\mathbf{W}$. We can also define the *vorticity vector* field as $\vec{\boldsymbol{\omega}} = 2\vec{\boldsymbol{w}}$. Moreover, as we saw in the chapter on tensors: given an antisymmetric tensor, the following holds:

$$\mathsf{W}_{ij} = -w_k \epsilon_{kij} \quad or \quad w_k = -\frac{1}{2}\epsilon_{kij}\mathsf{W}_{ij} \tag{2.86}$$

In Chapter 1 we proved that $2\vec{\boldsymbol{w}} = \mathrm{rot}(\vec{\boldsymbol{v}}) = \vec{\nabla}_{\vec{x}} \wedge \vec{\boldsymbol{v}}$ where $\vec{\boldsymbol{w}}$ is the axial vector associated with the antisymmetric tensor $(\nabla_{\vec{x}}\vec{\boldsymbol{v}})^{skew}$. Therefore, the vorticity vector can be expressed as:

$$\boxed{\vec{\boldsymbol{\omega}} = 2\vec{w} = \mathrm{rot}(\vec{v}) = \vec{\nabla}_{\vec{x}} \wedge \vec{v}} \quad \textit{Vorticity vector} \tag{2.87}$$

Also in Chapter 1 we showed the following relationship was satisfied:

$$\mathbf{W} \cdot \vec{v} = \vec{w} \wedge \vec{v} = \frac{1}{2}\left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v} \tag{2.88}$$

When $\mathbf{D} = \mathbf{0}$ motion is characterized by rigid body motion, *i.e.* the distance between particles does not change. Furthermore, the condition $\frac{D(d\vec{x})}{Dt} = \vec{w} \wedge d\vec{x}$ is satisfied which is proved by:

$$\underbrace{\frac{D(d\vec{x})}{Dt} = \boldsymbol{\ell} \cdot d\vec{x}}_{\text{(See Problem 2.7)}} = (\mathbf{D} + \mathbf{W}) \cdot d\vec{x} = \underbrace{\mathbf{W} \cdot d\vec{x} = \vec{w} \wedge d\vec{x}}_{\substack{\text{Antisymmetric} \\ \text{tensor property}}} \tag{2.89}$$

To prove that $\mathbf{D} = \mathbf{0}$ characterized by rigid body motion our starting point is the definition of rigid body motion in which the distance between particles does not change, hence the magnitude of $d\vec{x}$ does not change over time. Taking the material time derivative of $\|d\vec{x}\|^2$ we obtain:

$$\begin{aligned}
\frac{D}{Dt}\|d\vec{x}\|^2 &= \frac{D}{Dt}(d\vec{x} \cdot d\vec{x}) = \frac{D}{Dt}(d\vec{x}) \cdot d\vec{x} + d\vec{x} \cdot \frac{D}{Dt}(d\vec{x}) \\
&= 2d\vec{x} \cdot \frac{D}{Dt}(d\vec{x}) \quad \text{(see Problem 2.7)} \\
&= 2d\vec{x} \cdot \boldsymbol{\ell} \cdot d\vec{x} = 2d\vec{x} \cdot (\mathbf{D} + \mathbf{W}) \cdot d\vec{x} \\
&= 2d\vec{x} \cdot \mathbf{D} \cdot d\vec{x} + 2d\vec{x} \cdot \mathbf{W} \cdot d\vec{x} = 2d\vec{x} \cdot \mathbf{D} \cdot d\vec{x} + 2\mathbf{W} : (d\vec{x} \otimes d\vec{x}) \\
&= 2d\vec{x} \cdot \mathbf{D} \cdot d\vec{x}
\end{aligned} \tag{2.90}$$

where we have used the property $\mathbf{A}^{skew} : \mathbf{B}^{sym} = 0 \Rightarrow \mathbf{W} : (d\vec{x} \otimes d\vec{x}) = 0$. So, according to (2.90), the magnitude of $d\vec{x}$ does not change over time if $\mathbf{D} = \mathbf{0}$.

If the spin tensor is a zero tensor, $\mathbf{W} = \mathbf{0}$, the velocity field is said to be irrotational, thus $\vec{\nabla}_{\vec{x}} \wedge \vec{v} = \vec{\mathbf{0}}$. In **Problem 2.3** the following relationship was validated $\nabla_{\vec{x}}\vec{v} \cdot \vec{v} = \nabla_{\vec{x}}\left(\frac{v^2}{2}\right) + \frac{1}{2}\left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v}$ which can contrast with:

$$\begin{aligned}
\nabla_{\vec{x}}\vec{v} \cdot \vec{v} &= (\mathbf{D} + \mathbf{W}) \cdot \vec{v} = \mathbf{D} \cdot \vec{v} + \mathbf{W} \cdot \vec{v} = \mathbf{D} \cdot \vec{v} + \mathbf{W} \cdot \vec{v} + (\mathbf{W} \cdot \vec{v} - \mathbf{W} \cdot \vec{v}) \\
&= [\mathbf{D} - \mathbf{W}] \cdot \vec{v} + 2\mathbf{W} \cdot \vec{v} = \frac{1}{2}\left[\left(\boldsymbol{\ell} + \boldsymbol{\ell}^T\right) - \left(\boldsymbol{\ell} - \boldsymbol{\ell}^T\right)\right] \cdot \vec{v} + 2\mathbf{W} \cdot \vec{v} \\
&= \frac{1}{2}\left[2\boldsymbol{\ell}^T\right] \cdot \vec{v} + 2\mathbf{W} \cdot \vec{v} = \frac{1}{2}\left[2(\nabla_{\vec{x}}\vec{v})^T\right] \cdot \vec{v} + \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v}
\end{aligned} \tag{2.91}$$

The term $2(\nabla_{\vec{x}}\vec{v})^T \cdot \vec{v}$ can be written in indicial notation as $2v_{j,i}v_j$, which is equivalent to $(\|\vec{v}\|^2)_{,i} = (v^2)_{,i} = (\vec{v} \cdot \vec{v})_{,i} = (v_j v_j)_{,i} = v_{j,i}v_j + v_j v_{j,i} = 2v_j v_{j,i}$, Thus:

$$\nabla_{\vec{x}}\vec{v} \cdot \vec{v} = \nabla_{\vec{x}}\left(\frac{v^2}{2}\right) + \left(\vec{\nabla}_{\vec{x}} \wedge \vec{v}\right) \wedge \vec{v} \tag{2.92}$$

2.5.5.3 The Material Time Derivative of $\boldsymbol{F}^{-1}$

The material time derivative of the spatial deformation gradient ($\boldsymbol{F}^{-1}$) is obtained from the material time derivative of $\boldsymbol{F}^{-1} \cdot \boldsymbol{F} = \mathbf{1}$, *i.e.*:

$$
\begin{aligned}
\frac{D}{Dt}(\boldsymbol{F} \cdot \boldsymbol{F}^{-1}) &= \frac{D}{Dt}\mathbf{1} \\
\frac{D\boldsymbol{F}}{Dt} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F} \cdot \frac{D\boldsymbol{F}^{-1}}{Dt} &= \mathbf{0} \\
\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F} \cdot \dot{\boldsymbol{F}}^{-1} &= \mathbf{0}
\end{aligned}
\tag{2.93}
$$

Therefore:

$$
\boldsymbol{F} \cdot \dot{\boldsymbol{F}}^{-1} = -\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \quad \Rightarrow \quad \underbrace{\boldsymbol{F}^{-1} \cdot \boldsymbol{F}}_{\mathbf{1}} \cdot \dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1} \cdot \underbrace{\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}}_{\boldsymbol{\ell}}
\tag{2.94}
$$

which leads to:

Tensorial notation	Indicial notation	
$\boxed{\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell}}$	$\dot{F}^{-1}_{ij} = -F^{-1}_{ik} \ell_{kj}$	(2.95)

NOTE: In this publication we adopt the following notation to represent the material time derivative of the inverse of a tensor: $\dfrac{D\boldsymbol{F}^{-1}}{Dt} \equiv \overset{\bullet}{\left(\boldsymbol{F}^{-1}\right)} \equiv \dot{\boldsymbol{F}}^{-1}$. ■

2.5.5.4 The Material Time Derivative of the Jacobian Determinant

The material time derivative of the Jacobian determinant can be evaluated by the definition of the second-order tensor determinant:

$$
\begin{aligned}
& J = |\boldsymbol{F}| = \left|\frac{\partial x_i}{\partial X_j}\right| = \epsilon_{PQR} \frac{\partial x_1}{\partial X_P} \frac{\partial x_2}{\partial X_Q} \frac{\partial x_3}{\partial X_R} = \epsilon_{PQR} x_{1,P} x_{2,Q} x_{3,R} \\
& \Rightarrow \frac{D(J)}{Dt} \equiv \dot{J} = \epsilon_{PQR} \left(\dot{x}_{1,P} x_{2,Q} x_{3,R} + x_{1,P} \dot{x}_{2,Q} x_{3,R} + x_{1,P} x_{2,Q} \dot{x}_{3,R}\right)
\end{aligned}
\tag{2.96}
$$

According to equations (2.79) and (2.80) the following relationships are valid:

$$
\begin{aligned}
& \dot{x}_{1,P} \equiv \frac{\partial \dot{x}_1(\vec{\boldsymbol{X}},t)}{\partial X_p} = \frac{\partial v_1(\vec{\boldsymbol{X}},t)}{\partial X_p} = \frac{\partial v_1(\vec{\boldsymbol{x}},t)}{\partial x_s} \frac{\partial x_s(\vec{\boldsymbol{X}},t)}{\partial X_P} = v_{1,s} x_{s,P} \\
& \dot{x}_{2,Q} = \frac{\partial v_2}{\partial x_s} \frac{\partial x_s}{\partial X_Q} = v_{2,s} x_{s,Q} \quad ; \quad \dot{x}_{3,R} = \frac{\partial v_3}{\partial x_s} \frac{\partial x_s}{\partial X_R} = v_{3,s} x_{s,R}
\end{aligned}
\tag{2.97}
$$

By substituting $\dot{x}_{1,P}$, $\dot{x}_{2,Q}$, $\dot{x}_{3,R}$, given by (2.97), into equation (2.96), we obtain:

$$
\begin{aligned}
\dot{J} &= \epsilon_{PQR} \left(\dot{x}_{1,P} x_{2,Q} x_{3,R} + x_{1,P} \dot{x}_{2,Q} x_{3,R} + x_{1,P} x_{2,Q} \dot{x}_{3,R}\right) \\
&= \epsilon_{PQR} v_{1,s} x_{s,P} x_{2,Q} x_{3,R} + \epsilon_{PQR} x_{1,P} v_{2,s} x_{s,Q} x_{3,R} + \epsilon_{PQR} x_{1,P} x_{2,Q} v_{3,s} x_{s,R}
\end{aligned}
\tag{2.98}
$$

The first term on the right hand side of the equation in (2.98) can be expressed as:

$$
\begin{aligned}
\epsilon_{PQR} v_{1,s} x_{s,P} x_{2,Q} x_{3,R} &= v_{1,1} \underbrace{\epsilon_{PQR} x_{1,P} x_{2,Q} x_{3,R}}_{=J} + v_{1,2} \underbrace{\epsilon_{PQR} x_{2,P} x_{2,Q} x_{3,R}}_{=0} + v_{1,3} \underbrace{\epsilon_{PQR} x_{3,P} x_{2,Q} x_{3,R}}_{=0} \\
&= v_{1,1} J
\end{aligned}
\tag{2.99}
$$

in which the following was validated: $\epsilon_{PQR} x_{2,P} x_{2,Q} x_{3,R} = \epsilon_{PQR} x_{3,P} x_{2,Q} x_{3,R} = 0$, since these relationships represent a matrix determinant that has two equal rows (linearly dependent). Similarly, we obtain: $\epsilon_{PQR} x_{1,P} v_{2,s} x_{s,Q} x_{3,R} = v_{2,2} J$ and $\epsilon_{PQR} x_{1,P} x_{2,Q} v_{3,s} x_{s,R} = v_{3,3} J$ after which the equation in (2.98) can be rewritten as:

$$\dot{J} = v_{1,1} J + v_{2,2} J + v_{3,3} J = v_{k,k} J \tag{2.100}$$

which is the same as:

$$\boxed{\begin{aligned} \frac{D}{Dt}|\boldsymbol{F}| \equiv \dot{J} &= |\boldsymbol{F}| \nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} \\ &= J \nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} \\ &= J \,\mathrm{Tr}(\nabla_{\vec{x}} \vec{\boldsymbol{v}}) \\ &= J \,\mathrm{Tr}(\boldsymbol{\ell}) \\ &= J \,\mathrm{Tr}(\mathbf{D}) \end{aligned}} \quad \textit{Material time derivative of the Jacobian determinant} \tag{2.101}$$

where we have used the equation in which the trace of an antisymmetric tensor is zero, $\mathrm{Tr}(\boldsymbol{\ell}) = \mathrm{Tr}(\mathbf{D} + \mathbf{W}) = \mathrm{Tr}(\mathbf{D}) + \mathrm{Tr}(\mathbf{W}) = \mathrm{Tr}(\mathbf{D})$.

The material time derivative of the Jacobian determinant could also have been obtained as, (see Chapter 1):

$$\frac{D III_{\boldsymbol{F}}}{Dt} = \frac{D III_{\boldsymbol{F}}}{D\boldsymbol{F}} : \frac{D\boldsymbol{F}}{Dt} = (III_{\boldsymbol{F}} \boldsymbol{F}^{-T}) : (\boldsymbol{\ell} \cdot \boldsymbol{F}) = III_{\boldsymbol{F}} F_{ji}^{-1} \ell_{ik} F_{kj} = III_{\boldsymbol{F}} \ell_{ik} \delta_{ik} = III_{\boldsymbol{F}} \ell_{kk} \tag{2.102}$$

Problem 2.8: Starting from the definition $\frac{D}{Dt}[\det(\boldsymbol{F})] = \frac{DF_{ij}}{Dt} \mathrm{cof}(F_{ij})$, show that the equation in (2.101), $\dot{J} = J \nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}$, is valid.

Solution: Considering that $F_{ij} = \frac{\partial x_i}{\partial X_j}$, the material time derivative of $|\boldsymbol{F}| \equiv \det(\boldsymbol{F})$ is given by:

$$\frac{D}{Dt}[\det(\boldsymbol{F})] = \frac{D}{Dt}\left(\frac{\partial x_i(\boldsymbol{X},t)}{\partial X_j}\right) \mathrm{cof}(F_{ij}) = \frac{D}{\partial X_j}\left(\frac{\partial x_i(\boldsymbol{X},t)}{Dt}\right) \mathrm{cof}(F_{ij}) = \frac{D}{\partial X_j}(v_i) \mathrm{cof}(F_{ij})$$

and considering that $v_i(\vec{\boldsymbol{x}}(\boldsymbol{X},t),t)$, we can state that:

$$\frac{D}{Dt}[\det(\boldsymbol{F})] = \frac{\partial v_i}{\partial x_k} \frac{\partial x_k}{\partial X_j} \mathrm{cof}(F_{ij})$$

By referring to the definition of the cofactor: $[\mathrm{cof}(F_{ij})]^T = (F_{ij})^{-1} \det(F_{ij})$, we can also state the following is valid:

$$\begin{aligned} \frac{D}{Dt}[\det(\boldsymbol{F})] &= \frac{\partial v_i}{\partial x_k} \frac{\partial x_k}{\partial X_j} (F_{ij})^{-T} \det(F_{ij}) = \frac{\partial v_i}{\partial x_k} F_{kj} (F_{ji})^{-1} \det(F_{ij}) = \frac{\partial v_i}{\partial x_k} \delta_{ki} \det(F_{ij}) \\ &= \frac{\partial v_i}{\partial x_i} \det(F_{ij}) = J v_{i,i} \end{aligned}$$

2.6 Finite Strain Tensors

Before outlining the different ways we can define the strain tensors, it must be stressed that displacement is a measurable quantity, whereas strain is based on concepts that have been introduced for convenience. The strain definition used in this section is the dimensionless quantity $\frac{(ds)^2-(dS)^2}{(dS)^2}$ (material description) or $\frac{(ds)^2-(dS)^2}{(ds)^2}$ (spatial configuration).

Let us consider, once again, two particles P and Q, connected by the vector $d\vec{\boldsymbol{X}}$ in the reference configuration. After motion, the particles occupying the points P and Q are moved to the points P' and Q', respectively, and the new vector joining these material points is defined by $d\vec{\boldsymbol{x}}$, (see Figure 2.16). The magnitudes of these vectors squared are:

$$\| d\vec{\boldsymbol{X}} \|^2 = (dS)^2 = d\vec{\boldsymbol{X}} \cdot d\vec{\boldsymbol{X}} \tag{2.103}$$

and

$$\| d\vec{\boldsymbol{x}} \|^2 = (ds)^2 = d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}} \tag{2.104}$$

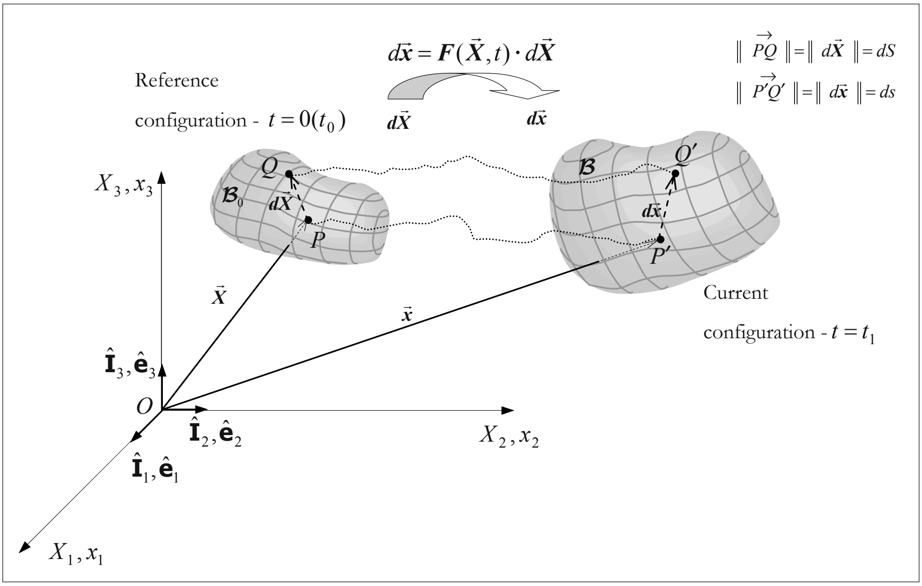


Figure 2.16: Deformation of the continuum.

2.6.1 The Material Finite Strain Tensor

The relationship $(ds)^2 - (dS)^2$ can be expressed in the material description as:

$$
\begin{aligned}
(ds)^2 - (dS)^2 &= d\vec{x}\cdot d\vec{x} - d\vec{X}\cdot d\vec{X} \\
&= \boldsymbol{F}\cdot d\boldsymbol{X}\cdot\boldsymbol{F}\cdot d\vec{X} - d\vec{X}\cdot d\vec{X} \\
&= d\vec{X}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot d\vec{X} - d\vec{X}\cdot d\vec{X} \\
&= d\vec{X}\cdot\left(\boldsymbol{F}^T\cdot\boldsymbol{F} - \mathbf{1}\right)\cdot d\vec{X} \\
&= d\vec{X}\cdot\left(\boldsymbol{C} - \mathbf{1}\right)\cdot d\vec{X} \\
&= d\vec{X}\cdot\left(2\boldsymbol{E}\right)\cdot d\vec{X}
\end{aligned}
\qquad
\begin{aligned}
(ds)^2 - (dS)^2 &= dx_i dx_i - dX_k dX_k \\
&= F_{ik} dX_k F_{ij} dX_j - \delta_{kj} dX_k dX_j \\
&= (F_{ik}F_{ij} - \delta_{kj}) dX_k dX_j \\
&= (C_{kj} - \delta_{kj}) dX_k dX_j \\
&= (2E_{kj}) dX_k dX_j
\end{aligned}
\tag{2.105}
$$

where $\boldsymbol{C}$ is known as the *right Cauchy-Green deformation tensor*, also known as the *Green deformation tensor*, and is defined as:

$$\boxed{\boldsymbol{C}(\vec{X},t) = \boldsymbol{F}^T\cdot\boldsymbol{F}} \quad \textit{The right Cauchy-Green deformation tensor} \tag{2.106}$$

$\boldsymbol{C}$ is a symmetric tensor, *i.e.* $\boldsymbol{C}^T = \left(\boldsymbol{F}^T\cdot\boldsymbol{F}\right)^T = \boldsymbol{F}^T\cdot\boldsymbol{F} = \boldsymbol{C}$, and is also a positive definite tensor, since $|\boldsymbol{F}| \neq 0$ (see **Problem 1.25** in Chapter 1). The inverse of $\boldsymbol{C}$, which is also in the reference configuration, is given by:

$$\boldsymbol{C}^{-1}(\vec{X},t) = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T} \tag{2.107}$$

We can now introduce the *left Cauchy-Green deformation tensor* $(\boldsymbol{b})$, also known as the *Finger deformation tensor*, which we can find in the spatial configuration, and is defined as:

$$\boxed{\boldsymbol{b}(\vec{x},t) = \boldsymbol{F}\cdot\boldsymbol{F}^T} \quad \textit{The left Cauchy-Green deformation tensor} \tag{2.108}$$

$\boldsymbol{b}$ is a positive definite symmetric tensor.

NOTE: The word *"right"* is always associated with *material configuration*, meanwhile *"left"* is related to *spatial configuration*. ■

The inverse of $\boldsymbol{b}$, which is also in the current configuration, is given by:

$$\boldsymbol{b}^{-1}(\vec{x},t) = \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1} \tag{2.109}$$

We can also define the *Piola deformation tensor* ($\boldsymbol{B}$) as:

$$\boldsymbol{B}(\vec{X},t) = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T} = \boldsymbol{C}^{-1} \xrightarrow{\text{the inverse}} \boldsymbol{B}^{-1} = \boldsymbol{F}^T\cdot\boldsymbol{F} = \boldsymbol{C} \tag{2.110}$$

In the subsection *Polar Decomposition* more details will be provided about the configurations in which these tensors appear.

We can now present some relationships and properties of $\boldsymbol{C}$ and $\boldsymbol{b}$:

- The tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ are related by:

$$
\begin{aligned}
&\boldsymbol{C} = \boldsymbol{F}^{-1}\cdot\boldsymbol{b}\cdot\boldsymbol{F} \quad ; \quad \boldsymbol{C}^{-1} = \boldsymbol{F}^{-1}\cdot\boldsymbol{b}^{-1}\cdot\boldsymbol{F} \\
&\boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{C}\cdot\boldsymbol{F}^{-1} \quad ; \quad \boldsymbol{b}^{-1} = \boldsymbol{F}\cdot\boldsymbol{C}^{-1}\cdot\boldsymbol{F}^{-1} \\
&\boldsymbol{F}\cdot\boldsymbol{C}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T = \boldsymbol{b}\cdot\boldsymbol{b} = \boldsymbol{b}^2
\end{aligned}
\tag{2.111}
$$

- The determinant of $\boldsymbol{b}$ is:

$$\det(\boldsymbol{b}) = \det(\boldsymbol{F}\cdot\boldsymbol{F}^T) = \det(\boldsymbol{F})\det(\boldsymbol{F}^T) = \det(\boldsymbol{F})\det(\boldsymbol{F}) = J^2 = \det(\boldsymbol{C}) \tag{2.112}$$

Then the Jacobian determinant can also be expressed as $J = \sqrt{\det(\boldsymbol{C})} = \sqrt{III_{\boldsymbol{C}}}$.

- The invariants of $\boldsymbol{C}$ and $\boldsymbol{b}$:

$$
\begin{aligned}
&I_{\boldsymbol{b}} = I_{\boldsymbol{C}} = \mathrm{Tr}(\boldsymbol{C}) = C_{ii} \\
&I\!I_{\boldsymbol{b}} = I\!I_{\boldsymbol{C}} = \frac{1}{2}\left[I_{\boldsymbol{C}}^2 - \mathrm{Tr}(\boldsymbol{C}^2)\right] \\
&I\!I\!I_{\boldsymbol{b}} = I\!I\!I_{\boldsymbol{C}} = \det(\boldsymbol{C}) = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}C_{ip}C_{jq}C_{kr} = J^2 = \frac{1}{3}\left[\mathrm{Tr}\boldsymbol{C}^3 - \frac{3}{2}\mathrm{Tr}\boldsymbol{C}^3\mathrm{Tr}\boldsymbol{C} + \frac{1}{2}(\mathrm{Tr}\boldsymbol{C})^3\right]
\end{aligned}
\tag{2.113}
$$

The relation $I_{\boldsymbol{b}} = I_{\boldsymbol{C}}$ is proven by applying the trace property, (see Chapter 1), $\mathrm{Tr}(\boldsymbol{C}) = \mathrm{Tr}(\boldsymbol{F}^T \cdot \boldsymbol{F}) = \mathrm{Tr}(\boldsymbol{F} \cdot \boldsymbol{F}^T) = \mathrm{Tr}(\boldsymbol{b})$. Furthermore, the relationship $\mathrm{Tr}(\boldsymbol{C}^n) = \mathrm{Tr}(\boldsymbol{b}^n)$ is also valid.

$I\!I\!I_{\boldsymbol{b}} = I\!I\!I_{\boldsymbol{C}}$ can be proved by using the determinant property, *i.e.*:

$$I\!I\!I_{\boldsymbol{C}} = \det(\boldsymbol{C}) = \det(\boldsymbol{F}^T \cdot \boldsymbol{F}) = \det(\boldsymbol{F} \cdot \boldsymbol{F}^T) = \det(\boldsymbol{F}^T)\det(\boldsymbol{F}) = [\det(\boldsymbol{F})]^2 = I\!I\!I_{\boldsymbol{b}}$$

- The tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ are positive definite symmetric tensors which was proven in **Problem 1.25** in Chapter 1.

Returning to equation (2.105), we now introduce the *Green-Lagrange strain tensor* denoted by $\boldsymbol{E}$, also called the *Lagrangian finite strain tensor* or the *Green-St_Venant strain tensor*, and defined as:

	Tensorial notation	Indicial notation	
The Green-Lagrange strain tensor	$\boxed{\begin{aligned}\boldsymbol{E}(\vec{\boldsymbol{X}},t) &= \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}) \\ &= \frac{1}{2}(\boldsymbol{C} - \mathbf{1})\end{aligned}}$	$\begin{aligned}E_{ij} &= \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) \\ &= \frac{1}{2}(C_{ij} - \delta_{ij})\end{aligned}$	(2.114)

The Green-Lagrange strain tensor ($\boldsymbol{E}$) is a symmetric tensor, *i.e.:*

$$\boldsymbol{E}^T = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1})^T = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}) = \boldsymbol{E} \tag{2.115}$$

The Green-Lagrange strain tensor ($\boldsymbol{E}$) can also be expressed in function of the *material displacement gradient tensor*, $\boldsymbol{J} \equiv \nabla_{\vec{X}}\vec{\mathbf{u}}(\vec{X},t)$. To do this, we start from the equation in (2.73), *i.e.*:

$$\boldsymbol{F} = \boldsymbol{J} + \mathbf{1} \qquad\Big|\qquad F_{ij} = J_{ij} + \delta_{ij} = \frac{\partial \mathrm{u}_i}{\partial X_j} + \delta_{ij} = \mathrm{u}_{i,J} + \delta_{ij} \tag{2.116}$$

Afterwards, the right Cauchy-Green deformation tensor ($\boldsymbol{C}$) can be expressed in terms of $\boldsymbol{J}$ as:

$$
\begin{aligned}
\boldsymbol{C} &= \boldsymbol{F}^T \cdot \boldsymbol{F} \\
&= (\boldsymbol{J} + \mathbf{1})^T \cdot (\boldsymbol{J} + \mathbf{1}) \\
&= (\boldsymbol{J}^T + \mathbf{1}) \cdot (\boldsymbol{J} + \mathbf{1}) \\
&= \boldsymbol{J}^T \cdot \boldsymbol{J} + \boldsymbol{J}^T + \boldsymbol{J} + \mathbf{1}
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
C_{ij} &= F_{ki}F_{kj} \\
&= (\mathrm{u}_{k,I} + \delta_{ki})(\mathrm{u}_{k,J} + \delta_{kj}) \\
&= \mathrm{u}_{k,I}\mathrm{u}_{k,J} + \mathrm{u}_{k,I}\delta_{kj} + \delta_{ki}\mathrm{u}_{k,J} + \delta_{ki}\delta_{kj} \\
&= \mathrm{u}_{k,I}\mathrm{u}_{k,J} + \mathrm{u}_{j,I} + \mathrm{u}_{i,J} + \delta_{ij}
\end{aligned}
\tag{2.117}
$$

or

$$\underbrace{\boldsymbol{C} - \mathbf{1}}_{2\boldsymbol{E}} = \boldsymbol{J}^T \cdot \boldsymbol{J} + \boldsymbol{J}^T + \boldsymbol{J} \qquad\Big|\qquad \underbrace{C_{ij} - \delta_{ij}}_{2E_{ij}} = \mathrm{u}_{k,I}\mathrm{u}_{k,J} + \mathrm{u}_{j,I} + \mathrm{u}_{i,J} \tag{2.118}$$

then:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{J} + \boldsymbol{J}^T + \boldsymbol{J}^T \cdot \boldsymbol{J}\right) \qquad \Bigg| \qquad E_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i} + \frac{\partial \mathrm{u}_k}{\partial X_i}\frac{\partial \mathrm{u}_k}{\partial X_j}\right) \tag{2.119}$$

Explicitly, the components of $\boldsymbol{E}$ are given by:

$$E_{11} = \frac{\partial \mathrm{u}_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial \mathrm{u}_1}{\partial X_1}\right)^2 + \left(\frac{\partial \mathrm{u}_2}{\partial X_1}\right)^2 + \left(\frac{\partial \mathrm{u}_3}{\partial X_1}\right)^2\right],$$

$$E_{22} = \frac{\partial \mathrm{u}_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial \mathrm{u}_1}{\partial X_2}\right)^2 + \left(\frac{\partial \mathrm{u}_2}{\partial X_2}\right)^2 + \left(\frac{\partial \mathrm{u}_3}{\partial X_2}\right)^2\right]$$

$$E_{33} = \frac{\partial \mathrm{u}_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial \mathrm{u}_1}{\partial X_3}\right)^2 + \left(\frac{\partial \mathrm{u}_2}{\partial X_3}\right)^2 + \left(\frac{\partial \mathrm{u}_3}{\partial X_3}\right)^2\right]$$

$$E_{12} = \frac{1}{2}\left[\frac{\partial \mathrm{u}_1}{\partial X_2} + \frac{\partial \mathrm{u}_2}{\partial X_1} + \frac{\partial \mathrm{u}_1}{\partial X_1}\frac{\partial \mathrm{u}_1}{\partial X_2} + \frac{\partial \mathrm{u}_2}{\partial X_1}\frac{\partial \mathrm{u}_2}{\partial X_2} + \frac{\partial \mathrm{u}_3}{\partial X_1}\frac{\partial \mathrm{u}_3}{\partial X_2}\right] = E_{21}$$

$$E_{13} = \frac{1}{2}\left[\frac{\partial \mathrm{u}_1}{\partial X_3} + \frac{\partial \mathrm{u}_3}{\partial X_1} + \frac{\partial \mathrm{u}_1}{\partial X_1}\frac{\partial \mathrm{u}_1}{\partial X_3} + \frac{\partial \mathrm{u}_2}{\partial X_1}\frac{\partial \mathrm{u}_2}{\partial X_3} + \frac{\partial \mathrm{u}_3}{\partial X_1}\frac{\partial \mathrm{u}_3}{\partial X_3}\right] = E_{31}$$

$$E_{23} = \frac{1}{2}\left[\frac{\partial \mathrm{u}_2}{\partial X_3} + \frac{\partial \mathrm{u}_3}{\partial X_2} + \frac{\partial \mathrm{u}_1}{\partial X_2}\frac{\partial \mathrm{u}_1}{\partial X_3} + \frac{\partial \mathrm{u}_2}{\partial X_2}\frac{\partial \mathrm{u}_2}{\partial X_3} + \frac{\partial \mathrm{u}_3}{\partial X_2}\frac{\partial \mathrm{u}_3}{\partial X_3}\right] = E_{32}$$

Problem 2.9: Let us consider the equations of motion:

$$\begin{cases} x_1 = X_1 + 4X_1X_2 \\ x_2 = X_2 + X_2^2 \\ x_3 = X_3 + X_3^2 \end{cases}$$

Find the Green-Lagrange strain tensor ($\boldsymbol{E}$).

Solution:

Referring to the $\boldsymbol{E}$ equation given in (2.114):

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}) \quad ; \quad E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) \tag{2.120}$$

where the components of $\boldsymbol{F}$ are derived as:

$$F_{kj} = \frac{\partial x_k}{\partial X_j} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} (1+4X_2) & 4X_1 & 0 \\ 0 & 1+2X_2 & 0 \\ 0 & 0 & 1+2X_3 \end{bmatrix}$$

And,

$$F_{ki}F_{kj} = \begin{bmatrix} (1+4X_2) & 0 & 0 \\ 4X_1 & 1+2X_2 & 0 \\ 0 & 0 & 1+2X_3 \end{bmatrix}\begin{bmatrix} (1+4X_2) & 4X_1 & 0 \\ 0 & 1+2X_2 & 0 \\ 0 & 0 & 1+2X_3 \end{bmatrix}$$
$$= \begin{bmatrix} (1+4X_2)^2 & (1+4X_2)4X_1 & 0 \\ (1+4X_2)4X_1 & (4X_1)^2+(1+2X_2)^2 & 0 \\ 0 & 0 & (1+2X_3)^2 \end{bmatrix}$$

Then substituting the above into the equation in (2.120) we obtain:

$$E_{ij} = \frac{1}{2}\begin{bmatrix} (1+4X_2)^2-1 & (1+4X_2)4X_1 & 0 \\ (1+4X_2)4X_1 & (4X_1)^2+(1+2X_2)^2-1 & 0 \\ 0 & 0 & (1+2X_3)^2-1 \end{bmatrix}$$

Problem 2.10: Obtain the principal invariants of $\boldsymbol{E}$ in terms of the principal invariants of $\boldsymbol{C}$ and $\boldsymbol{b}$.

Solution:

The principal invariants of $\boldsymbol{E}$ are given by:

$$I_{\boldsymbol{E}} = \mathrm{Tr}(\boldsymbol{E}) \quad ; \quad II_{\boldsymbol{E}} = \frac{1}{2}\left[I_{\boldsymbol{E}}^2 - \mathrm{Tr}(\boldsymbol{E}^2)\right] \quad ; \quad III_{\boldsymbol{E}} = \det(\boldsymbol{E})$$

Referring to the fact $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{1})$, the principal invariants can also be expressed as:

The First Invariant:

$$I_{\boldsymbol{E}} = \mathrm{Tr}(\boldsymbol{E}) = \mathrm{Tr}\left[\frac{1}{2}(\boldsymbol{C}-\boldsymbol{1})\right] = \frac{1}{2}\mathrm{Tr}(\boldsymbol{C}-\boldsymbol{1}) = \frac{1}{2}\left[\mathrm{Tr}(\boldsymbol{C}) - \mathrm{Tr}(\boldsymbol{1})\right] = \frac{1}{2}\left(I_{\boldsymbol{C}} - 3\right)$$

The Second Invariant:

$$II_{\boldsymbol{E}} = \frac{1}{2}\left[I_{\boldsymbol{E}}^2 - \mathrm{Tr}(\boldsymbol{E}^2)\right]$$

where

$$I_{\boldsymbol{E}}^2 = \left[\frac{1}{2}\left(I_{\boldsymbol{C}} - 3\right)\right]^2 = \frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 6I_{\boldsymbol{C}} + 9\right)$$

$$\mathrm{Tr}(\boldsymbol{E}^2) = \mathrm{Tr}\left[\frac{1}{2}(\boldsymbol{C}-\boldsymbol{1})\right]^2 = \frac{1}{4}\mathrm{Tr}\left[(\boldsymbol{C}-\boldsymbol{1})^2\right] = \frac{1}{4}\mathrm{Tr}\left(\boldsymbol{C}^2 - 2\boldsymbol{C} + \boldsymbol{1}\right) = \frac{1}{4}\left[\mathrm{Tr}\left(\boldsymbol{C}^2\right) - 2\mathrm{Tr}\left(\boldsymbol{C}\right) + \mathrm{Tr}\left(\boldsymbol{1}\right)\right]$$
$$= \frac{1}{4}\left[\mathrm{Tr}\left(\boldsymbol{C}^2\right) - 2I_{\boldsymbol{C}} + 3\right]$$

The term $\mathrm{Tr}\left(\boldsymbol{C}^2\right)$ can be obtained as follows:

$$\boldsymbol{C}\cdot\boldsymbol{C} = \boldsymbol{C}^2 \quad \Rightarrow \quad C'^2_{ij} = \begin{bmatrix} C_1^2 & 0 & 0 \\ 0 & C_2^2 & 0 \\ 0 & 0 & C_3^2 \end{bmatrix} \Rightarrow \mathrm{Tr}\left(\boldsymbol{C}^2\right) = C_1^2 + C_2^2 + C_3^2$$

It is also true that:

$$I_{\boldsymbol{C}}^2 = \left(C_1 + C_2 + C_3\right)^2 = C_1^2 + C_2^2 + C_3^2 + 2\underbrace{\left(C_1C_2 + C_1C_3 + C_2C_3\right)}_{II_{\boldsymbol{C}}}$$

$$\Rightarrow C_1^2 + C_2^2 + C_3^2 = I_{\boldsymbol{C}}^2 - 2\,II_{\boldsymbol{C}}$$

Therefore we have:

$$\mathrm{Tr}(\boldsymbol{E}^2) = \frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 2\,II_{\boldsymbol{C}} - 2I_{\boldsymbol{C}} + 3\right)$$

Whereupon, the second invariant can also be expressed as:

$$I\!I_{\boldsymbol{E}} = \frac{1}{2}\left[\frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 6I_{\boldsymbol{C}} + 9\right) - \frac{1}{4}\left(I_{\boldsymbol{C}}^2 - 2\,I\!I_{\boldsymbol{C}} - 2I_{\boldsymbol{C}} + 3\right)\right] = \frac{1}{4}\left(-2I_{\boldsymbol{C}} + I\!I_{\boldsymbol{C}} + 3\right)$$

The Third Invariant:

$$I\!I\!I_{\boldsymbol{E}} = \det(\boldsymbol{E}) = \det\left[\frac{1}{2}(\boldsymbol{C} - \mathbf{1})\right] = \left(\frac{1}{2}\right)^3 \det[(\boldsymbol{C} - \mathbf{1})]$$

The term $\det[(\boldsymbol{C} - \mathbf{1})]$ can also be expressed as:

$$\det(\boldsymbol{C} - \mathbf{1}) = \begin{vmatrix} C_1 - 1 & 0 & 0 \\ 0 & C_2 - 1 & 0 \\ 0 & 0 & C_3 - 1 \end{vmatrix} = (C_1 - 1)(C_2 - 1)(C_3 - 1)$$

$$= C_1C_2C_3 - C_1C_2 - C_1C_3 - C_2C_3 + C_1 + C_2 + C_3 - 1 = I\!I\!I_{\boldsymbol{C}} - I\!I_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1$$

Then:

$$I\!I\!I_{\boldsymbol{E}} = \frac{1}{8}\left(I\!I\!I_{\boldsymbol{C}} - I\!I_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1\right)$$

In short we have:

$$\boxed{\begin{aligned} I_{\boldsymbol{E}} &= \frac{1}{2}\left(I_{\boldsymbol{C}} - 3\right) \\ I\!I_{\boldsymbol{E}} &= \frac{1}{4}\left(-2I_{\boldsymbol{C}} + I\!I_{\boldsymbol{C}} + 3\right) \\ I\!I\!I_{\boldsymbol{E}} &= \frac{1}{8}\left(I\!I\!I_{\boldsymbol{C}} - I\!I_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1\right) \end{aligned}} \quad ; \quad \boxed{\begin{aligned} I_{\boldsymbol{C}} &= 2I_{\boldsymbol{E}} + 3 \\ I\!I_{\boldsymbol{C}} &= 4\,I\!I_{\boldsymbol{E}} + 4I_{\boldsymbol{E}} + 3 \\ I\!I\!I_{\boldsymbol{C}} &= 8\,I\!I\!I_{\boldsymbol{E}} + 4\,I\!I_{\boldsymbol{E}} + 2I_{\boldsymbol{E}} + 1 \end{aligned}}$$

2.6.2 The Spatial Finite Strain Tensor (The Almansi Strain Tensor)

In the previous subsection, $(ds)^2 - (dS)^2$ was expressed in the material description. Alternatively, it can be expressed in spatial description, *i.e.*:

$$\begin{aligned} (ds)^2 - (dS)^2 &= d\vec{x}\cdot d\vec{x} - d\vec{X}\cdot d\vec{X} \\ &= d\vec{x}\cdot d\vec{x} - \boldsymbol{F}^{-1}\cdot d\vec{x}\cdot \boldsymbol{F}^{-1}\cdot d\vec{x} \\ &= d\vec{x}\cdot d\vec{x} - d\vec{x}\cdot \boldsymbol{F}^{-T}\cdot \boldsymbol{F}^{-1}\cdot d\vec{x} \\ &= d\vec{x}\cdot\left(\mathbf{1} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right)\cdot d\vec{x} \\ &= d\vec{x}\cdot(\mathbf{1} - \boldsymbol{c})\cdot d\vec{x} \\ &= d\vec{x}\cdot(2\boldsymbol{e})\cdot d\vec{x} \end{aligned} \qquad \begin{aligned} (ds)^2 - (dS)^2 &= dx_i dx_i - dX_k dX_k \\ &= \delta_{ij} dx_i dx_j - X_{k,i} dx_i X_{k,j} dx_j \\ &= (\delta_{ij} - X_{k,i}X_{k,j}) dx_i dx_j \\ &= (\delta_{ij} - c_{ij}) dx_i dx_j \\ &= 2e_{ij} dx_i dx_j \end{aligned} \tag{2.121}$$

where we have introduced $\boldsymbol{c}$ known as the *Cauchy deformation tensor*, and defined as:

$$\boxed{\boldsymbol{c}(\vec{\boldsymbol{x}}, t) = \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}} \quad \textit{The Cauchy deformation tensor} \tag{2.122}$$

Additionally, it holds that $\boldsymbol{c}^{-1} = \boldsymbol{b}$, where $\boldsymbol{b}(\vec{\boldsymbol{x}}, t)$ is the *left Cauchy-Green deformation tensor*, defined in (2.108).

We can also define the *Almansi strain tensor* or *Eulerian finite strain tensor*, $\boldsymbol{e}$, as:

$$\boxed{\boldsymbol{e}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}(\mathbf{1} - \boldsymbol{c}) = \frac{1}{2}(\mathbf{1} - \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}) = \frac{1}{2}(\mathbf{1} - \boldsymbol{b}^{-1})} \quad \textit{The Almansi strain tensor} \tag{2.123}$$

The components of $\boldsymbol{e}$ are given by:

$$e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij}) = \frac{1}{2}(\delta_{ij} - F^{-1}_{ki} F^{-1}_{kj}) \tag{2.124}$$

It is also true that:

$$\begin{aligned} \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{b} = \mathbf{1} \quad &\Rightarrow \quad \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1} = \mathbf{1} \\ \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{C} \cdot \boldsymbol{B} = \mathbf{1} \quad &\Rightarrow \quad \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T} = \mathbf{1} \end{aligned} \tag{2.125}$$

The Almansi strain tensor ($\boldsymbol{e}$) can also be expressed in terms of the spatial displacement gradient tensor, $\nabla_{\vec{x}} \vec{\mathbf{u}}(\vec{x},t) \equiv \boldsymbol{j} = \mathbf{1} - \boldsymbol{F}^{-1}$:

$$\begin{aligned} \boldsymbol{e} &= \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}\right) = \frac{1}{2}\left[\mathbf{1} - (\mathbf{1} - \boldsymbol{j})^T \cdot (\mathbf{1} - \boldsymbol{j})\right] = \frac{1}{2}\left[\mathbf{1} - (\mathbf{1} - \boldsymbol{j}^T) \cdot (\mathbf{1} - \boldsymbol{j})\right] \\ &= \frac{1}{2}\left[\mathbf{1} - \left(\mathbf{1} - \boldsymbol{j} - \boldsymbol{j}^T + \boldsymbol{j}^T \cdot \boldsymbol{j}\right)\right] = \frac{1}{2}\left(\boldsymbol{j} + \boldsymbol{j}^T - \boldsymbol{j}^T \cdot \boldsymbol{j}\right) \end{aligned} \tag{2.126}$$

After which we have:

Tensorial notation	Indicial notation
$\boxed{\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{j} + \boldsymbol{j}^T - \boldsymbol{j}^T \cdot \boldsymbol{j}\right)}$	$e_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i} - \frac{\partial \mathrm{u}_k}{\partial x_i}\frac{\partial \mathrm{u}_k}{\partial x_j}\right)$ (2.127)

Both tensors $\boldsymbol{E}$ and $\boldsymbol{e}$ are symmetric second-order tensors and the relationship between them can be obtained starting from the definition in (2.123), $2\boldsymbol{e} = (\mathbf{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1})$:

$$\begin{aligned} 2\boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F} &= \boldsymbol{F}^T \cdot (\mathbf{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}) \cdot \boldsymbol{F} \\ &= \boldsymbol{F}^T \cdot \mathbf{1} \cdot \boldsymbol{F} - \boldsymbol{F}^T \cdot \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{F} \\ &= \boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1} \\ &= 2\boldsymbol{E} \end{aligned} \tag{2.128}$$

Thus, we can draw the conclusion that:

$$\boxed{\boldsymbol{E} = \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F}} \tag{2.129}$$

and:

$$\boxed{\boldsymbol{e} = \boldsymbol{F}^{-T} \cdot \boldsymbol{E} \cdot \boldsymbol{F}^{-1}} \tag{2.130}$$

OBS.: In rigid body motion the relation $(ds)^2 - (dS)^2$ is zero, so the strain tensors ($\boldsymbol{E}$, $\boldsymbol{e}$) must be zero tensors at any time during motion.

Figure 2.17 summarizes some equations by the use of deformation and strain tensors.

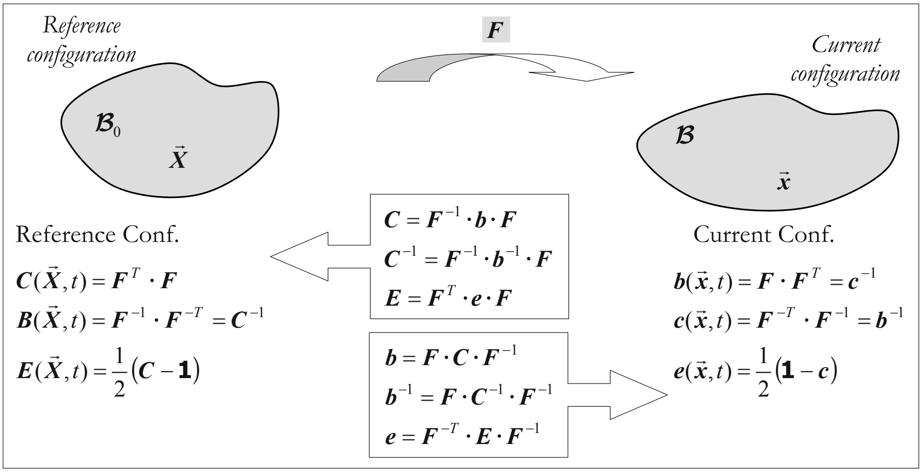


Figure 2.17: Deformation and strain tensors (Kinematic tensors).

Problem 2.11: Show that the Green-Lagrange strain tensor ($\boldsymbol{E}$) and the right Cauchy-Green deformation tensor ($\boldsymbol{C}$) are coaxial tensors.
Solution:
Two tensors are coaxial if they have the same principal directions. Coaxiality can also be demonstrated if the relation $\boldsymbol{C}\cdot\boldsymbol{E}=\boldsymbol{E}\cdot\boldsymbol{C}$ holds.
Starting with the definition $\boldsymbol{C}=\boldsymbol{1}+2\boldsymbol{E}$, we can conclude that:

$$\boldsymbol{C}\cdot\boldsymbol{E}=(\boldsymbol{1}+2\boldsymbol{E})\cdot\boldsymbol{E}=\boldsymbol{1}\cdot\boldsymbol{E}+2\boldsymbol{E}\cdot\boldsymbol{E}=\boldsymbol{E}\cdot(\boldsymbol{1}+2\boldsymbol{E})=\boldsymbol{E}\cdot\boldsymbol{C}$$

Thus, we can prove that $\boldsymbol{E}$ and $\boldsymbol{C}$ are coaxial tensors.

2.6.3 The Material Time Derivative of Strain Tensors

2.6.3.1 The Material Time Derivative of the Right Cauchy-Green Deformation Tensor

The material time derivative of the right Cauchy-Green deformation tensor, $\dot{\boldsymbol{C}}$, is obtained as follows:

$$\begin{aligned}\frac{D}{Dt}(\boldsymbol{C})\equiv\dot{\boldsymbol{C}}&=\frac{D}{Dt}(\boldsymbol{F}^T\cdot\boldsymbol{F})=\dot{\boldsymbol{F}}^T\cdot\boldsymbol{F}+\boldsymbol{F}^T\cdot\dot{\boldsymbol{F}}\\&=\boldsymbol{F}^T\cdot\boldsymbol{\ell}^T\cdot\boldsymbol{F}+\boldsymbol{F}^T\cdot\boldsymbol{\ell}\cdot\boldsymbol{F}=\boldsymbol{F}^T\cdot\underbrace{(\boldsymbol{\ell}^T+\boldsymbol{\ell})}_{=2\boldsymbol{\ell}^{sym}}\cdot\boldsymbol{F}\\&=2\boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F}\end{aligned}\tag{2.131}$$

2.6.3.2 The Material Time Derivative of the Green-Lagrange Strain Tensor

The material time derivative of the Green-Lagrange strain tensor, $\dot{\boldsymbol{E}}$, is obtained by means of the equation in (2.114), the result of which is:

$$\frac{D}{Dt}(\boldsymbol{E})\equiv\dot{\boldsymbol{E}}=\frac{D}{Dt}\left[\frac{1}{2}(\boldsymbol{F}^T\cdot\boldsymbol{F}-\boldsymbol{1})\right]=\frac{D}{Dt}\left[\frac{1}{2}(\boldsymbol{C}-\boldsymbol{1})\right]=\frac{1}{2}(\dot{\boldsymbol{F}}^T\cdot\boldsymbol{F}+\boldsymbol{F}^T\cdot\dot{\boldsymbol{F}})=\frac{1}{2}\dot{\boldsymbol{C}}\tag{2.132}$$

By comparing the equation in (2.132) with (2.131), we can conclude that:

$$\boxed{\dot{\boldsymbol{E}} = \frac{1}{2}\dot{\boldsymbol{C}} = \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F}} \tag{2.133}$$

and after some algebraic work we can obtain the inverse relationship:

$$\boxed{\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1} = \frac{1}{2}\boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{C}} \cdot \boldsymbol{F}^{-1}} \tag{2.134}$$

The equation in (2.133) could have been obtained by means of the equation in (2.90), *i.e.*:

$$\begin{aligned}
\frac{D}{Dt}\left[(ds)^2 - (dS)^2\right] &= \frac{D}{Dt}\left[(ds)^2\right] = \frac{D}{Dt}\left[d\vec{\boldsymbol{X}} \cdot 2\boldsymbol{E} \cdot d\vec{\boldsymbol{X}}\right] \\
&= \frac{D}{Dt}\left[d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}}\right] = 2\underbrace{d\dot{\vec{\boldsymbol{X}}}}_{=\vec{\boldsymbol{0}}} \cdot \boldsymbol{E} \cdot d\vec{\boldsymbol{X}} + 2d\vec{\boldsymbol{X}} \cdot \dot{\boldsymbol{E}} \cdot d\vec{\boldsymbol{X}} + 2d\vec{\boldsymbol{X}} \cdot \boldsymbol{E} \cdot \underbrace{d\dot{\vec{\boldsymbol{X}}}}_{=\vec{\boldsymbol{0}}} \\
&= \frac{D}{Dt}\left[d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}}\right] = 2d\vec{\boldsymbol{X}} \cdot \dot{\boldsymbol{E}} \cdot d\vec{\boldsymbol{X}}
\end{aligned} \tag{2.135}$$

Then we have:

$$2d\vec{\boldsymbol{X}} \cdot \dot{\boldsymbol{E}} \cdot d\vec{\boldsymbol{X}} = \frac{D}{Dt}\left[d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}}\right] = 2d\vec{\boldsymbol{x}} \cdot \mathbf{D} \cdot d\vec{\boldsymbol{x}} = 2d\vec{\boldsymbol{X}} \cdot \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} \tag{2.136}$$

Therefore, we can conclude that $\dot{\boldsymbol{E}} = \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F}$.

2.6.3.3 The Material Time Derivative of $\boldsymbol{C}^{-1}$

The material time derivative of the inverse of the right Cauchy-Green deformation tensor can be obtained by considering that if $\boldsymbol{C}^{-1} \cdot \boldsymbol{C} = \mathbf{1}$, it follows that:

$$\begin{gathered}
\frac{D}{Dt}(\boldsymbol{C}^{-1} \cdot \boldsymbol{C}) = \frac{D}{Dt}(\mathbf{1}) \Rightarrow \dot{\boldsymbol{C}}^{-1} \cdot \boldsymbol{C} + \boldsymbol{C}^{-1} \cdot \dot{\boldsymbol{C}} = \mathbf{0} \Rightarrow \dot{\boldsymbol{C}}^{-1} \cdot \boldsymbol{C} = -\boldsymbol{C}^{-1} \cdot \dot{\boldsymbol{C}} \\
\dot{\boldsymbol{C}}^{-1} = -\boldsymbol{C}^{-1} \cdot \dot{\boldsymbol{C}} \cdot \boldsymbol{C}^{-1}
\end{gathered} \tag{2.137}$$

Also by referring to $\boldsymbol{C}^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T}$ and $\dot{\boldsymbol{C}} = 2\boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F}$, (see Eq. (2.131)), we obtain:

$$\boxed{\dot{\boldsymbol{C}}^{-1} = -2\boldsymbol{F}^{-1} \cdot \mathbf{D} \cdot \boldsymbol{F}^{-T}} \tag{2.138}$$

Note that the tensors $\dot{\boldsymbol{C}}$ and $\dot{\boldsymbol{C}}^{-1}$ are still symmetric tensors.

2.6.3.4 Material Time Derivative of the Left Cauchy-Green Deformation Tensor

The material time derivative of the left Cauchy-Green deformation tensor ($\boldsymbol{b}$), (see equation (2.108)), is given by:

$$\frac{D}{Dt}(\boldsymbol{b}) \equiv \dot{\boldsymbol{b}} = \frac{D}{Dt}(\boldsymbol{F} \cdot \boldsymbol{F}^T) = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \dot{\boldsymbol{F}}^T = \boldsymbol{\ell} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{\ell}^T = \boldsymbol{\ell} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{\ell}^T \tag{2.139}$$

The material time derivative of $(\boldsymbol{b}^{-1})$ is obtained as follows:

$$\frac{D}{Dt}(\boldsymbol{b}^{-1}) = \frac{D}{Dt}(\boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}) \Rightarrow \dot{\boldsymbol{F}}^{-T} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^{-1} \Rightarrow \dot{\boldsymbol{b}}^{-1} = -\boldsymbol{\ell}^T \cdot \boldsymbol{b}^{-1} - \boldsymbol{b}^{-1} \cdot \boldsymbol{\ell} \tag{2.140}$$

So, we can see the tensors $\dot{\boldsymbol{b}}$ and $\dot{\boldsymbol{b}}^{-1}$ are still symmetric tensors.

The material time derivative of the *Piola deformation tensor* $\boldsymbol{B}$ is given by:

$$\frac{D}{Dt}(\boldsymbol{B}) \equiv \dot{\boldsymbol{B}} = \frac{D}{Dt}(\boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T}) = -\boldsymbol{F}^{-1} \cdot 2\mathbf{D} \cdot \boldsymbol{F}^{-T} \tag{2.141}$$

2.6.3.5 The Material Time Derivative of the Almansi Strain Tensor

The material time derivative of the Almansi strain tensor, $\dot{\boldsymbol{e}}$, can be obtained by means of equation (2.123), the outcome of which is:

$$\begin{aligned}\frac{D}{Dt}(\boldsymbol{e}) \equiv \dot{\boldsymbol{e}} &= \frac{D}{Dt}\left[\frac{1}{2}(\mathbf{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1})\right] &&= \frac{D}{Dt}\left[\frac{1}{2}(\mathbf{1} - \boldsymbol{c})\right] \\ &= \frac{1}{2}(-\dot{\boldsymbol{F}}^{-T} \cdot \boldsymbol{F}^{-1} - \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^{-1}) &&= \frac{1}{2}(\dot{\boldsymbol{c}})\end{aligned} \tag{2.142}$$

We can also obtain the relationship between ($\dot{\boldsymbol{e}}$) and ($\mathbf{D}$). In order to do so we consider the material time derivative of the equation in (2.129), $\boldsymbol{E} = \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F}$:

$$\dot{\boldsymbol{E}} = \dot{\boldsymbol{F}}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{e}} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \dot{\boldsymbol{F}} \tag{2.143}$$

Referring to the fact that $\dot{\boldsymbol{E}} = \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F}$ and $\dot{\boldsymbol{F}} = \boldsymbol{\ell} \cdot \boldsymbol{F}$, we obtain:

$$\begin{aligned}&\boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F} = \dot{\boldsymbol{F}}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \dot{\boldsymbol{e}} \cdot \boldsymbol{F} + \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \dot{\boldsymbol{F}} \\ &\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^T \cdot \boldsymbol{e} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^T \cdot \dot{\boldsymbol{e}} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^T \cdot \boldsymbol{e} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \\ &\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{F}}^T \cdot \boldsymbol{e} + \dot{\boldsymbol{e}} + \boldsymbol{e} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \\ &\mathbf{D} = \boldsymbol{F}^{-T} \cdot \left(\boldsymbol{\ell} \cdot \boldsymbol{F}\right)^T \cdot \boldsymbol{e} + \dot{\boldsymbol{e}} + \boldsymbol{e} \cdot \left(\boldsymbol{\ell} \cdot \boldsymbol{F}\right) \cdot \boldsymbol{F}^{-1}\end{aligned} \tag{2.144}$$

Thus,

$$\boxed{\mathbf{D} = \dot{\boldsymbol{e}} + \boldsymbol{e} \cdot \boldsymbol{\ell} + \boldsymbol{\ell}^T \cdot \boldsymbol{e}} \tag{2.145}$$

Problem 2.12: Obtain the material time derivative of the Jacobian determinant ($\dot{J}$) in terms of ($\dot{\boldsymbol{E}}$), ($\dot{\boldsymbol{C}}$), ($\dot{\boldsymbol{F}}$).

Solution:

This was obtained in (2.101) when $\dot{J} = J\,\mathrm{Tr}(\mathbf{D})$, where $\mathbf{D}$ is the rate-of-deformation tensor which is related to $\dot{\boldsymbol{E}}$ by means of the relationship $\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}$, (see equation (2.134)), then:

$$\dot{J} = J\,\mathrm{Tr}(\mathbf{D}) = J\,\mathrm{Tr}\left(\boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}\right) = J\left(\boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}\right) : \mathbf{1}$$

In indicial notation we have:

$$\dot{J} = J\,F_{ki}^{-1} \dot{E}_{kp} F_{pj}^{-1} \delta_{ij} = J\,F_{ki}^{-1} F_{pi}^{-1} \dot{E}_{kp} = J\,(\boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-T}) : \dot{\boldsymbol{E}} = J\,\boldsymbol{C}^{-1} : \dot{\boldsymbol{E}} = \frac{J}{2}\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}}$$

The $\dot{J}$ can still be expressed in terms of $\dot{\boldsymbol{F}}$. To this end let us consider the following equation $\dot{E}_{kp} = \frac{1}{2}\left(\dot{F}_{sk} F_{sp} + F_{sk} \dot{F}_{sp}\right)$, (see Eq. (2.132)). Then, $\dot{J}$ can also be expressed by:

$$\begin{aligned}\dot{J} &= J\,F_{ki}^{-1} F_{pi}^{-1} \dot{E}_{kp} = J\,F_{ki}^{-1} F_{pi}^{-1} \frac{1}{2}\left(\dot{F}_{sk} F_{sp} + F_{sk} \dot{F}_{sp}\right) = \frac{J}{2}\left(F_{ki}^{-1} F_{pi}^{-1} \dot{F}_{sk} F_{sp} + F_{ki}^{-1} F_{pi}^{-1} F_{sk} \dot{F}_{sp}\right) \\ &= \frac{J}{2}\left(\delta_{si} F_{ki}^{-1} \dot{F}_{sk} + \delta_{si} F_{pi}^{-1} \dot{F}_{sp}\right) = \frac{J}{2}\left(F_{ks}^{-1} \dot{F}_{sk} + F_{ps}^{-1} \dot{F}_{sp}\right) = J F_{ts}^{-1} \dot{F}_{st} = J \dot{F}_{st} F_{ts}^{-1} \\ &= J\boldsymbol{F}^{-T} : \dot{\boldsymbol{F}} = J\dot{\boldsymbol{F}} : \boldsymbol{F}^{-T}\end{aligned}$$

In short, there are various different ways to express the material time derivative of the Jacobian determinant:

$$\boxed{\begin{aligned}\dot{J} = J\,\mathrm{Tr}(\mathbf{D}) &= J\,\boldsymbol{C}^{-1}:\dot{\boldsymbol{E}} &&= \frac{J}{2}\boldsymbol{C}^{-1}:\dot{\boldsymbol{C}} &&= J\dot{\boldsymbol{F}}:\boldsymbol{F}^{-T}\\ &= J\,\mathrm{Tr}(\boldsymbol{C}^{-1}\cdot\dot{\boldsymbol{E}}) &&= \frac{J}{2}\mathrm{Tr}(\boldsymbol{C}^{-1}\cdot\dot{\boldsymbol{C}}) &&= J\,\mathrm{Tr}(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1})\end{aligned}}$$

where we have used the trace property: $\mathbf{A}:\mathbf{B} = \mathrm{Tr}(\mathbf{A}\cdot\mathbf{B}^T) = \mathrm{Tr}(\mathbf{A}^T\cdot\mathbf{B})$ in which $\mathbf{A}$ and $\mathbf{B}$ are arbitrary second-order tensors.

2.6.4 Interpreting Deformation/Strain Tensors

Now we can consider two vectors in the reference configuration defined by $d\vec{\boldsymbol{X}}^{(1)} = dS^{(1)}\hat{\boldsymbol{M}}$ and $d\vec{\boldsymbol{X}}^{(2)} = dS^{(2)}\hat{\boldsymbol{N}}$, where Θ is the angle formed between them, (see Figure 2.18). After motion, these vectors are transformed into $d\vec{\boldsymbol{x}}^{(1)} = ds^{(1)}\hat{\boldsymbol{m}}$ and $d\vec{\boldsymbol{x}}^{(2)} = ds^{(2)}\hat{\boldsymbol{n}}$, respectively.

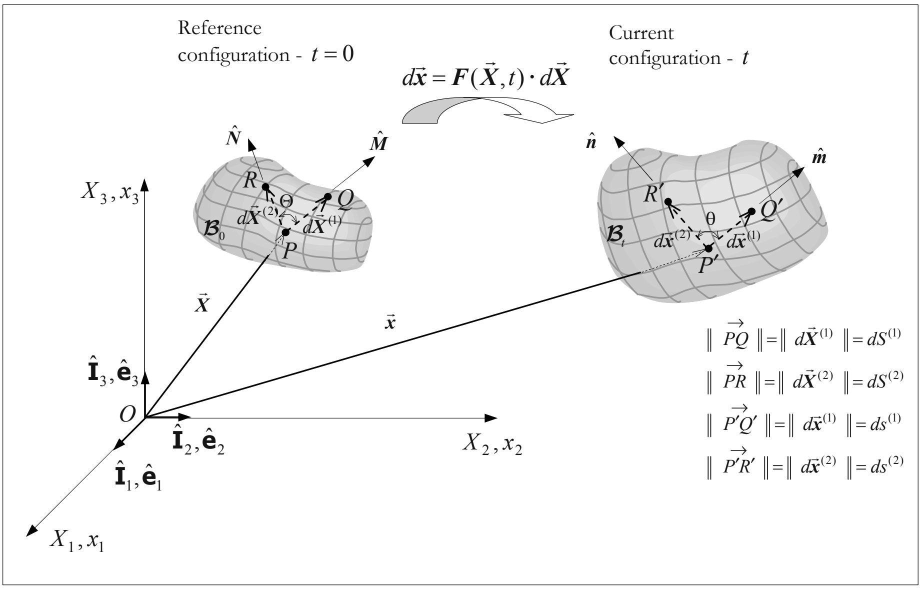


Figure 2.18: Change of angle.

The vectors $d\vec{\boldsymbol{x}}^{(1)}$ and $d\vec{\boldsymbol{x}}^{(2)}$ are given, respectively, by:

$$d\vec{\boldsymbol{x}}^{(1)} = \boldsymbol{F}\big|_P\cdot d\vec{\boldsymbol{X}}^{(1)} \qquad\Big|\qquad (d\vec{\boldsymbol{x}}^{(1)})_j = F_{jk}\big|_P\, d\vec{X}_k^{(1)} \tag{2.146}$$

and

$$d\vec{\boldsymbol{x}}^{(2)} = \boldsymbol{F}\big|_P\cdot d\vec{\boldsymbol{X}}^{(2)} \qquad\Big|\qquad (d\vec{\boldsymbol{x}}^{(2)})_j = F_{jk}\big|_P\, d\vec{X}_k^{(2)} \tag{2.147}$$

where $F_{jk}\big|_P$ show us that the deformation gradient is evaluated at the material point P. Afterwards, the scalar product $(d\vec{\boldsymbol{x}}^{(1)}\cdot d\vec{\boldsymbol{x}}^{(2)})$ is expressed in the following manner:

$$
\begin{aligned}
d\vec{x}^{(1)} \cdot d\vec{x}^{(2)} &= \boldsymbol{F} \cdot d\vec{X}^{(1)} \cdot \boldsymbol{F} \cdot d\vec{X}^{(2)} \\
&= d\vec{X}^{(1)} \cdot \underbrace{\boldsymbol{F}^T \cdot \boldsymbol{F}}_{\boldsymbol{C}} \cdot d\vec{X}^{(2)} \\
&= d\vec{X}^{(1)} \cdot \boldsymbol{C} \cdot d\vec{X}^{(2)} \\
&= dS^{(1)} dS^{(2)} \hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}} \\
&= dS^{(1)} dS^{(2)} \hat{\boldsymbol{M}} \cdot (\mathbf{1} + 2\boldsymbol{E}) \cdot \hat{\boldsymbol{N}}
\end{aligned}
\qquad
\begin{aligned}
dx_k^{(1)} dx_k^{(2)} &= (F_{ki} dX_i^{(1)})(F_{kj} dX_j^{(2)}) \\
&= dX_i^{(1)} \underbrace{F_{ki} F_{kj}}_{C_{ij}} dX_j^{(2)} \\
&= dX_i^{(1)} C_{ij} dX_j^{(2)} \\
&= dS^{(1)} dS^{(2)} \hat{M}_i C_{ij} \hat{N}_j \\
&= dS^{(1)} dS^{(2)} \hat{M}_i (\delta_{ij} + 2E_{ij}) \hat{N}_j
\end{aligned}
\tag{2.148}
$$

Additionally, the scalar product $(d\vec{X}^{(1)} \cdot d\vec{X}^{(2)})$ can be expressed as:

$$
\begin{aligned}
d\vec{X}^{(1)} \cdot d\vec{X}^{(2)} &= \boldsymbol{F}^{-1} \cdot d\vec{x}^{(1)} \cdot \boldsymbol{F}^{-1} \cdot d\vec{x}^{(2)} \\
&= d\vec{x}^{(1)} \cdot \underbrace{\boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}}_{\boldsymbol{c}} \cdot d\vec{x}^{(2)} \\
&= d\vec{x}^{(1)} \cdot \boldsymbol{c} \cdot d\vec{x}^{(2)} \\
&= ds^{(1)} ds^{(2)} \hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}} \\
&= ds^{(1)} ds^{(2)} \hat{\boldsymbol{m}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{n}}
\end{aligned}
\qquad
\begin{aligned}
dX_k^{(1)} dX_k^{(2)} &= (F_{ki}^{-1} dx_i^{(1)})(F_{kj}^{-1} dx_j^{(2)}) \\
&= dx_i^{(1)} \underbrace{F_{ki}^{-1} F_{kj}^{-1}}_{c_{ij}} dx_j^{(2)} \\
&= dx_i^{(1)} c_{ij} dx_j^{(2)} \\
&= ds^{(1)} ds^{(2)} \hat{m}_i c_{ij} \hat{n}_j \\
&= ds^{(1)} ds^{(2)} \hat{m}_i (\delta_{ij} - 2e_{ij}) \hat{n}_j
\end{aligned}
\tag{2.149}
$$

We use the equation in (2.148) to evaluate the magnitude of $d\vec{x}^{(1)}$ and $d\vec{x}^{(2)}$ in terms of the deformation tensors. To do this, in equation (2.148) we enforce that $d\vec{x}^{(2)} = d\vec{x}^{(1)}$, which leads to:

$$
d\vec{x}^{(1)} \cdot d\vec{x}^{(1)} = \left\| d\vec{x}^{(1)} \right\|^2 \left.\begin{aligned} &= dS^{(1)} dS^{(1)} \hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}} \\ &= dS^{(1)} dS^{(1)} \hat{\boldsymbol{M}} \cdot (\mathbf{1} + 2\boldsymbol{E}) \cdot \hat{\boldsymbol{M}} \end{aligned}\right\} \Rightarrow \left\| d\vec{x}^{(1)} \right\| = dS^{(1)} \sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}} \tag{2.150}
$$

Similarly, we can obtain the magnitude of $d\vec{x}^{(2)}$ as:

$$
d\vec{x}^{(2)} \cdot d\vec{x}^{(2)} = \left\| d\vec{x}^{(2)} \right\|^2 \left.\begin{aligned} &= dS^{(2)} dS^{(2)} \hat{\boldsymbol{N}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}} \\ &= dS^{(2)} dS^{(2)} \hat{\boldsymbol{N}} \cdot (\mathbf{1} + 2\boldsymbol{E}) \cdot \hat{\boldsymbol{N}} \end{aligned}\right\} \Rightarrow \left\| d\vec{x}^{(2)} \right\| = dS^{(2)} \sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}} \tag{2.151}
$$

Now by using the definition in (2.149) we can express the magnitude of $d\vec{X}^{(1)}$ and $d\vec{X}^{(2)}$ as:

$$
d\vec{X}^{(1)} \cdot d\vec{X}^{(1)} = \left\| d\vec{X}^{(1)} \right\|^2 \left.\begin{aligned} &= ds^{(1)} ds^{(1)} \hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{m}} \\ &= ds^{(1)} ds^{(1)} \hat{\boldsymbol{m}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{m}} \end{aligned}\right\} \Rightarrow \left\| d\vec{X}^{(1)} \right\| = ds^{(1)} \sqrt{\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{m}}} \tag{2.152}
$$

and

$$
d\vec{X}^{(2)} \cdot d\vec{X}^{(2)} = \left\| d\vec{X}^{(2)} \right\|^2 \left.\begin{aligned} &= ds^{(2)} ds^{(2)} \hat{\boldsymbol{n}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}} \\ &= ds^{(2)} ds^{(2)} \hat{\boldsymbol{n}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{n}} \end{aligned}\right\} \Rightarrow \left\| d\vec{X}^{(2)} \right\| = ds^{(2)} \sqrt{\hat{\boldsymbol{n}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}} \tag{2.153}
$$

2.6.4.1 The Relationship between the Strain and Stretch Tensors

Next we can establish the relationship between the stretch, unit extension and strain tensors. To do so we can start by defining the stretch (see equation (2.44)). Then the stretch along direction $\hat{\boldsymbol{M}}$, (see Figure 2.18), can be obtained by means of Lagrangian variables as:

$$\lambda_{\hat{M}} = \frac{\|d\vec{x}^{(1)}\|}{\|d\vec{X}^{(1)}\|} = \frac{dS^{(1)}\sqrt{\hat{M}\cdot C\cdot\hat{M}}}{dS^{(1)}} = \sqrt{\hat{M}\cdot C\cdot\hat{M}} = \sqrt{\hat{M}\cdot(\mathbf{1}+2E)\cdot\hat{M}} = \sqrt{1+2\hat{M}\cdot E\cdot\hat{M}} \tag{2.154}$$

where we have used the term $d\vec{x}^{(1)}$ given in (2.150). If we now use the Eulerian variable, the stretch along direction $\hat{m}$, (see Figure 2.18), is defined as:

$$\lambda_{\hat{m}} = \frac{\|d\vec{x}^{(1)}\|}{\|d\vec{X}^{(1)}\|} = \frac{ds^{(1)}}{ds^{(1)}\sqrt{\hat{m}\cdot c\cdot\hat{m}}} = \frac{1}{\sqrt{\hat{m}\cdot c\cdot\hat{m}}} = \frac{1}{\sqrt{\hat{m}\cdot(\mathbf{1}-2e)\cdot\hat{m}}} = \frac{1}{\sqrt{1-2\hat{m}\cdot e\cdot\hat{m}}} \tag{2.155}$$

Later on, we will show that $\lambda_{\hat{m}} = \lambda_{\hat{M}}$. Once the stretch has been defined in terms of strain tensors, and bearing in mind that the unit extension and stretch are related by the definition in (2.46), *i.e.* $\varepsilon = \lambda - 1$, we can express the unit extension along direction $\hat{M}$ in terms of Lagrangian variables as:

$$\varepsilon_{\hat{M}} = \lambda_{\hat{M}} - 1 = \sqrt{\hat{M}\cdot C\cdot\hat{M}} - 1 = \sqrt{\hat{M}\cdot(\mathbf{1}+2E)\cdot\hat{M}} - 1 = \sqrt{1+2\hat{M}\cdot E\cdot\hat{M}} - 1 \tag{2.156}$$

We can also evaluate the unit extension in terms of Eulerian variables as:

$$\varepsilon_{\hat{m}} = \lambda_{\hat{m}} - 1 = \frac{1}{\sqrt{\hat{m}\cdot c\cdot\hat{m}}} - 1 = \frac{1}{\sqrt{\hat{m}\cdot(\mathbf{1}-2e)\cdot\hat{m}}} - 1 = \frac{1}{\sqrt{1-2\hat{m}\cdot e\cdot\hat{m}}} - 1 \tag{2.157}$$

In short, we can state:

$$\boxed{\begin{cases}\lambda_{\hat{M}} = \sqrt{1+2\,\hat{M}\cdot E\cdot\hat{M}} = \sqrt{\hat{M}\cdot(\mathbf{1}+2E)\cdot\hat{M}} = \sqrt{\hat{M}\cdot C\cdot\hat{M}} \\ \varepsilon_{\hat{M}} = \lambda_{\hat{M}} - 1 = \sqrt{1+2\,\hat{M}\cdot E\cdot\hat{M}} - 1\end{cases}} \tag{2.158}$$

Stretch and unit extension according to the $\hat{M}$-direction, in terms of C and E

and

$$\boxed{\begin{cases}\lambda_{\hat{m}} = \dfrac{1}{\sqrt{1-2\,\hat{m}\cdot e\cdot\hat{m}}} = \dfrac{1}{\sqrt{\hat{m}\cdot c\cdot\hat{m}}} = \dfrac{1}{\sqrt{\hat{m}\cdot b^{-1}\cdot\hat{m}}} \\ \varepsilon_{\hat{m}} = \lambda_{\hat{m}} - 1 = \dfrac{1}{\sqrt{1-2\,\hat{m}\cdot e\cdot\hat{m}}} - 1 = \dfrac{1}{\sqrt{\hat{m}\cdot c\cdot\hat{m}}} - 1\end{cases}} \tag{2.159}$$

Stretch and unit extension according to the $\hat{m}$-direction, in terms of c and e

Notice that, for any given motion, if there is no stretch ($\lambda_{\hat{m}} = 1$) in a particular direction ($\hat{m}$), it holds that $\hat{m}\cdot c\cdot\hat{m} = 1$ or $\hat{m}\cdot e\cdot\hat{m} = 0$.

2.6.4.2 Change of Angle

The angle between the vectors $d\vec{x}^{(1)}$ and $d\vec{x}^{(2)}$, (see Figure 2.18), can be obtained by means of the definition of the scalar product $d\vec{x}^{(1)}\cdot d\vec{x}^{(2)} = \| d\vec{x}^{(1)} \| \| d\vec{x}^{(2)} \| \cos\theta$, the outcome of which is:

$$\cos\theta = \frac{d\vec{x}^{(1)} \cdot d\vec{x}^{(2)}}{\| d\vec{x}^{(1)} \| \| d\vec{x}^{(2)} \|} = \frac{\left(dS^{(1)} dS^{(2)}\right) \hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{dS^{(1)} \sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}}\, dS^{(2)} \sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}}$$
$$= \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}} \sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}} = \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{M}} \lambda_{\hat{N}}} \tag{2.160}$$

where we have used the equations in (2.148), (2.150) and (2.151). We can summarize the different ways of expressing $\cos\theta$ as:

$$\boxed{\begin{aligned}\cos\theta &= \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}} \sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}} = \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{M}} \lambda_{\hat{N}}} \\ &= \frac{\hat{\boldsymbol{M}} \cdot (\mathbf{1} + 2\boldsymbol{E}) \cdot \hat{\boldsymbol{N}}}{\sqrt{\hat{\boldsymbol{M}} \cdot (\mathbf{1} + 2\boldsymbol{E}) \cdot \hat{\boldsymbol{M}}} \sqrt{\hat{\boldsymbol{N}} \cdot (\mathbf{1} + 2\boldsymbol{E}) \cdot \hat{\boldsymbol{N}}}} = \frac{\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{N}} + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{E} \cdot \hat{\boldsymbol{N}}}{\sqrt{1 + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{E} \cdot \hat{\boldsymbol{M}}} \sqrt{1 + 2\hat{\boldsymbol{N}} \cdot \boldsymbol{E} \cdot \hat{\boldsymbol{N}}}}\end{aligned}} \tag{2.161}$$

Likewise, we can evaluate the angle in the reference configuration as:

$$\cos\Theta = \frac{d\vec{X}^{(1)} \cdot d\vec{X}^{(2)}}{\| d\vec{X}^{(1)} \| \| d\vec{X}^{(2)} \|} = \frac{ds^{(1)} ds^{(2)} \hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}}{ds^{(1)} \sqrt{\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{m}}}\, ds^{(2)} \sqrt{\hat{\boldsymbol{n}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}}}$$
$$= \frac{\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}}{\sqrt{\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{m}}} \sqrt{\hat{\boldsymbol{n}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}}} = \frac{\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}}{\lambda_{\hat{m}} \lambda_{\hat{n}}} \tag{2.162}$$
$$\frac{\hat{\boldsymbol{m}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{n}}}{\sqrt{\hat{\boldsymbol{m}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{m}}} \sqrt{\hat{\boldsymbol{n}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{n}}}}$$

where we have used the equations in (2.149), (2.152) and (2.153). Then, we can summarize $\cos\Theta$ as:

$$\boxed{\begin{aligned}\cos\Theta &= \frac{\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}}{\sqrt{\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{m}}} \sqrt{\hat{\boldsymbol{n}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}}} = \lambda_{\hat{m}} \lambda_{\hat{n}} (\hat{\boldsymbol{m}} \cdot \boldsymbol{c} \cdot \hat{\boldsymbol{n}}) \\ &= \frac{\hat{\boldsymbol{m}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{n}}}{\sqrt{\hat{\boldsymbol{m}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{m}}} \sqrt{\hat{\boldsymbol{n}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{n}}}} = \frac{\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{n}} - 2\hat{\boldsymbol{m}} \cdot \boldsymbol{e} \cdot \hat{\boldsymbol{n}}}{\sqrt{1 - 2\hat{\boldsymbol{m}} \cdot \boldsymbol{e} \cdot \hat{\boldsymbol{m}}} \sqrt{1 - 2\hat{\boldsymbol{n}} \cdot \boldsymbol{e} \cdot \hat{\boldsymbol{n}}}}\end{aligned}} \tag{2.163}$$

Taking into account that $\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{N}} = \cos\Theta$ and $\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{n}} = \cos\theta$, the equations in (2.161) and (2.163) become:

$$\cos\theta = \frac{\hat{\boldsymbol{M}} \cdot (\mathbf{1} + 2\boldsymbol{E}) \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{M}} \lambda_{\hat{N}}} = \frac{\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{N}} + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{E} \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{M}} \lambda_{\hat{N}}} = \frac{\cos\Theta + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{E} \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{M}} \lambda_{\hat{N}}} \tag{2.164}$$

and

$$\cos\Theta = [\hat{\boldsymbol{m}} \cdot (\mathbf{1} - 2\boldsymbol{e}) \cdot \hat{\boldsymbol{n}}] \lambda_{\hat{m}} \lambda_{\hat{n}} = [\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{n}} - 2\hat{\boldsymbol{m}} \cdot \boldsymbol{e} \cdot \hat{\boldsymbol{n}}] \lambda_{\hat{m}} \lambda_{\hat{n}} = [\cos\theta - 2\hat{\boldsymbol{m}} \cdot \boldsymbol{e} \cdot \hat{\boldsymbol{n}}] \lambda_{\hat{m}} \lambda_{\hat{n}} \tag{2.165}$$

2.6.4.3 The Physical Interpretation of the Deformation/Strain Tensor Components. The Right Stretch Tensor

2.6.4.3.1 The Normal Components

Let us consider the Cartesian components of the right Cauchy-Green deformation tensor at the material point P (particle), (see Figure 2.18).

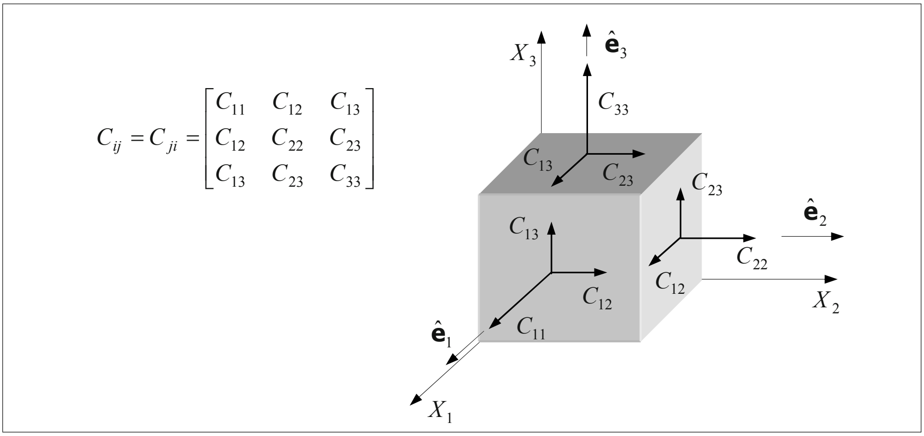


Figure 2.19: Cartesian components of $\boldsymbol{C}$.

Now let us state that the unit vector $\hat{\boldsymbol{M}}$, shown in Figure 2.18, has the same direction as the X_1-axis, *i.e.* $\hat{\boldsymbol{M}} = \hat{\mathbf{e}}_1$. So, the product $\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}$ becomes:

$$\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}} = \hat{M}_i C_{ij} \hat{M}_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = C_{11} \tag{2.166}$$

Referring to the definition of stretch given in (2.44), we can conclude that:

$$\lambda_{X_1} = \frac{\| d\vec{\boldsymbol{x}}^{(1)} \|}{\| d\vec{\boldsymbol{X}}^{(1)} \|} = \sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}} = \sqrt{C_{11}} = \sqrt{1 + 2E_{11}} \quad ; \quad \lambda_{X_1} > 0 \tag{2.167}$$

As we can see, C_{11} is the stretch measurement along the X_1-axis. Similarly, C_{22} and C_{33} show the stretch along X_2 and X_3, respectively, *i.e.*:

$$\begin{aligned} \lambda_{X_1} &= \sqrt{C_{11}} = \sqrt{1+2E_{11}} \\ \lambda_{X_2} &= \sqrt{C_{22}} = \sqrt{1+2E_{22}} \\ \lambda_{X_3} &= \sqrt{C_{33}} = \sqrt{1+2E_{33}} \end{aligned} \quad \Bigg| \quad \begin{aligned} E_{11} &= \frac{1}{2}\left(\lambda_{X_1}^2 - 1\right) \\ E_{22} &= \frac{1}{2}\left(\lambda_{X_2}^2 - 1\right) \\ E_{33} &= \frac{1}{2}\left(\lambda_{X_3}^2 - 1\right) \end{aligned} \tag{2.168}$$

Therefore, the conclusion is that the diagonal terms $\boldsymbol{E}$ and $\boldsymbol{C}$ are related to the stretches. Notice that, $\boldsymbol{C}$ is a symmetric positive definite tensor, and if we are working in the $\boldsymbol{C}$ principal space, it follows that:

$$C'_{ij} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \quad \Rightarrow \quad E'_{ij} = \begin{bmatrix} \frac{1}{2}\left(\lambda_1^2 - 1\right) & 0 & 0 \\ 0 & \frac{1}{2}\left(\lambda_2^2 - 1\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(\lambda_3^2 - 1\right) \end{bmatrix} \tag{2.169}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$ are the *principal stretches*, which by definition are positive real numbers, (see equation (2.44)). Then, the spectral representations of $\boldsymbol{C}$ and $\boldsymbol{E}$ are expressed as follows:

$\boxed{\boldsymbol{C} = \sum_{a=1}^{3} \lambda_a^2 \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}}$	*The spectral representation of the right Cauchy-Green deformation tensor*	(2.170)
$\boxed{\boldsymbol{E} = \sum_{a=1}^{3} \frac{1}{2}(\lambda_a^2 - 1) \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}}$	*The spectral representation of the Green- Lagrange strain tensor*	(2.171)

In addition, by means of the spectral representation of $\boldsymbol{C}$ we can define a new tensor such that $\mathbf{U}^2 = \boldsymbol{C}$, where $\mathbf{U}$ denotes the *right stretch tensor*, and where the only possible solution for $\mathbf{U}$ is $\mathbf{U} = +\sqrt{\boldsymbol{C}}$. Since the stretches are by definition positive real numbers, it follows that the tensor $\mathbf{U}$ is definite positive. We can then define the right stretch tensor as:

$\boxed{\mathbf{U}(\vec{X}, t) = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}}$	*The spectral representation of the right stretch tensor*	(2.172)

2.6.4.3.2 The Tangential Components

Now let us state that $\hat{\boldsymbol{M}} = \hat{\mathbf{e}}_1$ and $\hat{\boldsymbol{N}} = \hat{\mathbf{e}}_2$. So, the product $\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}$ becomes:

$$\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}} = \hat{M}_i C_{ij} \hat{N}_j = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = C_{12} \tag{2.173}$$

With the following we can verify that the above term is related to $\cos\theta$, (see equation (2.161)):

$$\cos\theta = \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\sqrt{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{M}}}\sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}} = \frac{\hat{\boldsymbol{M}} \cdot \boldsymbol{C} \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{M}} \lambda_{\hat{N}}} = \frac{C_{12}}{\sqrt{C_{11}}\sqrt{C_{22}}} = \frac{1 + 2E_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}} \tag{2.174}$$

So, C_{12} measures the angle change between two differential line elements in the reference configuration. Therefore, the off-diagonal terms $\boldsymbol{E}$ and $\boldsymbol{C}$ contain information about the angle change.

2.7 Particular Cases of Motion

2.7.1 Homogeneous Deformation

If we consider an example in which the motion of all the particles is characterized by the same deformation gradient, it follows that $\boldsymbol{F}$ is independent of the position vector $\vec{\boldsymbol{X}}$, and it is therefore only dependent on time, $\boldsymbol{F} = \boldsymbol{F}(t)$. This type of motion is an example of *homogeneous deformation*. By integrating the equation $d\vec{\boldsymbol{x}} = \boldsymbol{F}(t) \cdot d\vec{\boldsymbol{X}}$, we obtain:

$$\vec{\boldsymbol{x}} = \boldsymbol{F}(t) \cdot \vec{\boldsymbol{X}} + \vec{\mathbf{c}}(t) \tag{2.175}$$

where the constant of integration $\vec{\mathbf{c}}$ shows translational motion, which is only dependent on time.

Motion characterized by homogeneous deformation has the following characteristics:

- A material surface defined by a plane in the reference configuration will remain a plane in the current (deformed) configuration. Therefore, any material line in the reference configuration will remain a line in the deformed configuration;
- A material surface defined by a sphere in the reference configuration, will appear as an ellipsoid in the current configuration. Therefore, any material curve defined by a circle in the reference configuration will become an ellipse in the deformed configuration.

2.7.2 Rigid Body Motion

We can state that a body undergoes rigid body motion when the distance between particles are constant during motion. Under these conditions we can conclude that rigid body motion is a specific case of homogenous deformation. Let us consider a vector in the reference configuration $\vec{\boldsymbol{A}}$. After motion, this vector is represented by $\vec{\boldsymbol{a}}$ (deformed configuration). Then according to equation (2.175), it follows that $\vec{\boldsymbol{a}} = \boldsymbol{F} \cdot \vec{\boldsymbol{A}}$. Moreover, as the distances between particles do not change it holds that: $\|\vec{\boldsymbol{a}}\| = \|\vec{\boldsymbol{A}}\|$. In this situation, we can conclude that $\boldsymbol{F}(t)$ is an orthogonal tensor, *i.e.* $\boldsymbol{F}^{-1}(t) = \boldsymbol{F}^{T}(t) \Rightarrow \boldsymbol{F}(t) = \mathbf{Q}(t)$, (see orthogonal tensor, Chapter 1). Hence, here the left and right Cauchy-Green deformation tensors become:

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{F}^{T} \cdot \boldsymbol{F} = \mathbf{Q}^{T} \cdot \mathbf{Q} = \mathbf{1} \\ \boldsymbol{b} &= \boldsymbol{F} \cdot \boldsymbol{F}^{T} = \mathbf{Q} \cdot \mathbf{Q}^{T} = \mathbf{1} \end{aligned} \quad \text{The right and left Cauchy-Green deformation tensor related to rigid body motion} \quad (2.176)$$

In addition, we find that for rigid body motion the following is satisfied:

$$\begin{aligned} \boldsymbol{E} &= \frac{1}{2}(\boldsymbol{C} - \mathbf{1}) = \mathbf{0} \\ \boldsymbol{e} &= \frac{1}{2}(\mathbf{1} - \boldsymbol{b}^{-1}) = \mathbf{0} \\ \mathbf{D} &= \mathbf{0} \end{aligned} \quad \text{The strain tensors for rigid body motion} \quad (2.177)$$

In rigid body motion the stretches are unitary, since the distance between particles does not change, and it is possible to check the previous result by means of the spectral representation of $\boldsymbol{C}$ and $\boldsymbol{E}$, (see equations (2.170) and (2.171)):

$$\boldsymbol{C} = \sum_{a=1}^{3} \lambda_a^2 \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \mathbf{1} \quad (2.178)$$

$$\boldsymbol{E} = \sum_{a=1}^{3} \frac{1}{2}(\lambda_a^2 - 1) \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} 0_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \mathbf{0} \quad (2.179)$$

NOTE: To ensure that the continuum is subjected to rigid body motion, the equation $\dot{\boldsymbol{E}} = \mathbf{0}$ or $\mathbf{D} = \mathbf{0}$ must be valid for all material points throughout the continuum. ■

Problem 2.13: Let us consider the following equations of motion:

$$x_1 = X_1 + \frac{1}{2}X_2 \quad ; \quad x_2 = \frac{1}{2}X_1 + X_2 \quad ; \quad x_3 = X_3 \quad (2.180)$$

a) Obtain the displacement field ($\vec{\mathbf{u}}$) in the Lagrangian and Eulerian descriptions;
b) Determine the material curve in the current configuration for a material circle defined in the reference configuration as:

$$X_1^2 + X_2^2 = 2 \qquad X_3 = 0$$

c) Obtain the components of the right Cauchy-Green deformation tensor and the Green-Lagrange strain tensor;
d) Obtain the principal stretches.

Solution:

The deformation gradient is given by:

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad ; \quad J = |\boldsymbol{F}| = 0.75$$

And by comparing this with the equations of motion in (2.180) we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad ; \quad x_i = F_{ij} X_j$$

So, we can verify that the proposed example is a case of homogeneous deformation in which $\vec{\mathbf{c}} = \vec{\mathbf{0}}$. The inverse form of the above equation is given by:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} X_1 = \dfrac{4}{3}x_1 - \dfrac{2}{3}x_2 \\ X_2 = -\dfrac{2}{3}x_1 + \dfrac{4}{3}x_2 \\ X_3 = x_3 \end{cases} \tag{2.181}$$

The displacement field is defined by $\vec{\mathbf{u}} = \vec{\boldsymbol{x}} - \vec{\boldsymbol{X}}$, after which the components of the Lagrangian displacement become:

$$\mathrm{u}_i = x_i - X_i \Rightarrow \begin{cases} \mathrm{u}_1(\vec{\boldsymbol{X}},t) = x_1 - X_1 = X_1 + \dfrac{1}{2}X_2 - X_1 = \dfrac{1}{2}X_2 \\ \mathrm{u}_2(\vec{\boldsymbol{X}},t) = x_2 - X_2 = \dfrac{1}{2}X_1 + X_2 - X_2 = \dfrac{1}{2}X_1 \\ \mathrm{u}_3(\vec{\boldsymbol{X}},t) = x_3 - X_3 = 0 \end{cases} \tag{2.182}$$

The components of the Eulerian displacement can be obtained by substituting the Eulerian description of motion (2.181) into (2.182), the result of which is:

$$\begin{cases} \mathrm{u}_1(\boldsymbol{X}(\vec{\boldsymbol{x}},t),t) = \dfrac{1}{2}X_2(\vec{\boldsymbol{x}},t) = \dfrac{1}{2}\left[-\dfrac{2}{3}x_1 + \dfrac{4}{3}x_2\right] = \mathrm{u}_1(\vec{\boldsymbol{x}},t) \\ \mathrm{u}_2(\boldsymbol{X}(\vec{\boldsymbol{x}},t),t) = \dfrac{1}{2}X_1(\vec{\boldsymbol{x}},t) = \dfrac{1}{2}\left[-\dfrac{2}{3}x_1 + \dfrac{4}{3}x_2\right] = \mathrm{u}_2(\vec{\boldsymbol{x}},t) \\ \mathrm{u}_3(\boldsymbol{X}(\vec{\boldsymbol{x}},t),t) = \mathrm{u}_2(\vec{\boldsymbol{x}},t) = 0 \end{cases} \tag{2.183}$$

The particles belonging to the circle $X_1^2 + X_2^2 = 2$ in the reference configuration will form a new curve in the current configuration which is defined by:

$$X_1^2 + X_2^2 = 2 \Rightarrow \left[\frac{4}{3}x_1 - \frac{2}{3}x_2\right]^2 + \left[-\frac{2}{3}x_1 + \frac{4}{3}x_2\right]^2 = 2 \Rightarrow 20x_1^2 - 32x_1x_2 + 20x_2^2 = 18$$

which is an ellipse equation (Figure 2.20 shows the material curve in different configurations).

The components of $\boldsymbol{C}$ and $\boldsymbol{E}$ can be obtained by using the definitions $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$ and $\boldsymbol{E}=\frac{1}{2}(\boldsymbol{C}-\mathbf{1})$:

$$C_{ij}=F_{ki}F_{kj} \quad \Rightarrow \quad C_{ij}=\frac{1}{4}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=\begin{bmatrix}1.25 & 1 & 0\\ 1 & 1.25 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$E_{ij}=\frac{1}{2}\left(C_{ij}-\delta_{ij}\right) \quad \Rightarrow \quad E_{ij}=\frac{1}{2}\left(\begin{bmatrix}1.25 & 1 & 0\\ 1 & 1.25 & 0\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)=\begin{bmatrix}0.125 & 0.5 & 0\\ 0.5 & 0.125 & 0\\ 0 & 0 & 0\end{bmatrix}$$

In the principal space of $\boldsymbol{C}$ its components are given by:

$$C'_{ij}=\begin{bmatrix}\lambda_1^2 & 0 & 0\\ 0 & \lambda_2^2 & 0\\ 0 & 0 & \lambda_3^2\end{bmatrix} \quad \Rightarrow \quad \sqrt{C'_{ij}}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$$

where λ_i show the principal stretches. Therefore, to calculate these we need to obtain the $\boldsymbol{C}$ eigenvalues:

$$\begin{vmatrix}1.25-C & 1\\ 1 & 1.25-C\end{vmatrix}=0 \Rightarrow C^2-2.5C+0.5625=0 \Rightarrow \begin{cases}C_1=2.25\\ C_2=0.25\end{cases}$$

$$C'_{ij}=\begin{bmatrix}\lambda_1^2 & 0 & 0\\ 0 & \lambda_2^2 & 0\\ 0 & 0 & \lambda_3^2\end{bmatrix}=\begin{bmatrix}2.25 & 0 & 0\\ 0 & 0.25 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix}1.5 & 0 & 0\\ 0 & 0.5 & 0\\ 0 & 0 & 1\end{bmatrix}$$

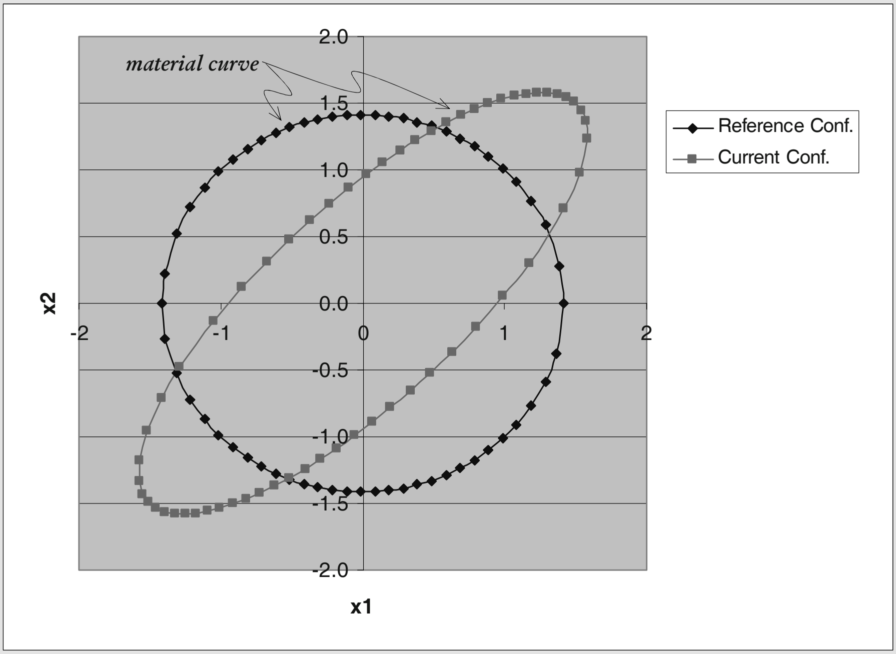


Figure 2.20: Material curve.

Problem 2.14: Let us consider the following velocity field:

$$\begin{cases} v_1 = -3x_2 + 1x_3 \\ v_2 = 3x_1 - 5x_3 \\ v_3 = -1x_1 + 5x_2 \end{cases}$$

Show that this motion corresponds to rigid body motion.

Solution: First we obtain the components of the spatial velocity gradient $(\boldsymbol{\ell})$:

$$\ell_{ij} = \frac{\partial v_i(\vec{x},t)}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ 3 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} = \ell_{ij}^{skew}$$

Taking into account that $\boldsymbol{\ell}$ can be decomposed into a symmetric ($\boldsymbol{\ell}^{sym} \equiv \mathbf{D}$) and an antisymmetric ($\boldsymbol{\ell}^{skew} \equiv \mathbf{W}$) part, *i.e.* $\boldsymbol{\ell} = \mathbf{D} + \mathbf{W}$, we can thus conclude that $\mathbf{D} = \mathbf{0}$, which is a characteristic of rigid body motion.

2.8 Polar Decomposition of *F*

As mentioned in Chapter 1, a non-singular second-order tensor can be decomposed multiplicatively by means of the *polar decomposition theorem*. By applying polar decomposition to the deformation gradient $\boldsymbol{F}$ which is a non-singular tensor $\det(\boldsymbol{F}) \neq 0$ and $\det(\boldsymbol{F}) > 0$, we obtain:

$$\boldsymbol{F} = \underbrace{\mathbf{R}\cdot\mathbf{U}}_{\substack{Right \\ polar\ \ decomposition}} = \overbrace{\mathbf{V}\cdot\mathbf{R}}^{\substack{Left \\ polar\ \ decomposition}} \tag{2.184}$$

where $\mathbf{R}$ is a proper orthogonal tensor (rotation tensor), which must meet:

$\underbrace{\mathbf{R}^T\cdot\mathbf{R} = \mathbf{R}\cdot\mathbf{R}^T = \mathbf{1} \Rightarrow \mathbf{R}^T = \mathbf{R}^{-1}}_{orthogonality}$ and $\underbrace{\det(\mathbf{R}) = 1}_{proper}$. $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive definite tensors, and are known as:

$\mathbf{U}$ - The *right stretch tensor,* the *Lagrangian stretch tensor,* or the *material stretch tensor.*

$\mathbf{V}$ - The *left stretch tensor,* the *Eulerian stretch tensor,* or the *spatial stretch tensor.*

In the right polar decomposition, we first carry out a transformation just with strain, and then we make a transformation characterized by a rotation, (see Figure 2.21), whereas, in the left polar decomposition, we first carry out an orthogonal transformation (rotation) and then transformation only with strain is applied.

With the right polar decomposition it holds that $\mathbf{U} = \mathbf{R}^T\cdot\boldsymbol{F}$, and also by applying the scalar product between $\boldsymbol{F}^T$ and the equation in (2.184) we obtain:

$$\underbrace{\boldsymbol{F}^T\cdot\boldsymbol{F}}_{\boldsymbol{C}} = \boldsymbol{F}^T\cdot\mathbf{R}\cdot\mathbf{U} \quad\Rightarrow\quad \boldsymbol{C} = \mathbf{U}^T\cdot\mathbf{U} = \mathbf{U}^2 \tag{2.185}$$

In addition, based on the spectral representation of $\boldsymbol{C}$, *i.e.* $\boldsymbol{C} = \sum_{a=1}^{3} \lambda_a^2 \,\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$, (see equation (2.170)), we can conclude that $\mathbf{U} = \sum_{a=1}^{3} \lambda_a \,\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$, where λ_a are the principal stretches, and are positive numbers by definition, and so the tensor $\mathbf{U}$ is a positive definite tensor.

Since the determinant of $\boldsymbol{F}$ is positive, $\det(\boldsymbol{F}) > 0$, and the determinant of a positive definite tensor is also positive $\det(\mathbf{U}) > 0$, we can conclude that $\mathbf{R}$ is a proper orthogonal tensor, *i.e.* a rotation tensor:

$$\det(\boldsymbol{F}) = \det(\mathbf{R} \cdot \mathbf{U}) = \underbrace{\det(\mathbf{R})}_{=1} \underbrace{\det(\mathbf{U})}_{>0} = \det(\mathbf{U}) > 0 \tag{2.186}$$

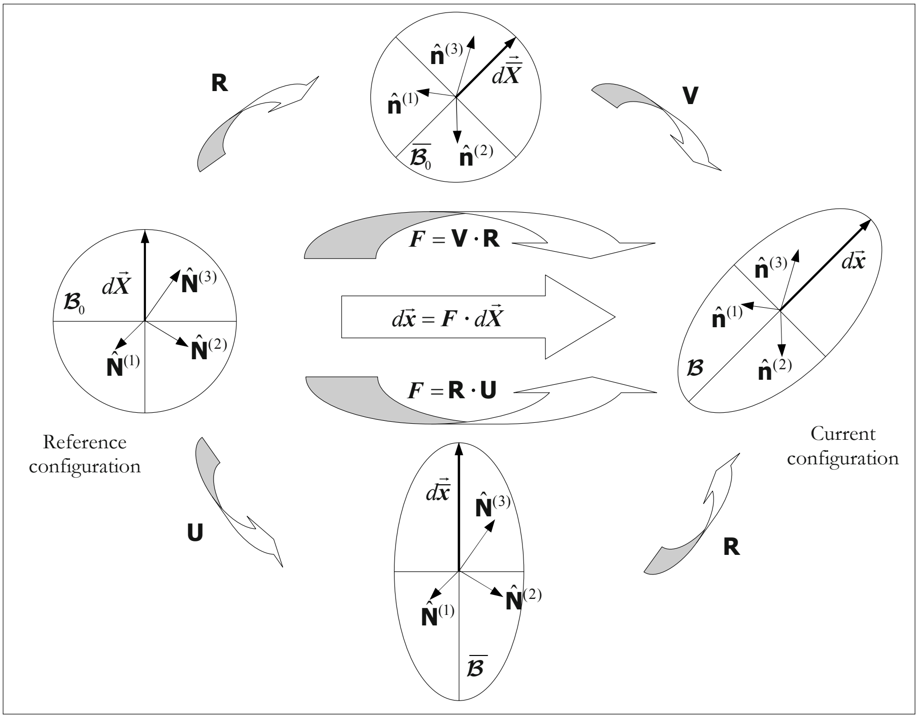


Figure 2.21: Polar decomposition of $\boldsymbol{F}$.

Taking into account the polar decomposition of $\boldsymbol{F}$, it is possible to express the right Cauchy-Green deformation tensor, $\boldsymbol{C}$, as:

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{F}^T \cdot \boldsymbol{F} \\ &= (\mathbf{R} \cdot \mathbf{U})^T \cdot (\mathbf{R} \cdot \mathbf{U}) \\ &= \mathbf{U}^T \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U} \\ &= \mathbf{U} \cdot \mathbf{U} = \mathbf{U}^2 \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \boldsymbol{C} &= \boldsymbol{F}^T \cdot \boldsymbol{F} \\ &= (\mathbf{V} \cdot \mathbf{R})^T \cdot (\mathbf{V} \cdot \mathbf{R}) \\ &= \mathbf{R}^T \cdot \mathbf{V} \cdot \mathbf{V} \cdot \mathbf{R} \\ &= \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R} \end{aligned} \tag{2.187}$$

and the left Cauchy-Green deformation tensor, $\boldsymbol{b}$, as:

$$
\begin{aligned}
\boldsymbol{b}=\boldsymbol{c}^{-1} &= \boldsymbol{F}\cdot\boldsymbol{F}^T \\
&= (\mathbf{V}\cdot\mathbf{R})\cdot(\mathbf{V}\cdot\mathbf{R})^T \\
&= \mathbf{V}\cdot\mathbf{R}\cdot\mathbf{R}^T\cdot\mathbf{V}^T \\
&= \mathbf{V}\cdot\mathbf{V}=\mathbf{V}^2
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
\boldsymbol{b} &= \boldsymbol{F}\cdot\boldsymbol{F}^T \\
&= (\mathbf{R}\cdot\mathbf{U})\cdot(\mathbf{R}\cdot\mathbf{U})^T \\
&= \mathbf{R}\cdot\mathbf{U}\cdot\mathbf{U}\cdot\mathbf{R}^T \\
&= \mathbf{R}\cdot\boldsymbol{C}\cdot\mathbf{R}^T
\end{aligned}
\tag{2.188}
$$

where $\boldsymbol{c}$ is the Cauchy deformation tensor. And the tensors $\boldsymbol{C}$ and $\boldsymbol{b}$ are interrelated to each other by:

$$
\boxed{\mathbf{U}=\mathbf{R}^T\cdot\mathbf{V}\cdot\mathbf{R} \quad ; \quad \mathbf{V}=\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{R}^T}
\tag{2.189}
$$

2.8.1 Spectral Representation of Kinematic Tensors

As we have seen before, the eigenvalues of $\mathbf{U}$ represent the principal stretches, λ_i. Each principal stretch (λ_i) is associated with a principal direction ($\hat{\mathbf{N}}^{(i)}$), *i.e.*:

$$
\begin{aligned}
\text{for } \lambda_1 \quad &\Rightarrow \quad \hat{\mathbf{N}}^{(1)} = \left[\hat{\mathrm{N}}_1^{(1)} \quad \hat{\mathrm{N}}_2^{(1)} \quad \hat{\mathrm{N}}_3^{(1)}\right] \\
\text{for } \lambda_2 \quad &\Rightarrow \quad \hat{\mathbf{N}}^{(2)} = \left[\hat{\mathrm{N}}_1^{(2)} \quad \hat{\mathrm{N}}_2^{(2)} \quad \hat{\mathrm{N}}_3^{(2)}\right] \\
\text{for } \lambda_3 \quad &\Rightarrow \quad \hat{\mathbf{N}}^{(3)} = \left[\hat{\mathrm{N}}_1^{(3)} \quad \hat{\mathrm{N}}_2^{(3)} \quad \hat{\mathrm{N}}_3^{(3)}\right]
\end{aligned}
\tag{2.190}
$$

Then, the spectral representation of $\mathbf{U}$ is given by:

$$
\boxed{\mathbf{U}=\sum_{a=1}^{3}\lambda_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}}
\tag{2.191}
$$

$$
\begin{aligned}
\boldsymbol{C}=\mathbf{U}^2 &= \left(\sum_{a=1}^{3}\lambda_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\lambda_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right) \\
&= \sum_{a=1}^{3}\lambda_a\lambda_a\,\hat{\mathbf{N}}^{(a)}\otimes\underbrace{\hat{\mathbf{N}}^{(a)}\cdot\hat{\mathbf{N}}^{(a)}}_{=1}\otimes\hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3}\lambda_a^2\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}
\end{aligned}
\tag{2.192}
$$

Thus,

$$
\boxed{\boldsymbol{C}=\mathbf{U}^2=\sum_{a=1}^{3}\lambda_a^2\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}}
\tag{2.193}
$$

The spectral representation of the right Cauchy-Green deformation tensor

If we can verify that, $\mathbf{U}$ and $\boldsymbol{C}$ are coaxial tensors, then it holds that $\boldsymbol{C}\cdot\mathbf{U}=\mathbf{U}\cdot\boldsymbol{C}=\mathbf{U}^3$. Based on the principle that $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same principal invariants, (see equation (2.113)), then $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same eigenvalues. We can prove this after having defined the eigenvalues and eigenvectors of $\mathbf{U}$:

$$
\mathbf{U}\cdot\hat{\mathbf{N}}=\lambda_a\hat{\mathbf{N}}
\tag{2.194}
$$

By substituting the equation $\mathbf{U}=\mathbf{R}^T\cdot\mathbf{V}\cdot\mathbf{R}$ into that in (2.194), we obtain:

$$
\mathbf{U}\cdot\hat{\mathbf{N}}=\mathbf{R}^T\cdot\mathbf{V}\cdot\underbrace{\mathbf{R}\cdot\hat{\mathbf{N}}}_{\hat{\mathbf{n}}}=\lambda_a\hat{\mathbf{N}}
\tag{2.195}
$$

Now, by applying the scalar product between $\mathbf{R}$ and (2.195) we obtain:

$$
\begin{aligned}
\mathbf{R}\cdot\mathbf{R}^T\cdot\mathbf{V}\cdot\hat{\mathbf{n}} &= \lambda_a\underbrace{\mathbf{R}\cdot\hat{\mathbf{N}}}_{\hat{\mathbf{n}}} \\
\mathbf{V}\cdot\hat{\mathbf{n}} &= \lambda_a\hat{\mathbf{n}}
\end{aligned}
\tag{2.196}
$$

Thus, we find that $\mathbf{U}$ and $\mathbf{V}$ have the same eigenvalues λ_i, but different eigenvectors so the spectral representation of $\mathbf{V}$ and $\boldsymbol{b}$ are given by:

$$\boxed{\mathbf{V} = \sum_{a=1}^{3} \lambda_a \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}} \quad \text{The spectral representation of the left stretch tensor} \tag{2.197}$$

$$\boxed{\boldsymbol{b} = \mathbf{V}^2 = \sum_{a=1}^{3} \lambda_a^2 \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}} \quad \text{The spectral representation of the left Cauchy-Green deformation tensor} \tag{2.198}$$

Next, we show $\boldsymbol{F}$ in terms of the eigenvalues of $\mathbf{U}$, λ_a, and eigenvectors $\hat{\mathbf{N}}$, $\hat{\mathbf{n}}$. If we consider that $\mathbf{U} = \mathbf{R}^T \cdot \boldsymbol{F}$ and $\hat{\mathbf{N}} = \mathbf{R}^T \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \mathbf{R}$, it holds that:

$$\begin{aligned}
\mathbf{U} \cdot \hat{\mathbf{N}} &= \lambda_a \hat{\mathbf{N}} \\
\mathbf{R}^T \cdot \boldsymbol{F} \cdot \hat{\mathbf{N}} &= \lambda_a \hat{\mathbf{N}} \\
\mathbf{R}^T \cdot \left(\boldsymbol{F} \cdot \hat{\mathbf{N}}\right) &= \mathbf{R}^T \cdot \left(\lambda_a \hat{\mathbf{n}}\right)
\end{aligned} \tag{2.199}$$

Thus, we can conclude that:

$$\boldsymbol{F} \cdot \hat{\mathbf{N}} = \lambda_a \hat{\mathbf{n}} \tag{2.200}$$

Taking into account that $\hat{\mathbf{N}} = \hat{\mathbf{n}} \cdot \mathbf{R}$, the deformation gradient can also be expressed as:

$$\boldsymbol{F} = \mathbf{V} \cdot \mathbf{R} = \left(\sum_{a=1}^{3} \lambda_a \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) \cdot \mathbf{R} = \sum_{a=1}^{3} \lambda_a \,\hat{\mathbf{n}}^{(a)} \otimes \underbrace{\hat{\mathbf{n}}^{(a)} \cdot \mathbf{R}}_{=\hat{\mathbf{N}}^{(a)}} \tag{2.201}$$

whereas its inverse $\boldsymbol{F}^{-1} = (\mathbf{V} \cdot \mathbf{R})^{-1} = \mathbf{R}^T \cdot \mathbf{V}^{-1}$ is:

$$\boldsymbol{F}^{-1} = \mathbf{R}^T \cdot \mathbf{V}^{-1} = \mathbf{R}^T \cdot \left(\sum_{a=1}^{3} \frac{1}{\lambda_a} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) = \sum_{a=1}^{3} \frac{1}{\lambda_a} \underbrace{\mathbf{R}^T \cdot \hat{\mathbf{n}}^{(a)}}_{=\hat{\mathbf{N}}^{(a)}} \otimes \hat{\mathbf{n}}^{(a)} \tag{2.202}$$

Thus, we can conclude that:

$$\boxed{\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \quad ; \quad \boldsymbol{F}^{-1} = \sum_{a=1}^{3} \frac{1}{\lambda_a} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}} \quad \text{“Spectral” representation of the deformation gradient} \tag{2.203}$$

By means of the “spectral” representation of the deformation gradient, we can see that $\boldsymbol{F}$ is neither in the reference nor in the current configuration. It is as if it were straddling both of them.

By making use of the left polar decomposition, $\boldsymbol{F} = \mathbf{V} \cdot \mathbf{R} \Rightarrow \mathbf{R} = \mathbf{V}^{-1} \cdot \boldsymbol{F}$, the “spectral” representation of the orthogonal tensor of the polar decomposition can be obtained as follows:

$$\begin{aligned}
\mathbf{R} &= \left(\sum_{a=1}^{3} \frac{1}{\lambda_a} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) \cdot \left(\sum_{a=1}^{3} \lambda_a \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right) \\
&= \sum_{a=1}^{3} \frac{1}{\lambda_a} \lambda_a \,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}
\end{aligned} \tag{2.204}$$

Thus:

$$\boxed{\mathbf{R} = \sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \hat{\mathbf{n}}^{(i)} \otimes \hat{\mathbf{N}}^{(i)}}$$ *The "spectral" representation of the orthogonal tensor* (2.205)

Additionally, the spectral representation of $\boldsymbol{E}$ and $\boldsymbol{e}$ are given by:

$$\boxed{\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}) = \sum_{a=1}^{3} \frac{1}{2}(\lambda_a^2 - 1)\,\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}}$$ *The spectral representation of the Green-Lagrange strain tensor*

$$\boxed{\boldsymbol{e} = \frac{1}{2}(\mathbf{1} - \boldsymbol{b}^{-1}) = \frac{1}{2}(\mathbf{1} - \mathbf{V}^{-2}) = \sum_{a=1}^{3} \frac{1}{2}(1 - \lambda_a^{-2})\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}}$$ *Spectral representation of the Almansi strain tensor* (2.206)

Next, we can establish a connection between the configurations $\mathcal{B}_0$, $\overline{\overline{\mathcal{B}}}_0$, $\overline{\mathcal{B}}$ and $\mathcal{B}$, (see Figure 2.22). To start with we can observe:

$$\begin{aligned} d\vec{x} &= \boldsymbol{F} \cdot d\vec{X} = \mathbf{V} \cdot \mathbf{R} \cdot d\vec{X} = \mathbf{V} \cdot d\vec{\overline{\overline{X}}} \\ d\vec{x} &= \boldsymbol{F} \cdot d\vec{X} = \mathbf{R} \cdot \mathbf{U} \cdot d\vec{X} = \mathbf{R} \cdot d\vec{\overline{x}} \end{aligned} \tag{2.207}$$

then:

$$\mathbf{V} \cdot d\vec{\overline{\overline{X}}} = \mathbf{R} \cdot d\vec{\overline{x}} \quad \Rightarrow \quad d\vec{\overline{x}} = \underbrace{(\mathbf{R}^T \cdot \mathbf{V})}_{=\boldsymbol{F}^T} \cdot d\vec{\overline{\overline{X}}} \quad \Rightarrow \quad d\vec{\overline{\overline{X}}} = \underbrace{(\mathbf{V}^{-1} \cdot \mathbf{R})}_{=\boldsymbol{F}^{-T}} \cdot d\vec{\overline{x}} \tag{2.208}$$

Using the equation shown in (2.111) we obtain:

$$\begin{aligned} \boldsymbol{b} &= \boldsymbol{F} \cdot \boldsymbol{C} \cdot \boldsymbol{F}^{-1} \\ &= \mathbf{V} \cdot \mathbf{R} \cdot \boldsymbol{C} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} \\ &= \mathbf{V} \cdot \overline{\overline{\boldsymbol{C}}} \cdot \mathbf{V}^{-1} \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \boldsymbol{b} &= \boldsymbol{F} \cdot \boldsymbol{C} \cdot \boldsymbol{F}^{-1} \\ &= \mathbf{R} \cdot \mathbf{U} \cdot \boldsymbol{C} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T \\ &= \mathbf{R} \cdot \overline{\boldsymbol{C}} \cdot \mathbf{R}^T \end{aligned} \tag{2.209}$$

Notice also that $\overline{\boldsymbol{C}} = \mathbf{U} \cdot \boldsymbol{C} \cdot \mathbf{U}^{-1} = \mathbf{U} \cdot \mathbf{U}^2 \cdot \mathbf{U}^{-1} = \mathbf{U}^2 = \boldsymbol{C}$. So we can conclude that:

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{F}^{-1} \cdot \boldsymbol{b} \cdot \boldsymbol{F} \\ &= \mathbf{R}^T \cdot \mathbf{V}^{-1} \cdot \boldsymbol{b} \cdot \mathbf{V} \cdot \mathbf{R} \\ &= \mathbf{R}^T \cdot \mathbf{V}^{-1} \cdot \mathbf{V}^2 \cdot \mathbf{V} \cdot \mathbf{R} \\ &= \mathbf{R}^T \cdot \mathbf{V}^2 \cdot \mathbf{R} \\ &= \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R} \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \boldsymbol{C} &= \boldsymbol{F}^{-1} \cdot \boldsymbol{b} \cdot \boldsymbol{F} \\ &= \mathbf{U}^{-1} \cdot \mathbf{R}^T \cdot \boldsymbol{b} \cdot \mathbf{R} \cdot \mathbf{U} \\ &= \mathbf{U}^{-1} \cdot \overline{\boldsymbol{b}} \cdot \mathbf{U} \end{aligned} \tag{2.210}$$

All the equations obtained above can be appreciated in Figure 2.22. We leave the reader to make the necessary algebraic operations with the inverse tensors.

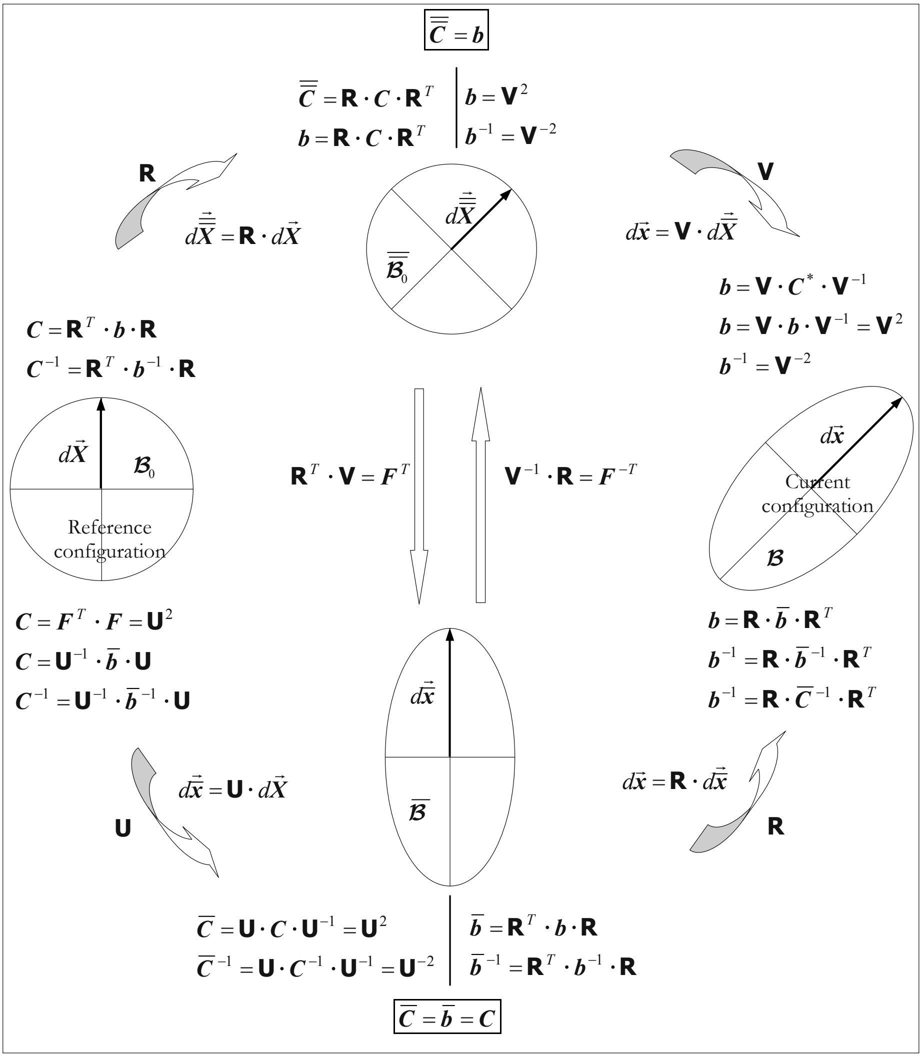


Figure 2.22: Polar decomposition of $\boldsymbol{F}$.

Problem 2.15: Let us consider the Cartesian components of the deformation gradient:

$$F_{ij} = \begin{bmatrix} 5 & 3 & 3 \\ 2 & 6 & 3 \\ 2 & 2 & 4 \end{bmatrix}$$

obtain the tensors $\mathbf{U}$ (right stretch tensor), $\mathbf{V}$ (left stretch tensor), and $\mathbf{R}$ (rotation tensor).

Solution:

Before obtaining the tensors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{R}$, we analyze the deformation gradient $\boldsymbol{F}$.
The motion is possible if the determinant of $\boldsymbol{F}$ is greater than zero, $\det(\boldsymbol{F}) = 60 > 0$. The eigenvalues and eigenvectors of $\boldsymbol{F}$ are given by:

$F'_{11} = 10$ associated with eigenvector $\hat{m}_i^{(1)} = [0.6396021491; \quad 0.6396021491; \quad 0.4264014327]$

$F'_{22} = 3$ associated with $\hat{m}_i^{(2)} = [-0.5570860145; \quad 0.7427813527; \quad -0.3713906764]$

$F'_{33}=2$ associated with $\hat{m}_i^{(3)}=[-0.4082482905;\quad -0.4082482905;\quad 0.8164965809]$

It is easy to check that the basis formed by these eigenvectors does not form an orthogonal basis, *i.e.* $\hat{m}_i^{(1)}\hat{m}_i^{(2)}\neq 0$, $\hat{m}_i^{(1)}\hat{m}_i^{(3)}\neq 0$, $\hat{m}_i^{(2)}\hat{m}_i^{(3)}\neq 0$. We can also verify that if $\mathcal{D}$ is the matrix containing the eigenvectors of $\boldsymbol{F}$:

$$\mathcal{D}=\begin{bmatrix}\hat{m}_i^{(1)}\\ \hat{m}_i^{(2)}\\ \hat{m}_i^{(3)}\end{bmatrix}=\begin{bmatrix}0.6396021491; & 0.6396021491; & 0.4264014327\\ -0.5570860145; & 0.7427813527; & -0.3713906764\\ -0.4082482905; & -0.4082482905; & 0.8164965809\end{bmatrix}$$

we find that $\det(\mathcal{D})=0.905\neq 1$, and $\mathcal{D}^{-1}\neq\mathcal{D}^T$. However, it holds that:

$$\mathcal{D}^{-1}\begin{bmatrix}10 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{bmatrix}\mathcal{D}=\begin{bmatrix}5 & 2 & 2\\ 3 & 6 & 2\\ 3 & 3 & 4\end{bmatrix}=(\boldsymbol{F}^T)_{ij}\qquad and\qquad \mathcal{D}\begin{bmatrix}5 & 2 & 2\\ 3 & 6 & 2\\ 3 & 3 & 4\end{bmatrix}\mathcal{D}^{-1}=\begin{bmatrix}10 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{bmatrix}$$

The right Cauchy-Green deformation tensor components, $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$, are given by:

$$C_{ij}=F_{ki}F_{kj}=\begin{bmatrix}33 & 31 & 29\\ 31 & 49 & 35\\ 29 & 35 & 34\end{bmatrix}$$

Then the eigenvalues and eigenvectors of $\boldsymbol{C}$ are given by:

$C'_{11}=9.274739 \xrightarrow{eigenvector} \hat{\mathsf{N}}_i^{(1)}=[0.6861511933;\quad -0.7023576528;\quad 0.1894472683]$

$C'_{22}=3.770098 \xrightarrow{eigenvector} \hat{\mathsf{N}}_i^{(2)}=[0.5105143234;\quad 0.2793856273;\quad -0.8132215099]$

$C'_{33}=102.955163 \xrightarrow{eigenvector} \hat{\mathsf{N}}_i^{(3)}=[-0.518239;\quad -0.65470405;\quad -0.550264423]$

These eigenvectors constitute an orthogonal basis, so, it holds that $\mathcal{A}_C^{-1}=\mathcal{A}_C^T$, and $\det(\mathcal{A}_C)=-1$ (improper orthogonal tensor):

$$\mathcal{A}_C=\begin{bmatrix}\hat{\mathsf{N}}_i^{(1)}\\ \hat{\mathsf{N}}_i^{(2)}\\ \hat{\mathsf{N}}_i^{(3)}\end{bmatrix}=\begin{bmatrix}0.6861511933 & -0.7023576528 & 0.1894472683\\ 0.5105143234 & 0.2793856273 & -0.8132215099\\ -0.518239 & -0.65470405 & -0.550264423\end{bmatrix}$$

Furthermore, it holds that:

$$\mathcal{A}_C^T\begin{bmatrix}C'_{11} & 0 & 0\\ 0 & C'_{22} & 0\\ 0 & 0 & C'_{33}\end{bmatrix}\mathcal{A}_C=\begin{bmatrix}33 & 31 & 29\\ 31 & 49 & 35\\ 29 & 35 & 34\end{bmatrix}=C_{ij}\quad ;\quad \mathcal{A}_C\begin{bmatrix}33 & 31 & 29\\ 31 & 49 & 35\\ 29 & 35 & 34\end{bmatrix}\mathcal{A}_C^T=\begin{bmatrix}C'_{11} & 0 & 0\\ 0 & C'_{22} & 0\\ 0 & 0 & C'_{33}\end{bmatrix}$$

In the $\boldsymbol{C}$ principal space we obtain the components of the right stretch tensor, $\mathbf{U}$, as:

$$\mathcal{U}'=\mathsf{U}'_{ij}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix}\sqrt{C'_{11}} & 0 & 0\\ 0 & \sqrt{C'_{22}} & 0\\ 0 & 0 & \sqrt{C'_{33}}\end{bmatrix}=\begin{bmatrix}3.0454455 & 0 & 0\\ 0 & 1.9416741 & 0\\ 0 & 0 & 10.1466824\end{bmatrix}$$

and its inverse:

$$\mathcal{U}'^{-1}=\mathsf{U}'^{-1}_{ij}=\begin{bmatrix}\frac{1}{\lambda_1} & 0 & 0\\ 0 & \frac{1}{\lambda_2} & 0\\ 0 & 0 & \frac{1}{\lambda_3}\end{bmatrix}=\begin{bmatrix}\frac{1}{3.0454455} & 0 & 0\\ 0 & \frac{1}{1.9416741} & 0\\ 0 & 0 & \frac{1}{10.1466824}\end{bmatrix}$$

We can evaluate the components of the tensor $\mathbf{U}$ in the original space by means of the transformation law:

$$\mathcal{A}_C^T \mathcal{U}' \mathcal{A}_C = \begin{bmatrix} 4.66496626 & 2.25196988 & 2.48328843 \\ 2.25196988 & 6.00314487 & 2.80907159 \\ 2.48328843 & 2.80907159 & 4.46569091 \end{bmatrix} = \mathsf{U}_{ij}$$

and

$$\mathcal{A}_C^T \mathcal{U}'^{-1} \mathcal{A}_C = \begin{bmatrix} 0.31528844 & -0.05134777 & -0.14302659 \\ 2.25196988 & 0.24442627 & -0.12519889 \\ -0.14302659 & -0.12519889 & 0.38221833 \end{bmatrix} = \mathsf{U}_{ij}^{-1}$$

Then, the rotation tensor of the polar decomposition is given by the equation $\mathbf{R} = \boldsymbol{F} \cdot \mathbf{U}^{-1}$, which is a proper orthogonal tensor, *i.e.* $\det(\mathbf{R}) = 1$.

$$\mathsf{R}_{ij} = F_{ik} \mathsf{U}_{kj}^{-1} = \begin{bmatrix} 0.9933191 & 0.10094326 & 0.05592536 \\ -0.10658955 & 0.98826538 & 0.10940847 \\ -0.04422505 & -0.11463858 & 0.9924224 \end{bmatrix}$$

The left Cauchy-Green deformation tensor components, $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$, are given by:

$$b_{ij} = F_{ik} F_{jk} = \begin{bmatrix} 43 & 37 & 28 \\ 37 & 49 & 28 \\ 28 & 28 & 24 \end{bmatrix}$$

Next, the eigenvalues and eigenvectors of $\boldsymbol{b}$ are given by:

$b'_{11} = 9.274739 \xrightarrow{eigenvector} \hat{\mathsf{n}}_i^{(1)} = [0.6212637156 \quad -0.7465251613 \quad 0.238183919]$

$b'_{22} = 3.770098 \xrightarrow{eigenvector} \hat{\mathsf{n}}_i^{(2)} = [0.4898263742 \quad 0.1327190337 \quad -0.8616587383]$

$b'_{33} = 102.95516 \xrightarrow{eigenvector} \hat{\mathsf{n}}_i^{(3)} = [-0.611638389 \quad -0.6519860747 \quad -0.448121233]$

Note that, the tensors $\boldsymbol{b}$ and $\boldsymbol{C}$ have the same eigenvalues but different eigenvectors. If the eigenvectors of $\boldsymbol{b}$ constitute an orthogonal basis then it holds that $\mathcal{A}_b^{-1} = \mathcal{A}_b^T$, and $\det(\mathcal{A}_b) = -1$:

$$\mathcal{A}_b = \begin{bmatrix} \hat{\mathsf{n}}_i^{(1)} \\ \hat{\mathsf{n}}_i^{(2)} \\ \hat{\mathsf{n}}_i^{(3)} \end{bmatrix} = \begin{bmatrix} 0.6212637156 & -0.7465251613 & 0.238183919 \\ 0.4898263742 & 0.1327190337 & -0.8616587383 \\ -0.611638389 & -0.6519860747 & -0.448121233 \end{bmatrix}$$

and, it also holds that:

$$\mathcal{A}_b^T \begin{bmatrix} b'_{11} & 0 & 0 \\ 0 & b'_{22} & 0 \\ 0 & 0 & b'_{33} \end{bmatrix} \mathcal{A}_b = \begin{bmatrix} 43 & 37 & 28 \\ 37 & 49 & 28 \\ 28 & 28 & 24 \end{bmatrix} = b_{ij} \quad ; \quad \mathcal{A}_b \begin{bmatrix} 43 & 37 & 28 \\ 37 & 49 & 28 \\ 28 & 28 & 24 \end{bmatrix} \mathcal{A}_b^T = \begin{bmatrix} b'_{11} & 0 & 0 \\ 0 & b'_{22} & 0 \\ 0 & 0 & b'_{33} \end{bmatrix}$$

Since $\boldsymbol{C}$ and $\boldsymbol{b}$ have the same eigenvalues, it follows that $\mathsf{U}'_{ij} = \mathsf{V}'_{ij}$, *i.e.* they have the same components in their respectively principal space. Additionally, it holds that $\mathsf{U}'^{-1}_{ij} = \mathsf{V}'^{-1}_{ij}$.

The components of the tensor $\mathbf{V}$ in the original space can be evaluated by:

$$\mathcal{A}_b^T \mathcal{V}' \mathcal{A}_b = \mathcal{A}_b^T \mathcal{U}' \mathcal{A}_b = \begin{bmatrix} 5.3720129 & 2.76007379 & 2.41222612 \\ 2.76007379 & 6.04463857 & 2.20098553 \\ 2.41222612 & 2.20098553 & 3.6519622 \end{bmatrix} = \mathsf{V}_{ij}$$

and

$$\mathcal{A}_b^T \mathcal{V}'^{-1} \mathcal{A}_b = \mathcal{A}_b^T \mathcal{U}'^{-1} \mathcal{A}_b = \begin{bmatrix} 0.28717424 & -0.07950684 & -0.14176921 \\ -0.07950684 & 0.23396031 & -0.08848799 \\ -0.14176921 & -0.08848799 & 0.42079849 \end{bmatrix} = \mathsf{V}_{ij}^{-1}$$

The polar decomposition rotation tensor obtained previously has to be the same as the one obtained by $\mathbf{R} = \mathbf{V}^{-1} \cdot \boldsymbol{F}$.

We could also have obtained the tensors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{R}$, by means of their spectral representation. That is, if we know the principal stretches, λ_i, and the eigenvectors of $\boldsymbol{C}$ ($\hat{\mathbf{N}}^{(i)}$), and the eigenvectors of $\boldsymbol{b}$ ($\hat{\mathbf{n}}^{(i)}$), it is easy to show that:

$$\mathsf{U}_{ij} = \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \lambda_1 \hat{\mathsf{N}}_i^{(1)} \hat{\mathsf{N}}_j^{(1)} + \lambda_2 \hat{\mathsf{N}}_i^{(2)} \hat{\mathsf{N}}_j^{(2)} + \lambda_3 \hat{\mathsf{N}}_i^{(3)} \hat{\mathsf{N}}_j^{(3)}$$

$$\mathsf{V}_{ij} = \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right)_{ij} = \lambda_1 \hat{\mathsf{n}}_i^{(1)} \hat{\mathsf{n}}_j^{(1)} + \lambda_2 \hat{\mathsf{n}}_i^{(2)} \hat{\mathsf{n}}_j^{(2)} + \lambda_3 \hat{\mathsf{n}}_i^{(3)} \hat{\mathsf{n}}_j^{(3)}$$

$$\mathsf{R}_{ij} = \left(\sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \hat{\mathsf{n}}_i^{(1)} \hat{\mathsf{N}}_j^{(1)} + \hat{\mathsf{n}}_i^{(2)} \hat{\mathsf{N}}_j^{(2)} + \hat{\mathsf{n}}_i^{(3)} \hat{\mathsf{N}}_j^{(3)}$$

$$F_{ij} = \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right)_{ij} = \lambda_1 \hat{\mathsf{n}}_i^{(1)} \hat{\mathsf{N}}_j^{(1)} + \lambda_2 \hat{\mathsf{n}}_i^{(2)} \hat{\mathsf{N}}_j^{(2)} + \lambda_3 \hat{\mathsf{n}}_i^{(3)} \hat{\mathsf{N}}_j^{(3)}$$

$$\begin{aligned} \boldsymbol{F} &= \sum_{a=1}^{3} \lambda_a \mathbf{R} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \cdot \mathbf{R} \\ &= \mathbf{R} \cdot \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \right) = \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \right) \cdot \mathbf{R} \\ &= \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R} \end{aligned}$$

As we can verify, the representations of the tensors $\mathbf{R}$ and $\boldsymbol{F}$ are not the spectral representations in the strict sense of the word, *i.e.*, λ_i are not eigenvalues of $\boldsymbol{F}$, and neither $\hat{\mathbf{n}}^{(i)}$ nor $\hat{\mathbf{N}}^{(i)}$ are eigenvectors of $\boldsymbol{F}$.

2.8.2 Evolution of the Polar Decomposition

Using the right polar decomposition ($\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U}$) as seen in (2.184), the material time derivative of the deformation gradient ($\boldsymbol{F}$) can also be evaluated by:

$$\dot{\boldsymbol{F}} = \dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}} \tag{2.211}$$

By considering equation (2.81), *i.e.* $\dot{\boldsymbol{F}} = \boldsymbol{\ell} \cdot \boldsymbol{F} = \boldsymbol{\ell} \cdot \mathbf{R} \cdot \mathbf{U}$, and by incorporating it into the above equation we can obtain an equation for the spatial velocity gradient ($\boldsymbol{\ell}$):

$$\begin{aligned} \boldsymbol{\ell} \cdot \mathbf{R} \cdot \mathbf{U} &= \dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}} \\ \boldsymbol{\ell} \cdot \mathbf{R} &= \dot{\mathbf{R}} \cdot \mathbf{U} \cdot \mathbf{U}^{-1} + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \\ \boldsymbol{\ell} &= \dot{\mathbf{R}} \cdot \underbrace{\mathbf{U} \cdot \mathbf{U}^{-1}}_{\mathbf{1}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T \end{aligned} \tag{2.212}$$

Thus,

$$\boldsymbol{\ell} = \dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T \tag{2.213}$$

Notice that, in rigid body motion, $\mathbf{U} = \mathbf{1} \Rightarrow \dot{\mathbf{U}} = \mathbf{0}$, the spatial velocity gradient becomes $\boldsymbol{\ell} = \dot{\mathbf{R}} \cdot \mathbf{R}^T$. This is a prompt for us to introduce an antisymmetric second-order tensor, the *rate of the material rotation tensor* (also called the *angular-velocity tensor)*, and defined as:

$$\boxed{\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^T = -\boldsymbol{\Omega}^T} \quad \textit{The rate of the material rotation tensor} \tag{2.214}$$

Additionally, the axial vector associated with $\boldsymbol{\Omega}$ is called the *angular-velocity vector* and is denoted by $\vec{\boldsymbol{\omega}}$.

We want to show that $\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T$. To do so, we start from the orthogonality condition $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{1}$, it then follows that:

$$
\begin{aligned}
\frac{D}{Dt}(\mathbf{R} \cdot \mathbf{R}^T) &= \frac{D}{Dt}(\mathbf{1}) \\
\dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{R}}^T &= \mathbf{0} \\
\boldsymbol{\Omega} + \boldsymbol{\Omega}^T &= \mathbf{0} \qquad \Rightarrow \boldsymbol{\Omega} = -\boldsymbol{\Omega}^T
\end{aligned}
\tag{2.215}
$$

It is also true that:

$$
\left.
\begin{aligned}
\frac{D}{Dt}(\dot{\mathbf{R}} \cdot \mathbf{R}^T) &= \frac{D}{Dt}(\dot{\mathbf{R}}) \cdot \mathbf{R}^T + \dot{\mathbf{R}} \cdot \frac{D}{Dt}(\mathbf{R}^T) \\
\dot{\boldsymbol{\Omega}} &= \ddot{\mathbf{R}} \cdot \mathbf{R}^T + \dot{\mathbf{R}} \cdot \dot{\mathbf{R}}^T \\
&= \ddot{\mathbf{R}} \cdot \mathbf{R}^T + \dot{\mathbf{R}} \cdot \mathbf{1} \cdot \dot{\mathbf{R}}^T \\
&= \ddot{\mathbf{R}} \cdot \mathbf{R}^T + \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \dot{\mathbf{R}}^T \\
&= \ddot{\mathbf{R}} \cdot \mathbf{R}^T - \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}
\end{aligned}
\right\} \Rightarrow \boxed{\dot{\boldsymbol{\Omega}} = \ddot{\mathbf{R}} \cdot \mathbf{R}^T - \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}}
\tag{2.216}
$$

Taking into account (2.213), the rate-of-deformation tensor, $\mathbf{D}$, can also be expressed as:

$$
\begin{aligned}
\mathbf{D} = \frac{1}{2}(\boldsymbol{\ell} + \boldsymbol{\ell}^T) \quad &= \frac{1}{2}\left[\left(\dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T\right) + \left(\dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T\right)^T\right] \\
&= \frac{1}{2}\left[\dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T + \left(\dot{\mathbf{R}} \cdot \mathbf{R}^T\right)^T + \mathbf{R} \cdot \mathbf{U}^{-T} \cdot \dot{\mathbf{U}}^T \cdot \mathbf{R}^T\right]
\end{aligned}
\tag{2.217}
$$

Notice that $\dot{\mathbf{R}} \cdot \mathbf{R}^T = -\left(\dot{\mathbf{R}} \cdot \mathbf{R}^T\right)^T$ (antisymmetric tensor) and $\mathbf{U} = \mathbf{U}^T$ (symmetric tensor). Therefore, the above relationship becomes:

$$
\mathbf{D} = \frac{1}{2}\mathbf{R} \cdot \left[\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}\right] \cdot \mathbf{R}^T
\tag{2.218}
$$

Following the same reasoning, $\mathbf{W}$ can be expressed in terms of $\mathbf{R}$ and $\mathbf{U}$ as:

$$
\mathbf{W} = \frac{1}{2}(\boldsymbol{\ell} - \boldsymbol{\ell}^T) = \dot{\mathbf{R}} \cdot \mathbf{R}^T + \frac{1}{2}\mathbf{R} \cdot \left[\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}\right] \cdot \mathbf{R}^T
\tag{2.219}
$$

We can now attempt to graphically visualize the tensors obtained above. To do so, let us consider Figure 2.23, in which the time domain is discretized by means of time increments Δt. And, at each time step we represent the right polar decomposition.

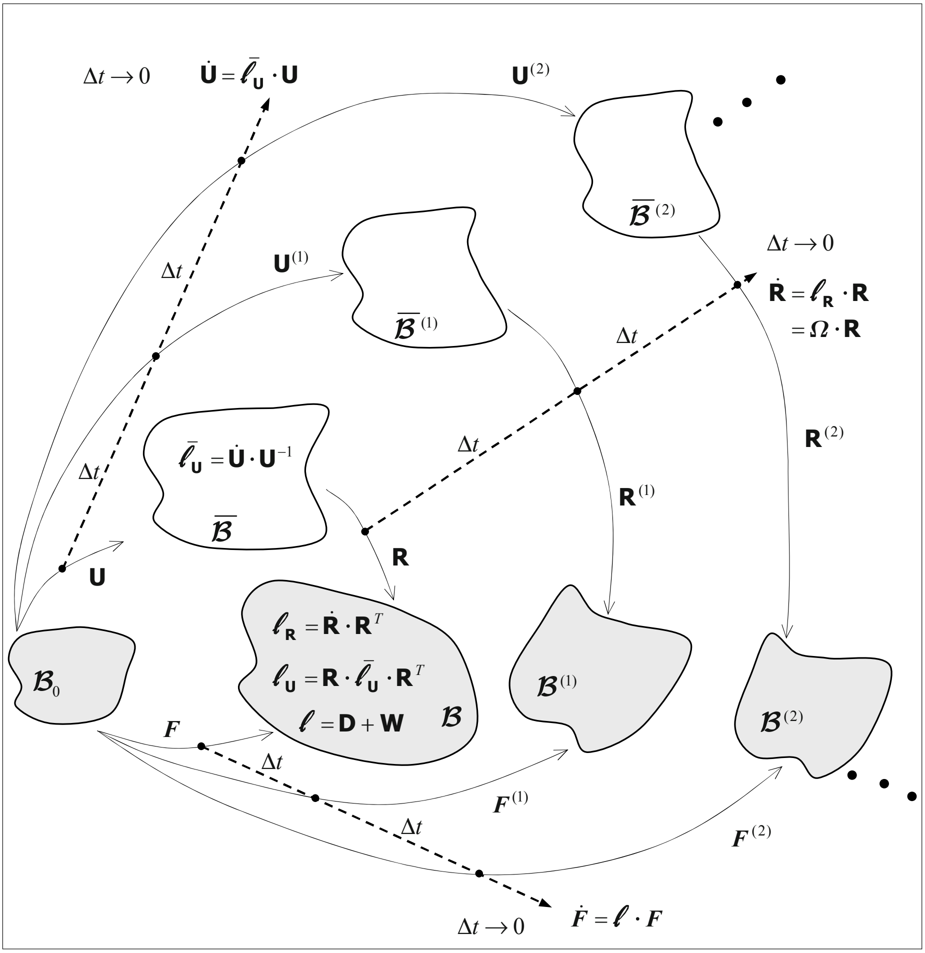


Figure 2.23: Evolution of the right polar decomposition.

As we can verify in Figure 2.23 we have represented the rate of change of $\mathbf{U}$ by means of the tensor $\bar{\boldsymbol{\ell}}_{\mathbf{U}}$, which is in the intermediate configuration $\bar{\mathcal{B}}^{(t)}$:

$$\dot{\mathbf{U}} = \bar{\boldsymbol{\ell}}_{\mathbf{U}} \cdot \mathbf{U} \quad \Rightarrow \quad \bar{\boldsymbol{\ell}}_{\mathbf{U}} = \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \tag{2.220}$$

Moreover, we can see this in the current configuration ($\boldsymbol{\ell}_{\mathbf{U}}$) by means of an orthogonal transformation, *i.e.* $\boldsymbol{\ell}_{\mathbf{U}} = \mathbf{R} \cdot \bar{\boldsymbol{\ell}}_{\mathbf{U}} \cdot \mathbf{R}^T$, (see Figure 2.23). In general, $\bar{\boldsymbol{\ell}}_{\mathbf{U}}$ is not a symmetric tensor.

Likewise, we have represented the material time derivative of $\mathbf{R}$ by means of $\boldsymbol{\ell}_{\mathbf{R}}$ (current configuration), and it follows that:

$$\dot{\mathbf{R}} = \boldsymbol{\ell}_{\mathbf{R}} \cdot \mathbf{R} \quad \Rightarrow \quad \boldsymbol{\ell}_{\mathbf{R}} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1} \quad \Rightarrow \quad \boldsymbol{\ell}_{\mathbf{R}} = \dot{\mathbf{R}} \cdot \mathbf{R}^T = \boldsymbol{\Omega} \tag{2.221}$$

Note that $\boldsymbol{\ell}_{\mathbf{R}}$ is the rate of the material rotation tensor $\boldsymbol{\Omega}$ (antisymmetric tensor). It is also true that:

$$\boldsymbol{\ell} = \boldsymbol{\ell}_{\mathbf{R}} + \boldsymbol{\ell}_{\mathbf{U}} = \dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \underbrace{\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}}_{=\bar{\boldsymbol{\ell}}_{\mathbf{U}}} \cdot \mathbf{R}^T \tag{2.222}$$

which is the same as that obtained in (2.213). The symmetric part of $\boldsymbol{\ell}$ can also be expressed as:

$$\begin{aligned}
\mathbf{D} \equiv \boldsymbol{\ell}^{sym} = \boldsymbol{\ell}_{\mathbf{U}}^{sym} + \underbrace{\boldsymbol{\ell}_{\mathbf{R}}^{sym}}_{=\mathbf{0}} = \boldsymbol{\ell}_{\mathbf{U}}^{sym} &= \frac{1}{2}\left[\left(\mathbf{R} \cdot \bar{\boldsymbol{\ell}}_{\mathbf{U}} \cdot \mathbf{R}^T\right) + \left(\mathbf{R} \cdot \bar{\boldsymbol{\ell}}_{\mathbf{U}} \cdot \mathbf{R}^T\right)^T\right] \\
&= \frac{1}{2}\left[\left(\mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T\right) + \left(\mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T\right)^T\right] \\
&= \frac{1}{2}\mathbf{R} \cdot \left[\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}\right] \cdot \mathbf{R}^T
\end{aligned} \tag{2.223}$$

which is the same as that obtained in (2.218). The antisymmetric part of $\boldsymbol{\ell}$ can also be expressed as:

$$\begin{aligned}
\mathbf{W} \equiv \boldsymbol{\ell}^{skew} = \boldsymbol{\ell}_{\mathbf{U}}^{skew} + \boldsymbol{\ell}_{\mathbf{R}}^{skew} = \boldsymbol{\ell}_{\mathbf{U}}^{skew} + \boldsymbol{\ell}_{\mathbf{R}} &= \frac{1}{2}\left[\left(\mathbf{R} \cdot \bar{\boldsymbol{\ell}}_{\mathbf{U}} \cdot \mathbf{R}^T\right) - \left(\mathbf{R} \cdot \bar{\boldsymbol{\ell}}_{\mathbf{U}} \cdot \mathbf{R}^T\right)^T\right] + \dot{\mathbf{R}} \cdot \mathbf{R}^T \\
&= \frac{1}{2}\left[\left(\mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T\right) - \left(\mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} \cdot \mathbf{R}^T\right)^T\right] + \dot{\mathbf{R}} \cdot \mathbf{R}^T \\
&= \frac{1}{2}\mathbf{R} \cdot \left[\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}\right] \cdot \mathbf{R}^T + \dot{\mathbf{R}} \cdot \mathbf{R}^T
\end{aligned} \tag{2.224}$$

which matches the equation in (2.219).

Now, if we refer to the left polar decomposition, $\boldsymbol{F} = \mathbf{V} \cdot \mathbf{R}$, the material time derivative of $\boldsymbol{F}$ is:

$$\dot{\boldsymbol{F}} = \dot{\mathbf{V}} \cdot \mathbf{R} + \mathbf{V} \cdot \dot{\mathbf{R}} \tag{2.225}$$

Using $\dot{\boldsymbol{F}} = \boldsymbol{\ell} \cdot \boldsymbol{F} = \boldsymbol{\ell} \cdot \mathbf{V} \cdot \mathbf{R}$, the above equation can be rewritten as:

$$\boldsymbol{\ell} \cdot \mathbf{V} \cdot \mathbf{R} = \dot{\mathbf{V}} \cdot \mathbf{R} + \mathbf{V} \cdot \dot{\mathbf{R}} \tag{2.226}$$

In addition, by applying the dot product with $\mathbf{R}^T \cdot \mathbf{V}^{-1}$ we obtain:

$$\boldsymbol{\ell} = \dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} \tag{2.227}$$

Based on (2.227), it is also true that:

$$\begin{aligned}
\mathbf{D} = \frac{1}{2}\left(\boldsymbol{\ell} + \boldsymbol{\ell}^T\right) &= \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \dot{\mathbf{V}} + \mathbf{V}^{-1} \cdot \mathbf{R} \cdot \dot{\mathbf{R}}^T \cdot \mathbf{V}\right) \\
&= \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}\right) + \frac{1}{2}\left(\mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}\right) \\
&= \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}\right) + \frac{1}{2}\left(\mathbf{V} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}\right)
\end{aligned} \tag{2.228}$$

and

$$\begin{aligned}
\mathbf{W} = \frac{1}{2}\left(\boldsymbol{\ell} - \boldsymbol{\ell}^T\right) &= \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \dot{\mathbf{V}} - \mathbf{V}^{-1} \cdot \mathbf{R} \cdot \dot{\mathbf{R}}^T \cdot \mathbf{V}\right) \\
&= \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \dot{\mathbf{V}} + \mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}\right) \\
&= \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}\right) + \frac{1}{2}\left(\mathbf{V} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}\right)
\end{aligned} \tag{2.229}$$

The tensors obtained above can be appreciated in Figure 2.24, where we represent the rate of change of $\mathbf{R}$ by means of $\overline{\overline{\boldsymbol{\ell}}}_{\mathbf{R}} = \dot{\mathbf{R}} \cdot \mathbf{R}^T = \boldsymbol{\Omega}$, which is in the intermediate configuration $\overline{\overline{\mathcal{B}}}_0^{(t)}$.

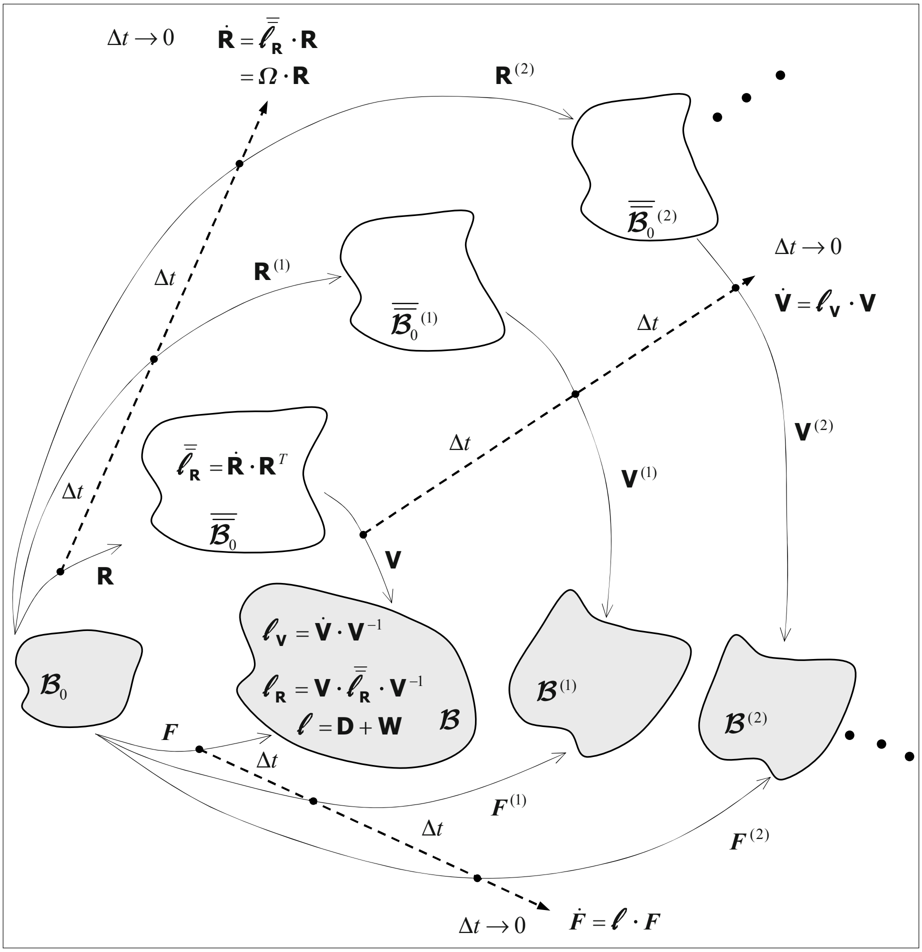


Figure 2.24: Evolution of the left polar decomposition.

Additionally, the representation of $\overline{\overline{\boldsymbol{\ell}}}_{\mathbf{R}}$ in the current configuration is given by:

$$\boldsymbol{\ell}_{\mathbf{R}} = \mathbf{V} \cdot \overline{\overline{\boldsymbol{\ell}}}_{\mathbf{R}} \cdot \mathbf{V}^{-1} = \mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} = \mathbf{V} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}^{-1} \tag{2.230}$$

The rate of change of $\mathbf{V}$ is represented by means of $\boldsymbol{\ell}_{\mathbf{V}}$, which is in the current configuration, and is given by:

$$\boldsymbol{\ell}_{\mathbf{V}} = \dot{\mathbf{V}} \cdot \mathbf{V}^{-1} \tag{2.231}$$

Then, $\boldsymbol{\ell}$ can be represented as:

$$\boldsymbol{\ell} = \boldsymbol{\ell}_{\mathbf{V}} + \boldsymbol{\ell}_{\mathbf{R}} = \dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V} \cdot \overline{\overline{\boldsymbol{\ell}}}_{\mathbf{R}} \cdot \mathbf{V}^{-1} = \dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V} \cdot \dot{\mathbf{R}} \cdot \mathbf{R}^T \cdot \mathbf{V}^{-1} \tag{2.232}$$

which is the same equation as that obtained in (2.227). The symmetric part of $\boldsymbol{\ell}$ can be obtained as:

$$\mathbf{D} \equiv \boldsymbol{\ell}^{sym} = \boldsymbol{\ell}_{\mathbf{V}}^{sym} + \boldsymbol{\ell}_{\mathbf{R}}^{sym} = \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}\right) + \frac{1}{2}\left(\mathbf{V} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}^{-1} + \mathbf{V}^{-1} \cdot \boldsymbol{\Omega}^{T} \cdot \mathbf{V}\right) \tag{2.233}$$

And the antisymmetric part of $\boldsymbol{\ell}$ is given by:

$$\mathbf{W} \equiv \boldsymbol{\ell}^{skew} = \boldsymbol{\ell}_{\mathbf{V}}^{skew} + \boldsymbol{\ell}_{\mathbf{R}}^{skew} = \frac{1}{2}\left(\dot{\mathbf{V}} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \dot{\mathbf{V}}\right) + \frac{1}{2}\left(\mathbf{V} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \boldsymbol{\Omega}^{T} \cdot \mathbf{V}\right) \tag{2.234}$$

which matches the equation in (2.229).

2.8.2.1 The Alternative Way to Express the Rate of Kinematic Tensors

Let us consider the motion and evolution of the polar decomposition as shown in Figure 2.25. We can introduce a new configuration $\overline{\mathcal{B}}_0$, which does not change over time. So, we denote by $\mathcal{R}_0$ the orthogonal transformation between the fixed basis $\hat{\mathbf{N}}_0$ and the principal direction of the right stretch tensor $\hat{\mathbf{N}}$, (see Figure 2.25). In fact, the basis $\hat{\mathbf{N}}_0$ is a system that is fixed in $\mathcal{B}_0$. Here we have separated these configurations, *i.e.* $\mathcal{B}_0$ and $\overline{\mathcal{B}}_0$, so as to have a better understanding of the process involved.

Remember that:

$$d\vec{\boldsymbol{x}} = \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} \xrightarrow{\text{rate of change}} \frac{D}{Dt}(d\vec{\boldsymbol{x}}) = \boldsymbol{\ell} \cdot d\vec{\boldsymbol{x}} \quad ; \quad \boxed{\dot{\boldsymbol{F}} = \boldsymbol{\ell} \cdot \boldsymbol{F}} \tag{2.235}$$

Then, by comparison we have:

$$\hat{\mathbf{N}} = \mathcal{R}_0 \cdot \hat{\mathbf{N}}_0 \xrightarrow{\text{rate of change}} \dot{\hat{\mathbf{N}}} = \overline{\boldsymbol{\ell}}_{\mathcal{R}_0} \cdot \hat{\mathbf{N}} = \boldsymbol{\omega}_0 \cdot \hat{\mathbf{N}} \quad ; \quad \boxed{\dot{\mathcal{R}}_0 = \boldsymbol{\omega}_0 \cdot \mathcal{R}_0} \tag{2.236}$$

$$\hat{\mathbf{n}} = \mathcal{R} \cdot \hat{\mathbf{N}}_0 \xrightarrow{\text{rate of change}} \dot{\hat{\mathbf{n}}} = \overline{\boldsymbol{\ell}}_{\mathcal{R}} \cdot \hat{\mathbf{n}} = \boldsymbol{\omega} \cdot \hat{\mathbf{n}} \quad ; \quad \boxed{\dot{\mathcal{R}} = \boldsymbol{\omega} \cdot \mathcal{R}} \tag{2.237}$$

where we have made a name change: $\overline{\boldsymbol{\ell}}_{\mathcal{R}_0} \equiv \boldsymbol{\omega}_0$, $\overline{\boldsymbol{\ell}}_{\mathcal{R}} \equiv \boldsymbol{\omega}$. It is now possible to show that $\mathcal{R} = \mathbf{R} \cdot \mathcal{R}_0$, (see Figure 2.25) and the following condition is satisfied:

$$\begin{aligned}
&\dot{\mathcal{R}} = \dot{\mathbf{R}} \cdot \mathcal{R}_0 + \mathbf{R} \cdot \dot{\mathcal{R}}_0 \\
&\Rightarrow \boldsymbol{\omega} \cdot \mathcal{R} = \boldsymbol{\Omega} \cdot \mathbf{R} \cdot \mathcal{R}_0 + \mathbf{R} \cdot \boldsymbol{\omega}_0 \cdot \mathcal{R}_0 \\
&\Rightarrow \boldsymbol{\omega} \cdot \mathbf{R} \cdot \mathcal{R}_0 = \boldsymbol{\Omega} \cdot \mathbf{R} \cdot \mathcal{R}_0 + \mathbf{R} \cdot \boldsymbol{\omega}_0 \cdot \mathcal{R}_0 \\
&\Rightarrow (\boldsymbol{\omega} \cdot \mathbf{R}) \cdot \mathcal{R}_0 = (\boldsymbol{\Omega} \cdot \mathbf{R} + \mathbf{R} \cdot \boldsymbol{\omega}_0) \cdot \mathcal{R}_0
\end{aligned} \tag{2.238}$$

where we have considered the following relationships: $\dot{\mathbf{R}} = \boldsymbol{\Omega} \cdot \mathbf{R}$, $\dot{\mathcal{R}}_0 = \boldsymbol{\omega}_0 \cdot \mathcal{R}_0$, $\dot{\mathcal{R}} = \boldsymbol{\omega} \cdot \mathcal{R}$. According to (2.238) we can conclude that:

$$\boxed{\boldsymbol{\omega} \cdot \mathbf{R} = \boldsymbol{\Omega} \cdot \mathbf{R} + \mathbf{R} \cdot \boldsymbol{\omega}_0 \quad \Leftrightarrow \quad \boldsymbol{\omega} = \boldsymbol{\Omega} + \mathbf{R} \cdot \boldsymbol{\omega}_0 \cdot \mathbf{R}^{T}} \tag{2.239}$$

Starting from (2.239) and referring to the fact that $\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{T}$, we can obtain the following equation for $\dot{\mathbf{R}}$:

$$\boldsymbol{\omega} \cdot \mathbf{R} = (\dot{\mathbf{R}} \cdot \mathbf{R}^{T}) \cdot \mathbf{R} + \mathbf{R} \cdot \boldsymbol{\omega}_0 \quad \Rightarrow \quad \boxed{\dot{\mathbf{R}} = \boldsymbol{\omega} \cdot \mathbf{R} - \mathbf{R} \cdot \boldsymbol{\omega}_0} \tag{2.240}$$

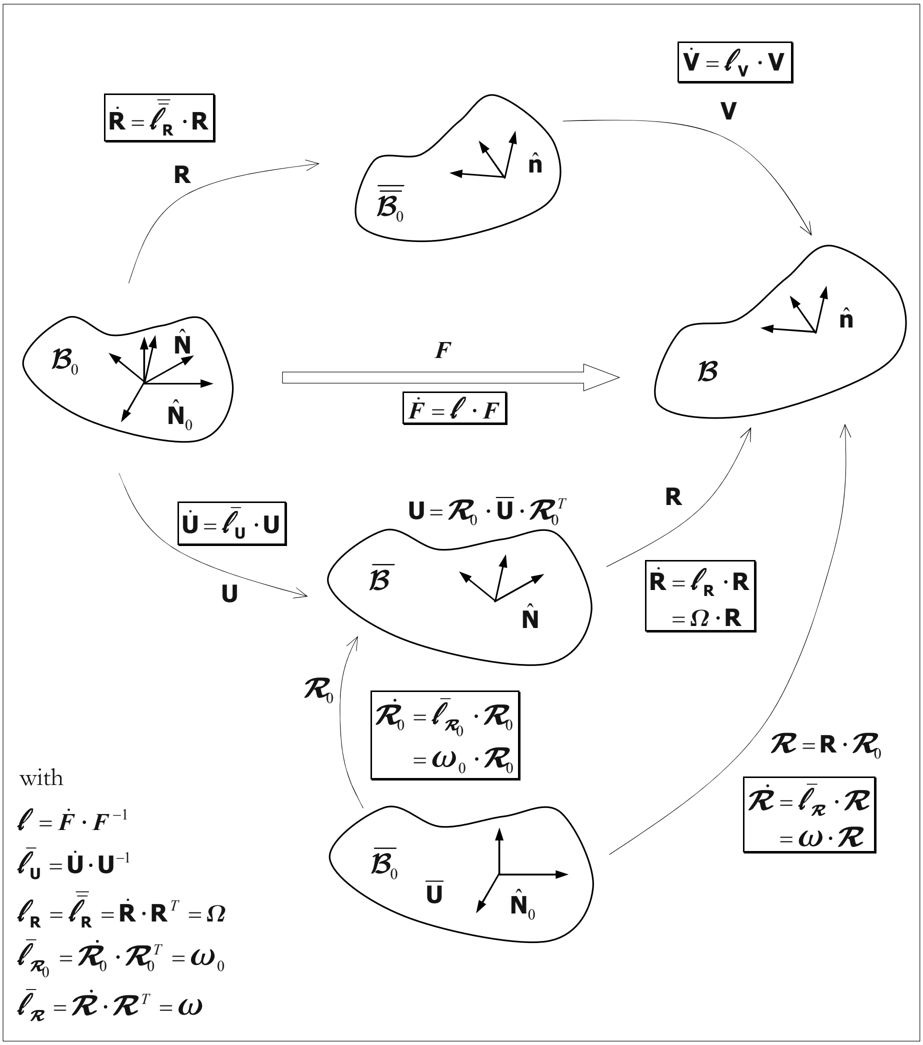


Figure 2.25: Evolution of the polar decomposition.

The material time derivative of $\hat{\mathbf{n}} = \mathbf{R} \cdot \hat{\mathbf{N}}$ can be evaluated as follows:

$$
\begin{aligned}
\dot{\hat{\mathbf{n}}} &= \dot{\mathbf{R}} \cdot \hat{\mathbf{N}} + \mathbf{R} \cdot \dot{\hat{\mathbf{N}}} \\
&= \Omega \cdot \mathbf{R} \cdot \hat{\mathbf{N}} + \mathbf{R} \cdot \boldsymbol{\omega}_0 \cdot \hat{\mathbf{N}} \\
&= (\Omega \cdot \mathbf{R} + \mathbf{R} \cdot \boldsymbol{\omega}_0) \cdot \hat{\mathbf{N}} \\
&= \boldsymbol{\omega} \cdot \mathbf{R} \cdot \hat{\mathbf{N}} \\
&= \boldsymbol{\omega} \cdot \hat{\mathbf{n}}
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
\dot{\hat{\mathbf{n}}} &= \dot{\mathbf{R}} \cdot \hat{\mathbf{N}} + \mathbf{R} \cdot \dot{\hat{\mathbf{N}}} \\
&= \Omega \cdot \mathbf{R} \cdot \hat{\mathbf{N}} + \mathbf{R} \cdot \boldsymbol{\omega}_0 \cdot \hat{\mathbf{N}} \\
&= (\Omega \cdot \mathbf{R} + \mathbf{R} \cdot \boldsymbol{\omega}_0) \cdot \mathbf{R}^T \cdot \hat{\mathbf{n}} \\
&= \left(\Omega + \mathbf{R} \cdot \boldsymbol{\omega}_0 \cdot \mathbf{R}^T\right) \cdot \hat{\mathbf{n}} \\
&= \boldsymbol{\omega} \cdot \hat{\mathbf{n}}
\end{aligned}
\tag{2.241}
$$

And, that of $\hat{\mathbf{N}} = \mathcal{R}_0 \cdot \hat{\mathbf{N}}_0$ is given by:

$$\frac{D}{Dt}\hat{\mathbf{N}} \equiv \dot{\hat{\mathbf{N}}} = \dot{\boldsymbol{\mathcal{R}}}_0 \cdot \hat{\mathbf{N}}_0 + \boldsymbol{\mathcal{R}}_0 \cdot \underbrace{\dot{\hat{\mathbf{N}}}_0}_{=\mathbf{0}} = \dot{\boldsymbol{\mathcal{R}}}_0 \cdot \hat{\mathbf{N}}_0 = \boldsymbol{\omega}_0 \cdot \boldsymbol{\mathcal{R}}_0 \cdot \hat{\mathbf{N}}_0 = \boldsymbol{\omega}_0 \cdot \hat{\mathbf{N}} \tag{2.242}$$

By referring to the spectral representation of the second-order unit tensor, *i.e.* $\mathbf{1} = \sum_{a=1}^{3} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$, and the equation in (2.236), $\dot{\hat{\mathbf{N}}} = \boldsymbol{\omega}_0 \cdot \hat{\mathbf{N}}$, we can conclude that:

$$\boldsymbol{\omega}_0 \cdot \mathbf{1} = \boldsymbol{\omega}_0 \cdot \sum_{a=1}^{3} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \boldsymbol{\omega}_0 \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \dot{\hat{\mathbf{N}}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \boldsymbol{\omega}_0 \tag{2.243}$$

Similarly,

$$\boldsymbol{\omega} \cdot \mathbf{1} = \boldsymbol{\omega} \cdot \sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \boldsymbol{\omega} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \dot{\hat{\mathbf{n}}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \boldsymbol{\omega} \tag{2.244}$$

The antisymmetric tensors $\boldsymbol{\omega}_0$, $\boldsymbol{\omega}$ can also be expressed by means of the summation symbol, (see **Problem 1.34**), as

$$\boldsymbol{\omega}_0 = \sum_{\substack{a,b=1\\ a\neq b}}^{3} \omega_{0ab} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)} \qquad \Bigg| \qquad \boldsymbol{\omega} = \sum_{\substack{a,b=1\\ a\neq b}}^{3} \omega_{ab} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \tag{2.245}$$

Then, the equation in (2.240) can also be expressed as:

$$\begin{aligned}\dot{\mathbf{R}} &= \left(\sum_{\substack{a,b=1\\ a\neq b}}^{3} \omega_{ab} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \right) \cdot \mathbf{R} - \mathbf{R} \cdot \left(\sum_{\substack{a,b=1\\ a\neq b}}^{3} \omega_{0ab} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)} \right) \\ &= \sum_{\substack{a,b=1\\ a\neq b}}^{3} \omega_{ab} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)} - \sum_{\substack{a,b=1\\ a\neq b}}^{3} \omega_{0ab} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)} = \sum_{\substack{a,b=1\\ a\neq b}}^{3} (\omega_{ab} - \omega_{0ab}) \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)}\end{aligned} \tag{2.246}$$

With the above, the term $\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^T$ can also be shown as:

$$\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^T = \left(\sum_{\substack{a,b=1\\ a\neq b}}^{3} (\omega_{ab} - \omega_{0ab}) \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)} \right) \cdot \mathbf{R}^T = \sum_{\substack{a,b=1\\ a\neq b}}^{3} (\omega_{ab} - \omega_{0ab}) \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)} \tag{2.247}$$

Moreover, we can express $\mathbf{U}$ by starting from its spectral representation:

$$\begin{aligned}\mathbf{U} &= \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \lambda_a \boldsymbol{\mathcal{R}}_0 \cdot \hat{\mathbf{N}}_0^{(a)} \otimes \boldsymbol{\mathcal{R}}_0 \cdot \hat{\mathbf{N}}_0^{(a)} = \sum_{a=1}^{3} \lambda_a \boldsymbol{\mathcal{R}}_0 \cdot \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)} \cdot \boldsymbol{\mathcal{R}}_0^T \\ &= \boldsymbol{\mathcal{R}}_0 \cdot \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)} \right) \cdot \boldsymbol{\mathcal{R}}_0^T \\ &= \boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T\end{aligned} \tag{2.248}$$

where we have introduced the tensor $\overline{\mathbf{U}}$, which is in the configuration $\overline{\mathcal{B}}_0$, (see Figure 2.25), and $\overline{\mathbf{U}}$ is given by:

$$\overline{\mathbf{U}} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)} \quad \xrightarrow{\text{rate of change}} \quad \dot{\overline{\mathbf{U}}} = \sum_{a=1}^{3} \dot{\lambda}_a \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)} \tag{2.249}$$

Then, the material time derivative of $\mathbf{U} = \boldsymbol{\mathcal{R}}_0 \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T$ becomes:

$$
\begin{aligned}
\dot{\mathbf{U}} &= \dot{\boldsymbol{\mathcal{R}}}_0 \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \dot{\bar{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \bar{\mathbf{U}} \cdot \dot{\boldsymbol{\mathcal{R}}}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \boldsymbol{\mathcal{R}}_0 \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \dot{\bar{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T \cdot \boldsymbol{\omega}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \mathbf{U} + \boldsymbol{\mathcal{R}}_0 \cdot \dot{\bar{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}_0^T + \mathbf{U} \cdot \boldsymbol{\omega}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \mathbf{U} + \mathbf{U} \cdot \boldsymbol{\omega}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \left(\sum_{a=1}^{3} \dot{\lambda}_a \, \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)} \right) \cdot \boldsymbol{\mathcal{R}}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \mathbf{U} - \mathbf{U} \cdot \boldsymbol{\omega}_0 + \left(\sum_{a=1}^{3} \dot{\lambda}_a \, \boldsymbol{\mathcal{R}}_0 \cdot \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)} \cdot \boldsymbol{\mathcal{R}}_0^T \right) \\
&= \boldsymbol{\omega}_0 \cdot \mathbf{U} - \mathbf{U} \cdot \boldsymbol{\omega}_0 + \sum_{a=1}^{3} \dot{\lambda}_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}
\end{aligned}
\tag{2.250}
$$

In **Problem 1.34** we demonstrated that:

$$
\begin{aligned}
\boldsymbol{\omega}_0 \cdot \mathbf{U} - \mathbf{U} \cdot \boldsymbol{\omega}_0 &= \sum_{\substack{a,b=1 \\ a \neq b}}^{3} \omega_{0ab} (\lambda_b - \lambda_a) \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)} \\
\boldsymbol{\omega}_0 \cdot \mathbf{U}^2 - \mathbf{U}^2 \cdot \boldsymbol{\omega}_0 &= \sum_{\substack{a,b=1 \\ a \neq b}}^{3} \omega_{0ab} (\lambda_b^2 - \lambda_a^2) \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)}
\end{aligned}
\tag{2.251}
$$

and

$$
\boldsymbol{\omega} \cdot \mathbf{V}^2 - \mathbf{V}^2 \cdot \boldsymbol{\omega} = \sum_{\substack{a,b=1 \\ a \neq b}}^{3} \omega_{ab} (\lambda_b^2 - \lambda_a^2) \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}
\tag{2.252}
$$

Then, the equation in (2.250) can be rewritten as:

$$
\boxed{\dot{\mathbf{U}} = \sum_{a=1}^{3} \dot{\lambda}_a \, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \sum_{\substack{a,b=1 \\ a \neq b}}^{3} \omega_{0ab} (\lambda_b - \lambda_a) \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)}}
\tag{2.253}
$$

Moreover, the left stretch tensor $\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T$ can also be expressed as:

$$
\begin{aligned}
\mathbf{V} &= \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T \\
&= \mathbf{R} \cdot \boldsymbol{\mathcal{R}}_0 \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T \cdot \mathbf{R}^T \\
&= \boldsymbol{\mathcal{R}} \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T
\end{aligned}
\tag{2.254}
$$

and, the material time derivative of $\mathbf{V} = \boldsymbol{\mathcal{R}} \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T$ becomes:

$$
\begin{aligned}
\dot{\mathbf{V}} &= \dot{\boldsymbol{\mathcal{R}}} \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T + \boldsymbol{\mathcal{R}} \cdot \dot{\bar{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}^T + \boldsymbol{\mathcal{R}} \cdot \bar{\mathbf{U}} \cdot \dot{\boldsymbol{\mathcal{R}}}^T \\
&= \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{R}} \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T + \boldsymbol{\mathcal{R}} \cdot \dot{\bar{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}^T + \boldsymbol{\mathcal{R}} \cdot \bar{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T \cdot \boldsymbol{\omega}^T \\
&= \boldsymbol{\omega} \cdot \mathbf{V} + \boldsymbol{\mathcal{R}} \cdot \dot{\bar{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}^T + \mathbf{V} \cdot \boldsymbol{\omega}^T = \boldsymbol{\omega} \cdot \mathbf{V} + \boldsymbol{\mathcal{R}} \cdot \left(\sum_{a=1}^{3} \dot{\lambda}_a \, \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)} \right) \cdot \boldsymbol{\mathcal{R}}^T + \mathbf{U} \cdot \boldsymbol{\omega}^T \\
&= \boldsymbol{\omega} \cdot \mathbf{V} - \mathbf{V} \cdot \boldsymbol{\omega} + \sum_{a=1}^{3} \dot{\lambda}_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}
\end{aligned}
\tag{2.255}
$$

or

$$\boxed{\dot{\mathbf{V}} = \sum_{a=1}^{3} \dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} + \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}(\lambda_b - \lambda_a)\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}} \tag{2.256}$$

The right Cauchy-Green deformation tensor ($\boldsymbol{C}$) is also expressed as:

$$\boldsymbol{C} = \mathbf{U}^2 = \left(\boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T\right)^2 = \left(\boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T\right) \cdot \left(\boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}_0^T\right) = \boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}}^2 \cdot \boldsymbol{\mathcal{R}}_0^T \tag{2.257}$$

and its material time derivative is given by:

$$\begin{aligned}
\dot{\boldsymbol{C}} &= \dot{\boldsymbol{\mathcal{R}}}_0 \cdot \overline{\mathbf{U}}^2 \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot 2\overline{\mathbf{U}} \cdot \dot{\overline{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}}^2 \cdot \dot{\boldsymbol{\mathcal{R}}}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}}^2 \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot 2\overline{\mathbf{U}} \cdot \dot{\overline{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \overline{\mathbf{U}}^2 \cdot \boldsymbol{\mathcal{R}}_0^T \cdot \boldsymbol{\omega}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \boldsymbol{C} + \boldsymbol{\mathcal{R}}_0 \cdot 2\overline{\mathbf{U}} \cdot \dot{\overline{\mathbf{U}}} \cdot \boldsymbol{\mathcal{R}}_0^T + \boldsymbol{C} \cdot \boldsymbol{\omega}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \boldsymbol{C} + \boldsymbol{C} \cdot \boldsymbol{\omega}_0^T + \boldsymbol{\mathcal{R}}_0 \cdot \left(\sum_{a=1}^{3} 2\lambda_a \dot{\lambda}_a \hat{\mathbf{N}}_0^{(a)} \otimes \hat{\mathbf{N}}_0^{(a)}\right) \cdot \boldsymbol{\mathcal{R}}_0^T \\
&= \boldsymbol{\omega}_0 \cdot \boldsymbol{C} - \boldsymbol{C} \cdot \boldsymbol{\omega}_0 + \sum_{a=1}^{3} 2\lambda_a \dot{\lambda}_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}
\end{aligned} \tag{2.258}$$

or

$$\boxed{\dot{\boldsymbol{C}} = \sum_{a=1}^{3} 2\lambda_a \dot{\lambda}_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{0ab}(\lambda_b^2 - \lambda_a^2)\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(b)}} \tag{2.259}$$

Similarly, it is possible to define the left Cauchy-Green deformation tensor as:

$$\boldsymbol{b} = \mathbf{V}^2 = \left(\boldsymbol{\mathcal{R}} \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T\right)^2 = \left(\boldsymbol{\mathcal{R}} \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T\right) \cdot \left(\boldsymbol{\mathcal{R}} \cdot \overline{\mathbf{U}} \cdot \boldsymbol{\mathcal{R}}^T\right) = \boldsymbol{\mathcal{R}} \cdot \overline{\mathbf{U}}^2 \cdot \boldsymbol{\mathcal{R}}^T \tag{2.260}$$

and, its material time derivative as:

$$\dot{\boldsymbol{b}} = \boldsymbol{\omega} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{\omega} + \sum_{a=1}^{3} 2\lambda_a \dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{2.261}$$

By fixing $\boldsymbol{\omega} \cdot \mathbf{V}^2 - \mathbf{V}^2 \cdot \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{\omega}$, and by referring to (2.252) we obtain:

$$\boxed{\dot{\boldsymbol{b}} = \sum_{a=1}^{3} 2\lambda_a \dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} + \sum_{\substack{a,b=1\\a\neq b}}^{3} \omega_{ab}(\lambda_b^2 - \lambda_a^2)\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(b)}} \tag{2.262}$$

The material time derivative of the deformation gradient, $\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}$, becomes:

$$\begin{aligned}
\dot{\boldsymbol{F}} &= \sum_{a=1}^{3}\left(\dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \lambda_a \dot{\hat{\mathbf{n}}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \dot{\hat{\mathbf{N}}}^{(a)}\right) \\
&= \sum_{a=1}^{3}\left(\dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \lambda_a \boldsymbol{\omega} \cdot \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \boldsymbol{\omega}_0 \cdot \hat{\mathbf{N}}^{(a)}\right) \\
&= \sum_{a=1}^{3} \dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \boldsymbol{\omega} \cdot \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}\right) + \left(\sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}\right) \cdot \boldsymbol{\omega}_0^T \\
&= \sum_{a=1}^{3} \dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \boldsymbol{\omega} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\omega}_0^T = \sum_{a=1}^{3} \dot{\lambda}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \boldsymbol{\omega} \cdot \boldsymbol{F} - \boldsymbol{F} \cdot \boldsymbol{\omega}_0
\end{aligned} \tag{2.263}$$

Using the same reasoning we made to solve **Problem 1.34**, we can state that:

$$\boldsymbol{\omega}\cdot\boldsymbol{F}=\left(\sum_{\substack{a,b=1\\a\neq b}}^{3}\omega_{ab}\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(b)}\right)\cdot\left(\sum_{b=1}^{3}\lambda_b\,\hat{\mathbf{n}}^{(b)}\otimes\hat{\mathbf{N}}^{(b)}\right)=\sum_{\substack{a,b=1\\a\neq b}}^{3}\omega_{ab}\lambda_b\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(b)} \tag{2.264}$$

$$\boldsymbol{F}\cdot\boldsymbol{\omega}_0=\left(\sum_{a=1}^{3}\lambda_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\left(\sum_{\substack{a,b=1\\a\neq b}}^{3}\omega_{0ab}\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(b)}\right)=\sum_{\substack{a,b=1\\a\neq b}}^{3}\omega_{0ab}\lambda_b\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(b)} \tag{2.265}$$

So, the equation in (2.263) can also be written as:

$$\boxed{\dot{\boldsymbol{F}}=\sum_{a=1}^{3}\dot{\lambda}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}+\sum_{\substack{a,b=1\\a\neq b}}^{3}\left(\lambda_b\omega_{ab}-\lambda_a\omega_{0ab}\right)\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(b)}} \tag{2.266}$$

Then, the spatial velocity gradient $\boldsymbol{\ell}=\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}$ can also be shown as:

$$\begin{aligned}\boldsymbol{\ell}=&\left(\sum_{a=1}^{3}\dot{\lambda}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)+\\&+\left(\sum_{\substack{a,b=1\\a\neq b}}^{3}\left(\lambda_b\omega_{ab}-\lambda_a\omega_{0ab}\right)\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(b)}\right)\cdot\left(\sum_{b=1}^{3}\frac{1}{\lambda_b}\hat{\mathbf{N}}^{(b)}\otimes\hat{\mathbf{n}}^{(b)}\right)\end{aligned} \tag{2.267}$$

which becomes:

$$\boxed{\boldsymbol{\ell}=\sum_{a=1}^{3}\frac{\dot{\lambda}_a}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}+\sum_{\substack{a,b=1\\a\neq b}}^{3}\left(\omega_{ab}-\frac{\lambda_a}{\lambda_b}\omega_{0ab}\right)\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(b)}} \tag{2.268}$$

By referring to $\mathbf{D}=\boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{E}}\cdot\boldsymbol{F}^{-1}=\frac{1}{2}\boldsymbol{F}^{-T}\cdot\dot{\boldsymbol{C}}\cdot\boldsymbol{F}^{-1}$, (see equation (2.134)), and using the expression of $\dot{\boldsymbol{C}}$ given in (2.259), we can also verify that:

$$\begin{aligned}\mathbf{D}=&\frac{1}{2}\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}2\lambda_a\dot{\lambda}_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\\&+\frac{1}{2}\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\left(\sum_{\substack{a,b=1\\a\neq b}}^{3}\omega_{0ab}(\lambda_b^2-\lambda_a^2)\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(b)}\right)\cdot\left(\sum_{b=1}^{3}\frac{1}{\lambda_b}\hat{\mathbf{N}}^{(b)}\otimes\hat{\mathbf{n}}^{(b)}\right)\end{aligned} \tag{2.269}$$

which becomes:

$$\boxed{\mathbf{D}=\sum_{a=1}^{3}\frac{\dot{\lambda}_a}{\lambda_a}\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}+\sum_{\substack{a,b=1\\a\neq b}}^{3}\omega_{0ab}\frac{(\lambda_b^2-\lambda_a^2)}{2\lambda_a\lambda_b}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(b)}} \tag{2.270}$$

By referring to $\mathbf{W}=\boldsymbol{\ell}-\mathbf{D}$, and the equation in (2.268) and (2.270) we obtain:

$$\boxed{\mathbf{W}=\sum_{\substack{a,b=1\\a\neq b}}^{3}\left(\omega_{ab}-\omega_{0ab}\frac{\lambda_b^2+\lambda_a^2}{2\lambda_a\lambda_b}\right)\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(b)}} \tag{2.271}$$

Problem 2.16: A rigid body motion is characterized by the following equation:

$$\vec{x} = \vec{\mathbf{c}}(t) + \mathbf{Q}(t) \cdot \vec{X} \tag{2.272}$$

Find the velocity and the acceleration fields as a function of $\vec{\omega}$, where $\vec{\omega}$ is the axial vector associated with the antisymmetric tensor ($\boldsymbol{\Omega} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T$).

Solution:

The material time derivative of $\vec{x} = \vec{\mathbf{c}}(t) + \mathbf{Q}(t) \cdot \vec{X}$ is given by

$$\vec{v} = \frac{D}{Dt}\vec{x} \equiv \dot{\vec{x}} = \dot{\vec{\mathbf{c}}} + \dot{\mathbf{Q}} \cdot \vec{X}$$

Let us consider that $\boldsymbol{\Omega} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T \Rightarrow \dot{\mathbf{Q}} = \boldsymbol{\Omega} \cdot \mathbf{Q}$. The above equation can also be expressed as:

$$\vec{v} = \dot{\vec{\mathbf{c}}} + \boldsymbol{\Omega} \cdot \mathbf{Q} \cdot \vec{X}$$

$$\vec{v} = \dot{\vec{\mathbf{c}}} + \boldsymbol{\Omega} \cdot (\vec{x} - \vec{\mathbf{c}})$$

If $\boldsymbol{\Omega}$ is an antisymmetric tensor, it holds that $\boldsymbol{\Omega} \cdot \vec{\mathbf{a}} = \vec{\omega} \wedge \vec{\mathbf{a}}$, where $\vec{\omega}$ (*angular velocity vector*) is the axial vector associated with the antisymmetric tensor $\boldsymbol{\Omega}$, (see equation (2.88)). Then, the associated velocity can be expressed as:

$$\boxed{\begin{aligned} \vec{v} &= \dot{\vec{\mathbf{c}}} + \boldsymbol{\Omega} \cdot (\vec{x} - \vec{\mathbf{c}}) \\ &= \dot{\vec{\mathbf{c}}} + \vec{\omega} \wedge (\vec{x} - \vec{\mathbf{c}}) \end{aligned}} \tag{2.273}$$

Note that $\mathbf{Q}(t)$ is only dependent on time, hence the axial vector (angular velocity) associated with $\boldsymbol{\Omega}$ is also time-dependent, *i.e.* $\vec{\omega} = \vec{\omega}(t)$.

Then, its acceleration is given by:

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{x}} = \ddot{\vec{\mathbf{c}}} + \ddot{\mathbf{Q}} \cdot \vec{X}$$

By referring to $\ddot{\mathbf{Q}} = \dot{\boldsymbol{\Omega}} \cdot \mathbf{Q} + \boldsymbol{\Omega} \cdot \dot{\mathbf{Q}}$, the above equation can also be expressed as:

$$\begin{aligned} \vec{a} &= \ddot{\vec{\mathbf{c}}} + (\dot{\boldsymbol{\Omega}} \cdot \mathbf{Q} + \boldsymbol{\Omega} \cdot \dot{\mathbf{Q}}) \cdot \vec{X} \\ &= \ddot{\vec{\mathbf{c}}} + \dot{\boldsymbol{\Omega}} \cdot \mathbf{Q} \cdot \vec{X} + \boldsymbol{\Omega} \cdot \dot{\mathbf{Q}} \cdot \vec{X} \\ &= \ddot{\vec{\mathbf{c}}} + \dot{\boldsymbol{\Omega}} \cdot \mathbf{Q} \cdot \vec{X} + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \cdot \mathbf{Q} \cdot \vec{X} \\ &= \ddot{\vec{\mathbf{c}}} + \dot{\boldsymbol{\Omega}} \cdot (\vec{x} - \vec{\mathbf{c}}) + \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} \cdot (\vec{x} - \vec{\mathbf{c}}) \end{aligned}$$

Then by using the property in (2.88) again we can state that:

$$\boxed{\vec{a} = \ddot{\vec{\mathbf{c}}} + \dot{\vec{\omega}} \wedge (\vec{x} - \vec{\mathbf{c}}) + \vec{\omega} \wedge \left[\vec{\omega} \wedge (\vec{x} - \vec{\mathbf{c}})\right]} \tag{2.274}$$

where $\vec{\alpha} \equiv \dot{\vec{\omega}}$ shows the *angular acceleration*.

For a rigid body motion where $\vec{\mathbf{c}} = \vec{\mathbf{0}}$, the velocity becomes $\vec{v} = \vec{\omega} \wedge \vec{x}$ whose components are $v_i = \epsilon_{ipq}\omega_p x_q$, and the rate-of-deformation tensor $\mathbf{D}$ becomes:

$$\mathrm{D}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial(\epsilon_{ipq}\omega_p x_q)}{\partial x_j} + \frac{\partial(\epsilon_{jpq}\omega_p x_q)}{\partial x_i}\right) = \frac{1}{2}\left(\epsilon_{ipq}\omega_p \frac{\partial x_q}{\partial x_j} + \epsilon_{jpq}\omega_p \frac{\partial x_q}{\partial x_i}\right)$$

$$= \frac{1}{2}\left(\epsilon_{ipq}\omega_p \delta_{qj} + \epsilon_{jpq}\omega_p \delta_{qi}\right) = \frac{1}{2}\left(\epsilon_{ipj}\omega_p + \epsilon_{jpi}\omega_p\right) = \frac{1}{2}\left(\epsilon_{ipj}\omega_p - \epsilon_{ipj}\omega_p\right) = 0_{ij}$$

So, once again we have proved that $\mathbf{D} = \mathbf{0}$ for a rigid body motion.

2.9 Area and Volume Elements Deformation

2.9.1 Area Element Deformation

Let us consider two line elements $d\vec{\boldsymbol{X}}^{(1)}$ and $d\vec{\boldsymbol{X}}^{(2)}$ in the reference configuration that define the area element $d\vec{\boldsymbol{A}}$, (see Figure 2.26). After motion, theses vectors are transformed into $d\vec{\boldsymbol{x}}^{(1)}$ and $d\vec{\boldsymbol{x}}^{(2)}$, thus defining the new area element $d\vec{\boldsymbol{a}}$, (see Figure 2.26).

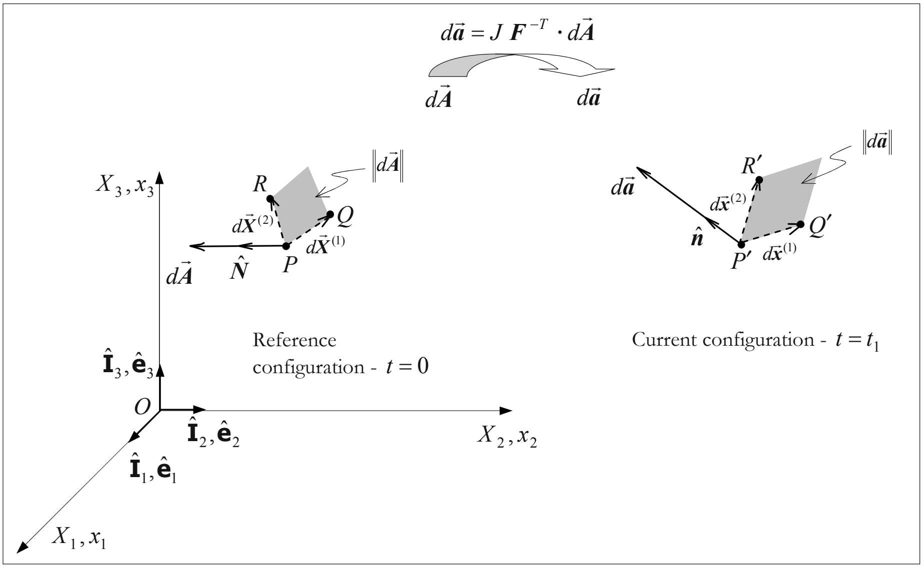


Figure 2.26: Area element deformation.

The area element $d\vec{\boldsymbol{A}}$ can be found using the definition of the vector product (the cross product) used in Chapter 1, *i.e.*:

$$d\vec{\boldsymbol{A}}(\vec{\boldsymbol{X}}) = \overrightarrow{PQ} \wedge \overrightarrow{PR} = d\vec{\boldsymbol{X}}^{(1)} \wedge d\vec{\boldsymbol{X}}^{(2)} = \left\|d\vec{\boldsymbol{A}}\right\| \hat{\boldsymbol{N}} = dA\,\hat{\boldsymbol{N}} \tag{2.275}$$

where $\left\|d\vec{\boldsymbol{A}}\right\| \equiv dA$ is the magnitude of $d\vec{\boldsymbol{A}}$, and $\hat{\boldsymbol{N}}$ is the unit vector which is normal to the area element, *i.e.* codirectional with $d\vec{\boldsymbol{A}}$. In indicial notation, the cross product is represented via the permutation symbol as:

$$dA_i = \epsilon_{ijk} dX_j^{(1)} dX_k^{(2)} \tag{2.276}$$

The deformed area element (current configuration) is given by:

$$d\vec{\boldsymbol{a}} = \overrightarrow{P'Q'} \wedge \overrightarrow{P'R'} = d\vec{\boldsymbol{x}}^{(1)} \wedge d\vec{\boldsymbol{x}}^{(2)} = \left\|d\vec{\boldsymbol{a}}\right\| \hat{\boldsymbol{n}} = da\,\hat{\boldsymbol{n}} \tag{2.277}$$

where $\left\|d\vec{\boldsymbol{a}}\right\| \equiv da$ is the magnitude of $d\vec{\boldsymbol{a}}$, and $\hat{\boldsymbol{n}}$ is the unit vector associated with the $d\vec{\boldsymbol{a}}$ - direction. In indicial notation, the deformed area element can be expressed as:

$$da_i = \epsilon_{ijk} dx_j^{(1)} dx_k^{(2)} \tag{2.278}$$

From the equation in (2.278), we obtain:

$$
\begin{aligned}
d\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}},t) &= d\vec{\boldsymbol{x}}^{(1)} \wedge d\vec{\boldsymbol{x}}^{(2)} \\
&= \boldsymbol{F}\cdot d\vec{\boldsymbol{X}}^{(1)} \wedge \boldsymbol{F}\cdot d\vec{\boldsymbol{X}}^{(2)} \\
&= \mathrm{cof}(\boldsymbol{F})\cdot(d\vec{\boldsymbol{X}}^{(1)} \wedge d\vec{\boldsymbol{X}}^{(2)}) \\
&= |\boldsymbol{F}|\boldsymbol{F}^{-T}\cdot d\vec{\boldsymbol{A}} \\
&= J\,\boldsymbol{F}^{-T}\cdot d\vec{\boldsymbol{A}}
\end{aligned}
\qquad
\begin{aligned}
da_i &= \epsilon_{ijk}\,dx_j^{(1)}dx_k^{(2)} \\
&= \epsilon_{ijk}F_{jp}dX_p^{(1)}F_{kq}dX_q^{(2)} \\
&= \epsilon_{rjk}\delta_{ri}F_{jp}F_{kq}dX_p^{(1)}dX_q^{(2)} \\
&= \epsilon_{rjk}\left(F_{rt}F_{ti}^{-1}\right)F_{jp}F_{kq}dX_p^{(1)}dX_q^{(2)} \\
&= \epsilon_{rjk}F_{rt}F_{jp}F_{kq}F_{ti}^{-1}dX_p^{(1)}dX_q^{(2)}
\end{aligned}
\tag{2.279}
$$

In the above demonstration, by using tensor notation, we applied the tensor cofactor definition, *i.e.* given a tensor $\mathbf{T}$ and two vector $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, it holds that $\mathrm{cof}(\mathbf{T})\cdot(\vec{\mathbf{a}}\wedge\vec{\mathbf{b}}) = \mathbf{T}\cdot\vec{\mathbf{a}}\wedge\mathbf{T}\cdot\vec{\mathbf{b}}$. Then, $\mathbf{T}^{-1} = |\mathbf{T}|^{-1}[\mathrm{cof}(\mathbf{T})]^T$, (see Chapter 1). We also proved in the same chapter that $\epsilon_{rjk}F_{rt}F_{jp}F_{kq} = |\boldsymbol{F}|\epsilon_{tpq}$, with which the equation in (2.279), in indicial notation, can be expressed as:

$$da_i = \epsilon_{rjk}F_{rt}F_{jp}F_{kq}F_{ti}^{-1}dX_p^{(1)}dX_q^{(2)} = |\boldsymbol{F}|F_{ti}^{-1}\epsilon_{tpq}dX_p^{(1)}dX_q^{(2)} = J\,F_{ti}^{-1}dA_t \tag{2.280}$$

or in tensor notation as:

$$\boxed{d\vec{\boldsymbol{a}} = J\,d\vec{\boldsymbol{A}}\cdot\boldsymbol{F}^{-1} = J\,\boldsymbol{F}^{-T}\cdot d\vec{\boldsymbol{A}}}\quad \textit{Nanson's formula} \tag{2.281}$$

which is known as *Nanson's formula*, and can also be written in terms of $\hat{\boldsymbol{N}}$ and $\hat{\boldsymbol{n}}$ as:

$$d\vec{\boldsymbol{a}} = da\,\hat{\boldsymbol{n}} = J\,\boldsymbol{F}^{-T}\cdot\hat{\boldsymbol{N}}\,dA = J\,dA\,\hat{\boldsymbol{N}}\cdot\boldsymbol{F}^{-1} \tag{2.282}$$

whose inverse relationship is given by:

$$d\vec{\boldsymbol{A}} = dA\,\hat{\boldsymbol{N}} = \frac{1}{J}d\vec{\boldsymbol{a}}\cdot\boldsymbol{F} = \frac{1}{J}\boldsymbol{F}^T\cdot d\vec{\boldsymbol{a}} \tag{2.283}$$

The magnitude of $d\vec{\boldsymbol{a}}$ is evaluated as follows:

$$\|d\vec{\boldsymbol{a}}\| = J\,dA\left\|\hat{\boldsymbol{N}}\cdot\boldsymbol{F}^{-1}\right\| \tag{2.284}$$

Using equations (2.281) and (2.283), the magnitudes of da and dA are interrelated by:

$$
\begin{aligned}
da^2 &= d\vec{\boldsymbol{a}}\cdot d\vec{\boldsymbol{a}} \\
&= J^2\,\boldsymbol{F}^{-T}\cdot d\vec{\boldsymbol{A}}\cdot\boldsymbol{F}^{-T}\cdot d\vec{\boldsymbol{A}} \\
&= J^2\,d\vec{\boldsymbol{A}}\cdot\underbrace{\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}}_{\boldsymbol{B}=\boldsymbol{C}^{-1}}\cdot d\vec{\boldsymbol{A}} \\
&= J^2 dA^2\;\hat{\boldsymbol{N}}\cdot\boldsymbol{B}\cdot\hat{\boldsymbol{N}}
\end{aligned}
\qquad
\begin{aligned}
dA^2 &= d\vec{\boldsymbol{A}}\cdot d\vec{\boldsymbol{A}} \\
&= \frac{1}{J^2}d\vec{\boldsymbol{a}}\cdot\boldsymbol{F}\cdot d\vec{\boldsymbol{a}}\cdot\boldsymbol{F} \\
&= \frac{1}{J^2}d\vec{\boldsymbol{a}}\cdot\underbrace{\boldsymbol{F}\cdot\boldsymbol{F}^T}_{\boldsymbol{b}}\cdot d\vec{\boldsymbol{a}} \\
&= \frac{1}{J^2}da^2\;\hat{\boldsymbol{n}}\cdot\boldsymbol{b}\cdot\hat{\boldsymbol{n}}
\end{aligned}
\tag{2.285}
$$

Then:

$$da = J\,dA\sqrt{\hat{\boldsymbol{N}}\cdot\boldsymbol{B}\cdot\hat{\boldsymbol{N}}} \qquad dA = \frac{1}{J}da\sqrt{\hat{\boldsymbol{n}}\cdot\boldsymbol{b}\cdot\hat{\boldsymbol{n}}} \tag{2.286}$$

Thus, it is also valid that:

$$\left(\frac{da}{dA}\right)^2 = J^2\hat{\boldsymbol{N}}\cdot\boldsymbol{B}\cdot\hat{\boldsymbol{N}} = J^2\frac{1}{(\hat{\boldsymbol{n}}\cdot\boldsymbol{b}\cdot\hat{\boldsymbol{n}})} \tag{2.287}$$

Taking into account the equation in (2.282) the expression of $\hat{\boldsymbol{n}}$ is obtained as:

$$\hat{\boldsymbol{n}} = \frac{J\, d\vec{\boldsymbol{A}} \cdot \boldsymbol{F}^{-1}}{da} = \frac{J\, \boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}}}{da} = \frac{J\, dA\, \hat{\boldsymbol{N}} \cdot \boldsymbol{F}^{-1}}{J\, dA \sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}}} = \frac{J\, dA\, \boldsymbol{F}^{-T} \cdot \hat{\boldsymbol{N}}}{J\, dA \sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}}} \tag{2.288}$$

or

$$\boxed{\hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{N}} \cdot \boldsymbol{F}^{-1}}{\sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}}} = \frac{\boldsymbol{F}^{-T} \cdot \hat{\boldsymbol{N}}}{\sqrt{\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}}}} \tag{2.289}$$

The material time derivative of $\hat{\boldsymbol{n}}$ can be evaluated as follows:

$$\begin{aligned}
\frac{D(\hat{\boldsymbol{n}})}{Dt} &= \frac{\hat{\boldsymbol{N}} \cdot \dot{\boldsymbol{F}}^{-1}}{\left(\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}\right)^{\frac{1}{2}}} - \frac{1}{2} \frac{\hat{\boldsymbol{N}} \cdot \boldsymbol{F}^{-1}}{\left(\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}\right)^{\frac{3}{2}}} \cdot \left(\hat{\boldsymbol{N}} \cdot \dot{\boldsymbol{B}} \cdot \hat{\boldsymbol{N}}\right) \\
&= -\frac{\hat{\boldsymbol{N}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{\ell}}{\left(\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}\right)^{\frac{1}{2}}} + \frac{\hat{\boldsymbol{N}} \cdot \boldsymbol{F}^{-1}}{\left(\hat{\boldsymbol{N}} \cdot \boldsymbol{B} \cdot \hat{\boldsymbol{N}}\right)^{\frac{3}{2}}} \cdot \left(\hat{\boldsymbol{N}} \cdot \boldsymbol{F}^{-1} \cdot \mathbf{D} \cdot \dot{\boldsymbol{F}}^{-T} \cdot \hat{\boldsymbol{N}}\right)
\end{aligned} \tag{2.290}$$

or

$$\boxed{\begin{aligned}
\frac{D(\hat{\boldsymbol{n}})}{Dt} &= -\hat{\boldsymbol{n}} \cdot \boldsymbol{\ell} + \hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}} \cdot \mathbf{D} \cdot \hat{\boldsymbol{n}}\right) \\
&= -\hat{\boldsymbol{n}} \cdot \boldsymbol{\ell} + \hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{\ell} \cdot \hat{\boldsymbol{n}} - \underbrace{\hat{\boldsymbol{n}} \cdot \mathbf{W} \cdot \hat{\boldsymbol{n}}}_{=0}\right) \\
&= -\hat{\boldsymbol{n}} \cdot \boldsymbol{\ell} + \hat{\boldsymbol{n}}\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{\ell} \cdot \hat{\boldsymbol{n}}\right) \\
&= \left[\left(\hat{\boldsymbol{n}} \cdot \boldsymbol{\ell} \cdot \hat{\boldsymbol{n}}\right)\mathbf{1} - \boldsymbol{\ell}^{T}\right] \cdot \hat{\boldsymbol{n}}
\end{aligned}} \tag{2.291}$$

2.9.1.1 The Material Time Derivative of the Area Element

Let us consider the undeformed and deformed area elements:

$$d\vec{\boldsymbol{A}}(\vec{\boldsymbol{X}},t) = \left\| d\vec{\boldsymbol{A}} \right\| \hat{\boldsymbol{N}} \quad ; \quad d\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}},t) = \left\| d\vec{\boldsymbol{a}} \right\| \hat{\boldsymbol{n}} = J\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}} \tag{2.292}$$

The material time derivative of the deformed are element, $d\vec{\boldsymbol{a}}$, is given by:

$$\begin{aligned}
\frac{D}{Dt}(d\vec{\boldsymbol{a}}) &= \frac{D}{Dt}\left[J\,\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}}\right] \\
&= \frac{DJ}{Dt}\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}} + J\frac{D}{Dt}\left[\boldsymbol{F}^{-T}\right] \cdot d\vec{\boldsymbol{A}} + J\,\boldsymbol{F}^{-T} \cdot \underbrace{\frac{D}{Dt}\left[d\vec{\boldsymbol{A}}\right]}_{=\vec{\mathbf{0}}} \\
&= (\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}})\underbrace{J\,\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}}}_{d\vec{\boldsymbol{a}}} - \boldsymbol{\ell}^{T} \cdot \underbrace{J\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}}^{-1}}_{d\vec{\boldsymbol{a}}}
\end{aligned} \tag{2.293}$$

Then:

$$\boxed{\begin{aligned}
\frac{D}{Dt}(d\vec{\boldsymbol{a}}) &= (\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}})\, d\vec{\boldsymbol{a}} - \boldsymbol{\ell}^{T} \cdot d\vec{\boldsymbol{a}} \\
&= \mathrm{Tr}(\mathbf{D})\, d\vec{\boldsymbol{a}} - \boldsymbol{\ell}^{T} \cdot d\vec{\boldsymbol{a}} \\
&= \left[\mathrm{Tr}(\mathbf{D})\,\mathbf{1} - \boldsymbol{\ell}^{T}\right] \cdot d\vec{\boldsymbol{a}}
\end{aligned}} \tag{2.294}$$

2.9.2 The Volume Element Deformation

Let us consider a parallelepiped formed by the line elements $d\vec{\boldsymbol{X}}^{(1)}$, $d\vec{\boldsymbol{X}}^{(2)}$, $d\vec{\boldsymbol{X}}^{(3)}$, (in the reference configuration) whose volume is denoted by dV_0. After motion, the vectors $d\vec{\boldsymbol{X}}^{(1)}$, $d\vec{\boldsymbol{X}}^{(2)}$, $d\vec{\boldsymbol{X}}^{(3)}$ are transformed into the line elements $d\vec{\boldsymbol{x}}^{(1)}$, $d\vec{\boldsymbol{x}}^{(2)}$, $d\vec{\boldsymbol{x}}^{(3)}$, respectively, and describe a new parallelepiped volume denoted by dV, (see Figure 2.27). Next, we can establish the relationship between $dV_0(\vec{\boldsymbol{X}})$ and $dV(\vec{\boldsymbol{x}},t)$.

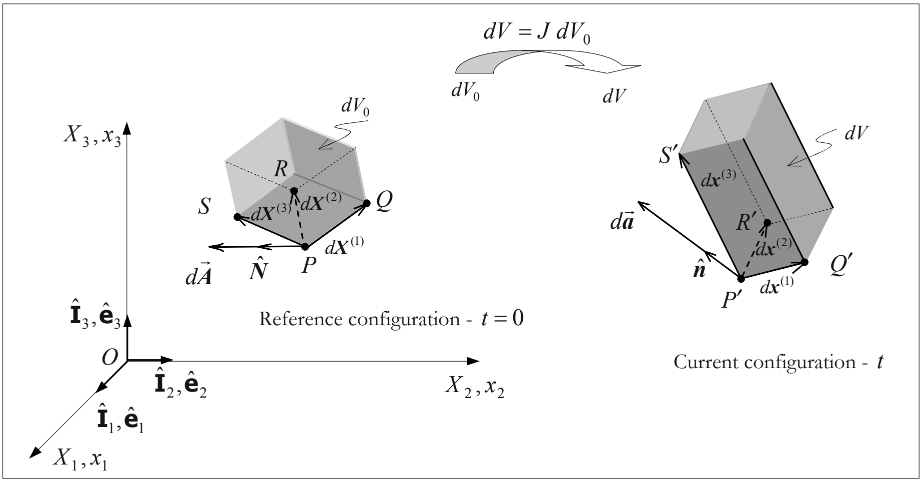


Figure 2.27: Volume element deformation.

The parallelepiped volume, in the reference configuration, can be obtained by means of the scalar triple product by including the three vectors $d\vec{\boldsymbol{X}}^{(1)}, d\vec{\boldsymbol{X}}^{(2)}, d\vec{\boldsymbol{X}}^{(3)}$, *i.e.*:

$$dV_0 = \left(d\vec{\boldsymbol{X}}^{(1)} \wedge d\vec{\boldsymbol{X}}^{(2)}\right) \cdot d\vec{\boldsymbol{X}}^{(3)} = \epsilon_{ijk} dX_i^{(1)} dX_j^{(2)} dX_k^{(3)} \tag{2.295}$$

Similarly, the deformed volume element can be evaluated as:

$$dV = \left(d\vec{\boldsymbol{x}}^{(1)} \wedge d\vec{\boldsymbol{x}}^{(2)}\right) \cdot d\vec{\boldsymbol{x}}^{(3)} = \epsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)} \tag{2.296}$$

The element volume dV can also be expressed as:

$$\begin{aligned} dV &= \left(d\vec{\boldsymbol{x}}^{(1)} \wedge d\vec{\boldsymbol{x}}^{(2)}\right) \cdot d\vec{\boldsymbol{x}}^{(3)} \\ &= \epsilon_{ijk} dx_i^{(1)} dx_j^{(2)} dx_k^{(3)} \\ &= \epsilon_{ijk} F_{iB} dX_B^{(1)} F_{jC} dX_C^{(2)} F_{kD} dX_D^{(3)} \\ &= \underbrace{\epsilon_{ijk} F_{iB} F_{jC} F_{kD}}_{|\boldsymbol{F}|\epsilon_{BCD}} dX_B^{(1)} dX_C^{(2)} dX_D^{(3)} \\ &= |\boldsymbol{F}| \underbrace{\epsilon_{BCD} dX_B^{(1)} dX_C^{(2)} dX_D^{(3)}}_{dV_0} \end{aligned} \qquad \begin{aligned} dV &= \left(d\vec{\boldsymbol{x}}^{(1)} \wedge d\vec{\boldsymbol{x}}^{(2)}\right) \cdot d\vec{\boldsymbol{x}}^{(3)} \\ &= \left(\boldsymbol{F} \cdot d\vec{\boldsymbol{X}}^{(1)} \wedge \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}^{(2)}\right) \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}^{(3)} \\ &= \left[\mathrm{cof}(\boldsymbol{F}) \cdot (d\vec{\boldsymbol{X}}^{(1)} \wedge d\vec{\boldsymbol{X}}^{(2)})\right] \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}^{(3)} \\ &= \left[J\boldsymbol{F}^{-T} \cdot (d\vec{\boldsymbol{X}}^{(1)} \wedge d\vec{\boldsymbol{X}}^{(2)})\right] \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}^{(3)} \\ &= J\left[(d\vec{\boldsymbol{X}}^{(1)} \wedge d\vec{\boldsymbol{X}}^{(2)}) \cdot \boldsymbol{F}^{-1}\right] \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}^{(3)} \\ &= J(d\vec{\boldsymbol{X}}^{(1)} \wedge d\vec{\boldsymbol{X}}^{(2)}) \cdot d\vec{\boldsymbol{X}}^{(3)} \\ &= J dV_0 \end{aligned} \tag{2.297}$$

Thus proving that the relationship between dV and dV_0 is given by:

$$\boxed{dV = |\boldsymbol{F}|\; dV_0 = J\; dV_0} \quad \textit{Deformation of the volume element} \tag{2.298}$$

OBS.: If $|\boldsymbol{F}| = J = 1$, the volume is preserved during motion. If $J > 1$, the volume expands. If $0 < J < 1$, the volume shrinks. If $J \leq 0$, there is particle penetration, thus violating the *Axiom of impenetrability*, *i.e.* without any physical meaning in classical Mechanics.

Let $\rho_0(\vec{\boldsymbol{X}})$ and $\rho(\vec{\boldsymbol{x}},t)$ be mass densities in the reference and current configurations, respectively. The differential mass in the reference configuration (dm_0) and in the current configuration (dm) are related by:

$$dm_0 = \rho_0\, dV_0 \qquad ; \qquad dm = \rho\, dV \tag{2.299}$$

Based on the principle of conservation of mass, we find that:

$$\begin{aligned} dm_0 &= dm \\ \rho_0\, dV_0 &= \rho\, dV \\ \frac{\rho_0}{\rho} &= \frac{dV}{dV_0} = |\boldsymbol{F}| = J \end{aligned} \tag{2.300}$$

$$\boxed{\rho_0(\vec{\boldsymbol{X}}) = J\,\rho(\vec{\boldsymbol{x}},t)} \tag{2.301}$$

Problem 2.17: Obtain an equation for mass density in terms of the third invariant of the right Cauchy-Green deformation tensor, *i.e.* $\rho_0 = \rho_0(I\!I\!I_{\boldsymbol{C}})$.

Solution:

From the equation in (2.301) we obtain:

$$\rho_0(\vec{\boldsymbol{X}}) = \rho(\vec{\boldsymbol{x}},t) J$$

and considering that the third invariant $I\!I\!I_{\boldsymbol{C}} = \det(\boldsymbol{C}) = \det(\boldsymbol{F}^T \cdot \boldsymbol{F}) = J^2$, we obtain $J = \sqrt{I\!I\!I_{\boldsymbol{C}}}$, then:

$$\boxed{\rho_0 = \rho\sqrt{I\!I\!I_{\boldsymbol{C}}}} \tag{2.302}$$

2.9.2.1 The Material Time Derivative of the Volume Element

The material time derivative of dV is given by:

$$\frac{D}{Dt}(dV) = \frac{D}{Dt}(J\ dV_0) = dV_0\frac{D}{Dt}(J) + J\underbrace{\frac{D}{Dt}(dV_0)}_{=0} \tag{2.303}$$

Then we have:

$$\boxed{\begin{aligned} \frac{D}{Dt}(dV) &= \frac{DJ}{Dt}dV_0 \\ &= J\,\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\,dV_0 \\ &= \nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\,dV \\ &= \mathrm{Tr}(\mathbf{D})\,dV \end{aligned}} \tag{2.304}$$

2.9.2.2 Dilatation

Dilatation is the relative variation of the volume element, *i.e.*:

$$D_V(\vec{X},t) = \frac{dV(\vec{x},t) - dV(\vec{X},0)}{dV(\vec{X},0)} = \frac{dV - dV_0}{dV_0} \tag{2.305}$$

If we considering that $dV = |\boldsymbol{F}| \, dV_0 = J \, dV_0$, the above equation becomes:

$$D_V(\vec{X},t) = \frac{dV - dV_0}{dV_0} = \frac{J \, dV_0 - dV_0}{dV_0} = J - 1 \tag{2.306}$$

2.9.2.3 Isochoric Motion. Incompressibility

If during motion the volume element remains unchanged this implies that the Jacobian determinant field is unitary, since:

$$|\boldsymbol{F}| = J = 1 = \frac{dV}{dV_0} \quad \Rightarrow \quad dV = dV_0 \quad \Rightarrow \quad \frac{D}{Dt}(dV) = 0 \tag{2.307}$$

Then, if during motion the volume of every particle remains unchanged the motion is said to be isochoric, *i.e.*:

$$\rho_0(\vec{X}) = \rho(\vec{x},t) \ \text{(Isochoric motion)} \tag{2.308}$$

The continuous medium characterized by isochoric motion is said to be *incompressible*. An incompressible medium can also be characterized by:

$$\frac{D}{Dt}(dV) = \frac{D}{Dt}(J dV_0) = \frac{DJ}{Dt} dV_0 = \underbrace{J}_{\neq 0} \nabla_{\vec{x}} \cdot \vec{v} \, \underbrace{dV_0}_{\neq 0} = 0 \quad \Rightarrow \quad \nabla_{\vec{x}} \cdot \vec{v} = 0 \tag{2.309}$$

where we have taken into account that $dV_0 \neq 0$, $J \neq 0$. By using the equation in (2.102), it is possible to express the incompressibility of the form:

$$\nabla_{\vec{x}} \cdot \vec{v} = v_{k,k} = \mathrm{Tr}(\nabla_{\vec{x}} \vec{v}) = \mathrm{Tr}(\boldsymbol{\ell}) = \mathrm{Tr}(\mathbf{D}) = 0 \tag{2.310}$$

NOTE: It is interesting to point out that incompressibility is an approximation. That is, all continuous media are compressible, but depending on the material, such as liquids in general, the compressibility can be insignificant. ■

2.10 Material and Control Domains

2.10.1 The Material Domain

The Material Curve

The material curve is a moving line that is always made up of the same particles.

The Material Surface

We can define the material surface, (see Figure 2.28), as a moving surface which is always made up of the same particles. The material surfaces in the reference and current configuration are represented, respectively, by:

$$\Phi(\vec{X}) = C \quad ; \quad \phi(\vec{x},t) = \Phi(\vec{X}(\vec{x},t)) = c \tag{2.311}$$

By applying the chain rule of differentiation to the equation in (2.311) we obtain:

$$\frac{\partial \phi(\vec{x},t)}{\partial x_i} = \frac{\partial \Phi(\vec{x},t)}{\partial X_k}\frac{\partial X_k}{\partial x_i} = \frac{\partial \phi(\vec{x},t)}{\partial X_k} F_{ki}^{-1} \quad \Bigg| \quad \nabla_{\vec{x}} \phi(\vec{x},t) = \boldsymbol{F}^{-T} \cdot \nabla_{\vec{X}} \Phi(\vec{X}) \tag{2.312}$$

The normal to the surface $\Phi(\vec{X}) = C$ is given by:

$$\hat{\boldsymbol{N}} = \frac{\nabla_{\vec{X}} \Phi(\vec{X})}{\left\| \nabla_{\vec{X}} \Phi(\vec{X}) \right\|} \tag{2.313}$$

and the derivative of C with respect to $\hat{\boldsymbol{N}}$ is given by:

$$\frac{dC}{d\hat{\boldsymbol{N}}} = \left\| \nabla_{\vec{X}} \Phi(\vec{X}) \right\| \tag{2.314}$$

Then, the unit vector $\hat{\boldsymbol{n}}$ can be expressed by:

$$\hat{\boldsymbol{n}} = \frac{\nabla_{\vec{x}} \phi(\vec{x},t)}{\left\| \nabla_{\vec{x}} \phi(\vec{x},t) \right\|} \tag{2.315}$$

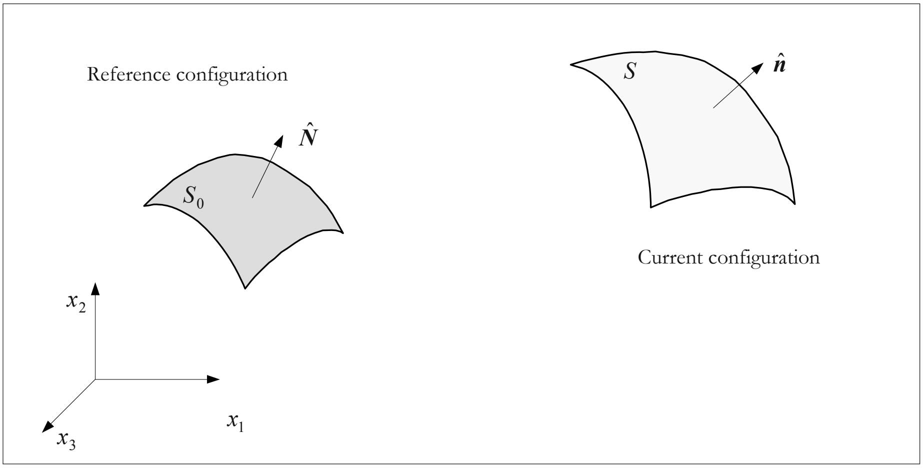


Figure 2.28: Material surface.

The Material Volume

The material volume is a moving volume that is always made up of the same particles, (see Figure 2.29).

2.10.2 The Control Domain

A spatial domain that is fixed in space is denoted by the control domain. Then, a *control surface* is a fixed spatial surface and a *control volume* is a fixed spatial volume, (see Figure 2.29).

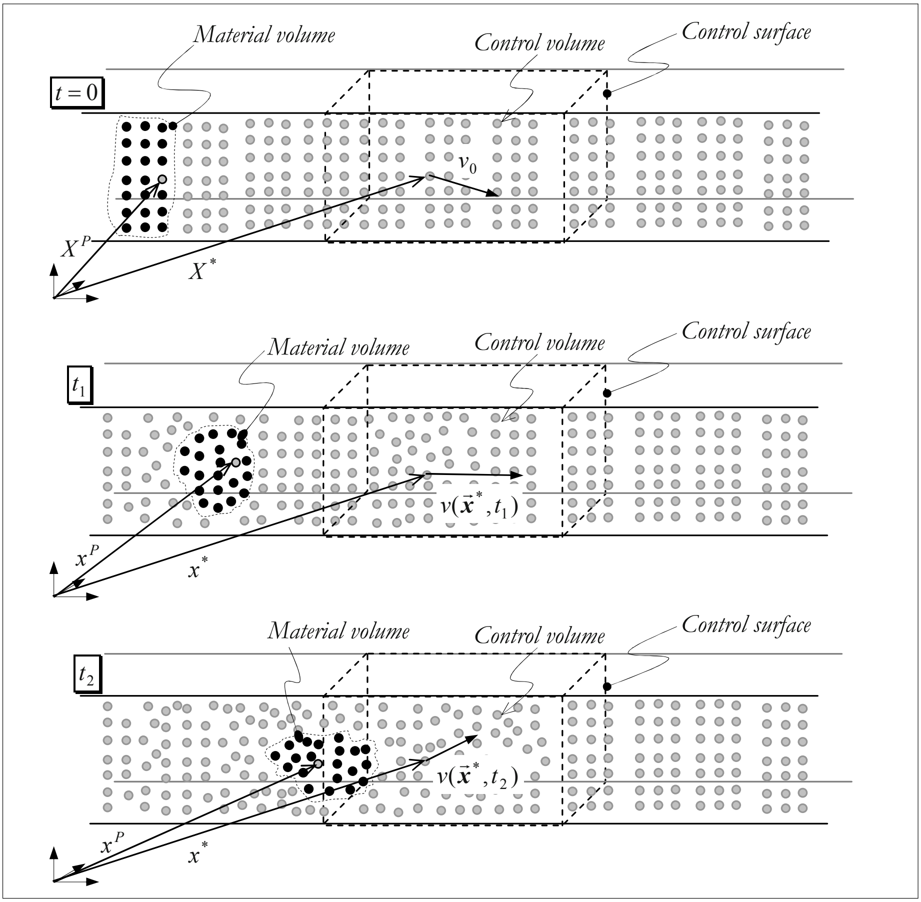


Figure 2.29: Material volume and control volume.

2.11 Transport Equations

Now, we can establish how an intrinsic property of a *material curve*, *material surface* or *material volume*, changes over time. A material curve is always made up of the same particles, so if we consider that $\phi(\vec{x},t)$ is a property of each particle belonging to this material curve as well as being a continuous and differentiable field, we obtain:

$$\frac{D}{Dt}\int_C \phi \, d\vec{x} = \int_C \left(\dot{\phi} \, d\vec{x} + \phi \frac{D}{Dt}(d\vec{x}) \right) = \int_C \left(\dot{\phi} \, d\vec{x} + \phi \boldsymbol{\ell} \cdot d\vec{x} \right) = \int_C \left[\left(\dot{\phi} \, \mathbf{1} + \phi \boldsymbol{\ell} \right) \cdot d\vec{x} \right] \tag{2.316}$$

Likewise, for a material surface we have:

$$\frac{D}{Dt}\int_S \phi\, d\vec{a} = \int_S \left(\dot{\phi}\, d\vec{a} + \phi \frac{D}{Dt}(d\vec{a})\right) = \int_S \left(\dot{\phi}\, d\vec{a} + \phi\left(\mathrm{Tr}(\mathbf{D})d\vec{a} - \boldsymbol{\ell}^T \cdot d\vec{a}\right)\right)$$
$$= \int_S \left[\left(\dot{\phi}\,\mathbf{1} + \phi\mathrm{Tr}(\mathbf{D})\mathbf{1} - \phi\boldsymbol{\ell}^T\right)\cdot d\vec{a}\right] \tag{2.317}$$

Similarly, for a material volume we have:

$$\begin{aligned}\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV &= \int_V \left[dV \frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)\frac{D}{Dt}(dV)\right] \\ &= \int_V \left[dV \frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\, dV\right] \\ &= \int_V \left[\frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right]dV\end{aligned} \tag{2.318}$$

These equations are known as *transport equations*:

Transport Equations

a) Material curve $\boxed{\frac{D}{Dt}\int_C \phi\, d\vec{x} = \int_C \left(\dot{\phi}\,\mathbf{1} + \phi\boldsymbol{\ell}\right)\cdot d\vec{x}}$

b) Material surface $\boxed{\frac{D}{Dt}\int_S \phi\, d\vec{a} = \int_S \left(\dot{\phi}\,\mathbf{1} + \phi\mathrm{Tr}(\mathbf{D})\mathbf{1} - \phi\boldsymbol{\ell}^T\right)\cdot d\vec{a}}$ (2.319)

c) Material volume $\boxed{\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV = \int_V \left[\frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right]dV}$

Next, we can apply the material time derivative to the equation in (2.319)(c) to obtain:

$$\begin{aligned}\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV &= \int_V \left[\frac{D}{Dt}\phi(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right]dV \\ &= \int_V \left[\frac{\partial}{\partial t}\phi(\vec{\boldsymbol{x}},t) + \frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial\vec{\boldsymbol{x}}}\cdot\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) + \phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right]dV \\ &= \int_V \left[\frac{\partial}{\partial t}\phi(\vec{\boldsymbol{x}},t)\right]dV + \int_V \left[\frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial\vec{\boldsymbol{x}}}\cdot\vec{\boldsymbol{v}} + \phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right]dV \\ &= \int_V \left[\frac{\partial}{\partial t}\phi(\vec{\boldsymbol{x}},t)\right]dV + \int_V \left[\nabla_{\vec{x}}\cdot(\phi\vec{\boldsymbol{v}})\right]dV\end{aligned} \tag{2.320}$$

Now we apply the divergence theorem to the second integral on the right side of the equation and we obtain:

$$\frac{D}{Dt}\int_V \phi(\vec{\boldsymbol{x}},t)dV = \int_V \underbrace{\frac{\partial\phi(\vec{\boldsymbol{x}},t)}{\partial t}dV}_{\text{local}} + \overbrace{\int_S \underbrace{(\phi\vec{\boldsymbol{v}})}_{\text{flux of }\phi}\cdot\hat{\boldsymbol{n}}\,dS}^{\text{flux of }\phi\text{ acrossing the surface S}} \tag{2.321}$$

As we can verify the volume integral on the right of the equation is a control volume and the surface integral is a control surface. The term $(\phi\vec{\boldsymbol{v}})$, (as discussed in Chapter 5), represents the property flux (ϕ) that crosses the control surface. We can also see that only the normal component of the flux ($\vec{\mathbf{q}}_n$) crosses the surface, whereas the tangential component remains on the surface. Moreover, when the property is mass density, the equation in (2.320) is known as the mass continuity equation, which is also discussed in Chapter 5.

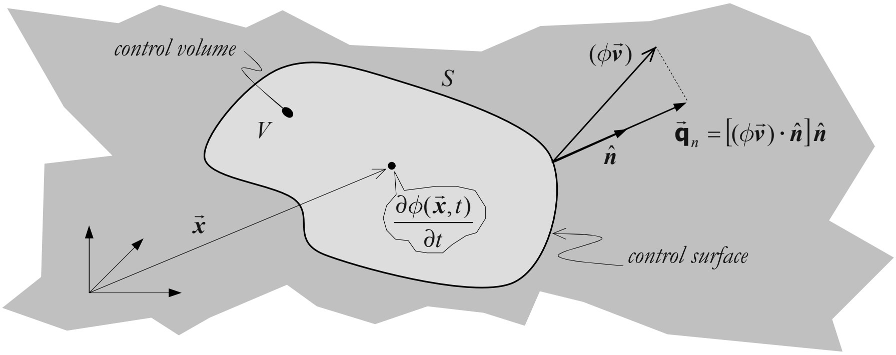


Figure 2.30: Surface and volume control.

2.12 Circulation and Vorticity

Let us consider a circuit Γ (closed curve) where the Eulerian velocity line integral around the circuit Γ is given by:

$$\boxed{C(\Gamma) = \oint_\Gamma \vec{v}(\vec{x},t)\cdot d\vec{x}} \quad \textit{Circulation around } \Gamma \tag{2.322}$$

The line integral (2.322) is denoted by the circulation around Γ, (Chadwick (1976)) and by using the Stokes' Theorem we can obtain:

$$\begin{aligned} C(\Gamma) = \oint_\Gamma \vec{v}(\vec{x},t)\cdot d\vec{x} &= \int_\Omega (\vec{\nabla}_{\vec{x}} \wedge \vec{v})\cdot\hat{n}\,d\Omega \\ &= \int_\Omega \vec{\boldsymbol{\omega}}\cdot\hat{n}\,d\Omega \end{aligned} \tag{2.323}$$

where $\vec{\boldsymbol{\omega}}$ is the vorticity vector defined in (2.87) for which the physical interpretation in (2.323) follows. Let us consider the region Ω, which consists of molecules subject to circulatory motion, (see Figure 2.31). The equation in (2.323) ensures that the vorticity contribution of all molecules in the region Ω is equal to the total circulation of the circuit Γ (the boundary of Ω). We can also verify that if the circulation around any closed curve is zero, it holds that $\vec{\nabla}_{\vec{x}} \wedge \vec{v} = \vec{\mathbf{0}}$. If this is the case, then the flow is said to be irrotational.

The rate of change of circulation around Γ can be obtained directly from (2.319)(b) by the following change of variables $\phi \leftarrow \vec{\boldsymbol{\omega}}(\vec{x},t)$, $d\vec{a} = d\vec{\Omega}$, so, we obtain:

$$\frac{D}{Dt}C(\Gamma) = \frac{D}{Dt}\int_\Omega \vec{\boldsymbol{\omega}}(\vec{x},t)\cdot d\vec{\Omega} = \int_\Omega \left[\left(\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}}\mathrm{Tr}(\mathbf{D}) - \vec{\boldsymbol{\omega}}\cdot\boldsymbol{\ell}^T\right)\cdot d\vec{\Omega}\right] \tag{2.324}$$

The motion is said to be *circulation preserving* if and only if $\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}}\mathrm{Tr}(\mathbf{D}) - \vec{\boldsymbol{\omega}}\cdot\boldsymbol{\ell}^T = \vec{\mathbf{0}}$, where the following is satisfied:

$$
\begin{aligned}
\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}}\mathrm{Tr}(\mathbf{D}) - \vec{\boldsymbol{\omega}} \cdot \boldsymbol{\ell}^T &= \dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}}\mathrm{Tr}(\boldsymbol{\ell}) - \boldsymbol{\ell} \cdot \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&= \dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}}\frac{\dot{J}}{J} + \boldsymbol{F} \cdot \dot{\boldsymbol{F}}^{-1} \cdot \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&\Rightarrow (J\boldsymbol{F}^{-1}) \cdot \dot{\vec{\boldsymbol{\omega}}} + (J\boldsymbol{F}^{-1}) \cdot \vec{\boldsymbol{\omega}}\frac{\dot{J}}{J} + (J\boldsymbol{F}^{-1}) \cdot \boldsymbol{F} \cdot \dot{\boldsymbol{F}}^{-1} \cdot \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&\Rightarrow J\boldsymbol{F}^{-1} \cdot \dot{\vec{\boldsymbol{\omega}}} + \dot{J}\boldsymbol{F}^{-1} \cdot \vec{\boldsymbol{\omega}} + J \cdot \dot{\boldsymbol{F}}^{-1} \cdot \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\
&\Rightarrow \frac{D}{Dt}\left(J\boldsymbol{F}^{-1} \cdot \vec{\boldsymbol{\omega}}\right) = \vec{\mathbf{0}}
\end{aligned}
\qquad (2.325)
$$

where we have used $\dot{J} = J\mathrm{Tr}(\boldsymbol{\ell})$, $\boldsymbol{\ell} = -\boldsymbol{F} \cdot \dot{\boldsymbol{F}}^{-1}$. From the equation in (2.325) we can conclude that circulation which preserves the particle vorticity can be expressed as:

$$
\boxed{\vec{\boldsymbol{\omega}}_0 = J\boldsymbol{F}^{-1} \cdot \vec{\boldsymbol{\omega}} \quad \Rightarrow \quad \vec{\boldsymbol{\omega}} = \frac{1}{J}\boldsymbol{F} \cdot \vec{\boldsymbol{\omega}}_0} \quad \textit{Cauchy's vorticity formula} \qquad (2.326)
$$

which is known as Cauchy's vorticity formula, and links the vorticity between the reference and current configurations with the circulation preserving motion, (Chadwick (1976)).

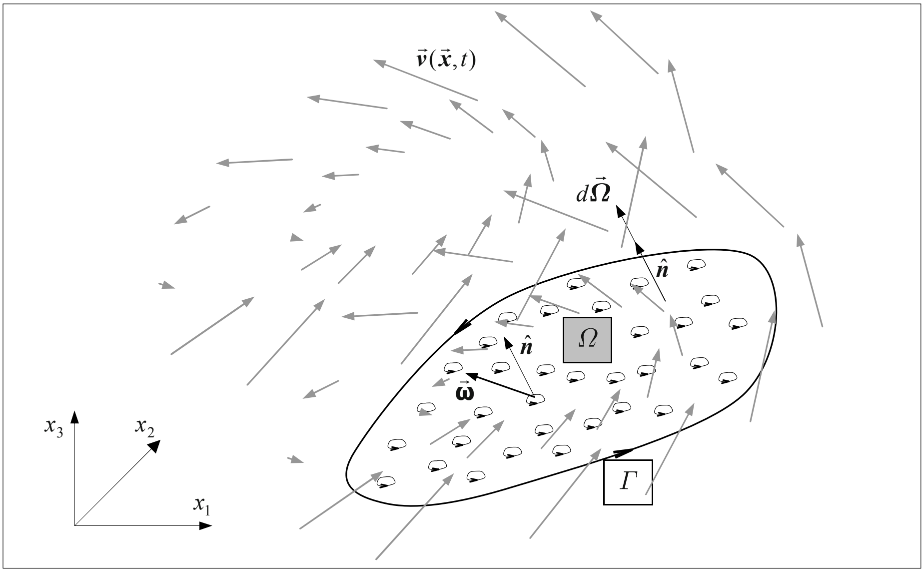


Figure 2.31: Circulation and vorticity.

2.13 Motion Decomposition: Volumetric and Isochoric Motions

Sometimes, when establishing certain constitutive equations, it may be convenient (at the time of the numerical implementation) to separate the motion into a volumetric and an isochoric part. Then, we can introduce the multiplicative decomposition of the deformation gradient as:

$$\boldsymbol{F} = \boldsymbol{F}^{iso} \cdot \boldsymbol{F}^{vol} = \tilde{\boldsymbol{F}} \cdot \boldsymbol{F}^{vol} \tag{2.327}$$

where $\tilde{\boldsymbol{F}} \equiv \boldsymbol{F}^{iso}$ is the transformation characterized by a volume-preserving (isochoric) transformation, and $\boldsymbol{F}^{vol}$ characterizes a volume-changing (dilatational) transformation, (see Figure 2.32), where:

$$\tilde{\boldsymbol{F}} = J^{\frac{-1}{3}} \boldsymbol{F} \quad ; \quad \boldsymbol{F}^{vol} = J^{\frac{1}{3}} \mathbf{1} \tag{2.328}$$

According to Simo&Hughes (1998) this decomposition was originally introduced by Flory in 1961.

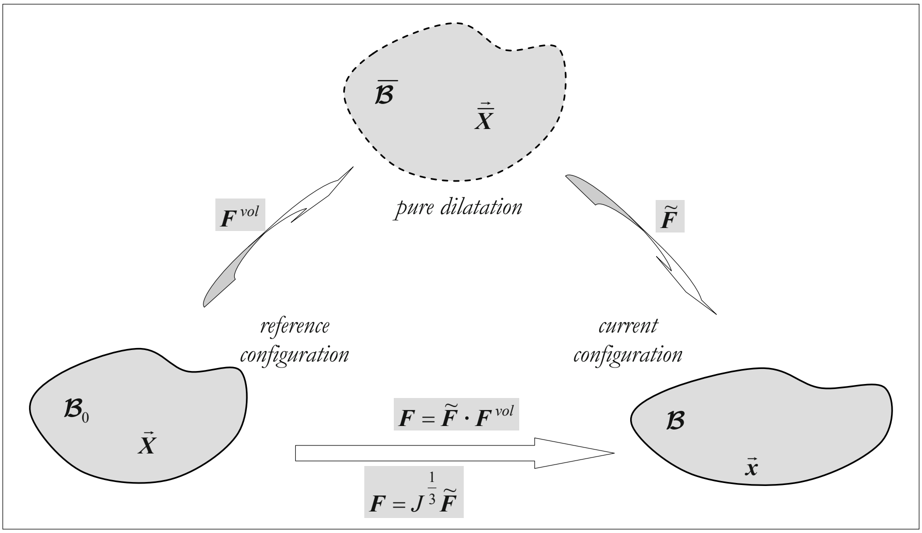


Figure 2.32: Multiplicative decomposition of the deformation gradient into a volumetric and an isochoric part.

If we consider the right Cauchy-Green deformation tensor, $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$, the isochoric part of $\boldsymbol{C}$ is represented by $\tilde{\boldsymbol{C}}$, and the volumetric part by $\boldsymbol{C}^{vol}$. There are given respectively by:

$$\begin{aligned}
\tilde{\boldsymbol{C}} &= \tilde{\boldsymbol{F}}^T \cdot \tilde{\boldsymbol{F}} \\
&= \left(J^{\frac{-1}{3}} \boldsymbol{F} \right)^T \cdot \left(J^{\frac{-1}{3}} \boldsymbol{F} \right) \\
&= J^{\frac{-1}{3}} J^{\frac{-1}{3}} \underbrace{\boldsymbol{F}^T \cdot \boldsymbol{F}}_{\boldsymbol{C}} \\
&= J^{\frac{-2}{3}} \boldsymbol{C}
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{C}^{vol} &= \boldsymbol{F}^{vol^T} \cdot \boldsymbol{F}^{vol} \\
&= \left(J^{\frac{1}{3}} \mathbf{1} \right)^T \cdot \left(J^{\frac{1}{3}} \mathbf{1} \right) \\
&= J^{\frac{1}{3}} J^{\frac{1}{3}} \underbrace{\mathbf{1}^T \cdot \mathbf{1}}_{\mathbf{1}} \\
&= J^{\frac{2}{3}} \mathbf{1}
\end{aligned}
\tag{2.329}$$

Similarly, the isochoric part of the left Cauchy-Green deformation tensor $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ is expressed as:

$$\tilde{\boldsymbol{b}} = J^{\frac{-2}{3}} \boldsymbol{b} \quad \Rightarrow \quad \boldsymbol{b} = (J^{\frac{2}{3}} \mathbf{1}) \cdot \tilde{\boldsymbol{b}} = \boldsymbol{b}^{vol} \cdot \tilde{\boldsymbol{b}} \tag{2.330}$$

where $\boldsymbol{b}^{vol} = (J^{\frac{2}{3}}\mathbf{1})$ is the volumetric part of $\boldsymbol{b}$.

We can now obtain the material time derivative of $\tilde{\boldsymbol{C}} = J^{\frac{-2}{3}}\boldsymbol{C}$:

$$\dot{\tilde{\boldsymbol{C}}} = -\frac{2}{3}J^{\frac{-5}{3}}\dot{J}\boldsymbol{C} + J^{\frac{-2}{3}}\dot{\boldsymbol{C}} \tag{2.331}$$

Taking into account that $\dot{J} = J\,\mathrm{Tr}(\mathbf{D})$ and $\dot{\boldsymbol{C}} = 2\boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F}$, the equation in (2.331) becomes:

$$\begin{aligned}
\dot{\tilde{\boldsymbol{C}}} &= -\frac{2}{3}J^{\frac{-5}{3}}\dot{J}\boldsymbol{C} + J^{\frac{-2}{3}}\dot{\boldsymbol{C}} = -\frac{2}{3}J^{\frac{-5}{3}}J\,\mathrm{Tr}(\mathbf{D})\underbrace{\boldsymbol{C}}_{=\boldsymbol{F}^T\cdot\boldsymbol{F}} + J^{\frac{-2}{3}}2\boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F} \\
&= 2J^{\frac{-2}{3}}\left(-\frac{1}{3}\mathrm{Tr}(\mathbf{D})\boldsymbol{F}^T\cdot\mathbf{1}\cdot\boldsymbol{F} + \boldsymbol{F}^T\cdot\mathbf{D}\cdot\boldsymbol{F}\right) = 2J^{\frac{-2}{3}}\boldsymbol{F}^T\cdot\left(\mathbf{D} - \frac{1}{3}\mathrm{Tr}(\mathbf{D})\mathbf{1}\right)\cdot\boldsymbol{F} \\
&= 2\tilde{\boldsymbol{F}}^T\cdot\mathbf{D}^{dev}\cdot\boldsymbol{F}
\end{aligned} \tag{2.332}$$

where it holds that $\mathbf{D} = \mathbf{D}^{sph} + \mathbf{D}^{dev} = \frac{1}{3}\mathrm{Tr}(\mathbf{D})\mathbf{1} + \mathbf{D}^{dev}$. In addition, it holds that:

$$\begin{aligned}
\dot{\tilde{\boldsymbol{C}}}:\boldsymbol{C}^{-1} &= \left[2J^{\frac{-2}{3}}\boldsymbol{F}^T\cdot\mathbf{D}^{dev}\cdot\boldsymbol{F}\right]:\left[\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\right] = \left[2J^{\frac{-2}{3}}F_{ki}\mathrm{D}^{dev}_{kp}F_{pj}\right]\left[F^{-1}_{iq}F^{-1}_{jq}\right] \\
&= 2J^{\frac{-2}{3}}F_{ki}F^{-1}_{iq}\mathrm{D}^{dev}_{kp}F_{pj}F^{-1}_{jq} = 2J^{\frac{-2}{3}}\delta_{kq}\mathrm{D}^{dev}_{kp}\delta_{pq} = 2J^{\frac{-2}{3}}\mathrm{D}^{dev}_{qq} = 2J^{\frac{-2}{3}}\mathrm{Tr}(\mathbf{D}^{dev}) \\
&= 0
\end{aligned} \tag{2.333}$$

2.13.1 The Principal Invariants

The principal invariants of $\tilde{\boldsymbol{F}}$ and $\boldsymbol{F}^{vol}$ are given, respectively, by:

$$\begin{cases}
I_{\tilde{\boldsymbol{F}}} = \mathrm{Tr}(\tilde{\boldsymbol{F}}) = \mathrm{Tr}\left(J^{\frac{-1}{3}}\boldsymbol{F}\right) = J^{\frac{-1}{3}}\mathrm{Tr}(\boldsymbol{F}) = J^{\frac{-1}{3}}I_{\boldsymbol{F}} \\
I\!I_{\tilde{\boldsymbol{F}}} = \frac{1}{2}\left[I^2_{\tilde{\boldsymbol{F}}} - \mathrm{Tr}(\tilde{\boldsymbol{F}}^2)\right] = \frac{1}{2}\left[J^{\frac{-2}{3}}I^2_{\boldsymbol{F}} - \mathrm{Tr}\left(J^{\frac{-2}{3}}\boldsymbol{F}^2\right)\right] = J^{\frac{-2}{3}}\frac{1}{2}\left[I^2_{\boldsymbol{F}} - \mathrm{Tr}(\boldsymbol{F}^2)\right] = J^{\frac{-2}{3}}I\!I_{\boldsymbol{F}} \\
I\!I\!I_{\tilde{\boldsymbol{F}}} = \det(\tilde{\boldsymbol{F}}) = \det\left(J^{\frac{-1}{3}}\boldsymbol{F}\right) = \left(J^{\frac{-1}{3}}\right)^3\det(\boldsymbol{F}) = J^{-1}J = 1
\end{cases} \tag{2.334}$$

and

$$\begin{cases}
I_{\boldsymbol{F}^{vol}} = \mathrm{Tr}(\boldsymbol{F}^{vol}) = \mathrm{Tr}\left(J^{\frac{1}{3}}\mathbf{1}\right) = 3J^{\frac{1}{3}} \\
I\!I_{\boldsymbol{F}^{vol}} = \frac{1}{2}\left[I^2_{\boldsymbol{F}^{vol}} - \mathrm{Tr}(\boldsymbol{F}^{vol^2})\right] = \frac{1}{2}\left[9J^{\frac{2}{3}} - 3J^{\frac{2}{3}}\right] = 3J^{\frac{2}{3}} \\
I\!I\!I_{\boldsymbol{F}^{vol}} = \det(\boldsymbol{F}^{vol}) = \det\left(J^{\frac{1}{3}}\mathbf{1}\right) = J
\end{cases} \tag{2.335}$$

If we consider that the Jacobian determinant can be expressed as $J = \sqrt{\mathit{III}_{\boldsymbol{C}}}$, the principal invariants of $\tilde{\boldsymbol{C}}$ are:

$$I_{\tilde{\boldsymbol{C}}} = \mathrm{Tr}(\tilde{\boldsymbol{C}}) = \mathrm{Tr}\left(J^{\frac{-2}{3}}\boldsymbol{C}\right) = J^{\frac{-2}{3}} I_{\boldsymbol{C}} = \frac{I_{\boldsymbol{C}}}{\sqrt[3]{\mathit{III}_{\boldsymbol{C}}}} \tag{2.336}$$

$$\mathit{II}_{\tilde{\boldsymbol{C}}} = \frac{1}{2}\left[I_{\tilde{\boldsymbol{C}}}^2 - \mathrm{Tr}(\tilde{\boldsymbol{C}}^2)\right] = \frac{1}{2}\left[J^{\frac{-4}{3}} I_{\boldsymbol{C}}^2 - \mathrm{Tr}\left(J^{\frac{-4}{3}}\boldsymbol{C}^2\right)\right] = J^{\frac{-4}{3}}\mathit{II}_{\boldsymbol{C}} = \frac{\mathit{II}_{\boldsymbol{C}}}{\sqrt[3]{\mathit{III}_{\boldsymbol{C}}^2}} \tag{2.337}$$

$$\mathit{III}_{\tilde{\boldsymbol{C}}} = \det(\tilde{\boldsymbol{C}}) = \det\left(J^{\frac{-2}{3}}\boldsymbol{C}\right) = \left(J^{\frac{-2}{3}}\right)^3 \det(\boldsymbol{C}) = J^{-2}J^2 = 1 \tag{2.338}$$

Likewise, we obtain:

$$I_{\tilde{\boldsymbol{b}}} = \frac{I_{\boldsymbol{b}}}{\sqrt[3]{\mathit{III}_{\boldsymbol{b}}}} \quad ; \quad \mathit{II}_{\tilde{\boldsymbol{b}}} = \frac{\mathit{II}_{\boldsymbol{b}}}{\sqrt[3]{\mathit{III}_{\boldsymbol{b}}^2}} \quad ; \quad \mathit{III}_{\tilde{\boldsymbol{b}}} = 1 \tag{2.339}$$

Taking into account that $I_{\boldsymbol{b}} = I_{\boldsymbol{C}}$, $\mathit{II}_{\boldsymbol{b}} = \mathit{II}_{\boldsymbol{C}}$, $\mathit{III}_{\boldsymbol{b}} = \mathit{III}_{\boldsymbol{C}}$, we can also conclude that:

$$I_{\tilde{\boldsymbol{b}}} = I_{\tilde{\boldsymbol{C}}} \quad ; \quad \mathit{II}_{\tilde{\boldsymbol{b}}} = \mathit{II}_{\tilde{\boldsymbol{C}}} \quad ; \quad \mathit{III}_{\tilde{\boldsymbol{b}}} = \mathit{III}_{\tilde{\boldsymbol{C}}} \tag{2.340}$$

2.14 The Small Deformation Regime

2.14.1 Introduction

Given a quadratic ($y^C = ax^2 + bx + c$) and a linear ($y^L = bx + c$) function, we can ask the following question: In which situation are these two functions approximately the same? To answer this, let us consider the following numerical values for the constants $a = 2$, $b = 1$, $c = 0$. We can verify that for very small values of x, these functions are very close to each other, $x \ll 1 \Rightarrow y^C \approx y^L$, (see Figure 2.33). That is, if x is very small compared with unity, the quadratic or higher terms can be discarded.

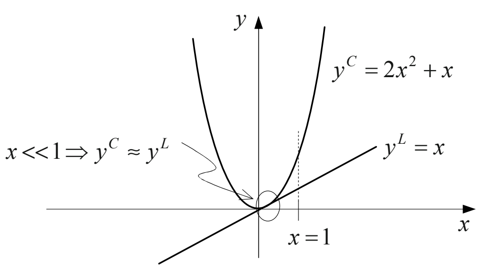


Figure 2.33: Linear and quadratic functions.

We can extend this reasoning to tensors. For example, let us consider the material strain tensor $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{J} + \boldsymbol{J}^T + \boldsymbol{J}^T \cdot \boldsymbol{J})$. If the material displacement gradient components J_{ij} are much smaller than the unity, *i.e.* $\boldsymbol{J}_{ij} \ll 1$, the quadratic terms $(\boldsymbol{J}^T \cdot \boldsymbol{J})$ are even smaller and can be discarded.

There are many cases in engineering in which the displacement gradient components are small compared with the unity:

$$J_{ij} = \frac{\partial \mathrm{u}_i}{\partial X_j} \ll 1 \tag{2.341}$$

The approximation in (2.341) takes place when the continuum (structure) is made up of very rigid material and the loads (forces) to which theses structures are subjected produce small displacements. In such cases, motion can be approximated by the *infinitesimal strain theory,* also known as the *small deformation theory*, or the *small displacement theory*.

2.14.2 Infinitesimal Strain and Spin Tensors

As previously discussed, if the displacement gradient is already small, higher-order terms are even smaller. In this case, the Green-Lagrange strain tensor can be approached by means of the *linear Green-Lagrange strain tensor* (E^L_{ij}), which is defined as:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i} + \frac{\partial \mathrm{u}_k}{\partial X_i}\frac{\partial \mathrm{u}_k}{\partial X_j}\right) \xrightarrow[\textit{regime}]{\textit{small deformation}} E^L_{ij} \approx \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i}\right) \tag{2.342}$$

or in tensorial notation:

$$\begin{aligned} \boldsymbol{E}^L &= \frac{1}{2}\left(\vec{\mathbf{u}} \otimes \nabla_{\vec{X}} + \nabla_{\vec{X}} \otimes \vec{\mathbf{u}}\right) = \frac{1}{2}\left[\nabla_{\vec{X}}\vec{\mathbf{u}} + (\nabla_{\vec{X}}\vec{\mathbf{u}})^T\right] = \frac{1}{2}\left(\boldsymbol{J} + \boldsymbol{J}^T\right) \\ &= \nabla^{sym}\vec{\mathbf{u}}(\vec{X},t) \equiv \left[\nabla_{\vec{X}}\vec{\mathbf{u}}(\vec{X},t)\right]^{sym} \end{aligned} \tag{2.343}$$

If we now consider the Almansi strain tensor in terms of the gradient of the displacements, (see Eq. (2.128)), we can define the *linear Almansi strain tensor*, e^L_{ij}, as:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i} + \frac{\partial \mathrm{u}_k}{\partial x_i}\frac{\partial \mathrm{u}_k}{\partial x_j}\right) \xrightarrow[\textit{regime}]{\textit{small deformation}} e^L_{ij} \approx \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) \tag{2.344}$$

or in tensorial notation:

$$\begin{aligned} \boldsymbol{e}^L &= \frac{1}{2}\left(\vec{\mathbf{u}} \otimes \nabla_{\vec{x}} + \nabla_{\vec{x}} \otimes \vec{\mathbf{u}}\right) = \frac{1}{2}\left[\nabla_{\vec{x}}\vec{\mathbf{u}} + (\nabla_{\vec{x}}\vec{\mathbf{u}})^T\right] = \frac{1}{2}\left(\boldsymbol{j} + \boldsymbol{j}^T\right) \\ &= \nabla^{sym}\vec{\mathbf{u}}(\vec{x},t) \equiv \left[\nabla_{\vec{x}}\vec{\mathbf{u}}(\vec{x},t)\right]^{sym} \end{aligned} \tag{2.345}$$

NOTE: To verify that the material time derivative of the linear Almansi strain tensor $\dot{\boldsymbol{e}}^L$ is equal to the rate-of-deformation tensor:

$$\begin{aligned} \dot{e}^L_{ij} &= \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial t}\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial}{\partial t}\frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\frac{\partial \mathrm{u}_i}{\partial t} + \frac{\partial}{\partial x_i}\frac{\partial \mathrm{u}_j}{\partial t}\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial x_j}v_i + \frac{\partial}{\partial x_i}v_j\right) = \mathrm{D}_{ij} \end{aligned} \tag{2.346}$$

■

If both the displacement gradient and the displacement are small, it means there is very little difference between the spatial and material configurations, so the linear strain tensors can be considered equal, *i.e.*:

$$\boldsymbol{E}^L(\vec{\boldsymbol{X}} \approx \vec{\boldsymbol{x}}, t) \approx \boldsymbol{e}^L(\vec{\boldsymbol{x}} \approx \vec{\boldsymbol{X}}, t) = \boldsymbol{\varepsilon}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}\left(\boldsymbol{J} + \boldsymbol{J}^T\right) = \nabla^{sym}\vec{\mathbf{u}} \tag{2.347}$$

which defines the *infinitesimal strain tensor:*

$$\boxed{\boldsymbol{\varepsilon}(\vec{\boldsymbol{x}}, t) = \nabla^{sym}\vec{\mathbf{u}} = \varepsilon_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j} \quad \textit{The infinitesimal strain tensor} \tag{2.348}$$

Explicitly, the components of $\boldsymbol{\varepsilon}$ are given by:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right)$$

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathrm{u}_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_2} + \dfrac{\partial \mathrm{u}_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_3} + \dfrac{\partial \mathrm{u}_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_2} + \dfrac{\partial \mathrm{u}_2}{\partial x_1}\right) & \dfrac{\partial \mathrm{u}_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_2}{\partial x_3} + \dfrac{\partial \mathrm{u}_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_3} + \dfrac{\partial \mathrm{u}_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_2}{\partial x_3} + \dfrac{\partial \mathrm{u}_3}{\partial x_2}\right) & \dfrac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} \tag{2.349}$$

The displacement gradient $\nabla\vec{\mathbf{u}}$ can be split into a symmetric and antisymmetric part as:

$$\nabla\vec{\mathbf{u}} = \frac{1}{2}\left[\nabla\vec{\mathbf{u}} + (\nabla\vec{\mathbf{u}})^T\right] + \frac{1}{2}\left[\nabla\vec{\mathbf{u}} - (\nabla\vec{\mathbf{u}})^T\right] = \nabla^{sym}\vec{\mathbf{u}} + \nabla^{skew}\vec{\mathbf{u}} = \boldsymbol{\varepsilon} + \boldsymbol{\omega} \tag{2.350}$$

where the symmetric part is the infinitesimal strain tensor, and the antisymmetric part is known as the *infinitesimal spin tensor*, which is also called the *infinitesimal rotation tensor*. For rigid body motion the condition that strain tensor is zero, *i.e.* $\boldsymbol{\varepsilon} = \mathbf{0}$ must be satisfied and for motion characterized only by strain $\boldsymbol{\omega} = \mathbf{0}$ must be true.

Notice that, the tensor $\boldsymbol{\varepsilon}$ does not accurately measure strain, since it is affected by rigid body motion. To illustrate this, let us consider that a material body is subjected to a rotation as indicated in Figure 2.34. In this situation, the equations of motion are given by:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = X_1\cos\theta - X_2\sin\theta \\ x_2 = X_1\sin\theta + X_2\cos\theta \\ x_3 = X_3 \end{cases} \tag{2.351}$$

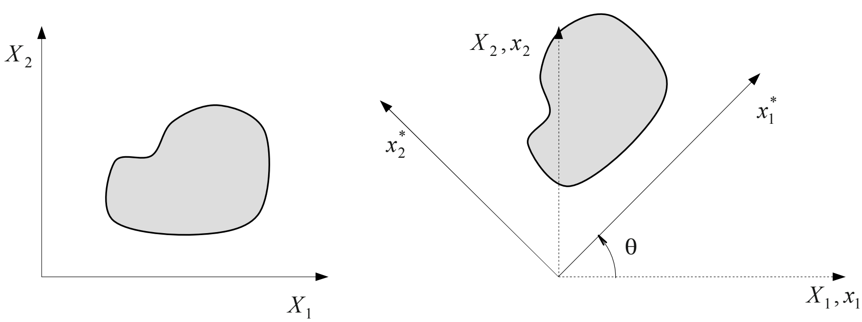


Figure 2.34: A material body subjected to a rotation.

The displacement field, $\mathrm{u}_i = x_i - X_i$, can be obtained as follows:

$$\begin{cases} \mathrm{u}_1 = x_1 - X_1 = X_1(\cos\theta - 1) - X_2 \sin\theta \\ \mathrm{u}_2 = x_2 - X_2 = X_1 \sin\theta + X_2(\cos\theta - 1) \\ \mathrm{u}_3 = x_3 - X_3 = 0 \end{cases} \tag{2.352}$$

If we consider the equation in (2.349) we can obtain the components of the infinitesimal strain tensor as:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial X_j} + \frac{\partial \mathrm{u}_j}{\partial X_i}\right) = \begin{bmatrix} (\cos\theta - 1) & 0 & 0 \\ 0 & (\cos\theta - 1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.353}$$

As we can see ε_{11} and ε_{22} are not equal to zero, but with small rotations it is true that $\cos\theta \approx 1$, thus the terms ε_{11} and ε_{22} are insignificant. As discussed in the chapter on linear elasticity, the small deformation approximation is widely used for various engineering problems and such problems are subjected to *small displacements* and *small rotations*. Note that, for the rigid body motion described in Figure 2.34, the Green-Lagrange strain tensor components are equal to zero. For instance, the component E_{11} works out as:

$$E_{11} = \frac{\partial \mathrm{u}_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial \mathrm{u}_1}{\partial X_1}\right)^2 + \left(\frac{\partial \mathrm{u}_2}{\partial X_1}\right)^2 + \left(\frac{\partial \mathrm{u}_3}{\partial X_1}\right)^2\right] = (\cos\theta - 1) + \frac{1}{2}\left[(\cos\theta - 1)^2 + \sin^2\theta\right] = 0 \tag{2.354}$$

2.14.3 Stretch and Unit Extension

Let us consider the relationship between the stretch and the unit extension in terms of the Green-Lagrange strain tensor $\boldsymbol{E}$, (see equation (2.158)). Then, if we are dealing with a small strain regime ($\boldsymbol{E} \approx \boldsymbol{\varepsilon}$) the stretch and the unity extension become:

$$\begin{cases} \lambda_{\hat{\boldsymbol{m}}} = \sqrt{1 + 2\,\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}}} \\ \varepsilon_{\hat{\boldsymbol{m}}} = \lambda_{\hat{\boldsymbol{m}}} - 1 = \sqrt{1 + 2\,\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}}} - 1 \end{cases} \tag{2.355}$$

Remainder: Taking into account the binomial series we have:

$$\begin{aligned} &(a + x)^n = a^n + n a^{n-1} x + \frac{n(n-1)}{2!} a^{n-2} x^2 + \cdots \\ &\Rightarrow (1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \end{aligned} \tag{2.356}$$

In the event that x is very small, we can discard higher order terms, *i.e.*:

$$(1 + x)^{\frac{1}{2}} \cong 1 + \frac{1}{2}x \quad \blacksquare \tag{2.357}$$

Taking into account the Remainder above, the stretch and the unit extension, in a small strain regime, are represented by:

$$\begin{cases} \lambda_{\hat{\boldsymbol{m}}} = \sqrt{1 + 2\,\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}}} \cong 1 + \hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}} \\ \varepsilon_{\hat{\boldsymbol{m}}} = \lambda_{\hat{\boldsymbol{m}}} - 1 = \sqrt{1 + 2\,\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}}} - 1 \cong \hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}} = \varepsilon_N^{(\hat{\boldsymbol{m}})} \end{cases} \tag{2.358}$$

which verifies that in a small strain regime the unit extension is equal to the normal engineering strain.

2.14.4 Change of Angle

By applying the equation obtained in (2.161) for a small strain regime, $\boldsymbol{E} \approx \boldsymbol{\varepsilon}$, the angle in the reference configuration becomes:

$$\begin{aligned}\cos\theta &= \frac{\hat{\boldsymbol{M}} \cdot (\mathbf{1} + 2\boldsymbol{\varepsilon}) \cdot \hat{\boldsymbol{N}}}{\sqrt{\hat{\boldsymbol{M}} \cdot (\mathbf{1} + 2\boldsymbol{\varepsilon}) \cdot \hat{\boldsymbol{M}}}\ \sqrt{\hat{\boldsymbol{N}} \cdot (\mathbf{1} + 2\boldsymbol{\varepsilon}) \cdot \hat{\boldsymbol{N}}}} \\ &= \frac{\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{N}}}{\sqrt{1 + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}}}\ \sqrt{1 + 2\hat{\boldsymbol{N}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}}} + \frac{2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}}{\sqrt{1 + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}}}\ \sqrt{1 + 2\hat{\boldsymbol{N}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}}}\end{aligned} \tag{2.359}$$

If we consider that $2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}} << 1$, $2\hat{\boldsymbol{N}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} << 1$, we can conclude that:

$$\cos\theta = \hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{N}} + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} = \frac{\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{N}}}{\|\hat{\boldsymbol{M}}\|\|\hat{\boldsymbol{N}}\|} + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} = \cos\Theta + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \tag{2.360}$$

where Θ is the angle between $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{N}}$, and is defined in the initial configuration. The equation in (2.360) could have been obtained directly by applying the equation in (2.164) and by considering that:

$$\cos\theta = \frac{\cos\Theta + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{E} \cdot \hat{\boldsymbol{N}}}{\lambda_{\hat{\boldsymbol{M}}}\lambda_{\hat{\boldsymbol{N}}}} \approx \cos\Theta + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \tag{2.361}$$

Additionally, if we consider that $\Delta\theta = \theta - \Theta$ is the angle variation, then the equation in (2.360) can be rewritten as:

$$\cos(\Theta + \Delta\theta) = \cos\Theta + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \tag{2.362}$$

Moreover, the term on the left of the equation can be rewritten by the following trigonometric relationship:

$$\cos(\Theta + \Delta\theta) = \cos\Theta\underbrace{\cos\Delta\theta}_{\approx 1} - \sin\Theta\underbrace{\sin\Delta\theta}_{\approx\Delta\theta} = \cos\Theta - \Delta\theta\sin\Theta \tag{2.363}$$

in which we have considered that the angle $\Delta\theta$ is very small. If we substitute the previous result into (2.362) we obtain:

$$\begin{aligned}&\cos\Theta - \Delta\theta\sin\Theta = \cos\Theta + 2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \\ &\Rightarrow \Delta\theta\sin\Theta = -2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}\end{aligned} \tag{2.364}$$

Then the angle change for the small strain regime is given by:

$$\boxed{\Delta\theta = \frac{-2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}}{\sin\Theta}} \quad \textit{The angle change for a small strain regime} \tag{2.365}$$

2.14.5 The Physical Interpretation of the Infinitesimal Strain Tensor

First, let us consider the components of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{2.366}$$

The stretch λ_x and the unit extension $\varepsilon_{\boldsymbol{x}}$ according to the $x_1 \equiv x$-direction can be evaluated by considering the equations (2.358) and $\hat{\boldsymbol{M}} = \hat{\mathbf{e}}_1$, thus:

$$\begin{aligned} \lambda_x &\cong 1 + \hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{M}} = 1 + \varepsilon_{xx} \\ \varepsilon_{\boldsymbol{x}} &= \lambda_x - 1 \cong \varepsilon_{xx} \end{aligned} \tag{2.367}$$

Hence, we can conclude that the diagonal terms are related to the unit extension as follows:

$$\varepsilon_{xx} \cong \varepsilon_{\boldsymbol{x}} \quad ; \quad \varepsilon_{yy} \cong \varepsilon_{\boldsymbol{y}} \quad ; \quad \varepsilon_{zz} \cong \varepsilon_{\boldsymbol{z}} \tag{2.368}$$

Now, let us consider that $\hat{\boldsymbol{M}} = \hat{\mathbf{e}}_1$ and $\hat{\boldsymbol{N}} = \hat{\mathbf{e}}_2$, (see Figure 2.35). In this particular case we obtain:

$$\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} = [1 \quad 0 \quad 0] \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \varepsilon_{xy} \tag{2.369}$$

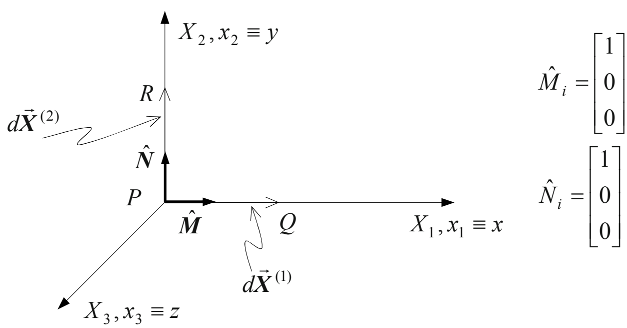


Figure 2.35: Angle change.

Then by using the equation in (2.365) and by considering (2.369) the angle variation becomes:

$$\Delta\theta = \frac{-2\hat{\boldsymbol{M}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}}{\sin\Theta} \quad \Rightarrow \quad \varepsilon_{xy} = \frac{-1}{2}\Delta\theta_{xy} \tag{2.370}$$

Afterwards we can interpret ε_{xy} as the angular distortion between two of the line elements, whereas if we consider all three directions we obtain:

$$\varepsilon_{xy} = \frac{-1}{2}\Delta\theta_{xy} \quad ; \quad \varepsilon_{xz} = \frac{-1}{2}\Delta\theta_{xz} \quad ; \quad \varepsilon_{yz} = \frac{-1}{2}\Delta\theta_{yz} \tag{2.371}$$

2.14.5.1 Engineering Strain

Traditionally in engineering we have the following notation for the axes $x \equiv x_1$, $y \equiv x_2$, $z \equiv x_3$, for the displacement components $u \equiv \mathsf{u}_1$, $v \equiv \mathsf{u}_2$, $w \equiv \mathsf{u}_3$.

Now, let us consider a segment $\overline{AB}$ whose end is displaced by Δu as shown in Figure 2.36. In one dimension, the strain is defined as:

$$\varepsilon = \lim_{\Delta x \to 0} \frac{(u + \Delta u) - u}{\Delta x} = \frac{du}{dx} \tag{2.372}$$

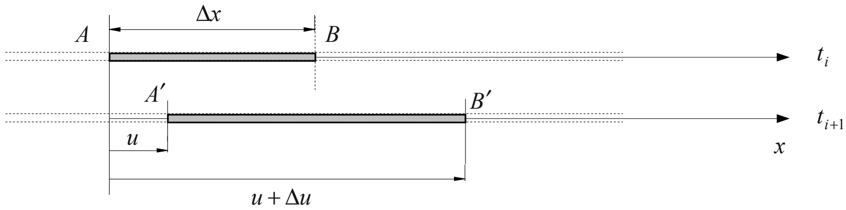


Figure 2.36: Strain in one dimension.

Now let us consider a differential element $dxdy$ and the displacement field, $u(x,y)$ and $v(x,y)$, as shown in Figure 2.37. The normal strains according to the x-direction and y-direction, respectively, are given by:

$$\varepsilon_x = \frac{\left(u + \frac{\partial u}{\partial x} dx\right) - u}{dx} = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\left(v + \frac{\partial v}{\partial y} dy\right) - v}{dy} = \frac{\partial v}{\partial y} \tag{2.373}$$

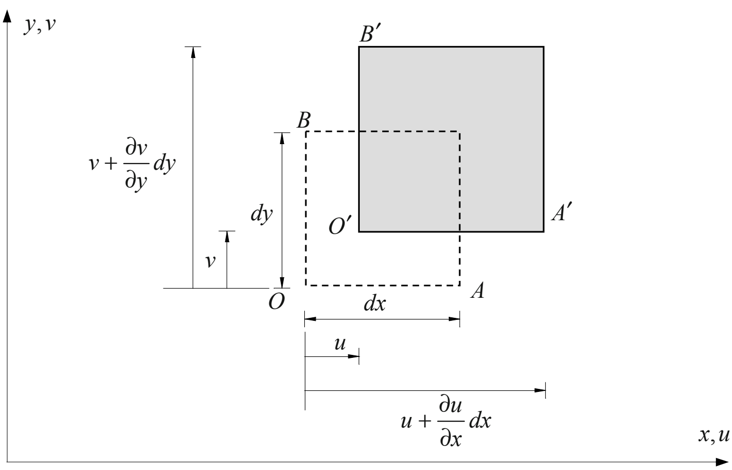


Figure 2.37: Normal strain.

Similarly, if we take into account all three dimensions, the normal strain components are given by:

$$\varepsilon_x = \frac{\partial u(x,y,z)}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v(x,y,z)}{\partial y} \quad ; \quad \varepsilon_z = \frac{\partial w(x,y,z)}{\partial z} \tag{2.374}$$

To find the tangential strain (or the shear strain), let us consider that the differential element is only distorted by the angle shown in Figure 2.38. For small angles it holds that $\tan\theta \cong \theta$, then:

$$\tan\theta_1 \cong \theta_1 = \frac{\partial v}{\partial x} \quad ; \quad \tan\theta_2 \cong \theta_2 = \frac{\partial u}{\partial y} \tag{2.375}$$

So, the shear strain γ_{xy} is defined as:

$$\gamma_{xy} = \theta_1 + \theta_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2.376}$$

Similarly, if we consider the other dimensions, we can obtain:

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad ; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad ; \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{2.377}$$

Note that $\Delta\theta_{xy} = \theta - \Theta = -\gamma_{xy}$, and if we compare that with the equation in (2.371) we can conclude that $\gamma_{xy} = 2\varepsilon_{xy}$. Similarly, we can say that $\gamma_{yz} = 2\varepsilon_{yz}$ and $\gamma_{xz} = 2\varepsilon_{xz}$, thus,

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{z} \end{bmatrix}}_{\text{Engineering Notation}} \tag{2.378}$$

Then, the strain components in terms of displacement in engineering notation are given by:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{z} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \tag{2.379}$$

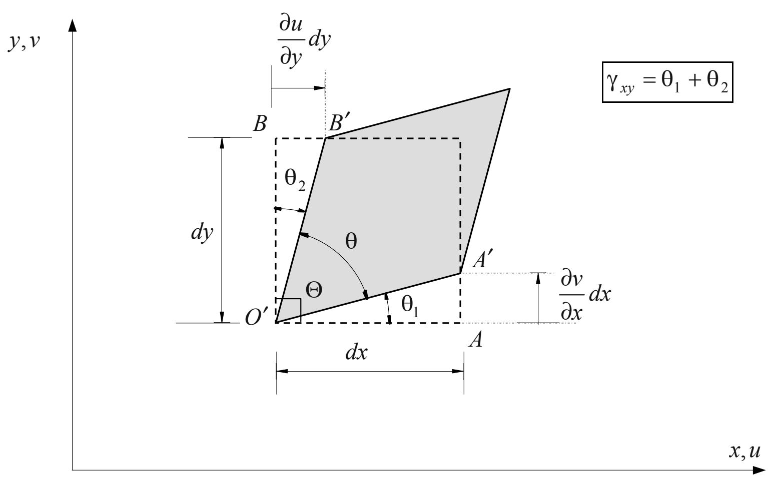


Figure 2.38: Shear strain (small deformation regime).

2.14.6 The Volume Ratio (Dilatation)

Let us consider a cube defined by the line elements dX_1, dX_2, dX_3, (see Figure 2.39).

The volume variation is given by:

$$\begin{aligned}\Delta V = dV - dV_0 &= \lambda_{x_1} dX_1 \lambda_{x_2} dX_2 \lambda_{x_3} dX_3 - dX_1 dX_2 dX_3 \\ &= (1+\varepsilon_{x_1}) dX_1 \,(1+\varepsilon_{x_2}) dX_2 \,(1+\varepsilon_{x_3}) dX_3 - dX_1 dX_2 dX_3 \\ &= \left[(1+\varepsilon_{x_1})(1+\varepsilon_{x_2})(1+\varepsilon_{x_3}) - 1\right] dX_1 dX_2 dX_3\end{aligned} \tag{2.380}$$

In a small strain regime, $\varepsilon_{ij} << 1$, the higher order terms can be discarded without there being any significant change in the outcome. It is also true that with small strains unit extensions are in keeping with the normal components of the strains $\varepsilon_{x_1} = \varepsilon_{11}$, $\varepsilon_{x_2} = \varepsilon_{22}$, $\varepsilon_{x_3} = \varepsilon_{33}$, (see equation in (2.368)), thus:

$$\Delta V = [\varepsilon_{x_1} + \varepsilon_{x_2} + \varepsilon_{x_3}] \; dV_0 = [\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}] \, dV_0 \tag{2.381}$$

In this case, the dilatation (volume ratio) becomes:

$$D_V^L(\vec{x},t) \equiv \varepsilon_V = \frac{\Delta V}{dV_0} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon}) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{2.382}$$

Additionally, if the continuum is incompressible, it is valid that:

$$\frac{\Delta V}{dV_0} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = I_{\boldsymbol{\varepsilon}} = 0 \tag{2.383}$$

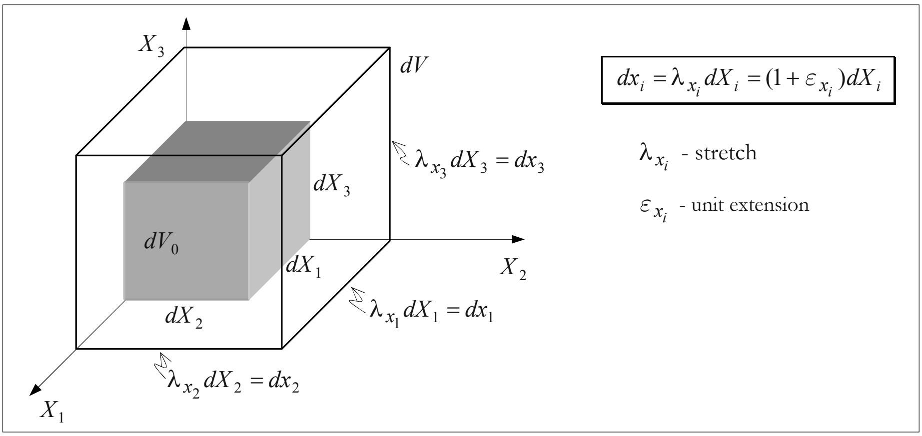


Figure 2.39: Dilatation.

2.14.7 The Plane Strain

When the strain tensor field is independent of any one direction we say that the continuum represents a *plane strain state*. In general, the independent direction is adopted by the x_3-one. So, in this situation, the infinitesimal strain tensor components are given by:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Plane Strain}} \varepsilon'_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix} \qquad (i,j = 1,2) \tag{2.384}$$

The displacement field for a plane strain state is only a function of x_1 and x_2, *i.e.*:

$$\mathrm{u}_1 = \mathrm{u}_1(x_1, x_2) \quad ; \quad \mathrm{u}_2 = \mathrm{u}_2(x_1, x_2) \quad ; \quad \mathrm{u}_3 = C(\text{constant}) \tag{2.385}$$

Problem 2.18: Consider a material body in a small deformation regime, which is subjected to the following displacement field:

$$\mathrm{u}_1 = (-2x_1 + 7x_2)\times 10^{-3} \quad ; \quad \mathrm{u}_2 = (-10x_2 - x_1)\times 10^{-3} \quad ; \quad \mathrm{u}_3 = x_3 \times 10^{-3}$$

a) Find the infinitesimal spin and strain tensor;
b) Find the principal invariants of the infinitesimal strain tensor, as well as the correspondent characteristic equation;
c) Draw the Mohr's circle for strain, and obtain the maximum shear strain;
d) Find the dilatation and the deviatoric infinitesimal strain tensor.

Solution

a) For the displacement gradient we obtain:

$$(\nabla\vec{\mathbf{u}})_{ij} = \frac{\partial \mathrm{u}_i}{\partial x_j} = \begin{bmatrix} \frac{\partial \mathrm{u}_1}{\partial x_1} & \frac{\partial \mathrm{u}_1}{\partial x_2} & \frac{\partial \mathrm{u}_1}{\partial x_3} \\ \frac{\partial \mathrm{u}_2}{\partial x_1} & \frac{\partial \mathrm{u}_2}{\partial x_2} & \frac{\partial \mathrm{u}_2}{\partial x_3} \\ \frac{\partial \mathrm{u}_3}{\partial x_1} & \frac{\partial \mathrm{u}_3}{\partial x_2} & \frac{\partial \mathrm{u}_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -2 & 7 & 0 \\ -1 & -10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times 10^{-3} \qquad \left[\frac{m}{m}\right]$$

In the International System of Units the displacement gradient is dimensionless, *i.e.* $[\nabla\vec{\mathbf{u}}] = \left[\frac{\partial \vec{\mathbf{u}}}{\partial \vec{x}}\right] = \frac{m}{m}$.

As for the infinitesimal spin tensor we obtain:

$$\omega_{ij} = (\nabla^{skew}\vec{\mathbf{u}})_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} - \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \begin{bmatrix} 0 & 4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

Then for the infinitesimal strain tensor we have:

$$\varepsilon_{ij} = (\nabla^{sym}\vec{\mathbf{u}})_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) = \begin{bmatrix} -2 & 3 & 0 \\ 3 & -10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times 10^{-3}$$

b) The principal invariants are defined as $I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon})$, $I\!I_{\boldsymbol{\varepsilon}} = \frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\}$, $I\!I\!I_{\boldsymbol{\varepsilon}} = \det(\boldsymbol{\varepsilon})$, (see Chapter 1). Then, it follows that:

$$I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon}) = (-2 - 10 + 1)\times 10^{-3} = -11\times 10^{-3}$$

$$I\!I_{\boldsymbol{\varepsilon}} = \frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\} = \left(\begin{vmatrix} -10 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -2 & 3 \\ 3 & -10 \end{vmatrix} \right) \times 10^{-6} = -1\times 10^{-6}$$

$$I\!I\!I_{\boldsymbol{\varepsilon}} = \det(\boldsymbol{\varepsilon}) = 11\times 10^{-9}$$

Then, the characteristic determinant is:

$$\left| \begin{bmatrix} -2-\varepsilon & 3 & 0 \\ 3 & -10-\varepsilon & 0 \\ 0 & 0 & 1-\varepsilon \end{bmatrix} \times 10^{-3} \right| = 0$$

whilst the characteristic equation is:

$$\varepsilon^3 - I_{\boldsymbol{\varepsilon}}\varepsilon^2 + I\!I_{\boldsymbol{\varepsilon}}\varepsilon - I\!I\!I_{\boldsymbol{\varepsilon}} = 0 \quad \Rightarrow \quad \varepsilon^3 + 11\times 10^{-3}\varepsilon^2 + \varepsilon \times 11\times 10^{-6} - 11\times 10^{-9} = 0$$

c) To draw the Mohr's circle for strain, (see Appendix A), we need to evaluate the eigenvalues of $\boldsymbol{\varepsilon}$. But, if we take a look at the components of $\boldsymbol{\varepsilon}$ we can verify that $\varepsilon = 1$ is already an eigenvalue associated with the direction $\hat{n}_i = [0 \quad 0 \quad \pm 1]$. So, to obtain the remaining eigenvalues one only need solve the following system:

$$\left\| \begin{bmatrix} -2-\varepsilon & 3 \\ 3 & -10-\varepsilon \end{bmatrix} \times 10^{-3} \right\| = 0 \quad \Rightarrow \quad \varepsilon^2 + 12\times 10^{-3}\varepsilon + 11\times 10^{-6} = 0 \Rightarrow \begin{cases} \varepsilon_1 = -1.0\times 10^{-3} \\ \varepsilon_2 = -11.0\times 10^{-3} \end{cases}$$

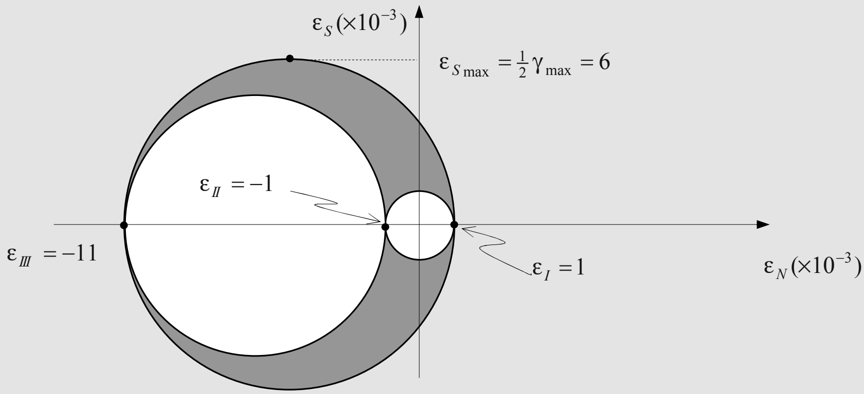
Then by restructuring the eigenvalues such that $\varepsilon_I > \varepsilon_{II} > \varepsilon_{III}$, we obtain:

$$\varepsilon_I = 1.0\times 10^{-3} \quad ; \quad \varepsilon_{II} = -1.0\times 10^{-3} \quad ; \quad \varepsilon_{III} = -11.0\times 10^{-3}$$

Then the maximum shear (tangential) strain is evaluated as follows:

$$\varepsilon_{S\max} = \frac{\varepsilon_I - \varepsilon_{III}}{2} = 6\times 10^{-3}$$

Finally, the Mohr's circle for strain can be depicted as:



d) The volumetric strain (dilatation) - ε_V is:

$$\varepsilon_V = I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon}) = -12\times 10^{-3}$$

The additive decomposition of $\boldsymbol{\varepsilon}$ into a spherical and a deviatoric part is denoted by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}$, where the spherical part is given by:

$$\varepsilon_{ij}^{sph} = \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\delta_{ij} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \times 10^{-3}$$

And, the deviatoric part is given by:

$$\varepsilon_{ij}^{dev} = \varepsilon_{ij} - \varepsilon_{ij}^{sph} = \left(\begin{bmatrix} -2 & 3 & 0 \\ 3 & -10 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} \right) \times 10^{-3} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & -6 & 0 \\ 0 & 0 & 4 \end{bmatrix} \times 10^{-3}$$

2.15 Other Ways to Define Strain

2.15.1 Motivation

In this section we will define other "measures" of strain that may be useful in addressing the problem incrementally. We shall see that the strain tensors defined previously cannot be obtained by adding incremental strains caused by successive motions.

Let us consider a continuum which is subjected to successive configurations, (see Figure 2.40).

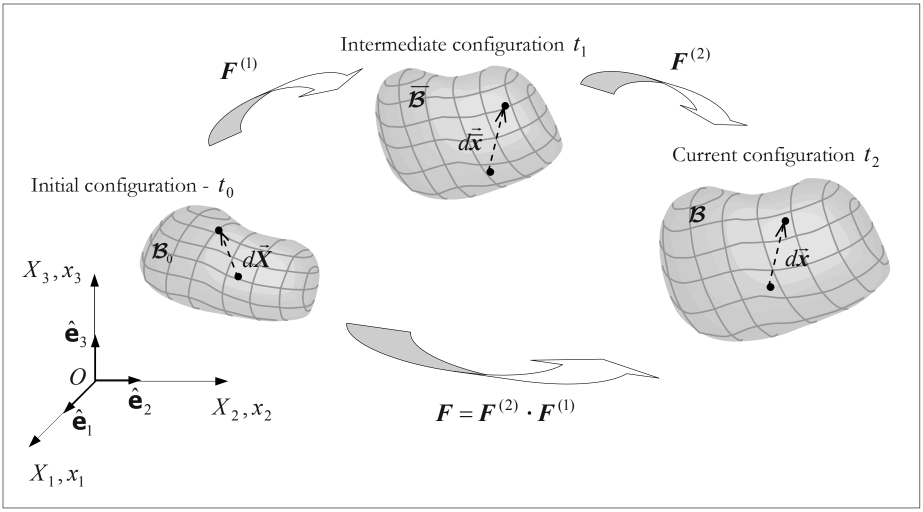


Figure 2.40: Motion defined by successive configurations.

According to Figure 2.40 the following conditions are satisfied:

$$d\vec{x} = \boldsymbol{F} \cdot d\vec{X} \quad ; \quad d\vec{\bar{x}} = \boldsymbol{F}^{(1)} \cdot d\vec{X} \quad ; \quad d\vec{x} = \boldsymbol{F}^{(2)} \cdot d\vec{\bar{x}} \tag{2.386}$$

By substituting the $d\vec{\bar{x}}$ given by the second expression into the third one, we obtain:

$$d\vec{x} = \boldsymbol{F}^{(2)} \cdot d\vec{\bar{x}} = \boldsymbol{F}^{(2)} \cdot \left(\boldsymbol{F}^{(1)} \cdot d\vec{X}\right) = \boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)} \cdot d\vec{X} \tag{2.387}$$

And by comparing (2.387) with $d\vec{x} = \boldsymbol{F} \cdot d\vec{X}$, we can conclude that:

$$\boldsymbol{F} = \boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)} \tag{2.388}$$

The Green-Lagrange strain tensor $\boldsymbol{E}$, (reference configuration $\mathcal{B}_0$) is defined as $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}\right)$. We defined new tensors in a similar fashion because of the transformations $\boldsymbol{F}^{(1)}$ and $\boldsymbol{F}^{(2)}$, *i.e.*:

Initial configuration $\mathcal{B}_0$ *:*	*Intermediate configuration* $\overline{\mathcal{B}}$ *:*	
$\boldsymbol{E}^{(1)}(\vec{X},t) = \frac{1}{2}\left(\boldsymbol{F}^{(1)^T} \cdot \boldsymbol{F}^{(1)} - \mathbf{1}\right)$	$\boldsymbol{E}^{(2)}(\vec{\bar{x}},t) = \frac{1}{2}\left(\boldsymbol{F}^{(2)^T} \cdot \boldsymbol{F}^{(2)} - \mathbf{1}\right)$	(2.389)

The Green-Lagrange strain tensor ($\boldsymbol{E}$) can be written in terms of $\boldsymbol{F}^{(1)}$ and $\boldsymbol{F}^{(2)}$ as:

$$
\begin{aligned}
\boldsymbol{E} &= \frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}\right) = \frac{1}{2}\left[\left(\boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)}\right)^T \cdot \left(\boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)}\right) - \mathbf{1}\right] \\
&= \frac{1}{2}\left(\boldsymbol{F}^{(1)^T} \cdot \boldsymbol{F}^{(2)^T} \cdot \boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)} - \mathbf{1}\right)
\end{aligned} \tag{2.390}
$$

Taking into account the $\boldsymbol{E}^{(2)}$ equation given in (2.389), we obtain $\boldsymbol{F}^{(2)^T} \cdot \boldsymbol{F}^{(2)} = 2\boldsymbol{E}^{(2)} + \mathbf{1}$, and by substituting this into (2.390) we obtain:

$$
\begin{aligned}
\boldsymbol{E} &= \frac{1}{2}\left(\boldsymbol{F}^{(1)^T} \cdot \boldsymbol{F}^{(2)^T} \cdot \boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)} - \mathbf{1}\right) = \frac{1}{2}\left[\boldsymbol{F}^{(1)^T} \cdot \left(2\boldsymbol{E}^{(2)} + \mathbf{1}\right) \cdot \boldsymbol{F}^{(1)} - \mathbf{1}\right] \\
&= \boldsymbol{F}^{(1)^T} \cdot \boldsymbol{E}^{(2)} \cdot \boldsymbol{F}^{(1)} + \underbrace{\frac{1}{2}\left[\boldsymbol{F}^{(1)^T} \cdot \boldsymbol{F}^{(1)} - \mathbf{1}\right]}_{=\boldsymbol{E}^{(1)}} \\
&= \boldsymbol{E}^{(1)} + \boldsymbol{F}^{(1)^T} \cdot \boldsymbol{E}^{(2)} \cdot \boldsymbol{F}^{(1)}
\end{aligned} \tag{2.391}
$$

Thus, we can verify that $\boldsymbol{E} \neq \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)}$, *i.e.* the Green-Lagrange strain tensor is not additive for increments of motions. We can apply the same reasoning to the Almansi strain tensor, $\boldsymbol{e} = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}\right)$, (current configuration). To demonstrate this, we define the following tensors in the intermediate and current configurations because of the transformations $\boldsymbol{F}^{(1)}$ and $\boldsymbol{F}^{(2)}$:

intermediate configuration $\overline{\mathcal{B}}$:	*current configuration* $\mathcal{B}$:
$\boldsymbol{e}^{(1)}(\vec{\bar{x}}, t) = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{(1)^{-T}} \cdot \boldsymbol{F}^{(1)^{-1}}\right)$	$\boldsymbol{e}^{(2)}(\vec{x}, t) = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{(2)^{-T}} \cdot \boldsymbol{F}^{(2)^{-1}}\right)$

(2.392)

The Almansi strain tensor can be written in terms of $\boldsymbol{F}^{(1)}$ and $\boldsymbol{F}^{(2)}$ as:

$$
\begin{aligned}
\boldsymbol{e} &= \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}\right) = \frac{1}{2}\left[\mathbf{1} - \left(\boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)}\right)^{-T} \cdot \left(\boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)}\right)^{-1}\right] \\
&= \frac{1}{2}\left(\mathbf{1} - \left(\boldsymbol{F}^{(1)^T} \cdot \boldsymbol{F}^{(2)^T}\right)^{-1} \cdot \left(\boldsymbol{F}^{(2)} \cdot \boldsymbol{F}^{(1)}\right)^{-1}\right) \\
&= \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{(2)^{-T}} \cdot \boldsymbol{F}^{(1)^{-T}} \cdot \boldsymbol{F}^{(1)^{-1}} \cdot \boldsymbol{F}^{(2)^{-1}}\right)
\end{aligned} \tag{2.393}
$$

Taking into account that $\boldsymbol{F}^{(1)^{-T}} \cdot \boldsymbol{F}^{(1)^{-1}} = \left(\mathbf{1} - 2\boldsymbol{e}^{(1)}\right)$, we obtain:

$$
\begin{aligned}
\boldsymbol{e} &= \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{(2)^{-T}} \cdot \boldsymbol{F}^{(1)^{-T}} \cdot \boldsymbol{F}^{(1)^{-1}} \cdot \boldsymbol{F}^{(2)^{-1}}\right) = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{(2)^{-T}} \cdot \left(\mathbf{1} - 2\boldsymbol{e}^{(1)}\right) \cdot \boldsymbol{F}^{(2)^{-1}}\right) \\
&= \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{(2)^{-T}} \cdot \boldsymbol{F}^{(2)^{-1}} + \boldsymbol{F}^{(2)^{-T}} \cdot 2\boldsymbol{e}^{(1)} \cdot \boldsymbol{F}^{(2)^{-1}}\right) \\
&= \boldsymbol{F}^{(2)^{-T}} \cdot \boldsymbol{e}^{(1)} \cdot \boldsymbol{F}^{(2)^{-1}} + \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{(2)^{-T}} \cdot \boldsymbol{F}^{(2)^{-1}}\right) \\
&= \boldsymbol{F}^{(2)^{-T}} \cdot \boldsymbol{e}^{(1)} \cdot \boldsymbol{F}^{(2)^{-1}} + \boldsymbol{e}^{(2)}
\end{aligned} \tag{2.394}
$$

Hence, we can see that $\boldsymbol{e} \neq \boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)}$ and Figure 2.41 shows the strain tensors defined by successive configurations.

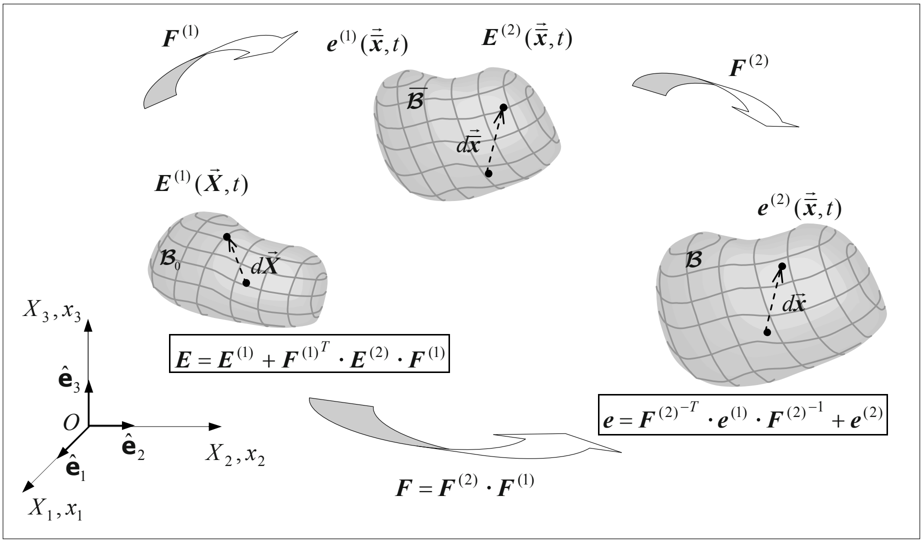


Figure 2.41: Motion defined by successive configurations.

2.15.2 The Logarithmic Strain Tensor

Let us consider a bar subjected to a stretching, in which we define the differential strain along the axis of the bar as:

$$d\varepsilon_{axial} = \frac{dL}{L_0} \xrightarrow{\text{by integrating}} \varepsilon_{axial} = \int_{L_0}^{L_f} \frac{1}{L_0}\, dL = \ln\left(\frac{L_f}{L_0}\right) = \ln(\lambda) \tag{2.395}$$

where ε_{axial} is known as the *logarithmic strain* or *true strain*. Note that, if there are successive increments of displacement, *i.e.* $L_0 \rightarrow L_f^{(1)}$, and $L_f^{(1)} \rightarrow L_f$, it follows that the logarithmic strain is additive, *i.e.*:

$$\varepsilon_{axial}^{Total} = \int_{L_0}^{L_f} \frac{1}{L_0}\, dL = \int_{L_0}^{L_f^{(1)}} \frac{1}{L_0}\, dL + \int_{L_f^{(1)}}^{L_f} \frac{1}{L_0}\, dL = \ln\left(\frac{L_f^{(1)}}{L_0}\right) + \ln\left(\frac{L_f}{L_f^{(1)}}\right) = \varepsilon_{axial}^{(1)} + \varepsilon_{axial}^{(2)} \tag{2.396}$$

Then, from the logarithmic strain definition in (2.395), we can define the *logarithmic strain tensor* as:

$$\boxed{\mathbf{U}^{(Ln)} = \ln(\mathbf{U}) \quad ; \quad \mathbf{U}^{(Ln)} = \sum_{a=1}^{3} \ln(\lambda_a)\, \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}} \quad \textit{The logarithmic strain tensor} \tag{2.397}$$

Likewise, we can define this tensor in the current configuration, which is also known as the *Hencky strain tensor*:

$$\boxed{\mathbf{V}^{(Ln)} = \ln(\mathbf{V}) \quad ; \quad \mathbf{V}^{(Ln)} = \sum_{a=1}^{3} \ln(\lambda_a)\, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}} \quad \textit{The Hencky strain tensor} \tag{2.398}$$

The tensors $\mathbf{V}^{(Ln)}$ and $\mathbf{U}^{(Ln)}$ have the same eigenvalues. Also, the following condition is satisfied:

$$\mathrm{Tr}\left(\mathbf{U}^{(Ln)}\right) = \mathrm{Tr}\left(\mathbf{V}^{(Ln)}\right) = \ln(\lambda_1) + \ln(\lambda_2) + \ln(\lambda_3) = \ln(\lambda_1 \lambda_2 \lambda_3) \tag{2.399}$$

2.15.3 The Biot Strain Tensor

We define the *Biot strain tensor* ($\mathbf{H}$), in the reference configuration, as:

$$\boxed{\mathbf{H} = \mathbf{U} - \mathbf{1} \quad ; \quad \mathbf{H} = \sum_{a=1}^{3} (\lambda_a - 1) \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}} \quad \text{\textit{The Biot strain tensor} (Reference configuration)} \tag{2.400}$$

and in the current configuration as:

$$\boxed{\mathbf{h} = \mathbf{1} - \mathbf{V} \quad ; \quad \mathbf{h} = \sum_{a=1}^{3} \left(1 - \lambda_a^{-1}\right) \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}} \quad \text{\textit{The Biot strain tensor} (Current configuration)} \tag{2.401}$$

2.15.4 Unifying the Strain Tensors

Using the above concepts, it is possible to define several tensors according to the basis of $\mathbf{U}$ or $\mathbf{V}$. Then, we can define the strain tensors in the *material configuration* as:

$$\boldsymbol{E}^m(\mathbf{U}) = \begin{cases} \dfrac{1}{m}\left(\mathbf{U}^m - \mathbf{1}\right) = \dfrac{1}{m}\left(\boldsymbol{C}^{\frac{m}{2}} - \mathbf{1}\right) & \textit{for} \quad m \neq 0 \\[2ex] \ln(\mathbf{U}) = \dfrac{1}{2}\ln(\boldsymbol{C}) & \textit{for} \quad m = 0 \end{cases} \tag{2.402}$$

where m is a positive integer number. The tensors defined above may be represented by the eigenvalues of $\mathbf{U}$ (principal stretches), then:

$$\boldsymbol{E}^m(\lambda_i) = \begin{cases} \dfrac{1}{m}\displaystyle\sum_{a=1}^{3}\left(\lambda_a^m - 1\right)\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} & \textit{for} \quad m \neq 0 \\[2ex] \displaystyle\sum_{a=1}^{3} \ln(\lambda_a)\,\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} & \textit{for} \quad m = 0 \end{cases} \tag{2.403}$$

Notice that, depending on the value of m we recover the following tensors

$$\begin{array}{lll} m = 2 \Rightarrow & \boldsymbol{E}^{(2)}(\mathbf{U}) = \boldsymbol{E} = \dfrac{1}{2}\left(\mathbf{U}^2 - \mathbf{1}\right) = \dfrac{1}{2}\left(\boldsymbol{C} - \mathbf{1}\right) & \text{Green-Lagrange strain tensor} \\ m = 1 \Rightarrow & \boldsymbol{E}^{(1)}(\mathbf{U}) = \mathbf{H} = \left(\mathbf{U} - \mathbf{1}\right) & \text{Biot strain tensor} \\ m = 0 \Rightarrow & \boldsymbol{E}^{(0)}(\mathbf{U}) = \mathbf{U}^{(Ln)} = \ln(\mathbf{U}) & \text{Logarithmic strain tensor} \end{array} \tag{2.404}$$

Next, we can define the strain tensors in the *spatial configuration* as:

$$\boldsymbol{e}^m(\mathbf{U}) = \begin{cases} \dfrac{1}{m}\left(\mathbf{V}^m - \mathbf{1}\right) & \textit{for} \quad m \neq 0 \\[2ex] \ln(\mathbf{V}) & \textit{for} \quad m = 0 \end{cases} \tag{2.405}$$

where m is a negative integer number. The above equations can be expressed in the spectral representation as:

$$\boldsymbol{e}^m(\lambda_i) = \begin{cases} \dfrac{1}{m}\sum_{a=1}^{3}\left(\lambda_a^m - 1\right)\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} & for \quad m \neq 0 \\ \sum_{a=1}^{3} \ln(\lambda_a)\,\hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} & for \quad m = 0 \end{cases} \tag{2.406}$$

Notice that, depending on the value of m we recover the following tensors:

$$\begin{array}{lll} m = -2 \Rightarrow & \begin{aligned} \boldsymbol{e}^{(-2)}(\mathbf{U}) = \boldsymbol{e} &= \frac{1}{2}\left(\mathbf{1} - \mathbf{V}^{-2}\right) \\ &= \frac{1}{2}\left(\mathbf{1} - \boldsymbol{b}^{-1}\right) \end{aligned} & \text{The Almansi strain tensor} \\ m = -1 \Rightarrow & \boldsymbol{e}^{(-1)}(\mathbf{V}) = \mathbf{h} = \left(\mathbf{1} - \mathbf{V}^{-1}\right) & \text{The Biot strain tensor} \\ m = 0 \Rightarrow & \boldsymbol{e}^{(0)}(\mathbf{V}) = \mathbf{V}^{(Ln)} = \ln(\mathbf{V}) & \text{The Hencky strain tensor} \end{array} \tag{2.407}$$

2.15.5 One Dimensional Measurements of Strain (1D)

2.15.5.1 Cauchy's strain or Engineering strain or the Linear strain

$$\varepsilon_C = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} = \lambda - 1 \tag{2.408}$$

where λ shows stretch.

2.15.5.2 The Logarithmic or True strain

$$\varepsilon_H = \int_{L_0}^{L} \frac{d\ell}{\ell} = \ln\left(\frac{L}{L_0}\right) = \ln(\lambda) \tag{2.409}$$

or:

$$\varepsilon_H = \ln(1 + \varepsilon_C) = \varepsilon_C - \frac{1}{2}(\varepsilon_C)^2 + \cdots \tag{2.410}$$

2.15.5.3 The Green-Lagrange strain

Generally speaking we obtain the following relationship:

$$(ds)^2 - (dS)^2 = d\vec{\boldsymbol{X}} \cdot (2\boldsymbol{E}) \cdot d\vec{\boldsymbol{X}} \qquad 1D \Rightarrow \qquad \varepsilon_G = \frac{1}{2}\frac{(ds)^2 - (dS)^2}{(dS)^2} \tag{2.411}$$

In the one dimensional (uniaxial case), we have:

$$\varepsilon_G = \frac{L^2 - L_0^2}{2L_0^2} = \frac{1}{2}\left(\lambda^2 - 1\right) = \frac{1}{2}(\lambda + 1)(\lambda - 1) = \frac{1}{2}\left(\lambda^2 - 1\right) = \frac{1}{2}(\lambda + 1)\varepsilon_C = \varepsilon_C + \frac{1}{2}\varepsilon_C^2 \tag{2.412}$$

2.15.5.4 The Almansi strain

Generally speaking we obtained the following relationship:

$$(ds)^2 - (dS)^2 = d\vec{\boldsymbol{x}} \cdot (2\boldsymbol{e}) \cdot d\vec{\boldsymbol{x}} \qquad 1D \Rightarrow \qquad \varepsilon_A = \frac{1}{2}\frac{(ds)^2 - (dS)^2}{(ds)^2} \tag{2.413}$$

In the one dimensional (uniaxial case), we have:

$$\varepsilon_A = \frac{L^2 - L_0^2}{2L^2} = \frac{1}{2}\left(1 - \lambda^{-2}\right) \tag{2.414}$$

Additionally, the following relationships are satisfied:

$$\varepsilon_A = \left(\varepsilon_C + \frac{1}{2}\varepsilon_C^2\right)\frac{1}{(1+\varepsilon_C)^2} = \varepsilon_G \frac{1}{\lambda^2} \quad \Rightarrow \lambda^2 = \frac{\varepsilon_G}{\varepsilon_A} \tag{2.415}$$

2.15.5.5 The Swaiger strain

$$\varepsilon_S = \frac{\Delta L}{L} = \frac{L - L_0}{L} = 1 - \lambda^{-1} \tag{2.416}$$

Additionally, the following relationship is satisfied:

$$\varepsilon_S = \frac{\varepsilon_C}{1+\varepsilon_C} \tag{2.417}$$

2.15.5.6 The Kuhn strain

$$\varepsilon_K = \frac{L^3 - L_0^3}{3L_0^2 L} = \frac{1}{3}\left(\lambda^2 - \lambda^{-1}\right) \tag{2.418}$$

In a small strain regime (infinitesimal strain) it holds that:

$$\varepsilon \cong \varepsilon_H \cong \varepsilon_C \cong \varepsilon_G \cong \varepsilon_A \cong \varepsilon_S \cong \varepsilon_K \tag{2.419}$$

We can draw a graph where the abscissa is represented by stretch (λ) and the ordinate represents the strains, (see Figure 2.42). We can also verify that for stretch values close to unity the relations in (2.419) are met.

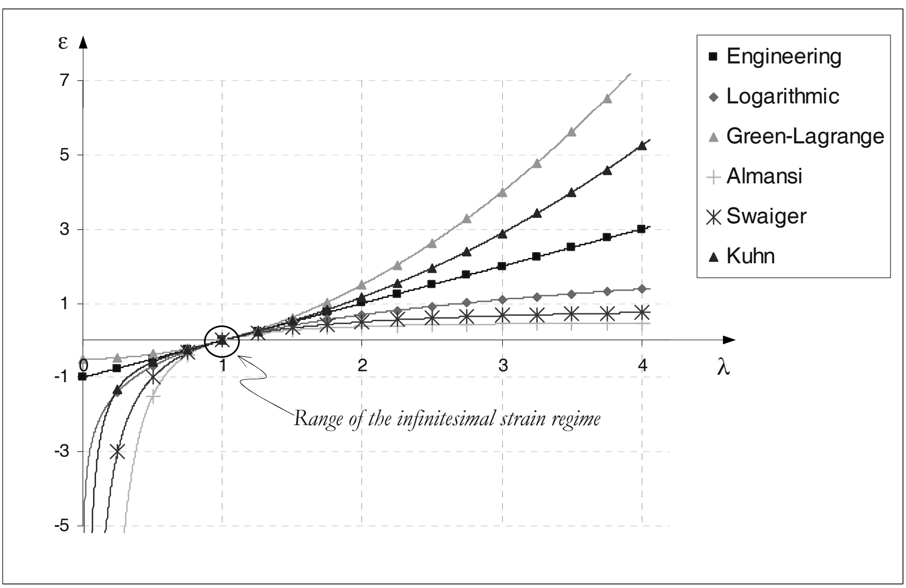


Figure 2.42: Curve stretch vs. strains.

3

Stress

3.1 Introduction

When an external force is acting on a body, the atoms or molecules that make up the continuum are affected and undergo a position change to achieve balance. Resistance to this movement depends on the characteristics of the atoms or molecules that make up the continuum. Internal resistance to movement is called *internal force*, and can be interpreted as the average of the interatomic forces of a handful of atoms, thereby characterizing internal force as a macroscopic variable. The internal force at each material point (particle) of the continuum is represented by the *traction vector field* (force per unit area), which is the starting point to establish the stress state at a material point.

3.2 Forces

When forces are applied directly to a body they are known as *surface forces* (*e.g.* contact forces between two bodies), whereas when the body is immersed in, for instance, a gravitational or electromagnetic field we have an indirect force. As regards forces, this chapter will only deal with surface and gravitational forces.

3.2.1 Surface Forces (Traction)

An example of a surface force is illustrated in Figure 3.1 in which the water pressure on the dam is substituted by the surface force $\vec{\mathbf{t}}^{*}(\vec{x})$ also called the *traction vector*. The total force acting on the dam wall can be obtained by means of the surface integral over the surface $S_{\boldsymbol{\sigma}}$:

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_5,

$$\vec{f} = \int_{S_\sigma} d\vec{f} = \int_{S_\sigma} \vec{\mathbf{t}}^*(\vec{x})\, dS \tag{3.1}$$

where $d\vec{f}$ is the differential force acting on the differential area dS_σ (surface element), where it holds that $d\vec{f} = \vec{\mathbf{t}}^*(\vec{x})\, dS_\sigma$. The unit of the surface forces in the International System of Units (SI) is $N/m^2 = Pa$ (a Pascal is equivalent to one Newton per square metre).

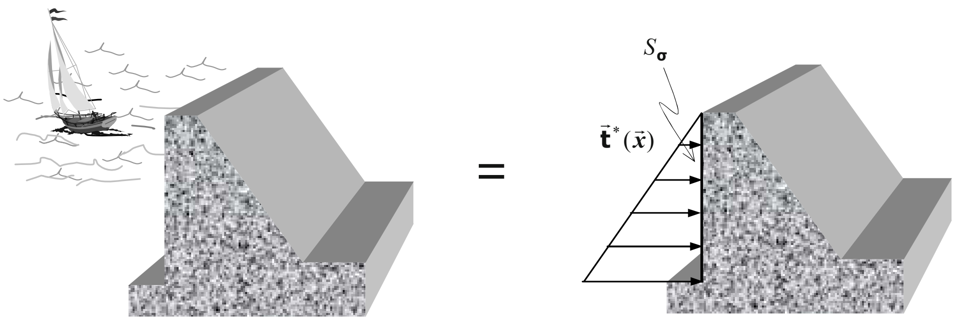


Figure 3.1: Surface force (traction).

3.2.2 Gravitational Force (Body Force)

When the continuous medium is submerged in a gravitational field, the continuum is also subjected to a force which can be represented either by force per unit mass (body force), $\vec{\mathbf{b}}$, or by force per unit volume (force density), $\vec{\mathbf{p}}$. These two forces are related to each other by means of the equation:

$$\rho\vec{\mathbf{b}} = \vec{\mathbf{p}} \qquad \Big| \qquad \rho \mathrm{b}_i = \mathrm{p}_i \tag{3.2}$$

where $\rho(\vec{x},t)$ is the mass density (mass per unit volume). The units of ρ, $\vec{\mathbf{b}}$, $\vec{\mathbf{p}}$, in the International System of Units (SI), are $[\rho] = \frac{kg}{m^3}$, $[\vec{\mathbf{b}}] = \frac{N}{kg} = \frac{m}{s^2}$, $[\vec{\mathbf{p}}] = \frac{N}{m^3}$. Then, the total force acting on the body defined by the domain $\mathcal{B}$ can be evaluated by means of the integral:

$$\vec{\mathbf{F}} = \int_{\mathcal{B}} d\vec{\mathbf{F}} = \int_{\mathcal{B}} \vec{\mathbf{b}}\, dm = \int_V \rho\, \vec{\mathbf{b}}\, dV \tag{3.3}$$

where we must have take into account that $dm = \rho\, dV$.

Problem 3.1: Ignoring the curvature of the earth's surface, the gravitational field can be assumed to be uniform as shown in Figure 3.2, where g is the acceleration caused by gravity (the gravity of the Earth). Find the resultant force acting on the body $\mathcal{B}$.

Solution:

All bodies immersed in a force field are subjected to the body force $\vec{\mathbf{b}}$, and in the special case presented in Figure 3.2 this is given by:

$$\mathsf{b}_i(\vec{\boldsymbol{x}},t) = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \qquad \left[\frac{m}{s^2}\right]$$

Hence, the total force acting on the body can be evaluated as follows:

$$\mathsf{F}_i = \int_V \rho \mathsf{b}_i(\vec{\boldsymbol{x}},t)\, dV = \begin{bmatrix} 0 \\ 0 \\ -\int_V \rho g\, dV \end{bmatrix}$$

We can also verify the F unit: $[\mathsf{F}] = \int_V \left[\frac{kg}{m^3}\right]\left[\frac{m}{s^2}\right] \overbrace{dV}^{[m^3]} = \frac{kg\, m}{s^2} = N(Newton)$.

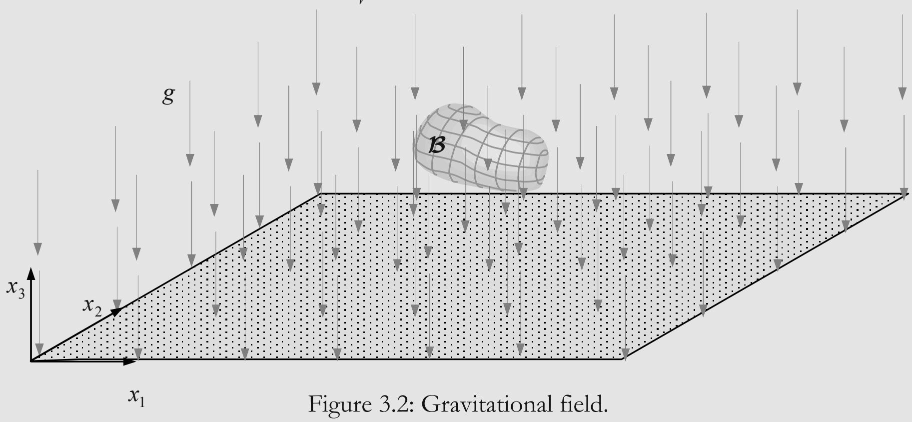


Figure 3.2: Gravitational field.

3.3 Stress Tensors

Let $\mathcal{B}_0$ be a material body in the reference configuration, which is defined by the volume V_0 and bounded by the surface S_0. After motion, the material body $\mathcal{B}_t$ occupies a new configuration characterized by volume V_t which is delimited by the surface S_t, (see Figure 3.3).

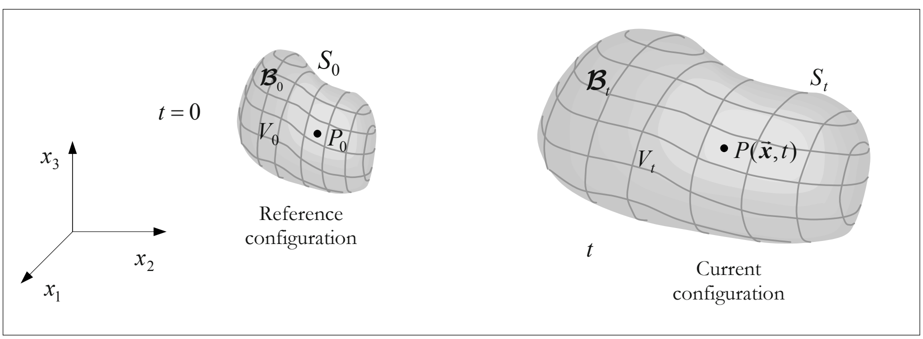


Figure 3.3: Reference and current configuration.

If the continuum is subjected to external forces (surface, gravitational or otherwise), the body is also subjected to internal forces. In this section, we will define a continuous and differentiable tensor field so as to represent the internal force, thereby characterizing the stress state in the continuous medium.

3.3.1 The Cauchy Stress Tensor

3.3.1.1 The Traction Vector

Let us consider a continuum in the current configuration (deformed) which has been divided into two parts by a plane Π passing through the point $P(\vec{\boldsymbol{x}},t)$, (see Figure 3.4). This plane is defined by said point and by the normal $\hat{\boldsymbol{n}}$ (unit vector). Let us also consider a deformed area Δa centered at the point $P(\vec{\boldsymbol{x}},t)$, so, the outcome of the internal force acting on this area element is denoted by $\overrightarrow{\Delta \boldsymbol{f}}$. Then, we can define the *traction vector* (also called the *stress vector*) at the point $P(\vec{\boldsymbol{x}},t)$ and which is associated with the normal $\hat{\boldsymbol{n}}$, as:

$$\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}(\vec{\boldsymbol{x}},t,\hat{\boldsymbol{n}}) = \lim_{\Delta a \to 0} \left(\frac{\overrightarrow{\Delta \boldsymbol{f}}}{\Delta a} \right) \qquad \left[\frac{N}{m^2} = Pa \right] \tag{3.4}$$

Note that, the traction vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ can vary from point to point and said variation defines the traction vector field. Additionally, at a point $P(\vec{\boldsymbol{x}},t)$ the traction vector is only dependent on the normal $\hat{\boldsymbol{n}}$. This traction vector represents the *force per unit deformed area* and its limit (3.4) exists because the medium was assumed to be continuous.

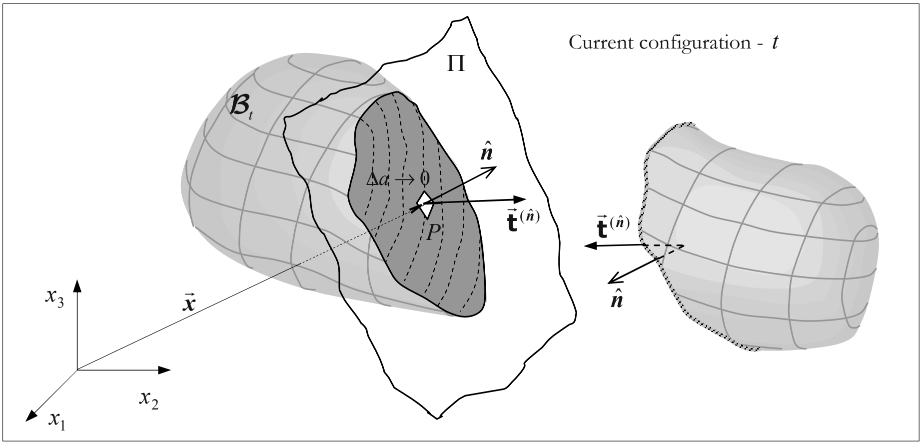


Figure 3.4: Traction vector.

3.3.1.2 Cauchy's Fundamental Postulate

Cauchy's Fundamental Postulate: the "traction $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}$ is a function of the position $(\vec{\boldsymbol{x}})$ and normal $\hat{\boldsymbol{n}}$".

For instance, let us consider a plane Π_1 defined by the normal $\hat{\boldsymbol{n}}^{(1)}$ that passes through the point $P(\vec{\boldsymbol{x}},t)$, thereby, defining the traction vector $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}}^{(1)})}$ at the point, which is associated with the normal $\hat{\boldsymbol{n}}^{(1)}$, (see Figure 3.5(a)). Now if we consider a second plane Π_2 defined by the normal $\hat{\boldsymbol{n}}^{(2)}$, which also passes through the point $P(\vec{\boldsymbol{x}},t)$, there will be another traction vector $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}}^{(2)})}$ which is associated with this new plane, (see Figure 3.5(a)). Then an immediate result of Cauchy's Fundamental Postulate is the *principle of action and reaction*, (see Figure 3.5(b)):

$$\vec{\mathbf{t}}(\vec{\boldsymbol{x}},-\hat{\boldsymbol{n}}) = -\vec{\mathbf{t}}(\vec{\boldsymbol{x}},\hat{\boldsymbol{n}}) \tag{3.5}$$

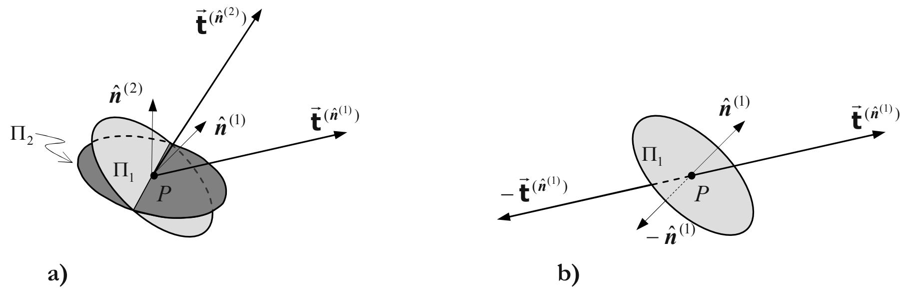


Figure 3.5: Traction vector.

The stress state at a point $P(\vec{\boldsymbol{x}},t)$ is completely described when the traction vector $\vec{\mathbf{t}}(\vec{\boldsymbol{x}},\hat{\boldsymbol{n}})$ can be obtained for any arbitrary plane passing through this point $P(\vec{\boldsymbol{x}},t)$. Cauchy showed that if we define the traction vector on three mutually perpendicular planes passing through the point $P(\vec{\boldsymbol{x}},t)$ we can fully describe the stress state at that point, (see Figure 3.6).

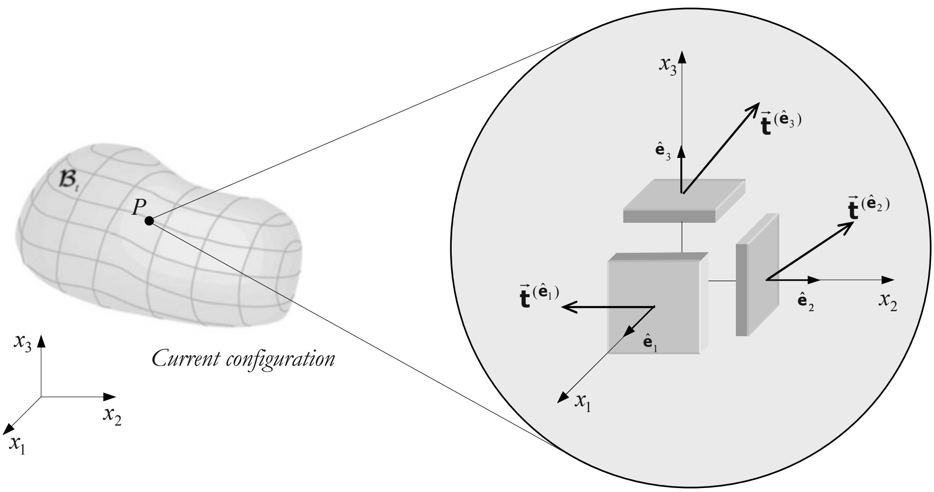


Figure 3.6: The stress state at the point *P*.

If we adopt three planes that are perpendicular to the unit vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$, we can obtain three traction vectors associated with each direction and represented by $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}$, $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_2)}$ and $\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_3)}$ respectively, (see Figure 3.6). Then by breaking each traction vector down according to the directions x_1, x_2 and x_3, (see Figure 3.7), we can obtain:

$$\begin{cases}\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}=t_1^{\hat{\mathbf{e}}_1}\hat{\mathbf{e}}_1+t_2^{\hat{\mathbf{e}}_1}\hat{\mathbf{e}}_2+t_3^{\hat{\mathbf{e}}_1}\hat{\mathbf{e}}_3\\ \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_2)}=t_1^{\hat{\mathbf{e}}_2}\hat{\mathbf{e}}_1+t_2^{\hat{\mathbf{e}}_2}\hat{\mathbf{e}}_2+t_3^{\hat{\mathbf{e}}_2}\hat{\mathbf{e}}_3\\ \vec{\mathbf{t}}^{(\hat{\mathbf{e}}_3)}=t_1^{\hat{\mathbf{e}}_3}\hat{\mathbf{e}}_1+t_2^{\hat{\mathbf{e}}_3}\hat{\mathbf{e}}_2+t_3^{\hat{\mathbf{e}}_3}\hat{\mathbf{e}}_3\end{cases} \tag{3.6}$$

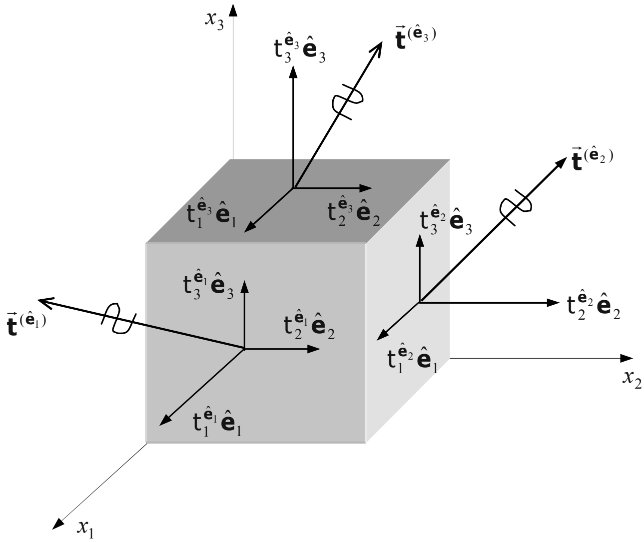


Figure 3.7: Representation of the three tractions in the Cartesian basis.

In order to consider these three vectors simultaneously we can define a second-order tensor as:

$$\begin{aligned}\boldsymbol{\sigma}&=\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_1)}\otimes\hat{\mathbf{e}}_1+\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_2)}\otimes\hat{\mathbf{e}}_2+\vec{\mathbf{t}}^{(\hat{\mathbf{e}}_3)}\otimes\hat{\mathbf{e}}_3\\ &=t_1^{\hat{\mathbf{e}}_1}\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_1+t_1^{\hat{\mathbf{e}}_2}\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_2+t_1^{\hat{\mathbf{e}}_3}\hat{\mathbf{e}}_1\otimes\hat{\mathbf{e}}_3+\\ &+t_2^{\hat{\mathbf{e}}_1}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_1+t_2^{\hat{\mathbf{e}}_2}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_2+t_2^{\hat{\mathbf{e}}_3}\hat{\mathbf{e}}_2\otimes\hat{\mathbf{e}}_3+\\ &+t_3^{\hat{\mathbf{e}}_1}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_1+t_3^{\hat{\mathbf{e}}_2}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_2+t_3^{\hat{\mathbf{e}}_3}\hat{\mathbf{e}}_3\otimes\hat{\mathbf{e}}_3\end{aligned} \tag{3.7}$$

Note that, in Chapter 1 we established that any second-order tensor can be represented by a linear combination of dyads.

Then we can rearrange the components of $\boldsymbol{\sigma}$ into matrix form, and make a change in the nomenclature so that we obtain:

$$\begin{bmatrix} t_1^{\hat{\mathbf{e}}_1} & t_1^{\hat{\mathbf{e}}_2} & t_1^{\hat{\mathbf{e}}_3}\\ t_2^{\hat{\mathbf{e}}_1} & t_2^{\hat{\mathbf{e}}_2} & t_2^{\hat{\mathbf{e}}_3}\\ t_3^{\hat{\mathbf{e}}_1} & t_3^{\hat{\mathbf{e}}_2} & t_3^{\hat{\mathbf{e}}_3}\end{bmatrix}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} \tag{3.8}$$

thus, defining σ_{ij} as the components of the *Cauchy stress tensor* also called the *true stress tensor*, $\boldsymbol{\sigma}$:

$$\boxed{\boldsymbol{\sigma}=\sigma_{ij}(\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j)}\quad \textit{The Cauchy stress tensor}\quad [Pa] \tag{3.9}$$

The representation of the Cauchy stress tensor components in the Cartesian system is shown in Figure 3.8(a).

OBS.: The Cauchy stress tensor is a symmetric tensor $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}^{sym}$. Therefore it holds that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. Proof of this is provided in Chapter 5.

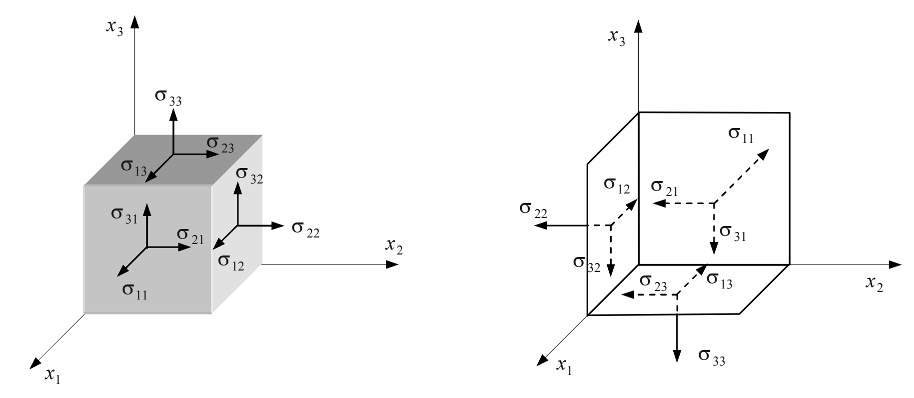


Figure 3.8: Stress state at a point P.

In the literature, we can find other nomenclature for the Cauchy stress tensor components, distinguishing the scientific and engineering notations, (see Figure 3.9):

$$\sigma_{ij} = \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}}_{\substack{\textit{Scientific} \\ \textit{Notation}}} = \underbrace{\begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}}_{\substack{\textit{Engineering} \\ \textit{Notation}}} \tag{3.10}$$

NOTE: It is important to note that many authors (mostly engineers) reverse the convention of the indices, so to avoid misunderstandings, when we refer to engineering notation the symmetry of the Cauchy stress tensor is already implicit. ■

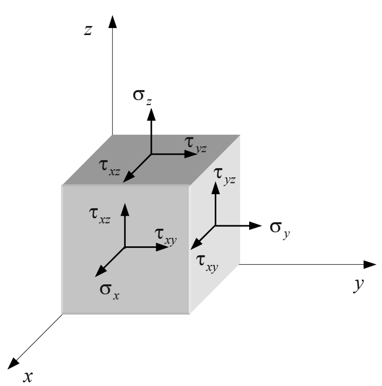


Figure 3.9: Stress state at a point P – Engineering notation.

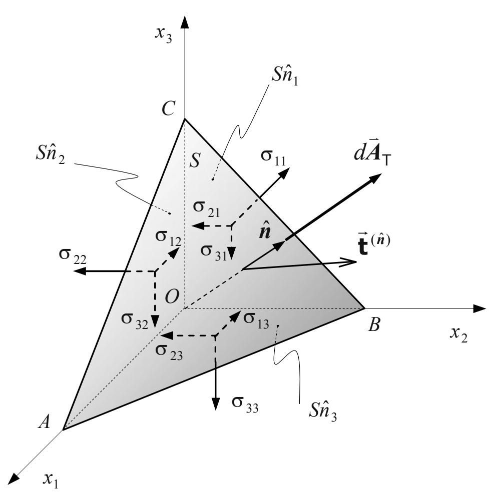
Taking into account the symmetry of the Cauchy stress tensor, the representation of its components in Voigt notation is given by:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \xrightarrow{Voigt} \{\boldsymbol{\sigma}\} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} \sigma_{x} \\ \sigma_{y} \\ \sigma_{z} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} \tag{3.11}$$

3.3.2 The Relationship between the Traction and the Cauchy Stress Tensor

Our goal now is that given the nine components of the Cauchy stress tensor, how can we find the traction vector associated with an arbitrary plane? It is very easy to answer this question if we consider that the projection of the second-order tensor ($\boldsymbol{\sigma}$) according to the direction ($\hat{\boldsymbol{n}}$) is given by $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$, (see Chapter 1).

We will prove we can obtain the same result by starting from the forces equilibrium at the material point. To do this, we define an arbitrary plane ABC with the normal $\hat{\boldsymbol{n}}$, (see Figure 3.10), where the plane ABC passes through the point P.



$$d\vec{A}_{\mathsf{T}} = \tfrac{1}{2}(\overrightarrow{AB} \wedge \overrightarrow{AC}) = \left\| d\vec{A}_{\mathsf{T}} \right\| \hat{\boldsymbol{n}} \equiv S\hat{\boldsymbol{n}}$$

$$(d\vec{A}_{\mathsf{T}})_i = S\hat{\boldsymbol{n}} \cdot \hat{\mathbf{e}}_i$$

$$(d\vec{A}_{\mathsf{T}})_1 = S\hat{\boldsymbol{n}} \cdot \hat{\mathbf{e}}_1 = S\hat{n}_1 = \triangle BOC$$

$$(d\vec{A}_{\mathsf{T}})_2 = S\hat{\boldsymbol{n}} \cdot \hat{\mathbf{e}}_2 = S\hat{n}_2 = \triangle AOC$$

$$(d\vec{A}_{\mathsf{T}})_3 = S\hat{\boldsymbol{n}} \cdot \hat{\mathbf{e}}_3 = S\hat{n}_3 = \triangle AOB$$

$$\triangle ABC = S$$

Figure 3.10: The traction vector in an arbitrary plane.

Associated with this plane is the traction vector $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}$. Taking into account the Cauchy stress tensor components in the rear faces of the tetrahedron, (see Figure 3.8(b)), and by considering that the point is in equilibrium, the balance of forces according to the x_1-direction is evaluated as follows:

$$\mathsf{t}_1^{(\hat{\boldsymbol{n}})} S - S\,\hat{n}_1 \sigma_{11} - S\,\hat{n}_2 \sigma_{12} - S\,\hat{n}_3 \sigma_{13} = 0 \tag{3.12}$$

where S is the triangle ABC area and the projection of the area S according to the planes $x_2 - x_3$, $x_1 - x_3$ and $x_1 - x_2$ is given respectively by $S\hat{n}_1$, $S\hat{n}_2$ and $S\hat{n}_3$, (see Figure 3.10). Then, if we simplify the equation (3.12) we obtain:

$$\mathrm{t}_1^{(\hat{\boldsymbol{n}})} = \hat{n}_1\sigma_{11} + \hat{n}_2\sigma_{12} + \hat{n}_3\sigma_{13} \tag{3.13}$$

Similarly, the balance of forces according to the directions x_2 and x_3 provide us, the following relationships respectively:

$$\mathrm{t}_2^{(\hat{\boldsymbol{n}})} = \hat{n}_1\sigma_{12} + \hat{n}_2\sigma_{22} + \hat{n}_3\sigma_{32} \tag{3.14}$$

$$\mathrm{t}_3^{(\hat{\boldsymbol{n}})} = \hat{n}_1\sigma_{13} + \hat{n}_2\sigma_{23} + \hat{n}_3\sigma_{33} \tag{3.15}$$

Then by rearranging the equations (3.13), (3.14) and (3.15) into matrix form we obtain:

$$\begin{bmatrix} \mathrm{t}_1^{(\hat{\boldsymbol{n}})} \\ \mathrm{t}_2^{(\hat{\boldsymbol{n}})} \\ \mathrm{t}_3^{(\hat{\boldsymbol{n}})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} \tag{3.16}$$

The above equations can still be represented as:

Indicial notation	Tensorial notation
$\mathrm{t}_i^{(\hat{\boldsymbol{n}})} = \sigma_{ij}\hat{n}_j$	$\boxed{\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}}$

(3.17)

Then, by referring to the symmetry of the Cauchy stress tensor ($\boldsymbol{\sigma}$), the traction vector components can be represented in Voigt notation as follows:

$$\begin{Bmatrix} \mathrm{t}_1^{(\hat{\boldsymbol{n}})} \\ \mathrm{t}_2^{(\hat{\boldsymbol{n}})} \\ \mathrm{t}_3^{(\hat{\boldsymbol{n}})} \end{Bmatrix} = \begin{bmatrix} \hat{n}_1 & 0 & 0 & \hat{n}_2 & 0 & \hat{n}_3 \\ 0 & \hat{n}_2 & 0 & \hat{n}_1 & \hat{n}_3 & 0 \\ 0 & 0 & \hat{n}_3 & 0 & \hat{n}_2 & \hat{n}_1 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} \quad \Rightarrow \quad \{\mathcal{T}\} = [\mathcal{N}]^T \{\boldsymbol{\sigma}\} \tag{3.18}$$

Problem 3.2: The Cauchy stress tensor components at a point P are given by:

$$\sigma_{ij} = \begin{bmatrix} 8 & -4 & 1 \\ -4 & 3 & 0.5 \\ 1 & 0.5 & 2 \end{bmatrix} Pa$$

a) Calculate the traction vector ($\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}$) at P which is associated with the plane ABC defined in Figure 3.11.
b) With reference to paragraph a).
Obtain the normal ($\vec{\boldsymbol{\sigma}}_N$) and tangential ($\vec{\boldsymbol{\sigma}}_S$) traction vectors at P (see Appendix A).

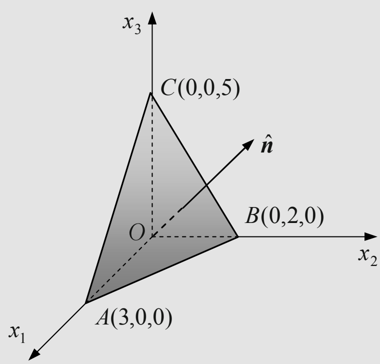


Figure 3.11: Plane ABC.

Solution:
First, we obtain the unit vector which is normal to the plane ABC. To do this we choose two vectors on the plane:

$$\vec{BA} = \vec{OA} - \vec{OB} = 3\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 0\hat{\mathbf{e}}_3$$
$$\vec{BC} = \vec{OC} - \vec{OB} = 0\hat{\mathbf{e}}_1 - 2\hat{\mathbf{e}}_2 + 5\hat{\mathbf{e}}_3$$

Then, the normal vector associated with the plane ABC is obtained by means of the cross product between $\vec{BA}$ and $\vec{BC}$, *i.e.*:

$$\vec{\boldsymbol{n}} = \vec{BC} \wedge \vec{BA} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & -2 & 5 \\ 3 & -2 & 0 \end{vmatrix} = 10\hat{\mathbf{e}}_1 + 15\hat{\mathbf{e}}_2 + 6\hat{\mathbf{e}}_3$$

Additionally, the unit vector codirectional with $\vec{\boldsymbol{n}}$ is given by:

$$\hat{\boldsymbol{n}} = \frac{\vec{\boldsymbol{n}}}{\|\vec{\boldsymbol{n}}\|} = \frac{10}{19}\hat{\mathbf{e}}_1 + \frac{15}{19}\hat{\mathbf{e}}_2 + \frac{6}{19}\hat{\mathbf{e}}_3$$

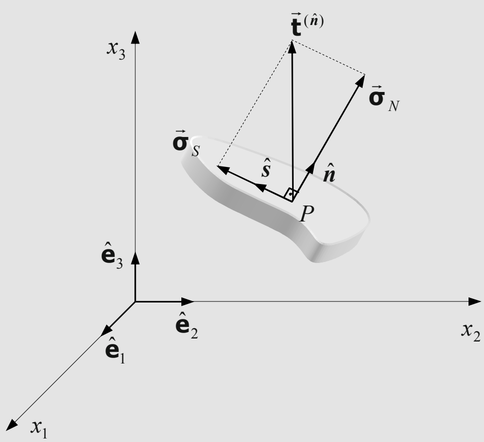
Then by using the equation in (3.16), we can obtain the traction components as:

$$\mathrm{t}_i^{(\hat{\boldsymbol{n}})} = \sigma_{ij}\hat{\mathrm{n}}_j \quad \Rightarrow \quad \begin{bmatrix} \mathrm{t}_1 \\ \mathrm{t}_2 \\ \mathrm{t}_3 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 8 & -4 & 1 \\ -4 & 3 & 0.5 \\ 1 & 0.5 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 15 \\ 6 \end{bmatrix} Pa \quad \Rightarrow \begin{bmatrix} \mathrm{t}_1 \\ \mathrm{t}_2 \\ \mathrm{t}_3 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 26 \\ 8 \\ 29.5 \end{bmatrix} Pa$$

b) The traction vector $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}$ associated with the normal $\hat{\boldsymbol{n}}$ can be broken down into a normal ($\vec{\boldsymbol{\sigma}}_N$) and a tangential ($\vec{\boldsymbol{\sigma}}_S$) vector as shown in Figure 3.12. Then,

$$\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} = \vec{\boldsymbol{\sigma}}_N + \vec{\boldsymbol{\sigma}}_S \qquad \text{or} \qquad \vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} = \sigma_N \hat{\boldsymbol{n}} + \sigma_S \hat{\boldsymbol{s}}$$

where σ_N and σ_S are the magnitudes of $\vec{\boldsymbol{\sigma}}_N$ and $\vec{\boldsymbol{\sigma}}_S$, respectively.



$$\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$$
$$\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} = \vec{\boldsymbol{\sigma}}_N + \vec{\boldsymbol{\sigma}}_S$$
$$\sigma_N = \vec{\boldsymbol{\sigma}}_N \cdot \hat{\boldsymbol{n}}$$
$$\left\|\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}\right\|^2 = \sigma_N^2 + \sigma_S^2$$

Figure 3.12: Normal and tangential stress vector.

As we have seen in Appendix A, the normal component, σ_N, can be evaluated as follows:

$$\sigma_N = \vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}\cdot\hat{\boldsymbol{n}} = (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\cdot\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \boldsymbol{\sigma}:(\hat{\boldsymbol{n}}\otimes\hat{\boldsymbol{n}}) = \mathrm{t}_i^{(\hat{\boldsymbol{n}})}\hat{n}_i = (\sigma_{ij}\hat{n}_j)\hat{n}_i = \hat{n}_i\sigma_{ij}\hat{n}_j = \sigma_{ij}(\hat{n}_i\hat{n}_j)$$

Thus:

$$\sigma_N = \mathrm{t}_i\hat{n}_i \quad \Rightarrow \quad \sigma_N = \frac{1}{19^2}\begin{bmatrix} 26 & 8 & 29.5 \end{bmatrix}\begin{bmatrix} 10 \\ 15 \\ 6 \end{bmatrix} \approx 1.54\, Pa$$

Then the tangential component, σ_S, can be obtained by means of the Pythagorean Theorem, *i.e.*:

$$\left\| \vec{\mathbf{t}}^{(\hat{n})} \right\|^2 = \sigma_N^2 + \sigma_S^2 \quad \Rightarrow \quad \sigma_S^2 = \mathrm{t}_i^{(\hat{n})}\mathrm{t}_i^{(\hat{n})} - \sigma_N^2$$

where

$$\mathrm{t}_i^{(\hat{n})}\mathrm{t}_i^{(\hat{n})} = \frac{1}{19^2}\begin{bmatrix} 26 & 8 & 29.5 \end{bmatrix}\begin{bmatrix} 26 \\ 8 \\ 29.5 \end{bmatrix} \approx 4.46$$

Thus,

$$\sigma_S = \sqrt{\mathrm{t}_i^{(\hat{n})}\mathrm{t}_i^{(\hat{n})} - \sigma_N^2} = \sqrt{4.46 - 2.3716} \approx 2.0884\, Pa$$

Problem 3.3: The stress state at a point in the continuum is represented by the components of the Cauchy stress tensor as:

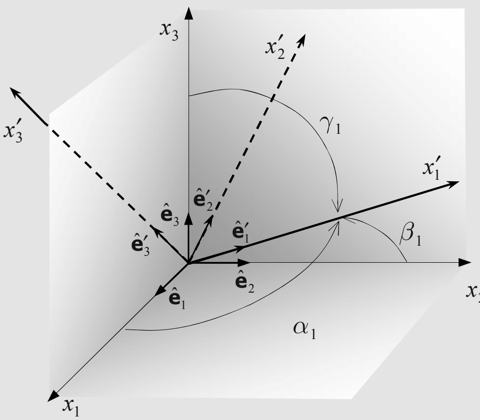
$$\sigma_{ij} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} Pa$$

a) Obtain the components of $\boldsymbol{\sigma}$ in a new system x_1', x_2', x_3', where the transformation matrix is given by:

$$a_{ij} = \mathcal{A} = \frac{1}{5}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

where

$a_{11} = \cos\alpha_1$
$a_{12} = \cos\beta_1$
$a_{13} = \cos\gamma_1$
$\vdots$



b) Obtain the principal invariants of $\boldsymbol{\sigma}$;
c) Obtain the eigenvalues and eigenvectors of $\boldsymbol{\sigma}$. Also verify if the eigenvectors form a basis transformation between the original and the principal space;
d) Illustrate the Cauchy stress tensor graphically, *i.e.* with the Mohr's circle in stress, (see Appendix A);
e) Obtain the spherical ($\boldsymbol{\sigma}^{sph}$) and the deviatoric ($\boldsymbol{\sigma}^{dev}$) part of $\boldsymbol{\sigma}$. Also, find the principal invariants of $\boldsymbol{\sigma}^{dev}$;
f) Obtain the octahedral normal (σ_N^{oct}) and tangential (σ_S^{oct}) components of $\boldsymbol{\sigma}$, (see Appendix A).
Solution:
a) As we have seen in Chapter 1, the transformation law for the components of a second-order tensor is given by:

$$\sigma_{ij}' = a_{ik} a_{jl} \sigma_{kl} \xrightarrow{\text{Matrix form}} \boldsymbol{\sigma}' = \mathcal{A}\,\boldsymbol{\sigma}\,\mathcal{A}^T$$

Thus,

$$\sigma'_{ij} = \frac{1}{5^2}\begin{bmatrix} 3 & 0 & -4 \\ 0 & 5 & 0 \\ 4 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0.6 & 0 \\ 0.6 & 2 & 0.8 \\ 0 & 0.8 & 2 \end{bmatrix}$$

These new components σ'_{ij} can be appreciated in Figure 3.13.

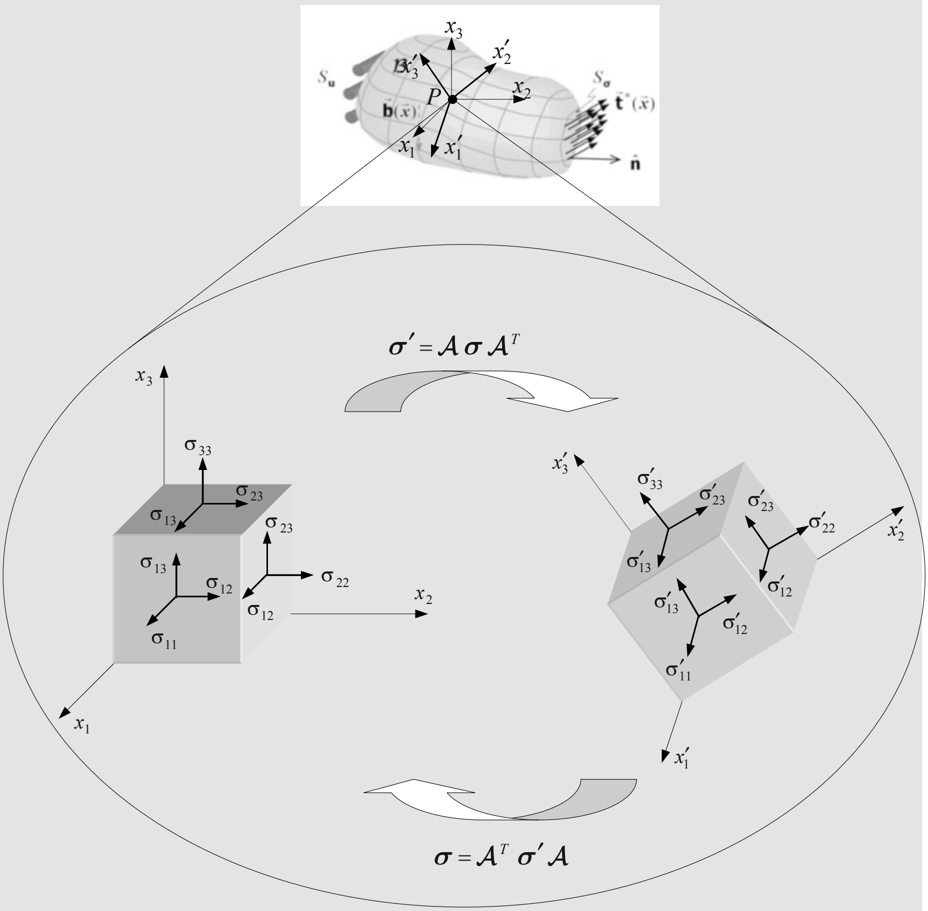


Figure 3.13: Basis transformation.

b) The principal invariants of the Cauchy stress tensor can be calculated as follows:

$$I_{\boldsymbol{\sigma}} = \mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$II_{\boldsymbol{\sigma}} = \frac{1}{2}\left[(\mathrm{Tr}\boldsymbol{\sigma})^2 - \mathrm{Tr}(\boldsymbol{\sigma}^2)\right] = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$
$$= \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{33}\sigma_{22} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2$$

$$III_{\boldsymbol{\sigma}} = \det(\boldsymbol{\sigma}) = \epsilon_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3} = \frac{1}{6}\left(\sigma_{ii}\sigma_{jj}\sigma_{kk} - 3\sigma_{ii}\sigma_{jk}\sigma_{jk} + 2\sigma_{ij}\sigma_{jk}\sigma_{ki}\right)$$
$$= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2$$

By substituting the values of σ_{ij} for those in the proposed problem we obtain:

$$I_{\boldsymbol{\sigma}} = 6 \quad ; \quad II_{\boldsymbol{\sigma}} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 11 \quad ; \quad III_{\boldsymbol{\sigma}} = 6$$

c) The principal stresses (σ_i) and principal directions ($\hat{\mathbf{n}}^{(i)}$) are obtained by solving the following set of equations:

$$\begin{bmatrix} 2-\sigma & 1 & 0 \\ 1 & 2-\sigma & 0 \\ 0 & 0 & 2-\sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To obtain the nontrivial solutions of $\hat{\mathbf{n}}^{(i)}$ we have to solve the characteristic determinant, which is a cubic equation for the unknown magnitude σ:

$$\left| \sigma_{ij} - \sigma \delta_{ij} \right| = 0 \quad \Rightarrow \quad \sigma^3 - I_{\boldsymbol{\sigma}} \sigma^2 + II_{\boldsymbol{\sigma}} \sigma - III_{\boldsymbol{\sigma}} = 0$$

However, if we look at the format of the Cauchy stress tensor components, we can notice that we already have one solution as in the x_3-direction the tangential components are equal to zero, then:

$$\sigma_3 = 2 \xrightarrow{\text{Principal direction}} n_1^{(3)} = n_2^{(3)} = 0 \,,\; n_3^{(3)} = 1$$

To obtain the other two eigenvalues, one only need solve:

$$\begin{vmatrix} 2-\sigma & 1 \\ 1 & 2-\sigma \end{vmatrix} = (2-\sigma)^2 - 1 = 0 \quad \Rightarrow \quad \begin{cases} \sigma_1 = 1 \\ \sigma_2 = 3 \end{cases}$$

Then we can express the Cauchy stress tensor components in the principal space as:

$$\sigma''_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} Pa$$

Additionally, the principal direction associated with $\sigma_1 = 1$ is calculated as follows:

$$\begin{bmatrix} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \\ 0 & 0 & 2-1 \end{bmatrix} \begin{bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} n_1^{(1)} + n_2^{(1)} = 0 \\ n_1^{(1)} + n_2^{(1)} = 0 \end{cases} \Rightarrow n_1^{(1)} = -n_2^{(1)}$$

with $n_3^{(1)} = 0$ and by using the condition ${n_1^{(1)}}^2 + {n_2^{(1)}}^2 = 1$ we obtain:

$$n_1^{(1)} = -n_2^{(1)} = \frac{1}{\sqrt{2}} \quad \text{then } \hat{n}_i^{(1)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

Since $\boldsymbol{\sigma}$ is a symmetric tensor, the principal space is formed by an orthogonal basis, so, it is valid that:

$$\hat{\mathbf{n}}^{(1)} \wedge \hat{\mathbf{n}}^{(2)} = \hat{\mathbf{n}}^{(3)} \quad ; \quad \hat{\mathbf{n}}^{(2)} \wedge \hat{\mathbf{n}}^{(3)} = \hat{\mathbf{n}}^{(1)} \quad ; \quad \hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)} = \hat{\mathbf{n}}^{(2)}$$

Thus, the second principal direction can be obtained by the cross product between $\hat{\mathbf{n}}^{(3)}$ and $\hat{\mathbf{n}}^{(1)}$, *i.e.*:

$$\hat{\mathbf{n}}^{(2)} = \hat{\mathbf{n}}^{(3)} \wedge \hat{\mathbf{n}}^{(1)} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \hat{\mathbf{e}}_1 + \frac{1}{\sqrt{2}} \hat{\mathbf{e}}_2$$

which can also be checked by the following analysis:

The Principal direction associated with $\sigma_2 = 3$:

$$\begin{bmatrix} 2-3 & 1 & 0 \\ 1 & 2-3 & 0 \\ 0 & 0 & 2-3 \end{bmatrix} \begin{bmatrix} n_1^{(2)} \\ n_2^{(2)} \\ n_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -n_1^{(2)} + n_2^{(2)} = 0 \\ n_1^{(2)} - n_2^{(2)} = 0 \end{cases} \Rightarrow n_1^{(2)} = n_2^{(2)}$$

With $\mathsf{n}_3^{(3)}=0$ and using the condition $\mathsf{n}_1^{(3)^2}+\mathsf{n}_2^{(3)^2}=1$ we obtain:

$$\mathsf{n}_1^{(2)}=\mathsf{n}_2^{(2)}=\frac{1}{\sqrt{2}} \quad \text{then} \quad \hat{\mathsf{n}}_i^{(2)}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}$$

As we have seen in Chapter 1, the eigenvectors of a symmetric tensor form the transformation matrix $\mathcal{D}$, from the original system to the principal space, *i.e.* $\boldsymbol{\sigma}''=\mathcal{D}\,\boldsymbol{\sigma}\,\mathcal{D}^T$, thus:

$$\begin{bmatrix}\sigma_1=1 & 0 & 0\\ 0 & \sigma_2=3 & 0\\ 0 & 0 & \sigma_3=2\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}$$

d) The graphical representation of a second-order tensor can be obtained from the description in Appendix A. To do this we have to restructure the eigenvalues of $\boldsymbol{\sigma}$ so that $\sigma_I>\sigma_{II}>\sigma_{III}$, thus:

$$\sigma_I=3 \quad ; \quad \sigma_{II}=2 \quad ; \quad \sigma_{III}=1$$

Then the three circumferences are defined by:

$$\text{Circle 1}\Rightarrow \quad ; \quad (center)C_1=\frac{1}{2}(\sigma_{II}+\sigma_{III})=1.5 \quad ; \quad (radius)R_1=\frac{1}{2}(\sigma_{II}-\sigma_{III})=0.5$$

$$\text{Circle 2}\Rightarrow \quad ; \quad (center)C_2=\frac{1}{2}(\sigma_I+\sigma_{III})=2.0 \quad ; \quad (radius)R_2=\frac{1}{2}(\sigma_I-\sigma_{III})=1.0$$

$$\text{Circle 3}\Rightarrow \quad ; \quad (center)C_3=\frac{1}{2}(\sigma_I+\sigma_{II})=2.5 \quad ; \quad (radius)R_3=\frac{1}{2}(\sigma_I-\sigma_{II})=0.5$$

Then, we can illustrate the Cauchy stress tensor at P by means of Mohr's circle in stress as shown in Figure 3.14.

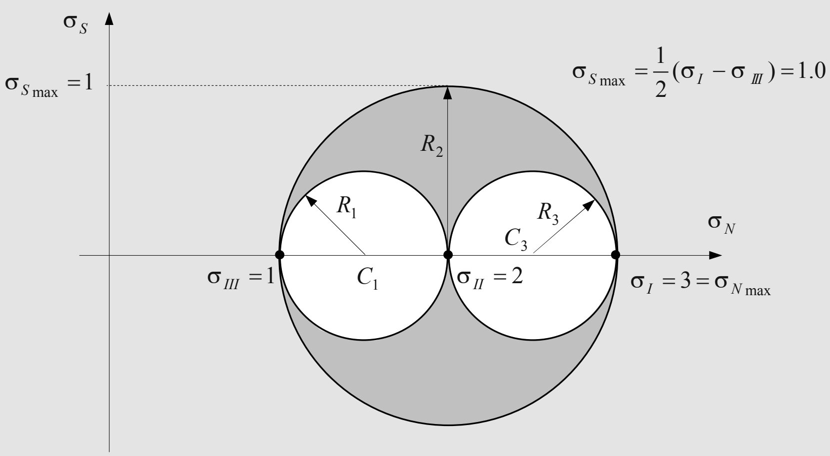


Figure 3.14: Mohr's circle in stress at the point P.

e) As defined in Chapter 1, a second-order tensor can be broken down additively into a spherical and a deviatoric part, *i.e.*:

Tensorial notation

$$\boxed{\begin{aligned}\boldsymbol{\sigma} &= \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} \\ &= \sigma_m \mathbf{1} + \boldsymbol{\sigma}^{dev}\end{aligned}}$$

Indicial notation

$$\begin{aligned}\sigma_{ij} &= \sigma_{ij}^{sph} + \sigma_{ij}^{dev} \\ &= \frac{1}{3}\sigma_{kk}\delta_{ij} + \sigma_{ij}^{dev} \\ &= \sigma_m \delta_{ij} + \sigma_{ij}^{dev}\end{aligned} \tag{3.19}$$

A schematic representation of these components in the Cartesian basis can be appreciated in Figure 3.15 and the value of the scalar σ_m is evaluated as follows:

$$\sigma_m = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{1}{3}\sigma_{kk} = \frac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{I_{\boldsymbol{\sigma}}}{3} = \frac{6}{3} = 2$$

Then the spherical part becomes:

$$\sigma_{ij}^{sph} = \sigma_m \delta_{ij} = 2\delta_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

And, the deviatoric part can be evaluated as follows:

$$\sigma_{ij}^{dev} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}(2\sigma_{11}-\sigma_{22}-\sigma_{33}) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}(2\sigma_{22}-\sigma_{11}-\sigma_{33}) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}(2\sigma_{33}-\sigma_{11}-\sigma_{22}) \end{bmatrix}$$

Thus,

$$\sigma_{ij}^{dev} = \begin{bmatrix} 2-2 & 1 & 0 \\ 1 & 2-2 & 0 \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now let us remember from Chapter 1 that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{dev}$ are coaxial tensors, *i.e.*, they have the same principal directions, so we can use this information to operate in the principal space of $\boldsymbol{\sigma}$ to obtain the eigenvalues of $\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{sph}$. With that we obtain:

$$\sigma_{ij}'^{dev} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the invariants of $\boldsymbol{\sigma}^{dev}$ are given by:

$$I_{\boldsymbol{\sigma}^{dev}} = \mathrm{Tr}(\boldsymbol{\sigma}^{dev}) = 0 \quad ; \quad II_{\boldsymbol{\sigma}^{dev}} = -1 \quad ; \quad III_{\boldsymbol{\sigma}^{dev}} = 0$$

Traditionally, in engineering, the invariants of the deviatoric stress tensor are represented by:

$$\boxed{\begin{aligned} \mathrm{J}_1 &= I_{\boldsymbol{\sigma}^{dev}} = 0 \\ \mathrm{J}_2 &= -II_{\boldsymbol{\sigma}^{dev}} = \frac{1}{3}\left(I_{\boldsymbol{\sigma}}^2 - 3\,II_{\boldsymbol{\sigma}}\right) \\ \mathrm{J}_3 &= III_{\boldsymbol{\sigma}^{dev}} = \frac{1}{27}\left(2I_{\boldsymbol{\sigma}}^3 - 9I_{\boldsymbol{\sigma}}II_{\boldsymbol{\sigma}} + 27\,III_{\boldsymbol{\sigma}}\right) \end{aligned}}$$

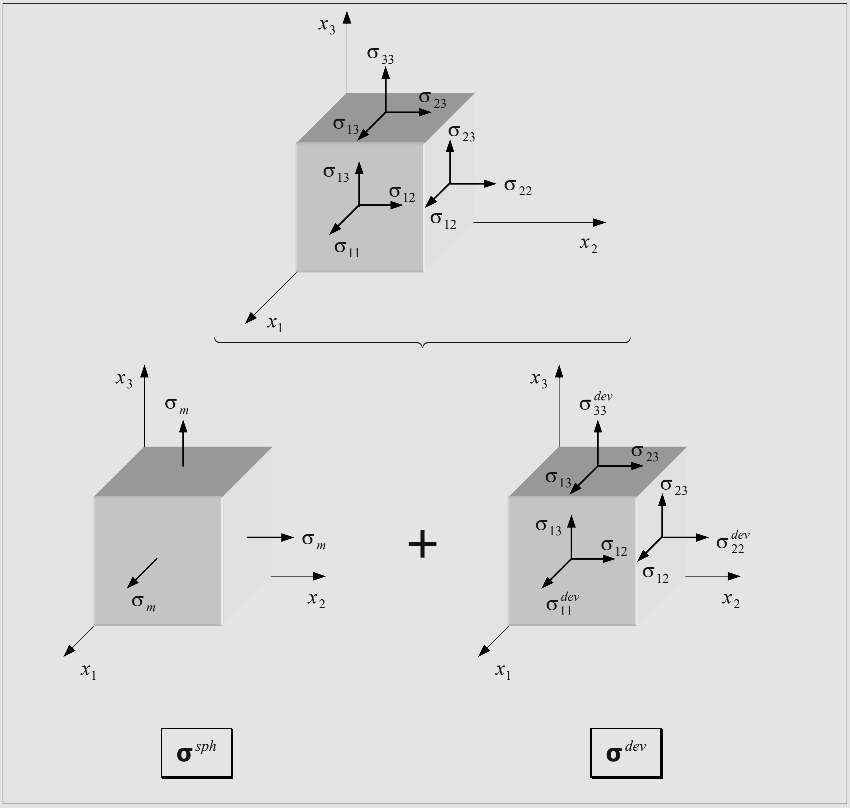


Figure 3.15: The spherical and deviatoric part of $\boldsymbol{\sigma}$.

f) The octahedral normal and tangential components, (see Appendix A), can be expressed as:

$$\boxed{\sigma_N^{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}\sigma_{ii} = \frac{I_{\boldsymbol{\sigma}}}{3} = \sigma_m}$$

$$\boxed{\sigma_S^{oct} \equiv \tau_{oct} = \frac{1}{3}\sqrt{2I_{\boldsymbol{\sigma}}^2 - 6\,II_{\boldsymbol{\sigma}}} = \sqrt{\frac{2}{3}J_2} = \sqrt{\frac{\left(\sigma_1^{dev}\right)^2 + \left(\sigma_2^{dev}\right)^2 + \left(\sigma_3^{dev}\right)^2}{3}}}$$

Then, by substituting the values of the proposed problem we obtain:

$$\sigma_N^{oct} = \sigma_m = 6 \quad ; \quad \tau_{oct} = \sqrt{\frac{2}{3}J_2} = \sqrt{\frac{2}{3}}$$

3.3.3 Other Measures of Stress

3.3.3.1 The First Piola-Kirchhoff Stress Tensor

As we have seen before, the Cauchy stress tensor was derived in the current configuration (deformed). In some cases we may wish to adopt the Lagrangian description for studying motion, and then it will be necessary to correlate the Cauchy stress tensor with a hypothetical stress tensor in the reference configuration, (see Figure 3.16).

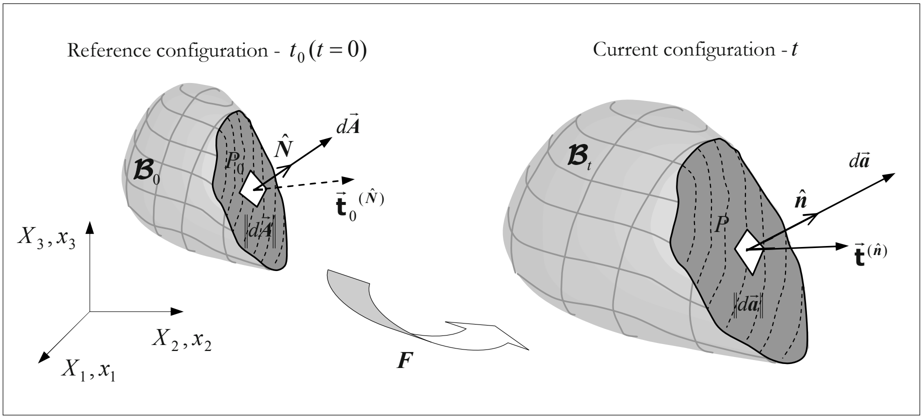


Figure 3.16: Traction vector – Current and reference configuration.

In the reference configuration, we adopt an area element $d\vec{A}$ with the normal $\hat{N}$ and associated with that plane we can define a *pseudo traction vector* $\vec{\mathbf{t}}_0^{(\hat{N})}$. After motion, this area element becomes $d\vec{a}$ in the deformed configuration, associated with which we have the traction vector $\vec{\mathbf{t}}^{(\hat{n})}$, (see Figure 3.16). Then by using the definition in (3.4) we can define $\vec{\mathbf{t}}_0^{(\hat{N})}$ and $\vec{\mathbf{t}}^{(\hat{n})}$, respectively, as:

$$\vec{\mathbf{t}}_0^{(\hat{N})}(\vec{X},t,\hat{N}) = \lim_{\Delta A \to 0}\left(\frac{\Delta F}{\Delta A}\right) = \frac{d\vec{\mathfrak{F}}}{\|d\vec{A}\|} \quad ; \quad \vec{\mathbf{t}}^{(\hat{n})}(\vec{x},t,\hat{n}) = \lim_{\Delta a \to 0}\left(\frac{\Delta f}{\Delta a}\right) = \frac{d\vec{f}}{\|d\vec{a}\|} \tag{3.20}$$

Based on the principle that:

$$d\vec{\mathfrak{F}} = d\vec{f} \qquad \Big| \qquad d\mathfrak{F}_i = df_i \tag{3.21}$$

we can conclude that:

$$\vec{\mathbf{t}}_0^{(\hat{N})}\|d\vec{A}\| = \vec{\mathbf{t}}^{(\hat{n})}\|d\vec{a}\| \qquad \Big| \qquad \mathrm{t}_{0\,i}^{(\hat{N})}\|d\vec{A}\| = \mathrm{t}_i^{(\hat{n})}\|d\vec{a}\| \tag{3.22}$$

$$\begin{aligned} &\Rightarrow \mathbf{P}\cdot\hat{N}\|d\vec{A}\| = \boldsymbol{\sigma}\cdot\hat{n}\|d\vec{a}\| \qquad &\Big|\qquad &\Rightarrow \mathrm{P}_{ik}\hat{N}_k\|d\vec{A}\| = \sigma_{ik}\hat{n}_k\|d\vec{a}\| \\ &\Rightarrow \mathbf{P}\cdot d\vec{A} = \boldsymbol{\sigma}\cdot d\vec{a} &\Big| &\Rightarrow \mathrm{P}_{ik}dA_k = \sigma_{ik}da_k \end{aligned} \tag{3.23}$$

where we have introduced a second-order tensor $\mathbf{P}$ so that the projection of $\mathbf{P}$ according to the $\hat{N}$-direction results in the traction vector $\vec{\mathbf{t}}_0^{(\hat{N})}$.

Then by referring to the Nanson's formula ($d\vec{a} = J\boldsymbol{F}^{-T}\cdot d\vec{A}$) obtained in Chapter 2, where J is the Jacobian determinant and is equal to the determinant of the deformation gradient $\boldsymbol{F}$, *i.e.* $J = |\boldsymbol{F}|$, the equation in (3.23) can be rewritten as:

$$\mathbf{P}\cdot d\vec{A} = \boldsymbol{\sigma}\cdot d\vec{a} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\cdot d\vec{A} = \left(J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\right)\cdot d\vec{A} \tag{3.24}$$

Thus, we can conclude that:

$$\mathbf{P} = J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} \quad \Leftrightarrow \quad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{P}\cdot\boldsymbol{F}^{T} \qquad \Big| \qquad \mathrm{P}_{ij} = J\,\sigma_{ik}F_{jk}^{-1} \quad \Leftrightarrow \quad \sigma_{ij} = \frac{1}{J}\mathrm{P}_{ik}F_{jk} \tag{3.25}$$

where $\mathbf{P}$ is the *first Piola-Kirchhoff stress tensor* which is also called the *nominal stress tensor*. This tensor represents the force in the current configuration per unit undeformed area, so, it is both a two-point second-order tensor and a non-symmetric tensor, *i.e.* $\mathbf{P} \neq \mathbf{P}^T$.

In addition to the traction vectors ($\vec{\mathbf{t}}_0^{(\hat{N})}$, $\vec{\mathbf{t}}^{(\hat{n})}$) defined previously, we can find in the literature other traction vectors that have a purely mathematical transformation, with no physical meaning, (see Figure 3.17), namely:

$$\vec{\mathbf{t}}_0^{\wedge} = \boldsymbol{F}^{-1} \cdot \vec{\mathbf{t}}_0 \quad ; \quad \vec{\mathbf{t}}_0^{\Theta} = \mathbf{R}^T \cdot \vec{\mathbf{t}}_0 \quad ; \quad \vec{\mathbf{t}}^* = J\,\vec{\mathbf{t}} \tag{3.26}$$

where $\mathbf{R}$ is the polar decomposition rotation tensor, (see Chapter 2). In addition, the following relationships are valid:

$$\begin{aligned}
\vec{\mathbf{t}}_0 &= \frac{da}{dA}\vec{\mathbf{t}} &&= \boldsymbol{F} \cdot \vec{\mathbf{t}}_0^{\wedge} &&= \mathbf{R} \cdot \vec{\mathbf{t}}_0^{\Theta} \\
\vec{\mathbf{t}}_0^{\wedge} &= \boldsymbol{F}^{-1} \cdot \vec{\mathbf{t}}_0 &&= \boldsymbol{F}^{-1} \cdot \mathbf{R} \cdot \vec{\mathbf{t}}_0^{\Theta} &&= \frac{da}{dA}\boldsymbol{F}^{-1} \cdot \vec{\mathbf{t}} \\
\vec{\mathbf{t}}_0^{\Theta} &= \mathbf{R}^T \cdot \vec{\mathbf{t}}_0 &&= \mathbf{R}^T \cdot \boldsymbol{F} \cdot \vec{\mathbf{t}}_0^{\wedge} &&= \frac{da}{dA}\mathbf{R}^T \cdot \vec{\mathbf{t}}
\end{aligned} \tag{3.27}$$

3.3.3.2 The Kirchhoff Stress Tensor

We define the *Kirchhoff stress tensor* $\boldsymbol{\tau}$, which is related to the traction vector $\vec{\mathbf{t}}^*$ (current configuration), as:

$$\vec{\mathbf{t}}^* = J\,\vec{\mathbf{t}} = J\,\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}} = \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}} \longrightarrow \boldsymbol{\tau} = J\,\boldsymbol{\sigma} \tag{3.28}$$

As we can verify the Kirchhoff stress tensor is a symmetric second-order tensor and is related to the Cauchy stress tensor and to the first Piola-Kirchhoff stress tensor by means of the following relationships:

$$\boldsymbol{\tau} = J\,\boldsymbol{\sigma} \quad ; \quad \boldsymbol{\tau} = \mathbf{P} \cdot \boldsymbol{F}^T \quad ; \quad \mathbf{P} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \tag{3.29}$$

3.3.3.3 The Second Piola-Kirchhoff Stress Tensor

We can also introduce the second Piola-Kirchhoff stress tensor, $\mathbf{S}$, which is defined in the reference configuration, as:

Tensorial notation	Indicial notation
$\mathbf{S} = \boldsymbol{F}^{-1} \cdot \mathbf{P} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$	$\mathrm{S}_{ij} = F_{ik}^{-1}\mathrm{P}_{kj} = F_{ik}^{-1}\tau_{kl}F_{jl}^{-1} = J\,F_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$

(3.30)

or

$$\mathbf{P} = \boldsymbol{F} \cdot \mathbf{S} \quad ; \quad \boldsymbol{\tau} = \mathbf{P} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T \tag{3.31}$$

The Cauchy stress tensor can be expressed in terms of the second Piola-Kirchhoff stress tensor as:

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T \tag{3.32}$$

Next, we can prove that $\mathbf{S}$ is a symmetric tensor, *i.e.* $\mathbf{S}^T = \mathbf{S}$:

$$\mathbf{S}^T = \left(J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}\right)^T = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \mathbf{S} \tag{3.33}$$

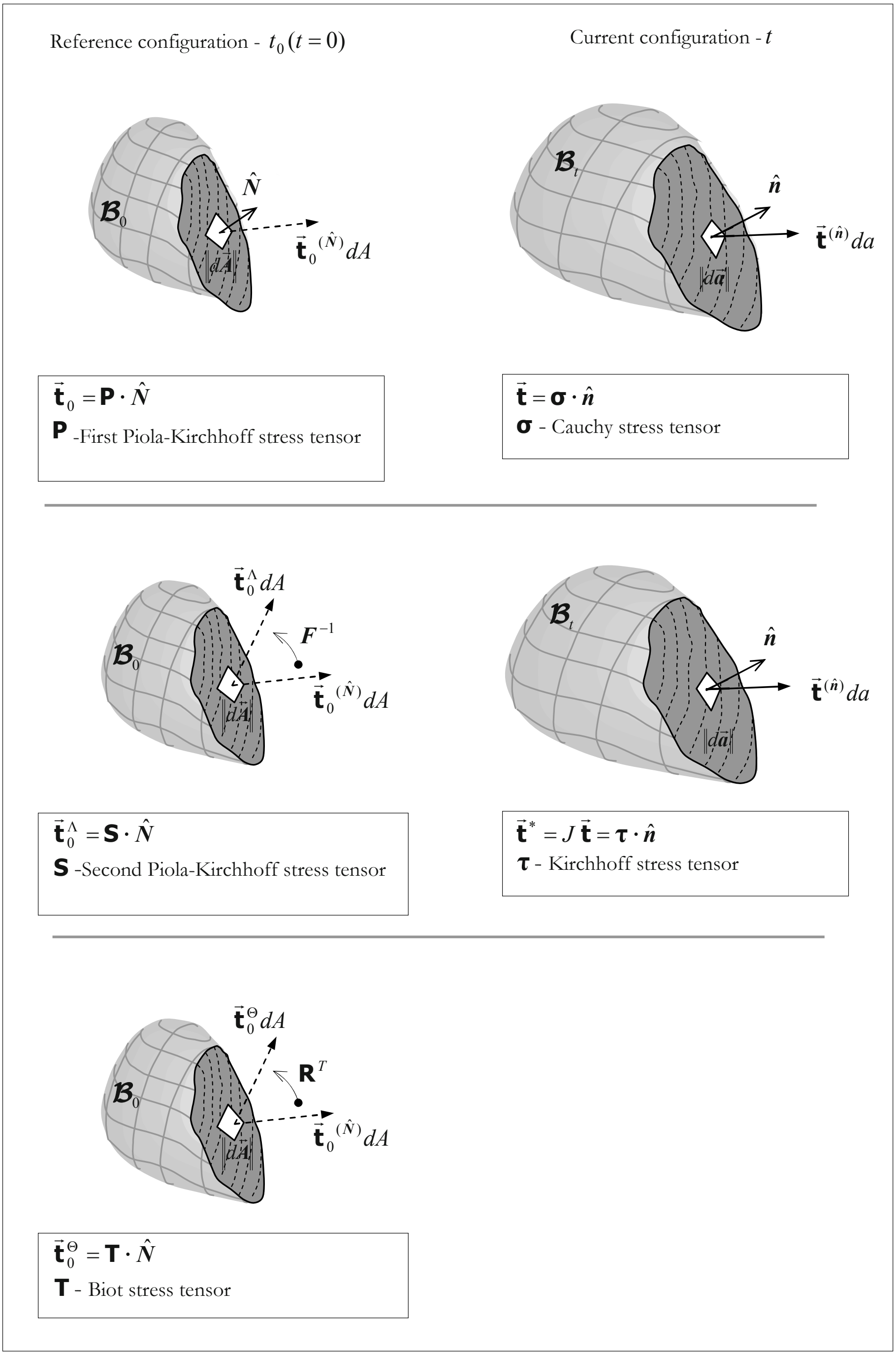


Figure 3.17: Traction vectors – Reference and current configuration.

3.3.3.4 The Biot Stress Tensor

We can also introduce the *Biot stress tensor*, $\mathbf{T}$, which in general is non-symmetric and is related to the traction vector $\vec{\mathbf{t}}_0^{\Theta}$:

$$\vec{\mathbf{t}}_0^{\Theta} = \mathbf{T} \cdot \hat{\boldsymbol{N}} \tag{3.34}$$

From (3.34) we can obtain:

$$\begin{aligned} \mathbf{T} \cdot \hat{\boldsymbol{N}} &= \mathbf{R}^T \cdot \vec{\mathbf{t}}_0 \\ \mathbf{T} \cdot \hat{\boldsymbol{N}} &= \mathbf{R}^T \cdot \mathbf{P} \cdot \hat{\boldsymbol{N}} \end{aligned} \tag{3.35}$$

Thus,

$$\mathbf{T} = \mathbf{R}^T \cdot \mathbf{P} \tag{3.36}$$

and by considering that $\mathbf{P} = J\,\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$ we can obtain:

$$\mathbf{T} = J\,\mathbf{R}^T \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \tag{3.37}$$

If we refer to the equations in (3.36) and (3.31), the tensor $\mathbf{T}$ can also be expressed as:

$$\mathbf{T} = \mathbf{R}^T \cdot \mathbf{P} = \mathbf{R}^T \cdot \boldsymbol{F} \cdot \mathbf{S} = \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{S} = \mathbf{U} \cdot \mathbf{S} \tag{3.38}$$

where we have applied the right polar decomposition $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U}$. Note that the tensor $\mathbf{T}$ will be symmetrical if $\mathbf{U}$ and $\mathbf{S}$ are coaxial tensors, hence $\mathbf{S} \cdot \mathbf{U} = \mathbf{U} \cdot \mathbf{S} \Rightarrow (\mathbf{U} \cdot \mathbf{S})^{skew} = \mathbf{0}$, (see Chapter 2 in subsection 1.5.9 Coaxial Tensors).

3.3.3.5 The Mandel Stress Tensor

We can also introduce the Mandel stress tensor denoted by $\mathbf{M}$, which in general is non-symmetric and is defined as:

$$\mathbf{M} = \boldsymbol{C} \cdot \mathbf{S} = \left(\boldsymbol{F}^T \cdot \boldsymbol{F}\right) \cdot \left(\boldsymbol{F}^{-1} \cdot \mathbf{P}\right) = \boldsymbol{F}^T \cdot \mathbf{P} = \boldsymbol{F}^T \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} \tag{3.39}$$

where $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor, which was defined in Chapter 2.

We can now summarize the relationships between stress tensors defined above as:

$$\boxed{\boldsymbol{\sigma} = \frac{1}{J}\mathbf{P} \cdot \boldsymbol{F}^T = \frac{1}{J}\boldsymbol{\tau} = \frac{1}{J}\boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T = \frac{1}{J}\mathbf{R} \cdot \mathbf{T} \cdot \boldsymbol{F}^T = \frac{1}{J}\boldsymbol{F}^{-T} \cdot \mathbf{M} \cdot \boldsymbol{F}^T} \quad \textit{The Cauchy stress tensor} \tag{3.40}$$

$$\boxed{\mathbf{P} = J\,\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} = \boldsymbol{F} \cdot \mathbf{S} = \mathbf{R} \cdot \mathbf{T} = \boldsymbol{F}^{-T} \cdot \mathbf{M}} \quad \textit{The first Piola-Kirchhoff stress tensor} \tag{3.41}$$

$$\boxed{\boldsymbol{\tau} = J\,\boldsymbol{\sigma} = \mathbf{P} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T = \mathbf{R} \cdot \mathbf{T} \cdot \boldsymbol{F}^T = \boldsymbol{F}^{-T} \cdot \mathbf{M} \cdot \boldsymbol{F}^T} \quad \textit{The Kirchhoff stress tensor} \tag{3.42}$$

$$\boxed{\mathbf{S} = J\,\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F}^{-1} \cdot \mathbf{P} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} = \mathbf{U}^{-1} \cdot \mathbf{T} = \boldsymbol{C}^{-1} \cdot \mathbf{M}} \quad \textit{The second Piola-Kirchhoff stress tensor} \tag{3.43}$$

$$\boxed{\mathbf{T} = J\,\mathbf{R}^T \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \mathbf{R}^T \cdot \mathbf{P} = \mathbf{R}^T \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} = \mathbf{U} \cdot \mathbf{S} = \mathbf{U}^{-1} \cdot \mathbf{M}} \quad \textit{The Biot stress tensor} \tag{3.44}$$

$$\boxed{\mathbf{M} = J\,\boldsymbol{F}^T \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F}^T \cdot \mathbf{P} = \boldsymbol{F}^T \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} = \boldsymbol{C} \cdot \mathbf{S} = \mathbf{U} \cdot \mathbf{T}} \quad \textit{The Mandel stress tensor} \tag{3.45}$$

Figure 3.18 shows the representation of tensors in different configurations.

NOTE: If the current configuration is very close to the reference configuration, the deformation gradient is approximately equal to the identity tensor $\mathbf{1}$, *i.e.*:

$$\boldsymbol{F} \approx \boldsymbol{F}^{-1} \approx \mathbf{1} \quad and \quad J = \det(\boldsymbol{F}) \approx 1 \tag{3.46}$$

In this condition all the stress tensors are equal, *i.e.* $\boldsymbol{\sigma} \approx \mathbf{P} \approx \boldsymbol{\tau} \approx \mathbf{S} \approx \mathbf{T} \approx \mathbf{M}$. ■

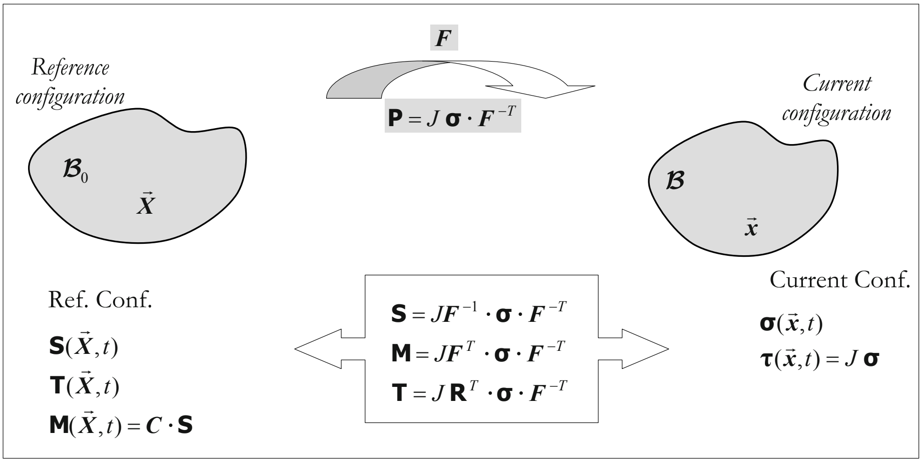


Figure 3.18: Stress tensors.

3.3.4 Spectral Representation of the Stress Tensors

For the next stress tensor representation let us consider that the Cauchy stress tensor ($\boldsymbol{\sigma}$) and the left stretch tensor ($\mathbf{V}$) are coaxial, *i.e.* they present the same principal directions. Then, the spectral representation of the Cauchy stress tensor is given by:

$$\boldsymbol{\sigma} = \sum_{a=1}^{3} \sigma_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{3.47}$$

where σ_a are the eigenvalues of $\boldsymbol{\sigma}$, and $\hat{\mathbf{n}}^{(a)}$ are the eigenvectors of $\boldsymbol{\sigma}$ or $\mathbf{V}$ or $\boldsymbol{b}$. Remember from Chapter 2 that the following representations are valid:

$$\boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \quad ; \quad \boldsymbol{F}^{-1} = \sum_{a=1}^{3} \frac{1}{\lambda_a} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \quad ; \quad \boldsymbol{F}^{-T} = \sum_{a=1}^{3} \frac{1}{\lambda_a} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \tag{3.48}$$

and also that the polar decomposition rotation tensor is represented by:

$$\mathbf{R} = \sum_{a=1}^{3} \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \tag{3.49}$$

where λ_a are the principal stretches, and $\hat{\mathbf{N}}^{(a)}$ are the eigenvectors of the right stretch tensor ($\mathbf{U}$). If we refer to the relationships between stress tensors given in (3.40) - (3.45), we can obtain the spectral representation of the stress tensor as follows:

- The Kirchhoff stress tensor:

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = J\sum_{a=1}^{3} \sigma_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} J\sigma_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \tau_a \, \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \tag{3.50}$$

- The first Piola-Kirchhoff stress tensor:

$$\begin{aligned}\mathbf{P} &= J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} = J\left(\sum_{a=1}^{3}\sigma_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\\ &= \sum_{a=1}^{3}\frac{J\sigma_a}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\cdot\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3}\frac{J\sigma_a}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3}\mathsf{P}_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\end{aligned} \tag{3.51}$$

As we can verify the first Piola-Kirchhoff stress tensor is neither in the current nor in the reference configuration, *i.e.* $\mathbf{P}$ is a two-point tensor, and P_a are not the eigenvalues of $\mathbf{P}$.

- The second Piola-Kirchhoff stress tensor is shown as:

$$\begin{aligned}\mathbf{S} &= J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\\ &= J\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\sigma_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\\ &= \sum_{a=1}^{3}\frac{J\sigma_a}{\lambda_a^2}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\cdot\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\cdot\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\\ &= \sum_{a=1}^{3}\frac{J\sigma_a}{\lambda_a^2}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3}\mathsf{S}_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\end{aligned} \tag{3.52}$$

As we can verify, the second Piola-Kirchhoff stress tensor has the same principal directions as the right stretch tensor ($\mathbf{U}$).

- The Biot stress tensor can be shown as:

$$\begin{aligned}\mathbf{T} &= J\,\mathbf{R}^{T}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} = \mathbf{U}\cdot\mathbf{S}\\ &= J\left(\sum_{a=1}^{3}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\sigma_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\\ &= \sum_{a=1}^{3}\frac{J\sigma_a}{\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} = \left(\sum_{a=1}^{3}\lambda_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\mathsf{S}_a\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\\ &= \sum_{a=1}^{3}\mathsf{T}_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3}\mathsf{S}_a\lambda_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\end{aligned} \tag{3.53}$$

- Then the Mandel stress tensor is shown as:

$$\begin{aligned}\mathbf{M} &= J\,\boldsymbol{F}^{T}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}\\ &= J\left(\sum_{a=1}^{3}\lambda_a\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\sigma_a\,\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}\right)\cdot\left(\sum_{a=1}^{3}\frac{1}{\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\\ &= \sum_{a=1}^{3}J\sigma_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3}\mathsf{M}_a\,\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\end{aligned} \tag{3.54}$$

Then, the following is valid:

$$\mathsf{S}_a = \frac{J}{\lambda_a^2}\sigma_a = \frac{1}{\lambda_a^2}\tau_a = \frac{1}{\lambda_a}\mathsf{P}_a = \frac{1}{\lambda_a}\mathsf{T}_a = \frac{1}{\lambda_a^2}\mathsf{M}_a \tag{3.55}$$

Problem 3.4: Prove that the following relationship are valid:

$$\mathbf{P} = J\,\boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\sigma_m \boldsymbol{F}^{-T} \quad ; \quad \mathbf{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\sigma_m \boldsymbol{C}^{-1}$$

where $\mathbf{P}$ and $\mathbf{S}$ are the first and second Piola-Kirchhoff stress tensors, respectively, $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor, $\boldsymbol{F}$ is the deformation gradient, J is the Jacobian determinant, and the scalar σ_m is the mean normal Cauchy tress. Also prove that the following relationships are true:

$$\mathbf{P} : \boldsymbol{F} = \mathbf{S} : \boldsymbol{C} = 3J\sigma_m$$

Solution:

First of all we prove that $\mathbf{P} : \boldsymbol{F} = \mathbf{S} : \boldsymbol{C}$:

$$\begin{aligned}
\mathbf{P} : \boldsymbol{F} &= \mathsf{P}_{ij} F_{ij} \\
&= (F_{ik}\mathsf{S}_{kj}) F_{ij} = \mathsf{S}_{kj}(F_{ik}F_{ij}) \\
&= \mathsf{S}_{kj}(\boldsymbol{F}^T \cdot \boldsymbol{F})_{kj} = \mathsf{S}_{kj}(\boldsymbol{C})_{kj} \\
&= \mathbf{S} : \boldsymbol{C}
\end{aligned}$$

Secondly, by referring to the definition $\mathbf{P} = J\,\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$, (see equation (3.25)), and the different components of $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev}$, we obtain:

$$\begin{aligned}
\mathbf{P} &= J\,(\boldsymbol{\sigma}^{dev} + \sigma_m \mathbf{1}) \cdot \boldsymbol{F}^{-T} \\
&= J\,\boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\sigma_m \mathbf{1} \cdot \boldsymbol{F}^{-T} \\
&= J\,\boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\sigma_m \boldsymbol{F}^{-T}
\end{aligned}$$

Thirdly, by taking into account the definition $\mathbf{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$, (see equation (3.30)), and by breaking down $\boldsymbol{\sigma}$ into $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev}$, we obtain:

$$\begin{aligned}
\mathbf{S} &= J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \\
&= J\boldsymbol{F}^{-1} \cdot (\boldsymbol{\sigma}^{dev} + \sigma_m \mathbf{1}) \cdot \boldsymbol{F}^{-T} \\
&= J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\boldsymbol{F}^{-1} \cdot \sigma_m \mathbf{1} \cdot \boldsymbol{F}^{-T} \\
&= J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\sigma_m (\boldsymbol{F}^T \cdot \boldsymbol{F})^{-1} \\
&= J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\sigma_m \boldsymbol{C}^{-1}
\end{aligned}$$

$$\begin{aligned}
\mathsf{S}_{ij} &= JF^{-1}_{ik}\sigma_{kp}F^{-1}_{jp} \\
&= JF^{-1}_{ik}(\sigma^{dev}_{kp} + \sigma_{(m)}\delta_{kp})F^{-1}_{jp} \\
&= JF^{-1}_{ik}\sigma^{dev}_{kp}F^{-1}_{jp} + JF^{-1}_{ik}\sigma_{(m)}\delta_{kp}F^{-1}_{jp} \\
&= JF^{-1}_{ik}\sigma^{dev}_{kp}F^{-1}_{jp} + J\sigma_{(m)}F^{-1}_{ik}F^{-1}_{jk} \\
&= JF^{-1}_{ik}\sigma^{dev}_{kp}F^{-1}_{jp} + J\sigma_{(m)}C^{-1}_{ij}
\end{aligned}$$

Then by applying the double scalar product between $\mathbf{S}$ and $\boldsymbol{C}$ we can obtain:

$$\begin{aligned}
\mathbf{S} : \boldsymbol{C} &= (J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} + J\sigma_m \boldsymbol{C}^{-1}) : \boldsymbol{C} \\
&= J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} : \boldsymbol{C} + J\sigma_m \boldsymbol{C}^{-1} : \boldsymbol{C}
\end{aligned}$$

where the term $J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} : \boldsymbol{C}$ becomes:

$$J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} : \boldsymbol{C} = (J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T}) : \underbrace{\boldsymbol{C}}_{\boldsymbol{F}^T \cdot \boldsymbol{F}}$$

$$\begin{aligned}
(J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T})_{ij}(\boldsymbol{F}^T \cdot \boldsymbol{F})_{ij} &= (F^{-1}_{ip}\sigma^{dev}_{pk}F^{-1}_{jk})(F_{qi}F_{qj}) \\
&= J\,\delta_{qp}\delta_{qk}\sigma^{dev}_{pk} \\
&= J\,\sigma^{dev}_{pk}\delta_{pk} = J\,\sigma^{dev}_{kk} \\
&= J\,\underbrace{\boldsymbol{\sigma}^{dev} : \mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\sigma}^{dev})=0} = 0
\end{aligned}$$

Thus:

$$\mathbf{S} : \boldsymbol{C} = J\sigma_m \boldsymbol{C}^{-1} : \boldsymbol{C} = J\sigma_m \mathrm{Tr}(\boldsymbol{C}^{-1} \cdot \boldsymbol{C}) = J\sigma_m \mathrm{Tr}(\mathbf{1}) = 3J\sigma_m$$

Now, by taking the double scalar product between $\mathbf{P}$ and $\boldsymbol{F}$ we obtain:

$$\mathbf{P} : \boldsymbol{F} = J\,\boldsymbol{\sigma}^{dev} \cdot \boldsymbol{F}^{-T} : \boldsymbol{F} + J\sigma_m \boldsymbol{F}^{-T} : \boldsymbol{F}$$

Then by analyzing the term $J\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{F}$ we can conclude that:

$$
\begin{aligned}
J\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T}:\boldsymbol{F} &= (J\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T})_{ij}(\boldsymbol{F})_{ij} \\
&= J\,\sigma^{dev}_{ik}\,F^{-1}_{jk}\,F_{ij} = J\sigma^{dev}_{ik}\delta_{ik} \\
&= J\underbrace{\boldsymbol{\sigma}^{dev}:\boldsymbol{1}}_{\mathrm{Tr}(\boldsymbol{\sigma}^{dev})=0} = 0
\end{aligned}
$$

Thus,

$$
\mathbf{P}:\boldsymbol{F} = J\sigma_m\boldsymbol{F}^{-T}:\boldsymbol{F} = J\sigma_m\mathrm{Tr}(\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{T}) = J\sigma_m\mathrm{Tr}(\boldsymbol{1}) = 3J\sigma_m
$$

4

The Objectivity of Tensors

4.1 Introduction

Any physical quantity must be invariant for different observers. For example, let us suppose that two observers are located at different positions, (see Figure 4.1), this means they must both detect the same stress state acting on the body for there to be physical meaning.

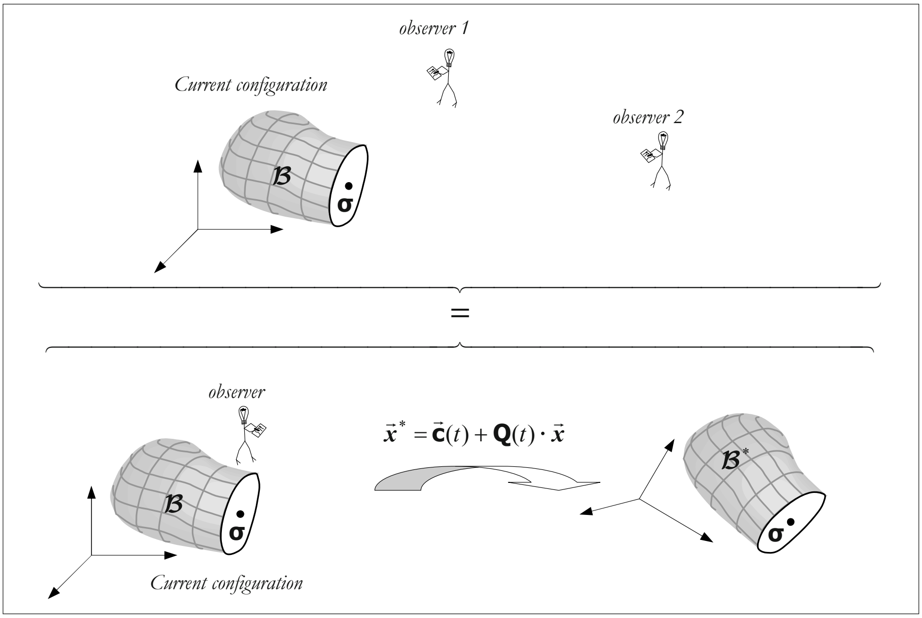


Figure 4.1: Superimposed rigid-body motion.

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_6,

The equivalent to the two observers is that one single observer that records the stress state in the current configuration must be able to compute the same stress state if the continuum undergoes rigid body motion.

When we are dealing with nonlinear problems it is necessary to approach the constitutive equations in rates. As we shall see, in general, the rate of change of the tensor, *e.g.* velocity, acceleration, etc., is not objective, which can be inconvenient when formulating the constitutive equation, which by definition must be objective. Therefore, to overcome this drawback, we will define some rates that are objective.

4.2 The Objectivity of Tensors

Let us consider two possible motions defined by $\boldsymbol{F}$ and $\boldsymbol{F}^*$, where the latter, only differs from the former, by a rigid body motion which in turn is characterized by a proper orthogonal tensor $\mathbf{Q}$, *i.e.* a rotation tensor $\det(\mathbf{Q}) = +1$, (see Figure 4.2). Then the tensor $(\bullet)$ is said to be objective, or frame-indifferent, when its counterpart $\bullet^*$ can be obtained by the corresponding orthogonal transformation. By virtue of the fact that the motion characterized by $\boldsymbol{F}$ generates the stress state $\boldsymbol{\sigma}$, and the motion $\boldsymbol{F}^*$ generates the stress state $\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T$ we have the *principle of objectivity* or *material frame indifference*.

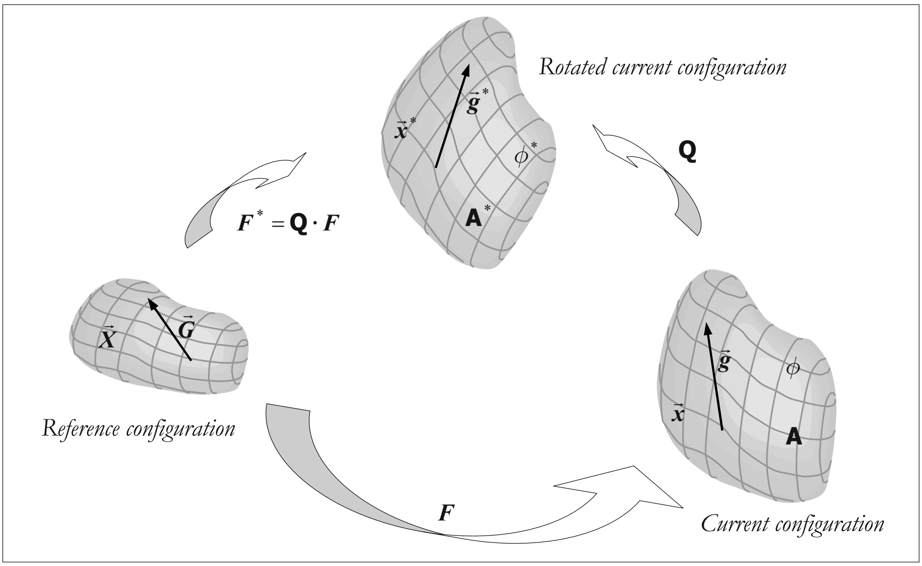


Figure 4.2: Motions.

Scalars

A scalar is objective if:

$$\phi^* = \phi \tag{4.1}$$

Then, all scalars are objective.

Vectors

If $\vec{g}$ is an Eulerian vector which is generated by the motion $\boldsymbol{F}$, then we can state that $\vec{g}$ is objective if its counterpart $\vec{g}^*$, which is generated by $\boldsymbol{F}^*$, is related to $\vec{g}$ by means of the equation:

$$\vec{g}^* = \mathbf{Q} \cdot \vec{g} \tag{4.2}$$

NOTE: We will take this opportunity to mention that the orthogonal transformation law for two-point tensors (pseudo-tensors) is the same as that for vectors. As examples of two-point tensor we can quote: the deformation gradient, $\boldsymbol{F}^* = \mathbf{Q} \cdot \boldsymbol{F}$; the first Piola-Kirchhoff stress tensor, $\mathbf{P}^* = \mathbf{Q} \cdot \mathbf{P}$; and the Polar Decomposition rotation tensor, $\mathbf{R}^* = \mathbf{Q} \cdot \mathbf{R}$. ■

As an example of an objective vector we can quote the area element vector $d\vec{a}$. To prove that, let us consider Figure 4.3, where the area element in the rotated current configuration is defined as:

$$\begin{aligned} d\vec{a}^* = d\vec{x}^{*(1)} \wedge d\vec{x}^{*(2)} \quad &= \boldsymbol{F}^* \cdot d\vec{X}^{(1)} \wedge \boldsymbol{F}^* \cdot d\vec{X}^{(2)} \\ &= (\mathbf{Q} \cdot \boldsymbol{F}) \cdot d\vec{X}^{(1)} \wedge (\mathbf{Q} \cdot \boldsymbol{F}) \cdot d\vec{X}^{(2)} = \mathbf{Q} \cdot d\vec{x}^{(1)} \wedge \mathbf{Q} \cdot d\vec{x}^{(2)} \\ &= \operatorname{cof}(\mathbf{Q})(d\vec{x}^{(1)} \wedge d\vec{x}^{(2)}) = |\mathbf{Q}|(\mathbf{Q}^{-T})(d\vec{x}^{(1)} \wedge d\vec{x}^{(2)}) \\ &= \mathbf{Q} \cdot d\vec{a} \end{aligned} \tag{4.3}$$

Hence, we have demonstrated that $d\vec{a}$ is objective.

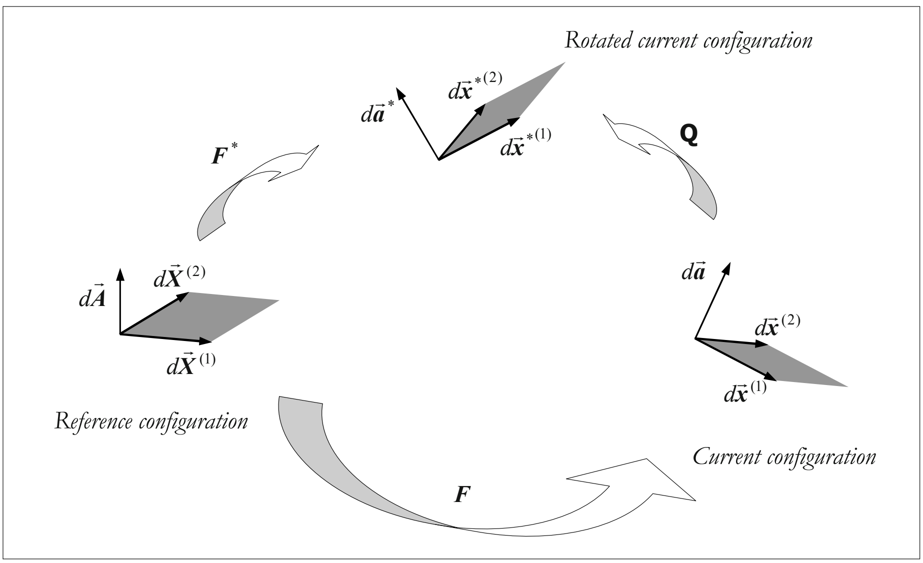


Figure 4.3: Differential area element.

The velocity is the rate of change of displacement and is not objective and neither is the acceleration. To prove this, let us consider a homogeneous motion represented by:

$$\vec{x} = \vec{\mathbf{c}} + \boldsymbol{F} \cdot \vec{X} \tag{4.4}$$

Its velocity is obtained as follows:

$$\dot{\vec{x}} \equiv \vec{v} = \dot{\vec{\mathbf{c}}} + \dot{\boldsymbol{F}} \cdot \vec{X} \tag{4.5}$$

and by applying an orthogonal transformation we obtain:

$$\mathbf{Q} \cdot \dot{\vec{x}} = \mathbf{Q} \cdot \vec{v} = \mathbf{Q} \cdot \dot{\vec{\mathbf{c}}} + \mathbf{Q} \cdot \dot{\boldsymbol{F}} \cdot \vec{X} \tag{4.6}$$

We can also define that

$$\begin{aligned} &\vec{x}^* = \vec{\mathbf{c}} + \boldsymbol{F}^* \cdot \vec{X} \quad \Rightarrow \quad \dot{\vec{x}}^* = \dot{\vec{\mathbf{c}}} + \dot{\boldsymbol{F}}^* \cdot \vec{X} = \dot{\vec{\mathbf{c}}} + \frac{D}{Dt}(\mathbf{Q} \cdot \boldsymbol{F}) \cdot \vec{X} \\ &\Rightarrow \dot{\vec{x}}^* = \dot{\vec{\mathbf{c}}} + \frac{D}{Dt}(\mathbf{Q}) \cdot \boldsymbol{F} \cdot \vec{X} + \mathbf{Q} \cdot \frac{D}{Dt}(\boldsymbol{F}) \cdot \vec{X} \\ &\Rightarrow \dot{\vec{x}}^* = \dot{\vec{\mathbf{c}}} + \dot{\mathbf{Q}} \cdot \boldsymbol{F} \cdot \vec{X} + \mathbf{Q} \cdot \dot{\boldsymbol{F}} \cdot \vec{X} \end{aligned} \tag{4.7}$$

If we compare the last line of the equation (4.7) with (4.6) we can conclude that the velocity is not objective as the only way to achieve this is when $\dot{\vec{\mathbf{c}}} = \vec{\mathbf{0}}$ and $\dot{\mathbf{Q}} = \mathbf{0}$.

Then the rate of change of (4.5) provides us the acceleration, *i.e.*:

$$\ddot{\vec{x}} \equiv \vec{a} = \ddot{\vec{\mathbf{c}}} + \ddot{\boldsymbol{F}} \cdot \vec{X} \tag{4.8}$$

Then by applying an orthogonal transformation we obtain:

$$\mathbf{Q} \cdot \ddot{\vec{x}} \equiv \mathbf{Q} \cdot \vec{a} = \mathbf{Q} \cdot \ddot{\vec{\mathbf{c}}} + \mathbf{Q} \cdot \ddot{\boldsymbol{F}} \cdot \vec{X} \tag{4.9}$$

Similarly, by applying the rate of change in (4.7) and we obtain:

$$\ddot{\vec{x}}^* = \ddot{\vec{\mathbf{c}}} + \ddot{\mathbf{Q}} \cdot \boldsymbol{F} \cdot \vec{X} + \dot{\mathbf{Q}} \cdot \dot{\boldsymbol{F}} \cdot \vec{X} + \dot{\mathbf{Q}} \cdot \dot{\boldsymbol{F}} \cdot \vec{X} + \mathbf{Q} \cdot \ddot{\boldsymbol{F}} \cdot \vec{X} \tag{4.10}$$

If we compare the equations (4.9) with (4.10) we conclude that the acceleration is not objective as this will only be so if and only if $\ddot{\vec{\mathbf{c}}} = \vec{\mathbf{0}}$ and $\dot{\mathbf{Q}} = \mathbf{0}$.

Second-order tensors

If $\mathbf{A}(\vec{x}, t)$ is an Eulerian second-order tensor produced by the motion $\boldsymbol{F}$, then $\mathbf{A}$ is objective if $\mathbf{A}^*$ is related to $\mathbf{A}$ by:

$$\mathbf{A}^*(\vec{x}^*, t) = \mathbf{Q} \cdot \mathbf{A}(\vec{x}, t) \cdot \mathbf{Q}^T \qquad \textit{The Eulerian second-order tensor} \tag{4.11}$$

Another special case is the two-point tensor which is neither in the current nor in the reference configuration. In this case the orthogonal transformation is characterized by:

$$\mathbf{A}^* = \mathbf{Q} \cdot \mathbf{A} \qquad \textit{The two-point tensor} \tag{4.12}$$

The Lagrangian second-order tensor, $\mathbf{A}(\vec{X}, t)$, is objective when the following condition is satisfied:

$$\mathbf{A}^*(\vec{X}, t) = \mathbf{A}(\vec{X}, t) \qquad \textit{The Lagrangian second-order tensor} \tag{4.13}$$

Note that, the reference configuration has not been rotated, (see Figure 4.4).

4.2.1 The Deformation Gradient

As we saw in Chapter 2, the deformation gradient (two-point tensor) relates line elements between reference and current configurations, *i.e.* $d\vec{x} = \boldsymbol{F} \cdot d\vec{X}$, (see Figure 4.4). With components this relation becomes:

$$dx_i = F_{ij} dX_j = \frac{\partial x_i}{\partial X_j} dX_j \tag{4.14}$$

Additionally, we can define the deformation gradient in the rotated current configuration as:

$$F^*_{ij} = \frac{\partial x^*_i}{\partial X_j} = \frac{\partial x^*_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} = \mathrm{Q}_{ik} F_{kj} \quad \Rightarrow \quad \boxed{\boldsymbol{F}^* = \mathbf{Q} \cdot \boldsymbol{F}} \tag{4.15}$$

Here we can notice that $\boldsymbol{F}$ is objective, since $\boldsymbol{F}$ is a two-point tensor.

Now, if we start from $\boldsymbol{F}^* = \mathbf{Q} \cdot \boldsymbol{F}$ we can prove that $d\vec{\boldsymbol{x}} = \boldsymbol{F} \cdot d\vec{\boldsymbol{X}}$ is also objective, *i.e.*:

$$d\vec{\boldsymbol{x}}^* = \boldsymbol{F}^* \cdot d\vec{\boldsymbol{X}} = \mathbf{Q} \cdot \boldsymbol{F} \cdot d\vec{\boldsymbol{X}} = \mathbf{Q} \cdot d\vec{\boldsymbol{x}} \tag{4.16}$$

It is now simple to show, (see Figure 4.4), that the following relationship is valid:

$$\boldsymbol{F} = \mathbf{Q}^T \cdot \boldsymbol{F}^* = \mathbf{Q}^T \cdot \mathbf{Q} \cdot \boldsymbol{F} = \boldsymbol{F} \tag{4.17}$$

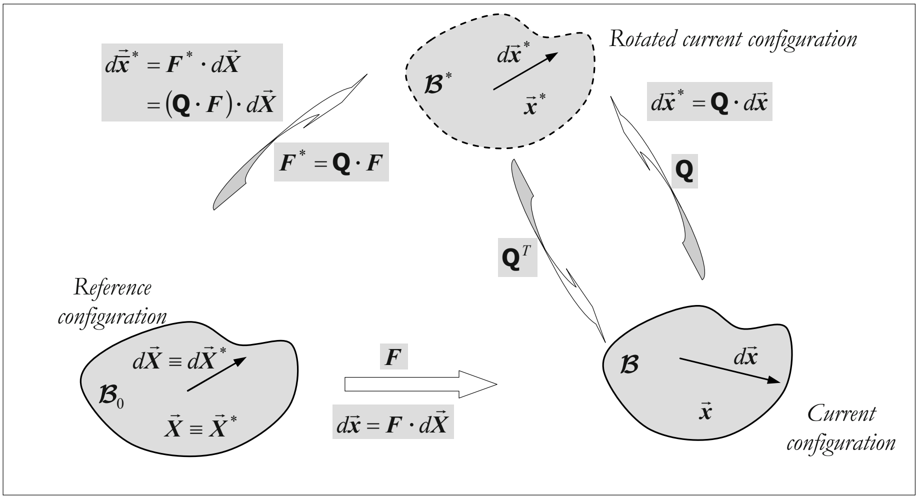


Figure 4.4: Deformation gradient, objectivity.

The inverse of (4.15) is given by:

$$\boldsymbol{F}^{*-1} = (\mathbf{Q} \cdot \boldsymbol{F})^{-1} \quad \Rightarrow \quad \boxed{\boldsymbol{F}^{*-1} = \boldsymbol{F}^{-1} \cdot \mathbf{Q}^T} \tag{4.18}$$

If we start from the definition of the Jacobian determinant $J = \det(\boldsymbol{F})$, then the following is valid:

$$J^* = \det(\boldsymbol{F}^*) = \det(\mathbf{Q} \cdot \boldsymbol{F}) = \underbrace{\det(\mathbf{Q})}_{=1}\det(\boldsymbol{F}) = \det(\boldsymbol{F}) = J \tag{4.19}$$

4.2.2 Kinematic Tensors

Taking into account the rotated current configuration, the right Cauchy-Green deformation tensor ($\boldsymbol{C}(\vec{\boldsymbol{X}}, t) = \boldsymbol{F}^T \cdot \boldsymbol{F}$) is defined as:

$$\boldsymbol{C}^* = \boldsymbol{F}^{*T} \cdot \boldsymbol{F}^* = (\mathbf{Q} \cdot \boldsymbol{F})^T \cdot (\mathbf{Q} \cdot \boldsymbol{F}) = \boldsymbol{F}^T \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \boldsymbol{F} = \boldsymbol{F}^T \cdot \boldsymbol{F} = \boldsymbol{C} \tag{4.20}$$

Then it follows from the above equation that the Green-Lagrange strain tensor, $\boldsymbol{E}(\vec{X},t)=\frac{1}{2}(\boldsymbol{C}-\mathbf{1})$, is given by the following transformation law:

$$\boldsymbol{E}^*=\frac{1}{2}\left(\boldsymbol{C}^*-\mathbf{1}^*\right)=\frac{1}{2}(\boldsymbol{C}-\mathbf{1})=\boldsymbol{E} \tag{4.21}$$

Note that the tensors $\boldsymbol{C}$ and $\boldsymbol{E}$ are objective, since they are defined in the reference configuration.

Given the left Cauchy-Green deformation tensor, $\boldsymbol{b}(\vec{x},t)=\boldsymbol{F}\cdot\boldsymbol{F}^T$, we can conclude that:

$$\boldsymbol{b}^*=\boldsymbol{F}^*\cdot\boldsymbol{F}^{*T}=(\mathbf{Q}\cdot\boldsymbol{F})\cdot(\mathbf{Q}\cdot\boldsymbol{F})^T=\mathbf{Q}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\mathbf{Q}^T=\mathbf{Q}\cdot\boldsymbol{b}\cdot\mathbf{Q}^T \tag{4.22}$$

We can find the same result if we start from the following definition, $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{C}\cdot\boldsymbol{F}^{-1}$, (see Chapter 2), *i.e.*:

$$\begin{aligned}
\boldsymbol{b}^* &= \boldsymbol{F}^*\cdot\boldsymbol{C}^*\cdot\boldsymbol{F}^{*-1}\\
&=\boldsymbol{F}^*\cdot\boldsymbol{F}^{*T}\cdot\boldsymbol{F}^*\cdot\boldsymbol{F}^{*-1}=\boldsymbol{F}^*\cdot\boldsymbol{F}^{*T}\\
&=\boldsymbol{F}^*\cdot\boldsymbol{C}\cdot\boldsymbol{F}^{*-1}\\
&=(\mathbf{Q}\cdot\boldsymbol{F})\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot(\mathbf{Q}\cdot\boldsymbol{F})^{-1}\\
&=\mathbf{Q}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\mathbf{Q}^{-1}\\
&=\mathbf{Q}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^T\cdot\mathbf{Q}^T\\
&=\mathbf{Q}\cdot\boldsymbol{b}\cdot\mathbf{Q}^T
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
\boldsymbol{b}^{*-1}&=(\mathbf{Q}\cdot\boldsymbol{b}\cdot\mathbf{Q}^T)^{-1}\\
&=(\mathbf{Q}^T)^{-1}\cdot\boldsymbol{b}^{-1}\cdot\mathbf{Q}^{-1}\\
&=\mathbf{Q}\cdot\boldsymbol{b}^{-1}\cdot\mathbf{Q}^T
\end{aligned}
\tag{4.23}$$

Consequently, the Almansi strain tensor $\boldsymbol{e}(\vec{x},t)=\frac{1}{2}\left(\mathbf{1}-\boldsymbol{b}^{-1}\right)$ in the rotated current configuration becomes:

$$\boldsymbol{e}^*=\mathbf{Q}\cdot\boldsymbol{e}\cdot\mathbf{Q}^T \tag{4.24}$$

Starting from the polar decomposition defined in Chapter 2, (see Figure 4.5), it follows that:

$$\boldsymbol{F}=\mathbf{R}\cdot\mathbf{U}=\mathbf{V}\cdot\mathbf{R} \quad ; \quad \boldsymbol{F}^*=\mathbf{R}^*\cdot\mathbf{U}^*=\mathbf{V}^*\cdot\mathbf{R}^* \tag{4.25}$$

where $\mathbf{R}$ is the polar decomposition (rotation tensor) proper orthogonal tensor, $\mathbf{U}(\vec{X},t)$ is the right stretch tensor, and $\mathbf{V}(\vec{x},t)$ is the left stretch tensor. Then by taking into account that $\boldsymbol{C}(\vec{X},t)=\boldsymbol{F}^T\cdot\boldsymbol{F}=\mathbf{U}^2$ and the equation in (4.20) we deduce that:

$$\boldsymbol{C}^*=\mathbf{U}^{*2}=\boldsymbol{C}=\mathbf{U}^2 \quad\Rightarrow\quad \mathbf{U}^*=\mathbf{U} \tag{4.26}$$

Then, taking as a starting point the equation $\boldsymbol{F}^*=\mathbf{Q}\cdot\boldsymbol{F}$ we can obtain:

$$\mathbf{R}^*\cdot\mathbf{U}^*=\mathbf{Q}\cdot\mathbf{R}\cdot\mathbf{U} \xrightarrow{\mathbf{U}=\mathbf{U}^*} \boxed{\mathbf{R}^*=\mathbf{Q}\cdot\mathbf{R}} \quad\Rightarrow\quad \mathbf{Q}=\mathbf{R}^*\cdot\mathbf{R}^T \tag{4.27}$$

Thus, $\mathbf{R}$ is an objective two-point tensor.

It is also true that:

$$\boldsymbol{b}^*=\mathbf{V}^{*2}=\mathbf{Q}\cdot\boldsymbol{b}\cdot\mathbf{Q}^T \quad\Rightarrow\quad \boldsymbol{b}^{*-1}=\mathbf{V}^{*-2}=\mathbf{Q}\cdot\boldsymbol{b}^{-1}\cdot\mathbf{Q}^T \tag{4.28}$$

If we use the left polar decomposition we have:

$$\boldsymbol{F}^*=\mathbf{V}^*\cdot\mathbf{R}^* \quad\Rightarrow\quad \mathbf{Q}\cdot\boldsymbol{F}=\mathbf{V}^*\cdot\mathbf{R}^* \quad\Rightarrow\quad \mathbf{Q}\cdot\mathbf{V}\cdot\mathbf{R}=\mathbf{V}^*\cdot\mathbf{R}^* \tag{4.29}$$

Moreover, if we bear in mind that $\mathbf{Q} = \mathbf{R}^* \cdot \mathbf{R}^T$, (see Eq. (4.27)), we can conclude that:

$$\begin{aligned}&\mathbf{Q}\cdot\mathbf{V}\cdot\mathbf{R} = \mathbf{V}^*\cdot\mathbf{R}^* \Rightarrow \mathbf{Q}\cdot\mathbf{V}\cdot\mathbf{R}\cdot\mathbf{R}^T = \mathbf{V}^*\cdot\mathbf{R}^*\cdot\mathbf{R}^T \Rightarrow \mathbf{Q}\cdot\mathbf{V} = \mathbf{V}^*\cdot\mathbf{Q}\\ &\Rightarrow \mathbf{V}^* = \mathbf{Q}\cdot\mathbf{V}\cdot\mathbf{Q}^T\end{aligned} \tag{4.30}$$

Therefore, tensors $\boldsymbol{F}$, $\mathbf{R}$, $\boldsymbol{C}$, $\mathbf{U}$, $\boldsymbol{E}$, $\boldsymbol{b}$, $\mathbf{V}$, $\boldsymbol{e}$ are objective.

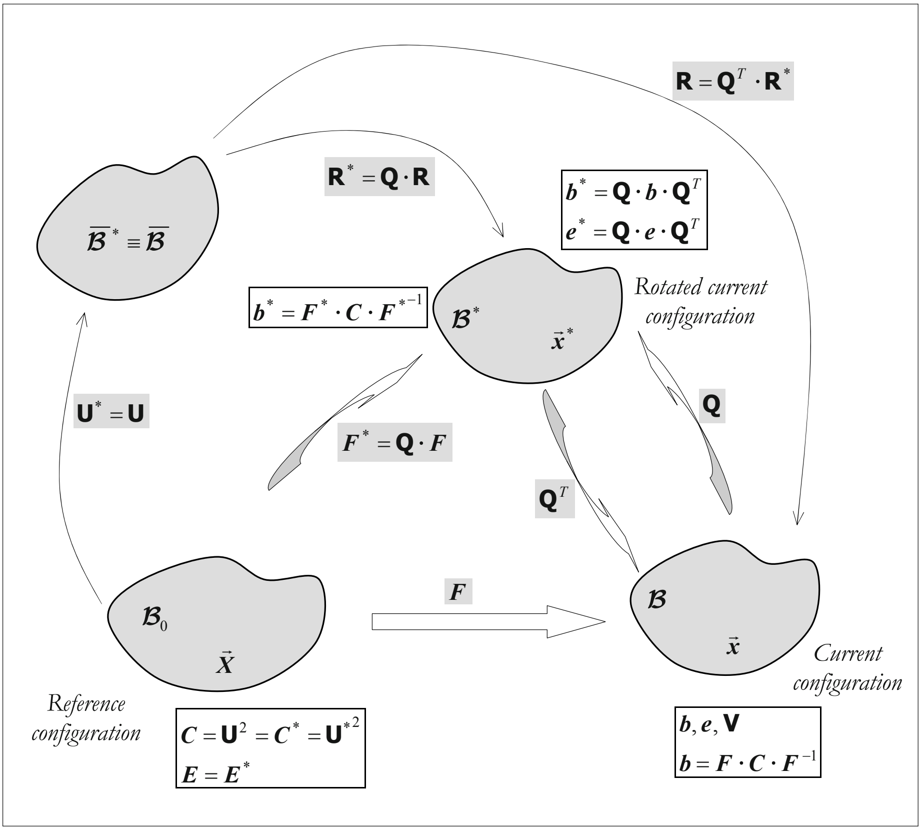


Figure 4.5: Objectivity of the kinematic tensors.

4.2.3 Stress Tensors

The Cauchy stress tensor, $\boldsymbol{\sigma}(\vec{x},t)$, is objective, since it is true that:

$$\boxed{\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^T} \tag{4.31}$$

We can prove the equation (4.31) based on the following equation $\boldsymbol{\sigma}^*\cdot\hat{\mathbf{n}}^* = \vec{\mathbf{t}}^{*(\hat{\mathbf{n}}^*)}$ defined in the rotated current configuration $\mathcal{B}_t^*$, (see Figure 4.6):

$$\begin{aligned}&\boldsymbol{\sigma}^*\cdot\hat{\mathbf{n}}^* = \vec{\mathbf{t}}^{*(\hat{\mathbf{n}}^*)} \quad\Rightarrow\quad \boldsymbol{\sigma}^*\cdot\mathbf{Q}\cdot\hat{\mathbf{n}} = \mathbf{Q}\cdot\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} \quad\Rightarrow\quad \underbrace{\mathbf{Q}^T\cdot\boldsymbol{\sigma}^*\cdot\mathbf{Q}}_{=\boldsymbol{\sigma}}\cdot\hat{\mathbf{n}} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\\ &\Rightarrow \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \vec{\mathbf{t}}^{(\hat{\mathbf{n}})}\end{aligned} \tag{4.32}$$

The first Piola-Kirchhoff stress tensor, defined in Chapter 3, is given by the equation $\mathbf{P} = \det(\boldsymbol{F})\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$. Then, $\mathbf{P}$ can be defined in the rotated current configuration as:

$$\mathbf{P}^* = \det(\boldsymbol{F}^*)\boldsymbol{\sigma}^* \cdot \boldsymbol{F}^{*-T} \tag{4.33}$$

Then if we consider that $\boldsymbol{\sigma}^* = \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T$ and $\boldsymbol{F}^* = \mathbf{Q} \cdot \boldsymbol{F}$, the equation in (4.33) becomes:

$$\left.\begin{aligned} \mathbf{P}^* &= \det(\boldsymbol{F}^*)\boldsymbol{\sigma}^* \cdot \boldsymbol{F}^{*-T} \\ &= \det(\mathbf{Q} \cdot \boldsymbol{F})\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T \cdot (\mathbf{Q} \cdot \boldsymbol{F})^{-T} \\ &= \det(\mathbf{Q})\det(\boldsymbol{F})\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T \cdot (\boldsymbol{F}^{-1} \cdot \mathbf{Q}^T)^T \\ &= \det(\boldsymbol{F})\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \boldsymbol{F}^{-T} \\ &= \mathbf{Q} \cdot \underbrace{\det(\boldsymbol{F})\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}}_{\mathbf{P}} \end{aligned}\right\} \Rightarrow \boxed{\mathbf{P}^* = \mathbf{Q} \cdot \mathbf{P}} \tag{4.34}$$

Note that the first Piola-Kirchhoff stress tensor is a two-point tensor whose transformation is defined according to the transformation law of vectors, (see Figure 4.6), hence, the first Piola-Kirchhoff stress tensor is objective.

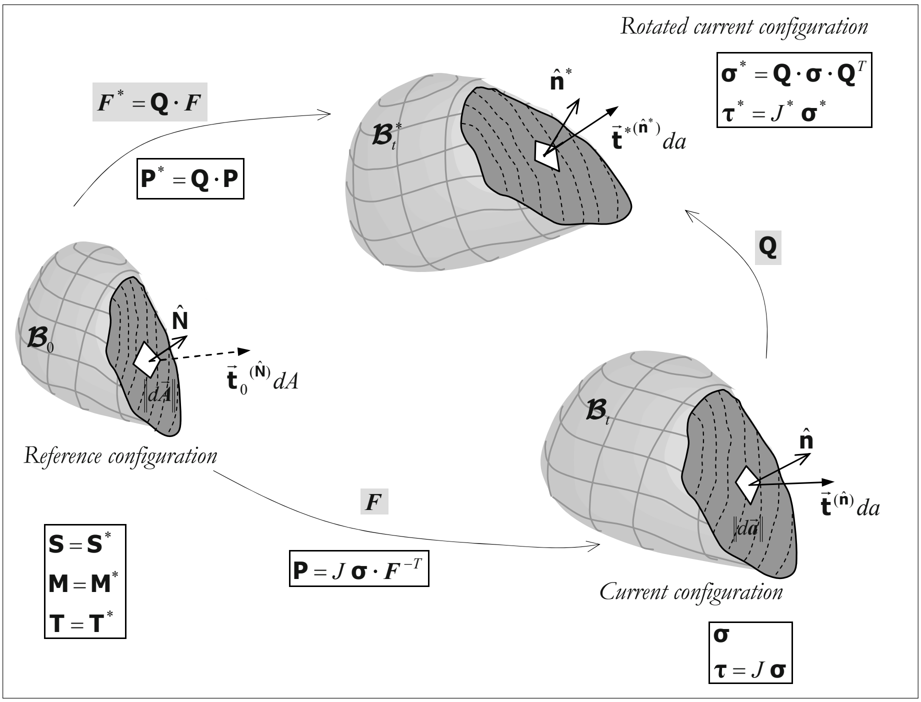


Figure 4.6: Objectivity of the stress tensors.

Then if we refer to the definition of the second Piola-Kirchhoff stress tensor, $\mathbf{S}(\vec{X}, t)$, we can conclude that:

$$\left.\begin{aligned} \mathbf{S}^* &= \det(\boldsymbol{F}^*)\boldsymbol{F}^{*-1} \cdot \boldsymbol{\sigma}^* \cdot \boldsymbol{F}^{*-T} \\ &= \det(\mathbf{Q} \cdot \boldsymbol{F})\boldsymbol{F}^{-1} \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \boldsymbol{F}^{-T} \\ &= \det(\boldsymbol{F})\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} \end{aligned}\right\} \Rightarrow \boxed{\mathbf{S}^* = \mathbf{S}} \tag{4.35}$$

The tensor $\mathbf{S}^* = \mathbf{S}$ is defined in the reference configuration, so it is objective.

Now, if we consider the Kirchhoff stress tensor, $\boldsymbol{\tau}(\vec{x},t) = J\,\boldsymbol{\sigma}$, it follows that in the rotated current configuration we obtain:

$$\boldsymbol{\tau}^* = J^*\,\boldsymbol{\sigma}^* = J\,\mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^T = \mathbf{Q}\cdot J\,\boldsymbol{\sigma}\cdot\mathbf{Q}^T = \mathbf{Q}\cdot\boldsymbol{\tau}\cdot\mathbf{Q}^T \tag{4.36}$$

We can notice from the above that $\boldsymbol{\tau}$ is also objective.

Similarly, we can verify the objectivity of the Mandel ($\mathbf{M} = \boldsymbol{F}^T\cdot\mathbf{P}$) and Biot ($\mathbf{T} = \mathbf{U}\cdot\mathbf{S}$) stress tensors:

$$\begin{aligned}\mathbf{M}^* &= \boldsymbol{F}^{*T}\cdot\mathbf{P}^* = (\mathbf{Q}\cdot\boldsymbol{F})^T\cdot(\mathbf{Q}\cdot\mathbf{P})\\ &= \boldsymbol{F}^T\cdot\mathbf{P}\\ &= \mathbf{M}(\vec{X},t)\end{aligned} \qquad\Bigg|\qquad \begin{aligned}\mathbf{T}^* &= \mathbf{U}^*\cdot\mathbf{S}^*\\ &= \mathbf{U}\cdot\mathbf{S}\\ &= \mathbf{T}(\vec{X},t)\end{aligned} \tag{4.37}$$

4.3 Tensor Rates

Before introducing objective rates we need to evaluate some equations that will be useful in generating these.

The material time derivative of $\boldsymbol{F}^* = \mathbf{Q}\cdot\boldsymbol{F}$ is given by:

$$\dot{\boldsymbol{F}}^* = \dot{\mathbf{Q}}\cdot\boldsymbol{F} + \mathbf{Q}\cdot\dot{\boldsymbol{F}} \tag{4.38}$$

Additionally, the material time derivative of the inverse $\boldsymbol{F}^{*-1} = \boldsymbol{F}^{-1}\cdot\mathbf{Q}^T$ becomes:

$$\dot{\boldsymbol{F}}^{*-1} = \boldsymbol{F}^{-1}\cdot\dot{\mathbf{Q}}^T + \dot{\boldsymbol{F}}^{-1}\cdot\mathbf{Q}^T \tag{4.39}$$

Then the spatial velocity gradient $\boldsymbol{\ell} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}$ can be evaluated in the rotated current configuration as follows:

$$\boldsymbol{\ell}^* = \dot{\boldsymbol{F}}^*\cdot\boldsymbol{F}^{*-1} = \left(\dot{\mathbf{Q}}\cdot\boldsymbol{F} + \mathbf{Q}\cdot\dot{\boldsymbol{F}}\right)\cdot\left(\boldsymbol{F}^{-1}\cdot\mathbf{Q}^T\right) = \dot{\mathbf{Q}}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\cdot\mathbf{Q}^T \tag{4.40}$$

$$\boxed{\boldsymbol{\ell}^* = \dot{\mathbf{Q}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\boldsymbol{\ell}\cdot\mathbf{Q}^T} \tag{4.41}$$

Hence, the tensor $\boldsymbol{\ell}$ is not objective due to the additional term ($\dot{\mathbf{Q}}\cdot\mathbf{Q}^T$) that appears in the equation (4.41) by means of which we can obtain $\dot{\mathbf{Q}}$ as:

$$\boxed{\dot{\mathbf{Q}} = \boldsymbol{\ell}^*\cdot\mathbf{Q} - \mathbf{Q}\cdot\boldsymbol{\ell} \quad\Rightarrow\quad \dot{\mathbf{Q}}^T = \mathbf{Q}^T\cdot\boldsymbol{\ell}^{*T} - \boldsymbol{\ell}^T\cdot\mathbf{Q}^T} \tag{4.42}$$

Then if we consider that $\mathbf{Q}\cdot\mathbf{Q}^T = \mathbf{1}$, we obtain:

$$\frac{D}{Dt}\left(\mathbf{Q}\cdot\mathbf{Q}^T\right) = \frac{D}{Dt}(\mathbf{1}) = \mathbf{0} \Rightarrow \dot{\mathbf{Q}}\cdot\mathbf{Q}^T = -\mathbf{Q}\cdot\dot{\mathbf{Q}}^T \tag{4.43}$$

Afterwards the equation in (4.41) can still be rewritten as:

$$\boxed{\boldsymbol{\ell}^* = -\mathbf{Q}\cdot\dot{\mathbf{Q}}^T + \mathbf{Q}\cdot\boldsymbol{\ell}\cdot\mathbf{Q}^T} \tag{4.44}$$

Then if we use (4.44) we can obtain another way to express the rate of $\mathbf{Q}$:

$$\boxed{\dot{\mathbf{Q}} = \mathbf{Q} \cdot \boldsymbol{\ell}^T - \boldsymbol{\ell}^{*T} \cdot \mathbf{Q} \quad ; \quad \dot{\mathbf{Q}}^T = \boldsymbol{\ell} \cdot \mathbf{Q}^T - \mathbf{Q}^T \cdot \boldsymbol{\ell}^*} \tag{4.45}$$

Let us now consider the symmetric part of $\boldsymbol{\ell}^*$, which by definition is $\mathbf{D}^*$, *i.e.*:

$$\boldsymbol{\ell}^{*sym} \equiv \mathbf{D}^* = \frac{1}{2}\left(\boldsymbol{\ell}^* + \boldsymbol{\ell}^{*T}\right) \tag{4.46}$$

Then by substituting $\boldsymbol{\ell}^*$, given in the equation (4.41), into the above equation (4.46), we obtain:

$$\mathbf{D}^* = \frac{1}{2}\left[\left(\dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \boldsymbol{\ell} \cdot \mathbf{Q}^T\right) + \left(\mathbf{Q} \cdot \dot{\mathbf{Q}}^T + \mathbf{Q} \cdot \boldsymbol{\ell}^T \cdot \mathbf{Q}^T\right)\right] \tag{4.47}$$

If we then bear in mind the equation in (4.43), the above becomes:

$$\mathbf{D}^* = \frac{1}{2}\left[\mathbf{Q} \cdot \boldsymbol{\ell} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \boldsymbol{\ell}^T \cdot \mathbf{Q}^T\right] = \mathbf{Q} \cdot \frac{1}{2}\left[\boldsymbol{\ell} + \boldsymbol{\ell}^T\right] \cdot \mathbf{Q}^T \tag{4.48}$$

$$\boxed{\mathbf{D}^* = \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T} \tag{4.49}$$

Thus, the rate-of-deformation tensor ($\mathbf{D}$) is objective.

If we now return to the equation in (4.41) and consider that $\boldsymbol{\ell}$ has been broken down into a symmetric ($\mathbf{D}$: rate-of-deformation tensor) and an antisymmetric ($\mathbf{W}$: spin tensor) part, *i.e.* $\boldsymbol{\ell} = \mathbf{D} + \mathbf{W}$, we can state that:

$$\begin{aligned} \boldsymbol{\ell}^* = \mathbf{D}^* + \mathbf{W}^* \quad &= \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \boldsymbol{\ell} \cdot \mathbf{Q}^T \\ &= \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot (\mathbf{D} + \mathbf{W}) \cdot \mathbf{Q}^T \\ &= \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T \end{aligned} \tag{4.50}$$

and if we consider $\mathbf{D}^* = \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T$, we can obtain $\mathbf{W}^*$ as:

$$\boxed{\mathbf{W}^* = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T} \tag{4.51}$$

Thus, $\mathbf{W}^*$ is not objective since $\mathbf{W}^* \neq \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T$. Therefore, from the equation in (4.51) we can obtain:

$$-\dot{\mathbf{Q}} \cdot \mathbf{Q}^T = \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T - \mathbf{W}^* \quad \Rightarrow \quad \dot{\mathbf{Q}} = \mathbf{W}^* \cdot \mathbf{Q} - \mathbf{Q} \cdot \mathbf{W} \tag{4.52}$$

Moreover, if we also consider that $-\dot{\mathbf{Q}} \cdot \mathbf{Q}^T = \left(\dot{\mathbf{Q}} \cdot \mathbf{Q}^T\right)^T = \mathbf{Q} \cdot \dot{\mathbf{Q}}^T$, the above equation becomes:

$$-\dot{\mathbf{Q}} \cdot \mathbf{Q}^T = \mathbf{Q} \cdot \dot{\mathbf{Q}}^T = \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T - \mathbf{W}^* \quad \Rightarrow \quad \dot{\mathbf{Q}}^T = \mathbf{W} \cdot \mathbf{Q}^T - \mathbf{Q}^T \cdot \mathbf{W}^* \tag{4.53}$$

4.3.1 Objective Rates

If we consider an arbitrary vector $\vec{\mathbf{a}}$, then the orthogonal transformation is given by:

$$\vec{\mathbf{a}}^* = \mathbf{Q} \cdot \vec{\mathbf{a}} \tag{4.54}$$

whose rate becomes:

$$\dot{\vec{\mathbf{a}}}^* = \dot{\mathbf{Q}} \cdot \vec{\mathbf{a}} + \mathbf{Q} \cdot \dot{\vec{\mathbf{a}}} \tag{4.55}$$

The above proves that the rate of change of $\vec{\mathbf{a}}$ is not objective, since an additional term ($\dot{\mathbf{Q}}\cdot\vec{\mathbf{a}}$) appears in the above equation.

As we have seen before, a second-order tensor that is defined in the current configuration is objective if it holds that:

$$\mathbf{A}^* = \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T \tag{4.56}$$

Then the material time derivative of $\mathbf{A}^* = \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T$ can be evaluated as follows:

$$\dot{\mathbf{A}}^* = \dot{\mathbf{Q}}\cdot\mathbf{A}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\mathbf{A}\cdot\dot{\mathbf{Q}}^T \tag{4.57}$$

Thus, we can conclude that $\dot{\mathbf{A}}$ is not objective, since $\dot{\mathbf{A}}^* \neq \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^T$. We can then define some objective rates.

4.3.1.1 The Convective Rate

Let us now consider the expression of $\dot{\mathbf{Q}}$, given in (4.45), and by substituting it into the equation in (4.55) we obtain:

$$\begin{aligned}
\dot{\vec{\mathbf{a}}}^* = \dot{\mathbf{Q}}\cdot\vec{\mathbf{a}} + \mathbf{Q}\cdot\dot{\vec{\mathbf{a}}} &= -\boldsymbol{\ell}^{*T}\cdot\mathbf{Q}\cdot\vec{\mathbf{a}} + \mathbf{Q}\cdot\boldsymbol{\ell}^T\cdot\vec{\mathbf{a}} + \mathbf{Q}\cdot\dot{\vec{\mathbf{a}}} \\
\dot{\vec{\mathbf{a}}}^* + \boldsymbol{\ell}^{*T}\cdot\mathbf{Q}\cdot\vec{\mathbf{a}} &= \mathbf{Q}\cdot\boldsymbol{\ell}^T\cdot\vec{\mathbf{a}} + \mathbf{Q}\cdot\dot{\vec{\mathbf{a}}} \\
\dot{\vec{\mathbf{a}}}^* + \boldsymbol{\ell}^{*T}\cdot\vec{\mathbf{a}}^* &= \mathbf{Q}\cdot\left(\dot{\vec{\mathbf{a}}} + \boldsymbol{\ell}^T\cdot\vec{\mathbf{a}}\right) \\
\overset{\mathrm{C}}{\vec{\mathbf{a}}}{}^* &= \mathbf{Q}\cdot\overset{\mathrm{C}}{\vec{\mathbf{a}}}
\end{aligned} \tag{4.58}$$

The rate ($\overset{\mathrm{C}}{\bullet}$) indicates the *convective rate* with which we can introduce a new vector rate $\overset{\mathrm{C}}{\vec{\mathbf{a}}}$, which is objective and is defined as:

$$\boxed{\overset{\mathrm{C}}{\vec{\mathbf{a}}} = \dot{\vec{\mathbf{a}}} + \boldsymbol{\ell}^T\cdot\vec{\mathbf{a}}} \quad \textit{The convective rate} \tag{4.59}$$

4.3.1.2 The Oldroyd Rate

If we use the equation given in (4.42), *i.e.* $\dot{\mathbf{Q}} = \boldsymbol{\ell}^*\cdot\mathbf{Q} - \mathbf{Q}\cdot\boldsymbol{\ell}$ and $\dot{\mathbf{Q}}^T = \mathbf{Q}^T\cdot\boldsymbol{\ell}^{*T} - \boldsymbol{\ell}^T\cdot\mathbf{Q}^T$, and by substituting them into the equation in (4.57) we obtain:

$$\dot{\mathbf{A}}^* = \left(\boldsymbol{\ell}^*\cdot\mathbf{Q} - \mathbf{Q}\cdot\boldsymbol{\ell}\right)\cdot\mathbf{A}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\mathbf{A}\cdot\left(\mathbf{Q}^T\cdot\boldsymbol{\ell}^{*T} - \boldsymbol{\ell}^T\cdot\mathbf{Q}^T\right) \tag{4.60}$$

or

$$\begin{aligned}
\dot{\mathbf{A}}^* - \boldsymbol{\ell}^*\cdot\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T - \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T\cdot\boldsymbol{\ell}^{*T} &= -\mathbf{Q}\cdot\boldsymbol{\ell}\cdot\mathbf{A}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^T - \mathbf{Q}\cdot\mathbf{A}\cdot\boldsymbol{\ell}^T\cdot\mathbf{Q}^T \\
\dot{\mathbf{A}}^* - \boldsymbol{\ell}^*\cdot\underbrace{\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T}_{\mathbf{A}^*} - \underbrace{\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T}_{\mathbf{A}^*}\cdot\boldsymbol{\ell}^{*T} &= \mathbf{Q}\cdot\left(\dot{\mathbf{A}} - \boldsymbol{\ell}\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\ell}^T\right)\cdot\mathbf{Q}^T \\
\dot{\mathbf{A}}^* - \boldsymbol{\ell}^*\cdot\mathbf{A}^* - \mathbf{A}^*\cdot\boldsymbol{\ell}^{*T} &= \mathbf{Q}\cdot\left(\dot{\mathbf{A}} - \boldsymbol{\ell}\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\ell}^T\right)\cdot\mathbf{Q}^T \\
\overset{\square}{\mathbf{A}}{}^* &= \mathbf{Q}\cdot\overset{\square}{\mathbf{A}}\cdot\mathbf{Q}^T
\end{aligned} \tag{4.61}$$

which defines a new objective rate known as the *Oldroyd rate:*

$$\boxed{\overset{\square}{\mathbf{A}} = \dot{\mathbf{A}} - \boldsymbol{\ell}\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\ell}^T} \quad \textit{The Oldroyd rate} \tag{4.62}$$

Problem 4.1: Obtain the Oldroyd rate of the left Cauchy-Green deformation tensor ($\boldsymbol{b}$).
Solution:
Based on the definition of the Oldroyd rate in (4.62), we can obtain the Oldroyd rate of $\boldsymbol{b}$ as $\overset{\square}{\boldsymbol{b}} = \dot{\boldsymbol{b}} - \boldsymbol{\ell}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{\ell}^T$. Then if we refer to $\boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{F}^T$, the material time derivative of $\boldsymbol{b}$ becomes $\dot{\boldsymbol{b}} = \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^T + \boldsymbol{F}\cdot\dot{\boldsymbol{F}}^T = (\boldsymbol{\ell}\cdot\boldsymbol{F})\cdot\boldsymbol{F}^T + \boldsymbol{F}\cdot(\boldsymbol{\ell}\cdot\boldsymbol{F})^T = \boldsymbol{\ell}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{\ell}^T$. Thus, we can conclude that $\overset{\square}{\boldsymbol{b}} = \dot{\boldsymbol{b}} - \boldsymbol{\ell}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{\ell}^T = \boldsymbol{\ell}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{\ell}^T - \boldsymbol{\ell}\cdot\boldsymbol{b} - \boldsymbol{b}\cdot\boldsymbol{\ell}^T = \mathbf{0}$.

4.3.1.3 The Cotter-Rivlin Rate

If instead of using the equation related to $\dot{\mathbf{Q}}$ and $\dot{\mathbf{Q}}^T$ given in (4.42), we use the equations given in (4.45), and by substituting them into (4.57) we obtain:

$$\begin{aligned}
&\dot{\mathbf{A}}^* = \left(\mathbf{Q}\cdot\boldsymbol{\ell}^T - \boldsymbol{\ell}^{*T}\cdot\mathbf{Q}\right)\cdot\mathbf{A}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\mathbf{A}\cdot\left(\boldsymbol{\ell}\cdot\mathbf{Q}^T - \mathbf{Q}^T\cdot\boldsymbol{\ell}^*\right)\\
&\dot{\mathbf{A}}^* + \boldsymbol{\ell}^{*T}\cdot\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T\cdot\boldsymbol{\ell}^* = \mathbf{Q}\cdot\boldsymbol{\ell}^T\cdot\mathbf{A}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\dot{\mathbf{A}}\cdot\mathbf{Q}^T + \mathbf{Q}\cdot\mathbf{A}\cdot\boldsymbol{\ell}\cdot\mathbf{Q}^T\\
&\dot{\mathbf{A}}^* + \boldsymbol{\ell}^{*T}\cdot\underbrace{\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T}_{\mathbf{A}^*} + \underbrace{\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T}_{\mathbf{A}^*}\cdot\boldsymbol{\ell}^* = \mathbf{Q}\cdot\left(\dot{\mathbf{A}} + \boldsymbol{\ell}^T\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\ell}\right)\cdot\mathbf{Q}^T\\
&\dot{\mathbf{A}}^* + \boldsymbol{\ell}^{*T}\cdot\mathbf{A}^* + \mathbf{A}^*\cdot\boldsymbol{\ell}^* = \mathbf{Q}\cdot\left(\dot{\mathbf{A}} + \boldsymbol{\ell}^T\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\ell}\right)\cdot\mathbf{Q}^T\\
&\overset{\Delta}{\mathbf{A}}{}^* = \mathbf{Q}\cdot\overset{\Delta}{\mathbf{A}}\cdot\mathbf{Q}^T
\end{aligned}\tag{4.63}$$

Thus, we can obtain a new objective rate, the Cotter-Rivlin rate:

$$\boxed{\overset{\Delta}{\mathbf{A}} = \dot{\mathbf{A}} + \boldsymbol{\ell}^T\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\ell}} \quad \textit{The Cotter-Rivlin rate} \tag{4.64}$$

Problem 4.2: Obtain the Cotter-Rivlin rate of the Almansi strain tensor ($\boldsymbol{e}$) in terms of the rate-of-deformation tensor ($\mathbf{D}$).
Solution: Based on the definition of the Cotter-Rivlin rate in (4.64), the Cotter-Rivlin rate of $\boldsymbol{e}$ is $\overset{\Delta}{\boldsymbol{e}} = \dot{\boldsymbol{e}} + \boldsymbol{\ell}^T\cdot\boldsymbol{e} + \boldsymbol{e}\cdot\boldsymbol{\ell}$. Remember that in Chapter 2 we saw that the rate of the Almansi strain tensor ($\dot{\boldsymbol{e}}$) is related to the rate-of-deformation tensor $\mathbf{D}$ by the equation:

$$\mathbf{D} = \dot{\boldsymbol{e}} + \boldsymbol{\ell}^T\cdot\boldsymbol{e} + \boldsymbol{e}\cdot\boldsymbol{\ell} \quad \Rightarrow \quad \dot{\boldsymbol{e}} = \mathbf{D} - \boldsymbol{\ell}^T\cdot\boldsymbol{e} - \boldsymbol{e}\cdot\boldsymbol{\ell}$$

And by substituting $\dot{\boldsymbol{e}}$ into the Cotter-Rivlin rate $\overset{\Delta}{\boldsymbol{e}}$ we obtain $\overset{\Delta}{\boldsymbol{e}} = \mathbf{D}$.

4.3.1.4 The Jaumann-Zaremba Rate

If we now consider $\dot{\mathbf{Q}}$ given in (4.52) and $\dot{\mathbf{Q}}^T$ given in (4.53), and substitute them into the equation in (4.57) we obtain:

$$\dot{\mathbf{A}}^* = \mathbf{Q}\cdot\left(\dot{\mathbf{A}} - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W}\right)\cdot\mathbf{Q}^T + \mathbf{W}^*\cdot\mathbf{A}^* - \mathbf{A}^*\cdot\mathbf{W}^* \tag{4.65}$$

Then by rearranging the above equation, we obtain:

$$\dot{\mathbf{A}}^* - \mathbf{W}^*\cdot\mathbf{A}^* + \mathbf{A}^*\cdot\mathbf{W}^* = \mathbf{Q}\cdot\left(\dot{\mathbf{A}} - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W}\right)\cdot\mathbf{Q}^T \quad \Rightarrow \quad \overset{\circ}{\mathbf{A}}{}^* = \mathbf{Q}\cdot\overset{\circ}{\mathbf{A}}\cdot\mathbf{Q}^T \tag{4.66}$$

We can conclude by the above that the rate $\overset{\circ}{\mathbf{A}}$, called the *Jaumann-Zaremba rate*, is objective and is given by:

$$\boxed{\overset{\circ}{\mathbf{A}} = \dot{\mathbf{A}} - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W}} \quad \textit{The Jaumann-Zaremba rate} \tag{4.67}$$

Next, we can interrelate the rates $\overset{\square}{\mathbf{A}}$, $\overset{\circ}{\mathbf{A}}$ and $\overset{\Delta}{\mathbf{A}}$. To do this let us consider the following equations:

$$\overset{\square}{\mathbf{A}} = \dot{\mathbf{A}} - \boldsymbol{\ell}\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\ell}^T \quad \Rightarrow \quad \dot{\mathbf{A}} = \overset{\square}{\mathbf{A}} + \boldsymbol{\ell}\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\ell}^T \tag{4.68}$$

$$\overset{\circ}{\mathbf{A}} = \dot{\mathbf{A}} - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W} \quad \Rightarrow \quad \dot{\mathbf{A}} = \overset{\circ}{\mathbf{A}} + \mathbf{W}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{W} \tag{4.69}$$

$$\overset{\Delta}{\mathbf{A}} = \dot{\mathbf{A}} + \boldsymbol{\ell}^T\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\ell} \quad \Rightarrow \quad \dot{\mathbf{A}} = \overset{\Delta}{\mathbf{A}} - \boldsymbol{\ell}^T\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\ell} \tag{4.70}$$

By combining (4.68) and (4.69) we obtain:

$$\begin{aligned}
&\overset{\circ}{\mathbf{A}} + \mathbf{W}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{W} = \overset{\square}{\mathbf{A}} + \boldsymbol{\ell}\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\ell}^T \\
&\overset{\circ}{\mathbf{A}} = \overset{\square}{\mathbf{A}} + \boldsymbol{\ell}\cdot\mathbf{A} + \mathbf{A}\cdot\boldsymbol{\ell}^T - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W} \\
&\overset{\circ}{\mathbf{A}} = \overset{\square}{\mathbf{A}} + (\mathbf{D}+\mathbf{W})\cdot\mathbf{A} + \mathbf{A}\cdot(\mathbf{D}+\mathbf{W})^T - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W}
\end{aligned} \tag{4.71}$$

Then, we can connect the Jaumann-Zaremba rate to the Oldroyd rate by:

$$\boxed{\overset{\circ}{\mathbf{A}} = \overset{\square}{\mathbf{A}} + \mathbf{D}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{D}} \quad \textit{Relationship between the Jaumann-Zaremba rate and the Oldroyd rate} \tag{4.72}$$

Afterwards, by combining (4.69) and (4.70) we obtain:

$$\begin{aligned}
&\overset{\circ}{\mathbf{A}} + \mathbf{W}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{W} = \overset{\Delta}{\mathbf{A}} - \boldsymbol{\ell}^T\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\ell} \Rightarrow \overset{\circ}{\mathbf{A}} = \overset{\Delta}{\mathbf{A}} - \boldsymbol{\ell}^T\cdot\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\ell} - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W} \\
&\Rightarrow \overset{\circ}{\mathbf{A}} = \overset{\Delta}{\mathbf{A}} - (\mathbf{D}+\mathbf{W})\cdot\mathbf{A} - \mathbf{A}\cdot(\mathbf{D}+\mathbf{W})^T - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W}
\end{aligned} \tag{4.73}$$

Then, we obtain the relationship between the Jaumann-Zaremba rate and the Cotter-Rivlin rate:

$$\boxed{\overset{\circ}{\mathbf{A}} = \overset{\Delta}{\mathbf{A}} - \mathbf{D}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{D}} \quad \textit{Relationship between the Jaumann-Zaremba rate and the Cotter-Rivlin rate} \tag{4.74}$$

Now by adding equations (4.72) and (4.74) we can reach the following conclusion:

$$\boxed{2\overset{\circ}{\mathbf{A}} = \overset{\square}{\mathbf{A}} + \overset{\Delta}{\mathbf{A}}} \tag{4.75}$$

Problem 4.3: Let $\mathbf{A}$ be a symmetric second-order tensor. Prove that $\frac{D}{Dt}(\mathbf{A}:\mathbf{A}) = 2\mathbf{A}:\overset{\circ}{\mathbf{A}}$, where $\overset{\circ}{\mathbf{A}}$ is the Jaumann-Zaremba rate of $\mathbf{A}$.

Solution:

$$\frac{D}{Dt}(\mathbf{A}:\mathbf{A}) = \dot{\mathbf{A}}:\mathbf{A} + \mathbf{A}:\dot{\mathbf{A}}$$

If we now incorporate $\dot{\mathbf{A}} = \overset{\circ}{\mathbf{A}} + \mathbf{W}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{W}$ into the above equation, we obtain:

$$\begin{aligned}
\frac{D}{Dt}(\mathbf{A}:\mathbf{A}) &= \dot{\mathbf{A}}:\mathbf{A} + \mathbf{A}:\dot{\mathbf{A}} = (\overset{\circ}{\mathbf{A}} + \mathbf{W}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{W}):\mathbf{A} + \mathbf{A}:(\overset{\circ}{\mathbf{A}} + \mathbf{W}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{W}) \\
&= \overset{\circ}{\mathbf{A}}:\mathbf{A} + (\mathbf{W}\cdot\mathbf{A}):\mathbf{A} - (\mathbf{A}\cdot\mathbf{W}):\mathbf{A} + \mathbf{A}:\overset{\circ}{\mathbf{A}} + \mathbf{A}:(\mathbf{W}\cdot\mathbf{A}) - \mathbf{A}:(\mathbf{A}\cdot\mathbf{W}) \\
&= 2\mathbf{A}:\overset{\circ}{\mathbf{A}} + 2\mathbf{A}:(\mathbf{W}\cdot\mathbf{A}) - 2\mathbf{A}:(\mathbf{A}\cdot\mathbf{W}) = 2\mathbf{A}:\overset{\circ}{\mathbf{A}}
\end{aligned}$$

where we have applied the commutative property of the double scalar product, *i.e.* $\mathbf{A}:\mathbf{B}=\mathbf{B}:\mathbf{A}$. Note that, due to the symmetry of $\mathbf{A}$ the following condition is satisfied:

$$\mathbf{A}:(\mathbf{W}\cdot\mathbf{A})=\mathrm{A}_{ij}(\mathrm{W}_{ik}\mathrm{A}_{kj})=\mathrm{A}_{jk}(\mathrm{A}_{ji}\mathrm{W}_{ik})=\mathbf{A}:(\mathbf{A}\cdot\mathbf{W})$$

4.3.1.5 The Green-Naghdi Rate (Polar Rate)

Let us now refer back to the Polar Decomposition of $\boldsymbol{F}$, (see Chapter 2), in which we obtained the following equation for the spin tensor:

$$\mathbf{W}=\frac{1}{2}\mathbf{R}\cdot\left(\dot{\mathbf{U}}\cdot\mathbf{U}^{-1}-\dot{\mathbf{U}}^{-1}\cdot\mathbf{U}\right)\cdot\mathbf{R}^{T}+\dot{\mathbf{R}}\cdot\mathbf{R}^{T} \tag{4.76}$$

If $\dot{\mathbf{U}}=\mathbf{0}$ the above equation is reduced to $\mathbf{W}=\dot{\mathbf{R}}\cdot\mathbf{R}^{T}$. Therefore $\boldsymbol{\ell}=\mathbf{W}=\dot{\mathbf{R}}\cdot\mathbf{R}^{T}$, and by substituting this into the equation in (4.64) we obtain:

$$\begin{aligned}&\overset{\Delta}{\mathbf{A}}\xrightarrow{\boldsymbol{\ell}=\mathbf{W}}\overset{\circ}{\mathbf{A}}\\ &\overset{\circ}{\mathbf{A}}=\dot{\mathbf{A}}+\mathbf{W}^{T}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{W}\\ &\quad=\dot{\mathbf{A}}-\mathbf{W}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{W}\end{aligned} \tag{4.77}$$

which is the same equation as that obtained in (4.67). Then, we can define the *Green-Naghdi rate,* also known as the *Polar rate* or *Green-McInnis rate*, by:

$$\boxed{\overset{\nabla}{\mathbf{A}}=\dot{\mathbf{A}}-\left(\mathbf{R}\cdot\dot{\mathbf{R}}^{T}\right)\cdot\mathbf{A}+\mathbf{A}\cdot\left(\dot{\mathbf{R}}\cdot\mathbf{R}^{T}\right)} \tag{4.78}$$

Green-Naghdi rate or *Polar rate* or *Green-McInnis rate*

4.3.2 The Objective Rate of Stress Tensors

The material time derivative of the first Piola-Kirchhoff stress tensor, (see Eq. (4.34)), becomes:

$$\mathbf{P}^{*}=\mathbf{Q}\cdot\mathbf{P}\quad\Rightarrow\quad\dot{\mathbf{P}}^{*}=\dot{\mathbf{Q}}\cdot\mathbf{P}+\mathbf{Q}\cdot\dot{\mathbf{P}} \tag{4.79}$$

Then by substituting $\dot{\mathbf{Q}}=\mathbf{Q}\cdot\boldsymbol{\ell}^{T}-\boldsymbol{\ell}^{*T}\cdot\mathbf{Q}$, (see equation (4.42)), into the above equation we obtain:

$$\begin{aligned}&\dot{\mathbf{P}}^{*}=\dot{\mathbf{Q}}\cdot\mathbf{P}+\mathbf{Q}\cdot\dot{\mathbf{P}}=\left(\mathbf{Q}\cdot\boldsymbol{\ell}^{T}-\boldsymbol{\ell}^{*T}\cdot\mathbf{Q}\right)\cdot\mathbf{P}+\mathbf{Q}\cdot\dot{\mathbf{P}}=\mathbf{Q}\cdot\boldsymbol{\ell}^{T}\cdot\mathbf{P}-\boldsymbol{\ell}^{*T}\cdot\underbrace{\mathbf{Q}\cdot\mathbf{P}}_{\mathbf{P}^{*}}+\mathbf{Q}\cdot\dot{\mathbf{P}}\\ &\Rightarrow\dot{\mathbf{P}}^{*}+\boldsymbol{\ell}^{*T}\cdot\mathbf{P}^{*}=\mathbf{Q}\cdot\left(\dot{\mathbf{P}}+\boldsymbol{\ell}^{T}\cdot\mathbf{P}\right)\end{aligned} \tag{4.80}$$

Note that the orthogonal transformation of $\mathbf{P}$ obeys the vector transformation law so the rate $\left(\dot{\mathbf{P}}+\boldsymbol{\ell}^{T}\cdot\mathbf{P}\right)$ is objective because it has the same structure as the convective rate presented in (4.59).

If we now apply the material time derivative to $\mathbf{P}=J\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}$ we obtain:

$$\frac{D}{Dt}(\mathbf{P})\equiv\dot{\mathbf{P}}=\dot{J}\,\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}+J\,\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{F}^{-T}+J\,\boldsymbol{\sigma}\cdot\dot{\boldsymbol{F}}^{-T} \tag{4.81}$$

Additionally, if we take into account the material time derivative of the Jacobian determinant, (see Chapter 2), we have:

$$\boxed{\dot{J}=J\,\mathrm{Tr}(\boldsymbol{\ell})=J\,\mathrm{Tr}(\mathbf{D}+\mathbf{W})=J\,\mathrm{Tr}(\mathbf{D})=J\,\boldsymbol{C}^{-1}:\dot{\boldsymbol{E}}=\frac{J}{2}\boldsymbol{C}^{-1}:\dot{\boldsymbol{C}}=J\boldsymbol{F}^{-T}:\dot{\boldsymbol{F}}} \tag{4.82}$$

and $\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell}$. Then, the equation in (4.81) becomes:

$$\dot{\mathbf{P}} = J\,\mathrm{Tr}(\boldsymbol{\ell})\,\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} + J\,\dot{\boldsymbol{\sigma}} \cdot \boldsymbol{F}^{-T} - J\,\boldsymbol{\sigma} \cdot \boldsymbol{\ell}^{T} \cdot \boldsymbol{F}^{-T} \tag{4.83}$$

or:

$$\dot{\mathbf{P}} = J\,\left[\mathrm{Tr}(\mathbf{D})\,\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^{T}\right] \cdot \boldsymbol{F}^{-T} \tag{4.84}$$

The Truesdell Stress Rate

Now let us consider the Kirchhoff stress tensor $\boldsymbol{\tau} = J\,\boldsymbol{\sigma}$ and $\boldsymbol{\tau}^* = J^*\,\boldsymbol{\sigma}^* = J\,\boldsymbol{\sigma}^*$, so the material time derivative of $\boldsymbol{\tau}^*$ becomes:

$$\begin{aligned}\dot{\boldsymbol{\tau}}^* &= \dot{J}\,\boldsymbol{\sigma}^* + J\,\dot{\boldsymbol{\sigma}}^* = J\,\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}^* + J\dot{\boldsymbol{\sigma}}^* \\ &= J\,\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}^* + J\left(\dot{\mathbf{Q}} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \dot{\mathbf{Q}}^T\right)\end{aligned} \tag{4.85}$$

where we have substituted $\dot{\boldsymbol{\sigma}}^*$ into the equation in (4.57). Then by substituting $\dot{\mathbf{Q}}$ and $\dot{\mathbf{Q}}^T$ given in the equation (4.42), *i.e.* $\dot{\mathbf{Q}} = \boldsymbol{\ell}^* \cdot \mathbf{Q} - \mathbf{Q} \cdot \boldsymbol{\ell}$ and $\dot{\mathbf{Q}}^T = \mathbf{Q}^T \cdot \boldsymbol{\ell}^{*T} - \boldsymbol{\ell}^T \cdot \mathbf{Q}^T$, we obtain:

$$\begin{aligned}\frac{\dot{\boldsymbol{\tau}}^*}{J} = \mathrm{Tr}(\mathbf{D})\,\underbrace{\boldsymbol{\sigma}^*}_{\mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^T} + \boldsymbol{\ell}^* \cdot \underbrace{\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T}_{\boldsymbol{\sigma}^*} - \mathbf{Q} \cdot \boldsymbol{\ell} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{Q}^T \\ + \underbrace{\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T}_{\boldsymbol{\sigma}^*} \cdot \boldsymbol{\ell}^{*T} - \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T \cdot \mathbf{Q}^T\end{aligned} \tag{4.86}$$

$$\frac{\dot{\boldsymbol{\tau}}^*}{J} - \boldsymbol{\ell}^* \cdot \boldsymbol{\sigma}^* - \boldsymbol{\sigma}^* \cdot \boldsymbol{\ell}^{*T} = \mathbf{Q} \cdot \left(\boldsymbol{\sigma}\,\mathrm{Tr}(\mathbf{D}) - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T\right) \cdot \mathbf{Q}^T \tag{4.87}$$

If we now bear in mind that $\dot{\boldsymbol{\tau}}^* = J\,\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}^* + J\dot{\boldsymbol{\sigma}}^*$, the above equation becomes:

$$\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}^* + \dot{\boldsymbol{\sigma}}^* - \boldsymbol{\ell}^{*T} \cdot \boldsymbol{\sigma}^* - \boldsymbol{\sigma}^* \cdot \boldsymbol{\ell}^* = \mathbf{Q} \cdot \left(\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T\right) \cdot \mathbf{Q}^T \tag{4.88}$$

Hence, we obtain a new objective stress rate called the *Truesdell stress rate:*

$$\boxed{\overset{\mathrm{T}}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T + \boldsymbol{\sigma}\,\mathrm{Tr}(\mathbf{D})} \quad \textit{The Truesdell stress rate} \tag{4.89}$$

Relationship between Objective Stress Rates

Let us consider the Oldroyd rate of the Cauchy stress tensor $\overset{\square}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T$, and if we use the Truesdell stress rate we can conclude that:

$$\boxed{\overset{\mathrm{T}}{\boldsymbol{\sigma}} = \overset{\square}{\boldsymbol{\sigma}} + \mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}} \tag{4.90}$$

We can also relate the Oldroyd rate of the Kirchhoff stress tensor $\overset{\square}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \boldsymbol{\ell} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\ell}^T$ with the Oldroyd rate of the Cauchy stress tensor $\overset{\square}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T$ as:

$$\boxed{\overset{\square}{\boldsymbol{\tau}} = J\,\overset{\mathrm{T}}{\boldsymbol{\sigma}}} \tag{4.91}$$

We can also prove the above equation is valid by starting from (4.87):

$$
\begin{aligned}
&\frac{\dot{\boldsymbol{\tau}}^*}{J} - \boldsymbol{\ell}^{*T} \cdot \boldsymbol{\sigma}^* - \boldsymbol{\sigma}^* \cdot \boldsymbol{\ell}^* = \mathbf{Q} \cdot \left(\boldsymbol{\sigma}\,\mathrm{Tr}(\mathbf{D}) - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T \right) \cdot \mathbf{Q}^T \\
&\frac{\dot{\boldsymbol{\tau}}^*}{J} - \boldsymbol{\ell}^{*T} \cdot \frac{\boldsymbol{\tau}^*}{J} - \frac{\boldsymbol{\tau}^*}{J} \cdot \boldsymbol{\ell}^* = \mathbf{Q} \cdot \left(\boldsymbol{\sigma}\,\mathrm{Tr}(\mathbf{D}) - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T \right) \cdot \mathbf{Q}^T \\
&\dot{\boldsymbol{\tau}}^* - \boldsymbol{\ell}^{*T} \cdot \boldsymbol{\tau}^* - \boldsymbol{\tau}^* \cdot \boldsymbol{\ell}^* = \mathbf{Q} \cdot J\left(\boldsymbol{\sigma}\,\mathrm{Tr}(\mathbf{D}) - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T \right) \cdot \mathbf{Q}^T
\end{aligned}
\tag{4.92}
$$

thus,

$$
\mathbf{Q}^T \cdot \underbrace{\left(\dot{\boldsymbol{\tau}}^* - \boldsymbol{\ell}^{*T} \cdot \boldsymbol{\tau}^* - \boldsymbol{\tau}^* \cdot \boldsymbol{\ell}^* \right)}_{\overset{\square}{\boldsymbol{\tau}}^*} \cdot \mathbf{Q} = J\left(\boldsymbol{\sigma}\,\mathrm{Tr}(\mathbf{D}) - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^T \right) \Rightarrow \overset{\square}{\boldsymbol{\tau}} = J \overset{\mathrm{T}}{\boldsymbol{\sigma}}
\tag{4.93}
$$

If we consider the equation of the second Piola-Kirchhoff stress tensor $\mathbf{S} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T}$, (see Chapter 3), the material time derivative becomes:

$$
\begin{aligned}
\dot{\mathbf{S}} &= \dot{\boldsymbol{F}}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} + \boldsymbol{F}^{-1} \cdot \dot{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-T} + \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \dot{\boldsymbol{F}}^{-T} \\
&= -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} + \boldsymbol{F}^{-1} \cdot \dot{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-T} - \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{\ell}^T \cdot \boldsymbol{F}^{-T} \\
&= \boldsymbol{F}^{-1} \cdot \left(\dot{\boldsymbol{\tau}} - \boldsymbol{\ell} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\ell}^T \right) \cdot \boldsymbol{F}^{-T}
\end{aligned}
\tag{4.94}
$$

where we have considered that $\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell}$. If we refer to the Oldroyd rate of the Kirchhoff stress tensor $\overset{\square}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \boldsymbol{\ell} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\ell}^T$ the equation in (4.94) becomes:

$$
\boxed{\dot{\mathbf{S}} = \boldsymbol{F}^{-1} \cdot \overset{\square}{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-T} = J\,\boldsymbol{F}^{-1} \cdot \overset{\mathrm{T}}{\boldsymbol{\sigma}} \cdot \boldsymbol{F}^{-T}}
\tag{4.95}
$$

Now, taking the material time derivative of $\boldsymbol{\tau}^* = J\boldsymbol{\sigma}^*$ we obtain:

$$
\dot{\boldsymbol{\tau}}^* = \dot{J}\boldsymbol{\sigma}^* + J\dot{\boldsymbol{\sigma}}^* = \dot{J}\boldsymbol{\sigma}^* + J\frac{D}{Dt}(\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T) = \dot{J}\boldsymbol{\sigma}^* + J\left(\dot{\mathbf{Q}} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \dot{\mathbf{Q}}^T \right)
\tag{4.96}
$$

Also if we consider the relationship $\dot{\mathbf{Q}} = \mathbf{W}^* \cdot \mathbf{Q} - \mathbf{Q} \cdot \mathbf{W}$ and $\dot{J} = J\,\mathrm{Tr}(\mathbf{D})$ in the above equation we obtain:

$$
\dot{\boldsymbol{\tau}}^* = J\,\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}^* + J\left(\left(\mathbf{W}^* \cdot \mathbf{Q} - \mathbf{Q} \cdot \mathbf{W} \right) \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \left(\mathbf{W}^* \cdot \mathbf{Q} - \mathbf{Q} \cdot \mathbf{W} \right)^T \right)
\tag{4.97}
$$

$$
\begin{aligned}
\dot{\boldsymbol{\tau}}^* &= J\,\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}^* + J\left(\mathbf{W}^* \cdot \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T - \mathbf{Q} \cdot \mathbf{W} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{Q}^T + \right. \\
&\qquad\qquad \left. \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T \cdot \mathbf{W}^{*T} - \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{W}^T \cdot \mathbf{Q}^T \right) \\
\dot{\boldsymbol{\tau}}^* - \mathbf{W}^* \cdot \mathbf{Q} \cdot J\boldsymbol{\sigma} \cdot \mathbf{Q}^T - \mathbf{Q} \cdot J\boldsymbol{\sigma} \cdot \mathbf{Q}^T \cdot \mathbf{W}^{*T} &= J\,\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}^* + J\left(-\mathbf{Q} \cdot \mathbf{W} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T + \right. \\
&\qquad\qquad \left. \mathbf{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{Q}^T - \mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{W}^T \cdot \mathbf{Q}^T \right)
\end{aligned}
\tag{4.98}
$$

or

$$
\underbrace{\dot{\boldsymbol{\tau}}^* - \mathbf{W}^* \cdot \boldsymbol{\tau}^* - \boldsymbol{\tau}^* \cdot \mathbf{W}^{*T}}_{\mathbf{Q} \cdot \overset{\circ}{\boldsymbol{\tau}} \cdot \mathbf{Q}^T} = J\,\mathbf{Q} \cdot \left[\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \mathbf{W} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbf{W}^T \right] \cdot \mathbf{Q}^T
\tag{4.99}
$$

which give us the relationship between the Jaumann-Zaremba rate of the Kirchhoff stress tensor and the Cauchy stress tensor:

$$
\boxed{\overset{\circ}{\boldsymbol{\tau}} = J\left[\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} - \mathbf{W} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbf{W}^T \right] = J\left[\mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma} + \overset{\circ}{\boldsymbol{\sigma}} \right]}
\tag{4.100}
$$

5

The Fundamental Equations of Continuum Mechanics

5.1 Introduction

The fundamental equations of continuum mechanics are based on the conservation principles of certain physical quantities. We consider five of these to establish the basic equations that govern the Initial Boundary Value Problem (IBVP), namely:

- The principle of conservation of mass;
- The principle of conservation of linear momentum;
- The principle of conservation of angular momentum;
- The principle of conservation of energy;
- The principle of irreversibility.

In this chapter we will address the fundamental principle of mechanics in the reference and current configurations. At the end of the Chapter we will show that these principles are insufficient to establish the IBVP set of partial differential equations, so, it is necessary to add certain equations to fully resolve this problem. Then, we will introduce some concepts and theorems to develop the concepts in this chapter.

5.2 Density

Density, denoted by $f(\vec{x},t)$, is a scalar function that measures the amount of a property per unit volume around a material point ($\vec{x}$) at a time t. One very important density function is *mass density*, denoted by $\rho(\vec{x},t)$, which measures the amount of mass per unit

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_7,

volume. Another density function we can quote is *energy density*, which measures stored energy per unit volume. The term "*specific*" will be used to denote the amount of the property *per unit mass*.

5.2.1 Mass Density

Any continuous medium is caused by a positive scalar quantity called *mass*. It is assumed that the mass is continuously distributed throughout the continuum.

We will next review the concept of mass density introduced in Chapter 2. Let us consider a sphere of infinitesimal radius centered at point P in the reference configuration, (see Figure 5.1). The material contained in this sphere is denoted by Δm and the sphere volume is represented by ΔV_0. Then, the mass density ρ_0, in the reference configuration, is defined by the limit:

$$\rho(\vec{\boldsymbol{x}}, t=0) = \rho_0(\vec{\boldsymbol{X}}) = \lim_{\Delta V_0 \to 0} \frac{\Delta m}{\Delta V_0} = \frac{dm}{dV_0} \tag{5.1}$$

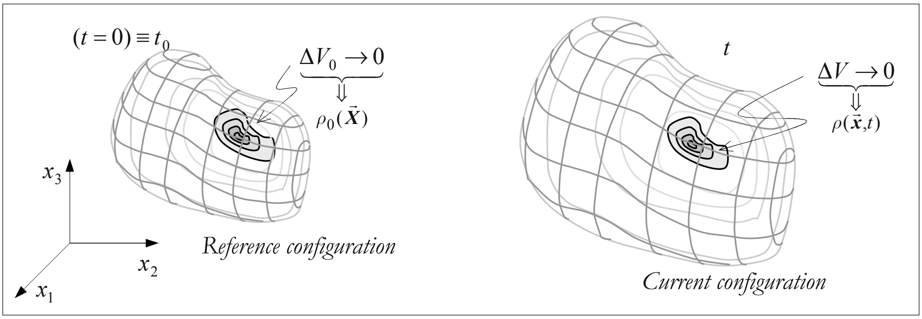


Figure 5.1: Mass density.

Likewise, mass density in the current configuration is given by:

$$\rho = \rho(\vec{\boldsymbol{x}}, t) = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV} \tag{5.2}$$

Then the functions $\rho(\vec{\boldsymbol{x}}, t)$ and $\rho_0(\vec{\boldsymbol{X}})$ are continuous density functions and are interrelated to each other, (see Chapter 2), by:

$$\boxed{\rho_0(\vec{X}) = J\rho(\vec{\boldsymbol{x}}, t)} \tag{5.3}$$

5.3 Flux

The properties conferred by density (*e.g.* mass, energy, entropy, etc.) are mobile and the rate of change and direction of these quantities are assigned by the flux vector, usually denoted by $\vec{\mathbf{q}}(\vec{\boldsymbol{x}}, t)$. With this information, we can define the amount of property that passes through a differential area element da per unit time, (see Figure 5.2), as:

$$\vec{\mathbf{q}} \cdot \hat{\mathbf{n}}\, da = \|\vec{\mathbf{q}}\| \cos\alpha\, da = \mathrm{q}_n\, da \tag{5.4}$$

where $\hat{\mathbf{n}}$ is the unit normal vector, and α is the angle formed between $\vec{\mathbf{q}}$ and $\vec{\mathbf{q}}_{\hat{\mathbf{n}}}$. Note that, only the normal vector $\vec{\mathbf{q}}_{\hat{\mathbf{n}}}$ crosses the surface, since the tangential vector $\vec{\mathbf{q}}_{\hat{\mathbf{s}}}$ remains on the surface da. As an example of flux, we can mention the mass flux vector which is represented by $\vec{\mathbf{q}} = \rho \vec{\boldsymbol{v}}$. With regard to the SI unit we have: $[\vec{\mathbf{q}}] = \frac{kg}{m^2 s}$ where $\vec{\mathbf{q}}$ represents the mass flux vector, and $[\vec{\mathbf{q}}] = \frac{J}{m^2 s}$ where $\vec{\mathbf{q}}$ refers to the energy flux vector.

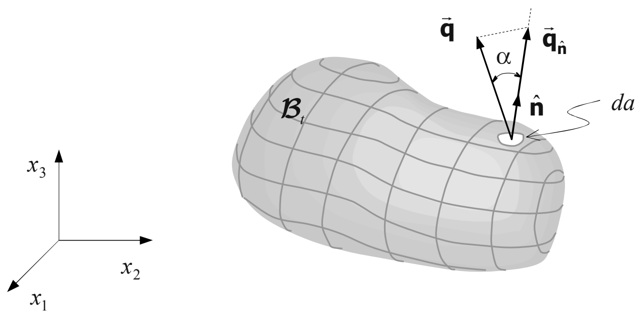


Figure 5.2: Flux vector.

5.4 The Reynolds' Transport Theorem

Let $\Phi(\vec{\boldsymbol{x}},t)$ be an Eulerian scalar field which describes a certain physical quantity *per unit volume*. If $\Phi(\vec{\boldsymbol{x}},t)$ is continuous and differentiable, we can state that:

$$\begin{aligned}\frac{D}{Dt}\int_V \Phi(\vec{\boldsymbol{x}},t)dV &= \int_V \left[dV \frac{D}{Dt}\Phi(\vec{\boldsymbol{x}},t) + \Phi(\vec{\boldsymbol{x}},t)\frac{D}{Dt}(dV) \right] \\ &= \int_V \left[dV \frac{D}{Dt}\Phi(\vec{\boldsymbol{x}},t) + \Phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}\, dV \right] \\ &= \int_V \left[\frac{D}{Dt}\Phi(\vec{\boldsymbol{x}},t) + \Phi(\vec{\boldsymbol{x}},t)\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} \right] dV \end{aligned} \tag{5.5}$$

whose equivalent in indicial notation is:

$$\frac{D}{Dt}\int_V \Phi(\vec{\boldsymbol{x}},t)dV = \int_V \left[\frac{D}{Dt}\Phi(\vec{\boldsymbol{x}},t) + \Phi(\vec{\boldsymbol{x}},t)\frac{\partial v_k}{\partial x_k} \right] dV \tag{5.6}$$

Then by using the material time derivative operator we can still state that:

$$\begin{aligned}\frac{D}{Dt}\int_V \Phi(\vec{\boldsymbol{x}},t)dV &= \int_V \left[\frac{\partial \Phi(\vec{\boldsymbol{x}},t)}{\partial t} + v_p \frac{\partial \Phi(\vec{\boldsymbol{x}},t)}{\partial x_p} + \Phi(\vec{\boldsymbol{x}},t)\frac{\partial v_k}{\partial x_k} \right] dV \\ &= \int_V \left[\frac{\partial \Phi(\vec{\boldsymbol{x}},t)}{\partial t} + \frac{\partial}{\partial x_p}\left(\Phi(\vec{\boldsymbol{x}},t)\, v_p \right) \right] dV \end{aligned} \tag{5.7}$$

This last equation is known as the *Reynolds' transport theorem* and can be represented by the following equations:

$$\boxed{\begin{aligned}\frac{D}{Dt}\int_V \Phi(\vec{x},t)dV &= \int_V\left[\frac{D}{Dt}\Phi(\vec{x},t)+\Phi(\vec{x},t)\nabla_{\vec{x}}\cdot\vec{v}\right]dV\\ &= \int_V\left[\frac{\partial\Phi(\vec{x},t)}{\partial t}+\nabla_{\vec{x}}\cdot\left(\Phi(\vec{x},t)\,\vec{v}\right)\right]dV\\ &= \int_{V_t}\frac{\partial\Phi(\vec{x},t)}{\partial t}dV+\int_{S_t}\Phi(\vec{x},t)\vec{v}\cdot\hat{\mathbf{n}}\,dS\end{aligned}}$$

The Reynolds' transport theorem (5.8)

where V_t is the control volume, S_t is the control surface, and $\hat{\mathbf{n}}$ is the outward unit normal to the boundary S_t of $\mathcal{B}_t$. The first term on the right of equation is the local rate of change of the property Φ in the domain V_t, while the second term characterizes the transport of $\Phi\vec{v}$, that leaves the domain V_t via the surface S_t, (see Figure 5.3).

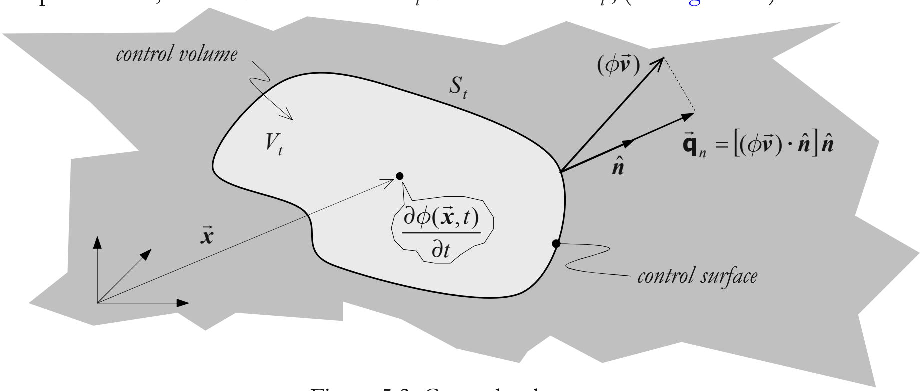


Figure 5.3: Control volume.

Problem 5.1: Prove that Reynolds' transport theorem is valid in the following equation:

$$\frac{D}{Dt}\int_V \Phi dV = \frac{D}{Dt}\int_{V_0}\Phi J dV_0 \tag{5.9}$$

Solution:

$$\frac{D}{Dt}\int_{V_0}\Phi J dV_0 = \int_{V_0}\left(J\frac{D\Phi}{Dt}+\Phi\frac{DJ}{Dt}\right)dV_0 = \int_{V_0}\left(J\frac{D\Phi}{Dt}+J\Phi\nabla_{\vec{x}}\cdot\vec{v}\right)dV_0 = \int_V\left(\frac{D\Phi}{Dt}+\Phi\nabla_{\vec{x}}\cdot\vec{v}\right)dV$$

5.4.1 Reynolds' Transport Theorem for Volumes with Discontinuities

Let us consider a material volume that is intersected by a discontinuous surface $\Sigma(t)$, a singular surface which is moving over time at velocity $\vec{\boldsymbol{\omega}}$, (see Figure 5.4). The surface $\Sigma(t)$ divides the material volume into two parts *viz.* $\mathcal{B}^+$ and $\mathcal{B}^-$. We can also define the boundaries Σ^+ and Σ^- situated ahead of and to the rear of the singular surface Σ as shown in Figure 5.4.

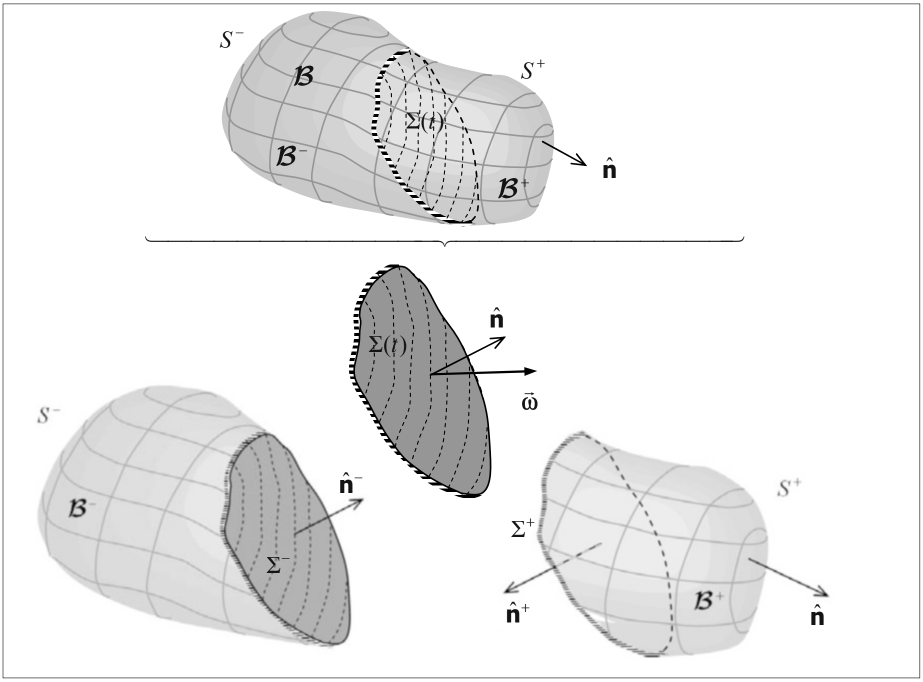


Figure 5.4: Material volume with discontinuity.

The mobile discontinuity $\Sigma(t)$ is defined by the surface equation:

$$f_\Sigma(\vec{x},t) = 0 \qquad \vec{x} \in \Sigma(t) \tag{5.10}$$

Then, the unit normal vector $\hat{\mathbf{n}}$ on the surface $\Sigma(t)$ is given by the equation:

$$\hat{\mathbf{n}}(\vec{x},t) = \frac{\nabla f_\Sigma}{\|\nabla f_\Sigma\|} \tag{5.11}$$

For material points belonging to the surface $\Sigma(t)$, the normal component of the velocity, ω_n, is defined as:

$$\omega_n = -\frac{\dfrac{\partial f_\Sigma}{\partial t}}{\|\nabla f_\Sigma\|} \tag{5.12}$$

Then by combining the equation in (5.11) with (5.12) we obtain:

$$\omega_n = \vec{\omega} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \vec{\omega} \quad \Rightarrow \quad -\frac{\dfrac{\partial f_\Sigma}{\partial t}}{\|\nabla f_\Sigma\|} = \frac{\nabla f_\Sigma}{\|\nabla f_\Sigma\|} \cdot \vec{\omega} \Rightarrow \frac{\partial f_\Sigma}{\partial t} + \nabla f_\Sigma \cdot \vec{\omega} = 0 \quad \Rightarrow \quad \frac{D f_\Sigma}{Dt} = 0 \tag{5.13}$$

Let $\mathbf{A}$ be a second-order tensor field and let us consider that $\mathbf{A}^+$ and $\mathbf{A}^-$ are the values of $\mathbf{A}$ in the boundaries Σ^+ and Σ^-, respectively. Then, we can define the *jump* of $\mathbf{A}$ as:

$$[\![\mathbf{A}]\!] = \mathbf{A}^+ - \mathbf{A}^- \tag{5.14}$$

Then by applying Gauss' theorem (the divergence theorem) for the two domains $\mathcal{B}^+$ and $\mathcal{B}^-$ we can obtain, respectively:

$$\int\limits_{V^+} \nabla \cdot \mathbf{A}\, dV = \int\limits_{S^+} \mathbf{A} \cdot \hat{\mathbf{n}}\, dS + \int\limits_{\Sigma^+} \mathbf{A}^+ \cdot \hat{\mathbf{n}}^+\, dS \quad ; \quad \int\limits_{V^-} \nabla \cdot \mathbf{A}\, dV = \int\limits_{S^-} \mathbf{A} \cdot \hat{\mathbf{n}}\, dS + \int\limits_{\Sigma^-} \mathbf{A}^- \cdot \hat{\mathbf{n}}^-\, dS \tag{5.15}$$

Then, by summing up these two above equations we obtain:

$$\int\limits_{V^+ + V^-} \nabla \cdot \mathbf{A}\, dV = \int\limits_{S^+ + S^-} \mathbf{A} \cdot \hat{\mathbf{n}}\, dS + \int\limits_{\Sigma^+} \mathbf{A}^+ \cdot \hat{\mathbf{n}}^+\, dS + \int\limits_{\Sigma^-} \mathbf{A}^- \cdot \hat{\mathbf{n}}^-\, dS \tag{5.16}$$

Additionally, if we bear in mind that $\hat{\mathbf{n}}^+ = -\hat{\mathbf{n}}^- = -\hat{\mathbf{n}}$ and $[[\mathbf{A}]] = \mathbf{A}^+ - \mathbf{A}^-$ the above equation becomes:

$$\int\limits_{V^+ + V^-} \nabla \cdot \mathbf{A}\, dV = \int\limits_{S^+ + S^-} \mathbf{A} \cdot \hat{\mathbf{n}}\, dS - \int\limits_{\Sigma} [[\mathbf{A}]] \cdot \hat{\mathbf{n}}\, dS \tag{5.17}$$

The Reynolds' transport theorem can be modified for the case in which there is a singular surface $\Sigma(t)$, which moves at the velocity $\vec{\omega}$, (see Figure 5.4). Then by applying the equation in (5.8) to the two domains $\mathcal{B}^+$ and $\mathcal{B}^-$, whose contours are $S^+ + \Sigma^+$ and $S^- + \Sigma^-$, respectively, we obtain:

$$\begin{aligned} \frac{D}{Dt} \int\limits_{V^+} \Phi(\vec{x},t) dV &= \int\limits_{V^+} \frac{\partial \Phi(\vec{x},t)}{\partial t} dV + \int\limits_{S^+} \Phi(\vec{x},t) \vec{v} \cdot \hat{\mathbf{n}}\, dS + \int\limits_{\Sigma^+} \Phi^+(\vec{x},t) \vec{\omega} \cdot \hat{\mathbf{n}}^+\, dS \\ \frac{D}{Dt} \int\limits_{V^-} \Phi(\vec{x},t) dV &= \int\limits_{V^-} \frac{\partial \Phi(\vec{x},t)}{\partial t} dV + \int\limits_{S^-} \Phi(\vec{x},t) \vec{v} \cdot \hat{\mathbf{n}}\, dS + \int\limits_{\Sigma^-} \Phi^-(\vec{x},t) \vec{\omega} \cdot \hat{\mathbf{n}}^-\, dS \end{aligned} \tag{5.18}$$

Then by adding the two equations above, and once again considering that $\hat{\mathbf{n}}^+ = -\hat{\mathbf{n}}^- = -\hat{\mathbf{n}}$ and $[[\Phi]] = \Phi^+ - \Phi^-$, we can conclude that:

$$\frac{D}{Dt} \int\limits_{V^+ + V^-} \Phi(\vec{x},t) dV = \int\limits_{V^+ + V^-} \frac{\partial \Phi(\vec{x},t)}{\partial t} dV + \int\limits_{S^+ + S^-} \Phi(\vec{x},t) \vec{v} \cdot \hat{\mathbf{n}}\, dS - \int\limits_{\Sigma} [[\Phi]] \left(\vec{\omega} \cdot \hat{\mathbf{n}}\right) dS \tag{5.19}$$

Additionally, by using the definition in (5.17) we can state that:

$$\int\limits_{S^+ + S^-} \Phi(\vec{x},t) \vec{v} \cdot \hat{\mathbf{n}}\, dS = \int\limits_{V^+ + V^-} \nabla_{\vec{x}} \cdot \left(\Phi(\vec{x},t) \vec{v}\right) dV + \int\limits_{\Sigma} [[\Phi(\vec{x},t) \vec{v}]] \cdot \hat{\mathbf{n}}\, dS \tag{5.20}$$

Then by combining the above equation with that in (5.19) we obtain:

$$\frac{D}{Dt} \int\limits_{V^+ + V^-} \Phi(\vec{x},t) dV = \int\limits_{V^+ + V^-} \left(\frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}} \cdot \left(\Phi(\vec{x},t) \vec{v}\right) \right) dV + \int\limits_{\Sigma} \left([[\Phi(\vec{x},t) \vec{v}]] - [[\Phi \vec{\omega}]] \right) \cdot \hat{\mathbf{n}}\, dS \tag{5.21}$$

which results in the Reynolds' transport theorem for domains with discontinuities:

$$\boxed{\begin{aligned} &\frac{D}{Dt} \int\limits_{V-\Sigma} \Phi(\vec{x},t) dV = \int\limits_{V-\Sigma} \left(\frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}} \cdot \left(\Phi(\vec{x},t) \otimes \vec{v}\right) \right) dV + \int\limits_{\Sigma} [[\Phi \otimes (\vec{v} - \vec{\omega})]] \cdot \hat{\mathbf{n}}\, dS \\ &\textit{or} \\ &\frac{D}{Dt} \int\limits_{V-\Sigma} \Phi(\vec{x},t) dV = \int\limits_{V-\Sigma} \left(\frac{D\Phi(\vec{x},t)}{Dt} + \Phi(\vec{x},t) \nabla_{\vec{x}} \cdot \vec{v} \right) dV + \int\limits_{\Sigma} [[\Phi \otimes (\vec{v} - \vec{\omega})]] \cdot \hat{\mathbf{n}}\, dS \end{aligned}} \tag{5.22}$$

5.5 Conservation Law

Conservation Law when applied to a particular physical quantity per unit volume, in a part of the domain, states that no physical quantity (mass density, energy density, etc.) can be created or destroyed, but merely moves from one place to another. The conservation law in global form (weak) is established from Reynolds' transport theorem:

$$\underbrace{\frac{D}{Dt}\int_V \Phi(\vec{x},t)dV}_{\neq 0 \Rightarrow \text{ source or sink}} = \int_V \left[\frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}} \cdot \left(\Phi(\vec{x},t)\,\vec{v}\right)\right] dV \tag{5.23}$$

If the term on the left of the equation is nonzero this means that somewhere in the domain there is a property source or sink, which can be represented locally by the variable Q. Then, $Q>0$ indicates that there is a source, and $Q<0$ that there is a sink. For example, if the property in question is mass density (mass per unit volume) in general $Q=0$. However, if there is a tumor (cells with abnormal growth) in a biological organism, we can establish a law (at the macroscopic level) that indicates how the mass changes over time (source), without regard to individual cells. Then, another example of a source we can cite is the internal heat generated by a chemical reaction, such as in cement hydration. The effect of the chemical reaction at the macroscopic level can be represented by a variable that provides the amount of heat generated per unit volume and per unit time (the internal heat source).

Note that the term $(\Phi\vec{v})$ shows the flux of the property Φ. Then, if $(\Phi\vec{v})$ represents the energy flux we have the following unit $[\Phi\vec{v}]=\frac{J}{m^2 s}$, and if we are dealing with mass transport we have $[\Phi\vec{v}]=\frac{kg}{m^2 s}$. As we have seen before, in general, the flux is represented by $\vec{\mathbf{q}}$, with which we can establish the local form (strong) of conservation law and which is denoted by the following continuity equation:

$$\boxed{Q = \frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}} \cdot \left(\vec{\mathbf{q}}(\vec{x},t)\right)} \quad \textit{Continuity equation} \qquad \frac{[\Phi]}{s} \tag{5.24}$$

where $[\Phi]$ is the SI unit of the physical quantity per unit volume.

5.6 The Principle of Conservation of Mass. The Mass Continuity Equation

The law of conservation of mass states that the total mass of a continuum does not change. This implies that the total mass in the reference configuration is equal to the total mass in the current configuration:

$$m = \int_{V_0} \rho_0 \, dV = \int_V \rho \, dV \qquad [kg] \tag{5.25}$$

As a result of conservation of mass, the material time derivative of the total mass is zero, *i.e.* $\frac{D}{Dt}m = 0$, then:

$$\begin{aligned}\frac{D}{Dt}m &= \frac{D}{Dt}\int_V \rho(\vec{x},t)\,dV = \int_V \frac{D}{Dt}[\rho(\vec{x},t)\,dV] = \int_V dV\frac{D}{Dt}[\rho(\vec{x},t)] + \rho(\vec{x},t)\frac{D}{Dt}[dV] = 0 \\ &= \int_V \left[\frac{D}{Dt}[\rho(\vec{x},t)] + \rho(\vec{x},t)\nabla_{\vec{x}}\cdot\vec{v}\right]dV = 0\end{aligned} \tag{5.26}$$

or in indicial notation:

$$\int_V \left[\frac{D}{Dt}[\rho(\vec{x},t)] + \rho\frac{\partial v_k}{\partial x_k}\right]dV = 0 \qquad \left[\frac{kg}{s}\right] \tag{5.27}$$

If the above equation is valid for the entire domain, then it must also be satisfied locally:

$$\frac{D\rho}{Dt} + \rho v_{k,k} = 0 \qquad \left[\frac{kg}{sm^3}\right] \tag{5.28}$$

which is the mass continuity equation in Eulerian description and is expressed in tensorial notation as:

$$\boxed{\frac{D\rho}{Dt} + \rho(\nabla_{\vec{x}}\cdot\vec{v}) = 0} \quad \textit{The mass continuity equation (Eulerian description)} \tag{5.29}$$

Then by applying the material time derivative operator, *i.e.* $\frac{D\bullet}{Dt} = \frac{\partial\bullet}{\partial t} + \frac{\partial\bullet}{\partial x_k}v_k$, the mass continuity equation (5.28) becomes:

$$\frac{D\rho}{Dt} + \rho v_{k,k} = \frac{\partial\rho}{\partial t} + v_p\frac{\partial\rho}{\partial x_p} + \rho\frac{\partial v_k}{\partial x_k} = \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_k) = \frac{\partial\rho}{\partial t} + (\rho v_k)_{,k} = 0 \tag{5.30}$$

Hence, we have another way to express the mass continuity equation:

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla_{\vec{x}}\cdot(\rho\,\vec{v}) = 0} \quad \textit{The mass continuity equation (Eulerian description)} \tag{5.31}$$

We could have obtained the same equation in (5.31) by means of Reynolds' transport theorem, *i.e.* in the equation (5.8) we substitute $\Phi(\vec{x},t)$ for $\rho(\vec{x},t)$, which means:

$$\frac{D}{Dt}\int_V \rho(\vec{x},t)dV = \int_V \left[\frac{\partial\rho(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\cdot\left(\rho(\vec{x},t)\,\vec{v}\right)\right]dV = 0 \Rightarrow \frac{\partial\rho(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\cdot\left(\rho(\vec{x},t)\,\vec{v}\right) = 0 \tag{5.32}$$

We could also have obtained the mass continuity equation in (5.29) by means of the principle of conservation of mass in a differential volume element $dx_1 dx_2 dx_3$, (see Figure 5.5), in which the following is satisfied:

Mass accumulation	=	Inward mass flux	-	Outward mass flux

The rate of mass entering through face A is represented by the mass flux $(\rho v_1)_{x_1} dx_2 dx_3$, while the rate of mass that goes through face B is given by $\left[\rho v_1 + \frac{\partial(\rho v_1)}{\partial x_1}dx_1\right]dx_2 dx_3$.

Likewise, we can obtain the rate of change of mass in other faces. Moreover, by applying the conservation of mass for the differential volume element we obtain:

$$\begin{aligned} dx_1 dx_2 dx_3 \frac{\partial \rho}{\partial t} = \left(\rho v_1 dx_2 dx_3 + \rho v_2 dx_1 dx_3 + \rho v_3 dx_1 dx_2\right) - \left\{ \left[\rho v_1 + \frac{\partial(\rho v_1)}{\partial x_1} dx_1\right] dx_2 dx_3 + \right. \\ \left. \left[\rho v_2 + \frac{\partial(\rho v_2)}{\partial x_2} dx_2\right] dx_1 dx_3 + \left[\rho v_3 + \frac{\partial(\rho v_3)}{\partial x_3} dx_3\right] dx_1 dx_2 \right\} \end{aligned} \tag{5.33}$$

Then by simplifying the above equation we obtain $\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_1)}{\partial x_1} - \frac{\partial(\rho v_2)}{\partial x_2} - \frac{\partial(\rho v_3)}{\partial x_3}$ and by using the chain rule of derivative we find that:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1} v_1 + \frac{\partial \rho}{\partial x_2} v_2 + \frac{\partial \rho}{\partial x_3} v_3 = -\rho\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) \\ &\Rightarrow \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} v_i}_{=\frac{D\rho}{Dt}} = -\rho \underbrace{\left(\frac{\partial v_i}{\partial x_i}\right)}_{=\nabla\cdot\vec{v}=\mathrm{Tr}(\nabla\vec{v})} \quad \Rightarrow \frac{D\rho}{Dt} + \rho\left(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}\right) = 0 \end{aligned} \tag{5.34}$$

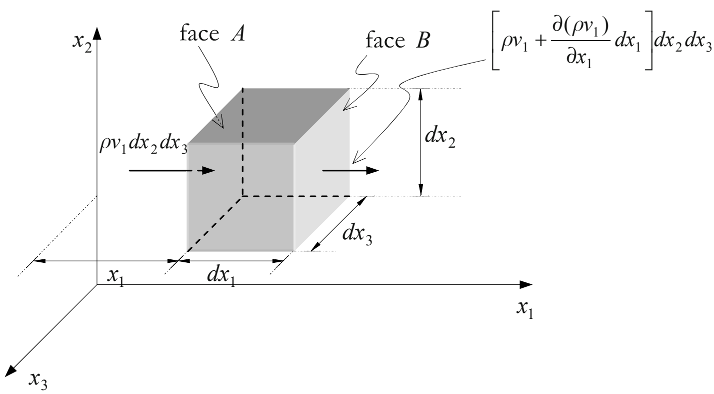


Figure 5.5: Conservation of mass in a differential volume element.

5.6.1 The Mass Continuity Equation in Lagrangian Description

The mass continuity equation in (5.29) can also be expressed in Lagrangian description (material). To do this, we can start from the conservation of mass which establishes:

$$\int_{V_0} \rho_0(\vec{\boldsymbol{X}})\, dV_0 = \int_{V} \rho(\vec{\boldsymbol{x}},t)\, dV = \int_{V_0} \underbrace{\rho(\vec{\boldsymbol{x}},t)\, J}_{f(\vec{\boldsymbol{X}})}\, dV_0 \tag{5.35}$$

Since the above equation is valid for any volume it means that it will be valid locally too, *i.e.*:

$$\rho_0(\vec{\boldsymbol{X}}) = J\,\rho \tag{5.36}$$

Note that ρ_0 is independent of time and so is $J\rho$ which results in the Lagrangian description of the mass continuity equation:

$$\boxed{\frac{D}{Dt}(\rho J) = 0} \quad \textit{The mass continuity equation (Lagrangian description)} \tag{5.37}$$

Problem 5.2: Show that

$$\boxed{\frac{D}{Dt}\int_V \rho\, P_{ij\ldots}(\vec{\boldsymbol{x}},t)\, dV = \int_V \rho \frac{DP_{ij\ldots}(\vec{\boldsymbol{x}},t)}{Dt} dV} \tag{5.38}$$

where $P_{ij\ldots}(\vec{\boldsymbol{x}},t)$ is a continuum property per unit mass, which can be a scalar, a vector or higher order tensor.

Solution: It was proven in equation (5.6) that:

$$\frac{D}{Dt}\int_V \Phi(\vec{\boldsymbol{x}},t) dV = \int_V \left[\frac{D}{Dt}\Phi(\vec{\boldsymbol{x}},t) + \Phi(\vec{\boldsymbol{x}},t)\frac{\partial v_p}{\partial x_p}\right] dV$$

Then by making $\Phi = \rho\, P_{ij\ldots}$, and by considering it in the above equation we obtain:

$$\frac{D}{Dt}\int_V \rho\, P_{ij\ldots} dV = \int_V \left[\frac{D}{Dt}(\rho P_{ij\ldots}) + \rho P_{ij\ldots}\frac{\partial v_p}{\partial x_p}\right] dV = \int_V \left[\rho \frac{D}{Dt}P_{ij\ldots} + P_{ij\ldots}\frac{D\rho}{Dt} + \rho P_{ij\ldots}\frac{\partial v_k}{\partial x_k}\right] dV$$

$$= \int_V \left[\rho \frac{D}{Dt}P_{ij\ldots} + P_{ij\ldots}\underbrace{\left(\frac{D\rho}{Dt} + \rho\frac{\partial v_k}{\partial x_k}\right)}_{\substack{=0 \\ mass\ continuity\ equation}}\right] dV$$

Thus, we can conclude that:

$$\frac{D}{Dt}\int_V \rho P_{ij\ldots} dV = \int_V \left[\rho \frac{DP_{ij\ldots}}{Dt}\right] dV$$

Problem 5.3: Prove that the following relationship is valid:

$$\boxed{\rho\, \vec{\boldsymbol{a}} = \frac{\partial}{\partial t}(\rho\, \vec{\boldsymbol{v}}) + \nabla_{\vec{x}} \cdot (\rho\, \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{v}})} \tag{5.39}$$

Solution: Based on the Reynolds' transport theorem:

$$\frac{D}{Dt}\int_V \Phi dV = \int_V \frac{\partial \Phi}{\partial t} dV + \int_S \Phi(\vec{\boldsymbol{v}} \cdot \hat{\mathbf{n}}) dS$$

and if we consider that $\Phi = \rho\, \vec{\boldsymbol{v}}$ we obtain:

$$\frac{D}{Dt}\int_V \rho\, \vec{\boldsymbol{v}}\, dV = \int_V \frac{\partial(\rho\, \vec{\boldsymbol{v}})}{\partial t} dV + \int_S \rho\, \vec{\boldsymbol{v}} \otimes (\vec{\boldsymbol{v}} \cdot \hat{\mathbf{n}}) dS$$

Then, the above equation in indicial notation becomes:

$$\frac{D}{Dt}\int_V \rho\, v_i\, dV = \int_V \frac{\partial(\rho\, v_i)}{\partial t} dV + \int_S \rho\, v_i (v_k \hat{\mathsf{n}}_k) dS \Rightarrow \int_V \rho \underbrace{\frac{D}{Dt} v_i}_{=a_i} dV = \int_V \frac{\partial(\rho\, v_i)}{\partial t} dV + \int_S (\rho\, v_i v_k) \hat{\mathsf{n}}_k\, dS$$

Additionally, by applying the divergence theorem to the surface integral we obtain:

$$\int_V \rho\, a_i\, dV = \int_V \frac{\partial(\rho\, v_i)}{\partial t} dV + \int_V (\rho\, v_i v_k)_{,k}\, dV = \int_V \left[\frac{\partial(\rho\, v_i)}{\partial t} + (\rho\, v_i v_k)_{,k}\right] dV$$

which in tensorial notation is:

$$\int_V \rho\, \vec{\boldsymbol{a}}\, dV = \int_V \left[\frac{\partial(\rho\, \vec{\boldsymbol{v}})}{\partial t} + \nabla_{\vec{x}} \cdot (\rho\, \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{v}})\right] dV \quad \Rightarrow \quad \rho\, \vec{\boldsymbol{a}} = \frac{\partial(\rho\, \vec{\boldsymbol{v}})}{\partial t} + \nabla_{\vec{x}} \cdot (\rho\, \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{v}})$$

5.6.2 Incompressibility

Compressibility is the ability to change the volume of a continuous medium. It is common knowledge that gases are more compressible than liquids, but for practical purposes, the liquid can be considered to be incompressible.

An *incompressible* medium is characterized by an *isochoric motion*, *i.e.* $J=1$, hence the mass density field (for all particle) is independent of time. In this case the mass continuity equation in (5.29) boils down to:

$$\frac{D\rho}{Dt}+\rho(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})=0 \quad\Rightarrow\quad \frac{D\rho}{Dt}=-\rho(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})=-\rho v_{k,k}=0 \tag{5.40}$$

thus

$$\boxed{\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}=0} \quad \textit{Mass continuity equation for a incompressible medium} \tag{5.41}$$

Thus, an incompressible medium can be characterized by:

$$\frac{D}{Dt}[\det(\boldsymbol{F})]\equiv\dot{J}=0 \quad ; \quad \frac{D\rho}{Dt}\equiv\dot{\rho}=0 \quad ; \quad \rho=\rho_0 \quad ; \quad J=1 \tag{5.42}$$

or

$$v_{k,k}=0 \quad\Rightarrow\quad \Rightarrow\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=\mathrm{Tr}(\boldsymbol{\ell})=\mathrm{Tr}(\mathbf{D})=0 \tag{5.43}$$

where $\boldsymbol{\ell}$ denotes the spatial velocity gradient, and $\mathbf{D}$ is the rate-of-deformation tensor which is equal to the symmetrical part of $\boldsymbol{\ell}$, (see Chapter 2).

5.6.3 The Mass Continuity Equation for Volume with Discontinuities

Now, let us consider a domain where there is a singular surface $\Sigma(t)$ as established in subsection 5.5.1, (see Figure 5.4). Based on the conservation of mass we have:

$$\frac{D}{Dt}\int_V \rho dV=0 \tag{5.44}$$

and if we consider the Reynolds' transport theorem with discontinuities, (see equation (5.22)), in which $\Phi=\rho$, we obtain:

$$\frac{D}{Dt}\int_{V-\Sigma}\rho dV=\int_{V-\Sigma}\left(\frac{D\rho}{Dt}+\rho\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}\right)dV+\int_{\Sigma}[\![\rho(\vec{\boldsymbol{v}}-\vec{\boldsymbol{\omega}})]\!]\cdot\hat{\mathbf{n}}\,dS=0 \tag{5.45}$$

where the mass density $\rho(\vec{\boldsymbol{x}},t)$, and the velocity $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)$ are continuous differentiable functions in $V-\Sigma$, and $[\![\rho(\vec{\boldsymbol{v}}-\vec{\boldsymbol{\omega}})]\!]$ is also a continuous differentiable function on Σ. The global balance law is valid for any arbitrary parts of the volume and for the discontinuous surface, hence it holds that:

$$\boxed{\begin{array}{ll}\dfrac{D\rho}{Dt}+\rho\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}}=0 & in \quad V-\Sigma \\ [\![\rho(\vec{\boldsymbol{v}}-\vec{\boldsymbol{\omega}})]\!]\cdot\hat{\mathbf{n}}=0 & on \quad \Sigma\end{array}} \quad \textit{The mass continuity equation with discontinuities (Eulerian description)} \tag{5.46}$$

Problem 5.4: Let us consider the following velocity field:

$$v_i = \frac{x_i}{1+t} \qquad \text{for } t \geq 0$$

1) Find the mass density field;

2) Prove that this motion satisfies $\rho\, x_1\, x_2\, x_3 = \rho_0\, X_1\, X_2\, X_3$.

Solution: 1) By applying the mass continuity equation we obtain:

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_k}{\partial x_k} = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} \equiv \frac{d\rho}{dt} = -\rho\frac{\partial v_k}{\partial x_k}$$

and by using the given velocity field, we find that:

$$\frac{\partial v_i}{\partial x_i} = \frac{1}{1+t}\frac{\partial x_i}{\partial x_i} = \frac{\delta_{ii}}{1+t} = \frac{3}{1+t}$$

Thus,

$$\frac{d\rho}{dt} = -\frac{3\rho}{1+t} \quad \Rightarrow \quad \frac{d\rho}{\rho} = -\frac{3dt}{1+t}$$

Then by integrating the both sides of the above equation we obtain:

$$\int \frac{d\rho}{\rho} = \int -\frac{3dt}{1+t} \quad \Rightarrow \quad \ln\rho = -3\ln(1+t) + C$$

The constant of integration C is obtained by means of the above equation if we refer to the initial condition $t = 0$, in which $\rho(\vec{\boldsymbol{x}}, t=0) = \rho_0$, thus

$$\ln\rho_0 = -3\ln(1+0) + C \quad \Rightarrow \quad C = \ln\rho_0$$

$$\ln\rho = -3\ln(1+t) + \ln\rho_0 = \ln\left(\frac{1}{(1+t)^3}\right) + \ln\rho_0 = \ln\left(\frac{\rho_0}{(1+t)^3}\right)$$

Thus, we can conclude that:

$$\rho = \frac{\rho_0}{(1+t)^3}$$

2) Then by using the velocity definition we obtain:

$$v_i = \frac{dx_i}{dt} = \frac{x_i}{1+t} \quad \Rightarrow \frac{dx_i}{x_i} = \frac{dt}{1+t}$$

Additionally, by integrating the both sides of the above equation we obtain:

$$\int \frac{dx_i}{x_i} = \int \frac{dt}{1+t} \quad \Rightarrow \quad \ln x_i = \ln(1+t) + K_i \tag{5.47}$$

Then by applying the initial condition, *i.e.* at time $t = 0 \;\Rightarrow x_i = X_i$, we obtain:

$$\ln X_i = \ln(1+0) + K_i \quad \Rightarrow \quad K_i = \ln X_i$$

Additionally, by substituting the value of K_i into the equation (5.47) we obtain:

$$\ln x_i = \ln(1+t) + \ln X_i \quad \Rightarrow \quad \ln(x_i) = \ln[X_i(1+t)]$$

Hence we can conclude that $x_i = X_i(1+t)$, which gives us $x_1 = X_1(1+t)$, $x_2 = X_2(1+t)$, $x_3 = X_3(1+t)$, and if we consider that $\rho = \dfrac{\rho_0}{(1+t)^3}$, we obtain:

$$\rho\underbrace{(1+t)}_{\frac{x_1}{X_1}}\underbrace{(1+t)}_{\frac{x_2}{X_2}}\underbrace{(1+t)}_{\frac{x_3}{X_3}} = \rho_0 \quad \Rightarrow \quad \rho\, x_1\, x_2\, x_3 = \rho_0\, X_1\, X_2\, X_3$$

5.7 The Principle of Conservation of Linear Momentum. The Equations of Motion

5.7.1 Linear Momentum

Let us consider the body, $\mathcal{B}_t$, in motion which is subjected both to body forces (per unit mass), $\vec{\mathbf{b}}(\vec{x},t)$, and to surface forces, $\vec{\mathbf{t}}^*(\vec{x},t)$, acting on the surface $S_{\boldsymbol{\sigma}}$, (see Figure 5.6). Let $\vec{v}(\vec{x},t)$ be the Eulerian velocity field, then we can define the *linear momentum* of the mass system $\mathcal{B}_t$ as:

$$\boxed{\vec{L} = \int_{\mathcal{B}_t} \vec{v}\, dm = \int_V \rho\, \vec{v}\, dV} \quad \textit{Linear momentum} \quad \left[\frac{kg\ m}{s}\right] \tag{5.48}$$

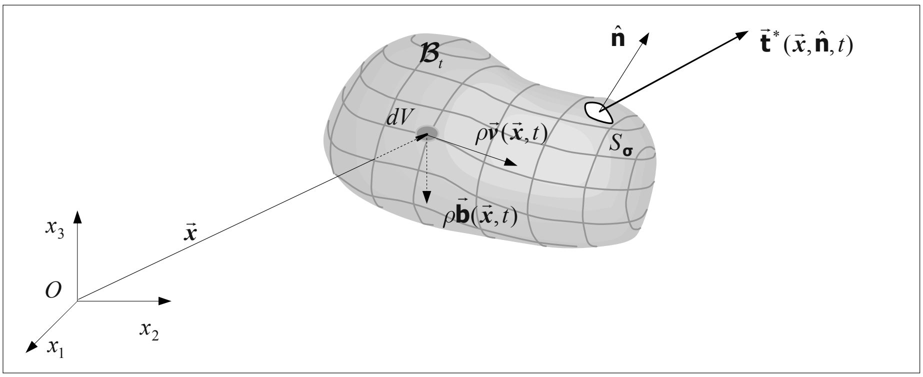


Figure 5.6: Continuum in motion.

5.7.2 The Principle of Conservation of Linear Momentum

The principle of conservation of linear momentum, based on Newton's second law, states that the rate of change of the linear momentum of an arbitrary part of a continuous medium is equal to the resultant force (body and surface forces) acting on the part in question, then:

$$\int_{S_{\boldsymbol{\sigma}}} \vec{\mathbf{t}}^* dS + \int_V \rho \vec{\mathbf{b}}\ dV = \frac{D}{Dt}\int_V \rho\, \vec{v}\ dV \qquad \Bigg| \qquad \int_{S_{\boldsymbol{\sigma}}} \mathrm{t}_i^* dS + \int_V \rho\, \mathrm{b}_i dV = \frac{D}{Dt}\int_V \rho\, v_i dV \tag{5.49}$$

The equation in (5.49) represents the global form of the principle of conservation of linear momentum and by applying $\mathrm{t}_i^* = \sigma_{ij}\hat{\mathrm{n}}_j$ we obtain:

$$\underbrace{\int_{S_{\boldsymbol{\sigma}}} \sigma_{ij}\hat{\mathrm{n}}_j dS}_{\overbrace{\int_V \sigma_{ij,j}\, dV}^{\substack{\text{Guass'} \\ \text{Theorem}}}} + \int_V \rho \mathrm{b}_i dV = \underbrace{\frac{D}{Dt}\int_V \rho v_i dV}_{\overbrace{\int_V \rho \dot{v}_i\, dV}} \tag{5.50}$$

thus,

$$\int_V (\sigma_{ij,j} + \rho \mathrm{b}_i - \rho \dot{v}_i) dV = 0_i \qquad \left[\frac{kg\, m}{s^2} = N \right] \tag{5.51}$$

If the above equation is valid for the entire volume, it is also valid locally, *i.e.*:

$$\sigma_{ij,j} + \rho \mathrm{b}_i - \rho \dot{v}_i = 0_i \qquad \left[\frac{kg}{s^2\, m^2} = \frac{N}{m^3} = \frac{Pa}{m} \right] \tag{5.52}$$

which are known as the *equations of motion* or *Cauchy's first equation of motion*:

$$\boxed{\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{v}} = \rho \vec{a}} \quad \textit{The equations of motion (Eulerian description)} \tag{5.53}$$

Sometimes it is useful to express the equations of motion in the reference configuration. To do this we can rewrite the equation in (5.49) in the undeformed configuration, *i.e.*:

$$\begin{gathered} \int_{S_0} \vec{\mathbf{t}}_0^* dS_0 + \int_{V_0} \rho_0 \vec{\mathbf{b}}_0 \, dV_0 = \frac{D}{Dt} \int_{V_0} \rho \vec{V} \, J \, dV_0 = \int_{V_0} \rho \vec{A} \, J \, dV_0 \\ \int_{S_0} \mathbf{P} \cdot \hat{\mathbf{N}} \, dS_0 + \int_{V_0} \rho_0 \vec{\mathbf{b}}_0 \, dV_0 = \frac{D}{Dt} \int_{V_0} \rho_0 \vec{V} \, dV_0 = \int_{V_0} \rho_0 \vec{A} \, dV_0 \end{gathered} \tag{5.54}$$

where $\vec{V} \equiv \vec{v}(\vec{X},t)$ is the Lagrangian velocity, $\vec{A} \equiv \vec{a}(\vec{X},t)$ is the Lagrangian acceleration field, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{P} = J \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$, and $\vec{\mathbf{b}}_0 (\vec{X},t)$ is the body forces vector per unit mass in the undeformed configuration. Note that $\mathbf{P} \cdot \hat{\mathbf{N}} \neq \hat{\mathbf{N}} \cdot \mathbf{P}$, since $\mathbf{P}$ is a non-symmetric tensor. Then by applying Gauss' theorem (the divergence theorem) to the surface integral we obtain:

$$\int_{V_0} \left(\nabla_{\vec{X}} \cdot \mathbf{P} + \rho_0 \vec{\mathbf{b}}_0 - \rho_0 \vec{A} \right) dV_0 = \vec{\mathbf{0}} \tag{5.55}$$

Then, the local form of the equations of motion in material description (Lagrangian) can be expressed as:

$$\boxed{\begin{aligned} \nabla_{\vec{X}} \cdot \mathbf{P} + \rho_0 \vec{\mathbf{b}}_0 &= \rho_0 \vec{A} \\ \nabla_{\vec{X}} \cdot (\boldsymbol{F} \cdot \mathbf{S}) + \rho_0 \vec{\mathbf{b}}_0 &= \rho_0 \vec{A} \end{aligned}} \quad \textit{The equations of motion (Lagrangian description)} \tag{5.56}$$

5.7.2.1 The Equilibrium Equations

In the exceptional cases when we have a static or quasi-static equilibrium, the acceleration components are zero, thus we obtain the equilibrium equations as:

$$\boxed{\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}} \quad \textit{The equilibrium equations (Eulerian description)} \tag{5.57}$$

Explicitly, the equations in (5.57), $\sigma_{ij,j} + \rho \mathrm{b}_i = 0_i$, can be expressed as:

$$\begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + \rho \mathrm{b}_1 = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} + \rho \mathrm{b}_2 = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + \rho \mathrm{b}_3 = 0 \end{cases} \tag{5.58}$$

Then by using both the engineering and Voigt notation, the equilibrium equations can be expressed as follows:

$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{[\boldsymbol{L}]^T} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} + \begin{bmatrix} \rho \mathrm{b}_1 \\ \rho \mathrm{b}_2 \\ \rho \mathrm{b}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [\boldsymbol{L}]^T \{\boldsymbol{\sigma}\} + \{\boldsymbol{M}\} = \{\boldsymbol{0}\} \tag{5.59}$$

Additionally, the equilibrium equations in the Lagrangian description are given by:

$$\boxed{\begin{aligned} &\nabla_{\vec{X}} \cdot \mathbf{P} + \rho_0 \vec{\mathbf{b}}_0 = \vec{\mathbf{0}} \\ &\nabla_{\vec{X}} \cdot (\boldsymbol{F} \cdot \mathbf{S}) + \rho_0 \vec{\mathbf{b}}_0 = \vec{\mathbf{0}} \end{aligned}} \quad \textit{The equilibrium equations (Lagrangian description)} \tag{5.60}$$

Problem 5.5: Find the equilibrium equations in engineering notation by means of the differential volume element equilibrium $(dxdydz)$. For this purpose consider that the Cauchy stress tensor field in the differential volume element varies as indicated in Figure 5.7.

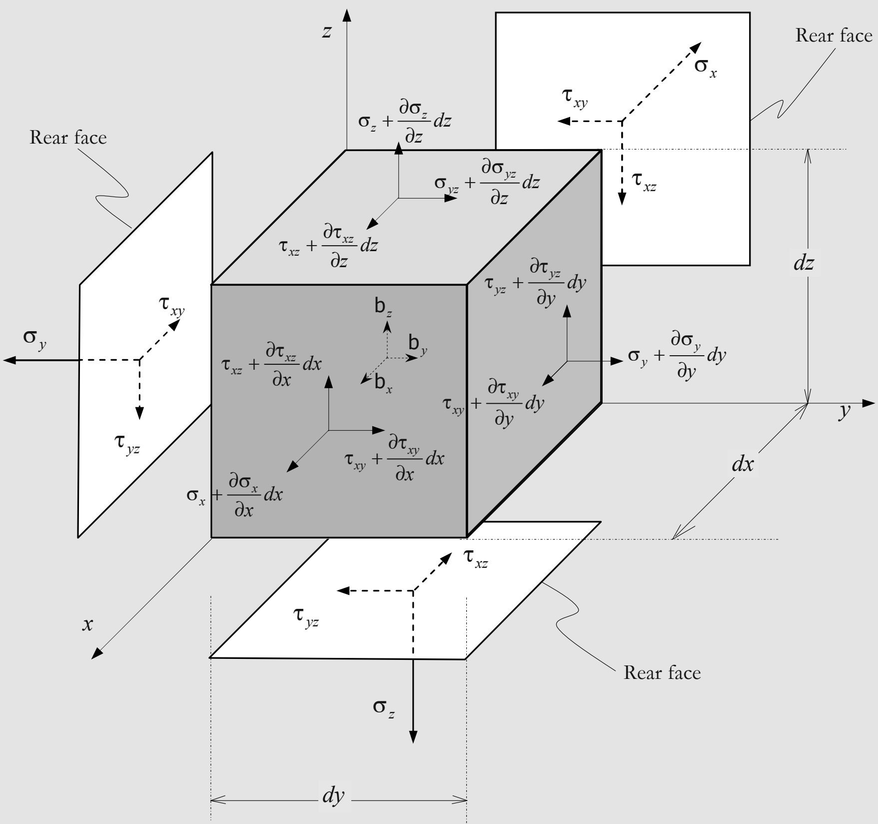


Figure 5.7: The stress field in the differential volume element.

Solution:
To obtain the equilibrium equations we apply the force equilibrium condition in the volume element. First, we evaluate the equilibrium force according to the x-direction:

$$\sum F_x = 0$$

$$\rho \mathrm{b}_x dxdydz + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dydz - \sigma_x dydz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right) dxdz$$
$$- \tau_{xy} dxdz + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz\right) dxdy - \tau_{xz} dxdy = 0$$

Then by simplifying the above equation we obtain:

$$\rho \mathrm{b}_x dxdydz + \frac{\partial \sigma_x}{\partial x} dxdydz + \frac{\partial \tau_{xy}}{\partial y} dxdydz + \frac{\partial \tau_{xz}}{\partial z} dxdydz = 0$$

$$\rho \mathrm{b}_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

The equilibrium force according to the y-direction, $\sum F_y = 0$, can be expressed as follows

$$\rho \mathrm{b}_y dxdydz + \left(\sigma_{22} + \frac{\partial \sigma_y}{\partial y} dy\right) dxdz - \sigma_y dxdz + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial z} dz\right) dxdy$$
$$- \tau_{yz} dxdy + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dydz - \tau_{xy} dydz = 0$$

Then by simplifying the above equation we obtain:

$$\rho \mathrm{b}_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial x_z} = 0$$

Finally, the equilibrium according to the z-direction, $\sum F_z = 0$, is given by:

$$\rho \mathrm{b}_z dxdydz + \left(\sigma_z + \frac{\partial \sigma_z}{\partial z} dz\right) dxdy - \sigma_z dxdy + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx\right) dzdy$$
$$- \tau_{xz} dzdy + \left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy\right) dxdz - \tau_{yz} dxdz = 0$$

Additionally, by simplifying the above equation we obtain:

$$\rho \mathrm{b}_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

Then, the equilibrium equations in engineering notation become:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} + \rho \mathrm{b}_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial x_z} + \rho \mathrm{b}_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + \rho \mathrm{b}_z = 0 \end{cases}$$

Problem 5.6: Let $\boldsymbol{\sigma}$ be the Cauchy stress tensor field, which is represented by its components in the Cartesian basis as:

$$\sigma_{11} = x_1^2; \qquad \sigma_{22} = x_2^2; \qquad \sigma_{33} = x_1^2 + x_2^2$$

$$\sigma_{12} = \sigma_{21} = 2x_1 x_2; \qquad \sigma_{23} = \sigma_{32} = \sigma_{31} = \sigma_{13} = 0$$

Considering that the body is in equilibrium, find the body forces acting on the continuum.

Solution: By applying the equilibrium equations, $\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \vec{\mathbf{0}}$, we obtain:

$$\sigma_{ij,j} + \rho \mathrm{b}_i = 0_i \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} + \rho \mathrm{b}_1 = 0 \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} + \rho \mathrm{b}_2 = 0 \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} + \rho \mathrm{b}_3 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} 2x_1 + 2x_1 + \rho \mathrm{b}_1 = 0 \\ 2x_2 + 2x_2 + \rho \mathrm{b}_2 = 0 \\ \rho \mathrm{b}_3 = 0 \end{cases}$$

Thus, to satisfy the equilibrium equations the following condition must be met:

$$4x_1 = -\rho \mathrm{b}_1 \Rightarrow \rho \mathrm{b}_1 = -4x_1$$
$$4x_2 = -\rho \mathrm{b}_2 \Rightarrow \rho \mathrm{b}_2 = -4x_2$$
$$\Rightarrow \rho \mathrm{b}_3 = 0$$
$$\rho \vec{\mathbf{b}} = -4(x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2)$$

Problem 5.7: The equations of motion of a body are given, in Lagrangian description, by:

$$\begin{cases} x_1 = X_1 + \alpha t X_3 \\ x_2 = X_2 + \alpha t X_3 \\ x_3 = X_3 - \alpha t (X_1 + X_2) \end{cases}$$

where α is a constant scalar. Find the mass density in the current configuration (ρ) in terms of the mass density of the reference configuration (ρ_0), *i.e.* $\rho = \rho(\rho_0)$.

Solution:

We can apply the equation $\rho_0 = J\rho$, where J is the Jacobian determinant and is given by:

$$J = |F| = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \alpha t \\ 0 & 1 & \alpha t \\ -\alpha t & -\alpha t & 1 \end{vmatrix} = 1 + 2(\alpha t)^2$$

Thus, we obtain $\rho = \dfrac{\rho_0}{J} = \dfrac{\rho_0}{1 + 2(\alpha t)^2}$

5.7.3 The Equations of Motion with Discontinuities

Let us consider again a domain with a singular surface $\Sigma(t)$ such as that discussed in subsection 5.5.1, (see Figure 5.4). Then, the principle of conservation of linear momentum becomes:

$$\frac{D}{Dt} \int_{V-\Sigma} \rho \vec{v} \, dV = \int_{S-\Sigma} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \, dS + \int_{V-\Sigma} \rho \vec{\mathbf{b}} \, dV \tag{5.61}$$

Then by applying the divergence theorem with discontinuities, (see Eq. (5.17)), we obtain:

$$\frac{D}{Dt} \int_{V-\Sigma} \rho \vec{v} \, dV = \int_{V-\Sigma} \left(\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} \right) dS + \int_{\Sigma} [[\boldsymbol{\sigma}]] \cdot \hat{\mathbf{n}} \, dV \tag{5.62}$$

Additionally, by using Reynolds' transport theorem, (see equation (5.22)), with $\Phi = \rho\vec{v}$, we obtain:

$$\frac{D}{Dt}\int_{V-\Sigma} \rho\vec{v}dV = \int_{V-\Sigma}\left(\frac{D(\rho\vec{v})}{Dt} + \rho\vec{v}\nabla_{\vec{x}}\cdot\vec{v}\right)dV + \int_{\Sigma} [[\rho\vec{v}\otimes(\vec{v}-\vec{\omega})]]\cdot\hat{\mathbf{n}}\,dS \tag{5.63}$$

Then by combining the above equation with the equation in (5.62) and by considering that $\frac{D(\rho\vec{v})}{Dt} = \vec{v}\frac{D(\rho)}{Dt} + \rho\frac{D(\vec{v})}{Dt}$ we obtain:

$$\int_{V-\Sigma} \vec{v}\left(\frac{D(\rho)}{Dt} + \rho\nabla_{\vec{x}}\cdot\vec{v}\right) + \left(\rho\frac{D(\vec{v})}{Dt} - \nabla_{\vec{x}}\cdot\boldsymbol{\sigma} - \rho\vec{\mathbf{b}}\right)dV + \int_{\Sigma} [[\rho\vec{v}\otimes(\vec{v}-\vec{\omega}) - \boldsymbol{\sigma}]]\cdot\hat{\mathbf{n}}\,dS = \vec{\mathbf{0}} \tag{5.64}$$

Bearing in mind the mass continuity equation, $\frac{D(\rho)}{Dt} + \rho\nabla_{\vec{x}}\cdot\vec{v} = 0$, the equation in (5.64) becomes:

$$\int_{V-\Sigma}\left(\rho\frac{D(\vec{v})}{Dt} - \nabla_{\vec{x}}\cdot\boldsymbol{\sigma} - \rho\vec{\mathbf{b}}\right)dV + \int_{\Sigma} [[\rho\vec{v}\otimes(\vec{v}-\vec{\omega}) - \boldsymbol{\sigma}]]\cdot\hat{\mathbf{n}}\,dS = \vec{\mathbf{0}} \tag{5.65}$$

Then, the local form can be expressed as:

$$\boxed{\begin{array}{lll} \nabla_{\vec{x}}\cdot\boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\vec{a} & in & V-\Sigma \\ [[\rho\vec{v}\otimes(\vec{v}-\vec{\omega}) - \boldsymbol{\sigma}]]\cdot\hat{\mathbf{n}} = \vec{\mathbf{0}} & on & \Sigma \end{array}} \quad \textit{The equations of motion with discontinuities (Eulerian description)} \tag{5.66}$$

For a static or quasi-static problem the equations in (5.66) become:

$$\boxed{\begin{array}{lll} \nabla_{\vec{x}}\cdot\boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \vec{\mathbf{0}} & in & V-\Sigma \\ \begin{cases} [[\boldsymbol{\sigma}]]\cdot\hat{\mathbf{n}} = \vec{\mathbf{0}} \\ \Rightarrow \boldsymbol{\sigma}^{+}\cdot\hat{\mathbf{n}} = \boldsymbol{\sigma}^{-}\cdot\hat{\mathbf{n}} \end{cases} & on & \Sigma \end{array}} \quad \textit{The equations of motion with discontinuities (static problem) (Eulerian description)} \tag{5.67}$$

5.8 The Principle of Conservation of Angular Momentum. Symmetry of the Cauchy Stress Tensor

5.8.1 Angular Momentum

Once again let us consider Figure 5.6, and we can define the angular momentum of a mass system with respect to the origin by:

$$\boxed{\begin{array}{l} \vec{\boldsymbol{H}}_O = \int_V (\vec{\boldsymbol{x}}\wedge\rho\vec{\boldsymbol{v}})\,dV \\ H_{O_i}(t) = \int_V (\epsilon_{ijk}x_j\rho\,v_k)\,dV \end{array}} \quad \textit{Angular momentum} \tag{5.68}$$

The SI unit of $\vec{\boldsymbol{H}}_O$ is $\left[\vec{\boldsymbol{H}}_O\right] = \frac{kg\,m^2}{s}$, and $\left[\dot{\vec{\boldsymbol{H}}}_O\right] = \frac{kg\,m^2}{s^2} = Nm = J$.

5.8.2 The Principle of Conservation of Angular Momentum

The principle of conservation of angular momentum states that the rate of change of angular momentum with respect to a point is equal to the resultant moment (with respect to this point) produced by all forces acting on the body under consideration.

Then by obtaining the resultant momentum with respect to the origin, (see Figure 5.6), and by applying the principle of angular momentum, we obtain:

$$\int_{S_{\boldsymbol{\sigma}}} (\vec{\boldsymbol{x}} \wedge \vec{\mathbf{t}}^{*}) dS + \int_{V} (\vec{\boldsymbol{x}} \wedge \rho \vec{\mathbf{b}}) dV = \frac{D}{Dt} \int_{V} (\vec{\boldsymbol{x}} \wedge \rho\, \vec{\boldsymbol{v}}) dV \qquad [Nm] \tag{5.69}$$

NOTE: The equation in (5.69) is valid for those continuous media in which the forces between particles are equal, opposite and collinear, and without any distributed moments.■

The equation in (5.69) can be rewritten in indicial notation as:

$$\begin{aligned} \int_{S_{\boldsymbol{\sigma}}} (\epsilon_{ijk} x_j \mathrm{t}_k^{*}) dS + \int_{V} (\epsilon_{ijk} x_j \rho \mathrm{b}_k) dV &= \frac{D}{Dt} \int_{V} (\epsilon_{ijk} x_j \rho\, v_k) dV = \int_{V} \rho \frac{D}{Dt} (\epsilon_{ijk} x_j\, v_k) dV \\ &= \int_{V} \rho (\underbrace{\epsilon_{ijk} \underbrace{\dot{x}_j}_{v_j} v_k}_{=0_i} + \epsilon_{ijk} x_j\, \dot{v}_k) dV \end{aligned} \tag{5.70}$$

Then by substituting $\mathrm{t}_k^{*} = \sigma_{kl} \hat{\mathrm{n}}_l$ into the first integral of (5.70), and by applying the Gauss' theorem, we obtain:

$$\int_{V} \epsilon_{ijk} (x_j \sigma_{kl})_{,l} dV + \int_{V} (\epsilon_{ijk} x_j \rho \mathrm{b}_k) dV = \int_{V} (\epsilon_{ijk} x_j \rho a_k) dV \tag{5.71}$$

$$\int_{V} (\epsilon_{ijk} \underbrace{x_{j,l}}_{\delta_{jl}} \sigma_{kl} + \epsilon_{ijk} x_j \sigma_{kl,l} + \epsilon_{ijk} x_j \rho \mathrm{b}_k) dV = \int_{V} (\epsilon_{ijk} x_j \rho a_k) dV \tag{5.72}$$

$$\int_{V} \left[\epsilon_{ijk} \sigma_{kj} + \epsilon_{ijk} x_j \underbrace{(\sigma_{kl,l} + \rho b_k - \rho a_k)}_{\substack{=0_k \\ \textit{Equations of motion}}} \right] dV = \mathbf{0}_i \quad \Rightarrow \quad \int_{V} \epsilon_{ijk} \sigma_{kj} dV = \mathbf{0}_i \tag{5.73}$$

$$\begin{gathered} \epsilon_{ijk} \sigma_{kj} = \mathbf{0}_i \\ \Updownarrow \\ \boxed{\sigma_{jk} = \sigma_{kj}} \end{gathered} \tag{5.74}$$

Thus obtaining Cauchy's second law of motion, also known as the *Boltzmann postulate*, the symmetry of the Cauchy stress tensor is:

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^T} \quad \textit{Cauchy's second law of motion} \tag{5.75}$$

Then bearing in mind the relationship $\boldsymbol{\sigma} = J^{-1} \mathbf{P} \cdot \boldsymbol{F}^T$, the Boltzmann postulate in the reference configuration becomes:

$$\left. \begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}^T \\ \frac{1}{J} \mathbf{P} \cdot \boldsymbol{F}^T &= \left(\frac{1}{J} \mathbf{P} \cdot \boldsymbol{F}^T \right)^T \end{aligned} \right\} \quad \Rightarrow \quad \boxed{\mathbf{P} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \mathbf{P}^T} \tag{5.76}$$

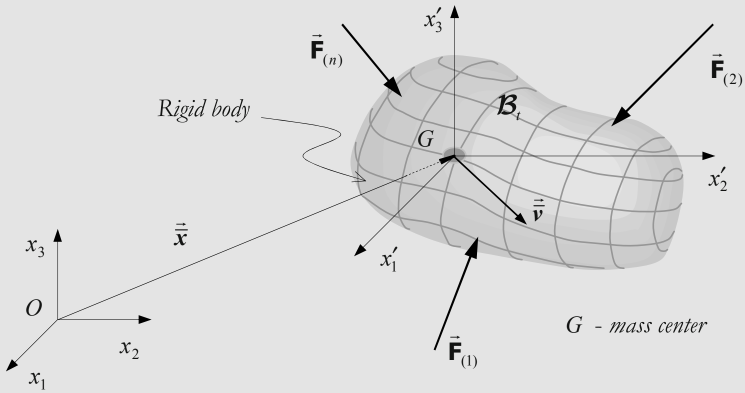
and considering that $\mathbf{P} = \boldsymbol{F} \cdot \mathbf{S}$, where $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, we obtain:

$$\boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T=\boldsymbol{F}\cdot(\boldsymbol{F}\cdot\mathbf{S})^T \quad ; \quad \boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T=\boldsymbol{F}\cdot\mathbf{S}^T\cdot\boldsymbol{F}^T \tag{5.77}$$

Thus

$$\boxed{\mathbf{S}=\mathbf{S}^T} \tag{5.78}$$

Problem 5.8: Find the linear and angular momentum for a solid subjected to rigid body motion.



Solution: According to **Problem 2.16** in Chapter 2, we obtained the velocity for rigid body motion as:

$$\vec{v}=\dot{\vec{\mathbf{c}}}+\vec{\boldsymbol{\omega}}\wedge(\vec{x}-\vec{\mathbf{c}})$$

where $\vec{\boldsymbol{\omega}}$ is the axial vector (angular velocity) associated with the antisymmetric tensor $\mathbf{W}$ (the spin tensor).

Linear momentum:

$$\vec{L}=\int_V \rho\,\vec{v}\,dV=\int_V \rho\left(\dot{\vec{\mathbf{c}}}+\vec{\boldsymbol{\omega}}\wedge(\vec{x}-\vec{\mathbf{c}})\right)dV=\int_V \rho\,\dot{\vec{\mathbf{c}}}\,dV+\int_V \rho\,\vec{\boldsymbol{\omega}}\wedge\vec{x}\,dV-\int_V \rho\,\vec{\boldsymbol{\omega}}\wedge\vec{\mathbf{c}}\,dV$$
$$=\dot{\vec{\mathbf{c}}}\int_V \rho\,dV+\vec{\boldsymbol{\omega}}\wedge\int_V \rho\,\vec{x}\,dV-\vec{\boldsymbol{\omega}}\wedge\vec{\mathbf{c}}\int_V \rho\,dV$$

By definition $\int_V \rho\,\vec{x}\,dV=m\bar{\vec{x}}$ is the first moment of inertia, where m is the total mass, and $\bar{\vec{x}}_k$ is the vector position of the center of mass G. The first moment of inertia is equal to zero if the Cartesian system originates at the center of mass, so, $\int_V \rho\,\vec{x}'\,dV=m\bar{\vec{x}}'=\vec{\mathbf{0}}$.

$$\boxed{\begin{aligned}\vec{L}&=m\left[\dot{\vec{\mathbf{c}}}+\vec{\boldsymbol{\omega}}\wedge(\bar{\vec{x}}-\vec{\mathbf{c}})\right]\\&=m\,\bar{\vec{v}}\end{aligned}} \quad \textit{(Linear momentum for rigid body motion)}$$

where $\bar{\vec{v}}=\dot{\vec{\mathbf{c}}}+\vec{\boldsymbol{\omega}}\wedge(\bar{\vec{x}}-\vec{\mathbf{c}})$ is the velocity of the center of mass.

Angular momentum:

$$\vec{H}_O=\int_V(\vec{x}\wedge\rho\vec{v})\,dV=\int_V\left[\vec{x}\wedge\rho\left(\dot{\vec{\mathbf{c}}}+\vec{\boldsymbol{\omega}}\wedge(\vec{x}-\vec{\mathbf{c}})\right)\right]\,dV$$

Thus

$$\begin{aligned}\vec{H}_O &= \int_V \rho\,\vec{x}\wedge\dot{\vec{\mathbf{c}}}\,dV + \int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV - \int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{\mathbf{c}})\,dV \\ &= \left[\int_V \rho\,\vec{x}\,dV\right]\wedge\dot{\vec{\mathbf{c}}} + \int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV - \left[\int_V \rho\,\vec{x}\,dV\right]\wedge(\vec{\omega}\wedge\vec{\mathbf{c}})\end{aligned} \tag{5.79}$$

Next, we discuss the second integral of the previous equation.

It was proven in Chapter 1 that given three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$, the relationship $\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{c}}) = (\vec{\mathbf{a}}\cdot\vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{c}}$ holds, thus when $\vec{\mathbf{a}}=\vec{\mathbf{c}}$ it holds that $\vec{\mathbf{a}}\wedge(\vec{\mathbf{b}}\wedge\vec{\mathbf{a}}) = (\vec{\mathbf{a}}\cdot\vec{\mathbf{a}})\vec{\mathbf{b}} - (\vec{\mathbf{a}}\cdot\vec{\mathbf{b}})\vec{\mathbf{a}}$, so, $\int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV = \int_V \rho\left[(\vec{x}\cdot\vec{x})\vec{\omega} - (\vec{x}\cdot\vec{\omega})\vec{x}\right]dV$, with which we obtain:

$$\begin{aligned}\int_V \rho\left[x_k x_k \omega_i - x_p\omega_p x_i\right]dV &= \int_V \rho\left[x_k x_k \omega_p \delta_{pi} - x_p\omega_p x_i\right]dV = \int_V \rho\left[x_k x_k \delta_{pi} - x_p x_i\right]\omega_p\,dV \\ &= \int_V \rho\left[x_k x_k \delta_{pi} - x_p x_i\right]dV\;\;\omega_p = \mathrm{I}_{Oip}\omega_p\end{aligned}$$

or in tensorial notation:

$$\int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV = \left[\int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1} - (\vec{x}\otimes\vec{x})\right]dV\right]\cdot\vec{\omega} = \mathbf{I}_O\cdot\vec{\omega}$$

where $\mathbf{I}_O = \int_V \rho\left[(\vec{x}\cdot\vec{x})\mathbf{1} - (\vec{x}\otimes\vec{x})\right]dV$ is the inertia tensor with respect to the origin O. As we can observe, $\mathbf{I}_O$ is a second-order pseudo-tensor, since it depends on the reference system, and the components $\mathrm{I}_{Oij} = \int_V \rho\left[x_k x_k \delta_{ij} - x_i x_j\right]dV$ can be expressed explicitly as:

$$\begin{aligned}&\mathrm{I}_{O11} = \int_V \rho\left[(x_1x_1 + x_2x_2 + x_3x_3)\delta_{11} - x_1x_1\right]dV = \int_V \rho\left[x_2^2 + x_3^2\right]dV \\ &\mathrm{I}_{O22} = \int_V \rho\left[x_1^2 + x_3^2\right]dV \quad ; \quad \mathrm{I}_{O33} = \int_V \rho\left[x_1^2 + x_2^2\right]dV \\ &\mathrm{I}_{O12} = \int_V \rho\left[(x_1x_1 + x_2x_2 + x_3x_3)\delta_{12} - x_1x_2\right]dV = -\int_V \rho\left[x_1x_2\right]dV = -\mathcal{I}_{O12} \\ &\mathrm{I}_{O13} = -\int_V \rho\left[x_1x_3\right]dV = -\mathcal{I}_{O13} \quad ; \quad \mathrm{I}_{O23} = -\int_V \rho\left[x_2x_3\right]dV = -\mathcal{I}_{O23}\end{aligned}$$

where I_{O11}, I_{O22}, I_{O33}, are *moments of inertia* of the body relative to the reference point O, and $\mathcal{I}_{O12}$, $\mathcal{I}_{O13}$, $\mathcal{I}_{O23}$, are the *products of inertia* of the body relative to the reference point O.

Returning to the equation in (5.79) we can state that:

$$\begin{aligned}\vec{H}_O &= \left[\int_V \rho\,\vec{x}\,dV\right]\wedge\dot{\vec{\mathbf{c}}} + \int_V \rho\,\vec{x}\wedge(\vec{\omega}\wedge\vec{x})\,dV - \left[\int_V \rho\,\vec{x}\,dV\right]\wedge(\vec{\omega}\wedge\vec{\mathbf{c}}) \\ &= m\,\vec{\bar{x}}\wedge\dot{\vec{\mathbf{c}}} + \mathbf{I}_O\cdot\vec{\omega} - m\,\vec{\bar{x}}\wedge(\vec{\omega}\wedge\vec{\mathbf{c}}) = m\,\vec{\bar{x}}\wedge\left[\dot{\vec{\mathbf{c}}} - (\vec{\omega}\wedge\vec{\mathbf{c}})\right] + \mathbf{I}_O\cdot\vec{\omega}\end{aligned}$$

Then by adding and subtracting the term $m\,\vec{\bar{x}}\wedge\vec{\omega}\wedge\vec{\bar{x}}$ in the above equation we obtain:

$$\begin{aligned}\vec{H}_O &= m\,\vec{\bar{x}}\wedge\left(\dot{\vec{\mathbf{c}}} - \vec{\omega}\wedge\vec{\mathbf{c}}\right) + \mathbf{I}_O\cdot\vec{\omega} = m\,\vec{\bar{x}}\wedge\left[\dot{\vec{\mathbf{c}}} + \vec{\omega}\wedge(\vec{\bar{x}} - \vec{\mathbf{c}})\right] - m\,\vec{\bar{x}}\wedge(\vec{\omega}\wedge\vec{\bar{x}}) + \mathbf{I}_O\cdot\vec{\omega} \\ &= m\,\vec{\bar{x}}\wedge\vec{\bar{v}} - m\left[(\vec{\bar{x}}\cdot\vec{\bar{x}})\mathbf{1} - (\vec{\bar{x}}\otimes\vec{\bar{x}})\right]\cdot\vec{\omega} + \mathbf{I}_O\cdot\vec{\omega} = m\,\vec{\bar{x}}\wedge\vec{\bar{v}} + \left\{m\left[(\vec{\bar{x}}\otimes\vec{\bar{x}}) - (\vec{\bar{x}}\cdot\vec{\bar{x}})\mathbf{1}\right] + \mathbf{I}_O\right\}\cdot\vec{\omega} \\ &= m\,\vec{\bar{x}}\wedge\vec{\bar{v}} + \bar{\mathbf{I}}\cdot\vec{\omega} \\ &= m\,\vec{\bar{x}}\wedge\vec{\bar{v}} + \vec{H}_G\end{aligned}$$

where $\bar{\mathbf{I}} = \mathbf{I}_O + m\left[(\vec{\bar{x}} \otimes \vec{\bar{x}}) - (\vec{\bar{x}} \cdot \vec{\bar{x}})\mathbf{1}\right]$ is the inertia pseudo-tensor, which is related to the reference system at the center of mass. By means of this equation we can calculate the inertia tensor in any reference system if we know the inertia tensor at the center of mass: $\mathrm{I}_{O_{ij}} = \bar{\mathrm{I}}_{ij} - m\left[\bar{x}_i\bar{x}_j - (\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2)\delta_{ij}\right]$. Explicitly, these components can be expressed as:

$$\boxed{\begin{array}{lll} \mathrm{I}_{O11} = \bar{\mathrm{I}}_{11} + m(\bar{x}_2^2 + \bar{x}_3^2) & ; & \mathrm{I}_{O12} = \bar{\mathrm{I}}_{12} - m(\bar{x}_1\bar{x}_2) \\ \mathrm{I}_{O22} = \bar{\mathrm{I}}_{22} + m(\bar{x}_1^2 + \bar{x}_3^2) & ; & \mathrm{I}_{O23} = \bar{\mathrm{I}}_{23} - m(\bar{x}_2\bar{x}_3) \\ \mathrm{I}_{O33} = \bar{\mathrm{I}}_{33} + m(\bar{x}_1^2 + \bar{x}_2^2) & ; & \mathrm{I}_{O13} = \bar{\mathrm{I}}_{13} - m(\bar{x}_1\bar{x}_3) \end{array}}$$

Note that, the above equations represent the parallel axis theorem (Steiner's theorem) from Classical Mechanics.

Problem 5.9: Obtain the principle of conservation of linear momentum and angular momentum for a solid subjected to rigid body motion.

Solution: We can start from the definition of the principle of conservation of linear momentum which states that:

$$\sum \vec{\mathbf{F}} = \frac{D}{Dt}\int_V \rho\,\vec{v}\; dV = \dot{\vec{L}}$$

Then we use the equation of linear momentum obtained in **Problem 5.8**, $\vec{L} = m\,\vec{\bar{v}}$, to obtain:

$$\sum \vec{\mathbf{F}} = \frac{D}{Dt}\int_V \rho\,\vec{v}\; dV = \dot{\vec{L}} = m\,\dot{\vec{\bar{v}}} = m\,\vec{\bar{a}}$$

Then we have:

$$\boxed{\sum \vec{\mathbf{F}} = m\,\vec{\bar{a}}}$$

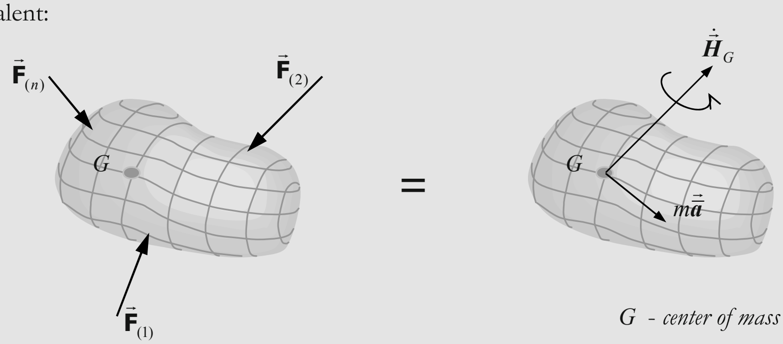
Now let us consider the principle of conservation of angular momentum which states:

$$\sum \vec{M}_O = \frac{D}{Dt}\int_V (\vec{x} \wedge \rho\,\vec{v})dV = \frac{D}{Dt}\vec{H}_O \equiv \dot{\vec{H}}_O$$

By which we obtain:

$$\boxed{\sum \vec{M}_O = \dot{\vec{H}}_O} \quad \text{or} \quad \boxed{\sum \vec{M}_G = \dot{\vec{H}}_G}$$

where the equation of angular momentum $\vec{H}_O$ was obtained in **Problem 5.8**. The set of equations $\sum \vec{\mathbf{F}} = m\,\vec{\bar{a}}$ and $\sum \vec{M}_G = \dot{\vec{H}}_G$ inform us that the following systems are equivalent:



5.9 The Principle of Conservation of Energy. The Energy Equation

The *principle of conservation of energy* states:

> *"The rate of change of the kinetic energy plus the rate of change of the internal energy is equal to the sum of the rate of change of the work done by the system plus the rate of change of any other energy supplied to, or removed from, the system".* (5.80)

The energy supplied to, or removed from, the system per unit time can be any of three kinds: thermal; chemical; or electromagnetic energy. In this publication we only consider thermal energy as the energy added to the system. In such circumstances, the principle of conservation of energy is known as the *first law of thermodynamics*. Mathematically, the principle of conservation of energy, for continuum thermodynamics, is given by:

$$\frac{D\mathcal{K}}{Dt}+\frac{DU}{Dt}=\frac{\text{Đ}\mathcal{W}}{Dt}+\frac{\text{Đ}Q}{Dt} \qquad \left[\frac{J}{s}=W\right] \tag{5.81}$$

where $\mathcal{K}$ is the kinetic energy, U is the internal energy, $\mathcal{W}$ is the work done by the system, and Q is the energy added to the system.

Next, we will introduce the types of energy involved in the energy equation.

5.9.1 Kinetic Energy

The kinetic energy of the system represented in Figure 5.6 is given by:

$$\boxed{\mathcal{K}(t)=\frac{1}{2}\int_V \rho(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}})dV=\frac{1}{2}\int_V \rho(v_i v_i)dV} \quad \textit{Kinetic energy} \qquad [J] \tag{5.82}$$

The SI unit of the energy is the joule: $[\mathcal{K}]=\int_V \frac{kg}{m^3}\frac{m}{s}\frac{m}{s}dV=\int_V \frac{Nm}{m^3}dV=Nm=J$.

Then, the rate of change of the kinetic energy becomes:

$$\frac{D}{Dt}\mathcal{K}(t)\equiv\dot{\mathcal{K}}=\frac{D}{Dt}\left[\frac{1}{2}\int_V \rho(v_i v_i)dV\right]=\frac{1}{2}\int_V \rho\frac{D}{Dt}(v_i v_i)dV=\frac{1}{2}\int_V \rho(\dot{v}_i v_i+v_i\dot{v}_i)dV \tag{5.83}$$

Thus

$$\frac{D}{Dt}\mathcal{K}(t)\equiv\dot{\mathcal{K}}=\int_V \rho\, v_i\dot{v}_i\, dV \tag{5.84}$$

5.9.2 External and Internal Mechanical Power

Let us consider the equations of motion $\sigma_{ij,j}+\rho \mathsf{b}_i=\rho\dot{v}_i$, and if we substituting those into the rate of kinetic energy given in (5.84) we obtain:

$$\dot{\mathcal{K}}=\int_V v_i(\sigma_{ij,j}+\rho \mathsf{b}_i)dV \tag{5.85}$$

Then the term $v_i\sigma_{ij,j}$ can be substituted by:

$$(v_i\sigma_{ij})_{,j} = v_{i,j}\sigma_{ij} + v_i\sigma_{ij,j} \quad \Rightarrow \quad v_i\sigma_{ij,j} = (v_i\sigma_{ij})_{,j} - \underbrace{v_{i,j}}_{\ell_{ij}}\sigma_{ij} \tag{5.86}$$

where $\nabla_{\vec{x}}\vec{v} \equiv \boldsymbol{\ell}$ is the spatial velocity gradient, which can be broken down into a symmetric and an antisymmetric part, *i.e.* $\boldsymbol{\ell} = \mathbf{D} + \mathbf{W}$, (see Chapter 2), where $\mathbf{D}$ is the rate-of-deformation tensor and $\mathbf{W}$ is the spin tensor. The components of these tensors can be expressed in terms of Eulerian velocity as:

$$\ell_{ij} = v_{i,j} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\mathsf{D}_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\mathsf{W}_{ij}} \qquad \left[\frac{m}{m\,s}\right] \tag{5.87}$$

Returning to the equation in (5.85), and considering the relationships in (5.86) and (5.87), the rate of change of the kinetic energy becomes:

$$\begin{aligned}\dot{\mathcal{K}} &= \int_V \left[\left(v_i\sigma_{ij}\right)_{,j} - v_{i,j}\sigma_{ij} + \rho\mathsf{b}_i v_i\right] dV = \int_V \left[(v_i\sigma_{ij})_{,j} - \sigma_{ij}(\mathsf{D}_{ij} + \mathsf{W}_{ij}) + \rho\mathsf{b}_i v_i\right] dV \\ &= \int_V \rho\mathsf{b}_i v_i\, dV + \int_V \left(v_i\sigma_{ij}\right)_{,j} dV - \int_V \sigma_{ij}\mathsf{D}_{ij}\, dV\end{aligned} \tag{5.88}$$

where we have taken into account that the double scalar product of a symmetric and antisymmetric tensor is equal to zero, *i.e.* $\sigma_{ij}\mathsf{W}_{ij} = 0$ or $\boldsymbol{\sigma}:\mathbf{W} = 0$. Then by applying the divergence theorem to the second integral of the right side of the equation in (5.88), we find that:

$$\int_V \left(v_i\sigma_{ij}\right)_{,j} dV = \int_{S_\sigma} v_i\sigma_{ij}\hat{\mathsf{n}}_j\, dS = \int_{S_\sigma} v_i\mathsf{t}_i^*\, dS \tag{5.89}$$

By combining the above relationship with the equation in (5.88), we can still express the rate of change of the kinetic energy as:

$$\dot{\mathcal{K}} = \underbrace{\int_V \rho\mathsf{b}_i v_i\, dV + \int_{S_\sigma} v_i\mathsf{t}_i^*\, dS}_{\substack{\mathcal{P}_{ext}(t) \\ \textit{External Mechanical Power}}} - \underbrace{\int_V \sigma_{ij}\mathsf{D}_{ij}\, dV}_{\substack{\mathcal{P}_{int}(t) \\ \textit{Internal Mechanical Power}}} \quad \Rightarrow \quad \frac{D}{Dt}\mathcal{K} = \mathcal{P}_{ext}(t) - \mathcal{P}_{int}(t) \tag{5.90}$$

or

$$\frac{D}{Dt}\mathcal{K} + \mathcal{P}_{int}(t) = \mathcal{P}_{ext}(t) \tag{5.91}$$

where we have introduced the *external mechanical power* $\mathcal{P}_{ext}(t)$, which is the rate of change of the work done by the external forces $\frac{DW}{Dt}$, as:

$$\boxed{\begin{aligned}\mathcal{P}_{ext}(t) &= \int_{S_\sigma} \vec{\mathbf{t}}^* \cdot \vec{v}\, dS + \int_V \rho\,\vec{\mathbf{b}} \cdot \vec{v}\, dV \\ \mathcal{P}_{ext}(t) &= \int_{S_\sigma} \mathsf{t}_i^* v_i\, dS + \int_V \rho\,\mathsf{b}_i v_i\, dV\end{aligned}} \quad \textit{The external mechanical power} \quad \left[\frac{J}{s} = W\right] \tag{5.92}$$

and the *internal mechanical power*, also known as the *stress power*, which is the rate of change of the work done by the internal forces:

$$\boxed{\mathcal{P}_{int} = \int_V \sigma_{ij}\mathsf{D}_{ij}\, dV = \int_V \boldsymbol{\sigma}:\mathbf{D}\, dV = \int_V \mathrm{Tr}\left(\boldsymbol{\sigma}^T \cdot \mathbf{D}\right) dV = \int_V \mathrm{Tr}\left(\boldsymbol{\sigma} \cdot \mathbf{D}\right) dV} \quad \textit{The stress power} \tag{5.93}$$

NOTE: The SI unit of power is the *watt*, $W \equiv J/s$, *i.e.* one joule (J) per second (s), which is equal to $[\mathcal{P}_{int}] = \int_V Pa\frac{m}{m\,s}dV = \int_V \frac{N}{m^2}\frac{m}{m\,s}dV = \int_V \frac{J}{m^3\,s}dV = \frac{J}{s} = W$. ■

We can also define the stress power per unit volume, denoted by $\mathsf{w}_{int}(t)$, as:

$$\boxed{\mathsf{w}_{int}(t) = \boldsymbol{\sigma} : \mathbf{D} = \mathrm{Tr}(\boldsymbol{\sigma} \cdot \mathbf{D})} \quad \textit{Stress power per unit volume} \tag{5.94}$$

Then by starting from the stress power in the current configuration we can also express the stress power as a function of the other stress tensors, *i.e.*:

$$\int_V \boldsymbol{\sigma} : \mathbf{D}\, dV = \int_{V_0} \underbrace{J\boldsymbol{\sigma}}_{\boldsymbol{\tau}} : \mathbf{D}\, dV_0 = \int_{V_0} \boldsymbol{\tau} : \mathbf{D}\, dV_0 \tag{5.95}$$

Bearing in mind that $\tau_{ij} = \mathsf{P}_{ik}F_{jk}$ (Kirchhoff stress tensor components), $\mathbf{P} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \mathbf{P}^T$ (Cauchy's second law of motion in the reference configuration), and $\mathsf{D}_{ij} = F^{-1}_{pi}\dot{E}_{pl}F^{-1}_{lj}$, we obtain:

$$\begin{aligned}
\int_{V_0} \boldsymbol{\tau} : \mathbf{D}\, dV_0 &= \int_{V_0} \mathsf{P}_{ik}F_{jk}F^{-1}_{pi}\dot{E}_{pl}F^{-1}_{lj}\, dV_0 & &= \int_{V_0} \mathsf{P}_{ik}F_{jk}\mathsf{D}_{ij}\, dV_0 \\
&= \int_{V_0} \mathsf{P}_{ik}F^{-1}_{pi}\dot{E}_{pl}\underbrace{F^{-1}_{lj}F_{jk}}_{\delta_{lk}}\, dV_0 & &= \int_{V_0} \mathsf{P}_{ik}F_{jk}(\ell_{ij} - \mathsf{W}_{ij})\, dV_0 \\
&= \int_{V_0} \mathsf{P}_{ik}F^{-1}_{pi}\dot{E}_{pk}\, dV_0 = \int_{V_0} F^{-1}_{pi}\mathsf{P}_{ik}\dot{E}_{pk}\, dV_0 & &= \int_{V_0} \mathsf{P}_{ik}\ell_{ij}F_{jk}\, dV_0 \\
&= \int_{V_0} \mathsf{S}_{pk}\dot{E}_{pk}\, dV_0 & &= \int_{V_0} \mathsf{P}_{ik}\dot{F}_{ik}\, dV_0 = \int_{V_0} \mathbf{P} : \dot{\boldsymbol{F}}\, dV_0 \\
&= \int_{V_0} \mathbf{S} : \dot{\boldsymbol{E}}\, dV_0 = \frac{1}{2}\int_{V_0} \mathbf{S} : \dot{\boldsymbol{C}}\, dV_0 & &= \int_V \frac{1}{J}\mathbf{P} : \dot{\boldsymbol{F}}\, dV = \int_V \frac{\rho}{\rho_0}\mathbf{P} : \dot{\boldsymbol{F}}\, dV
\end{aligned} \tag{5.96}$$

which proves that the rate of change of the deformation gradient and the first Piola-Kirchhoff stress tensor are conjugate quantities ($\mathbf{P} : \dot{\boldsymbol{F}}$). Other conjugate quantities are: the second Piola-Kirchhoff stress tensor and the rate of change of the Green-Lagrange strain tensor ($\mathbf{S} : \dot{\boldsymbol{E}}$); the Kirchhoff stress tensor and the rate-of-deformation tensor ($\boldsymbol{\tau} : \mathbf{D}$). Furthermore, we can show that $\mathbf{T} : \dot{\mathbf{U}}$ is already a conjugate pair. To prove this, let us consider the relationship $\mathbf{P} = \mathbf{R} \cdot \mathbf{T}$, where $\mathbf{T} = \mathbf{U} \cdot \mathbf{S}$ is the Biot stress tensor, and $\mathbf{R}$ is the orthogonal tensor from the polar decomposition, and $\mathbf{U} = \mathbf{U}^T$ is the right stretch tensor. Then if we refer to the right polar decomposition, *i.e.* $\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U} \Rightarrow \dot{\boldsymbol{F}} = \dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}}$, we obtain:

$$\begin{aligned}
\mathbf{P} : \dot{\boldsymbol{F}} &= (\mathbf{R} \cdot \mathbf{T}) : (\dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}}) & \mathsf{P}_{ij}\dot{F}_{ij} &= (\mathsf{R}_{ip}\mathsf{T}_{pj})(\dot{\mathsf{R}}_{ik}\mathsf{U}_{kj} + \mathsf{R}_{ik}\dot{\mathsf{U}}_{kj}) \\
&= (\mathbf{R} \cdot \mathbf{T}) : (\dot{\mathbf{R}} \cdot \mathbf{U}) + (\mathbf{R} \cdot \mathbf{T}) : (\mathbf{R} \cdot \dot{\mathbf{U}}) & &= \mathsf{R}_{ip}\mathsf{T}_{pj}\dot{\mathsf{R}}_{ik}\dot{\mathsf{U}}_{kj} + \mathsf{R}_{ip}\mathsf{T}_{pj}\mathsf{R}_{ik}\dot{\mathsf{U}}_{kj} \\
&= (\mathbf{T} \cdot \mathbf{U}^T) : (\mathbf{R}^T \cdot \dot{\mathbf{R}}) + \mathbf{T} : \dot{\mathbf{U}} & &= (\mathsf{T}_{pj}\mathsf{U}_{kj})(\mathsf{R}_{ip}\dot{\mathsf{R}}_{ik}) + \mathsf{T}_{kj}\dot{\mathsf{U}}_{kj} \\
&= (\mathbf{U} \cdot \mathbf{S} \cdot \mathbf{U}^T) : (\mathbf{R}^T \cdot \dot{\mathbf{R}}) + \mathbf{T} : \dot{\mathbf{U}} & &= (\mathsf{U}_{pq}\mathsf{S}_{qj}\mathsf{U}_{kj})(\mathsf{R}_{ip}\dot{\mathsf{R}}_{ik}) + \mathsf{T}_{kj}\dot{\mathsf{U}}_{kj} \\
&= \mathbf{T} : \dot{\mathbf{U}} & &= \mathsf{T}_{kj}\dot{\mathsf{U}}_{kj}
\end{aligned} \tag{5.97}$$

Note that $(\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{U}):(\mathbf{R}^T\cdot\dot{\mathbf{R}})=0$, since the tensor $(\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{U}^T)^T=\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{U}^T=\mathbf{U}\cdot\mathbf{S}\cdot\mathbf{U}$ is symmetrical and $(\mathbf{R}^T\cdot\dot{\mathbf{R}})$ is an antisymmetric tensor. Thus, the equation in (5.97) becomes:

$$\mathbf{P}:\dot{F}=\mathbf{T}:\dot{\mathbf{U}}=\mathbf{T}:\dot{\mathbf{H}} \tag{5.98}$$

where $\mathbf{H}=\mathbf{U}-\mathbf{1}$ is the Biot strain tensor, (see Chapter 2) and if we know that $\dot{\mathbf{U}}$ is symmetrical, it is also possible to express the above relationship as:

$$\mathbf{P}:\dot{F}=(\mathbf{T}^{sym}+\mathbf{T}^{skew}):\dot{\mathbf{U}}=\mathbf{T}^{sym}:\dot{\mathbf{U}} \tag{5.99}$$

Then, if we take into account all the equations obtained before, we can summarize the stress power per unit volume by:

$$\boxed{\mathsf{w}_{int}=\boldsymbol{\sigma}:\mathbf{D}=\frac{\rho}{\rho_0}\mathbf{P}:\dot{F}=\frac{1}{J}\mathbf{P}:\dot{F}} \quad \textit{The stress power per unit current volume} \tag{5.100}$$

$$\boxed{\mathsf{w}_{int}=\boldsymbol{\tau}:\mathbf{D}=\mathbf{S}:\dot{E}=\mathbf{P}:\dot{F}=\frac{1}{2}\mathbf{S}:\dot{C}=\mathbf{T}:\dot{\mathbf{U}}=\mathbf{T}:\dot{\mathbf{H}}} \quad \textit{The stress power per unit reference volume} \tag{5.101}$$

5.9.3 The Balance of Mechanical Energy

If we compare the equation given in (5.91) with the energy equation (5.81), *i.e.*:

$$\frac{D}{Dt}\mathcal{K}+\mathcal{P}_{int}(t)=\mathcal{P}_{ext}(t) \quad \Leftrightarrow \quad \frac{D\mathcal{K}}{Dt}+\frac{DU}{Dt}=\frac{\text{Đ}\mathcal{W}}{Dt}+\underbrace{\frac{\text{Đ}Q}{Dt}}_{=0} \tag{5.102}$$

we can observe that the equation in (5.91) is an exceptional case of the energy equation where only mechanical energy is considered. In this case the principle of conservation of energy is known as the *balance of mechanical energy* which is otherwise known as the *theorem of power extended:*

$$\boxed{\begin{aligned}&\frac{D}{Dt}\int_V\frac{1}{2}\rho v^2 dV+\int_V\sigma_{ij}\mathrm{D}_{ij}dV=\int_V\rho \mathrm{b}_i v_i dV+\int_{S_{\boldsymbol{\sigma}}}v_i\mathrm{t}_i^* dS\\&\underbrace{\frac{D}{Dt}\int_V\frac{1}{2}\rho v^2 dV}_{\text{Rate of change of the Kinetic energy}}+\underbrace{\int_V\boldsymbol{\sigma}:\mathbf{D}dV}_{\text{Stress power}}=\underbrace{\int_V\rho\vec{\mathbf{b}}\cdot\vec{v}dV+\int_{S_{\boldsymbol{\sigma}}}\vec{\mathbf{t}}^*\cdot\vec{v}dS}_{\text{External mechanical power}}\end{aligned}} \quad \textit{Balance of mechanical energy} \tag{5.103}$$

OBS.: In rigid body motion $\mathbf{D}=\mathbf{0}$ is satisfied, so, the stress power (internal mechanical power) is zero $\mathcal{P}_{int}(t)=0$, then it holds that $\dot{\mathcal{K}}=\mathcal{P}_{ext}(t)$.

If $\dot{\mathcal{K}}$ is discarded, which characterizes a static or quasi-static regime, it holds that $\mathcal{P}_{int}(t)=\mathcal{P}_{ext}(t)$.

Problem 5.10: Find the kinetic energy related to rigid body motion in terms of the inertia tensor, (see **Problem 5.8** and **Problem 5.9**).

Solution: The rigid body motion velocity can be expressed as $\vec{v} = \dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{x} - \vec{\mathbf{c}})$. Then, the kinetic energy becomes:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho(\vec{v}\cdot\vec{v})dV = \frac{1}{2}\int_V \rho\left\{ \left[\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{x} - \vec{\mathbf{c}})\right]\cdot\left[\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{x} - \vec{\mathbf{c}})\right] \right\} dV$$

Using the following vector sum $\vec{x} = \vec{\bar{x}} + \vec{x}'$, where $\vec{\bar{x}}$ is the mass center vector position, and $\vec{x}'$ is the particle vector position with respect to the system that has its origin in the center of mass, the energy equation becomes:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho\left\{\left[\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge ((\vec{\bar{x}} + \vec{x}') - \vec{\mathbf{c}})\right]\cdot\left[\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge ((\vec{\bar{x}} + \vec{x}') - \vec{\mathbf{c}})\right]\right\}dV$$

$$= \frac{1}{2}\int_V \rho\left\{\left[\left(\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\bar{x}} - \vec{\mathbf{c}})\right) + (\vec{\boldsymbol{\omega}} \wedge \vec{x}')\right]\cdot\left[\left(\dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\bar{x}} - \vec{\mathbf{c}})\right) + (\vec{\boldsymbol{\omega}} \wedge \vec{x}')\right]\right\}dV$$

Note that $\vec{\bar{v}} = \dot{\vec{\mathbf{c}}} + \vec{\boldsymbol{\omega}} \wedge (\vec{\bar{x}} - \vec{\mathbf{c}})$ is the center of mass velocity, thus:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho\left\{\left[\vec{\bar{v}} + (\vec{\boldsymbol{\omega}} \wedge \vec{x}')\right]\cdot\left[\vec{\bar{v}} + (\vec{\boldsymbol{\omega}} \wedge \vec{x}')\right]\right\}dV$$

or:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho\vec{\bar{v}}\cdot\vec{\bar{v}}\,dV + \frac{1}{2}\int_V \rho\vec{\bar{v}}\cdot(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\,dV + \frac{1}{2}\int_V \rho(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\cdot\vec{\bar{v}}\,dV + +\frac{1}{2}\int_V \rho(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\cdot(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\,dV$$

Then by simplifying the above equation we obtain:

$$\mathcal{K}(t) = \frac{1}{2}\int_V \rho\vec{\bar{v}}\cdot\vec{\bar{v}}\,dV + \int_V \rho\vec{\bar{v}}\cdot(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\,dV + \frac{1}{2}\int_V \rho(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\cdot(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\,dV$$

Next, we discuss separately the terms of the previous equation:

1) $\frac{1}{2}\int_V \rho\vec{\bar{v}}\cdot\vec{\bar{v}}\,dV = \frac{1}{2}\left\|\vec{\bar{v}}\right\|^2\int_V \rho\,dV = \frac{1}{2}m\bar{v}^2$

2) $\int_V \rho\vec{\bar{v}}\cdot(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\,dV = \vec{\bar{v}}\cdot\left[\vec{\boldsymbol{\omega}} \wedge \int_V \rho\vec{x}'\,dV\right] = \vec{\bar{v}}\cdot(\vec{\boldsymbol{\omega}} \wedge m\underbrace{\vec{\bar{x}}'}_{=\vec{\mathbf{0}}}) = 0$

Note that, the system $\vec{x}'$ is located at the center of mass (G), hence the center of mass vector position related to the system $\vec{x}'$ is zero.

3) $\int_V \rho(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\cdot(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\,dV$

$$\int_V \rho\left[(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\cdot(\vec{\boldsymbol{\omega}} \wedge \vec{x}')\right]dV = \int_V \rho\,\epsilon_{ijk}\omega_j x'_k \epsilon_{ipq}\omega_p x'_q\,dV = \int_V \rho\,(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})\omega_j x'_k \omega_p x'_q\,dV$$

$$= \int_V \rho\,\omega_j(\delta_{jp}\delta_{kq}x'_k x'_q - \delta_{jq}\delta_{kp}x'_k x'_q)\omega_p\,dV$$

$$= \int_V \rho\,\omega_j(\delta_{jp}x'_k x'_k - x'_p x'_j)\omega_p\,dV$$

$$= \omega_j\left(\int_V \rho\,(\delta_{jp}x'_k x'_k - x'_j x'_p)\,dV\right)\omega_p$$

$$= \omega_j \bar{\mathrm{I}}_{jp}\omega_p$$

or in tensorial notation as:

$$\int_V \rho\left[(\vec{\omega}\wedge\vec{x}')\cdot(\vec{\omega}\wedge\vec{x}')\right]dV = \vec{\omega}\cdot\left[\int_V \rho\left[(\vec{x}'\cdot\vec{x}')\mathbf{1}-(\vec{x}'\otimes\vec{x}')\right]dV\right]\cdot\vec{\omega}$$
$$=\vec{\omega}\cdot\bar{\mathbf{I}}\cdot\vec{\omega}$$

where $\bar{\mathbf{I}}$ is the inertia pseudo-tensor related to the system located at the center of mass, (see **Problem 5.8**).

Then if we bear in mind all the above considerations, the kinetic energy equation for rigid body motion becomes:

$$\mathcal{K}(t)=\frac{1}{2}\int_V \rho\bar{\vec{v}}\cdot\bar{\vec{v}}\,dV+\underbrace{\frac{1}{2}\int_V 2\rho\bar{\vec{v}}\cdot(\vec{\omega}\wedge\vec{x}')\,dV}_{=0}+\frac{1}{2}\int_V \rho(\vec{\omega}\wedge\vec{x}')\cdot(\vec{\omega}\wedge\vec{x}')\,dV$$

$$\boxed{\mathcal{K}(t)=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\vec{\omega}\cdot\bar{\mathbf{I}}\cdot\vec{\omega}}$$

Additionally, if we take into account that:

$$\bar{\mathrm{I}}_{ij}=\begin{bmatrix}\int_V \rho\left[x_2'^2+x_3'^2\right]dV & -\int_V \rho\left[x_1'x_2'\right]dV & -\int_V \rho\left[x_1'x_3'\right]dV\\ -\int_V \rho\left[x_1'x_2'\right]dV & \int_V \rho\left[x_1'^2+x_3'^2\right]dV & -\int_V \rho\left[x_2'x_3'\right]dV\\ -\int_V \rho\left[x_1'x_3'\right]dV & -\int_V \rho\left[x_2'x_3'\right]dV & \int_V \rho\left[x_1'^2+x_2'^2\right]dV\end{bmatrix}=\begin{bmatrix}\bar{\mathcal{I}}_{11} & -\bar{\mathcal{I}}_{12} & -\bar{\mathcal{I}}_{13}\\ -\bar{\mathcal{I}}_{12} & \bar{\mathcal{I}}_{22} & -\bar{\mathcal{I}}_{23}\\ -\bar{\mathcal{I}}_{13} & -\bar{\mathcal{I}}_{23} & \bar{\mathcal{I}}_{33}\end{bmatrix}$$

we obtain an explicit equation for the kinetic energy as:

$$\mathcal{K}(t)=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\omega_k\bar{\mathrm{I}}_{kj}\omega_j$$
$$=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}\begin{bmatrix}\bar{\mathcal{I}}_{11} & -\bar{\mathcal{I}}_{12} & -\bar{\mathcal{I}}_{13}\\ -\bar{\mathcal{I}}_{12} & \bar{\mathcal{I}}_{22} & -\bar{\mathcal{I}}_{23}\\ -\bar{\mathcal{I}}_{13} & -\bar{\mathcal{I}}_{23} & \bar{\mathcal{I}}_{33}\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$$
$$=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\left[\bar{\mathcal{I}}_{11}\omega_1^2+\bar{\mathcal{I}}_{22}\omega_2^2+\bar{\mathcal{I}}_{33}\omega_3^2-2\bar{\mathcal{I}}_{12}\omega_1\omega_2-2\bar{\mathcal{I}}_{13}\omega_1\omega_3-2\bar{\mathcal{I}}_{23}\omega_2\omega_3\right]$$

$$\boxed{\mathcal{K}(t)=\frac{1}{2}m\bar{v}^2+\frac{1}{2}\left[\bar{\mathcal{I}}_{11}\omega_1^2+\bar{\mathcal{I}}_{22}\omega_2^2+\bar{\mathcal{I}}_{33}\omega_3^2-2\bar{\mathcal{I}}_{12}\omega_1\omega_2-2\bar{\mathcal{I}}_{13}\omega_1\omega_3-2\bar{\mathcal{I}}_{23}\omega_2\omega_3\right]}$$

5.9.4 The Internal Energy

If we take a handful of atoms (the material point) and we evaluate the average of all forms of energy present in it we obtain what is known as the *internal energy*. Continuum thermodynamics usually presents the rate of change of the internal energy as:

$$\frac{DU}{Dt}=\frac{D}{Dt}\int_V \rho u\,dV=\int_V \rho\dot{u}\,dV \qquad \left[\frac{J}{s}\right] \qquad (5.104)$$

where u is the *specific internal energy, i.e.* energy per unit mass, $[u]=\frac{J}{kg}$. For example, for an ideal gas the specific internal energy is given by $u=c_vT=\underbrace{c_p}_{\eta}T-\frac{p}{\rho}$, where T is the temperature, c_v is the specific heat capacity at a constant volume, η is the specific entropy,

c_p is the specific heat capacity at a constant pressure, p is the thermodynamic pressure, and ρ is the mass density. We can give another example with the mechanical problem, which was discussed in the previous subsection, where the rate of change of the internal energy is given by $\frac{DU}{Dt} = \int_V \boldsymbol{\sigma} : \mathbf{D}\, dV$.

5.9.5 Thermal Power

We define thermal power as the rate of increase of total heat in the continuum, which is denoted by $\frac{ÐQ}{Dt}$. The contribution of thermal power considered here is caused by:

- The Cauchy heat flux (non convective, *i.e.* without mass transport);
- The heat sources.

1) The Cauchy heat flux

Let us assume that there is a temperature gradient in the continuum, so there is scientific evidence of energy transfer (heat) from the hottest to the colder region. Then, we can represent this transferred energy per unit area per unit time by the thermal flux vector $\vec{\mathbf{q}}(\vec{x},t)$, which is also known as the *Cauchy heat flux* or *true heat flux*. Now, let us consider the domain $\mathcal{B}$ bounded by the surface S, (see Figure 5.8). The amount of energy which is transferred through the surface dS per unit time, (see Figure 5.8), is represented by $\vec{\mathbf{q}}(\vec{x},t)\cdot\hat{\mathbf{n}}dS$, where $\hat{\mathbf{n}}$ is the outward unit normal to the area element dS. Meanwhile, the tangential component remains on the surface. Thus, the rate of increase of total heat, due to thermal flux, in the continuum is given by:

$$\int_S -\vec{\mathbf{q}}(\vec{x},t)\cdot\hat{\mathbf{n}}\, dS \qquad \left[\frac{J}{s} = W\right] \tag{5.105}$$

2) The heat sources

If in a continuum there is a nuclear or chemical reaction which results in the release of heat, we can represent this by means of the heat sources, (see Figure 5.8).

We represent the rate of increase of total heat in the continuum cause by the heat source as:

$$\int_V \rho\, r\, dV \qquad \left[\frac{J}{s} = W\right] \tag{5.106}$$

where $r(\vec{x},t)$ is the *radiant heat constant* (also called the *heat source*) per unit mass per unit time, a scalar function, and the SI unit is $[r(\vec{x},t)] = \frac{J}{s\,kg}$, and $\rho(\vec{x},t)$ is the mass density.

Then by considering the heat flux (incoming) and the heat source, we can define the *thermal power* (the *rate of thermal work*) as:

$$\boxed{\frac{ÐQ}{Dt} = \int_V \rho\, r\, dV - \int_S \vec{\mathbf{q}}\cdot\hat{\mathbf{n}}\, dS} \quad \textit{The thermal power} \qquad \left[\frac{J}{s} = W\right] \tag{5.107}$$

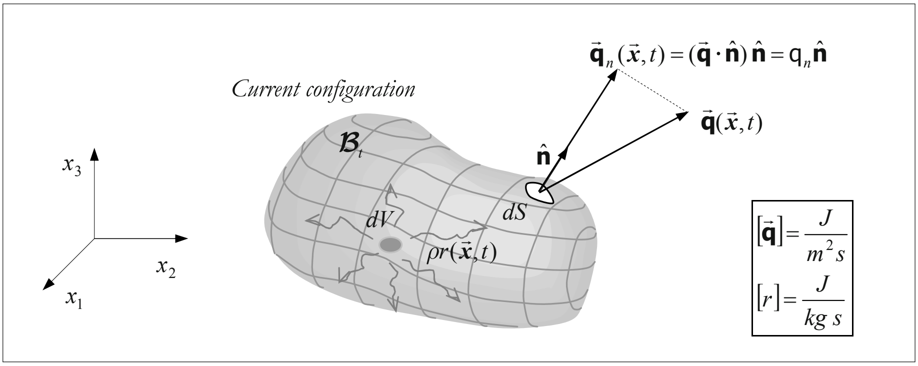


Figure 5.8: Heat flux and heat source.

5.9.6 The First Law of Thermodynamics. The Energy Equation

Once we know what forms of energy are involved in a system we can provide the energy equation by starting from that in (5.81):

$$\frac{D\mathcal{K}}{Dt} + \frac{DU}{Dt} = \frac{\text{Đ}\mathcal{W}}{Dt} + \frac{\text{Đ}Q}{Dt} \tag{5.108}$$

The mechanical power and the thermal power are not exact differentials ($\frac{\text{Đ}\bullet}{Dt}$), but there is experimental evidence showing that the sum of mechanical and thermal power is already an exact differential, (Mase(1977)).

Considering only the mechanical and thermal energy, the principle of conservation energy becomes what is known as the *first law of thermodynamics*, which postulates the interchangeability of mechanical and thermal energy. Then, the equation in (5.108) becomes:

$$\frac{D}{Dt}\int_V \rho \frac{v_i v_i}{2}\, dV + \int_V \rho\, \dot{u} dV = \int_{S_\sigma} \overbrace{\mathrm{t}_i^*}^{\sigma_{ij}\hat{\mathrm{n}}_j}\, v_i dS + \int_V \rho v_i \mathrm{b}_i dV + \int_V \rho\, r dV - \int_{S_\mathbf{q}} \mathrm{q}_i \hat{\mathrm{n}}_i dS \tag{5.109}$$

Then by using divergence theorem to transform the surface integral into the volume integral we obtain:

$$\begin{gathered}
\frac{D}{Dt}\int_V \rho \frac{v_i v_i}{2} dV + \int_V \rho\dot{u} dV = \int_V \left(\sigma_{ij} v_i\right)_{,j} dV + \int_V \rho v_i \mathrm{b}_i dV + \int_V \rho\, r dV - \int_V \mathrm{q}_{i,i}\, dV \\
\int_V \rho v_i \dot{v}_i dV + \int_V \rho\dot{u} dV = \int_V \left[\left(\sigma_{ij} v_i\right)_{,j} + \rho v_i \mathrm{b}_i + \rho\, r - \mathrm{q}_{i,i}\right] dV \\
\int_V \rho\dot{u} dV = \int_V \left(\sigma_{ij,j} v_i + \sigma_{ij} v_{i,j} + \rho \mathrm{b}_i v_i + \rho\, r - \mathrm{q}_{i,i} - \rho v_i \dot{v}_i\right) dV
\end{gathered} \tag{5.110}$$

Additionally, by rearranging the above equation we obtain:

$$\int_V \rho\dot{u} dV = \int_V \left[v_i \underbrace{\left(\sigma_{ij,j} + \rho \mathrm{b}_i - \rho\dot{v}_i\right)}_{\substack{=0_i \\ \text{the equations of motion}}} + \sigma_{ij} v_{i,j} + \rho r - \mathrm{q}_{i,i} \right] dV \tag{5.111}$$

Then if we bear in mind that $v_{i,j} = \mathsf{D}_{ij} + \mathsf{W}_{ij}$, the above equation becomes:

$$\int_V \rho\dot{u}\, dV = \int_V \left[\sigma_{ij}\left(\mathsf{D}_{ij} + \mathsf{W}_{ij}\right) + \rho r - \mathsf{q}_{i,i}\right] dV \;\Rightarrow\; \int_V \rho\dot{u}\, dV = \int_V \left(\sigma_{ij}\mathsf{D}_{ij} + \rho r - \mathsf{q}_{i,i}\right) dV \tag{5.112}$$

The local form of the above equation is known as the *energy equation*:

$$\rho\dot{u} = \sigma_{ij}\mathsf{D}_{ij} + \rho r - \mathsf{q}_{i,i} \qquad \left[\frac{J}{m^3 s} = \frac{W}{m^3}\right] \tag{5.113}$$

which is expressed in tensorial notation as:

$$\boxed{\rho\,\dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r} \quad \textit{The energy equation (current configuration)} \tag{5.114}$$

NOTE: For a purely mechanical problem in which there is no internal heat production ($r = 0$) nor heat flux $\vec{\mathbf{q}} = \vec{\mathbf{0}}$, the energy equation becomes:

$$\dot{u} = \frac{1}{\rho}\boldsymbol{\sigma} : \mathbf{D} \qquad \left[\frac{J}{s\,kg}\right] \tag{5.115}$$

where the SI unit can easily be verified $[\dot{u}] = \left[\frac{1}{\rho}\boldsymbol{\sigma} : \mathbf{D}\right] = \frac{m^3}{kg}\frac{N}{m^2}\frac{m}{m\,s} = \frac{N\,m}{s\,kg} = \frac{J}{s\,kg}$ ■

5.9.6.1 The Energy Equation in Lagrangian Description

The energy equation (5.114) can also be established in Lagrangian description (material description). From the equation in (5.112), the integral related to the integral energy can be written in the reference configuration as:

$$\int_V \rho(\vec{x},t)\dot{u}(\vec{x},t) dV = \int_{V_0} J\rho\dot{u}\, dV_0 = \int_{V_0} \rho_0(\vec{X})\dot{u}(\vec{X},t) dV_0 \tag{5.116}$$

The integral associated with stress power can be established in the reference and current configuration, (see equations (5.100) and (5.101)), as shown bellow:

$$\int_V \boldsymbol{\sigma} : \mathbf{D} dV = \int_{V_0} \underbrace{J\boldsymbol{\sigma}}_{\boldsymbol{\tau}} : \mathbf{D}\, dV_0 = \int_{V_0} \boldsymbol{\tau} : \mathbf{D} dV_0 = \int_{V_0} \mathbf{S} : \dot{\boldsymbol{E}} dV_0 = \int_{V_0} \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} dV_0 = \int_{V_0} \mathbf{P} : \dot{\boldsymbol{F}} dV_0 = \int_V \frac{\rho}{\rho_0}\mathbf{P} : \dot{\boldsymbol{F}}\, dV \tag{5.117}$$

Similarly for the integral related with the heat source, *i.e.*:

$$\underbrace{\int_V \rho(\vec{x},t)\, r(\vec{x},t)\, dV}_{current\ configuration} = \underbrace{\int_{V_0} J\rho\, r\, dV_0 = \int_{V_0} \rho_0(\vec{X})\, r(\vec{X},t)\, dV_0}_{reference\ configuration} \tag{5.118}$$

Finally, we can address the integral related to the heat flux. The amount of heat that passes through the area element da in the current configuration must in theory be the same as, that which passes through the area element dA in the reference configuration, (see Figure 5.9). Then the following relationship must be met:

$$\vec{\mathbf{q}}_0 \cdot d\vec{A} = \vec{\mathbf{q}} \cdot d\vec{a} \tag{5.119}$$

where $\vec{\mathbf{q}}_0$ is the heat flux in the reference configuration. Then if we use Nanson's formula $d\vec{a} = J\boldsymbol{F}^{-T} \cdot d\vec{A}$, obtained in Chapter 2, the equation in (5.119) becomes:

$$\vec{\mathbf{q}}_0 \cdot d\vec{A} = J\,\vec{\mathbf{q}} \cdot \boldsymbol{F}^{-T} \cdot d\vec{A} \;\Rightarrow\; \vec{\mathbf{q}}_0 = J\,\vec{\mathbf{q}} \cdot \boldsymbol{F}^{-T} \;\Rightarrow\; \vec{\mathbf{q}} = J^{-1}\vec{\mathbf{q}}_0 \cdot \boldsymbol{F}^{T} \tag{5.120}$$

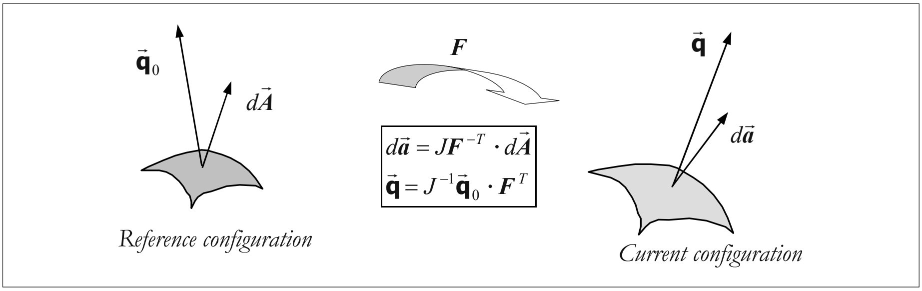


Figure 5.9: Heat flux.

Thus, the integral $\int_V \nabla_{\vec{x}} \cdot \vec{\mathbf{q}}\,dV$ can be written in the reference configuration as:

$$\begin{aligned}\int_V \nabla_{\vec{x}} \cdot \vec{\mathbf{q}}\,dV &= \int_V \mathrm{q}_{i,i}\,dV = \int_{V_0} J\frac{\partial \mathrm{q}_i}{\partial x_i}\,dV_0 = \int_{V_0} J\frac{\partial}{\partial x_i}\left(\frac{1}{J}\mathrm{q}_{0k}F_{ik}\right)dV_0 \\ &= \int_{V_0} J\frac{\partial \mathrm{q}_{0k}}{\partial x_i}\left(\frac{1}{J}F_{ik}\right) + J\,\mathrm{q}_{0k}\frac{\partial}{\partial x_i}\left(\frac{1}{J}F_{ik}\right)dV_0\end{aligned} \tag{5.121}$$

It was proven in Chapter 2 that $\nabla_{\vec{x}} \cdot \left(J^{-1}\boldsymbol{F}\right) = \vec{\mathbf{0}}$, thus, the above equation becomes:

$$\begin{aligned}\int_V \nabla_{\vec{x}} \cdot \vec{\mathbf{q}}\,dV &= \int_{V_0} J\frac{\partial \mathrm{q}_{0k}}{\partial x_i}\left(\frac{1}{J}F_{ik}\right)dV = \int_{V_0} \frac{\partial \mathrm{q}_{0k}}{\partial x_i}\left(\frac{\partial x_i}{\partial X_k}\right)dV_0 = \int_{V_0} \frac{\partial \mathrm{q}_{0k}}{\partial X_k}\,dV_0 \\ &= \int_{V_0} \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0\,dV_0\end{aligned} \tag{5.122}$$

Bearing in mind the equations in (5.116), (5.117), (5.118) and (5.122), the energy equation in the reference configuration can be established as:

$$\int_{V_0} \rho_0\,\dot{u}\,dV_0 = \int_{V_0} \left(\mathbf{S} : \dot{\boldsymbol{E}} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r\right)dV_0 \tag{5.123}$$

Additionally, the local form of the above equation is:

$$\boxed{\rho_0\dot{u}(\vec{X},t) = \mathbf{S} : \dot{\boldsymbol{E}} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r(\vec{X},t)} \quad \textit{The energy equation (reference configuration)} \tag{5.124}$$

5.9.7 The Energy Equation with Discontinuity

In this subsection we obtain the energy equation for a domain with a singular surface $\Sigma(t)$ as discussed in subsection 5.5.1, (see Figure 5.4). In this case the energy equation becomes:

$$\begin{gathered}\frac{D\mathcal{K}}{Dt} + \frac{DU}{Dt} = \frac{\text{Đ}\mathcal{W}}{Dt} + \frac{\text{Đ}Q}{Dt} \\ \frac{D}{Dt}\left[\frac{1}{2}\int_{V-\Sigma} \rho(\vec{v} \cdot \vec{v})dV + \int_{V-\Sigma} \rho u dV\right] = \int_{V-\Sigma} \rho\vec{\mathbf{b}} \cdot \vec{v}dV + \int_{S^+ + S^-} \vec{v} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}dS + \int_{V-\Sigma} \rho\,r dV - \int_{S^+ + S^-} \vec{\mathbf{q}} \cdot \hat{\mathbf{n}}\,dS\end{gathered} \tag{5.125}$$

For the terms on the left of the equation in (5.125) we can apply Reynolds' transport theorem, (see the equation in (5.22)), to $\Phi = \rho(\vec{v}\cdot\vec{v}) + \rho u$, thus:

$$\frac{D}{Dt}\left[\frac{1}{2}\int_{V-\Sigma}\rho(\vec{v}\cdot\vec{v}) + \rho u\,dV\right] =$$
$$= \int_{V-\Sigma}\left(\frac{D\left(\rho\frac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)}{Dt} + \left(\rho\frac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)\nabla_{\vec{x}}\cdot\vec{v}\right)dV + \int_{\Sigma}[\![\left(\rho\frac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)\otimes(\vec{v}-\vec{\omega})]\!]\cdot\hat{\mathbf{n}}\,dS \tag{5.126}$$

Then by mathematically manipulating the terms of the volume integral we can see that:

$$\begin{aligned}&\frac{D\left(\rho\frac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)}{Dt} + \left(\rho\frac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)\nabla_{\vec{x}}\cdot\vec{v} = \tfrac{1}{2}(\vec{v}\cdot\vec{v})\frac{D\rho}{Dt} + \rho\frac{D\left(\frac{1}{2}(\vec{v}\cdot\vec{v})\right)}{Dt} + \\ &+ u\frac{D\rho}{Dt} + \rho\frac{Du}{Dt} + \left(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)\nabla_{\vec{x}}\cdot\vec{v}\end{aligned} \tag{5.127}$$

Moreover, by reorganizing the above equation, we find that:

$$\begin{aligned}&\frac{D\left(\rho\frac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)}{Dt} + \left(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v}) + \rho \dot{u}\right)\nabla_{\vec{x}}\cdot\vec{v} = \rho\frac{D\left(\frac{1}{2}(\vec{v}\cdot\vec{v})\right)}{Dt} + \\ &+ \frac{D\rho}{Dt}\left(\tfrac{1}{2}(\vec{v}\cdot\vec{v}) + u\right) + \rho\frac{Du}{Dt} + \left(\tfrac{1}{2}(\vec{v}\cdot\vec{v}) + u\right)\rho\nabla_{\vec{x}}\cdot\vec{v} \\ &= \left(\tfrac{1}{2}(\vec{v}\cdot\vec{v}) + u\right)\underbrace{\left(\frac{D\rho}{Dt} + \rho\nabla_{\vec{x}}\cdot\vec{v}\right)}_{=0} + \rho\frac{Du}{Dt} + \rho\frac{D\left(\frac{1}{2}(\vec{v}\cdot\vec{v})\right)}{Dt}\end{aligned} \tag{5.128}$$

Thus,

$$\frac{D\left(\rho\frac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)}{Dt} + \left(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v}) + \rho\dot{u}\right)\nabla_{\vec{x}}\cdot\vec{v} = \rho\frac{Du}{Dt} + \rho\vec{v}\cdot\dot{\vec{v}} \tag{5.129}$$

Then if we return to the equation in (5.126), and if we refer to (5.129) we can conclude that:

$$\begin{aligned}&\frac{D}{Dt}\left[\frac{1}{2}\int_{V-\Sigma}\rho(\vec{v}\cdot\vec{v}) + \rho u\,dV\right] = \\ &\qquad = \int_{V-\Sigma}\rho\left(\frac{Du}{Dt} + \vec{v}\cdot\dot{\vec{v}}\right)dV + \int_{\Sigma}[\![\left(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v}) + \rho u\right)\otimes(\vec{v}-\vec{\omega})]\!]\cdot\hat{\mathbf{n}}\,dS\end{aligned} \tag{5.130}$$

For the surface integrals on the right side of the equation in (5.125) we can apply Gauss' theorem to a volume with discontinuity, (see equation (5.17)):

$$\int_{S^{+}+S^{-}}(\vec{v}\cdot\boldsymbol{\sigma} - \vec{\mathbf{q}})\cdot\hat{\mathbf{n}}\,dS = \int_{V^{+}+V^{-}}\nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma} - \vec{\mathbf{q}})\,dV + \int_{\Sigma}[\![\vec{v}\cdot\boldsymbol{\sigma} - \vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}\,dS \tag{5.131}$$

Then if we know that $\left(v_i\sigma_{ij}\right)_{,j} = \sigma_{ij,j}v_i + \sigma_{ij}v_{i,j} \Rightarrow \nabla_{\vec{x}}\cdot(\vec{v}\cdot\boldsymbol{\sigma}) = \vec{v}\cdot\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}:\nabla_{\vec{x}}\vec{v}$, and if we have observed the spatial velocity gradient has been broken down into a symmetric and an antisymmetric part, we obtain $\boldsymbol{\sigma}:\nabla_{\vec{x}}\vec{v} = \boldsymbol{\sigma}:\mathbf{D} + \boldsymbol{\sigma}:\mathbf{W} = \boldsymbol{\sigma}:\mathbf{D}$, so, we can conclude that:

$$\begin{aligned}\int_{S^{+}+S^{-}}(\vec{v}\cdot\boldsymbol{\sigma} - \vec{\mathbf{q}})\cdot\hat{n}\,dS &= \int_{V^{+}+V^{-}}(\vec{v}\cdot\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}:\nabla_{\vec{x}}\vec{v} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}})\,dV + \int_{\Sigma}[\![\vec{v}\cdot\boldsymbol{\sigma} - \vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}\,dS \\ &= \int_{V^{+}+V^{-}}(\vec{v}\cdot\nabla_{\vec{x}}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}:\mathbf{D} - \nabla_{\vec{x}}\cdot\vec{\mathbf{q}})\,dV + \int_{\Sigma}[\![\vec{v}\cdot\boldsymbol{\sigma} - \vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}\,dS\end{aligned} \tag{5.132}$$

Then by substituting the equations (5.130) and (5.132) into the energy expression in (5.125) we obtain:

$$\begin{aligned}&\int_{V-\Sigma}\rho\left(\frac{Du}{Dt}+\vec{v}\cdot\dot{\vec{v}}\right)dV+\int_{\Sigma}[\![(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v})+\rho u)\otimes(\vec{v}-\vec{\omega})]\!]\cdot\hat{\mathbf{n}}\,dS\\&=\int_{V-\Sigma}(\vec{v}\cdot\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}+\boldsymbol{\sigma}:\mathbf{D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}})dV+\int_{\Sigma}[\![\vec{v}\cdot\boldsymbol{\sigma}-\vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}\,dS+\int_V\rho\vec{\mathbf{b}}\cdot\vec{v}dV+\int_V\rho\, rdV\end{aligned}\tag{5.133}$$

or

$$\begin{aligned}&\int_{V-\Sigma}\rho\left(\frac{Du}{Dt}+\vec{v}\cdot\dot{\vec{v}}\right)+\left(-\vec{v}\cdot\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}-\boldsymbol{\sigma}:\mathbf{D}+\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}-\rho\vec{\mathbf{b}}\cdot\vec{v}-\rho\, r\right)dV\\&+\int_{\Sigma}[\![(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v})+\rho u)\otimes(\vec{v}-\vec{\omega})-\vec{v}\cdot\boldsymbol{\sigma}+\vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}\,dS=0\end{aligned}\tag{5.134}$$

$$\begin{aligned}&\int_{V-\Sigma}\rho\left(\frac{Du}{Dt}\right)+\left(-\boldsymbol{\sigma}:\mathbf{D}+\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}-\rho\, r\right)+\vec{v}\cdot\underbrace{\left(\rho\dot{\vec{v}}-\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}-\rho\vec{\mathbf{b}}\right)}_{=\vec{\mathbf{0}}}dV\\&+\int_{\Sigma}[\![(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v})+\rho u)\otimes(\vec{v}-\vec{\omega})-\vec{v}\cdot\boldsymbol{\sigma}+\vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}\,dS=0\end{aligned}\tag{5.135}$$

with which we can conclude that:

$$\int_{V-\Sigma}\rho\dot{u}-\boldsymbol{\sigma}:\mathbf{D}+\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}-\rho\, rdV+\int_{\Sigma}[\![(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v})+\rho u)\otimes(\vec{v}-\vec{\omega})-\vec{v}\cdot\boldsymbol{\sigma}+\vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}\,dS=0\tag{5.136}$$

which thereby results in the energy equation for volumes with discontinuity:

$$\boxed{\begin{aligned}&\rho\dot{u}=\boldsymbol{\sigma}:\mathbf{D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho\, r \qquad in\;\; V\\&[\![(\rho\tfrac{1}{2}(\vec{v}\cdot\vec{v})+\rho u)\otimes(\vec{v}-\vec{\omega})-\vec{v}\cdot\boldsymbol{\sigma}+\vec{\mathbf{q}}]\!]\cdot\hat{\mathbf{n}}=0 \quad on \;\;\Sigma\end{aligned}}\tag{5.137}$$

The energy equation with discontinuity

5.10 The Principle of Irreversibility. Entropy Inequality

5.10.1 The Second Law of Thermodynamics

Before applying the second law of thermodynamics, we define entropy which is a state function. In thermodynamics, entropy is the physical quantity that measures the energy that can not be used to produce work. In a broader sense, entropy is interpreted as the measurement of system disorder. The entropy unit is J/K, joules per Kelvin and a process characterized by constant entropy is called the *isentropic process.*

The second law of thermodynamics imposes restrictions on the possible direction of the thermodynamics process. For example, the first law of thermodynamics does not establish the direction of the heat flux.

The second law of thermodynamics states that "*the rate of change of the total entropy H is never less than the sum of the entropy flow $\vec{s}$ that enters through the surface of the continuum plus the entropy created inside the continuum $\boldsymbol{B}$*".

The total entropy of the system (H) is given by:

$$H(t)=\int_V \rho\,\eta(\vec{x},t)\,dV=\int_{V_0}\rho_0\,\eta(\vec{X},t)\,dV_0 \qquad \left[\frac{J}{K}\right] \tag{5.138}$$

where $\eta(\vec{x},t)$ is the *specific entropy* (per unit mass), $[\eta]=\dfrac{J}{kgK}$.

The entropy supplied to the system ($\boldsymbol{B}$) is given by:

$$\boldsymbol{B}=\int_V \rho\boldsymbol{b}(\vec{x},t)dV=\int_{V_0}\rho_0\,\boldsymbol{b}(\vec{X},t)\,dV_0 \qquad \left[\frac{J}{sK}\right] \tag{5.139}$$

where $\boldsymbol{b}$ is the source of local entropy per unit mass per unit time $[\boldsymbol{b}]=\dfrac{J}{kg\,s\,K}$.

Then the entropy flux that enters the system through the material surface is defined by:

$$-\int_{S_{\vec{s}}}\vec{s}\cdot\hat{\mathbf{n}}\,dS \qquad \left[\frac{J}{sK}\right] \tag{5.140}$$

Thus, we can set the entropy inequality as:

$$\begin{aligned}\Gamma(t)&=\frac{D}{Dt}\int_V \rho\,\eta(\vec{x},t)\,dV\geq\int_V \rho\boldsymbol{b}(\vec{x},t)dV-\int_{S_{\vec{s}}}\vec{s}\cdot\hat{\mathbf{n}}\,dS\\&=\int_V \rho\,\dot{\eta}(\vec{x},t)\,dV\geq\int_V \rho\boldsymbol{b}(\vec{x},t)dV-\int_{S_{\vec{s}}}\vec{s}\cdot\hat{\mathbf{n}}\,dS\end{aligned} \tag{5.141}$$

Then by applying the divergence theorem to the surface integral, we obtain:

$$\boxed{\Gamma(t)=\int_V \rho\,\dot{\eta}\,dV\geq\int_V \rho\boldsymbol{b}dV-\int_V \nabla_{\vec{x}}\cdot\vec{s}\,dV} \quad \textit{The second law of thermodynamics (Entropy inequality)} \tag{5.142}$$

NOTE: The global form of the entropy inequality in (5.142) implies that: if entropy occurs then the process is irreversible, that is, we can not return to the original system without adding work to the system. And, the equality of (5.142) represents a reversible process. ■

The local form of the equation in (5.142) is given by:

$$\rho\dot{\eta}(\vec{x},t)\geq\rho\boldsymbol{b}-\nabla_{\vec{x}}\cdot\vec{s} \tag{5.143}$$

and if we consider that:

$$\vec{s}=\frac{\vec{\mathbf{q}}}{T}+\vec{s}^{(1)} \quad ; \quad \boldsymbol{b}=\frac{r}{T}+\boldsymbol{b}^{(1)} \tag{5.144}$$

where $T(\vec{x},t)\geq 0$ is the absolute temperature, $[T]=K$, and by assuming that $\vec{s}^{(1)}$ and $\boldsymbol{b}^{(1)}$ are equal to zero, the entropy inequality in (5.143) becomes:

$$\rho\dot{\eta}\geq\rho\frac{r}{T}-\nabla_{\vec{x}}\cdot\left(\frac{\vec{\mathbf{q}}}{T}\right)=\rho\frac{r}{T}-\frac{1}{T}\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\frac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T \tag{5.145}$$

Thus,

$$\boxed{\begin{aligned}&\rho\dot{\eta}(\vec{x},t)-\rho\frac{r(\vec{x},t)}{T}+\nabla_{\vec{x}}\cdot\left(\frac{\vec{\mathbf{q}}}{T}\right)\geq 0\\&\rho\dot{\eta}(\vec{x},t)-\rho\frac{r(\vec{x},t)}{T}+\frac{1}{T}\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}-\frac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\geq 0\end{aligned}}\quad \textit{Entropy inequality (current configuration)} \tag{5.146}$$

We can also express the entropy inequality given in (5.143) in the reference configuration as:

$$\rho_0\dot{\eta}(\vec{X},t)\geq\rho_0 \boldsymbol{b}(\vec{X},t)-\nabla_{\vec{X}}\cdot\vec{\boldsymbol{S}}(\vec{X},t) \tag{5.147}$$

where $\vec{\boldsymbol{S}}$ is the entropy flux vector in Lagrangian description. For thermal processes, the entropy flux vector and entropy source can be established, respectively, as:

$$\vec{\boldsymbol{S}}(\vec{X},t)=\frac{\vec{\mathbf{q}}_0}{T}+\vec{\boldsymbol{S}}_1 \quad ; \quad \boldsymbol{b}(\vec{X},t)=\frac{r(\vec{X},t)}{T}+\boldsymbol{b}_1 \tag{5.148}$$

Then if we take into account the equation in (5.148) where $\vec{\boldsymbol{S}}_1$ and $\boldsymbol{b}_1$ are equal to zero, the equation in (5.147) becomes:

$$\boxed{\begin{aligned}&\rho_0\dot{\eta}(\vec{X},t)\geq\rho_0\frac{r(\vec{X},t)}{T}-\nabla_{\vec{X}}\cdot\left(\frac{\vec{\mathbf{q}}_0}{T}\right)\\&\rho_0\dot{\eta}(\vec{X},t)\geq\rho_0\frac{r(\vec{X},t)}{T}-\frac{1}{T}\nabla_{\vec{X}}\cdot\vec{\mathbf{q}}_0+\frac{1}{T^2}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\end{aligned}}\quad \textit{Entropy inequality (reference configuration)} \tag{5.149}$$

Then if we refer to Eq. (5.120), where we obtained $\vec{\mathbf{q}}_0=J\,\vec{\mathbf{q}}\cdot\boldsymbol{F}^{-T}$, or in indicial notation $\vec{\mathrm{q}}_{0_i}=J\;\mathrm{q}_k F^{-1}_{ik}$, it is true that:

$$\underbrace{\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T=\mathrm{q}_{0_i}\frac{\partial T}{\partial X_i}}_{Material}=\underbrace{J\;\mathrm{q}_kF^{-1}_{ik}\frac{\partial T}{\partial x_p}\frac{\partial x_p}{\partial X_i}=J\;\mathrm{q}_kF^{-1}_{ik}\frac{\partial T}{\partial x_p}F_{pi}=J\;\mathrm{q}_k\delta_{pk}\frac{\partial T}{\partial x_p}=J\;\mathrm{q}_k\frac{\partial T}{\partial x_k}}_{Spatial}$$

Then, we can prove the following relationship is valid:

$$\vec{\mathbf{q}}_0(\vec{X},t)\cdot\nabla_{\vec{X}}T(\vec{X},t)=J\;\vec{\mathbf{q}}(\vec{x},t)\cdot\nabla_{\vec{x}}T(\vec{x},t) \tag{5.150}$$

5.10.2 The Clausius-Duhem Inequality

If we combine the entropy inequality in (5.146) with the energy equation given in (5.114), $\rho\,\dot{u}=\boldsymbol{\sigma}:\mathbf{D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r\Rightarrow\rho\,\dot{u}+\boldsymbol{\sigma}:\mathbf{D}=\rho r-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}$, we obtain:

$$\begin{aligned}&\rho\dot{\eta}-\rho\frac{r}{T}+\frac{1}{T}\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}-\frac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T=\rho\dot{\eta}-\frac{1}{T}\left(\rho\,r-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}\right)-\frac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\geq 0\\&\Rightarrow\rho\dot{\eta}-\frac{1}{T}\left(\rho\,\dot{u}-\boldsymbol{\sigma}:\mathbf{D}\right)-\frac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\geq 0\end{aligned} \tag{5.151}$$

In this scenario, the entropy inequality is called the Clausius-Duhem inequality, and is given by:

$$\boxed{\rho\,\dot{\eta}(\vec{x},t)+\frac{1}{T}\boldsymbol{\sigma}:\mathbf{D}-\frac{1}{T}\rho\,\dot{u}-\frac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\geq 0}\quad \textit{The Clausius-Duhem inequality (current configuration)} \tag{5.152}$$

We can also express the Clausius-Duhem inequality in the reference configuration. From the equation in (5.124) we obtained $\rho_0 r(\vec{X},t) - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 = \rho_0 \dot{u} - \mathbf{S} : \dot{\boldsymbol{E}}$ and by substituting this into the entropy inequality given in (5.149) we obtain:

$$\begin{aligned} \rho_0 \dot{\eta}(\vec{X},t) &\geq \frac{1}{T}\left(\rho_0 r(\vec{X},t) - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0\right) + \frac{1}{T^2}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \\ \rho_0 \dot{\eta}(\vec{X},t) &\geq \frac{1}{T}\left(\rho_0 \dot{u} - \mathbf{S} : \dot{\boldsymbol{E}}\right) + \frac{1}{T^2}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \end{aligned} \tag{5.153}$$

or:

$$\boxed{\begin{aligned} &\rho_0 \dot{\eta}(\vec{X},t) + \frac{1}{T}\mathbf{S} : \dot{\boldsymbol{E}} - \frac{1}{T}\rho_0 \dot{u} - \frac{1}{T^2}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \\ &or \\ &\rho_0 \dot{\eta}(\vec{X},t) + \frac{1}{T}\mathbf{P} : \dot{\boldsymbol{F}} - \frac{1}{T}\rho_0 \dot{u} - \frac{1}{T^2}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \end{aligned}} \quad \textit{The Clausius-Duhem inequality (reference configuration)} \tag{5.154}$$

5.10.3 The Clausius-Planck Inequality

Note that the inequality $\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \leq 0$ is always valid, since the orientation of the heat flux vector ($\vec{\mathbf{q}}$) is always opposite to the temperature gradient ($\nabla_{\vec{x}} T$), (see Figure 5.10). Then, we can formulate the heat conduction inequality:

$$\textit{Heat conduction inequality} \quad \begin{aligned} &\boxed{-\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \geq 0} \quad \textit{(current configuration)} \\ &\boxed{-\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0} \quad \textit{(reference configuration)} \end{aligned} \tag{5.155}$$

If we now incorporate the restrictions in (5.155) into the Clausius-Duhem inequality (5.152) and in (5.154) we will have a less restrictive inequality known as the Clausius-Planck inequality:

$$\textit{Clausius-Planck inequality} \quad \begin{aligned} &\boxed{\mathcal{D}_{int} = \rho \dot{\eta}(\vec{x},t) + \frac{1}{T}\boldsymbol{\sigma} : \mathbf{D} - \frac{1}{T}\rho\, \dot{u}(\vec{x},t) \geq 0} \quad \textit{(current configuration)} \\ &\boxed{\mathcal{D}_{int} = \rho_0 \dot{\eta}(\vec{X},t) + \frac{1}{T}\mathbf{P} : \dot{\boldsymbol{F}} - \frac{1}{T}\rho_0 \dot{u}(\vec{X},t) \geq 0} \quad \textit{(reference configuration)} \end{aligned} \tag{5.156}$$

where $\mathcal{D}_{int}$ is the internal energy dissipation, which requires positiveness at any time, $\mathcal{D}_{int} \geq 0$.

5.10.4 The Alternative Form to Express the Clausius-Duhem Inequality

An alternative form of entropy inequality is that expressed in terms of the *Helmholtz free energy*, ψ, which is a *thermodynamic potential* per unit mass and is given in Eulerian description by:

$$\boxed{\psi = u - T\eta} \quad \textit{The Helmholtz free energy} \quad \left[\frac{J}{kg}\right] \tag{5.157}$$

NOTE: A thermodynamic potential indicates the amount of energy available in the system. In this chapter we only work with the potentials $u = u(\boldsymbol{E}, \eta)$ and $\psi(\boldsymbol{E}, T)$. We can also use another potential, *e.g.* *Gibbs free energy* ($\mathsf{G}(\mathbf{S}, T)$), or *Enthalpy* ($\mathsf{H}(\mathbf{S}, \eta)$), (see Chapter 10). The choice to adopt one or the other depends on the independent variables under consideration, ($\boldsymbol{E}$ -"volume", η -entropy, $\mathbf{S}$ -"pressure", T -temperature). For more details about these potentials see the chapter on Thermoelasticity. ■

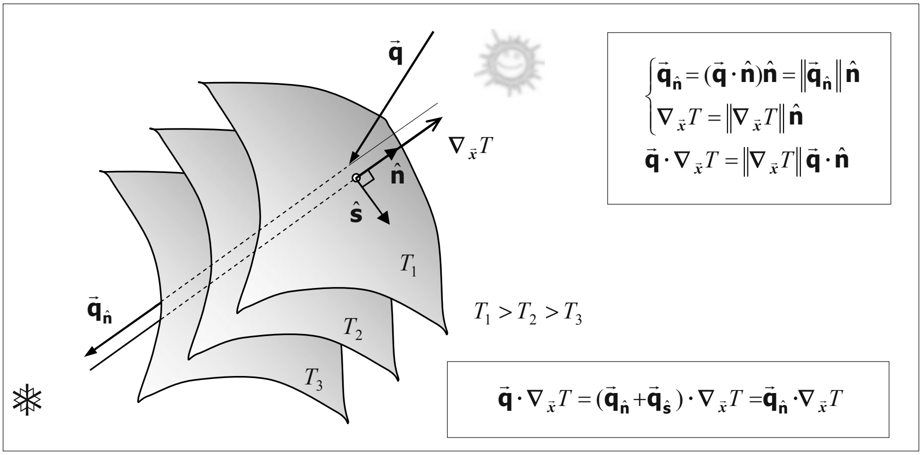


Figure 5.10: Temperature gradient and heat flux vector.

If we calculate the rate of change of the Helmholtz free energy, we obtain:

$$\dot{\psi} = \dot{u} - \eta \dot{T} - T\dot{\eta} \quad \Rightarrow \quad T\dot{\eta} = \dot{u} - \eta\dot{T} - \dot{\psi} \quad \Rightarrow \quad \begin{aligned} T\rho\dot{\eta} &= \rho\dot{u} - \rho\eta\dot{T} - \rho\dot{\psi} \\ &= \rho\dot{u} - \rho\left[\eta\dot{T} + \dot{\psi}\right] \end{aligned} \tag{5.158}$$

Then if we consider that $T > 0$ (absolute temperature) and the entropy inequality given in (5.145), we obtain:

$$\rho\dot{\eta} \geq \rho\frac{r}{T} - \frac{1}{T}\nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \frac{1}{T^2}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \quad \Rightarrow \quad \rho T\dot{\eta} \geq \rho\, r - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \frac{1}{T}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \tag{5.159}$$

Afterwards by combining the above inequality with the equation in (5.158) we obtain:

$$\rho\dot{u} - \rho\left[\eta\dot{T} + \dot{\psi}\right] \geq \rho\, r - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \frac{1}{T}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \tag{5.160}$$

Then by also considering the energy equation in (5.114), *i.e.* $\rho\,\dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r$, we obtain:

$$\begin{aligned} &\boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho\, r - \rho\left[\eta\dot{T} + \dot{\psi}\right] \geq \rho\, r - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \frac{1}{T}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \\ &\Rightarrow \boldsymbol{\sigma} : \mathbf{D} - \rho\left[\eta\dot{T} + \dot{\psi}\right] \geq \frac{1}{T}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \end{aligned} \tag{5.161}$$

by which we obtain the Clausius-Duhem inequality (current configuration) in terms of the Helmholtz free energy:

$$\boxed{\boldsymbol{\sigma} : \mathbf{D} - \rho\left[\eta\dot{T} + \dot{\psi}\right] - \frac{1}{T}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \geq 0} \quad \textit{Clausius-Duhem inequality (current configuration)} \tag{5.162}$$

The Clausius-Duhem inequality in the reference configuration, (see Eq. (5.154)),

$$\rho_0 T\dot{\eta} + \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0 \dot{u} - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \quad \Rightarrow \quad \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0 [\dot{u} - T\dot{\eta}] - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{5.163}$$

can also be written in terms of the Helmholtz free energy ψ. To do this let us consider the Helmholtz free energy in Lagrangian description $\psi = u(\vec{X},t) - T(\vec{X},t)\eta(\vec{X},t)$. Additionally, the rate of change is given by $\dot{\psi} = \dot{u} - \dot{T}\eta - T\dot{\eta} \Rightarrow \dot{u} - T\dot{\eta} = \dot{\psi} + \dot{T}\eta$ with that the Clausius-Duhem inequality in the reference configuration becomes:

$$\boxed{\begin{aligned} &\mathbf{S} : \dot{\boldsymbol{E}} - \rho_0 \left[\dot{\psi} + \dot{T}\eta\right] - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \\ &\mathbf{P} : \dot{\boldsymbol{F}} - \rho_0 \left[\dot{\psi} + \dot{T}\eta\right] - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \end{aligned}} \quad \textit{Clausius-Duhem inequality (reference configuration)} \tag{5.164}$$

The Helmholtz free energy *per unit reference volume* is denoted by $\Psi = \rho_0(\vec{X})\psi$, and it holds that $\dot{\Psi} = \rho_0 \dot{\psi}$. Proof of this can be shown by:

$$\dot{\Psi} \equiv \frac{D\Psi}{Dt} = \frac{D(\rho_0 \psi)}{Dt} = \psi \underbrace{\frac{D(\rho_0)}{Dt}}_{=0} + \rho_0 \frac{D(\psi)}{Dt} = \rho_0 \dot{\psi}.$$

5.10.5 The Alternative Form of the Clausius-Planck Inequality

The Clausius-Planck inequality can also be expressed in terms of Helmholtz free energy. Then, if we consider the heat conduction inequality, $-\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \geq 0$, the equation in (5.162) becomes:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \mathbf{D} - \rho\left[\eta\dot{T} + \dot{\psi}\right] \geq 0 \tag{5.165}$$

which in the reference configuration is given by:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0\left[\eta\dot{T} + \dot{\psi}\right] \geq 0 \tag{5.166}$$

5.10.6 Reversible Process

A thermodynamic process is said to be reversible if there is no dissipation of energy, *i.e.* $\Gamma(t) = 0$, (see equation (5.142)). A reversible process is characterized by:

- The work done by the forces between two points being independent of the path;
- The work done in a closed cycle being zero.

If we take into account that the dissipation of energy is equal to zero in a reversible process we obtain:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \mathbf{D} - \rho\dot{\psi} = 0 \quad \Rightarrow \quad \boxed{\rho\dot{\psi} = \boldsymbol{\sigma} : \mathbf{D}} \tag{5.167}$$

Then the equation in (5.167) in the reference configuration becomes:

$$\begin{aligned} \mathcal{D}_{int} &= \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0\dot{\psi} = 0 \\ &= \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} - \rho_0\dot{\psi} = 0 \end{aligned} \quad \Rightarrow \quad \boxed{\rho_0\dot{\psi} = \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} = \mathbf{S} : \dot{\boldsymbol{E}}} \tag{5.168}$$

5.10.7 Entropy Inequality for a Domain with Discontinuity

If we applying the entropy inequality in (5.141) for a volume with discontinuity we obtain:

$$\Gamma(t)=\frac{D}{Dt}\int_{V^{+}+V^{-}}\rho\,\eta(\vec{x},t)\,dV\geq\int_{V^{+}+V^{-}}\rho\boldsymbol{b}(\vec{x},t)dV+\int_{S^{+}+S^{-}}\vec{s}\cdot\hat{\mathbf{n}}\,dS \tag{5.169}$$

For the surface integral on the right side of the inequality in (5.169) we can apply the divergence theorem with discontinuity given in Eq. (5.17), the result of which is:

$$\int_{S^{+}+S^{-}}\vec{s}\cdot\hat{\mathbf{n}}\,dS=\int_{V^{+}+V^{-}}\nabla\cdot\vec{s}\,dV+\int_{\Sigma}[[\vec{s}]]\cdot\hat{\mathbf{n}}\,dS \tag{5.170}$$

For the volume integral on the left side of the inequality in (5.169) we can apply the Reynolds' transport theorem given in (5.22) in which $\Phi=\rho\,\eta$, then:

$$\frac{D}{Dt}\left[\int_{V-\Sigma}\rho\,\eta\,dV\right]=\int_{V-\Sigma}\left(\frac{D(\rho\,\eta)}{Dt}+(\rho\,\eta)\nabla_{\vec{x}}\cdot\vec{v}\right)dV+\int_{\Sigma}[[\rho\,\eta\otimes(\vec{v}-\vec{\omega})]]\cdot\hat{\mathbf{n}}\,dS \tag{5.171}$$

Then by substituting the equations in (5.171) and (5.170) into (5.169), we obtain:

$$\begin{aligned}\int_{V-\Sigma}\left(\frac{D(\rho\,\eta)}{Dt}+(\rho\,\eta)\nabla\cdot\vec{v}\right)dV+\int_{\Sigma}[[\rho\,\eta\otimes(\vec{v}-\vec{\omega})]]\cdot\hat{\mathbf{n}}\,dS\geq&\int_{V^{+}+V^{-}}\rho\boldsymbol{b}(\vec{x},t)dV+\\&+\int_{V-\Sigma}\nabla_{\vec{x}}\cdot\vec{s}\,dV+\int_{\Sigma}[[\vec{s}]]\cdot\hat{\mathbf{n}}\,dS\end{aligned} \tag{5.172}$$

Additionally by regrouping the integrands we obtain:

$$\int_{V-\Sigma}\left(\rho\frac{D\eta}{Dt}+\eta\frac{D\rho}{Dt}+(\rho\,\eta)\nabla_{\vec{x}}\cdot\vec{v}-\nabla_{\vec{x}}\cdot\vec{s}-\rho\boldsymbol{b}\right)dV+\int_{\Sigma}[[\rho\,\eta\otimes(\vec{v}-\vec{\omega})-\vec{s}]]\cdot\hat{\mathbf{n}}\,dS\geq 0 \tag{5.173}$$

Note that the equation $\eta\frac{D\rho}{Dt}+(\rho\,\eta)\nabla_{\vec{x}}\cdot\vec{v}=\eta\left(\frac{D\rho}{Dt}+\rho\nabla_{\vec{x}}\cdot\vec{v}\right)=0$ is valid due to the mass continuity equation. Then, the equation in (5.173) becomes:

$$\int_{V-\Sigma}\left(\rho\frac{D\eta}{Dt}-\nabla_{\vec{x}}\cdot\vec{s}-\rho\boldsymbol{b}\right)dV+\int_{\Sigma}[[\rho\,\eta\otimes(\vec{v}-\vec{\omega})-\vec{s}]]\cdot\hat{\mathbf{n}}\,dS\geq 0 \tag{5.174}$$

The local form of the above equation is expressed as:

$$\boxed{\begin{array}{ll}\rho\dfrac{D\eta}{Dt}\geq\nabla_{\vec{x}}\cdot\vec{s}+\rho\boldsymbol{b} & in\quad V-\Sigma\\ [[\rho\,\eta\otimes(\vec{v}-\vec{\omega})-\vec{s}]]\cdot\hat{\mathbf{n}}\geq 0 & on\quad\Sigma\end{array}}\quad \textit{Entropy inequality with discontinuity} \tag{5.175}$$

Problem 5.11: 1) Consider a continuum motion in which the stress power is equal to zero. Also, consider that the heat flux is given by $\vec{\mathbf{q}}=-\mathbf{K}(T)\cdot\nabla_{\vec{x}}T$, which is known as *Fourier's law of thermal conduction*, where $\mathbf{K}(T)$ is a second-order tensor called the *thermal conductivity tensor* (the thermal property of the material), and $c=\frac{\partial u(T)}{\partial T}$, where c is the *specific heat capacity* at a constant deformation (the thermal property of the material) and is expressed in

units of joule per kelvin, *i.e.* $[c]=\frac{J}{K}$. Taking into account all previous considerations, find the energy equation for this process. Then also provide the unit of $\mathbf{K}(T)$ in the International System of Units (SI).
2) Consider the stress power is equal to zero, and that there is a continuous medium with no internal heat source. Also consider that there is a heterogeneous material where $\mathbf{K}=\mathbf{K}(\vec{x})$ is an arbitrary second-order tensor (not necessarily symmetrical). a) Show that the thermal conductivity tensor is semi-definite positive, b) Check in which scenario the skew part of $\mathbf{K}(\vec{x})$ does not affect the outcome of the heat conduction problem. c) Taking into account that the material is isotropic, in what format is $\mathbf{K}$?
Solution: For this problem we know that the stress power is equal to zero, $\boldsymbol{\sigma}:\mathbf{D}=0$. It then follows that, the energy equation becomes:

$$\rho\,\dot{u}=\rho\frac{\partial u}{\partial T}\frac{\partial T}{\partial t}=\underbrace{\boldsymbol{\sigma}:\mathbf{D}}_{=0}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r=-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r$$

$$\Rightarrow \rho c\frac{\partial T}{\partial t}=-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r \qquad \Rightarrow \qquad \rho c\frac{\partial T}{\partial t}=-\nabla_{\vec{x}}\cdot\left[-\mathbf{K}(T)\cdot\nabla_{\vec{x}}T\right]+\rho r$$

or

$$\boxed{\nabla_{\vec{x}}\cdot\left[\mathbf{K}(T)\cdot\nabla_{\vec{x}}T\right]+\rho r=\rho c\frac{\partial T}{\partial t}}$$

The above equation is called the *heat flux equation* which is applied to the thermal conduction problem.

Then if we take into account the following units: $[\vec{\mathbf{q}}]=\frac{J}{m^2 s}=\frac{W}{m^2}$, $\left[\nabla_{\vec{x}}T\equiv\frac{\partial T}{\partial\vec{x}}\right]=\frac{K}{m}$, we can ensure that the units are consistent if the following is met:

$$\begin{array}{ccc}[\vec{\mathbf{q}}] & = & [\mathbf{K}]\cdot[\nabla_{\vec{x}}T] \\ \left[\frac{J}{m^2 s}=\frac{W}{m^2}\right] & = & \left[\frac{J}{s\,m\,K}=\frac{W}{m\,K}\right]\left[\frac{K}{m}\right]\end{array}$$

thus, we can draw the conclusion that $[\mathbf{K}]=\left[\frac{J}{s\,m\,K}=\frac{W}{m\,K}\right]$.

NOTE: As we will see later, when the stress power is equal to zero, we can decouple the thermal and mechanical problem. That is, we can study these problems separately. ■
2) a) We start from the heat conductivity inequality:

$$\begin{aligned}&-\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T=-(-\mathbf{K}(\vec{x})\cdot\nabla_{\vec{x}}T)\cdot\nabla_{\vec{x}}T\geq 0\\ &\nabla_{\vec{x}}T\cdot\mathbf{K}(\vec{x})\cdot\nabla_{\vec{x}}T\geq 0\end{aligned} \qquad \text{or} \qquad \begin{aligned}&-\mathsf{q}_i T_{,i}=-(-K_{ij}T_{,j})T_{,i}\geq 0\\ &T_{,i}K_{ij}T_{,j}\geq 0\end{aligned}$$

Remember that the arbitrary tensor $\mathbf{A}$ is semi-definite positive if it holds that $\vec{x}\cdot\mathbf{A}\cdot\vec{x}\geq 0$ for all $\vec{x}\neq\vec{\mathbf{0}}$ thereby demonstrating that $\mathbf{K}(\vec{x})$ is a semi-definite positive tensor. Then, as a result the eigenvalues of $\mathbf{K}(\vec{x})$ are all real values greater than or equal to zero, *i.e.* $\mathsf{K}_1\geq 0$, $\mathsf{K}_2\geq 0$, $\mathsf{K}_3\geq 0$. Also remember that since $\mathbf{K}(\vec{x})$ is not symmetric, the principal space of $\mathbf{K}(\vec{x})$ does not define an orthonormal basis. Moreover, it is noteworthy that: the antisymmetric part of $\mathbf{K}(\vec{x})$ does not affect the heat conduction inequality since:

$$\begin{aligned}\nabla_{\vec{x}}T\cdot\mathbf{K}(\vec{x})\cdot\nabla_{\vec{x}}T=\nabla_{\vec{x}}T\cdot\left[\mathbf{K}^{sym}+\mathbf{K}^{skew}\right]\cdot\nabla_{\vec{x}}T\geq 0\\ \nabla_{\vec{x}}T\cdot\mathbf{K}^{sym}\cdot\nabla_{\vec{x}}T+\nabla_{\vec{x}}T\cdot\mathbf{K}^{skew}\cdot\nabla_{\vec{x}}T\geq 0\\ \nabla_{\vec{x}}T\cdot\mathbf{K}^{sym}\cdot\nabla_{\vec{x}}T+\mathbf{K}^{skew}:(\nabla_{\vec{x}}T\otimes\nabla_{\vec{x}}T)\geq 0\end{aligned}$$

Notice that $\mathbf{K}^{skew} : (\nabla_{\vec{x}} T \otimes \nabla_{\vec{x}} T) = 0$, since the double scalar product between an antisymmetric tensor ($\mathbf{K}^{skew}$) and a symmetric one $(\nabla_{\vec{x}} T \otimes \nabla_{\vec{x}} T)$ is equal to zero, then:

$$0 \le \nabla_{\vec{x}} T \cdot \mathbf{K}(\vec{x}) \cdot \nabla_{\vec{x}} T = \nabla_{\vec{x}} T \cdot \mathbf{K}^{sym} \cdot \nabla_{\vec{x}} T \ge 0$$

That is, the above inequality is always true whether $\mathbf{K}(\vec{x})$ is symmetric or not.

b) For the proposed problem the only remaining governing equation is the energy equation: $\rho \frac{Du}{Dt} \equiv \rho \dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r = -\nabla_{\vec{x}} \cdot \vec{\mathbf{q}}$, where u is the specific internal energy, $\boldsymbol{\sigma} : \mathbf{D}$ is the stress power, and ρr is the internal heat source per unit volume. Then:

$$\begin{aligned}
\rho \dot{u} &= -\mathsf{q}_{i,i} = -(-\mathsf{K}_{ij} T_{,j})_{,i} = \mathsf{K}_{ij,i} T_{,j} + \mathsf{K}_{ij} T_{,ji} = (\nabla_{\vec{x}} \cdot \mathbf{K}^T) \cdot (\nabla_{\vec{x}} T) + \mathbf{K} : \nabla_{\vec{x}} (\nabla_{\vec{x}} T) \\
&= (\nabla_{\vec{x}} \cdot \mathbf{K}^T) \cdot (\nabla_{\vec{x}} T) + \left[\mathbf{K}^{sym} + \mathbf{K}^{skew}\right] : \nabla_{\vec{x}} (\nabla_{\vec{x}} T) \\
&= (\nabla_{\vec{x}} \cdot \mathbf{K}^T) \cdot (\nabla_{\vec{x}} T) + \mathbf{K}^{sym} : \nabla_{\vec{x}} (\nabla_{\vec{x}} T) + \mathbf{K}^{skew} : \nabla_{\vec{x}} (\nabla_{\vec{x}} T) \\
&= (\nabla_{\vec{x}} \cdot \mathbf{K}^T) \cdot (\nabla_{\vec{x}} T) + \mathbf{K}^{sym} : \nabla_{\vec{x}} (\nabla_{\vec{x}} T)
\end{aligned}$$

where we have considered the symmetry of $\left[\nabla_{\vec{x}} (\nabla_{\vec{x}} T)\right]_{ij} = T_{,ij} = T_{,ji}$. If the material is homogeneous the implication is that the $\mathbf{K}$ field does not depend on $(\vec{x})$, so $\mathsf{K}_{ij,i} = \mathsf{0}_j$. In this scenario the heat equation reduces to:

$$\rho \dot{u} = \mathbf{K}^{sym} : \nabla_{\vec{x}} (\nabla_{\vec{x}} T)$$

Therefore, when the material is homogeneous, the antisymmetric part of $\mathbf{K}$ does not affect the outcome.

c) The feature of isotropic materials is that their properties (at one material point) do not change if the coordinate system is changed. It follows then that $\mathbf{K}$ must be an isotropic tensor. An isotropic second-order tensor has the format of a spherical tensor, (see Chapter 1), then the tensor $\mathbf{K}$ must be of the type: $\mathbf{K} = \mathsf{K}\mathbf{1}$, where K is a scalar.

5.11 Fundamental Equations of Continuum Mechanics

Then, we sum up the fundamental equations of continuum mechanics in the current configuration as:

Fundamental Equations of the Continuum Mechanics (Current configuration)		
Mass Continuity Equation (Principle of conservation of mass)	$\frac{D\rho}{Dt} + \rho(\nabla_{\vec{x}} \cdot \vec{v}) = 0$ (1 equation)	(5.176)
Equation of Motion (Principle of conservation of linear momentum)	$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{v}}$ (3 equations)	(5.177)
Symmetry of the Cauchy Stress Tensor (Principle of conservation of angular momentum)	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ (6 unknowns)	(5.178)
Energy Equation (Principle of conservation of energy)	$\rho \dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r$ (1 equation)	(5.179)
Entropy Inequality (Principle of irreversibility)	$\rho \dot{\eta}(\vec{x}, t) + \frac{1}{T} \boldsymbol{\sigma} : \mathbf{D} - \frac{1}{T} \rho \dot{u} - \frac{1}{T^2} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \ge 0$	(5.180)

The entropy inequality is not one more problem equation. Rather, it is used to establish restrictions on the problem variables. The symmetry of the Cauchy stress tensor reduces the number of $\boldsymbol{\sigma}$ -unknown from 9 to 6.

The mass continuity equation, the equations of motion and the energy equation give us in total 5 equations. The unknowns are: the three components of velocity $\vec{v}$, temperature T, mass density ρ, six components of the Cauchy stress tensor $\boldsymbol{\sigma}$, the specific internal energy u, three components of the heat flux vector $\vec{\mathbf{q}}$, and the entropy η, with a total of 16 unknowns.

To achieve the well-posedness of the problem eleven equations must be added. We must add equations that connect the stress, heat and energy with other fields. These equations are called *constitutive equations*, which is the subject of the next chapter.

5.11.1 Particular Cases

5.11.1.1 Rigid Body Motion

When we are dealing with rigid body motion without the effect of temperature, the only principles needed to establish the set of equations are: the principle of conservation of the linear momentum and the principle of conservation of angular momentum. Then, the governing equations are characterized by $\sum \vec{\mathbf{F}} = m\vec{a}$ and $\sum \vec{M}_G = \dot{\vec{H}}_G$, (see **Problem 5.9**). The problem can then be solved by introducing the appropriate initial and boundary conditions.

5.11.1.2 Flux Problems

For problems which only involve the transport of a physical quantity (mass, energy, or otherwise) the only principle necessary to establish the governing equations is the conservation law of the physical quantity, or in its strong form: the physical quantity continuity equation, (see equation (5.24)):

$$Q = \frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}} \cdot (\Phi \vec{v}) \equiv \frac{\partial \Phi(\vec{x},t)}{\partial t} + \nabla_{\vec{x}} \cdot \left(\vec{\mathbf{q}}(\vec{x},t) \right) \qquad \frac{[\Phi]}{s} \tag{5.181}$$

The case in **Problem 5.11** was related to energy transport (not mass transport). Said energy transport exists due to the agitation of atoms, in which the degree of agitation at the macroscopic level is characterized by the temperature. If a particle starts to increase the degree of agitation, then neighboring particles also start behaving in a similar fashion. In this way the energy in solids is transported without any mass transport. This energy transport at the macroscopic level is represented by means of the flux, $\vec{\mathbf{q}} = \Phi \vec{v}$.

When we are working in the field of continuum mechanics we do not go down to the atomic level and measure the average velocity (vibration) of a handful of atoms to establish the flux. What we do is: we go to the laboratory with the material with which we want to establish the heat flux (energy flow), we vary the temperature and we verify macroscopically that the flux can be characterized by the following phenomenological law $q = -k\nabla T$ (in a one-dimensional case), where k is a thermal property of the material. This procedure was performed by Fourier, thereby establishing *Fourier's law of heat conduction*. Fourier also verified in the laboratory that heat flux is opposite to the temperature gradient, a fact already proven by the second law of thermodynamics. The law $q = -k\nabla T$ is a phenomenological law or constitutive equation of heat flux, and connects two thermal variables. It is also interesting to observe that the fundamental equations of continuum mechanics (5.176)-(5.180) do not have such a relationship.

In **Problem 5.11**, the physical quantity in question is given by $\Phi = \rho c T$. According to the SI units we have: $[T] = K$, $[\rho] = \dfrac{kg}{m^3}$, $[c] = \dfrac{J}{kg\, K}$ with which we can verify the following SI units:

$[\Phi] = [\rho c T] = \dfrac{kg}{m^3}\dfrac{J}{kg\, K}K = \dfrac{J}{m^3}$ (unit of energy per unit volume - energy density)

$[\vec{\mathbf{q}}] \equiv [\Phi \vec{v}] = \dfrac{J}{m^3}\dfrac{m}{s} = \dfrac{J}{m^2 s}$ (unit of energy flux)

There are several engineering problems which are characterized by the continuity equation, some of which are: heat conduction problems (energy flux); filtration problems in porous media (mass transport); diffusion problems (*e.g.* transport of contaminant in an aqueous medium); and the Saint-Venant torsion problem (stress flux).

5.12 Flux Problems

5.12.1 Heat Transfer

Heat flow is a form of energy transfer in a continuous medium which occurs in three ways, namely via: conduction; convection; radiation.

5.12.1.1 Thermal Conduction

Thermal Conduction: Transfer of energy in the form of heat, which is caused by the collision and vibration of molecules and atoms (no mass transport).

Temperature: The temperature ($T(\vec{x}, t) > 0$) is not a form of energy. Rather it is a measurement of how hot a particle is. In experiments it has been proven that the hot particles tend to give heat to cooler particles. The SI unit of absolute temperature is the kelvin, $[T] = K$. Absolute zero $T = 0K \approx -273{,}15^{\circ} C$ is a theoretical temperature when even atoms and electrons cease to move.

When a continuous medium undergoes a non-uniform temperature variation, heat is transferred from a higher to a lower temperature region. When this phenomenon occurs without mass transport, this is known as a *heat conduction problem*. The phenomenological law (*constitutive equation of heat flux*) that governs heat conduction behavior can be defined by means of *Fourier's law of heat conduction*, which states that the heat flux is proportional to the temperature gradient:

$$\boxed{\vec{\mathbf{q}} = -\mathbf{K} \cdot \frac{\partial T}{\partial \vec{x}} = -\mathbf{K} \cdot \nabla_{\vec{x}} T} \quad \textit{Fourier's law of heat conduction} \quad \left[\frac{J}{m^2 s}\right] \tag{5.182}$$

where $\vec{\mathbf{q}}$ is the heat flux per unit area per unit time, and its SI unit $[\vec{\mathbf{q}}]=\frac{J}{m^2 s}=\frac{W}{m^2}$; $\nabla_{\vec{x}}T$ is the temperature gradient whose SI unit is $[\nabla_{\vec{x}}T]=\frac{K}{m}$, and $\mathbf{K}$ is the thermal conductivity tensor, whose SI unit is $[\mathbf{K}]=\frac{W}{mK}$, (see **Problem 5.11**).

NOTE: Fourier's law of heat conduction is not universal as there are complex materials in which heat flow is governed by more complex laws. ■

The negative sign in Fourier's law is there because the heat flux vector is always opposite to the temperature gradient. The temperature gradient vector ($\nabla_{\vec{x}}T$) points from the coldest to the warmest region, while the heat flux vector ($\vec{\mathbf{q}}$) points from the warmest to the coldest region (physical fact), (see Figure 5.10).

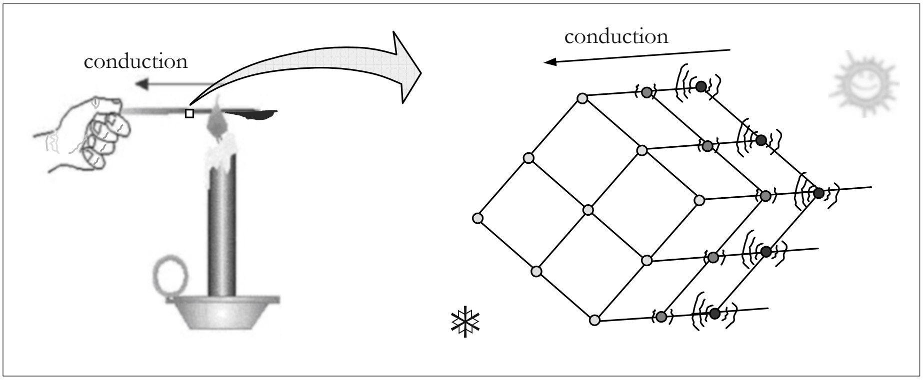


Figure 5.11: Heat conduction.

The thermal conductivity tensor contains the thermal properties of the material, which are obtained in the laboratory, and depends on porosity, mass density, composition, etc. Explicitly, the components of $\mathbf{K}$ are:

$$(\mathbf{K})_{ij}=\begin{bmatrix} \mathsf{K}_{11} & \mathsf{K}_{12} & \mathsf{K}_{13} \\ \mathsf{K}_{21} & \mathsf{K}_{22} & \mathsf{K}_{23} \\ \mathsf{K}_{31} & \mathsf{K}_{32} & \mathsf{K}_{33} \end{bmatrix} \xrightarrow{\textit{isotropic material}} (\mathbf{K})_{ij}=\mathsf{K}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.183}$$

For isotropic materials, *i.e.* those that have the same property in any direction, the thermal conductivity tensor is represented by a spherical tensor, (see **Problem 5.11**).

If we are dealing with homogenous material, $\mathbf{K}$ is not dependent on $\vec{x}$. For isotropic materials, the components of the heat flux vector ($\vec{\mathbf{q}}=-\mathbf{K}\cdot\nabla_{\vec{x}}T$) are obtained as follows:

$$(\vec{\mathbf{q}})_i=\begin{Bmatrix} \mathsf{q}_1 \\ \mathsf{q}_2 \\ \mathsf{q}_3 \end{Bmatrix}-\mathsf{K}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\partial T}{\partial x_1} \\ \frac{\partial T}{\partial x_2} \\ \frac{\partial T}{\partial x_3} \end{bmatrix}=-\mathsf{K}\begin{bmatrix} \frac{\partial T}{\partial x_1} \\ \frac{\partial T}{\partial x_2} \\ \frac{\partial T}{\partial x_3} \end{bmatrix} \tag{5.184}$$

and the normal component $\mathrm{q}_n = \vec{\mathbf{q}} \cdot \hat{\mathbf{n}}$, (see Figure 5.2), is evaluated as:

$$\mathrm{q}_n = \mathrm{q}_i \hat{\mathrm{n}}_i = \mathrm{q}_1 \hat{\mathrm{n}}_1 + \mathrm{q}_2 \hat{\mathrm{n}}_2 + \mathrm{q}_3 \hat{\mathrm{n}}_3 = -\mathrm{K}\frac{\partial T}{\partial x_1}\hat{\mathrm{n}}_1 - \mathrm{K}\frac{\partial T}{\partial x_2}\hat{\mathrm{n}}_2 - \mathrm{K}\frac{\partial T}{\partial x_3}\hat{\mathrm{n}}_3 \tag{5.185}$$

5.12.1.2 Thermal Convection Transfer

Heat transfer by convection occurs in a fluid environment where there are moving particles between regions with different temperatures, (see Figure 5.12). In other words it shows the transfer of energy (heat) due to the movement of fluid particles. This phenomenon is governed by *Newton's Law of Cooling*, which is:

$$\boxed{\mathrm{q} = \alpha(T - T_{ext})} \quad \textit{Newton's law of cooling} \tag{5.186}$$

where q is the thermal energy; α is the heat transfer coefficient per unit area; T is the temperature of the body's surface, and T_{ext} is the temperature of the surrounding environment.

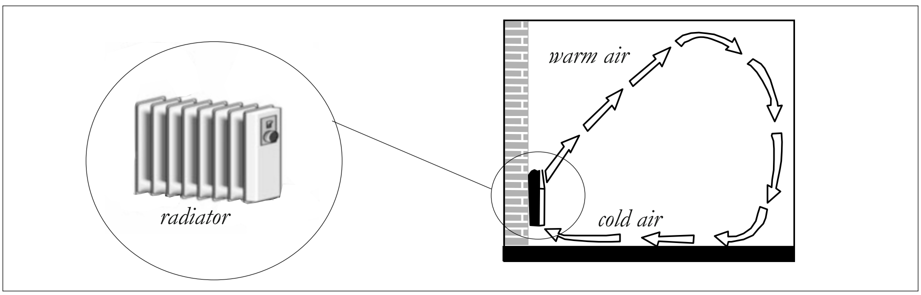


Figure 5.12: Thermal convection.

If we consider a room in which there is a radiator, the air particles in contact with the hot surface of the radiator increases their temperature and their mass density decreases, so that the hot ascending particles, displace the cooler particles moving downwards, (see Figure 5.12) and because of this movement, the heat will be transferred to the whole room.

5.12.1.3 Thermal Radiation

Thermal radiation is the process by which thermal energy is transferred between two surfaces, obeying the laws for electromagnetic radiation (photon transport). To give an example we can mention how heat is transferred from the Sun to the Earth. The phenomenological law governing this phenomenon is the *Stefan-Boltzmann law*.

5.12.1.4 The Heat Flux Equation

Next, we can obtain the partial differential equation that governs the heat transfer problem, by means of an energy balance, *i.e.*:

Heat that enters into the system	+	Heat generated internally	−	Heat that leaves the system	=	Change of internal energy

Let us consider a differential volume element, (see Figure 5.13), in which there are inflows and outflows of heat. In addition let us consider energy generated internally represented by $Q = \rho r$ (per unit volume per unit time), whose SI unit is $[Q] = \frac{J}{m^3 s}$. The scalar function r describes the heat generated which could be caused by a phenomenon such as a chemical, or nuclear reaction, and whose SI unit is $[r] = \frac{J}{kg\ s}$. As the temperature of the body increases, part of the thermal energy is stored in the body. For a differential volume element $(dx_1 dx_2 dx_3)$ this stored energy is governed by the expression:

$$\rho c_v \frac{\partial T}{\partial t} dx_1 dx_2 dx_3 \tag{5.187}$$

where ρ is the mass density; and the material property c_v is the specific heat capacity at a constant volume whose SI unit is $[c_v] = \frac{J}{kg\ K}$.

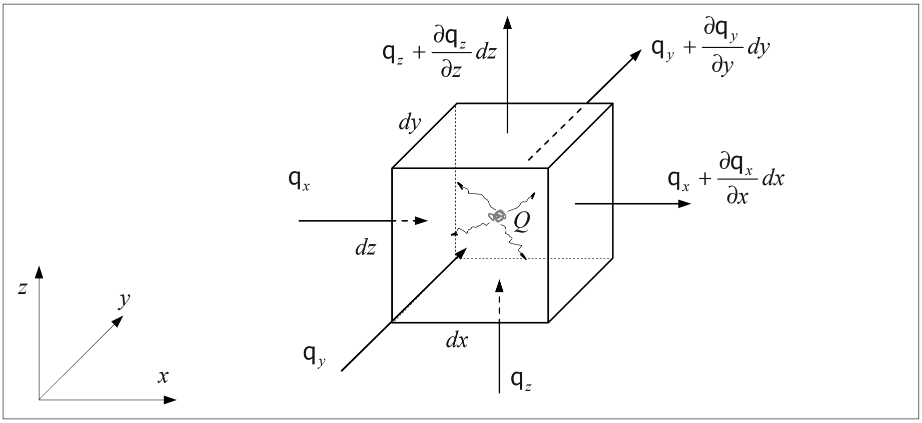


Figure 5.13: Source and heat flux in a differential volume element.

In the following demonstration, let us consider the following change of nomenclature: coordinates: $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$; heat flux components; $\mathsf{q}_1 \equiv \mathsf{q}_x$, $\mathsf{q}_2 \equiv \mathsf{q}_y$, $\mathsf{q}_3 \equiv \mathsf{q}_z$. Notice that we are employing the engineering notation.

Then by applying the energy conservation law throughout the differential element, we obtain:

$$\begin{aligned} &\mathsf{q}_x\, dy\, dz + \mathsf{q}_y\, dx\, dz + \mathsf{q}_z\, dx\, dy + Q\, dx\, dy\, dz - \left[\left(\mathsf{q}_x + \frac{\partial \mathsf{q}_x}{\partial x} dx \right) dydz \right. \\ &\qquad \left. + \left(\mathsf{q}_y + \frac{\partial \mathsf{q}_y}{\partial y} dy \right) dxdz + \left(\mathsf{q}_z + \frac{\partial \mathsf{q}_z}{\partial z} dz \right) dxdy \right] = c_v \frac{\partial T}{\partial t} \rho dxdydz \\ &\Rightarrow Q - \left[\frac{\partial \mathsf{q}_x}{\partial x} + \frac{\partial \mathsf{q}_y}{\partial y} + \frac{\partial \mathsf{q}_z}{\partial z} \right] = \rho c_v \frac{\partial T}{\partial t} \end{aligned} \tag{5.188}$$

which results in the heat equation:

$$\boxed{Q - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} = \rho c_v \frac{\partial T}{\partial t} \Rightarrow Q + \nabla_{\vec{x}} \cdot (\mathbf{K} \cdot \nabla_{\vec{x}} T) = \rho c_v \frac{\partial T}{\partial t}} \quad \textit{The heat flux equation} \tag{5.189}$$

where we have considered Fourier's law of heat conduction $\mathsf{q}_i = -\mathsf{K}_{ij} \dfrac{\partial T}{\partial x_j}$. Notice that the above equation was obtained in **Problem 5.11** and it should be pointed out that we have one equation in (5.189) and one unknown (temperature). The solution of equation (5.189) is unique if we are given the appropriate boundary and initial conditions. The governing equation in (5.189) together with the boundary and initial conditions are called the Initial Boundary Value Problem (IBVP) of thermal conduction.

Then by considering an isotropic homogeneous material, the heat equation in (5.189) becomes:

$$Q + \mathsf{K}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \mathsf{K}\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \mathsf{K}\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) = \rho\, c_v \frac{\partial T}{\partial t} \tag{5.190}$$

$$\Rightarrow \frac{Q}{\mathsf{K}} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{5.191}$$

where κ is known as the *thermal diffusivity*:

$$\kappa = \frac{\mathsf{K}}{\rho\, c_v} \qquad \left[\frac{m^2}{s}\right] \tag{5.192}$$

Particular Cases

- A steady state temperature field, *i.e.* $T = T(\vec{x})$:

$$\frac{\partial T}{\partial t} = 0 \tag{5.193}$$

The equation in (5.191) becomes:

$$\boxed{\frac{Q}{\mathsf{K}} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \Rightarrow \frac{Q}{\mathsf{K}} + \nabla^2_{\vec{x}} T = 0} \quad \textit{The Poisson's equation} \tag{5.194}$$

From a mathematical point of view, the above equation is known as the *Poisson's equation*.

- A steady state problem, and without internal heat generation:

$$\frac{\partial T}{\partial t} = 0 \quad ; \quad Q = 0 \tag{5.195}$$

In this scenario the equation in (5.191) becomes *Laplace's equation:*

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \Rightarrow \nabla^2_{\vec{x}} T = 0} \quad \textit{Laplace's equation} \tag{5.196}$$

- Transient problem, $T = T(\vec{x}, t)$ (time dependent), but in the absence of internal heat generation, $Q = 0$, the equation in (5.191) becomes *Fourier's equation*:

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \Rightarrow \nabla^2_{\vec{x}} T = \frac{1}{\kappa}\frac{\partial T}{\partial t}} \quad \textit{Fourier's equation} \tag{5.197}$$

Initial and Boundary Conditions

1. Prescribed value of the temperature:

$$T(x,y,z,t) = T^* \quad to \quad t>0 \quad on \ S_1 \tag{5.198}$$

Mathematically this condition is known as *Dirichlet* boundary condition.

2. Flux boundary condition:

$$Q + \mathsf{K}\frac{\partial T}{\partial x}\hat{\mathsf{n}}_x + \mathsf{K}\frac{\partial T}{\partial y}\hat{\mathsf{n}}_y + \mathsf{K}\frac{\partial T}{\partial z}\hat{\mathsf{n}}_z = 0 \quad to \quad t>0 \quad on \ S_2 \tag{5.199}$$

Mathematically this condition is known as a *Neumann* boundary condition.

A combination of the boundary conditions of Dirichlet and Neumann is known as the Robin boundary condition, *i.e.*:

$$Q + \mathsf{K}\frac{\partial T}{\partial x}\hat{\mathsf{n}}_x + \mathsf{K}\frac{\partial T}{\partial y}\hat{\mathsf{n}}_y + \mathsf{K}\frac{\partial T}{\partial z}\hat{\mathsf{n}}_z + \alpha(T - T_{ext}) = 0 \quad to \quad t>0 \quad on \ S_3 \tag{5.200}$$

where $\hat{\mathsf{n}}_x, \hat{\mathsf{n}}_y, \hat{\mathsf{n}}_z$ are the components of the outward unit normal vector on the surface.

Initial conditions $T(x,y,z,t=0) = \overline{T}_0$.

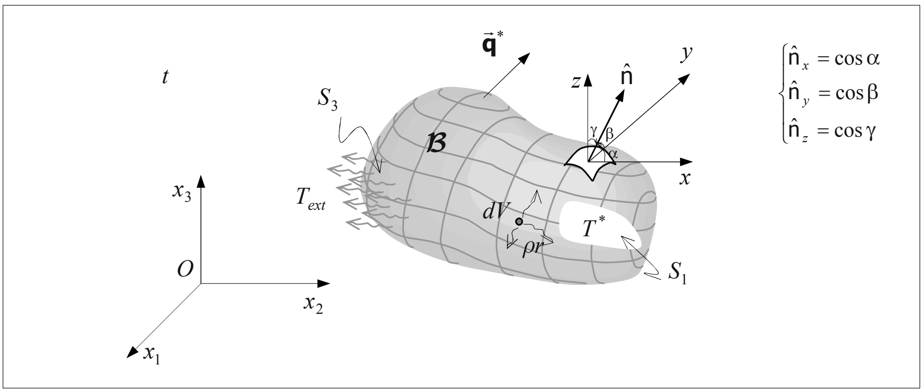


Figure 5.14: Heat flux problem and boundary conditions.

5.13 Fluid Flow in Porous Media (filtration)

Let us consider a reservoir as shown in Figure 5.15. To obtain the partial differential equation that governs the fluid flow in porous media we will make the same approach as that made to the heat flux problem, but in this case we will consider the two-dimensional case and steady state regime.

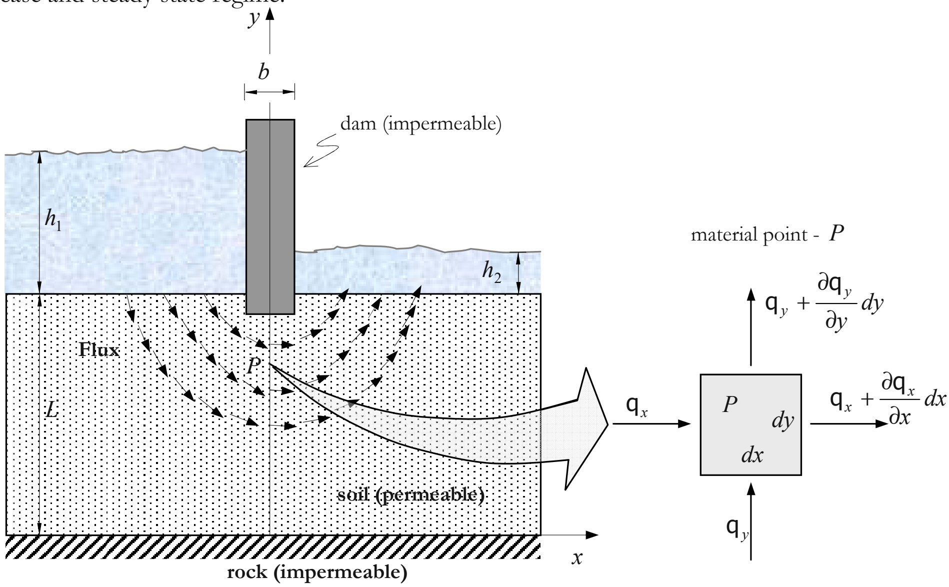


Figure 5.15: Fluid flow throughout the porous medium.

The partial differential equation governing the fluid flow in porous media for a steady state case can be obtained by means of the differential element equilibrium, (see Figure 5.15), *i.e.:*

$$\left[q_x - \left(q_x + \frac{\partial q_x}{\partial x} dx \right) \right] dy + \left[q_y - \left(q_y + \frac{\partial q_y}{\partial y} dy \right) \right] dx = 0 \Rightarrow -\frac{\partial q_x}{\partial x} dxdy - \frac{\partial q_y}{\partial y} dxdy = 0 \quad (5.201)$$

The phenomenological law of mass flux in porous media is governed by *Darcy's law*, $\vec{\mathbf{q}} = -\mathbf{K} \cdot \nabla_{\vec{x}} \phi$, where ϕ is the total potential (water level), and $\mathbf{K}$ is the permeability tensor which depends on the material. In the two-dimensional case, the components of $\vec{\mathbf{q}}$ are given by:

$$q_x = -K \frac{\partial \phi}{\partial x} \quad ; \quad q_y = -K \frac{\partial \phi}{\partial y} \quad (5.202)$$

where we have considered an isotropic material (one which has the same permeability in all directions). Then by substituting the components of the flux vector in Equation (5.201), we obtain:

$$\frac{\partial}{\partial x}\left(K \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y}\left(K \frac{\partial \phi}{\partial y} \right) = 0 \quad (5.203)$$

$$K \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0 \Rightarrow \nabla^2 \phi = 0 \quad \textit{Laplace's equation} \quad (5.204)$$

The boundary conditions are:

- There is no flow at $x=-\infty$ and $x=+\infty$: $\left.\dfrac{\partial\phi}{\partial x}\right|_{x=-\infty}=0 \quad ; \quad \left.\dfrac{\partial\phi}{\partial x}\right|_{x=+\infty}=0$
- There is no flow at the border (soil-rock interface): $\left.\dfrac{\partial\phi}{\partial y}\right|_{y=0}=0$
- There is no flow at the soil-dam interface: $\dfrac{\partial\phi}{\partial y}(x,L)=0 \quad ; \quad -\dfrac{b}{2}\le x\le\dfrac{b}{2}$

Additionally, the total potential is prescribed at the water-soil interface:

$$\phi(x,L)\,\Big|_{x<\frac{-b}{2}}=h_1 \qquad ; \qquad \phi(x,L)\,\Big|_{x>\frac{b}{2}}=h_2 \tag{5.205}$$

5.14 The Convection-Diffusion Equation

Diffusion: An irreversible physical process is one where particles which are in a high concentration region tend to move to a region of low concentration. In general this process is governed by Fick's law of diffusion:

$$\boxed{\vec{\mathbf{J}}=-\mathbf{D}\cdot\nabla_{\vec{x}}c} \quad \textit{Fick's law of diffusion} \qquad \left[\frac{mol}{m^2 s}\right] \tag{5.206}$$

where $\mathbf{D}>\mathbf{0}$ is the diffusion tensor (or diffusivity tensor), and $c(\vec{x},t)$ is the solute concentration whose SI unit is $[c]=\dfrac{mol}{m^3}$. This concentration is defined as follows:

$$c=\frac{\text{solute mass}}{\text{fluid mass}} \tag{5.207}$$

or we can express this concentration as:

$$c=\frac{1}{\rho_f}\frac{\text{solute mass}}{\text{volume}}=\frac{\rho_s}{\rho_f} \tag{5.208}$$

where ρ_f, ρ_s are the mass densities of the fluid and solute, respectively.

In general, when we have a process where there is mass transport (a fluid+solute) two mechanisms take place, namely: convection and diffusion. In this case the matter (solute) is defined by the concentration c and we must consider the matter to be diffused throughout the aqueous medium. Here, we can assume that the amount of the matter is too little to affect the fluid velocity field.

Let us consider the solute flux $\vec{\mathbf{q}}=c\,\vec{v}$ (a convective term) to which we add the diffusive term to obtain the total flux:

$$\vec{\mathbf{q}}=\underbrace{c\,\vec{v}}_{\substack{\textit{Convective}\\ \textit{term}}}\;\underbrace{-\mathbf{D}\cdot\frac{\partial c}{\partial\vec{x}}}_{\substack{\textit{Diffusive}\\ \textit{term}}} \tag{5.209}$$

Then, to obtain the partial differential equation for the convection-diffusion problem we consider the one-dimensional case, (see Figure 5.16).

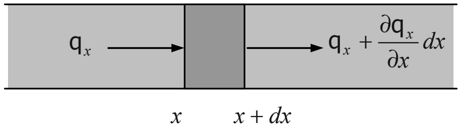


Figure 5.16: Mass transport (solute).

Here we can put the conservation law down to:

solute that enters into the system + solute generated internally − solute that leaves the system = Change of the solute internally

Then, mathematically, the above expression becomes:

$$\mathsf{q}_x dy + Q dxdy - \left(\mathsf{q}_x + \frac{\partial \mathsf{q}_x}{\partial x} dx \right) dy = \frac{\partial c}{\partial t} dxdy \quad \Rightarrow \quad Q - \frac{\partial \mathsf{q}_x}{\partial x} = \frac{\partial c}{\partial t} \tag{5.210}$$

Additionally, by substituting the flux given in (5.209) into the above equation, we obtain:

$$Q - \frac{\partial \left(c\, v_x - \mathsf{D} \frac{\partial c}{\partial x} \right)}{\partial x} = \frac{\partial c}{\partial t} \quad \Rightarrow \quad Q - \frac{\partial (c\, v_x)}{\partial x} + \frac{\partial}{\partial x} \left(\mathsf{D} \frac{\partial c}{\partial x} \right) = \frac{\partial c}{\partial t} \tag{5.211}$$

Therefore we can summarize the convection-diffusion equation in three dimensions as:

$$\boxed{Q - \nabla_{\vec{x}} \cdot (\vec{\boldsymbol{v}} c) + \nabla_{\vec{x}} \cdot (\mathbf{D} \cdot \nabla_{\vec{x}} c) = \frac{\partial c}{\partial t}} \quad \textit{Convection-diffusion equation} \tag{5.212}$$

where $\frac{\partial c}{\partial t}$ is the local variation of the concentration with respect to time, $\vec{\boldsymbol{v}} \cdot \nabla_{\vec{x}} c$ is the convection term caused by fluid motion, and $\nabla_{\vec{x}} \cdot (\mathbf{D} \cdot \nabla_{\vec{x}} c)$ is the diffusion.

Next, we assume that at a material point there are two types of material that are represented by a physical quantity per unit volume in such a way that $c = c^f + c^s$, and the following holds $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}^f + \vec{\boldsymbol{v}}^s$, (see Figure 5.17).

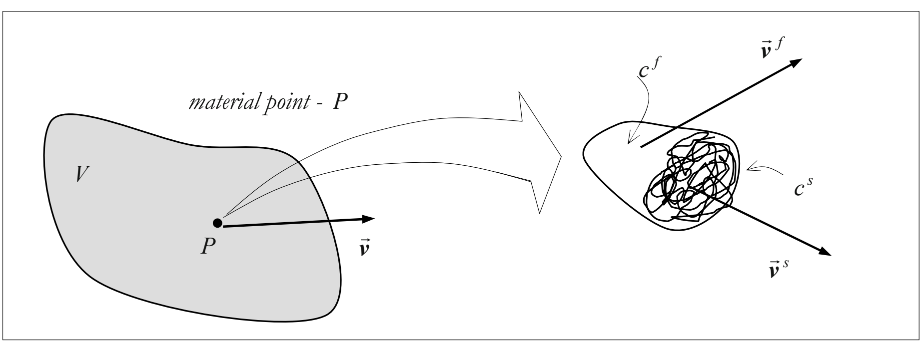


Figure 5.17: Heterogeneous medium.

Then, from the continuity equation for this physical quantity we obtain:

$$Q = \frac{\partial \Phi}{\partial t} + \nabla_{\vec{x}} \cdot (\Phi \vec{v}) \quad \Rightarrow \quad Q = \frac{\partial (c^f + c^s)}{\partial t} + \frac{\partial}{\partial \vec{x}}\left[(c^f + c^s)(\vec{v}^f + \vec{v}^s)\right] \tag{5.213}$$

thus

$$\begin{aligned}
&Q = \frac{\partial (c^f + c^s)}{\partial t} + \frac{\partial}{\partial \vec{x}}\left[(c^f + c^s)(\vec{v}^f + \vec{v}^s)\right] \\
&\Rightarrow Q = \frac{\partial (c^f + c^s)}{\partial t} + \frac{\partial}{\partial \vec{x}}\left[c^f \vec{v}^f + c^f \vec{v}^s + c^s \vec{v}^f + c^s \vec{v}^s\right] \\
&\Rightarrow Q = \frac{\partial c^f}{\partial t} + \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot \left[c^f \vec{v}^f + c^f \vec{v}^s + c^s \vec{v}^f + c^s \vec{v}^s\right] \\
&\Rightarrow Q = \left[\frac{\partial c^f}{\partial t} + \nabla_{\vec{x}} \cdot (c^f \vec{v}^f)\right] + \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot \left[c^f \vec{v}^s + c^s \vec{v}^s\right]
\end{aligned} \tag{5.214}$$

If we assume that there is no (f)-material source, then $\frac{\partial c^f}{\partial t} + \nabla_{\vec{x}} \cdot (c^f \vec{v}^f) = 0$ holds, which is the continuity equation of the quantity c^f with which the equation in (5.214) becomes:

$$\begin{aligned}
&Q = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot \left[c^f \vec{v}^s + c^s \vec{v}^s\right] \\
&\Rightarrow Q = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot (c^s \vec{v}^s) + \nabla_{\vec{x}} \cdot (c^f \vec{v}^s) \\
&\Rightarrow Q = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot (c^s \vec{v}^s) + \nabla_{\vec{x}} c^f \cdot \vec{v}^s + c^f \nabla_{\vec{x}} \cdot \vec{v}^s
\end{aligned} \tag{5.215}$$

If the physical quantity c^f does not change with $\vec{x}$, then the gradient of c^f becomes $\nabla_{\vec{x}} c^f = \vec{\mathbf{0}}$. In addition if we consider the medium (s) to be incompressible we obtain $\nabla_{\vec{x}} \cdot \vec{v}^s = 0$. These simplifications indicate that the material (s) does not affect the velocity field of the material (f). So, if the amount of the material (s) is significant, this approach is no longer valid. Then, with these approximations we obtain:

$$Q = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot (c^s \vec{v}^s) = \frac{\partial c^s}{\partial t} + \nabla_{\vec{x}} \cdot (c^s \vec{v}^f) + \nabla_{\vec{x}} \cdot \vec{\mathbf{q}}^{(D)} \tag{5.216}$$

Notice that the term $(c^s \vec{v}^s) \equiv \vec{\mathbf{q}}^{(D)}$ represents the flux caused by the (s)-material concentration, the diffusive term. The term $(c^s \vec{v}^f) \equiv \vec{\mathbf{q}}^{(C)}$ is related to mass transport, the convective term. Then, if $\vec{\mathbf{q}}^{(D)}$ is defined by Fick's law we refer back to the equation in (5.212).

5.14.1 The Generalization of the Flux Problem

Flux problems can be found in many branches of physics or engineering. These problems are only governed by the *continuity equation* (sometimes called the *transport equation*):

$$\boxed{Q + \nabla_{\vec{x}} \cdot (\mathbf{D} \cdot \nabla_{\vec{x}} \phi) = \rho\, c \frac{\partial \phi}{\partial t}} \tag{5.217}$$

where $\phi(x_1, x_2, x_3, t)$ is the scalar variable to be solved. Depending on the problem the variables take the following meanings:

$Q + \nabla_{\vec{x}} \cdot (\mathbf{D} \cdot \nabla_{\vec{x}} \phi) = 0$					
Equation	Scalar field ϕ	$\mathbf{D}$	Q	Flux vector $\vec{\mathbf{q}}$	Phenomenological law
Heat flux	Temperature T	Thermal conductivity tensor	Heat generated Q	Heat flux vector $\vec{\mathbf{q}}$	Fourier's law $\vec{\mathbf{q}} = -\mathbf{K} \cdot \nabla_{\vec{x}} T$
Fluid flow in the porous media	piezometric head (or hydraulic head) h	permeability tensor	water source	Volume flux vector	Darcy's law $\vec{\mathbf{q}} = -\mathbf{K} \cdot \nabla_{\vec{x}} \phi$
Diffusion	ion concentration c	Constitutive matrix for diffusion coefficient	Ion source	Ion flux vector $\vec{\mathbf{J}}$	Fick's law $\vec{\mathbf{J}} = -\mathbf{D} \cdot \nabla_{\vec{x}} c$
Saint-Venant torsion	Prandtl stress function θ	$\dfrac{1}{G}$	2θ		Hooke's law

5.15 Initial Boundary Value Problem (IBVP) and Computational Mechanics

An *Initial Boundary Value Problem* (IBVP) is defined by the governing equations (a set of *partial differential equations*-PDEs) and by boundary and initial conditions. Said conditions are restrictions on the governing equations. The IBVP solution is the one that is given by the solution of the equations and which also satisfies the boundary and initial conditions. The IBVP solution will be unique if the problem is well posed, *i.e.* given a boundary and initial conditions, there is only one solution to the problem. Then, the governing equations are defined by the fundamental and constitutive equations. In subsequent chapters of this publication we will fundamentally deal with the constitutive models (constitutive equations), which are used to complete the IBVP.

It must be stressed that until the fundamental equations were established, we did not discuss the type of material that the continuum is made up of. When we begin to specify this is the time when the concept of the constitutive equation appears. Then, we can automatically believe that every class of material has its particular constitutive equations.

In order to represent the real behavior of the material, the constitutive equation has to be calibrated with macroscopic parameters, *i.e.* these parameters, which are obtained in the laboratory, represent the material behavior at the *macroscopic level*. Remember that the scale of study of continuum mechanics is macroscopic, and then we need to obtain some representative macroscopic parameters of the phenomena that occur at the microscopic scale. We can consider this to be the Achilles' heel of Continuum Mechanics, because in some cases we are not able to obtain a macroscopic parameter that characterize phenomena that are taking place at the microscopic level. In fact, the evolution of the constitutive equation is directly linked to the precision of the instrumentation and new techniques used in laboratory testing of such materials. So in summary we can state that the constitutive equations used to characterize a material must be capable of simulating any phenomena that arise in the material during the loading/unloading/loading process, or at least the most significant.

As discussed earlier, the constitutive equations complete the set of equations that govern a particular physical problem. That is, they complement the IBVP. The solution of the problem can be analytical (the exact solution) or numerical (an approximate solution). In most cases it is impossible to obtain the analytical solution, so we turn to computational mechanics to obtain the numerical solution of the physical problem.

Computational Mechanics resolve specific problems by using numerical simulation tools incorporated in the computer. In general we can state that computational mechanics is not an independent block, *i.e.*, for its complete implementation it is directly dependent on three areas: *Theoretical Analysis* (IBVP establishment), *Experimental Analysis* (Lab), and *Numerical Analysis* (a numerical methodology incorporated into the computer to obtain the numerical solution for IBVP), (see Figure 5.18). From a very general point of view, we can also appreciate in Figure 5.18 how the constitutive models are embedded within the field of Computational Mechanics.

COMPUTATIONAL MECHANICS

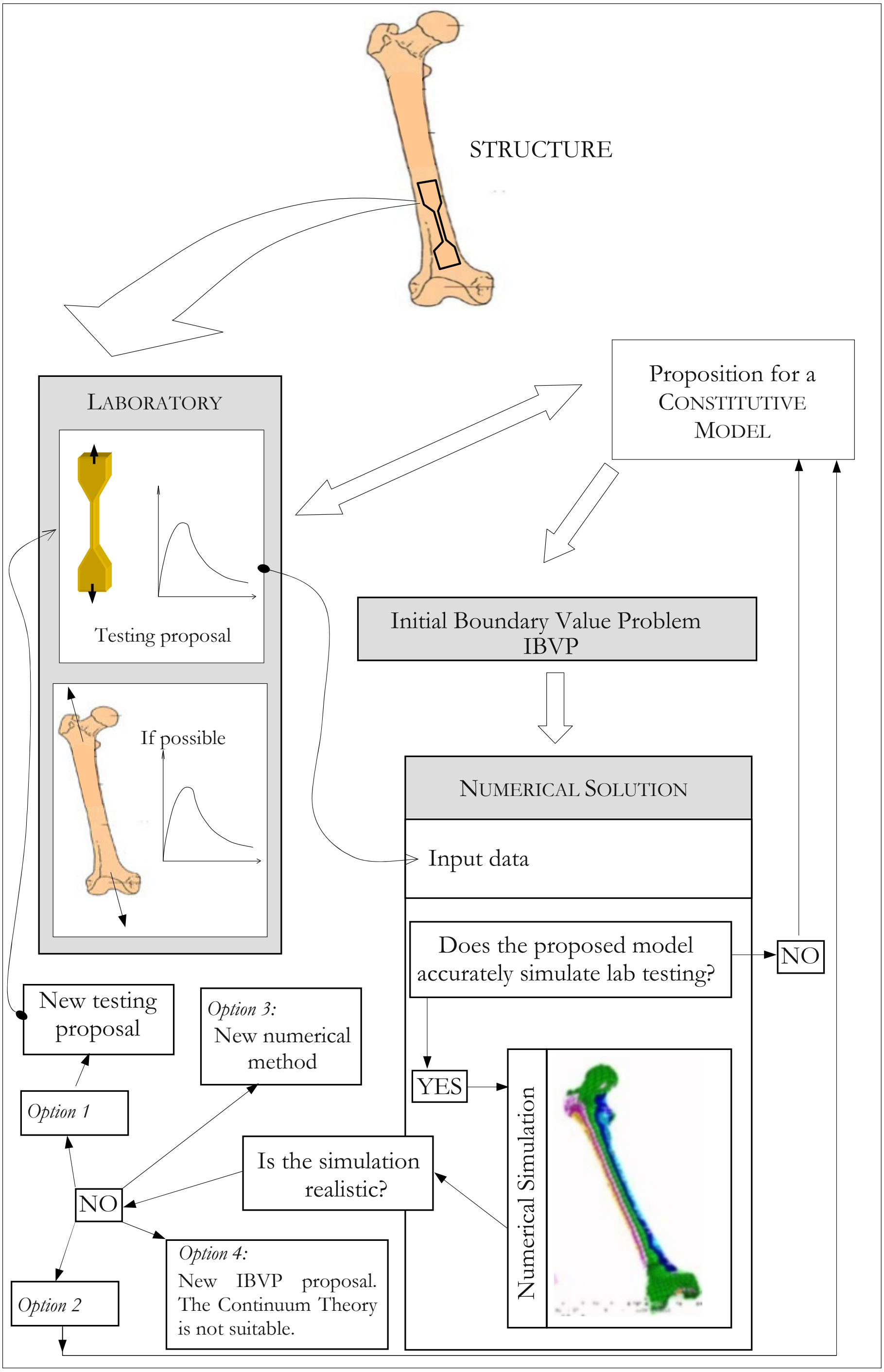


Figure 5.18: Role of the constitutive model in Computational Mechanics.

6 Introduction to Constitutive Equations

6.1 Introduction

Mathematically, the purpose of constitutive equations is to establish connections between kinematic, thermal and mechanical variables. As an example, Figure 6.1 shows the stress-strain relationship for a mechanical problem which represents the constitutive equation for stress. In this case, constitutive equations should be understood as being bijective relationships between stress and strain. Physically speaking, constitutive equations represent different ways of idealizing the response of a material.

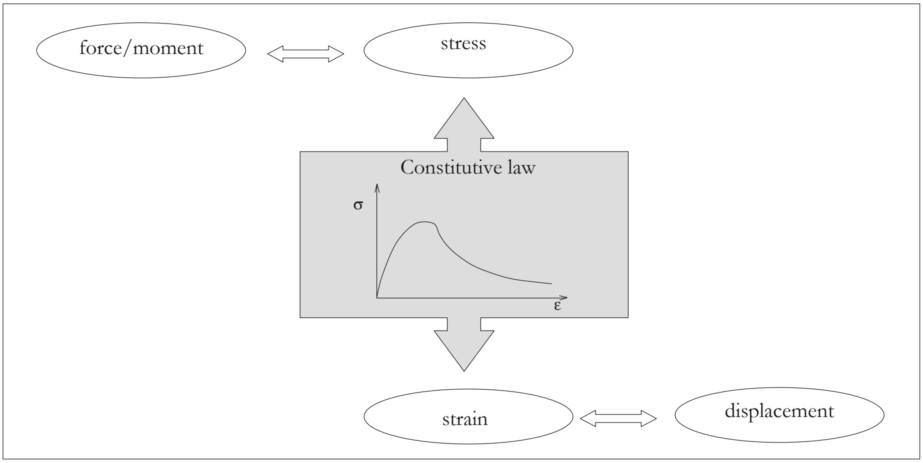


Figure 6.1: Stress-strain relationship (constitutive equation for stress).

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Next, we will summarize the fundamental equations of continuum mechanics obtained in Chapter 5:

The Fundamental Equations of Continuum Mechanics (Current configuration)		
The Mass Continuity Equation (The principle of conservation of mass)	$\frac{D\rho}{Dt} + \rho(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0$	(6.1)
The Equations of Motion (The principle of conservation of linear momentum)	$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\dot{\vec{\boldsymbol{v}}}$	(6.2)
Cauchy Stress Tensor symmetry (The principle of conservation of angular momentum)	$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$	(6.3)
The Energy Equation (The principle of conservation of energy)	$\rho\,\dot{u} = \boldsymbol{\sigma} : \mathbf{D} - \nabla_{\vec{x}} \cdot \vec{\mathbf{q}} + \rho r$	(6.4)
The Entropy Inequality (The principle of irreversibility)	$\rho\dot{\eta}(\vec{\boldsymbol{x}},t) + \frac{1}{T}\boldsymbol{\sigma} : \mathbf{D} - \frac{1}{T}\rho\,\dot{u} - \frac{1}{T^2}\vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \geq 0$	(6.5)

The Fundamental Equations of Continuum Mechanics (Reference Configuration)		
The Mass Continuity Equation	$\frac{D}{Dt}(J\rho) = 0$	(6.6)
The Equations of Motion	$\nabla_{\vec{X}} \cdot \mathbf{P} + \rho_0\vec{\mathbf{b}}_0 = \rho_0\dot{\vec{\boldsymbol{V}}}$ $\nabla_{\vec{X}} \cdot (\boldsymbol{F} \cdot \mathbf{S}) + \rho_0\vec{\mathbf{b}}_0 = \rho_0\dot{\vec{\boldsymbol{V}}}$	(6.7)
Second Piola-Kirchhoff Stress Tensor symmetry	$\mathbf{S} = \mathbf{S}^T$ or $\mathbf{P} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \mathbf{P}^T$	(6.8)
The Energy Equation	$\rho_0\dot{u}(\vec{\boldsymbol{X}},t) = \mathbf{S} : \dot{\boldsymbol{E}} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r(\vec{\boldsymbol{X}},t)$ or $\rho_0\dot{u}(\vec{\boldsymbol{X}},t) = \mathbf{P} : \dot{\boldsymbol{F}} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r(\vec{\boldsymbol{X}},t)$	(6.9)
The Entropy Inequality	$\rho_0\dot{\eta} + \frac{1}{T}\mathbf{S} : \dot{\boldsymbol{E}} - \frac{1}{T}\rho_0\dot{u} - \frac{1}{T^2}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0$ or $\rho_0\dot{\eta} + \frac{1}{T}\mathbf{P} : \dot{\boldsymbol{F}} - \frac{1}{T}\rho_0\dot{u} - \frac{1}{T^2}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0$	(6.10)

As we saw in Chapter 5, the mass continuity equation, the equations of motion and the energy equation give us in total 5 equations. The unknowns are: three components of velocity $\vec{\boldsymbol{v}}$, temperature T, mass density ρ, six components of the Cauchy stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, the specific internal energy u, three components of the heat flux vector $\vec{\mathbf{q}}$, and the entropy η, making a total of 16 unknowns.

For the problem to be well-posed eleven equations must be added which are ones that connect stress (the constitutive equation for stress), heat (the constitutive equation for heat conduction), energy, and entropy with other fields, *i.e.:*

The Constitutive Equations		
The constitutive equation for stress	The relationship between stress and state variables	(6.11)
The constitutive equation for heat conduction	The relationship between heat flux and state variables	(6.12)
The equations of state	These relate energy and/or entropy with state variables	(6.13)

where the constitutive equations for stress provide six equations, the heat conduction law provides three equations, and thermodynamic state laws provide two equations, which results in a well-posed system, with 16 unknowns and 16 equations.

The equations that relate state functions to state variables are called the *equations of state* or *constitutive equations* and state variables, the selection of which depends on the problem in hand, are those that depend only on themselves. For instance, when we are dealing with solids, in general, we use strain and temperature as independent variables. In this scenario, in thermodynamics, internal energy u (or Helmholtz free energy ψ), entropy η, heat flux $\vec{\mathbf{q}}$, and the Cauchy stress tensor $\boldsymbol{\sigma}$ are all considered to be state functions, which can be determined by the state variables. Then, how a material responds is fully defined by the fields (ψ, $\boldsymbol{\sigma}$, η and $\vec{\mathbf{q}}$). Depending on the problem it may be more appropriate to use other thermodynamic potentials among: $u(\boldsymbol{E},\eta)$ *-specific internal energy*, $\psi(\boldsymbol{E},T)$ *-Helmholtz free energy*, $\mathsf{H}(\mathbf{S},T)$ *-enthalpy* or $\mathsf{G}(\mathbf{S},T)$ *-Gibbs-free energy*. For further details concerning these potentials see Chapter 10. On a final note, the effects caused by an electromagnetic or chemical change will not be considered here.

NOTE: It must be emphasized that as constitutive equations describe material constitutions of systems from a macroscopic point of view, based on experimental evidence, then, constitutive equations are, by their nature, approximations. ■

6.2 The Constitutive Principles

Due to the complexity presented by constitutive equation formulation, it may be helpful to lay down certain principles (restrictions) when defining said constitutive equations, Chadwick(1976), Gurtin(1963), Truesdell&Noll(1965). These principles include:

- The Principle of Determinism;
- The Principle of Local Action;
- The Principle of Equipresence;
- The Principle of Objectivity;
- The Principle of Dissipation.

6.2.1 The Principle of Determinism

This principle states that the fields (ψ, $\boldsymbol{\sigma}$, η, $\vec{\mathbf{q}}$) at a material point ($\vec{X}$) depend on the entire history of the motion, $\vec{x}(\vec{X},t)$, and the temperature history, $T(\vec{X},t)$, *i.e.*, up until the present time t, but they never depend on the future values of ($\vec{x}$, T).

In certain types of thermodynamic processes, it may be unrealistic for the current field values (ψ, $\boldsymbol{\sigma}$, η, $\vec{\mathbf{q}}$) to depend on values that are too distant from the current ones, hence, we have the *principle of limited memory*. The history of fields that are too far removed from the current ones do not affect them. So, we must only consider the latest values of the fields, which implies we need to define what is meant by recent time.

6.2.2 The Principle of Local Action

This principle states that the current values of the fields (ψ, $\boldsymbol{\sigma}$, η, $\vec{\mathbf{q}}$) at a material point depend on the state of these field in the vicinity of the point. Motion information is given locally by the deformation gradient $\boldsymbol{F}(\vec{X},t)$, and temperature by its gradient ∇T.

Then, the materials that satisfy the determinism and local action principles are called *simple thermoelastic materials*.

6.2.3 The Principle of Equipresence

This principle states that: there is no reason to exclude a priori a state variable (independent variable) of the constitutive equations. For example, if we have a simple thermoelastic material it makes no sense if stress is only a function of the deformation gradient while the heat flux vector is only a function of temperature.

6.2.4 The Principle of Objectivity

This principle states that the constitutive equations must be the same for any observer. Therefore, any constitutive equation must satisfy the principle of objectivity, (see Chapter 4). That is, if an observer detects a stress state in the body $\mathcal{B}$ which undergoes a rigid body motion, then, this observer must detect the same stress state in the body $\mathcal{B}^*$, (see Figure 6.2).

6.2.5 The Principle of Dissipation

This principle states that: constitutive equations must satisfy the entropy inequality for any admissible thermodynamic process.

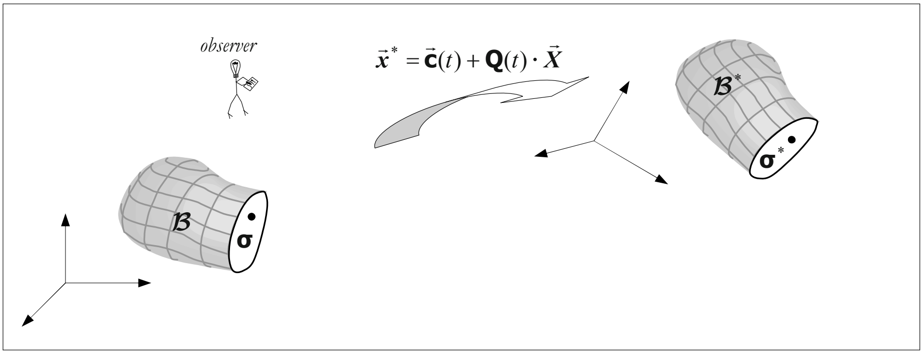


Figure 6.2: Rigid body motion.

6.3 Characterization of Constitutive Equations for Simple Thermoelastic Materials

For a simple thermoelastic material, the state variables (independent variables) are the deformation gradient $\boldsymbol{F}(\vec{X},t)$, temperature T, and the temperature gradient $\nabla_{\vec{X}}T$. We assume that ψ, η, $\vec{\mathbf{q}}_0$ and $\mathbf{P}$ (reference configuration) have been established by the history and current values of $\boldsymbol{F}$, T and $\nabla_{\vec{X}}T$. These quantities are expressed by means of the following set of functionals:

$$\begin{aligned}
\psi(t) &= \hat{\psi}(\vec{X},\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})\\
\eta(t) &= \hat{\eta}(\vec{X},\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})\\
\vec{\mathbf{q}}_0(t) &= \hat{\vec{\mathbf{q}}}_0(\vec{X},\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})\\
\mathbf{P}(t) &= \hat{\mathbf{P}}(\vec{X},\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})
\end{aligned} \tag{6.14}$$

where $\bullet^{(\tau)}$ represents the history of $\bullet$, until the present time t, $\tau\le t$. We can also verify that $\hat{\psi}$ and $\hat{\eta}$ are scalar-valued functionals, $\hat{\vec{\mathbf{q}}}$ is a vector-valued functional and $\hat{\mathbf{P}}$ is a second-order-valued functional. Then, taking into account the principle of dissipation, the Clausius-Duhem inequality must be satisfied for any thermodynamic process.

For a homogeneous simple thermoelastic material, the functionals described in (6.14) are independent of $\vec{X}$:

$$\begin{aligned}
\psi(t) &= \hat{\psi}(\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})\\
\eta(t) &= \hat{\eta}(\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})\\
\vec{\mathbf{q}}_0(t) &= \hat{\vec{\mathbf{q}}}_0(\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})\\
\mathbf{P}(t) &= \hat{\mathbf{P}}(\boldsymbol{F}^{(\tau)},T^{(\tau)},\nabla_{\vec{X}}T^{(\tau)})
\end{aligned} \tag{6.15}$$

NOTE: The functions with hat $\hat{\bullet}$ (functional) are distinct from those that are on the left of the equation, *i.e.*, $\hat{\bullet}$ provides the current value of $\bullet(t)$ taking into account the entire history of the arguments of $\hat{\bullet}$. ■

According to the principle of objectivity, the constitutive equations must be invariant under rigid body motion. Then the constitutive equations must satisfy the following:

$$\begin{aligned}
&\psi(t) = \hat{\psi}(\boldsymbol{F}^{(\tau)^*}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)^*}) \\
&\eta(t) = \hat{\eta}(\boldsymbol{F}^{(\tau)^*}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)^*}) \\
&\vec{\mathbf{q}}_0^*(t) = \hat{\vec{\mathbf{q}}}_0(\boldsymbol{F}^{(\tau)^*}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)^*}) \\
&\mathbf{P}^*(t) = \hat{\mathbf{P}}(\boldsymbol{F}^{(\tau)^*}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)^*})
\end{aligned} \tag{6.16}$$

where $\bullet^*$ represents the tensor in a new system which undergoes an orthogonal transformation, (see Chapter 4- The objectivity of tensors).

Then, by using the chain rule of derivative of the Helmholtz free energy (6.14) we obtain:

$$\begin{aligned}
&\psi = \psi(\boldsymbol{F}, T, \nabla T) \\
&\Rightarrow \dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial \nabla_{\vec{X}} T} \cdot \overset{\bullet}{\nabla}_{\vec{X}} T
\end{aligned} \tag{6.17}$$

Additionally, the alternative form of the Clausius-Duhem inequality (entropy inequality) in the reference configuration was obtained in Chapter 5 as:

$$\mathbf{P} : \dot{\boldsymbol{F}} - \rho_0 \left[\dot{\psi} + \dot{T}\eta\right] - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{6.18}$$

where ψ is the Helmholtz free energy per unit mass, η is the entropy per unit mass, and $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor. Note that, although the stress power $\mathbf{P} : \dot{\boldsymbol{F}}$ is defined in the reference configuration, neither $\mathbf{P}$ nor $\boldsymbol{F}$ are in the reference configuration.

Then, by combining the equation in (6.17) with the entropy inequality in (6.18) we obtain:

$$\begin{aligned}
&\mathbf{P} : \dot{\boldsymbol{F}} - \rho_0 \left[\frac{\partial \psi}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial \nabla T} \cdot \overset{\bullet}{\nabla T} + \dot{T}\eta\right] - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \\
&\Rightarrow \left[\mathbf{P} - \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \rho_0 \left[\frac{\partial \psi}{\partial T} + \eta\right]\dot{T} - \rho_0 \frac{\partial \psi}{\partial \nabla T} \cdot \overset{\bullet}{\nabla T} - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0
\end{aligned} \tag{6.19}$$

The above inequality must be satisfied for any admissible thermodynamic process.

Let us now consider the process such that $\dot{\boldsymbol{F}} = \mathbf{0}$, and a system with uniform temperature, thus $\nabla_{\vec{X}} T = \vec{\mathbf{0}}$, $\overset{\bullet}{\nabla}_{\vec{X}} T = \vec{\mathbf{0}}$. In this particular thermodynamic process the inequality in (6.19) becomes:

$$-\rho_0 \left[\frac{\partial \psi}{\partial T} + \eta\right]\dot{T} \geq 0 \tag{6.20}$$

Note that the inequality in (6.20) must also be met for any thermodynamic process. Then, if in the current process the condition in (6.20) is met, we can then apply another process such that $\dot{T} \to -\dot{T}$, in which the entropy inequality is violated. Thus, the only way in which the inequality in (6.20) is satisfied is when:

$$\frac{\partial \psi}{\partial T} + \eta = 0 \qquad \Rightarrow \qquad \eta = -\frac{\partial \psi}{\partial T} \tag{6.21}$$

Then if we take into account (6.21), the inequality in (6.19) becomes:

$$\left[\mathbf{P}-\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}}\right]:\dot{\boldsymbol{F}}-\rho_0\frac{\partial\psi}{\partial\nabla T}\cdot\dot{\overline{\nabla T}}-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0 \tag{6.22}$$

Now let us consider a process where $\dot{\boldsymbol{F}}=\boldsymbol{0}$, with which the inequality in (6.22) becomes:

$$-\rho_0\frac{\partial\psi}{\partial\nabla T}\cdot\dot{\overline{\nabla T}}-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0 \tag{6.23}$$

Note that the term $\frac{-1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0$ is always true, since the heat flux vector ($\vec{\mathbf{q}}_0$) is always opposite to the temperature gradient ($\nabla_{\vec{X}}T$). If $\frac{\partial\psi}{\partial\nabla_{\vec{X}}T}\neq 0$ we have an inconsistency, *i.e.* we can use $\dot{\overline{\nabla_{\vec{X}}T}}$ in such a way that the condition in (6.23) is violated, with which we can conclude that ψ should not depend on the temperature gradient, *i.e.* $\psi=\psi(\boldsymbol{F},T)$. Then the entropy inequality in (6.22) becomes:

$$\left[\mathbf{P}-\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}}\right]:\dot{\boldsymbol{F}}-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0 \tag{6.24}$$

Now let us consider a process where $\nabla_{\vec{X}}T=\vec{\boldsymbol{0}}$ (a uniform temperature field), then the inequality in (6.24) becomes:

$$\left[\mathbf{P}-\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}}\right]:\dot{\boldsymbol{F}}\geq 0 \tag{6.25}$$

Starting from this point, we could apply another process where $\dot{\boldsymbol{F}}\to-\dot{\boldsymbol{F}}$, thus:

$$-\left[\mathbf{P}-\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}}\right]:\dot{\boldsymbol{F}}\geq 0 \tag{6.26}$$

Then, the only way that the two equations (6.25) and (6.26) can remain valid is when:

$$\mathbf{P}-\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}}=\boldsymbol{0}\quad\Rightarrow\quad\mathbf{P}=\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}} \tag{6.27}$$

Afterwards, we can conclude that the constitutive equations for a simple thermoelastic material are given by:

$$\boxed{\begin{aligned}&\psi=\psi(\boldsymbol{F},T)\\&\mathbf{P}(\boldsymbol{F},T)=\rho_0\frac{\partial\psi(\boldsymbol{F},T)}{\partial\boldsymbol{F}}\\&\eta(\boldsymbol{F},T)=-\frac{\partial\psi(\boldsymbol{F},T)}{\partial T}\\&\vec{\mathbf{q}}_0=\vec{\mathbf{q}}_0(\boldsymbol{F},T,\nabla_{\vec{X}}T)\end{aligned}}\quad\textit{Constitutive equations for a simple thermoelastic material} \tag{6.28}$$

As we state before, the constitutive equations must satisfy the principle of objectivity and any scalar, for instance energy and entropy, satisfies this principle. However, we can take advantage of this principle in order to express these scalars in terms of other appropriate parameters. Then by applying the principle of objectivity to the energy we have:

$$\begin{aligned}\psi^*&=\psi(\boldsymbol{F}^*,T^*)\\&=\psi(\mathbf{Q}\cdot\boldsymbol{F},T)\end{aligned} \tag{6.29}$$

where we have taken into account that $\boldsymbol{F}^* = \mathbf{Q} \cdot \boldsymbol{F}$ and $T^* = T$ (see Chapter 4). Then as the principle of objectivity must be met for any orthogonal tensor $\mathbf{Q}$, we use the transpose rotation tensor ($\mathbf{Q} = \mathbf{R}^T$) of the polar decomposition ($\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U}$) as the orthogonal tensor, (see Figure 6.3), with which we obtain:

$$
\begin{aligned}
\psi &= \psi(\mathbf{Q} \cdot \boldsymbol{F}, T) \\
&= \psi(\mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U}, T) \\
&= \psi(\mathbf{U}, T)
\end{aligned}
\tag{6.30}
$$

That is, to satisfy the principle of objectivity, the energy must be a function of the right stretch tensor. Then, if we take into account the equations $\boldsymbol{C} = \mathbf{U}^2$ and $\boldsymbol{C} = 2\boldsymbol{E} + \mathbf{1}$, (see Chapter 2), we can still express the energy in terms of $\boldsymbol{C}$ or $\boldsymbol{E}$, *i.e.*:

$$
\overline{\psi} = \overline{\psi}(\boldsymbol{C}, T) \quad ; \quad \hat{\psi} = \hat{\psi}(\boldsymbol{E}, T)
\tag{6.31}
$$

Then, for the entropy we have:

$$
\begin{aligned}
\eta(\boldsymbol{F}^*, T^*) = \eta^*(\boldsymbol{F}, T) = -\frac{\partial \psi^*(\boldsymbol{F}, T)}{\partial T} \quad &= -\frac{\partial \psi(\boldsymbol{C}, T)}{\partial T} = \eta(\boldsymbol{C}, T) = \overline{\eta} \\
&= -\frac{\partial \psi(\boldsymbol{E}, T)}{\partial T} = \eta(\boldsymbol{E}, T) = \hat{\eta}
\end{aligned}
\tag{6.32}
$$

Likewise, the heat flux can be expressed as:

$$
\vec{\mathbf{q}}_0^* = \vec{\mathbf{q}}_0(\boldsymbol{F}^*, T^*, \nabla_{\vec{X}} T^*) \quad \Rightarrow \quad \vec{\mathbf{q}}_0(\boldsymbol{C}, T, \nabla_{\vec{X}} T) = \overline{\vec{\mathbf{q}}}_0 \quad or \quad \vec{\mathbf{q}}_0(\boldsymbol{E}, T, \nabla_{\vec{X}} T) = \hat{\vec{\mathbf{q}}}_0
\tag{6.33}
$$

To avoid excessive symbolism, we omit the symbols at the top part of the tensor.

As we saw in Chapter 3, $\mathbf{P}$ is related to the second Piola-Kirchhoff stress tensor $\mathbf{S}$ (reference configuration) by means of the equation $\mathbf{S} = \boldsymbol{F}^{-1} \cdot \mathbf{P}$, then, by taking into account (6.28) we can conclude that:

$$
\begin{aligned}
\mathsf{S}_{ij} &= F_{ik}^{-1} \mathsf{P}_{kj} = F_{ik}^{-1} \rho_0 \frac{\partial \psi(\boldsymbol{F}, T)}{\partial F_{kj}} = F_{ik}^{-1} \rho_0 \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{pq}} \frac{\partial C_{pq}}{\partial F_{kj}} \\
&= \rho_0 F_{ik}^{-1} \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{pq}} \frac{\partial (F_{rp} F_{rq})}{\partial F_{kj}} = \rho_0 F_{ik}^{-1} \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{pq}} \left[\frac{\partial F_{rp}}{\partial F_{kj}} F_{rq} + F_{rp} \frac{\partial F_{rq}}{\partial F_{kj}} \right] \\
&= \rho_0 F_{ik}^{-1} \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{pq}} \left[\delta_{rk} \delta_{pj} F_{rq} + F_{rp} \delta_{rk} \delta_{qj} \right] \\
&= \rho_0 \left[F_{ik}^{-1} \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{jq}} F_{kq} + F_{ik}^{-1} \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{pj}} F_{kp} \right] \\
&= \rho_0 \left[\delta_{iq} \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{jq}} + \delta_{ip} \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{pj}} \right] = \rho_0 \left[\frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{ji}} + \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{ij}} \right] \\
&= 2\rho_0 \frac{\partial \psi(\boldsymbol{C}, T)}{\partial C_{ij}}
\end{aligned}
\tag{6.34}
$$

Likewise, it is possible to show that the following is also true:

$$
\mathbf{S} = 2\rho_0 \frac{\partial \psi(\boldsymbol{C}, T)}{\partial \boldsymbol{C}} = \rho_0 \frac{\partial \psi(\boldsymbol{E}, T)}{\partial \boldsymbol{E}}
\tag{6.35}
$$

Thus, the constitutive equations can be expressed in the reference configuration as follows:

$$\boxed{\begin{aligned} &\hat{\psi} = \psi(\boldsymbol{E},T) \\ &\mathbf{S} = \rho_0 \frac{\partial \psi(\boldsymbol{E},T)}{\partial \boldsymbol{E}} \\ &\eta(\boldsymbol{E},T) = -\frac{\partial \psi(\boldsymbol{E},T)}{\partial T} \\ &\hat{\vec{\mathbf{q}}}_0 = \vec{\mathbf{q}}_0(\boldsymbol{E},T,\nabla_{\vec{X}} T) \end{aligned}} \quad \begin{aligned} &\textit{Constitutive equations for a simple} \\ &\textit{thermoelastic material} \\ &\textit{(Reference configuration)} \end{aligned} \tag{6.36}$$

It is also possible to express the constitutive equations in the current configuration (deformed). To do this, let us consider the relationship between the first Piola-Kirchhoff stress tensor ($\mathbf{P}$) and the Cauchy stress tensor, *i.e.* $\boldsymbol{\sigma} = \frac{1}{J}\mathbf{P}\cdot\boldsymbol{F}^T$ with which the constitutive equation for stress given in (6.28) can be rewritten as follows:

$$\begin{aligned} &\mathbf{P} = \rho_0 \frac{\partial \psi(\boldsymbol{F},T)}{\partial \boldsymbol{F}} \\ &\Rightarrow \frac{1}{J}\mathbf{P}\cdot\boldsymbol{F}^T = \frac{1}{J}\rho_0 \frac{\partial \psi(\boldsymbol{F},T)}{\partial \boldsymbol{F}}\cdot\boldsymbol{F}^T = \frac{\rho}{\rho_0}\rho_0 \frac{\partial \psi(\boldsymbol{F},T)}{\partial \boldsymbol{F}}\cdot\boldsymbol{F}^T \\ &\Rightarrow \boldsymbol{\sigma} = \rho \frac{\partial \psi(\boldsymbol{F},T)}{\partial \boldsymbol{F}}\cdot\boldsymbol{F}^T \end{aligned} \tag{6.37}$$

It is also true that $\vec{\mathbf{q}}_0 = J\vec{\mathbf{q}}\cdot\boldsymbol{F}^{-T} \Leftrightarrow \vec{\mathbf{q}} = J^{-1}\vec{\mathbf{q}}_0\cdot\boldsymbol{F}^T$, (see Chapter 2). Hence we can express the constitutive equations in the current configuration as:

$$\boxed{\begin{aligned} &\psi = \psi(\boldsymbol{F},T) \\ &\boldsymbol{\sigma} = \rho \frac{\partial \psi(\boldsymbol{F},T)}{\partial \boldsymbol{F}}\cdot\boldsymbol{F}^T \\ &\eta(\boldsymbol{F},T) = -\frac{\partial \psi(\boldsymbol{F},T)}{\partial T} \\ &\vec{\mathbf{q}} = J^{-1}\vec{\mathbf{q}}_0(\boldsymbol{F},T,\nabla_{\vec{X}} T)\cdot\boldsymbol{F}^T \\ &\quad = J^{-1}\boldsymbol{F}\cdot\vec{\mathbf{q}}_0(\boldsymbol{F},T,\nabla_{\vec{X}} T) \end{aligned}} \quad \begin{aligned} &\textit{Constitutive equations for a simple} \\ &\textit{thermoelastic material} \\ &\textit{(Current configuration)} \end{aligned} \tag{6.38}$$

Then, due to the principle of objectivity, the Helmholtz free energy can be written as a function of $\boldsymbol{C}$, *i.e.* $\psi(\boldsymbol{C},T)$. Additionally, the constitutive equation for stress $\boldsymbol{\sigma}_{ij} = \rho \frac{\partial \psi(\boldsymbol{F},T)}{\partial F_{ik}} F_{jk}$ can still be rewritten as:

$$\begin{aligned} \boldsymbol{\sigma}_{ij} &= \rho \frac{\partial \psi(\boldsymbol{F}(\boldsymbol{C}),T)}{\partial F_{ik}} F_{jk} \\ &= \rho \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} \frac{\partial C_{pq}}{\partial F_{ik}} F_{jk} \end{aligned} \tag{6.39}$$

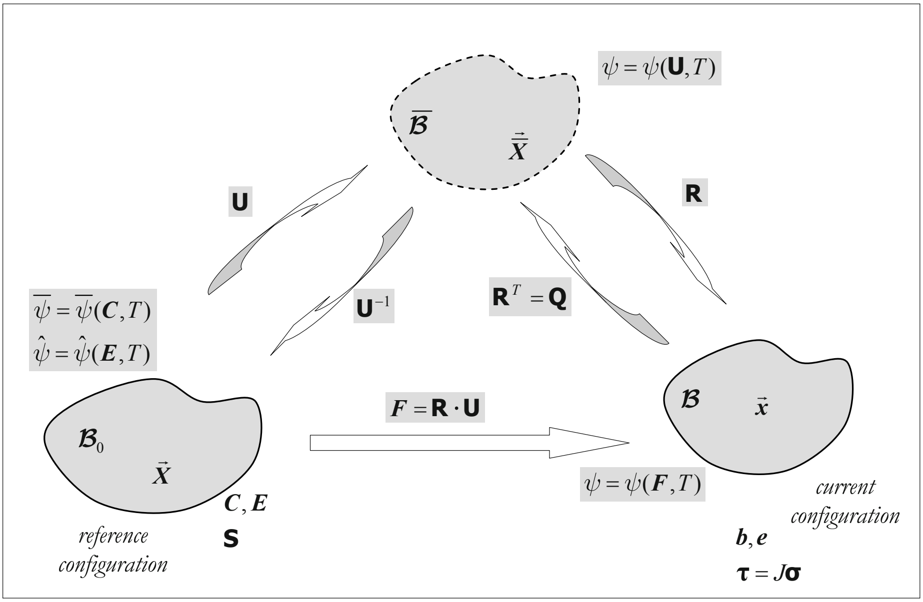


Figure 6.3: Right polar decomposition of the deformation gradient.

Then, by applying the definition of the right Cauchy-Green deformation tensor, $C_{pq} = F_{rp}F_{rq}$, the following is still valid:

$$
\begin{aligned}
\boldsymbol{\sigma}_{ij} &= \rho \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} \frac{\partial C_{pq}}{\partial F_{ik}} F_{jk} = \rho \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} \frac{\partial (F_{rp}F_{rq})}{\partial F_{ik}} F_{jk} \\
&= \rho \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} \left[\frac{\partial F_{rp}}{\partial F_{ik}} F_{rq} + F_{rp} \frac{\partial F_{rq}}{\partial F_{ik}} \right] F_{jk} \\
&= \rho \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} \left[\delta_{ri}\delta_{pk} F_{rq} F_{jk} + F_{rp} \delta_{ri} \delta_{qk} F_{jk} \right] \\
&= \rho \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} \left[F_{iq} F_{jp} + F_{ip} F_{jq} \right] \\
&= \rho F_{iq} \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} F_{jp} + \rho F_{ip} \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} F_{jq} \\
&= \rho F_{iq} \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{qp}} F_{jp} + \rho F_{ip} \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{pq}} F_{jq} \\
&= 2\rho F_{iq} \frac{\partial \psi(\boldsymbol{C},T)}{\partial C_{qp}} F_{jp}
\end{aligned}
\tag{6.40}
$$

where we have considered the symmetry of $\boldsymbol{C}$, *i.e.* $C_{pq} = C_{qp}$. In tensorial notation the constitutive equation for stress becomes:

$$
\boldsymbol{\sigma} = 2\rho \boldsymbol{F} \cdot \frac{\partial \psi(\boldsymbol{C},T)}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T \tag{6.41}
$$

Then if we take into account that $\rho_0 = J\rho$ in the above equation, we have:

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{\rho_0}{J} 2\boldsymbol{F} \cdot \frac{\partial \psi(\boldsymbol{C},T)}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T = \frac{1}{J} \boldsymbol{F} \cdot 2\rho_0 \frac{\partial \psi(\boldsymbol{C},T)}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T = \frac{1}{J} \boldsymbol{F} \cdot 2 \frac{\partial \Psi(\boldsymbol{C},T)}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T \\ &= \frac{1}{J} \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T\end{aligned} \tag{6.42}$$

where $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, and $\Psi(\boldsymbol{C},T)$ is the strain energy per unit reference volume.

The constitutive equation for heat conduction can also be expressed as:

$$\begin{aligned}\vec{\mathbf{q}}^* &= J^{-1} \vec{\mathbf{q}}_0^* (\boldsymbol{F}^*, T^*, \nabla_{\vec{X}} T^*) \cdot \boldsymbol{F}^{*T} \\ &= J^{-1} \mathbf{Q} \cdot \vec{\mathbf{q}}_0 (\mathbf{Q} \cdot \boldsymbol{F}, T, \nabla_{\vec{X}} T) \cdot [\mathbf{Q} \cdot \boldsymbol{F}]^T \\ &= J^{-1} \mathbf{Q} \cdot \vec{\mathbf{q}}_0 (\mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{U}, T, \nabla_{\vec{X}} T) \cdot [\mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{U}]^T\end{aligned} \tag{6.43}$$

Then by using $\mathbf{Q} = \mathbf{R}^T$, and by considering the symmetry of $\mathbf{U} = \mathbf{U}^T$, the above equation becomes:

$$\begin{aligned}\vec{\mathbf{q}}^* &= J^{-1} \mathbf{Q} \cdot \vec{\mathbf{q}}_0 (\mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{U}, T, \nabla_{\vec{X}} T) \cdot [\mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{U}]^T \\ &= J^{-1} \mathbf{R}^T \cdot \vec{\mathbf{q}}_0 (\mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U}, T, \nabla_{\vec{X}} T) \cdot [\mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{U}]^T \\ &= J^{-1} \mathbf{R}^T \cdot \vec{\mathbf{q}}_0 (\mathbf{U}, T, \nabla_{\vec{X}} T) \cdot \mathbf{U} \\ &= J^{-1} \vec{\mathbf{q}}_0 (\mathbf{U}, T, \nabla_{\vec{X}} T) \cdot \mathbf{R} \cdot \mathbf{U} \\ &= J^{-1} \vec{\mathbf{q}}_0 (\mathbf{U}, T, \nabla_{\vec{X}} T) \cdot \boldsymbol{F}\end{aligned} \tag{6.44}$$

After that, so as to satisfy the principle of objectivity, the constitutive equations can be expressed as:

$$\boxed{\begin{aligned}&\overline{\psi} = \psi(\boldsymbol{C},T) \\ &\boldsymbol{\sigma} = 2\rho \boldsymbol{F} \cdot \frac{\partial \psi(\boldsymbol{C},T)}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T \\ &\eta(\boldsymbol{C},T) = -\frac{\partial \psi(\boldsymbol{C},T)}{\partial T} \\ &\vec{\mathbf{q}} = J^{-1} \vec{\mathbf{q}}_0 (\mathbf{U}, T, \nabla_{\vec{X}} T) \cdot \boldsymbol{F} = J^{-1} \vec{\overline{\mathbf{q}}}_0 (\boldsymbol{C}, T, \nabla_{\vec{X}} T) \cdot \boldsymbol{F}\end{aligned}} \tag{6.45}$$

Constitutive equations for a simple thermoelastic material

(Current configuration)

6.4 Characterization of the Constitutive Equations for a Thermoviscoelastic Material

Let us consider a material, (see Romano *et al.* (2006)), which has the behavioral characteristics:

- The stress state depends on the local deformations ($\boldsymbol{F}$) and temperature (T);
- Phenomenon of energy dissipation (due to the internal friction) appears when one part of the system has a relative shearing motion with respect to other part of the

system, (see Romano *et al.* (2006)). In this way, the material response depends on the spatial velocity gradient ($\nabla_{\vec{x}}\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) \equiv \boldsymbol{\ell} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$).

Now we can observe that the functionals will also depend on the history of $\dot{\boldsymbol{F}}$:

$$
\begin{aligned}
&\psi(t) = \hat{\psi}(\boldsymbol{F}^{(\tau)}, \dot{\boldsymbol{F}}^{(\tau)}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)}) \\
&\eta(t) = \hat{\eta}(\boldsymbol{F}^{(\tau)}, \dot{\boldsymbol{F}}^{(\tau)}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)}) \\
&\vec{\mathbf{q}}_0(t) = \hat{\vec{\mathbf{q}}}_0(\boldsymbol{F}^{(\tau)}, \dot{\boldsymbol{F}}^{(\tau)}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)}) \\
&\mathbf{P}(t) = \hat{\mathbf{P}}(\boldsymbol{F}^{(\tau)}, \dot{\boldsymbol{F}}^{(\tau)}, T^{(\tau)}, \nabla_{\vec{X}} T^{(\tau)})
\end{aligned}
\tag{6.46}
$$

Then, to obtain the constitutive equations, we use alternative proof to that made on a simple thermoelastic material, (see Romano *et al.* (2006)).

Once again we can apply the Clausius-Duhem inequality:

$$
\mathbf{P} : \dot{\boldsymbol{F}} - \rho_0 \left[\dot{\psi} + \dot{T}\eta\right] - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{6.47}
$$

In addition we can calculate the rate of change of the energy $\psi(\boldsymbol{F}, \dot{\boldsymbol{F}}, T, \nabla_{\vec{X}} T)$:

$$
\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} + \frac{\partial \psi}{\partial \dot{\boldsymbol{F}}} : \ddot{\boldsymbol{F}} + \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial \nabla_{\vec{X}} T} \cdot \nabla_{\vec{X}} \overset{\bullet}{T} \tag{6.48}
$$

Then, by combining (6.48) with (6.47) we obtain:

$$
\begin{aligned}
&\left[\mathbf{P} - \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \rho_0 \frac{\partial \psi}{\partial \dot{\boldsymbol{F}}} : \ddot{\boldsymbol{F}} - \rho_0 \left[\frac{\partial \psi}{\partial T} + \eta\right]\dot{T} - \rho_0 \frac{\partial \psi}{\partial \nabla T} \cdot \nabla_{\vec{X}} \overset{\bullet}{T} - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \\
&\textit{or} \\
&\mathrm{Tr}\left\{\left[\mathbf{P} - \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}}\right] \cdot \dot{\boldsymbol{F}}^T\right\} - \rho_0 \frac{\partial \psi}{\partial \dot{\boldsymbol{F}}} : \ddot{\boldsymbol{F}} - \rho_0 \left[\frac{\partial \psi}{\partial T} + \eta\right]\dot{T} - \rho_0 \frac{\partial \psi}{\partial \nabla T} \cdot \nabla_{\vec{X}} \overset{\bullet}{T} - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0
\end{aligned}
\tag{6.49}
$$

where we have verified that given two tensors $\mathbf{A}$ and $\mathbf{B}$, the following holds $\mathbf{A} : \mathbf{B} = \mathrm{Tr}(\mathbf{A} \cdot \mathbf{B}^T)$. Then, we can restructure the above equation as follows:

$$
\{\boldsymbol{a}\}\{\boldsymbol{u}\}^T + b \geq 0 \tag{6.50}
$$

where

$$
\begin{aligned}
&\{\boldsymbol{a}\} = \left\{-\rho_0 \frac{\partial \psi}{\partial \dot{\boldsymbol{F}}}; -\rho_0 \left[\frac{\partial \psi}{\partial T} + \eta\right]; -\rho_0 \frac{\partial \psi}{\partial \nabla_{\vec{X}} T}\right\} \quad ; \quad \{\boldsymbol{u}\} = \left\{\ddot{\boldsymbol{F}}, \dot{T}, \nabla_{\vec{X}} \overset{\bullet}{T}\right\} \\
&b = \mathrm{Tr}\left\{\left[\rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}} - \mathbf{P}\right] \cdot \dot{\boldsymbol{F}}^T\right\} - \frac{1}{T}\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T
\end{aligned}
\tag{6.51}
$$

Since $\{\boldsymbol{a}\}$ and b are independent of $\{\boldsymbol{u}\}$, the inequality in (6.50) holds for any arbitrary value of $\{\boldsymbol{u}\}$, if and only if $\{\boldsymbol{a}\} = \{\mathbf{0}\}$ and $b \geq 0$ with which we can make the conclusion that:

$$
\rho_0 \frac{\partial \psi}{\partial \dot{\boldsymbol{F}}} = \mathbf{0} \quad \text{(The energy does not depend on } \dot{\boldsymbol{F}}\text{)} \tag{6.52}
$$

$$
\rho_0 \left[\frac{\partial \psi}{\partial T} + \eta\right] = 0 \quad \Rightarrow \quad \eta = -\frac{\partial \psi}{\partial T} \quad \text{(The constitutive equation for entropy)}
$$

$\rho_0 \dfrac{\partial \psi}{\partial \nabla_{\vec{X}} T} = \vec{\mathbf{0}}$ (The energy is not a function of the temperature gradient $\nabla_{\vec{X}} T$)

Then, if we take into account the considerations in (6.52), we can rewrite the Clausius-Duhem inequality as:

$$\left[\mathbf{P} - \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}} \right] : \dot{\boldsymbol{F}} - \frac{1}{T} \vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{6.53}$$

We can now break down the tensor $\mathbf{P}$ into static and dynamic equilibrium parts, *i.e.* $\mathbf{P} = \mathbf{P}^{(e)} + \mathbf{P}^{(d)}$ with which the above inequality becomes:

$$\begin{aligned} &\left[(\mathbf{P}^{(e)} + \mathbf{P}^{(d)}) - \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}} \right] : \dot{\boldsymbol{F}} - \frac{1}{T} \vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \\ &\left[\mathbf{P}^{(e)} - \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}} \right] : \dot{\boldsymbol{F}} + \mathbf{P}^{(d)} : \dot{\boldsymbol{F}} - \frac{1}{T} \vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \end{aligned} \tag{6.54}$$

Note that the above inequality must satisfy:

$$\mathbf{P}^{(e)} = \mathbf{P}^{(e)}(\boldsymbol{F}, \dot{\boldsymbol{F}} = \mathbf{0}, T, \nabla_{\vec{X}} T = \vec{\mathbf{0}}) = \rho_0 \frac{\partial \psi(\boldsymbol{F}, T)}{\partial \boldsymbol{F}} \tag{6.55}$$

Thus, we can rewrite the Clausius-Duhem inequality as:

$$\mathbf{P}^{(d)} : \dot{\boldsymbol{F}} - \frac{1}{T} \vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{6.56}$$

which must be met for any thermodynamic process. Note that $\mathbf{P}^{(d)}$ and $\vec{\mathbf{q}}_0$ cause the energy dissipation in the system. In this way, we can summarize all the constitutive equations for thermoviscoelastic materials as follows:

$$\begin{aligned} &\psi = \psi(\boldsymbol{F}, T) \\ &\eta = -\frac{\partial \psi(\boldsymbol{F}, T)}{\partial T} \\ &\mathbf{P}^{(e)} = \rho_0 \frac{\partial \psi(\boldsymbol{F}, T)}{\partial \boldsymbol{F}} \\ &\mathbf{P}^{(d)} : \dot{\boldsymbol{F}} - \frac{1}{T} \vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \end{aligned} \quad \textit{Constitutive equations for a thermoviscoelastic material and thermodynamic constraints (Reference configuration)} \tag{6.57}$$

Note that $\mathbf{P}^{(d)}$ is a function of $\mathbf{P}^{(d)}(\boldsymbol{F}, \dot{\boldsymbol{F}}, T, \nabla_{\vec{X}} T)$, and if we observe that $\dot{\boldsymbol{F}} = \boldsymbol{\ell} \cdot \boldsymbol{F} = (\mathbf{D} + \mathbf{W}) \cdot \boldsymbol{F}$, we can state that $\mathbf{P}^{(d)}$ is a function of $\mathbf{P}^{(d)}(\boldsymbol{F}, \mathbf{D}, \mathbf{W}, T, \nabla_{\vec{X}} T)$. Then, by applying the principle of objectivity we obtain:

$$\begin{aligned} &\mathbf{P}^{(d)}(\boldsymbol{F}^*, \mathbf{D}^*, \mathbf{W}^*, T, \nabla_{\vec{X}} T) = \mathbf{Q} \cdot \mathbf{P}^{(d)}(\boldsymbol{F}, \mathbf{D}, \mathbf{W}, T, \nabla_{\vec{X}} T) \cdot \mathbf{Q}^T \\ &\Rightarrow \mathbf{P}^{(d)}(\boldsymbol{F}, \mathbf{D}, \mathbf{W}, T, \nabla_{\vec{X}} T) = \mathbf{Q}^T \cdot \mathbf{P}^{(d)}(\boldsymbol{F}^*, \mathbf{D}^*, \mathbf{W}^*, T, \nabla_{\vec{X}} T) \cdot \mathbf{Q} \\ &\Rightarrow \mathbf{P}^{(d)}(\boldsymbol{F}, \mathbf{D}, \mathbf{W}, T, \nabla_{\vec{X}} T) = \mathbf{Q}^T \cdot \mathbf{P}^{(d)}(\mathbf{Q} \cdot \boldsymbol{F}, \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T, \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T, T, \nabla_{\vec{X}} T) \cdot \mathbf{Q} \end{aligned} \tag{6.58}$$

where we have considered that $\mathbf{W}^* = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T$, and $\mathbf{D}^* = \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T$, (see Chapter 4).

The equation in (6.58) must satisfy for any orthogonal tensor, including the particular case when $\mathbf{Q} = \mathbf{1}$. In this situation, the term $\mathbf{W}^* = \dot{\mathbf{Q}} \cdot \mathbf{Q}^T + \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T$ becomes $\mathbf{W}^* = \dot{\mathbf{Q}} + \mathbf{W} = -\mathbf{W} + \mathbf{W} = \mathbf{0}$, thus:

$$\mathbf{P}^{(d)}(\boldsymbol{F}, \mathbf{D}, \mathbf{W}, T, \nabla_{\vec{X}} T) = \mathbf{P}^{(d)}(\boldsymbol{F}, \mathbf{D}, \mathbf{0}, T, \nabla_{\vec{X}} T) \tag{6.59}$$

Therefore, we can proven that $\mathbf{P}^{(d)}$ is not a function of $\mathbf{W}$, *i.e.* $\mathbf{P}^{(d)} = \mathbf{P}^{(d)}(\boldsymbol{F}, \mathbf{D}, T, \nabla_{\vec{X}} T)$. We can also express the tensor $\mathbf{P}^{(d)}$ in the reference configuration by means of the tensor $\mathbf{S}^{(d)}(\boldsymbol{E}, \dot{\boldsymbol{E}}, T, \nabla_{\vec{X}} T)$, since the terms $\dot{\boldsymbol{E}}$ and $\mathbf{D}$ are interrelated by $\frac{1}{2}\dot{\boldsymbol{C}} = \dot{\boldsymbol{E}} = \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F}$.

As we saw in Chapter 5 the relationship $\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T = J\ \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T$ holds, which can be proved if we start from the equation $\vec{\mathbf{q}}_0 = J \vec{\mathbf{q}} \cdot \boldsymbol{F}^{-T}$ or $\mathrm{q}_{0_i} = J\ \mathrm{q}_k F^{-1}_{ik}$, *i.e.*:

$$\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T = \mathrm{q}_{0_i} \frac{\partial T}{\partial X_i} = J\,\mathrm{q}_k F^{-1}_{ik} \frac{\partial T}{\partial x_p}\frac{\partial x_p}{\partial X_i} = J\,\mathrm{q}_k F^{-1}_{ik} \frac{\partial T}{\partial x_p} F_{pi} = J\,\mathrm{q}_k \delta_{pk} \frac{\partial T}{\partial x_p} = J\,\mathrm{q}_k \frac{\partial T}{\partial x_k}$$

In Chapter 5 we obtained the following equation for stress power:

$$\int_V \boldsymbol{\sigma} : \mathbf{D} dV = \int_{V_0} \underbrace{J\boldsymbol{\sigma}}_{\boldsymbol{\tau}} : \mathbf{D}\, dV_0 = \int_{V_0} \boldsymbol{\tau} : \mathbf{D} dV_0 = \int_{V_0} \mathbf{S} : \dot{\boldsymbol{E}} dV_0 = \int_{V_0} \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} dV_0 = \int_{V_0} \mathbf{P} : \dot{\boldsymbol{F}} dV_0 = \int_V \frac{\rho}{\rho_0} \mathbf{P} : \dot{\boldsymbol{F}}\, dV \tag{6.60}$$

Hence we can express the constitutive equations for thermoviscoelastic materials in the current configurations as:

$$\boxed{\begin{aligned} &\psi = \psi(\boldsymbol{F}, T) \\ &\boldsymbol{\sigma}^{(e)} = \rho \frac{\partial \psi(\boldsymbol{F}, T)}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^T \\ &\eta(\boldsymbol{F}, T) = -\frac{\partial \psi(\boldsymbol{F}, T)}{\partial T} \\ &J\,\boldsymbol{\sigma}^{(d)} : \mathbf{D} - \frac{J}{T} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \geq 0 \end{aligned}} \quad \begin{aligned}&\textit{Constitutive equations for}\\ &\textit{thermoviscoelastic materials}\\ &\textit{(Current configuration)}\end{aligned} \tag{6.61}$$

The stress $\boldsymbol{\sigma}^{(d)}$ in the current configuration can be obtained by comparison with the equation in (6.42), $\boldsymbol{\sigma} = \frac{1}{J} \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T$, thus:

$$\boldsymbol{\sigma}^{(d)} = \frac{1}{J} \boldsymbol{F} \cdot \mathbf{S}^{(d)}(\boldsymbol{E}, \dot{\boldsymbol{E}}, T, \nabla_{\vec{X}} T) \cdot \boldsymbol{F}^T \tag{6.62}$$

Thus:

$$\boxed{\begin{aligned} &\psi = \psi(\boldsymbol{C}, T) \\ &\boldsymbol{\sigma}^{(e)} = 2\rho \boldsymbol{F} \cdot \frac{\partial \psi(\boldsymbol{C}, T)}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T \\ &\eta(\boldsymbol{C}, T) = -\frac{\partial \psi(\boldsymbol{C}, T)}{\partial T} \\ &\boldsymbol{\sigma}^{(d)} = \frac{1}{J} \boldsymbol{F} \cdot \mathbf{S}^{(d)}(\boldsymbol{E}, \dot{\boldsymbol{E}}, T, \nabla_{\vec{X}} T) \cdot \boldsymbol{F}^T \\ &\mathbf{q} = \hat{\mathbf{q}}_0(\boldsymbol{C}, \dot{\boldsymbol{E}}, T, \nabla_{\vec{X}} T) \cdot \boldsymbol{F} \end{aligned}} \quad \begin{aligned}&\textit{Constitutive equations for}\\ &\textit{thermoviscoelastic materials}\\ &\textit{(Current configuration)}\end{aligned} \tag{6.63}$$

or

$$\boxed{\begin{aligned}&\hat{\psi}=\psi(\boldsymbol{E},T)\\&\boldsymbol{\sigma}^{(e)}=\rho\boldsymbol{F}\cdot\frac{\partial\psi(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\cdot\boldsymbol{F}^T\\&\eta(\boldsymbol{E},T)=-\frac{\partial\psi(\boldsymbol{E},T)}{\partial T}\\&\boldsymbol{\sigma}^{(d)}=\frac{1}{J}\boldsymbol{F}\cdot\mathbf{S}^{(d)}(\boldsymbol{E},\dot{\boldsymbol{E}},T,\nabla_{\vec{X}}T)\cdot\boldsymbol{F}^T\\&\mathbf{q}=\hat{\mathbf{q}}_0(\boldsymbol{E},\dot{\boldsymbol{E}},T,\nabla_{\vec{X}}T)\cdot\boldsymbol{F}\end{aligned}}$$

Constitutive equations for thermoviscoelastic materials

(Current configuration) (6.64)

6.4.1 Constitutive Equations with Internal Variables

The constitutive equations in (6.14) written in terms of functionals by means of the histories of $\boldsymbol{F}$, T and $\nabla_{\vec{X}}T$ are very general. An effective alternative to using functionals is to apply the method known as "*thermodynamics with internal variables*". In this method it is postulated that the current state of an inelastic continuum can be characterized by the current values of $\boldsymbol{F}$, T, $\nabla_{\vec{X}}T$ and by a set of internal variables ($\boldsymbol{\alpha}_i$) whose evolution indirectly includes the deformation history. Hence, the constitutive equations can be defined as:

$$\begin{aligned}&\psi=\psi(\boldsymbol{F},T,\nabla_{\vec{X}}T,\boldsymbol{\alpha}_i)\\&\eta=\eta(\boldsymbol{F},T,\nabla_{\vec{X}}T,\boldsymbol{\alpha}_i)\\&\vec{\mathbf{q}}_0=\vec{\mathbf{q}}_0(\boldsymbol{F},T,\nabla_{\vec{X}}T,\boldsymbol{\alpha}_i)\\&\mathbf{P}=\mathbf{P}(\boldsymbol{F},T,\nabla_{\vec{X}}T,\boldsymbol{\alpha}_i)\end{aligned} \tag{6.65}$$

where $\boldsymbol{\alpha}_i$, $i=1,2,\cdots,n$, is a set of internal variables. These variables can be scalars, vectors or higher order tensors.

In a process in which energy dissipation takes place, the theory with internal variables together with the Clausius-Duhem inequality provides the conditions (restrictions) to the constitutive equations.

From now on we will assume that the Helmholtz free energy is independent of the temperature gradient, so, the Helmholtz free energy (6.65) is expressed by:

$$\psi=\psi(\boldsymbol{F},T,\boldsymbol{\alpha}_i) \tag{6.66}$$

where $\boldsymbol{\alpha}_i=\{\boldsymbol{\alpha}_1,\cdots,\boldsymbol{\alpha}_n\}$ are the internal variables which must be added in order to characterize the material behavior and whose presence requires that new equations be included in the model. These additional equations are only dependent on the thermodynamic state at the point in question, so they are local by nature.

Then, the rate of change of the equation in (6.66) is given by:

$$\dot{\psi}=\frac{\partial\psi}{\partial\boldsymbol{F}}:\dot{\boldsymbol{F}}+\frac{\partial\psi}{\partial T}\dot{T}+\frac{\partial\psi}{\partial\boldsymbol{\alpha}_i}\square\dot{\boldsymbol{\alpha}}_i \tag{6.67}$$

The operator $\square$ is substituting with the number of contractions of the $\boldsymbol{\alpha}$ order. That is, if $\boldsymbol{\alpha}$ is a scalar, $\square$ has no contractions, if $\boldsymbol{\alpha}$ is a vector, $\square=\cdot$ (scalar product), if $\boldsymbol{\alpha}$ is a second-order tensor, $\square=:$ (double scalar product) and so on.

Then, by combining the equation in (6.67) with the entropy inequality we obtain:

$$\left[\mathbf{P}-\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}}\right]:\dot{\boldsymbol{F}}-\rho_0\left[\frac{\partial\psi}{\partial T}+\eta\right]\dot{T}-\rho_0\frac{\partial\psi}{\partial\boldsymbol{\alpha}_i}\square\dot{\boldsymbol{\alpha}}_i-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0 \tag{6.68}$$

From the previous sections we have shown the following holds:

$$\mathbf{P}=\rho_0\frac{\partial\psi}{\partial\boldsymbol{F}}\quad;\quad\eta=-\frac{\partial\psi}{\partial T} \tag{6.69}$$

Then, the entropy inequality becomes:

$$\begin{aligned}&-\rho_0\frac{\partial\psi}{\partial\boldsymbol{\alpha}_i}\square\dot{\boldsymbol{\alpha}}_i-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\\&-\mathbf{A}_i\square\dot{\boldsymbol{\alpha}}_i-\frac{1}{T}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\end{aligned} \tag{6.70}$$

where we have introduced the *thermodynamic forces*, denoted by $(-\mathbf{A}_i)$:

$$-\mathbf{A}_i=-\rho_0\frac{\partial\psi}{\partial\boldsymbol{\alpha}_i} \tag{6.71}$$

Then, to fully characterize the constitutive equations, the complementary laws associated with the dissipative mechanism must be introduced, *i.e.* the equations for the variables $\frac{1}{T}\vec{\mathbf{q}}_0$ and $\dot{\boldsymbol{\alpha}}_i$. One way to ensure that the equations related to $\vec{\mathbf{q}}_0$ and $\dot{\boldsymbol{\alpha}}_i$ satisfy the condition in (6.70) is by using a *dissipative pseudo-potential* (scalar-valued tensor function), such that:

$$\Phi=\Phi(\mathbf{A}_i,\nabla_{\vec{X}}T) \tag{6.72}$$

which is a convex potential for any value of the pair ($\mathbf{A}_i$, $\nabla_{\vec{X}}T$). Then, the variables are determined by:

$$\dot{\boldsymbol{\alpha}}=-\frac{\partial\Phi}{\partial\mathbf{A}_i}\quad;\quad\frac{1}{T}\vec{\mathbf{q}}_0=-\frac{\partial\Phi}{\partial\nabla_{\vec{X}}T} \tag{6.73}$$

Problem 6.1: Find the governing equations for a continuum solid which has the following features: Isothermal and adiabatic processes; an infinitesimal strain regime and a linear elastic relationship between stress and strain.
b) Once the linear elastic, stress-strain relationship has been established, find the equation in which $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ is a tensor-valued isotropic tensor function.

Solution:

When we have isothermal and adiabatic processes temperature and entropy play no role.
In an infinitesimal strain regime, the following is satisfied:
Strain tensors: $\boldsymbol{E}\approx\boldsymbol{e}\approx\boldsymbol{\varepsilon}=\nabla^{sym}\vec{\mathbf{u}}$
Stress tensors: $\mathbf{P}\approx\mathbf{S}\approx\boldsymbol{\sigma}$
$\boldsymbol{F}\approx\mathbf{1}\quad;\quad\nabla_{\vec{X}}\approx\nabla_{\vec{x}}\approx\nabla\quad;\quad\rho\approx\rho_0$. If we take this approach, mass density is no longer unknown.
Then, taking into account the fundamental equations in (6.6)-(6.9), the remaining equations for the proposed problem are:
1) The equations of motion

$$\nabla\cdot\boldsymbol{\sigma}+\rho\vec{\mathbf{b}}=\rho\dot{\vec{\boldsymbol{v}}}$$

2) The energy equation

$$\rho_0 \dot{u}(\vec{X},t) = \mathbf{S} : \dot{E} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r(\vec{X},t) \quad \Rightarrow \quad \rho \dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$$

or in terms of the Helmholtz free energy: $\frac{Du}{Dt} = \frac{D}{Dt}[\psi + T\eta] = \dot{\psi}$:

$$\rho \dot{\psi} = \dot{\Psi}^e = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$$

where Ψ^e is the energy density (also known as *strain energy density*). Then if we bear in mind the entropy inequality, we can observe that the proposed problem is characterized by a process without any energy dissipation (an *elastic process*), *i.e.* all stored energy caused by $\boldsymbol{\varepsilon}$ will recover when $\boldsymbol{\varepsilon} = \mathbf{0}$.

3) In this problem, the constitutive equations in (6.36) become:

$$\psi = \psi(\boldsymbol{\varepsilon})$$

$$\mathbf{S} \approx \boldsymbol{\sigma} = \rho \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \Psi^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$$

Energy (ψ) and stress are only functions of strain. Then, if we calculate the rate of change of the Helmholtz free energy, *i.e.* $\dot{\psi}(\boldsymbol{\varepsilon}) = \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}$, and by substituting it with the equation $\rho \dot{\psi} = \dot{\Psi}^e = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$, we obtain:

$$\rho \frac{\partial \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \frac{\partial \dot{\Psi}^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \quad \Rightarrow \quad \boldsymbol{\sigma} = \frac{\partial \Psi^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}$$

Thus, we can conclude that the energy equation is a redundant one, *i.e.* if the stress is known the energy can be evaluated and vice-versa. So, we can summarize the governing equations for the problem proposed with:

The equations of motion:

$$\nabla \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{v}} = \rho \ddot{\vec{\mathbf{u}}} \quad \text{(3 equations)}$$

The constitutive equations for stress:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\partial \Psi^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \quad \text{(6 equations)} \tag{6.74}$$

Kinematic equations:

$$\boldsymbol{\varepsilon} = \nabla^{sym} \vec{\mathbf{u}} \quad \text{(6 equations)}$$

The unknowns of the proposed problem are: $\boldsymbol{\sigma}$ (6), $\vec{\mathbf{u}}$ (3) and $\boldsymbol{\varepsilon}$ (6), making a total of 15 unknowns and 15 equations, so the problem is well-posed. Then, to achieve the unique solution of the set of partial differential equations given in (6.74) one must introduce the initial and boundary conditions, hence defining the *Initial Boundary Value Problem* for the *linear elasticity problem*.

NOTE: Although the energy equation is a redundant one, at the time of establishing an analytical or numerical method for solving the problem, we will always start from energy principles, hence the importance of studying the energy equation in a system. ■

In subsection 1.6.1 The Tensor Series (Chapter 1), we saw that we can approach a tensor-valued tensor function by means of the following series:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \approx \frac{1}{0!}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0) + \frac{1}{1!}\frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \frac{1}{2!}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) : \frac{\partial^2 \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \cdots$$

$$\approx \boldsymbol{\sigma}_0 + \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) : \frac{\partial^2 \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \cdots$$

Then, by considering the application point $\boldsymbol{\varepsilon}_0 = \mathbf{0}$ and $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0) = \boldsymbol{\sigma}_0 = \mathbf{0}$, and also taking into account that the relationship $\boldsymbol{\sigma}$ - $\boldsymbol{\varepsilon}$ is linear, higher order terms can be discarded, thus:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} = \frac{\partial^2 \Psi^e(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} = \mathbb{C}^e : \boldsymbol{\varepsilon}$$

where $\mathbb{C}^e = \dfrac{\partial^2 \Psi^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}}$ is a symmetric fourth-order tensor which is known as the *elasticity tensor*, which contains the material mechanical properties.
Note that, the energy equation has to be quadratic with which we can guarantee that the relationship $\boldsymbol{\sigma}$ - $\boldsymbol{\varepsilon}$ is linear, since $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \dfrac{\partial \Psi^e(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}$. We can also use series expansion to represent the strain energy density as follows:

$$\begin{aligned}
\Psi^e(\boldsymbol{\varepsilon}) &= \frac{1}{0!}\Psi^e(\boldsymbol{\varepsilon}_0) + \frac{1}{1!}\frac{\partial \Psi^e(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \frac{1}{2!}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) : \frac{\partial^2 \Psi^e(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \cdots \\
&= \Psi^e_0 + \boldsymbol{\sigma}_0 : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) : \frac{\partial^2 \Psi^e(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \cdots \\
&= \frac{1}{2}\boldsymbol{\varepsilon} : \frac{\partial^2 \Psi^e(\boldsymbol{\varepsilon}_0)}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}
\end{aligned}$$

where we have also considered that $\boldsymbol{\varepsilon}_0 = \mathbf{0} \Rightarrow \Psi^e_0 = 0, \boldsymbol{\sigma}_0 = \mathbf{0}$.
To better illustrate the problem established here, let us consider a particular case (a one-dimensional case) where the stress and strain components are given by:

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ; \quad \varepsilon_{ij} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \sigma_{11} = \mathbb{C}^e_{1111}\varepsilon_{11} \Rightarrow \sigma = E\varepsilon$$

In this case, the stress-strain linear relationship becomes $\sigma = E\varepsilon$ (Hooke's law) and the strain energy density is given by $\Psi^e = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\varepsilon E \varepsilon$, (see Figure 6.4).

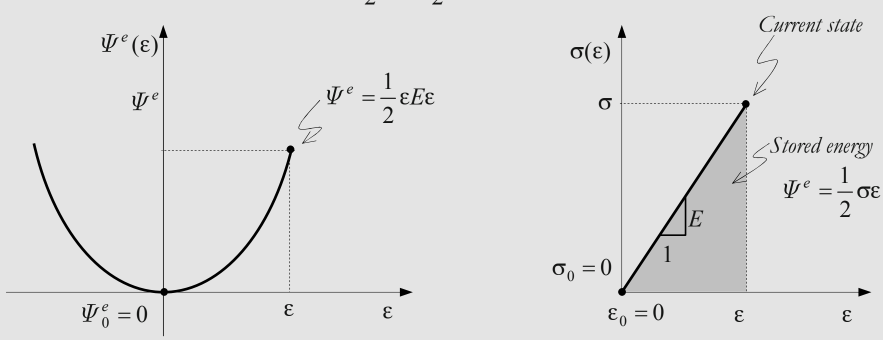


Figure 6.4: Stress-strain relationship (a one-dimensional case).

NOTE: Here it should be pointed out that in the case of elastic processes the constitutive equation $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ is only dependent on the current value of $\boldsymbol{\varepsilon}$, *i.e.* it is independent of the deformation history. ■
b) The tensor-valued tensor function $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ is isotropic if the following is satisfied:

$$\Psi^e(\varepsilon'_{kl}) = \Psi^e(\varepsilon_{kl}) \quad \Rightarrow \quad \sigma'_{ij}(\varepsilon'_{kl}) = \sigma_{ij}(\varepsilon'_{kl})$$

Then, taking into account that the relationship $\boldsymbol{\sigma}$ - $\boldsymbol{\varepsilon}$ is given by $\sigma_{ij}(\boldsymbol{\varepsilon}) = \mathbb{C}^e_{ijkl}\varepsilon_{kl}$ (indicial notation), we can conclude that:

$$\sigma'_{ij}(\varepsilon'_{kl}) = \sigma_{ij}(\varepsilon'_{kl}) \quad \Rightarrow \quad \mathbb{C}'^e_{ijkl}\varepsilon'_{kl} = \mathbb{C}^e_{ijkl}\varepsilon'_{kl} \quad \Rightarrow \quad \mathbb{C}'^e_{ijkl} = \mathbb{C}^e_{ijkl}$$

That is, the fourth-order tensor $\mathbb{C}^e$ is isotropic. An isotropic symmetric fourth-order tensor has the form $\mathbb{C}^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ or $\mathbb{C}^e = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}$, (see Chapter 1), and here the parameters λ and μ are known as *Lamé constants*. Figure 6.5 shows the stress-strain relationship for an isotropic material. Note that, for an isotropic linear elastic material in an infinitesimal strain regime the constitutive equation for stress becomes:

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \mathbb{C}^e : \boldsymbol{\varepsilon} \quad \longrightarrow \quad \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = (\lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}) : \boldsymbol{\varepsilon} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

It should be emphasized here that due to the fact that the $\mathbb{C}^e$-components are independent of the coordinate system, the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ share the same principal space (eigenvectors), (see Figure 6.5).

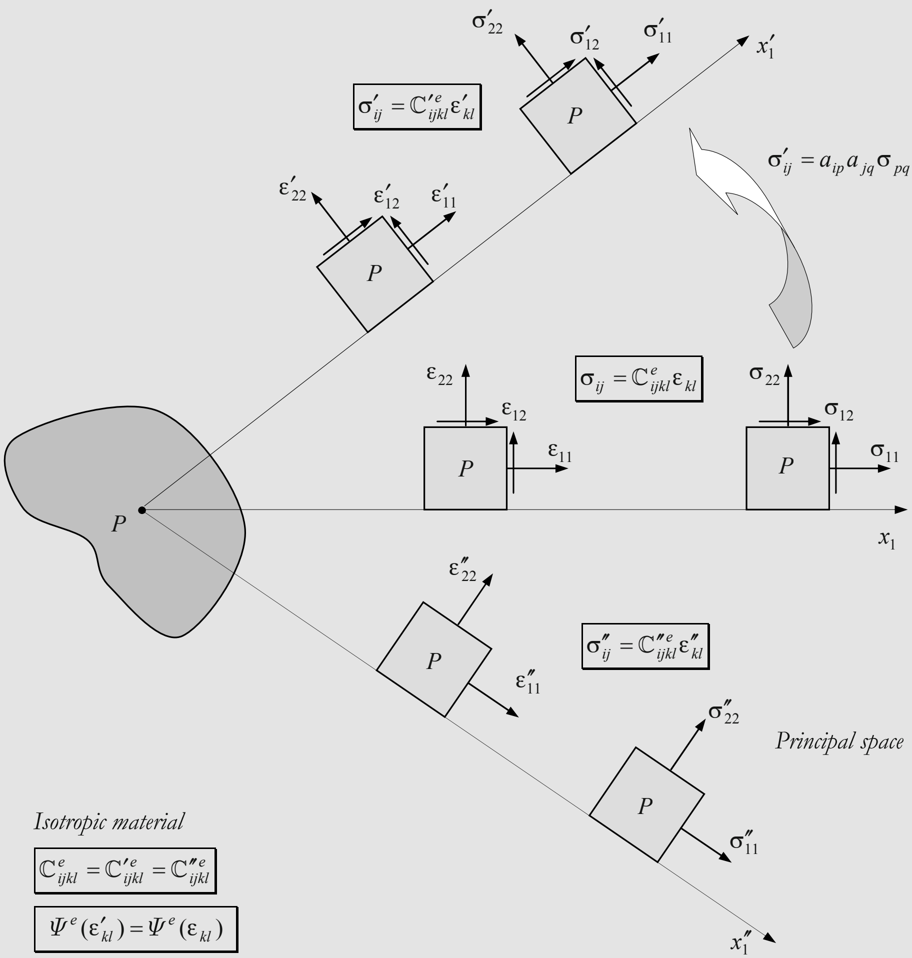


Figure 6.5: Stress-strain relationship (isotropic material).

6.5 Some Experimental Evidence

6.5.1 Behavior of Solids

In 1660, Robert Hooke discovered that for many materials (solids) displacement was proportional to the applied force, hence the notion of elasticity was established, but this was not the case in the sense of the stress-strain relationship. It was the Swiss mathematician Jacob Bernoulli who observed that the proper way to describe any change in length was by providing a force per unit area (stress) as a function of the elongation per unit length (strain), (see Figure 6.1).

Now, if we consider the one-dimensional stress-strain relationship seen in a loading /unloading process we can observe the following types of behavior (see Figure 6.6):

- Linear elastic behavior;
- Non-linear elastic behavior;
- Inelastic behavior.

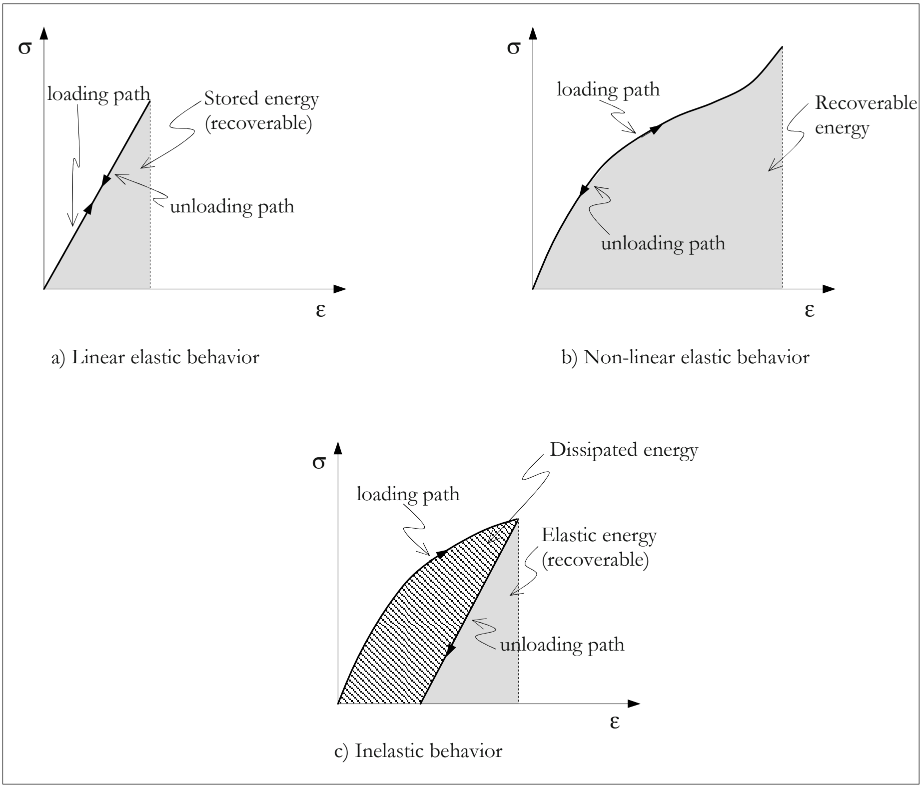


Figure 6.6: Behavior of solids (a one-dimensional case).

In *elastic* processes there is no energy dissipation, *i.e.*, all energy that there is during the loading process is stored, and after removing the entire load, that is, all the energy is completely recovered. In the linear elastic process stress-strain curve (see Figure 6.6(a)), the paths of the loading and unloading processes are the same.

In the case of small deformations and isothermal processes, materials that behave according to Figure 6.6(a) can be characterized by means of *Linear Elasticity* (Chapter 7).

A *non-linear elastic* process (Figure 6.6(b)) differs from a linear elastic one by the non-linearity of the stress-strain curve. In general, materials that behave in this way tend to show large deformations. These types of materials can be characterized by means of *Hyperelasticity* (*nonlinear elasticity*), which is the theme in Chapter 8. It should be stressed here that in the *elastic process* (whether this is linear or non-linear), the constitutive equation is only dependent on the current value of ε , *i.e.* it is independent of the deformation history.

In contrast to the above, *inelastic* behavior is characterized by involving energy dissipation, and this energy at an atomic level can be interpreted as being the energy released for restructuring atoms (dislocations). The tensile testing shown in Figure 6.7 is a typical example of inelastic behavior or elastoplastic behavior to be precise. Then, materials having these characteristics are analyzed by means of *plasticity models* (Chapter 9). At a macroscopic level, elastoplastic behavior is characterized by the fact that once the stress state exceeds a certain threshold (the yield stress) the material acquires a permanent strain, ε^p-plastic strain, *i.e.* when the material is stress free it no longer returns to its initial state.

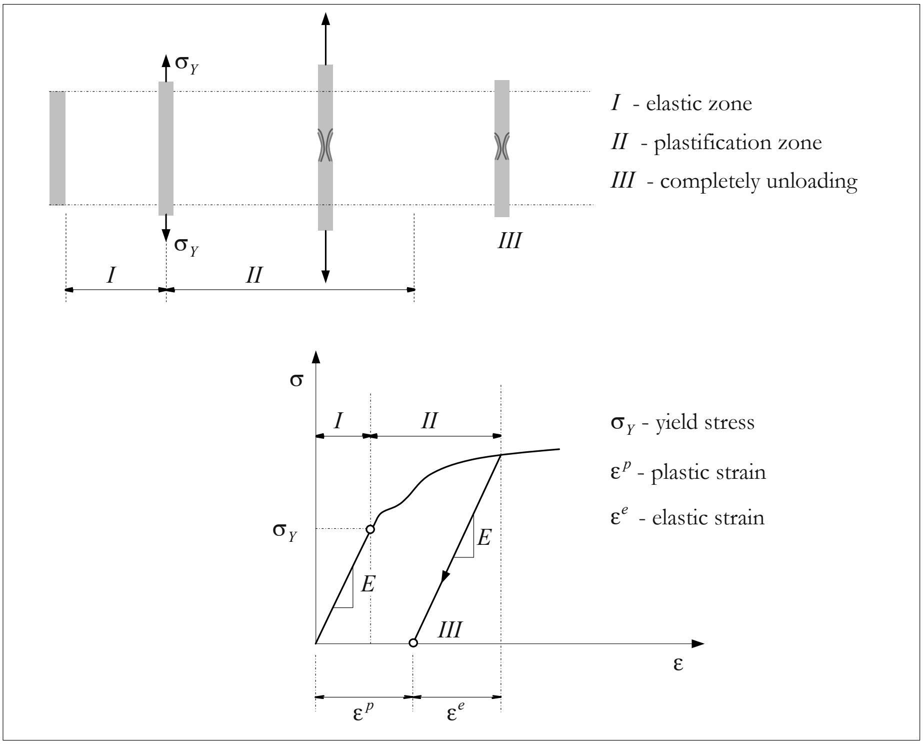


Figure 6.7: Elastoplastic behavior.

Another type of inelastic behavior is shown in Figure 6.8 and is characterized by *Damage Models* (Chapter 11), which are fundamentally used to show how the elastic modulus has degraded, (see Figure 6.8). In this type of behavior when the load is removed, the material has no permanent strain, but internally the material will have undergone internal degradation (an irreversible process). In Figure 6.8 we consider a loading/unloading/ loading process, where the steps 1-2-3 represent the loading process; the path 4 indicates the unloading process which if applied will proceed as indicated by the path 5. In general, brittle materials such as concrete, ice, ceramics and ice have these characteristics.

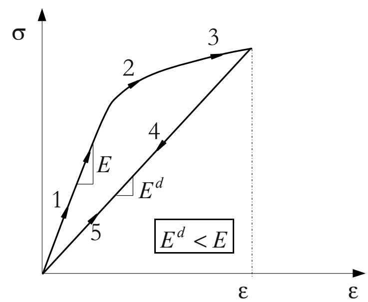


Figure 6.8: Inelastic behavior (damage model).

NOTE: Notice that materials are not strictly characterized by just one of the above classifications. In general, there are materials which need a combination of the models described above to be accurately represented, *i.e.* some materials exhibit both permanent deformation and elastic modulus degradation. So, to characterize this material a plastic-damage model can be employed. ■

6.5.1.1 Temperature Effect

When materials are subjected to a temperature change their mechanical properties change, *i.e.* they are temperature dependent. There are two possible scenarios to be aware of when analyzing the effect of temperature. The first is when the effect of temperature does not significantly affect the material mechanical properties. In this case we can decouple the problem, *i.e.* we can treat the thermal and mechanical effects independently. The second is when temperature has a significant effect on the mechanical properties. In this case the thermal and mechanical variables must be considered simultaneously in the constitutive equations.

6.5.1.2 Some Mechanical Properties of Solids

A simple method to determine some mechanical properties of solids is by means of testing the materials. There are destructive tests, which consist in using a material specimen and testing it. Other types of testing are the non-destructive tests which use special devices such as the ultra-sonic type with which we can obtain the material properties of the structure without causing any damage to it. Within the class of destructive tests we can cite: tensile testing, compressive testing, the triaxial compression test, to mention a few.

Tensile Testing

Tensile testing consists of a specimen whose ends are subjected to a tensile force as shown in Figure 6.9. Then, if we known the dimensions of the specimen cross section, b, h, and the tensile force F, it is possible to obtain the nominal stress ($\sigma_{nom} = \dfrac{F}{A_0}$). Furthermore, by means of a device, known as an extensometer, we can obtain the elongation of the specimen ($\Delta\ell$) during the test. As we known ℓ (which is measured before the beginning of the test) we can evaluate the engineering strain (ε), so, it is possible to define the stress-strain curve as shown in Figure 6.10.

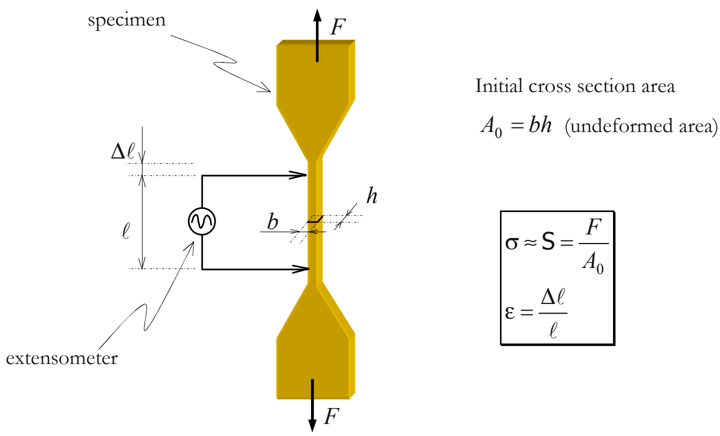


Figure 6.9: A specimen under tensile forces.

Now, as we have the stress-strain curve, (see Figure 6.10), we can evaluate *Young's modulus* (E) (also known as the *elastic* or *tensile modulus*). Then, if we bear in mind Figure 6.10 we can introduce the tangent modulus (E^{tan}) and the secant modulus (E^{s}), where at a given stress state it holds that $\sigma = E^{s}\varepsilon$ and $\dot{\sigma} = E^{tan}\dot{\varepsilon}$ (the slope of the stress-strain curve). Note that in the elastic zone theses modules coincide, *i.e.* $\sigma = E\varepsilon = E^{s}\varepsilon$ and $\dot{\sigma} = E^{tan}\dot{\varepsilon} = E\dot{\varepsilon}$.

If, in addition to the extensometer along to the tensile force direction we have a second extensometer to measure the contraction of the cross section we could have obtained other mechanical properties of the material: the Poisson's ration, ν. Remember that for an isotropic linear elastic material the elasticity tensor ($\mathbb{C}$) is a function of two independent variables, here these parameters are represented by E and ν. Later, in Chapter 7, we will link these variables to the Lamé constants (λ, μ).

Then, bearing in mind the curve $\sigma - \varepsilon$ we can emphasize some important points:

The proportionality limit – This point is denoted by the stress σ^{e}. The region between the stress-free state and the proportionality limit defines the linear elastic behavior area where there is no dissipation energy.

The yield point (the elastic limit) – This point is denoted by the stress σ_Y. In the region between σ^{e} and σ_Y the behavior of the material is assumed to be elastic, *i.e.* there is no dissipation of energy, but the relationship σ-ε is no longer linear. For some materials the proportionality point and the yield point coincide.

The ultimate strength point – This point is denoted by the stress σ^{u}. In the region between σ_Y and σ^{u} the material shows inelastic behavior, *i.e.* the internal structure of the material has suffered irreversible changes, that is, it is characterized by energy dissipation.

The rupture strength point – This point is defined by σ^{r}, which denotes the material rupture.

In the region between the points σ^u and σ^r the phenomenon of concentration of deformation in a body (inelastic process) occurs, this concentration is known as *strain localization*.

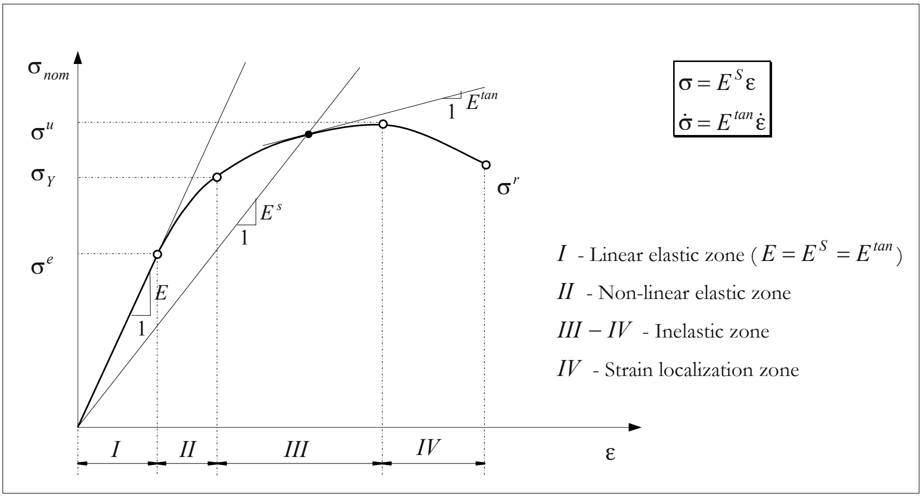


Figure 6.10: Stress-strain curve.

Figure 6.11 shows some typical stress-strain curves, for instance those of steel, iron and concrete.

There are materials in which the yield stress (σ_Y) is well defined, since the stress is maintained, while the strain increases in value, (see Figure 6.11 for steel). Such behavior is characteristic for some types of steel. For materials where the yield point is not well defined, we use an *offset method* in order to define σ_Y. For example, σ_Y can be used as the point of intersection between the $\sigma - \varepsilon$ curve and the slope line equal to Young's modulus and displaced by 0.2%, (see **Problem 6.2**).

When defining the material constitutive equations, we can make idealizations (simplifications) of the stress-strain curve, taking the curve that best fits the real material behavior, (see Figure 6.12). For example, in Figure 6.12(a) we focus on perfectly plastic behavior, in Figure 6.12(b) we idealize the elastoplastic stress-strain behavior by linear hardening, in Figure 6.12(c) we can appreciate an elastoplastic stress-strain behavior by bi-linear hardening, and Figure 6.12(d) shows behavior characterized by softening which is that typical detected in brittle materials.

Depending on how the solids behave, these have traditionally been classified into two categories, namely: Fragile and Ductile Materials. The most important characteristics of these types of materials are listed below:

- *Fragile Materials:* small deformation occurs; there is no previous warning of failure (abrupt rupture), *e.g.* as found in concrete, ceramics, glass, ice, rocks, etc.
- *Ductile Materials:* large deformation occurs; there is a previous warning of failure, *e.g.* as found in steel, aluminum, etc.

Depending on the manufacturing process and the amount of carbon involved in its manufacture, some steel may behave as if it were a brittle material.

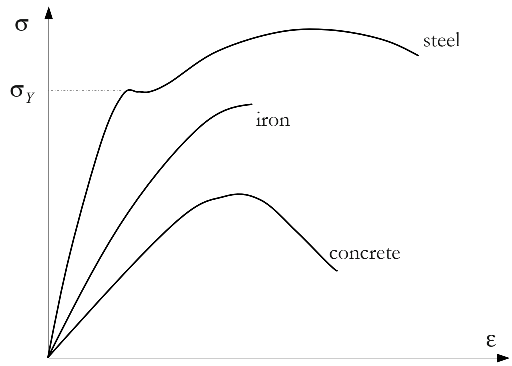


Figure 6.11: The stress-strain curve for a few materials.

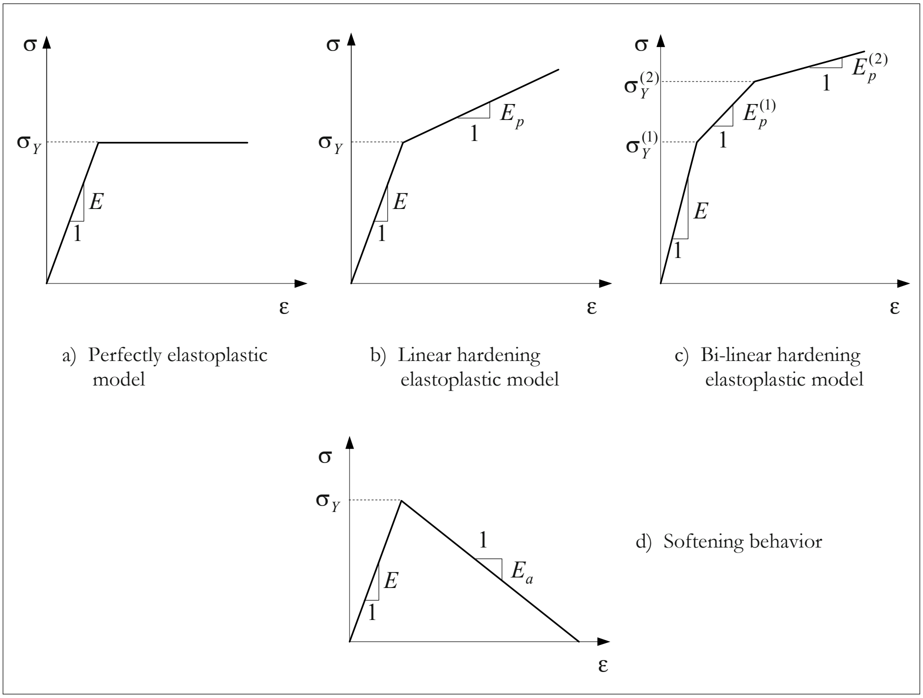


Figure 6.12: Some idealizations of the stress-strain curve.

Problem 6.2: In tensile testing we evaluated the following points:

Point	$\sigma(Pa)$	$\varepsilon(\times 10^{-3})$
1	6.67	0.667
2	13.3	1.33
3	20	2
4	24	3
5	22	3.6

Calculate Young's modulus (E) and define the stress-strain curve limit points.

Solution: First, we verify that the first three points maintain the same proportionalities:

$$E = \frac{\sigma^{(1)}}{\varepsilon^{(1)}} = \frac{\sigma^{(2)}}{\varepsilon^{(2)}} = \frac{\sigma^{(3)}}{\varepsilon^{(3)}} = \frac{20}{2\times 10^{-3}} = 10\,000 Pa = 10 kPa$$

The stress-strain curve can be appreciated in Figure 6.13, in which we define the following points: σ^e - the proportionality point; σ_Y - the yield point; σ^u - the ultimate strength point; and σ^r - the rupture strength point.

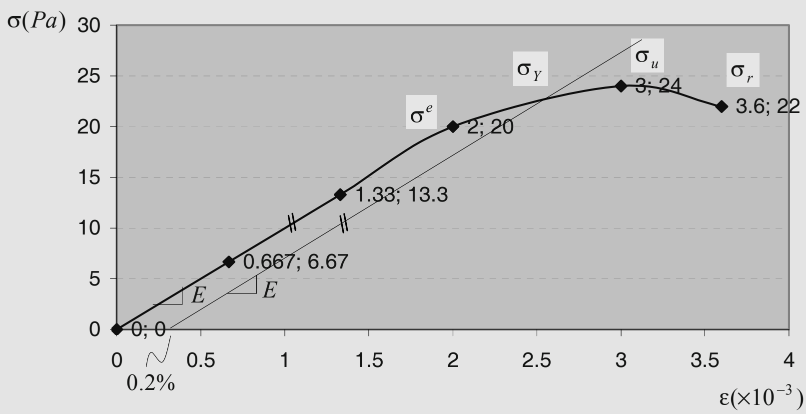


Figure 6.13: Stress-strain curve.

Brazilian Test

A direct test for tensile stress in brittle materials was put forward by Brazilian engineer Fernando Lobo Carneiro. In this test a compression cylinder was used as shown in Figure 6.14, which had an advantage when working with fragile materials (concrete, ceramics), since the manufacturing process of the specimen like the type described in Figure 6.9 can affect the material properties.

Unconfined Compression Test

Conversely, the compression test is the opposite of the tensile test as the specimen is in the shape of a solid circular cylinder, (see Figure 6.14). Here, the mechanical properties present include the elastic modulus for compression, the yield stress and the rupture strength point.

In some materials, such as steel, the properties present in both the tensile and compression tests are identical, whereas other materials, such as concrete, exhibit different properties depending on which test is carried out. Figure 6.14 shows how concrete typically behave. We can clearly see that it is a low tensile strength material.

Although tensile yield stress in metals is equal to compression, it was observed experimentally that when these materials were subjected to cyclic loads their yield stress changed. For example, let us assume that a metal which originally has as tensile and compressive yield stresses the following values σ_Y and $-\sigma_Y$, respectively, (see Figure 6.15). Once it exceeds the tensile yield stress σ_Y and then is subjected to an unloading process, the compressive yield stress changes to $-\sigma_Y^*$. This phenomenon was first detected by Bauschinger, hence it is known as the *"Bauschinger effect"*.

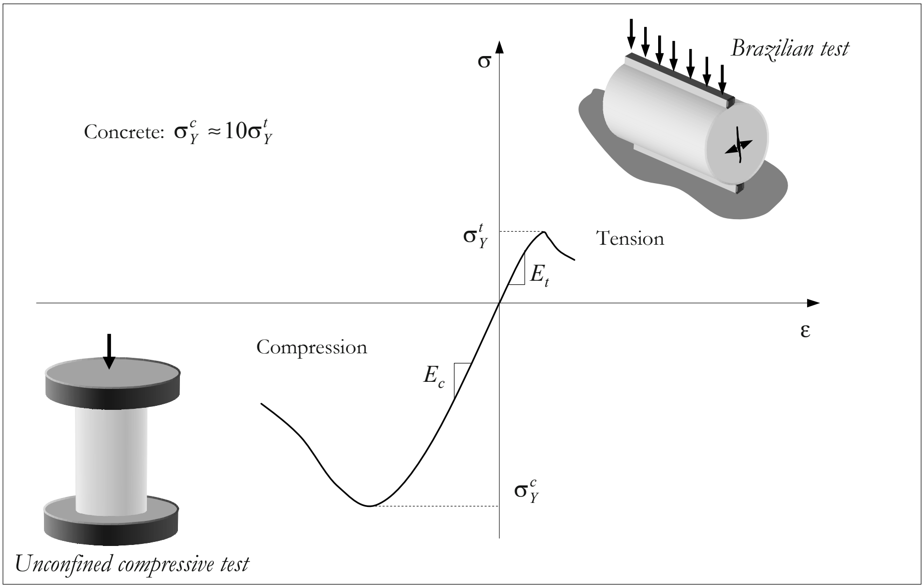


Figure 6.14: The stress-strain curve for concrete.

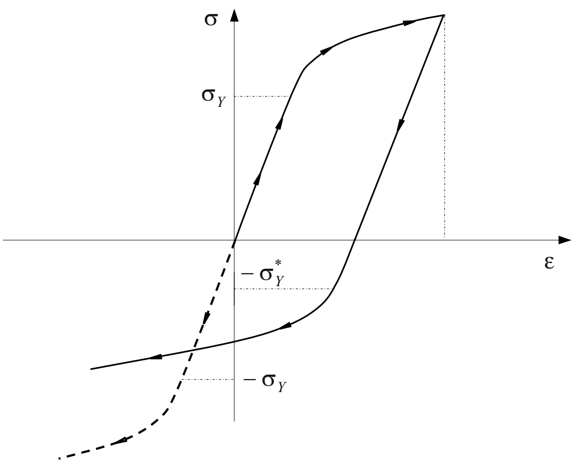


Figure 6.15: The Bauschinger effect.

Therefore, when establishing a constitutive model to show how a material behaves, it is important to consider the whole process, *i.e. loading/unloading/loading process*.

Another interesting phenomenon observed in metals occurs when these materials are subjected to cyclic loads of tension-compression. Although they do not reach the yield stress in these cycles, they can reach a state of complete rupture (depending on the number of cycles). This phenomenon is known as *fatigue*. It has become a subject of great interest since the time of the First and Second World War, when ship hulls suffered unexplained ruptures.

Triaxial Compression Test

The triaxial compression test, (see Figure 6.16), is used to obtain properties of cohesive saturated (or unsaturated) soils. A triaxial test is outlined below:

1. The specimen is a cylindrical sample.
2. The specimen is enclosed within an elastic membrane (rubber), and both ends of the specimen are supported by rigid plates.
3. The specimen is placed in a pressure chamber and confined to pressure denoted by σ_3.
4. A device is previously fixed to the specimen in order to measure how its length varies and which is used to calculate strain.

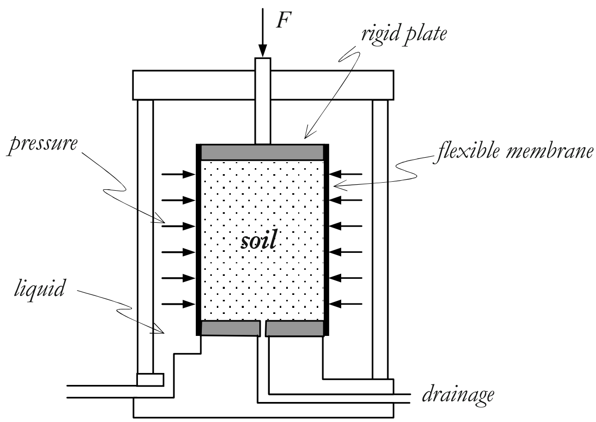


Figure 6.16: Cross section of the triaxial compression test apparatus.

With a fixed hydrostatic pressure during the test, the normal force increases gradually until the soil sample fails by shear. The same test is repeated by varying the hydrostatic pressure value. Each trial stress state at the time of failure is represented by the Mohr's circle. The envelope curve to the circles is used to define some parameters, namely: the angle of internal friction (ϕ) and cohesion (c), (see Figure 6.17).

NOTE: The sign convention in soil and rock mechanics is the reverse of the one adopted in solid mechanics, *i.e.* in soil mechanics compression is considered to be positive since tensile strength in soils is very low or non-existence. ■

Some soils are formed by sediments that are linked by electrostatic forces between fine particles, *e.g.* clay-water. Negatively charged clays are cohesive with water that has a strong electrical polarity and cohesion has the same unit of measurement as stress.

Another important parameter for cohesive materials is dilatancy (ψ), which can be obtained from a test as indicated in Figure 6.18.

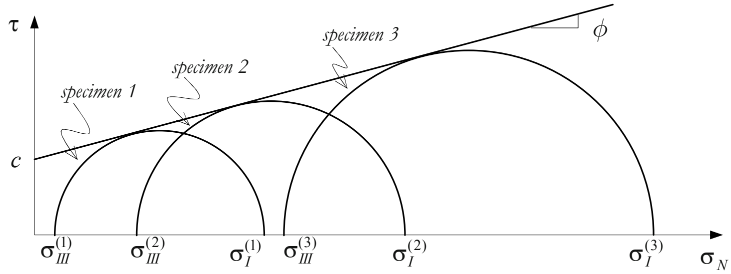


Figure 6.17: Mohr's circle for three specimens – Triaxial test.

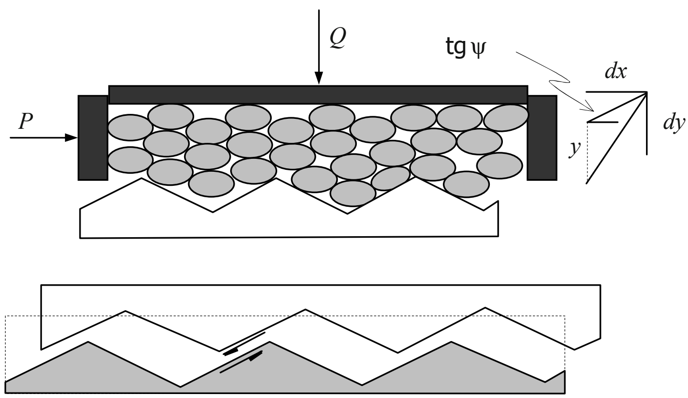


Figure 6.18: Dilatancy.

Pore Pressure

Soils (formed by sediments) have gaps which can be filled with liquid or gas (usually air), (see Figure 6.19). Pore pressure acts by reducing contact between grains, (see Figure 6.19). When this material is subjected to load, stress appears due to the presence of water and depending on the permeability of the material this stress will cease to exist or be a function of time. In this case we can define the effective stress $\boldsymbol{\sigma}^{eff}$ with:

$$\boldsymbol{\sigma}^{eff} = \boldsymbol{\sigma} - P_p \tag{6.75}$$

where $\boldsymbol{\sigma}$ is total stress, and P_p is the pore pressure.

6.5.2 Behavior of Fluids

Gases as well as liquids are materials made up of molecules (agglomerations of two or more atoms). Fundamentally, we can state that solids can resist shear stress and consequently have the ability to store mechanical energy while liquids have low or no resistance to shear stress and have no capacity to store energy. Additionally, resistance to shear stress in fluids is directly linked to fluid properties, namely, viscosity and fluids are classified as non-viscous (*e.g.* water) or viscous (*e.g.* oil). In the case of viscous fluids all dissipated energy is caused by viscosity.

Viscosity is highly temperature dependent, decreasing in value as temperature increases.

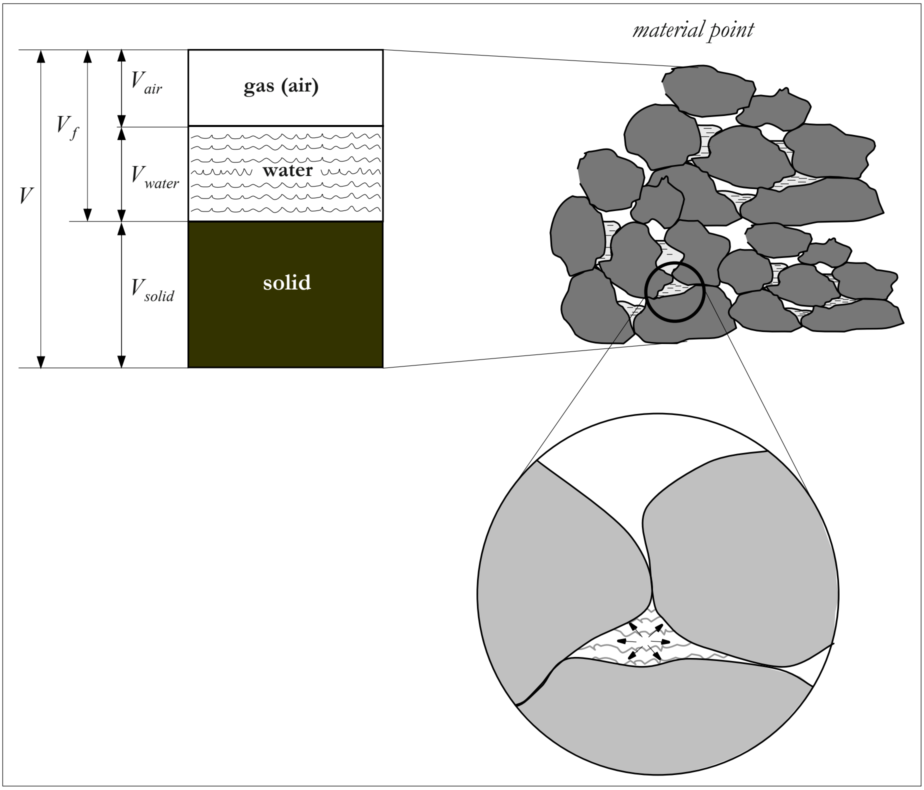


Figure 6.19: Porous material.

6.5.2.1 Viscosity

Viscosity can be kinematic or dynamic. Kinematic viscosity (ϑ) is not dependent on the fluid mass density whereas dynamic viscosity is highly dependent on it. Thus, we can define dynamic viscosity η_v as:

$$\eta_v = \frac{\tau}{\dot{\gamma}} \qquad \left[\frac{N}{m^2} s = \frac{kgm}{s^2} \frac{s}{m^2} = Pa \times s \right] \quad (6.76)$$

where τ is shear stress, and $\dot{\gamma}$ is the rate of change of the shear strain. The SI unit of dynamic viscosity is (Pascal x second) $Pa \times s$ or $\dfrac{kg\,m}{s}$.

The most accurate way to measure dynamic viscosity is by viscometers (also called viscosimeter), which are devices that measure the time it takes for a fluid to pass through a very precise capillary diameter. The kinematic and dynamic viscosities are related by:

$$\vartheta = \frac{\eta_v}{\rho} \qquad \left[\frac{m^2}{s} \right] \quad (6.77)$$

where ϑ is kinematic viscosity, η_v is dynamic viscosity, and ρ is mass density.

Depending on the type of the constitutive equation for stress used, fluids can be classified as follows:

- *Newtonian Fluids* (Chapter 12)

  A Newtonian fluid is characterized by one that shows a linear relation between the viscous stress tensor ($\boldsymbol{\tau}$) and the rate-of-deformation tensor ($\mathbf{D}$). Some examples of Newtonian fluids are: water and oil.

- *Non-Newtonian Fluids (Stokesian Fluids)*

  A Non-Newtonian fluid is characterized by one that shows a non-linear relation between the viscous stress and the rate-of-deformation tensors. Some examples of Non-Newtonian fluids are: blood, sauces, honey, toothpaste, heavy oil.

6.5.3 Behavior of Viscoelastic Materials

To understand viscoelastic behavior, we can carry out a simple experiment. For example, we can take a gum (used) and stretch it in such a way that most of the gum is concentrated at one end. Then, we place it in a vertical position so that the only force acting on it is the gravity, (see Figure 6.20 at time t_0). Without any force added to the system, we will observe that over time the gum will start to deform, (see Figure 6.20 during time $t_1 \to t_3$). After it has been deforming for a while, we cut its end off, *i.e.* we remove the force, and we will see that part of the deformation recovers instantly, and we will also verify that over time another part of the deformation recovers slowly.

That is, these materials have the ability to store mechanical energy as elastic solids and can also dissipate energy due to their viscosity. Hence, when we are working with how to approach the constitutive equation for these materials we have to take into account these phenomena simultaneously, (see Findley *et al.* (1976), Christensen (1982)).

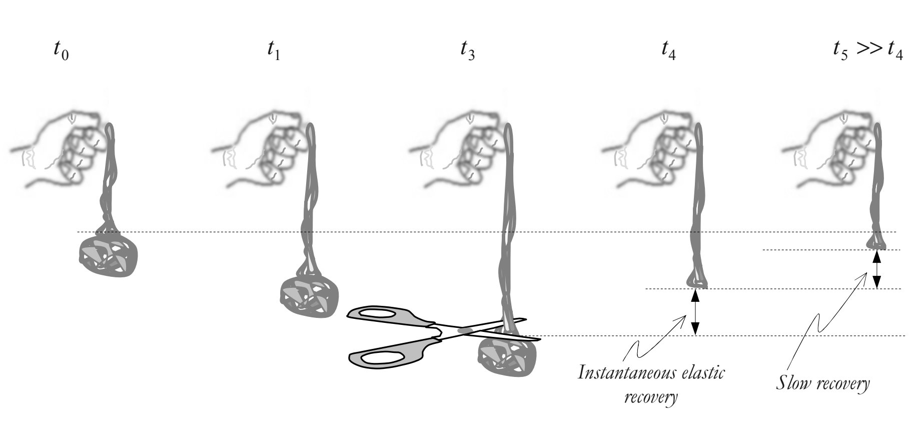


Figure 6.20: Viscoelastic behavior.

In other words, viscoelastic materials are those in which the stress-strain relationship is time dependent. The most relevant viscoelastic phenomena are listed below:

Creep – When stress is constant, strain increases over time. For example we can mention a building column, which, when force is first applied shows an initial strain, which increases over time with no corresponding increase in stress, (see Figure 6.21).

Relaxation – When strain is constant, stress decreases over time. As an example we can cite a prestressed cable bridge whose cable is initially subjected to an initial strain causing an initial stress and over time this stress decreases while the strain remains constant, (see Figure 6.22).

On a final note, creep and relaxation are reciprocal phenomena.

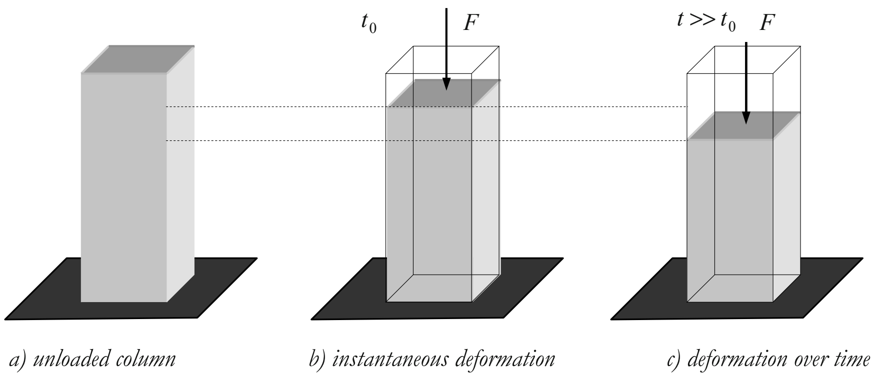


Figure 6.21: Creep phenomenon.

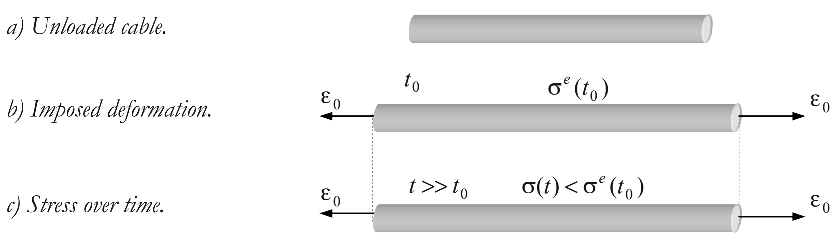


Figure 6.22: Relaxation phenomenon.

6.5.4 Rheological Models

Now we can introduce some simple devices that will help us to interpret constitutive models and which will also help us to formulate more complex constitutive models.

Let us consider a rod (a one-dimensional case) subjected to tension where the stress state at a material point is represented by σ. If we are working with a linear elastic material the stress-strain relationship is given by $\sigma = E\varepsilon$ (Hooke's law), (see **Problem 6.1**). If we then compare this with the governing law of a spring given by $F = ku$, where k is the spring constant and u is the displacement, we can state that the linear elastic model, $\sigma = E\varepsilon$, can be represented by the spring device, (see Figure 6.23).

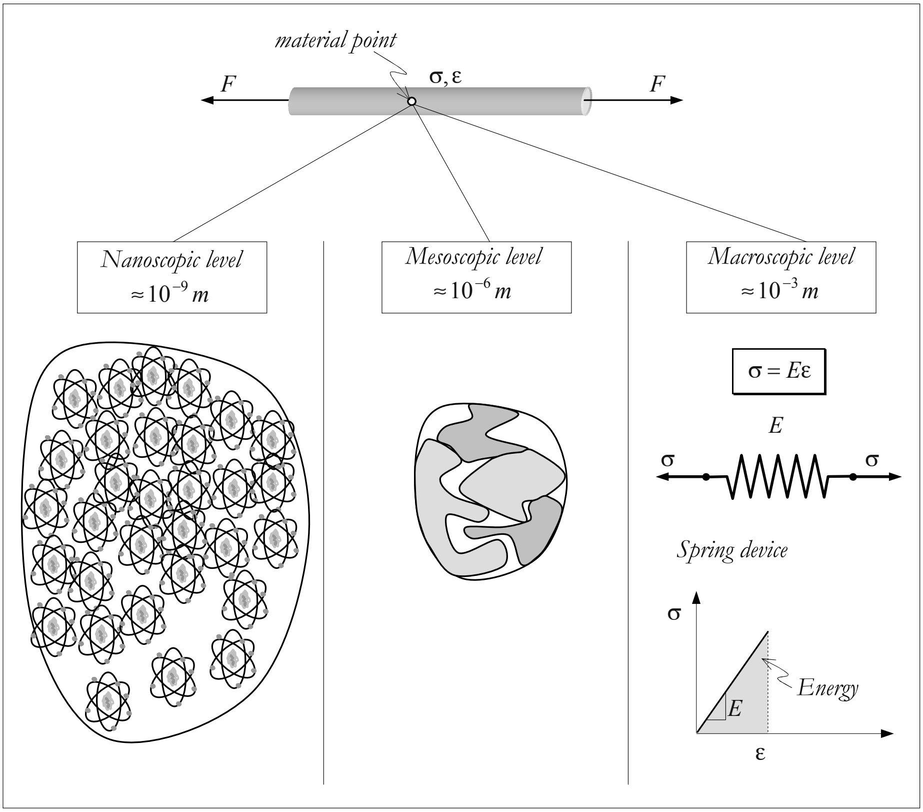


Figure 6.23: Spring device.

Let us suppose now that when a material reaches a certain stress value σ_Y permanent deformation appears, *i.e.* if we remove the load, the material does not recover its initial state. At a macroscopic level we can represent this phenomenon by means of a variable called plastic strain (ε^p) and the mechanical device that represents this phenomenon is the *coulombic frictional device* which has a mechanical parameter equal to σ_Y, (see Figure 6.24).

We have seen before that the viscosity phenomenon is characterized by one that maintains constant stress when the material undergoes strain that evolves over time. One mechanical device that represents this phenomenon is the *dashpot device* which has a mechanical property equal to η_v, *i.e.* viscosity, (see Figure 6.25).

A rheological model characterizes the behavior of the material by means of combining simple mechanical devices. For example, let us consider such a phenomenon that is initially linear elastic and after the material reaches the threshold σ_Y it shows perfect plastic behavior. The rheological model that represents this phenomenon can be made up of a spring device connected in series with a coulombic frictional device, (see Figure 6.26). Note that nothing happens on the coulombic frictional device if the stress is less than σ_Y. In the elastic range $[0, \sigma_Y)$ all the stress is absorbed by the spring device. Then, when the material reaches the value σ_Y the frictional device begins to deform freely and this deformation can not be recovered.

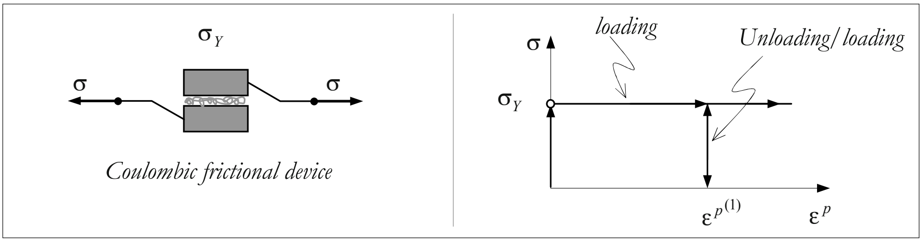


Figure 6.24: Coulombic frictional device.

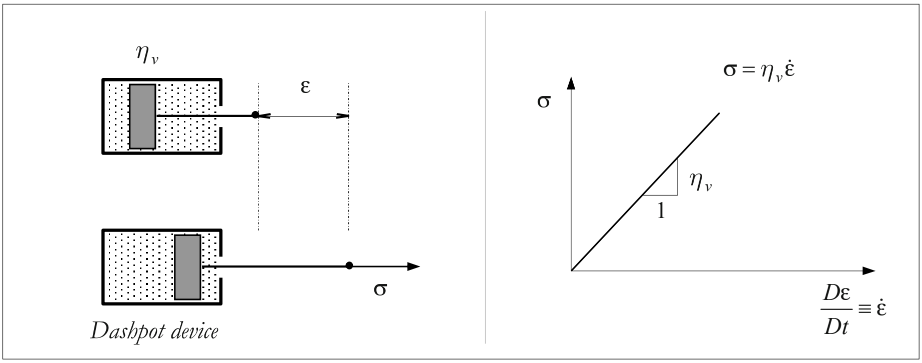


Figure 6.25: Viscous dashpot device.

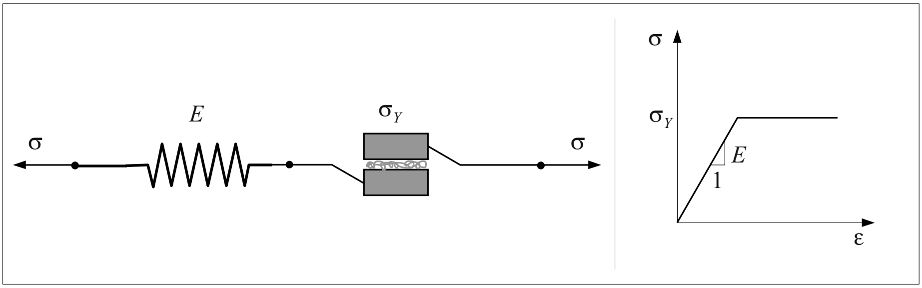


Figure 6.26: The rheological model for perfect elastoplasticity.

The type of device employed and their arrangements (series and/or parallel) depends on the type of material and also how these materials behave during the *loading/unloading/loading process*. This chapter will not go into detail on the mathematical formulation of these models, since each representative model will be established in the relevant chapter.

NOTE: As we have seen in this brief introduction to material behavior, we can start from simple models to develop more complex ones by combining them. Therefore, from now on we will start by studying simple constitutive equations and then go on to more complex ones. ■

7 Linear Elasticity

7.1 Introduction

Approaching the problem via the linear elasticity theory is perfectly acceptable in many practical cases in engineering. Linear elasticity is used when the displacement gradient is sufficiently small when compared with the unity. In this scenario we can apply the infinitesimal strain regime (small deformation) which was discussed in Chapter 2 (see subsection 2.14). In this approach the material strain (Green-Lagrange) and the spatial strain tensor (Almansi) collapse into:

$$\boldsymbol{\varepsilon}(\vec{x},t)=\frac{1}{2}\left[(\nabla\vec{\mathbf{u}})+(\nabla\vec{\mathbf{u}})^{T}\right]=\nabla^{sym}\vec{\mathbf{u}} \qquad \Bigg| \qquad \varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j}+\frac{\partial \mathrm{u}_j}{\partial x_i}\right)=\frac{1}{2}\left(\mathrm{u}_{i,j}+\mathrm{u}_{j,i}\right) \tag{7.1}$$

where $\boldsymbol{\varepsilon}(\vec{x},t)$ is a symmetric second-order tensor known as the *small strain* or *infinitesimal strain tensor*, whose components are explicitly given by:

$$\varepsilon_{ij}=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial \mathrm{u}_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_2}+\dfrac{\partial \mathrm{u}_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_3}+\dfrac{\partial \mathrm{u}_3}{\partial x_1}\right)\\ \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_2}+\dfrac{\partial \mathrm{u}_2}{\partial x_1}\right) & \dfrac{\partial \mathrm{u}_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_2}{\partial x_3}+\dfrac{\partial \mathrm{u}_3}{\partial x_2}\right)\\ \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_1}{\partial x_3}+\dfrac{\partial \mathrm{u}_3}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial \mathrm{u}_2}{\partial x_3}+\dfrac{\partial \mathrm{u}_3}{\partial x_2}\right) & \dfrac{\partial \mathrm{u}_3}{\partial x_3}\end{bmatrix} \tag{7.2}$$

Then, taking into account the following nomenclature used in engineering notation: displacement: $\mathrm{u}_1=u$, $\mathrm{u}_2=v$, $\mathrm{u}_3=w$, and the strain field: $\varepsilon_{11}=\varepsilon_x$, $\varepsilon_{22}=\varepsilon_y$, $\varepsilon_{33}=\varepsilon_z$, $2\varepsilon_{12}=\gamma_{xy}$, $2\varepsilon_{23}=\gamma_{yz}$, $2\varepsilon_{13}=\gamma_{xz}$, the equations in (7.2) can be rewritten as follows:

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_9,

$$\varepsilon_{ij}=\begin{bmatrix}\varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz}\\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz}\\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z\end{bmatrix}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\\ \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z}\end{bmatrix} \tag{7.3}$$

7.2 Initial Boundary Value Problem of Linear Elasticity

Let us consider a three-dimensional body $\mathcal{B}$ (deformed configuration) which has a volume V and mass density ρ. Let S be the boundary of $\mathcal{B}$ and $\hat{\mathbf{n}}$ be the outward unit normal to the surface S. Then, we shall consider that the body is moving under the action of body forces $\vec{\mathbf{b}}(\vec{x})$ and under traction forces $\vec{\mathbf{t}}^*(\vec{x})$ (prescribed value). The boundary consists of a part $S_{\mathbf{u}}$ in which the displacements are prescribed and a part $S_{\boldsymbol{\sigma}}$ where the traction vector is prescribed (surface force), such that $\overline{S_{\mathbf{u}}\cup S_{\boldsymbol{\sigma}}}=S$ and $S_{\mathbf{u}}\cap S_{\boldsymbol{\sigma}}=\varnothing$, (see Figure 7.1).

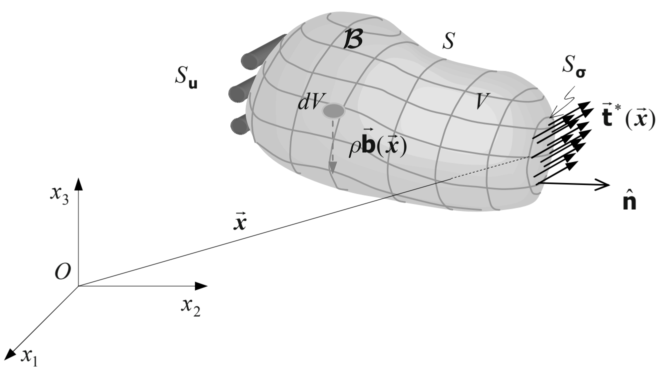


Figure 7.1: Body in motion.

7.2.1 Governing Equations

In **Problem 6.1** we established the governing equations for the linear elasticity problem, *i.e.*:

- The equations of motion:

$$\nabla\cdot\boldsymbol{\sigma}(\vec{x},t)+\rho\vec{\mathbf{b}}(\vec{x},t)=\rho\frac{\partial^2\vec{\mathbf{u}}(\vec{x},t)}{\partial t^2} \qquad \frac{\partial\sigma_{ij}}{\partial x_j}+\rho\mathrm{b}_i=\rho\frac{\partial^2\mathrm{u}_i}{\partial t^2} \tag{7.4}$$

which provide three equations.

- The constitutive equations for stress:

$$\boldsymbol{\sigma}=\mathbb{C}:\boldsymbol{\varepsilon} \qquad \sigma_{ij}=\mathbb{C}_{ijkl}\,\varepsilon_{kl} \tag{7.5}$$

which provide six equations.

- The kinematic equations (or strain-displacement equations):

$$\boldsymbol{\varepsilon}(\vec{x},t) = \nabla^{sym}\vec{\mathbf{u}}(\vec{x},t) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) \tag{7.6}$$

which provide six equations.

For the problem in hand there are a total of 15 equations whose unknowns are: the displacement vector $\vec{\mathbf{u}}(\vec{x},t)$ (with 3 unknowns); the strain tensor $\boldsymbol{\varepsilon}(\vec{x},t)$ (with 6 unknowns) and the stress tensor $\boldsymbol{\sigma}$ (with 6 unknowns) which results in a total of 15 unknowns, so, we have a fully established problem when given the appropriate initial and boundary conditions.

7.2.2 Initial and Boundary Conditions

The displacement boundary condition on the part $S_{\mathbf{u}}$ of the surface:

$$\vec{\mathbf{u}}(\vec{x},t) = \vec{\mathbf{u}}^*(\vec{x},t) \qquad \mathrm{u}_i(\vec{x},t) = \mathrm{u}_i^{\,*}(\vec{x},t) \tag{7.7}$$

The stress boundary condition on the part $S_{\boldsymbol{\sigma}}$ of the surface:

$$\boldsymbol{\sigma}(\vec{x},t)\cdot\hat{\mathbf{n}} = \vec{\mathbf{t}}^*(\vec{x},\hat{\mathbf{n}},t) \qquad \sigma_{jk}\mathrm{n}_k = \mathrm{t}_j^{\,*}(\vec{x},t) \tag{7.8}$$

Initial condition (at $t=0$):

$$\vec{\mathbf{u}}(\vec{x},t=0) = \vec{\mathbf{u}}_0 \qquad \mathrm{u}_i(\vec{x},t=0) = \mathrm{u}_{0i}(\vec{x})$$

$$\left.\frac{\partial \vec{\mathbf{u}}_0(\vec{x},t)}{\partial t}\right|_{t=0} = \dot{\vec{\mathbf{u}}}_0(\vec{x},t) = \vec{v}_0(\vec{x}) \qquad \dot{\mathrm{u}}_{0i}(\vec{x}) = v_{0i} \tag{7.9}$$

In the particular case when we have a static or quasi-static problem, the equation of motion becomes the equilibrium equations ($\nabla\cdot\boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \vec{\mathbf{0}}$), and the initial conditions become redundant.

7.3 Generalized Hooke's Law

The stress-strain linear elastic relationship ($\boldsymbol{\sigma}-\boldsymbol{\varepsilon}$) in its broadest form is known as the *generalized Hooke's law* which is given by the following equations:

Tensorial notation	Indicial notation	
$\boxed{\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}}$	$\sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{kl}$	(7.10)

where $\mathbb{C}$ is known as the *elasticity tensor* (or *elastic stiffness tensor*), which is of the symmetric fourth-order type and contains the elastic constants (the material properties). In **Problem 6.1** we showed that $\mathbb{C}$ has both minor ($\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{ijlk}$), and major ($\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$) symmetry, so the tensor features 21 independent components. It is said that a material is homogeneous when its elastic properties do not vary from point to point throughout the continuum, *i.e.* $\mathbb{C}$ is independent of the position vector. Moreover, a material is said to be isotropic at any point when the components of $\mathbb{C}$ do not change if the reference system undergoes a base change.

7.3.1 The Generalized Hooke's Law in Voigt Notation

By referring to the Cauchy stress tensor symmetry we can use Voigt notation (see Chapter 1) to store the tensor components as follows:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \xrightarrow{Voigt} \{\boldsymbol{\sigma}\} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{x} \\ \sigma_{y} \\ \sigma_{z} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}}_{\substack{Engineering \\ Notation}} \tag{7.11}$$

Each Cauchy stress tensor component (σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{23}, σ_{13}) can be obtained by using the constitutive equation in (7.10). Then by expanding said equation for the component σ_{11}, we find:

$$\sigma_{11} = \mathbb{C}_{11kl}\,\varepsilon_{kl} \Rightarrow \quad \sigma_{11} = \underbrace{\mathbb{C}_{111l}\,\varepsilon_{1l}}_{\substack{\mathbb{C}_{1111}\,\varepsilon_{11} \\ + \\ \mathbb{C}_{1112}\,\varepsilon_{12} \\ + \\ \mathbb{C}_{1113}\,\varepsilon_{13}}} + \underbrace{\mathbb{C}_{112l}\,\varepsilon_{2l}}_{\substack{\mathbb{C}_{1121}\,\varepsilon_{21} \\ + \\ \mathbb{C}_{1122}\,\varepsilon_{22} \\ + \\ \mathbb{C}_{1123}\,\varepsilon_{23}}} + \underbrace{\mathbb{C}_{113l}\,\varepsilon_{3l}}_{\substack{\mathbb{C}_{1131}\,\varepsilon_{31} \\ + \\ \mathbb{C}_{1132}\,\varepsilon_{32} \\ + \\ \mathbb{C}_{1133}\,\varepsilon_{33}}} \tag{7.12}$$

Then as we have the minor symmetry ($\varepsilon_{kl} = \varepsilon_{lk} \leftrightarrow \mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}$) the above equation becomes $\sigma_{11} = \mathbb{C}_{1111}\,\varepsilon_{11} + \mathbb{C}_{1122}\,\varepsilon_{22} + \mathbb{C}_{1133}\,\varepsilon_{33} + 2\mathbb{C}_{1112}\varepsilon_{12} + 2\mathbb{C}_{1123}\,\varepsilon_{23} + 2\mathbb{C}_{1113}\,\varepsilon_{13}$.

Likewise, we can obtain the expressions for σ_{22}, σ_{33}, σ_{12}, σ_{23} and σ_{13}, and if we then reorder them in matrix form we can obtain the generalized Hooke's law in Voigt notation, *i.e.*:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} \mathbb{C}_{1111} & \mathbb{C}_{1122} & \mathbb{C}_{1133} & \mathbb{C}_{1112} & \mathbb{C}_{1123} & \mathbb{C}_{1113} \\ \mathbb{C}_{2211} & \mathbb{C}_{2222} & \mathbb{C}_{2233} & \mathbb{C}_{2212} & \mathbb{C}_{2223} & \mathbb{C}_{2213} \\ \mathbb{C}_{3311} & \mathbb{C}_{3322} & \mathbb{C}_{3333} & \mathbb{C}_{3312} & \mathbb{C}_{3323} & \mathbb{C}_{3313} \\ \mathbb{C}_{1211} & \mathbb{C}_{1222} & \mathbb{C}_{1233} & \mathbb{C}_{1212} & \mathbb{C}_{1223} & \mathbb{C}_{1213} \\ \mathbb{C}_{2311} & \mathbb{C}_{2322} & \mathbb{C}_{2333} & \mathbb{C}_{2312} & \mathbb{C}_{2323} & \mathbb{C}_{2313} \\ \mathbb{C}_{1311} & \mathbb{C}_{1322} & \mathbb{C}_{1333} & \mathbb{C}_{1312} & \mathbb{C}_{1323} & \mathbb{C}_{1313} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} \quad \Rightarrow \quad \{\boldsymbol{\sigma}\} = [\mathcal{C}]\{\boldsymbol{\varepsilon}\} \tag{7.13}$$

where $[\mathcal{C}]$ is the matrix with mechanical elastic properties. Then by application of major symmetry, *i.e.* $\mathbb{C}_{ijkl} = \mathbb{C}_{klij}$, the elasticity tensor components can be expressed in Voigt notation as follows:

$$[\mathcal{C}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} & \mathcal{C}_{16} \\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} & \mathcal{C}_{26} \\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} & \mathcal{C}_{36} \\ \mathcal{C}_{14} & \mathcal{C}_{24} & \mathcal{C}_{34} & \mathcal{C}_{44} & \mathcal{C}_{45} & \mathcal{C}_{46} \\ \mathcal{C}_{15} & \mathcal{C}_{25} & \mathcal{C}_{35} & \mathcal{C}_{45} & \mathcal{C}_{55} & \mathcal{C}_{56} \\ \mathcal{C}_{16} & \mathcal{C}_{26} & \mathcal{C}_{36} & \mathcal{C}_{46} & \mathcal{C}_{56} & \mathcal{C}_{66} \end{bmatrix} \tag{7.14}$$

The equation in (7.13) indicates that the strain tensor components in Voigt notation are shown in the following format:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \xrightarrow{Voigt} \{\boldsymbol{\varepsilon}\} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{bmatrix} = \underbrace{\begin{bmatrix} \varepsilon_{x} \\ \varepsilon_{y} \\ \varepsilon_{z} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}}_{\substack{Engineering \\ Notation}} \tag{7.15}$$

Note that as the double scalar product $\boldsymbol{\sigma}:\boldsymbol{\varepsilon}$ has the unit of stored energy (see **Problem 6.1**), then, this energy must be the same as when it is obtained either by $\boldsymbol{\sigma}:\boldsymbol{\varepsilon}$ or when $\{\boldsymbol{\sigma}\}^T\{\boldsymbol{\varepsilon}\}$, *i.e.*:

$$\frac{1}{2}\underbrace{\boldsymbol{\sigma}:\boldsymbol{\varepsilon}}_{Tensorial} = \frac{1}{2}(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{23}\varepsilon_{23} + 2\sigma_{13}\varepsilon_{13}) = \frac{1}{2}\underbrace{\{\boldsymbol{\sigma}\}^T\{\boldsymbol{\varepsilon}\}}_{Voigt} \tag{7.16}$$

7.3.2 The Component Transformation Law for the Generalized Hooke's Law

If we consider the Cartesian coordinate system x_1, x_2, x_3, the stress-strain relationship is set by means of the generalized Hooke's law:

Indicial notation	Voigt notation	
$\sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{kl}$	$\{\boldsymbol{\sigma}\} = [\mathcal{C}]\{\boldsymbol{\varepsilon}\}$	(7.17)

These components are affected by any change to the coordinate system. Then, given the new coordinate system x'_1, x'_2, x'_3, the generalized Hooke's law therein is given by:

Indicial notation	Voigt notation	
$\sigma'_{ij} = \mathbb{C}'_{ijkl}\,\varepsilon'_{kl}$	$\{\boldsymbol{\sigma}'\} = [\mathcal{C}']\{\boldsymbol{\varepsilon}'\}$	(7.18)

where σ'_{ij}, ε'_{kl} and $\mathbb{C}'_{ijkl}$ show the stress, the strain, and the elasticity tensor components in the new system x'_1, x'_2, x'_3 respectively which are explicitly given in Voigt notation, by:

$$\{\boldsymbol{\varepsilon}'\} = \begin{bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ 2\varepsilon'_{12} \\ 2\varepsilon'_{23} \\ 2\varepsilon'_{13} \end{bmatrix} = \begin{bmatrix} \varepsilon'_{xx} \\ \varepsilon'_{yy} \\ \varepsilon'_{zz} \\ 2\varepsilon'_{xy} \\ 2\varepsilon'_{yz} \\ 2\varepsilon'_{xz} \end{bmatrix} = \begin{bmatrix} \varepsilon'_{x} \\ \varepsilon'_{y} \\ \varepsilon'_{z} \\ \gamma'_{xy} \\ \gamma'_{yz} \\ \gamma'_{xz} \end{bmatrix} \quad ; \quad \{\boldsymbol{\sigma}'\} = \begin{bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{13} \end{bmatrix} = \begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{xz} \end{bmatrix} = \begin{bmatrix} \sigma'_{x} \\ \sigma'_{y} \\ \sigma'_{z} \\ \tau'_{xy} \\ \tau'_{yz} \\ \tau'_{xz} \end{bmatrix} \tag{7.19}$$

and

$$[\mathcal{C}']=\begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & C'_{14} & C'_{15} & C'_{16} \\ C'_{12} & C'_{22} & C'_{23} & C'_{24} & C'_{25} & C'_{26} \\ C'_{13} & C'_{23} & C'_{33} & C'_{34} & C'_{35} & C'_{36} \\ C'_{14} & C'_{24} & C'_{34} & C'_{44} & C'_{45} & C'_{46} \\ C'_{15} & C'_{25} & C'_{35} & C'_{45} & C'_{55} & C'_{56} \\ C'_{16} & C'_{26} & C'_{36} & C'_{46} & C'_{56} & C'_{66} \end{bmatrix} \tag{7.20}$$

We will next establish the transformation law for these tensor components in Voigt notation.

7.3.2.1 The Matrix Transformation for Stress and Strain Components

At a given point in the continuum, the stress and the strain tensor components related to the system x_1, x_2, x_3, are represented by σ_{ij} and ε_{ij}, respectively. Then, the components of these second-order tensors (σ'_{ij}, ε'_{ij}), in the new system, can be obtained as follows:

Indicial notation	Matrix notation	
$\sigma'_{ij}=a_{ik}a_{jl}\sigma_{kl}$	$\boldsymbol{\sigma}'=\mathcal{A}\,\boldsymbol{\sigma}\,\mathcal{A}^T$	(7.21)

Indicial notation	Matrix notation	
$\varepsilon'_{ij}=a_{ik}a_{jl}\varepsilon_{kl}$	$\boldsymbol{\varepsilon}'=\mathcal{A}\,\boldsymbol{\varepsilon}\,\mathcal{A}^T$	(7.22)

where $\mathcal{A}$ is the transformation matrix which is denoted by:

$$a_{ij}=\mathcal{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{7.23}$$

In Voigt notation the transformation laws in (7.21) and (7.22), (see Chapter 1), are given respectively by:

$$\{\boldsymbol{\sigma}'\}=[\mathcal{M}]\{\boldsymbol{\sigma}\}\xrightarrow{inverse}\{\boldsymbol{\sigma}\}=[\mathcal{M}]^{-1}\{\boldsymbol{\sigma}'\} \tag{7.24}$$

$$\{\boldsymbol{\varepsilon}'\}=[\mathcal{N}]\{\boldsymbol{\varepsilon}\}\xrightarrow{inverse}\{\boldsymbol{\varepsilon}\}=[\mathcal{N}]^{-1}\{\boldsymbol{\varepsilon}'\} \tag{7.25}$$

where $[\mathcal{M}]$ is the transformation matrix of the stress tensor components in Voigt notation, which is given explicitly by:

$$[\mathcal{M}]=\begin{bmatrix} a_{11}{}^2 & a_{12}{}^2 & a_{13}{}^2 & 2a_{11}a_{12} & 2a_{12}a_{13} & 2a_{11}a_{13} \\ a_{21}{}^2 & a_{22}{}^2 & a_{23}{}^2 & 2a_{21}a_{22} & 2a_{22}a_{23} & 2a_{21}a_{23} \\ a_{31}{}^2 & a_{32}{}^2 & a_{33}{}^2 & 2a_{31}a_{32} & 2a_{32}a_{33} & 2a_{31}a_{33} \\ a_{21}a_{11} & a_{22}a_{12} & a_{13}a_{23} & (a_{11}a_{22}+a_{12}a_{21}) & (a_{13}a_{22}+a_{12}a_{23}) & (a_{13}a_{21}+a_{11}a_{23}) \\ a_{31}a_{21} & a_{32}a_{22} & a_{33}a_{23} & (a_{31}a_{22}+a_{32}a_{21}) & (a_{33}a_{22}+a_{32}a_{23}) & (a_{33}a_{21}+a_{31}a_{23}) \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & (a_{31}a_{12}+a_{32}a_{11}) & (a_{33}a_{12}+a_{32}a_{13}) & (a_{33}a_{11}+a_{31}a_{13}) \end{bmatrix} \tag{7.26}$$

and $[\mathcal{N}]$ is the transformation matrix of the strain tensor components in Voigt notation:

$$[\mathcal{N}] = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{11}a_{12} & a_{12}a_{13} & a_{11}a_{13} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & a_{21}a_{22} & a_{22}a_{23} & a_{21}a_{23} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & a_{31}a_{32} & a_{32}a_{33} & a_{31}a_{33} \\ 2a_{21}a_{11} & 2a_{22}a_{12} & 2a_{13}a_{23} & (a_{11}a_{22}+a_{12}a_{21}) & (a_{13}a_{22}+a_{12}a_{23}) & (a_{13}a_{21}+a_{11}a_{23}) \\ 2a_{31}a_{21} & 2a_{32}a_{22} & 2a_{33}a_{23} & (a_{31}a_{22}+a_{32}a_{21}) & (a_{33}a_{22}+a_{32}a_{23}) & (a_{33}a_{21}+a_{31}a_{23}) \\ 2a_{31}a_{11} & 2a_{32}a_{12} & 2a_{33}a_{13} & (a_{31}a_{12}+a_{32}a_{11}) & (a_{33}a_{12}+a_{32}a_{13}) & (a_{33}a_{11}+a_{31}a_{13}) \end{bmatrix} \tag{7.27}$$

Additionally, it can be shown that $[\mathcal{M}]$ and $[\mathcal{N}]$ are not orthogonal matrices, *i.e.* $[\mathcal{M}]^{-1} \neq [\mathcal{M}]^T$ and $[\mathcal{N}]^{-1} \neq [\mathcal{N}]^T$, and that:

$$[\mathcal{M}]^T = [\mathcal{N}]^{-1} \tag{7.28}$$

7.3.2.2 The Transformation Matrix of the Elasticity Tensor Components

As we saw in Chapter 1, the transformation law of the fourth-order tensor components is given by $\mathbb{C}'_{ijkl} = a_{ip}a_{jq}a_{kr}a_{ls}\mathbb{C}_{pqrs}$. Our goal now is to express the transformation law of the elasticity tensor components in Voigt notation. To do this we will use the equation in (7.17) as a starting point in order to obtain:

$$\begin{aligned} \{\boldsymbol{\sigma}\} &= [\mathcal{C}]\{\boldsymbol{\varepsilon}\} \\ [\mathcal{M}]\{\boldsymbol{\sigma}\} &= [\mathcal{M}][\mathcal{C}]\overbrace{[\mathcal{N}]^{-1}\{\boldsymbol{\varepsilon}'\}}^{\{\boldsymbol{\varepsilon}\}} \\ \{\boldsymbol{\sigma}'\} &= [\mathcal{M}][\mathcal{C}][\mathcal{N}]^{-1}\{\boldsymbol{\varepsilon}'\} \\ \{\boldsymbol{\sigma}'\} &= [\mathcal{M}][\mathcal{C}]\,[\mathcal{M}]^T\{\boldsymbol{\varepsilon}'\} \\ \{\boldsymbol{\sigma}'\} &= [\mathcal{C}']\{\boldsymbol{\varepsilon}'\} \end{aligned} \tag{7.29}$$

where $[\mathcal{C}']$ is the elasticity matrix in the new system $(x'_1,\ x'_2,\ x'_3)$. Therefore, we can define the transformation law of the elasticity tensor components in Voigt notation as:

$$\boxed{[\mathcal{C}'] = [\mathcal{M}][\mathcal{C}]\,[\mathcal{M}]^T} \quad \textit{Transformation law of the elasticity tensor components in Voigt notation} \tag{7.30}$$

7.4 The Elasticity Tensor

7.4.1 Anisotropy and Isotropy

Materials in general are anisotropic, *i.e.*, material properties have different values for different directions at any given point. Certain kinds of these at the microscopic and mesoscopic scale have anisotropic properties, such as: concrete (made by mixing different materials), but at a macroscopic level these properties can be considered as an average of these properties at the mesoscopic scale, so, it is possible to consider material as macroscopically isotropic, *i.e.*, material properties are independent of the coordinates system adopted. There are materials such as wood or man-made materials, *e.g.* composite materials, which are made up of fibers that are directionally oriented and embedded in a matrix, hence these materials exhibit a clear anisotropy even at the macroscopic level.

Most materials have some kind of symmetry along one or more axes, *i.e.* these axes can be reversed without changing the material properties. For example, Figure 7.2(b) shows one plane of symmetry, the plane $x_1 - x_2$, which implies that if we are dealing with just one of these we can change a coordinate system from say x_1, x_2, x_3 to x_1', x_2', x_3' without altering the elastic properties of the material. Then we can see another example of symmetry in Figure 7.2(c) in which two planes of symmetry are displayed, namely: $x_1 - x_2$ and $x_2 - x_3$. Remember that the transformation law from x_1-x_2-x_3 to x_1'-x_2'-x_3' system is given by the equation:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{7.31}$$

Next, we will study the different types of symmetry that appear in materials which may include: one plane of symmetry (monoclinic symmetry), two planes of symmetry (orthotropic symmetry), tetragonal symmetry, transversely isotropic symmetry, cubic symmetry, and finally symmetry in all orientation (isotropy).

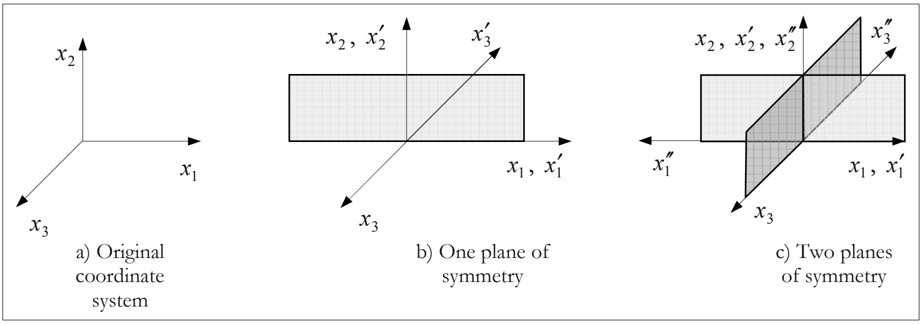


Figure 7.2: Symmetry planes.

7.4.2 Types of Elasticity Tensor Symmetry

7.4.2.1 Triclinic Materials

Triclinic materials are the most generic of anisotropic materials, *i.e.* there are no symmetry planes. Then, the elasticity tensor features 21 independent components to be determined in the laboratory:

$$[\mathcal{C}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & \mathcal{C}_{15} & \mathcal{C}_{16} \\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & \mathcal{C}_{25} & \mathcal{C}_{26} \\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & \mathcal{C}_{34} & \mathcal{C}_{35} & \mathcal{C}_{36} \\ \mathcal{C}_{14} & \mathcal{C}_{24} & \mathcal{C}_{34} & \mathcal{C}_{44} & \mathcal{C}_{45} & \mathcal{C}_{46} \\ \mathcal{C}_{15} & \mathcal{C}_{25} & \mathcal{C}_{35} & \mathcal{C}_{45} & \mathcal{C}_{55} & \mathcal{C}_{56} \\ \mathcal{C}_{16} & \mathcal{C}_{26} & \mathcal{C}_{36} & \mathcal{C}_{46} & \mathcal{C}_{56} & \mathcal{C}_{66} \end{bmatrix} \qquad \begin{array}{l} \textit{Triclinic materials} \\ \text{21 independent components} \end{array} \tag{7.32}$$

NOTE: The main drawback when dealing with high material anisotropy is the extreme complexity that appears at the time of obtaining the constants (material properties) in the laboratory. ■

7.4.2.2 Monoclinic Symmetry (One Plane of Symmetry)

Let us now consider a material that has a single plane of symmetry (plane $x_1 - x_2$) as illustrated in Figure 7.2(b). Then, the transformation law between the systems defined in Figure 7.2(a) and Figure 7.2(b) is given by:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{7.33}$$

with which we can obtain the transformation matrix ($[\mathcal{M}]$), previously defined in (7.26), as:

$$[\mathcal{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{7.34}$$

Then, to obtain the elasticity matrix in this new system, we can carry out the following matrix operation:

$$[\mathcal{C}'] = [\mathcal{M}][\mathcal{C}]\,[\mathcal{M}]^T \tag{7.35}$$

the result of which is:

$$[\mathcal{C}'] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & -\mathcal{C}_{15} & -\mathcal{C}_{16} \\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & -\mathcal{C}_{25} & -\mathcal{C}_{26} \\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & \mathcal{C}_{34} & -\mathcal{C}_{35} & -\mathcal{C}_{36} \\ \mathcal{C}_{14} & \mathcal{C}_{24} & \mathcal{C}_{34} & \mathcal{C}_{44} & -\mathcal{C}_{45} & -\mathcal{C}_{46} \\ -\mathcal{C}_{15} & -\mathcal{C}_{25} & -\mathcal{C}_{35} & -\mathcal{C}_{45} & \mathcal{C}_{55} & \mathcal{C}_{56} \\ -\mathcal{C}_{16} & -\mathcal{C}_{26} & -\mathcal{C}_{36} & -\mathcal{C}_{46} & \mathcal{C}_{56} & \mathcal{C}_{66} \end{bmatrix} \tag{7.36}$$

Since in this specific transformation, the elasticity matrix must provide symmetry, *i.e.* $[\mathcal{C}'] = [\mathcal{C}]$, we can draw the conclusion that the terms in which negative signs appear should be zero so as to satisfy the symmetry condition. Then for materials that exhibit one plane of symmetry, the elasticity matrix has 13 independent components, namely:

$$\boxed{[\mathcal{C}] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{14} & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & \mathcal{C}_{24} & 0 & 0 \\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & \mathcal{C}_{34} & 0 & 0 \\ \mathcal{C}_{14} & \mathcal{C}_{24} & \mathcal{C}_{34} & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & \mathcal{C}_{56} \\ 0 & 0 & 0 & 0 & \mathcal{C}_{56} & \mathcal{C}_{66} \end{bmatrix}} \quad \begin{array}{l} \textit{Monoclinic Symmetry} \\ \text{13 independent constants} \end{array} \tag{7.37}$$

7.4.2.3 Orthotropic Symmetry (Two Planes of Symmetry)

We will start from monoclinic symmetry to define the elasticity matrix format for a material with two planes of symmetry. Then, by means of the transformation law between the systems defined in Figure 7.2(b) and Figure 7.2(c) we obtain:

$$\mathcal{A}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\longrightarrow[\mathcal{M}]=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\end{bmatrix} \tag{7.38}$$

where we have considered the equation in (7.26) in order to evaluate the matrix $[\mathcal{M}]$. Then, to obtain the elasticity tensor components in the system $\boldsymbol{x}''$, we can carry out the following matrix operation $[\mathcal{C}'']=[\mathcal{M}][\mathcal{C}'][\mathcal{M}]^T$ the result of which is:

$$[\mathcal{C}'']=\begin{bmatrix}\mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & -\mathcal{C}_{14} & 0 & 0\\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & -\mathcal{C}_{24} & 0 & 0\\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & -\mathcal{C}_{34} & 0 & 0\\ -\mathcal{C}_{14} & -\mathcal{C}_{24} & -\mathcal{C}_{34} & \mathcal{C}_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & -\mathcal{C}_{56}\\ 0 & 0 & 0 & 0 & -\mathcal{C}_{56} & \mathcal{C}_{66}\end{bmatrix} \tag{7.39}$$

In this specific transformation the following must be satisfied $[\mathcal{C}'']=[\mathcal{C}']=[\mathcal{C}]$, with which we can draw the conclusion that the elasticity matrix features 9 independent constants to be determined:

$$\boxed{[\mathcal{C}]=\begin{bmatrix}\mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & 0 & 0 & 0\\ \mathcal{C}_{12} & \mathcal{C}_{22} & \mathcal{C}_{23} & 0 & 0 & 0\\ \mathcal{C}_{13} & \mathcal{C}_{23} & \mathcal{C}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{66}\end{bmatrix}} \quad \begin{array}{l}\textit{Orthotropic Symmetry}\\ \text{9 independent constants}\end{array} \tag{7.40}$$

NOTE: Materials such as bones present a high degree of anisotropy. Nevertheless, some researchers consider two planes of symmetry (orthotropic symmetry) to simulate numerically the bone behavior. ■

7.4.2.4 Tetragonal Symmetry

Materials with tetragonal symmetry have 5 planes of symmetry one of which is the x_1-x_2 plane and the other 4 are indicated in Figure 7.3. Note that this type also includes orthotropic symmetry. So, to determine the format in which the matrix $[\mathcal{C}]$ is presented we start from the elasticity matrix given in (7.40). Next, we apply the symmetry condition according to the transformation from $x_1-x_2-x_3$ to $x_1'-x_2'-x_3'=x_3$, in which the transformation matrix is given by:

$$\mathcal{A}=\begin{bmatrix}\cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) & 0\\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) & 0\\ 0 & 0 & 1\end{bmatrix}\longrightarrow[\mathcal{M}]=\begin{bmatrix}0.5 & 0.5 & 0 & 1 & 0 & 0\\ 0.5 & 0.5 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -0.5 & 0.5 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} \tag{7.41}$$

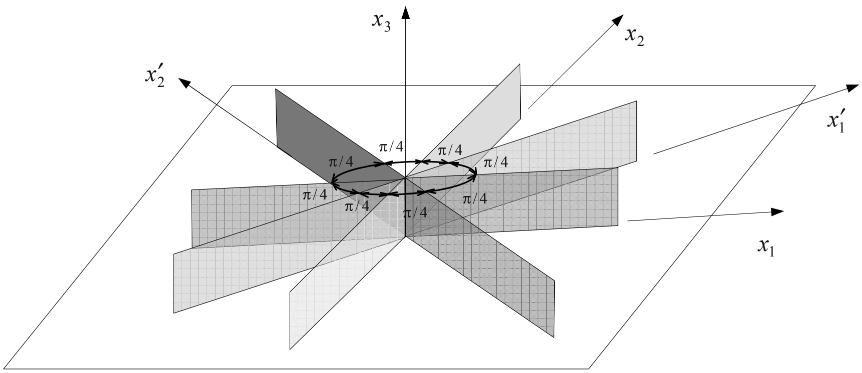


Figure 7.3: Tetragonal symmetry.

The elasticity matrix components in this new system can then be obtained by the transformation law $[\mathcal{C}']=[\mathcal{M}][\mathcal{C}]\,[\mathcal{M}]^T$ which becomes:

$$\underbrace{\begin{bmatrix}\left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)+\mathcal{C}_{44} & \left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)-\mathcal{C}_{44} & \frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2} & \frac{\mathcal{C}_{22}-\mathcal{C}_{11}}{4} & 0 & 0\\ \left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)-\mathcal{C}_{44} & \left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)+\mathcal{C}_{44} & \frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2} & \frac{\mathcal{C}_{22}-\mathcal{C}_{11}}{4} & 0 & 0\\ \frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2} & \left(\frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2}\right) & \mathcal{C}_{33} & \frac{\mathcal{C}_{23}-\mathcal{C}_{13}}{2} & 0 & 0\\ \frac{\mathcal{C}_{22}-\mathcal{C}_{11}}{4} & \left(\frac{\mathcal{C}_{22}-\mathcal{C}_{11}}{4}\right) & \frac{\mathcal{C}_{23}-\mathcal{C}_{13}}{2} & \frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\mathcal{C}_{55}+\mathcal{C}_{66}}{2} & \frac{\mathcal{C}_{55}-\mathcal{C}_{66}}{2}\\ 0 & 0 & 0 & 0 & \frac{\mathcal{C}_{55}-\mathcal{C}_{66}}{2} & \frac{\mathcal{C}_{55}+\mathcal{C}_{66}}{2}\end{bmatrix}}_{[\mathcal{C}']} \tag{7.42}$$

Note that the plane $(x_1'-x_3)$ is also a plane of symmetry, (see Figure 7.3), so, the components of (7.42) must be equal to the components obtained from the following coordinate transformation:

$$\mathcal{A}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\longrightarrow[\mathcal{M}]=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \tag{7.43}$$

Then, by once again applying the transformation seen in (7.30), we can obtain the following matrix:

$$\underbrace{\begin{bmatrix}\left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)+\mathcal{C}_{44} & \left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)-\mathcal{C}_{44} & \frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2} & \frac{\mathcal{C}_{11}-\mathcal{C}_{22}}{4} & 0 & 0\\ \left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)-\mathcal{C}_{44} & \left(\frac{\mathcal{C}_{11}+\mathcal{C}_{22}+2\mathcal{C}_{12}}{4}\right)+\mathcal{C}_{44} & \frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2} & \frac{\mathcal{C}_{11}-\mathcal{C}_{22}}{4} & 0 & 0\\ \frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2} & \left(\frac{\mathcal{C}_{13}+\mathcal{C}_{23}}{2}\right) & \mathcal{C}_{33} & \frac{\mathcal{C}_{13}-\mathcal{C}_{23}}{2} & 0 & 0\\ \frac{\mathcal{C}_{22}-\mathcal{C}_{11}}{4} & \left(\frac{\mathcal{C}_{22}-\mathcal{C}_{11}}{4}\right) & \frac{\mathcal{C}_{13}-\mathcal{C}_{23}}{2} & \frac{\mathcal{C}_{11}+\mathcal{C}_{22}-2\mathcal{C}_{12}}{4} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\mathcal{C}_{55}+\mathcal{C}_{66}}{2} & \frac{\mathcal{C}_{66}-\mathcal{C}_{55}}{2}\\ 0 & 0 & 0 & 0 & \frac{\mathcal{C}_{66}-\mathcal{C}_{55}}{2} & \frac{\mathcal{C}_{55}+\mathcal{C}_{66}}{2}\end{bmatrix}}_{[\mathcal{C}']} \tag{7.44}$$

and by comparing the two matrices given in (7.42) and in (7.44), we can conclude that $\mathcal{C}_{11}=\mathcal{C}_{22}$, $\mathcal{C}_{55}=\mathcal{C}_{66}$, $\mathcal{C}_{13}=\mathcal{C}_{23}$. Then, the elasticity matrix for tetragonal symmetry material features 6 independent constants to be determined:

$$\boxed{[\mathcal{C}]=\begin{bmatrix}\mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & 0 & 0 & 0\\ \mathcal{C}_{12} & \mathcal{C}_{11} & \mathcal{C}_{13} & 0 & 0 & 0\\ \mathcal{C}_{13} & \mathcal{C}_{13} & \mathcal{C}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55}\end{bmatrix}} \quad \begin{array}{l}\textit{Tetragonal symmetry}\\ \text{6 independent constants}\end{array} \tag{7.45}$$

7.4.2.5 Transversely Isotropic Symmetry (Hexagonal Symmetry)

Material with transversely isotropic symmetry already includes orthotropic symmetry, (see Eq. (7.40)). In addition, any transformation on the plane x_1-x_2 is also a plane of symmetry, (see Figure 7.4).

For these kinds of material x_1-x_2 and x_2-x_3 are planes of symmetry, *i.e.* they have orthotropic symmetry, so, by starting from the elasticity matrix for orthotropic symmetry in (7.40) and by some transformations on the plane x_1-x_2 we can obtain the constants. Initially, let us consider a transformation on the plane x_1-x_2 characterized by the angle $\alpha=90^\circ$, (see Figure 7.4), with which we can obtain the following transformation matrices:

$$\mathcal{A}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\longrightarrow[\mathcal{M}]=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \tag{7.46}$$

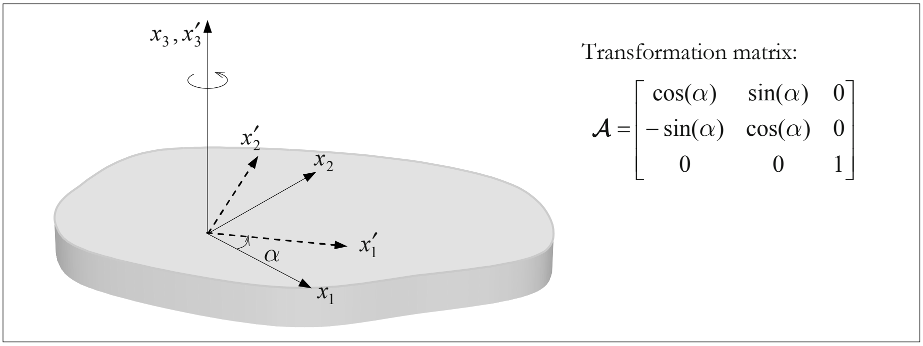


Figure 7.4: Transversely isotropic symmetry.

Then, using the equation in (7.30) we can obtain the elasticity matrix in this new system:

$$[\mathcal{C}']=\begin{bmatrix}\mathcal{C}_{22} & \mathcal{C}_{12} & \mathcal{C}_{23} & 0 & 0 & 0\\ \mathcal{C}_{12} & \mathcal{C}_{11} & \mathcal{C}_{13} & 0 & 0 & 0\\ \mathcal{C}_{23} & \mathcal{C}_{13} & \mathcal{C}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{C}_{66} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55}\end{bmatrix} \tag{7.47}$$

and by comparing (7.40) and (7.47) we can deduce that: $\mathcal{C}_{11}=\mathcal{C}_{22}$, $\mathcal{C}_{23}=\mathcal{C}_{13}$, $\mathcal{C}_{55}=\mathcal{C}_{66}$.

Now, if we consider a transformation characterized by the angle $\alpha=45^{\circ}$, the transformation matrix $[\mathcal{M}]$ becomes:

$$\mathcal{A}=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\longrightarrow[\mathcal{M}]=\begin{bmatrix}0.5 & 0.5 & 0 & 1 & 0 & 0\\ 0.5 & 0.5 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -0.5 & 0.5 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} \tag{7.48}$$

Then, by using the equation in (7.30) we obtain $[\mathcal{C}']$ as:

$$[\mathcal{C}']=\begin{bmatrix}\frac{1}{2}(\mathcal{C}_{11}+\mathcal{C}_{12})+\mathcal{C}_{44} & \frac{1}{2}(\mathcal{C}_{22}+\mathcal{C}_{12})-\mathcal{C}_{44} & \mathcal{C}_{13} & 0 & 0 & 0\\ \frac{1}{2}(\mathcal{C}_{22}+\mathcal{C}_{12})-\mathcal{C}_{44} & \frac{1}{2}(\mathcal{C}_{22}+\mathcal{C}_{12})+\mathcal{C}_{44} & \mathcal{C}_{13} & 0 & 0 & 0\\ \mathcal{C}_{13} & \mathcal{C}_{13} & \mathcal{C}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}(\mathcal{C}_{11}-\mathcal{C}_{12}) & 0 & 0\\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55}\end{bmatrix} \tag{7.49}$$

Afterwards, if we compare (7.49) with (7.47) we can draw the conclusion that $\mathcal{C}_{44}=\frac{1}{2}(\mathcal{C}_{11}-\mathcal{C}_{12})$, hence the matrix $[\mathcal{C}]$ features 5 independent constants. Note that, any other transformation on the plane x_1-x_2 will not reduce the number of constants. Thus, matrices with the elastic mechanical properties for transversely isotropic symmetry material appear in the following format:

$$
[\mathcal{C}]=\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{11} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{13} & \mathcal{C}_{13} & \mathcal{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\mathcal{C}_{11}-\mathcal{C}_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55} \end{bmatrix} \quad \begin{array}{l} \textit{Transversely Isotropic Symmetry} \\ \text{5 independent constants} \end{array} \tag{7.50}
$$

7.4.2.6 Cubic Symmetry

Some metals are formed by crystals which can be classified as cubic symmetry materials. These exhibit two planes of symmetry (orthotropic symmetry) and also have the same properties if we make a rotation along the x_3-axis at an angle $\alpha=90^\circ$, and along the axis x'_1 with $\beta=90^\circ$, as shown in Figure 7.5.

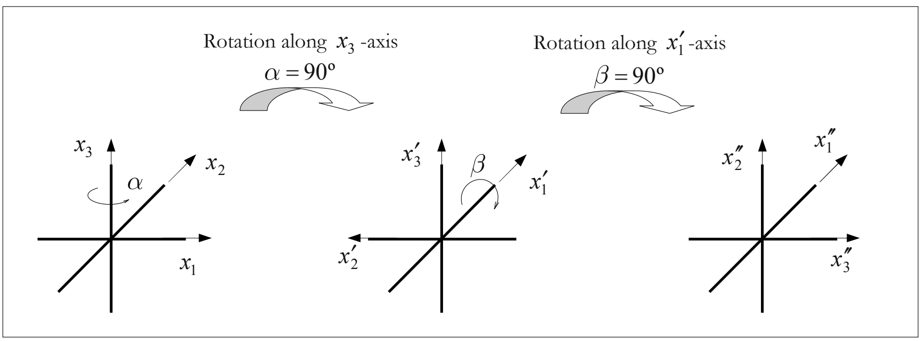


Figure 7.5: Cubic symmetry.

As a starting point we can use the elasticity matrix for orthotropic symmetry defined in (7.40). The transformation matrix from the x-system to the x'-system, (see Figure 7.5), is defined as:

$$
\mathcal{A}=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow [\mathcal{M}]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{7.51}
$$

By substituting (7.51) in the equation in (7.30) we obtain:

$$[\mathcal{C}]=\begin{bmatrix} \mathcal{C}_{22} & \mathcal{C}_{21} & \mathcal{C}_{23} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{11} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{32} & \mathcal{C}_{31} & \mathcal{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55} \end{bmatrix} \tag{7.52}$$

Then, by comparing the above matrix with (7.40) we can conclude that:

$$[\mathcal{C}]=\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{11} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{13} & \mathcal{C}_{13} & \mathcal{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{55} \end{bmatrix} \tag{7.53}$$

Then using the equation in (7.53), we rotate the x_1'-axis at an angle $\beta=90^\circ$, resulting in the transformation matrix:

$$\mathcal{A}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \longrightarrow [\mathcal{M}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{7.54}$$

After that, if we substitute (7.54) into the transformation law (7.30), we can obtain the elasticity matrix $[\mathcal{C}']$:

$$[\mathcal{C}']=\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{13} & \mathcal{C}_{12} & 0 & 0 & 0 \\ \mathcal{C}_{13} & \mathcal{C}_{33} & \mathcal{C}_{13} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{C}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{44} \end{bmatrix} \tag{7.55}$$

Additionally, by comparing (7.55) with (7.53) we can conclude that $\mathcal{C}_{33}=\mathcal{C}_{11}$; $\mathcal{C}_{55}=\mathcal{C}_{44}$; $\mathcal{C}_{13}=\mathcal{C}_{12}$, since $[\mathcal{C}']=[\mathcal{C}]$. Then, the elasticity matrix is defined by three independent constants:

$$\boxed{[\mathcal{C}]=\begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{12} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{11} & \mathcal{C}_{12} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{12} & \mathcal{C}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{C}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{44} \end{bmatrix}} \quad \begin{array}{l} \textit{Cubic Symmetry} \\ \text{3 independent constants} \end{array} \tag{7.56}$$

7.4.2.7 Symmetry in All Directions (Isotropy)

Finally, if the material has the same property in all directions it is called *isotropic material.* Note that if we compare (7.56) with (7.52) we can conclude that $\mathcal{C}_{44} = \mathcal{C}_{55} = \frac{1}{2}(\mathcal{C}_{11} - \mathcal{C}_{12})$ to fulfill symmetry in all direction. Then, the elasticity matrix features 2 elastic constants to be determined:

$$\boxed{\left[\mathcal{C}^e\right] = \begin{bmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{12} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{11} & \mathcal{C}_{12} & 0 & 0 & 0 \\ \mathcal{C}_{12} & \mathcal{C}_{12} & \mathcal{C}_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\mathcal{C}_{11} - \mathcal{C}_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(\mathcal{C}_{11} - \mathcal{C}_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(\mathcal{C}_{11} - \mathcal{C}_{12}) \end{bmatrix}}$$

Isotropic Symmetry
2 independent constants (7.57)

Then substituting some variables so that: $\lambda = \mathcal{C}_{12}$ and $\mu = \frac{1}{2}(\mathcal{C}_{11} - \mathcal{C}_{12})$, the elasticity matrix can be represented as follows:

$$\left[\mathcal{C}^e\right] = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{7.58}$$

where the constants λ, μ, are known as the *Lamé constants*. We then split the matrix $\left[\mathcal{C}^e\right]$ as follows:

$$\left[\mathcal{C}^e\right] = \lambda \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}} + 2\mu \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}}_{\mathcal{I}} \tag{7.59}$$

Note that $\mathcal{I}$ is the matrix in Voigt notation that represents the symmetric fourth-order unit tensor components ($\mathbf{I}_{ijkl} = \frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]$), (see Chapter 1). Then the tensor with the elastic properties for isotropic materials is represented in tensorial and indicial notations as follows:

The elasticity tensor for isotropic material	Tensorial notation	Indicial notation	
	$\boxed{\mathbb{C}^e = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}}$	$\mathbb{C}^e_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]$	(7.60)

NOTE: Remember that in Chapter 1 we saw that any isotropic fourth-order tensor can be written in terms of the following tensors: $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$, *i.e.*:

$$\mathbb{C}_{ijkl} = a_0\,\delta_{ij}\delta_{kl} + a_1\,\delta_{ik}\delta_{jl} + a_2\,\delta_{il}\delta_{jk} \tag{7.61}$$

Then, because of the $\mathbb{C}^e$ symmetry we can prove that $a_1 = a_2$. ■

Additionally, the inverse of the elasticity matrix in (7.58) is given by:

$$\left[\mathcal{C}^e\right]^{-1} = \begin{bmatrix} \frac{1}{\mu}\frac{\lambda+\mu}{(3\lambda+2\mu)} & \frac{-1}{2\mu}\frac{\lambda}{(3\lambda+2\mu)} & \frac{-1}{2\mu}\frac{\lambda}{(3\lambda+2\mu)} & 0 & 0 & 0 \\ \frac{-1}{2\mu}\frac{\lambda}{(3\lambda+2\mu)} & \frac{1}{\mu}\frac{\lambda+\mu}{(3\lambda+2\mu)} & \frac{-1}{2\mu}\frac{\lambda}{(3\lambda+2\mu)} & 0 & 0 & 0 \\ \frac{-1}{2\mu}\frac{\lambda}{(3\lambda+2\mu)} & \frac{-1}{2\mu}\frac{\lambda}{(3\lambda+2\mu)} & \frac{1}{\mu}\frac{\lambda+\mu}{(3\lambda+2\mu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu} \end{bmatrix} \tag{7.62}$$

Then, we can split the matrix $\left[\mathcal{C}^e\right]^{-1}$ as follows:

$$\left[\mathcal{C}^e\right]^{-1} = \frac{-\lambda}{2\mu(3\lambda+2\mu)} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 0 & 0 & 0\end{bmatrix}} + \frac{1}{2\mu} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}}_{\mathcal{I}^{-1}} \tag{7.63}$$

If we can verify firstly, that the second matrix of (7.63) is the Voigt notation representation of the inverse of the symmetric fourth-order unit tensor components, and secondly, $\mathbf{I}_{ijkl} = \mathbf{I}^{-1}_{ijkl}$ holds, then the inverse of the isotropic elasticity tensor is given as follows:

Tensorial notation	Indicial notation
$\boxed{\mathbb{C}^{e^{-1}} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{1}\otimes\mathbf{1} + \frac{1}{2\mu}\mathbf{I}}$	$\left(\mathbb{C}^e_{ijkl}\right)^{-1} = \frac{-\lambda\delta_{ij}\delta_{kl}}{2\mu(3\lambda+2\mu)} + \frac{1}{4\mu}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]$ (7.64)

where $\mathbb{C}^{e^{-1}}$ is known as the *elastic compliance tensor*. Here, we will left the reader work out whether $\mathbb{C}^e : \mathbb{C}^{e^{-1}} = \mathbb{I}^{sym} \equiv \mathbf{I}$.

7.5 Isotropic Materials

7.5.1 Constitutive Equations

The generalized Hooke's law (7.10) for isotropic linear elastic materials can be written using the equation in (7.60) as follows:

$$\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon} = (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) : \boldsymbol{\varepsilon} = \lambda \mathbf{1} \otimes \underbrace{\mathbf{1} : \boldsymbol{\varepsilon}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + 2\mu \underbrace{\mathbf{I} : \boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}^{sym} = \boldsymbol{\varepsilon}} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon} \tag{7.65}$$

thus

Tensorial notation	Indicial notation	
$\boxed{\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}}$	$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$	(7.66)

Then, the inverse of (7.66) is obtained as follows:

Tensorial notation	Indicial notation	
$\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}$	$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$	
$\Rightarrow 2\mu \boldsymbol{\varepsilon} = \boldsymbol{\sigma} - \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1}$	$\Rightarrow 2\mu \varepsilon_{ij} = \sigma_{ij} - \lambda \varepsilon_{kk} \delta_{ij}$	(7.67)
$\Rightarrow \boldsymbol{\varepsilon} = \frac{1}{2\mu} \boldsymbol{\sigma} - \frac{\lambda}{2\mu} \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1}$	$\Rightarrow \varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{\lambda}{2\mu} \varepsilon_{kk} \delta_{ij}$	

We now need to evaluate the trace ε_{kk} and to do so we need to obtain the trace of σ_{ij}, *i.e.*:

Tensorial notation	Indicial notation	
$\boldsymbol{\sigma} : \mathbf{1} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} : \mathbf{1} + 2\mu \boldsymbol{\varepsilon} : \mathbf{1}$	$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$	
$\mathrm{Tr}(\boldsymbol{\sigma}) = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon}) 3 + 2\mu \mathrm{Tr}(\boldsymbol{\varepsilon})$	$\Rightarrow \sigma_{ii} = \lambda \varepsilon_{kk} \delta_{ii} + 2\mu \varepsilon_{ii} = \lambda \varepsilon_{kk} 3 + 2\mu \varepsilon_{kk}$	
$\Rightarrow \mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{1}{(3\lambda + 2\mu)} \mathrm{Tr}(\boldsymbol{\sigma})$	$\Rightarrow \sigma_{kk} = (3\lambda + 2\mu) \varepsilon_{kk}$	(7.68)
	$\Rightarrow \varepsilon_{kk} = \frac{1}{(3\lambda + 2\mu)} \sigma_{kk}$	

Then, by substituting the $\mathrm{Tr}(\boldsymbol{\varepsilon})$ value given in (7.68) into the equation in (7.67) we obtain:

Tensorial notation	Indicial notation	
$\boxed{\boldsymbol{\varepsilon} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \mathrm{Tr}(\boldsymbol{\sigma}) \mathbf{1} + \frac{1}{2\mu} \boldsymbol{\sigma}}$	$\varepsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \sigma_{kk} \delta_{ij} + \frac{1}{2\mu} \sigma_{ij}$	(7.69)

Furthermore, the above equation could easily have been obtained by means of the following relationship:

$$\boldsymbol{\varepsilon} = \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma} = \left(\frac{-\lambda}{2\mu(3\lambda + 2\mu)} \mathbf{1} \otimes \mathbf{1} + \frac{1}{2\mu} \mathbf{I} \right) : \boldsymbol{\sigma} = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \mathbf{1} \mathrm{Tr}(\boldsymbol{\sigma}) + \frac{1}{2\mu} \boldsymbol{\sigma} \tag{7.70}$$

where we have applied the compliance elasticity tensor given in (7.64). Then, from the definition of the Cauchy stress tensor eigenvalues and eigenvectors, *i.e.* $\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = \gamma_{\boldsymbol{\sigma}} \hat{\mathbf{n}}$, we can obtain:

$$\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}=\gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \quad \Rightarrow (\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon})\cdot\hat{\mathbf{n}}=\gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \quad \Rightarrow \quad \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}\cdot\hat{\mathbf{n}}+2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}}=\gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}}$$
$$\Rightarrow \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\hat{\mathbf{n}}+2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}}=\gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}} \quad \Rightarrow \quad 2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}}=\gamma_{\boldsymbol{\sigma}}\hat{\mathbf{n}}-\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\hat{\mathbf{n}}$$
$$\Rightarrow 2\mu\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}}=(\gamma_{\boldsymbol{\sigma}}-\lambda \mathrm{Tr}(\boldsymbol{\varepsilon}))\hat{\mathbf{n}} \quad \Rightarrow \quad \boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}}=\left(\frac{\gamma_{\boldsymbol{\sigma}}-\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}\right)\hat{\mathbf{n}}$$

$$\boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}}=\left(\frac{\gamma_{\boldsymbol{\sigma}}-\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}\right)\hat{\mathbf{n}} \quad \Rightarrow \quad \boldsymbol{\varepsilon}\cdot\hat{\mathbf{n}}=\gamma_{\boldsymbol{\varepsilon}}\hat{\mathbf{n}} \tag{7.71}$$

Thus, as expected, we can see that the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ have the same principal directions (eigenvectors), and their eigenvalues (principal values) are connected by $\gamma_{\boldsymbol{\varepsilon}}=\frac{\gamma_{\boldsymbol{\sigma}}-\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}$.

If we denote by $\gamma_{\boldsymbol{\varepsilon}}^{(1)}=\varepsilon_1$, $\gamma_{\boldsymbol{\varepsilon}}^{(2)}=\varepsilon_2$, $\gamma_{\boldsymbol{\varepsilon}}^{(3)}=\varepsilon_3$ and $\gamma_{\boldsymbol{\sigma}}^{(1)}=\sigma_1$, $\gamma_{\boldsymbol{\sigma}}^{(2)}=\sigma_2$, $\gamma_{\boldsymbol{\sigma}}^{(3)}=\sigma_3$, the eigenvalues of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ can also be evaluated as follows:

$$\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}=\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+2\mu\begin{bmatrix}\varepsilon_1 & 0 & 0\\ 0 & \varepsilon_2 & 0\\ 0 & 0 & \varepsilon_3\end{bmatrix} \tag{7.72}$$

$$\begin{aligned}\begin{bmatrix}\varepsilon_1 & 0 & 0\\ 0 & \varepsilon_2 & 0\\ 0 & 0 & \varepsilon_3\end{bmatrix}&=\frac{1}{2\mu}\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}-\frac{\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2\mu}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\\ &=\frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+\frac{1}{2\mu}\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}\end{aligned} \tag{7.73}$$

7.5.2 Experimental Determination of Elastic Constants

7.5.2.1 Young's Modulus and Poisson's Ratio

For a homogeneous and isotropic body, the following assumptions (experimentally observed), Sechler (1952), are valid:

- The normal stresses $(\sigma_x, \sigma_y, \sigma_z)$ can not produce angular shear with respect to the same coordinate system;
- Shear stress only produces shear strain;
- Pure shear stress produces deformation only in the plane where shear is applied.

Based on these assumptions, we can conclude that the strain functions are:

$$\begin{array}{lll}\varepsilon_x=\varepsilon_x(\sigma_x,\sigma_y,\sigma_z) & \varepsilon_y=\varepsilon_y(\sigma_x,\sigma_y,\sigma_z) & \varepsilon_z=\varepsilon_z(\sigma_x,\sigma_y,\sigma_z)\\ \gamma_{xy}=\gamma_{xy}(\tau_{xy}) & \gamma_{yz}=\gamma_{yz}(\tau_{yz}) & \gamma_{zx}=\gamma_{zx}(\tau_{zx})\end{array} \tag{7.74}$$

We can also assume that the normal strain is a linear function of the normal stresses, *i.e.*:

$$\varepsilon_x=\alpha_1\sigma_x+\alpha_2\sigma_y+\alpha_3\sigma_z \tag{7.75}$$

Then, because of material isotropy, the effect of σ_y upon ε_x is the same as the effect of σ_z upon ε_x, (see Figure 7.6(a)), thus:

$$\alpha_2=\alpha_3<0 \qquad \text{(experimentally observed)} \tag{7.76}$$

with

$$\alpha_1 = \frac{1}{E} \quad ; \quad \alpha_2 = \alpha_3 = -\frac{\nu}{E} \tag{7.77}$$

where E is *Young's modulus* (unit of stress) and ν is *Poisson's ratio* (dimensionless). The values of E and ν can then be obtained by laboratory experiments, (see Chapter 6). Afterwards, all of these values will enable us to write the first three equations in (7.74) as, (see Figure 7.6):

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \; ; \; \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \; ; \; \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \tag{7.78}$$

7.5.2.2 The Shear and Bulk Moduli

The relationship between the types of shear stress and shear strain are given by:

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \quad ; \quad \gamma_{yz} = \frac{1}{G}\tau_{yz} \quad ; \quad \gamma_{zx} = \frac{1}{G}\tau_{zx} \tag{7.79}$$

where G is known as the *shear modulus*, which is related to the parameters E and ν by the equation:

$$\boxed{G = \frac{E}{2(1+\nu)}} \tag{7.80}$$

Then, to define the *bulk modulus* (κ) we can start from the strain tensor trace:

$$\begin{aligned} I_{\boldsymbol{\varepsilon}} = \varepsilon_x + \varepsilon_y + \varepsilon_z &= \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right) + \sigma_y - \nu\left(\sigma_x + \sigma_z\right) + \sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \\ &= \frac{(1-2\nu)}{E}\left[\sigma_x + \sigma_y + \sigma_z\right] \\ &= \frac{3(1-2\nu)}{E}\sigma_m \end{aligned} \tag{7.81}$$

In infinitesimal strain theory, (see Chapter 2), the strain tensor trace is equal to the volume ratio, *i.e.*:

$$\varepsilon_v = \frac{\Delta V}{dV_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z = I_{\boldsymbol{\varepsilon}} \tag{7.82}$$

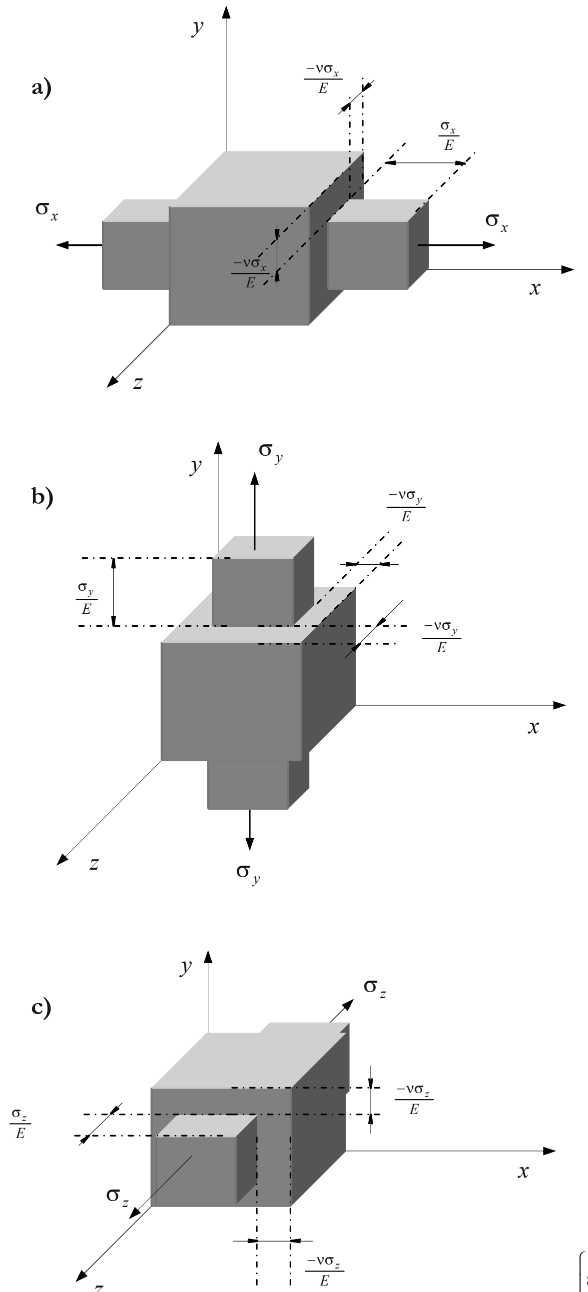
Then, if we compare the equation in (7.81) with (7.82) we can draw the conclusion that:

$$\varepsilon_v = \frac{3(1-2\nu)}{E}\sigma_m \quad \Rightarrow \quad \sigma_m = -\bar{p} = \frac{E}{3(1-2\nu)}\varepsilon_v \tag{7.83}$$

where $\frac{3(1-2\nu)}{E}$ is the *compressibility factor*. We can then define the *bulk modulus* (or *compressibility modulus*) (κ), (see Figure 7.7) as the inverse of the compressibility factor:

$$\kappa = \frac{E}{3(1-2\nu)} \tag{7.84}$$

If we then look at said compressibility factor, (see equation (7.83)), when working with incompressible material the following is satisfied: $\frac{3(1-2\nu)}{E} = 0$, which is equivalent to $\nu = 0.5$. Then, material in which Poisson's ratio equals $\nu = 0.5$ is considered to be incompressible.



$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} \\ \varepsilon_y = -\dfrac{\nu\sigma_x}{E} \\ \varepsilon_z = -\dfrac{\nu\sigma_x}{E} \end{cases}$$

+

$$\begin{cases} \varepsilon_x = -\dfrac{\nu\sigma_y}{E} \\ \varepsilon_y = \dfrac{\sigma_y}{E} \\ \varepsilon_z = -\dfrac{\nu\sigma_y}{E} \end{cases}$$

+

$$\begin{cases} \varepsilon_x = -\dfrac{\nu\sigma_z}{E} \\ \varepsilon_y = -\dfrac{\nu\sigma_z}{E} \\ \varepsilon_z = \dfrac{\sigma_z}{E} \end{cases}$$

=

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \\ \varepsilon_y = \dfrac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] \\ \varepsilon_z = \dfrac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \end{cases}$$

Figure 7.6: The normal strains at a material point.

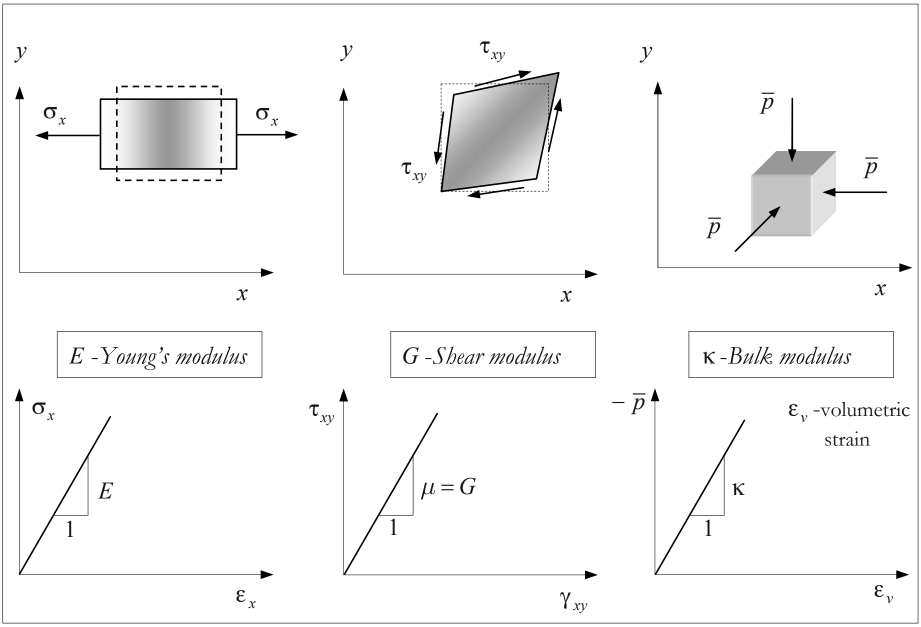


Figure 7.7: Some material mechanical properties.

We can now regroup the equations in (7.78) and (7.79) into matrix form as follows:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \tag{7.85}$$

where $G = \dfrac{E}{2(1+\nu)}$. Then, the inverse of (7.85) is given by:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \tag{7.86}$$

Now, the above relationship can be rewritten in tensorial or indicial notations as:

Tensorial notation	Indicial notation	
$\boxed{\boldsymbol{\sigma} = \dfrac{\nu E \,\mathrm{Tr}(\boldsymbol{\varepsilon})}{(1+\nu)(1-2\nu)}\mathbf{1} + \dfrac{E}{(1+\nu)}\boldsymbol{\varepsilon}}$	$\sigma_{ij} = \dfrac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} + \dfrac{E}{(1+\nu)}\varepsilon_{ij}$	(7.87)

If we then compare the equations in (7.87) and (7.66) we can conclude that the Lamé constants (λ, μ) are connected to the parameters (ν, E) via the following relationships:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad ; \quad \mu = G = \frac{E}{2(1+\nu)} \tag{7.88}$$

Additionally, the inverse of (7.87) is:

Tensorial notation	Indicial notation	
$\boxed{\boldsymbol{\varepsilon} = \dfrac{-\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \dfrac{1+\nu}{E}\boldsymbol{\sigma}}$	$\varepsilon_{ij} = \dfrac{-\nu}{E}\sigma_{kk}\delta_{ij} + \dfrac{1+\nu}{E}\sigma_{ij}$	(7.89)

Then if we compare the equations in (7.89) with those in (7.69) we can draw the conclusion that:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad ; \quad \nu = \frac{\lambda}{2(\lambda + \mu)} \tag{7.90}$$

Table 7.1 provides us with the different equations among the following mechanical properties: E - Young's modulus; ν - Poisson's ratio; κ - bulk modulus; G - shear modulus, and the Lamé constants (λ, μ).

Now, using the information in the table, we can show the elasticity tensor in terms of the parameters $(E;\nu)$, $(\lambda;\mu = G)$, $(\kappa;\mu = G)$, *i.e.*:

$$\boxed{\begin{aligned} \mathbb{C}^e &= \frac{\nu E}{(1+\nu)(1-2\nu)}\mathbf{1}\otimes\mathbf{1} + \frac{E}{(1+\nu)}\mathbf{I} \\ \mathbb{C}^e &= \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I} \\ \mathbb{C}^e &= \kappa\mathbf{1}\otimes\mathbf{1} + 2\mu\left[\mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right] \end{aligned}} \quad \textit{Elasticity tensor} \tag{7.91}$$

and respectively the inverse tensors:

$$\boxed{\begin{aligned} \mathbb{C}^{e^{-1}} &\equiv \mathbb{D}^e = \frac{-\nu}{E}\mathbf{1}\otimes\mathbf{1} + \frac{(1+\nu)}{E}\mathbf{I} \\ \mathbb{C}^{e^{-1}} &\equiv \mathbb{D}^e = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{1}\otimes\mathbf{1} + \frac{1}{2\mu}\mathbf{I} \\ \mathbb{C}^{e^{-1}} &\equiv \mathbb{D}^e = \frac{1}{9\kappa}\mathbf{1}\otimes\mathbf{1} + \frac{1}{2\mu}\left[\mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right] \end{aligned}} \quad \textit{Elastic compliance tensor} \tag{7.92}$$

Table 7.1: Equations among material mechanical properties.

	$G \equiv \mu =$	$E =$	$\kappa =$	$\lambda =$	$\nu =$
$f(G;E)$	G	E	$\dfrac{GE}{9G-3E}$	$\dfrac{G(E-2G)}{3G-E}$	$\dfrac{E-2G}{2G}$
$f(G;\kappa)$	G	$\dfrac{9G\kappa}{3\kappa+G}$	κ	$\kappa-\dfrac{2G}{3}$	$\dfrac{3\kappa-2G}{2(3\kappa+G)}$
$f(G;\lambda)$	G	$\dfrac{G(3\lambda+2G)}{\lambda+G}$	$\lambda+\dfrac{2}{3}G$	λ	$\dfrac{\lambda}{2(\lambda+G)}$
$f(G;\nu)$	G	$2G(1+\nu)$	$\dfrac{2G(1+\nu)}{3(1-2\nu)}$	$\dfrac{2G}{1-2\nu}$	ν
$f(E;\kappa)$	$\dfrac{3\kappa E}{9\kappa-E}$	E	κ	$\dfrac{\kappa(9\kappa-3E)}{9\kappa-E}$	$\dfrac{3\kappa-E}{6\kappa}$
$f(E;\nu)$	$\dfrac{E}{2(1+\nu)}$	E	$\dfrac{E}{3(1-2\nu)}$	$\dfrac{\nu E}{(1+\nu)(1-2\nu)}$	ν
$f(\kappa;\lambda)$	$\dfrac{3(\kappa-\lambda)}{2}$	$\dfrac{9\kappa(\kappa-\lambda)}{3\kappa-\lambda}$	κ	λ	$\dfrac{\lambda}{3\kappa-\lambda}$
$f(\kappa;\nu)$	$\dfrac{3\kappa(1-2\nu)}{2(1+\nu)}$	$3\kappa(1-2\nu)$	κ	$\dfrac{3\kappa\nu}{1+\nu}$	ν

7.5.3 Restrictions on Elastic Mechanical Properties

One significant tensor in elasticity is the *elastic acoustic tensor* ($\boldsymbol{Q}^e(\mathbf{N})$) defined as:

$$\boldsymbol{Q}^e(\hat{\mathbf{N}}) = \hat{\mathbf{N}} \cdot \mathbb{C}^e \cdot \hat{\mathbf{N}} \tag{7.93}$$

which is used to find restrictions on elastic mechanical properties. The components of (7.93) in isotropic materials are given by:

$$\begin{aligned}
(\boldsymbol{Q}^e)_{jl} &= \hat{\mathsf{N}}_i \mathbb{C}^e_{ijkl} \hat{\mathsf{N}}_k = \hat{\mathsf{N}}_i \left[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right]\hat{\mathsf{N}}_k \\
&= \hat{\mathsf{N}}_i \lambda\delta_{ij}\delta_{kl}\hat{\mathsf{N}}_k + \mu\hat{\mathsf{N}}_i\delta_{ik}\delta_{jl}\hat{\mathsf{N}}_k + \mu\hat{\mathsf{N}}_i\delta_{il}\delta_{jk}\hat{\mathsf{N}}_k = \lambda\hat{\mathsf{N}}_j\hat{\mathsf{N}}_l + \mu\hat{\mathsf{N}}_k\hat{\mathsf{N}}_k\delta_{jl} + \mu\hat{\mathsf{N}}_l\hat{\mathsf{N}}_j \\
&= \mu\delta_{jl} + (\lambda+\mu)\hat{\mathsf{N}}_j\hat{\mathsf{N}}_l
\end{aligned} \tag{7.94}$$

which in tensorial notation becomes $\boldsymbol{Q}^e(\hat{\mathbf{N}}) = \mu\mathbf{1} + (\lambda+\mu)\hat{\mathbf{N}}\otimes\hat{\mathbf{N}}$, and whose inverse form is given by:

$$\boldsymbol{Q}^{e^{-1}} = \frac{1}{\mu}\left[\mathbf{1} - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\hat{\mathbf{N}}\otimes\hat{\mathbf{N}}\right] \qquad Q^{e^{-1}}_{jl} = \frac{1}{\mu}\left[\delta_{jl} - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\hat{\mathsf{N}}_j\hat{\mathsf{N}}_l\right] \tag{7.95}$$

or in terms of the variables E, ν as:

$$\boldsymbol{Q}^{e^{-1}} = \frac{2(1+\nu)}{E}\left[\mathbf{1} - \frac{1}{2(1-\nu)}\hat{\mathbf{N}}\otimes\hat{\mathbf{N}}\right] \tag{7.96}$$

Next, the isotropic elastic acoustic tensor determinant can be evaluated as follows:

$$\left|\boldsymbol{Q}^e\right| = \mu^2(\lambda+2\mu) \tag{7.97}$$

In two-dimensional cases (2D) the determinant of $\boldsymbol{Q}^e$ becomes $\left|\boldsymbol{Q}^e\right| = \mu(\lambda+2\mu)$.

Then to obtain the eigenvalues of $\boldsymbol{Q}^e$ one need only solve the following characteristic determinant:

$$\begin{vmatrix} [(\lambda+\mu)\mathsf{N}_1\mathsf{N}_1+\mu]-\varsigma & (\lambda+\mu)\mathsf{N}_1\mathsf{N}_2 & (\lambda+\mu)\mathsf{N}_1\mathsf{N}_3 \\ (\lambda+\mu)\mathsf{N}_1\mathsf{N}_2 & [(\lambda+\mu)\mathsf{N}_2\mathsf{N}_2+\mu]-\varsigma & (\lambda+\mu)\mathsf{N}_2\mathsf{N}_3 \\ (\lambda+\mu)\mathsf{N}_1\mathsf{N}_3 & (\lambda+\mu)\mathsf{N}_2\mathsf{N}_3 & [(\lambda+\mu)\mathsf{N}_3\mathsf{N}_3+\mu]-\varsigma \end{vmatrix} = 0 \tag{7.98}$$

which gives rise to the characteristic equation. Then, if we use the constraint $\mathsf{N}_1^2+\mathsf{N}_2^2+\mathsf{N}_3^2=1$, we can obtain the following eigenvalues:

$$(\boldsymbol{Q}^e)'_{ij} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda+2\mu) \end{bmatrix} \tag{7.99}$$

The system for small perturbation is unstable in the presence of zero or negative roots. In other words, $\boldsymbol{Q}^e$ must be a positive definite tensor in order to guarantee the stability of the system. Now, the necessary and sufficient conditions for *strong ellipticity* occurs when $\mu > 0$ and $\lambda + 2\mu > 0$, but if said strong ellipticity conditions are violated, the material is subjected to instability shown by a homogeneous deformation band. These conditions can also be expressed as follows:

$$\mu = \frac{E}{2(1+\nu)} > 0 \Rightarrow \begin{cases} \begin{cases} E > 0 \\ \nu > -1 \end{cases} \\ \begin{cases} E < 0 \\ \nu < -1 \end{cases} \end{cases} \quad \text{and} \quad \lambda + 2\mu = 2\mu\frac{(1-\nu)}{(1-2\nu)} > 0 \Rightarrow \begin{cases} \nu < 0.5 \\ \nu > 1 \end{cases} \tag{7.100}$$

So, the material is stable if:

$$E > 0 \Rightarrow \nu \subset]-1\,;0.5[\;\cup\;]1\,;\infty[\qquad \textit{and} \qquad E < 0 \Rightarrow \nu \subset]-\infty\,;-1[\tag{7.101}$$

Now, in order to have physical meaning the bulk modulus (κ) (Truesdell&Noll – 1965) must be positive and the stability condition point-to-point is ensured by:

$$\mu > 0 \quad ; \quad \kappa = \lambda + \tfrac{2}{3}\mu = \frac{E}{3(1-2\nu)} > 0 \tag{7.102}$$

Then for isotropic linear elastic material, strain energy is positive when it holds that:

$$E > 0 \quad ; \quad -1 < \nu < 0.5 \tag{7.103}$$

For most materials, Poisson's ratio is between the range $0 < \nu < 0.5$ and materials with negative Poisson's ratio are called *auxetic materials*.

7.6 Strain Energy Density

As we saw in **Problem 6.1**, the energy equation is a redundant one for the elasticity problem. However, the starting points for devising a strategy which can be used to obtain the solution of the Initial Boundary Value Problem, whether analytically or numerically, are the energy principles, hence the importance of studying the strain energy density. The energy equation, for isotropic linear elastic material, was already obtained in **Problem 6.1**. Here, we will show strain energy from an engineering standpoint.

In order to physically interpret the strain energy we will consider a differential volume element $dxdydz$ in which we have the normal stress σ_x, (see Figure 7.8).

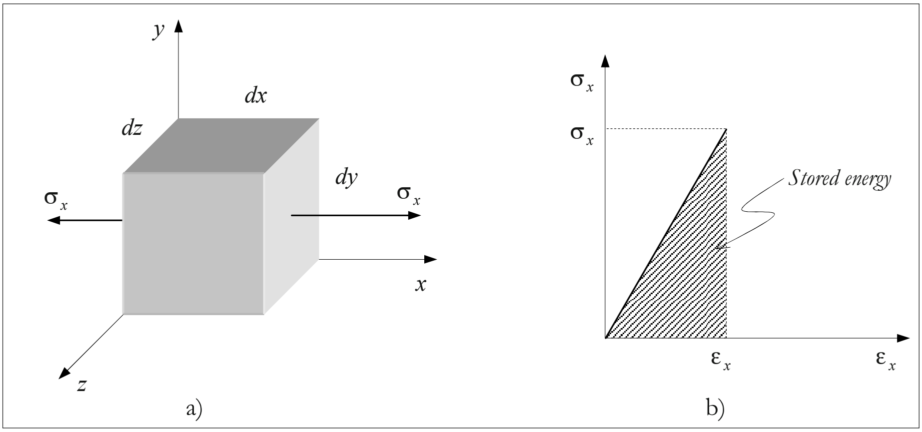


Figure 7.8: Stored energy – normal stress component.

The stored energy caused by the normal stress σ_x is equal to the triangular area of the graph defined in Figure 7.8(b), *i.e.*:

$$U_0 = \left(\frac{0+\sigma_x}{2}\right) dydz(\varepsilon_x dx) = \frac{1}{2}\sigma_x \varepsilon_x dxdydz \tag{7.104}$$

Likewise, we can obtain the strain energy caused by the normal stress types σ_y and σ_z.

Now, if we consider the shear stress τ_{xy}, (see Figure 7.9), the strain energy is given as follows:

$$U_0 = Moment \times Angle \tag{7.105}$$

$$U_0 = \left(\frac{0+\tau_{xy}}{2}\right) dxdz \times dy \times \gamma_{xy} = \frac{1}{2}\tau_{xy}\gamma_{xy} dxdydz \tag{7.106}$$

Next, if we consider the 6 Cauchy stress tensor components, the strain energy stored in a differential volume element is:

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right) dxdydz \tag{7.107}$$

We can now introduce the *strain energy per unit volume*, Ψ^e, which is known as the *strain energy density* and is given by:

$$\Psi^e = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right) \qquad \left[\frac{J}{m^3}\right] \tag{7.108}$$

Then, the total strain energy (U) in the entire continuum can be evaluated by integrating the strain energy density over the volume, *i.e.*:

$$U = \int_V \Psi^e dV \qquad [Nm \equiv J] \tag{7.109}$$

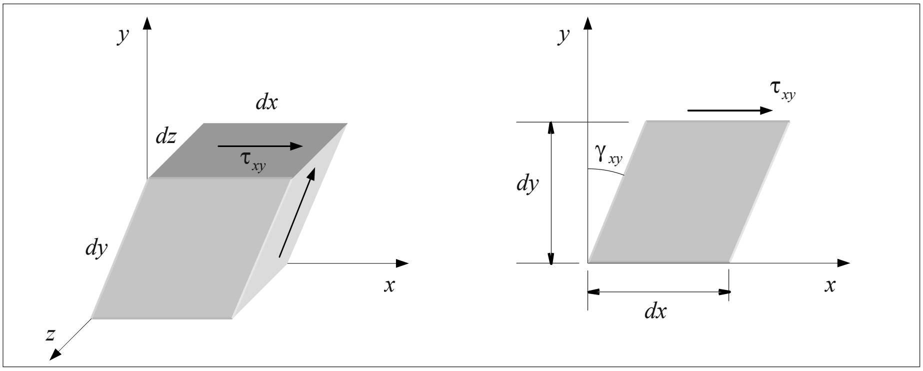


Figure 7.9: Strain energy – shear stress.

Note also that Ψ^e is the elastic potential (the Helmholtz free energy per unit volume):

Tensorial notation	Indicial notation	
$\boxed{\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma}}$	$\Psi^e = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$	(7.110)

Then, using the generalized Hooke's law $\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon}$, the elastic potential becomes:

Tensorial notation	Indicial notation	Voigt notation	
$\boxed{\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}^e:\boldsymbol{\varepsilon}}$	$\Psi^e = \frac{1}{2}\varepsilon_{ij}\varepsilon_{kl}\mathbb{C}_{ijkl}$	$\Psi^e = \frac{1}{2}\{\boldsymbol{\varepsilon}\}^T[\mathcal{C}]\{\boldsymbol{\varepsilon}\}$	(7.111)

Next, if we consider the equations in (7.87) and substitute them into equation (7.110), we obtain:

$$
\begin{aligned}
\Psi^e &= \frac{E}{2}\left[\frac{\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)^2 + \frac{1}{1+\nu}\left(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2\right) + \frac{1}{2(1+\nu)}\left(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2\right)\right] \\
\Psi^e &= \frac{\lambda}{2}\left[\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)^2 + \mu\left(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2\right) + 2\mu\left(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2\right)\right]
\end{aligned}
\tag{7.112}
$$

Additionally, if we take the derivative of the above equation with respect to the strain ε_x, we can obtain:

$$
\frac{\partial\Psi^e}{\partial\varepsilon_x} = \frac{E}{2}\left[\frac{\nu}{(1+\nu)(1-2\nu)}2\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right) + \frac{2}{1+\nu}\varepsilon_x\right] = \sigma_x \tag{7.113}
$$

Likewise, we can obtain:

$$
\frac{\partial\Psi^e}{\partial\varepsilon_x} = \sigma_x; \quad \frac{\partial\Psi^e}{\partial\varepsilon_y} = \sigma_y; \quad \frac{\partial\Psi^e}{\partial\varepsilon_z} = \sigma_z; \quad \frac{\partial\Psi^e}{\partial\gamma_{xy}} = \tau_{xy}; \quad \frac{\partial\Psi^e}{\partial\gamma_{yz}} = \tau_{yz}; \quad \frac{\partial\Psi^e}{\partial\gamma_{zx}} = \tau_{zx} \tag{7.114}
$$

thus, we can draw the conclusion that:

Tensorial notation	Indicial notation	
$\boxed{\frac{\partial\Psi^e}{\partial\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}}$	$\frac{\partial\Psi^e}{\partial\varepsilon_{ij}} = \sigma_{ij}$	(7.115)

Then, the second derivative of Ψ^e give us the elasticity tensor, *i.e.*:

$$\frac{\partial^2 \Psi^e}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} = \frac{\partial^2}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \left(\frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon} \right) = \mathbb{C}^e \tag{7.116}$$

7.6.1 Decoupling Strain Energy Density

Next, we can split the strain energy density into deviatoric and spherical parts. To do so, let us consider the strain energy density, (see Eq. (7.110)), for an isotropic linear elastic material:

$$\begin{aligned} \Psi^e &= \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}:[\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}] = \frac{\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2}\underbrace{\boldsymbol{\varepsilon}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} \\ &= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^T) = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}) = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}^2) \end{aligned} \tag{7.117}$$

where we have taken into account the constitutive equation in (7.66). The strain tensor can be split into spherical and deviatoric parts, *i.e.* $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{dev} + \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1}$, which can be substituted in the equation in (7.117), to obtain:

$$\begin{aligned} \Psi^e &= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\left(\boldsymbol{\varepsilon}^{dev} + \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1}\right):\left(\boldsymbol{\varepsilon}^{dev} + \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\mathbf{1}\right) \\ &= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\left(\boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev} + \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\underbrace{\boldsymbol{\varepsilon}^{dev}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\varepsilon}^{dev})=0} + \frac{\mathrm{Tr}(\boldsymbol{\varepsilon})}{3}\underbrace{\mathbf{1}:\boldsymbol{\varepsilon}^{dev}}_{\mathrm{Tr}(\boldsymbol{\varepsilon}^{dev})=0} + \frac{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{9}\underbrace{\mathbf{1}:\mathbf{1}}_{3}\right) \\ &= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev} + \mu\frac{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{3} = \underbrace{\left(\lambda + \frac{2\mu}{3}\right)}_{\kappa}\frac{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev} \\ &= \frac{\kappa[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev} \end{aligned} \tag{7.118}$$

where:

$$\boxed{\Psi^e = \underbrace{\frac{\kappa}{2}[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}_{\textit{purely volumetric energy}} + \underbrace{\mu\,\boldsymbol{\varepsilon}^{dev}:\boldsymbol{\varepsilon}^{dev}}_{\textit{purely distortional energy}}} \tag{7.119}$$

Then, we can conclude that the strain energy density allows for additive decomposition into purely volumetric and distortional parts.

Next, if we consider that instead of substituting the equation of stress in (7.117) we substitute the strain equation given in (7.69), the strain energy density becomes:

$$\Psi^e = \frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\sigma}:\left(\frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma}\right) = \frac{-\lambda}{4\mu(3\lambda+2\mu)}\mathrm{Tr}(\boldsymbol{\sigma})\underbrace{\boldsymbol{\sigma}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\sigma})} + \frac{1}{4\mu}\boldsymbol{\sigma}:\boldsymbol{\sigma} \tag{7.120}$$

The double scalar product $\boldsymbol{\sigma}:\boldsymbol{\sigma}$ is a scalar and an invariant. Now, using the stress principal space we can obtain $\boldsymbol{\sigma}:\boldsymbol{\sigma} = \sigma_{ij}\sigma_{ij} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$. Then, given that $\mathrm{Tr}(\boldsymbol{\sigma}) = I_{\boldsymbol{\sigma}} = \sigma_1 + \sigma_2 + \sigma_3$, we can conclude that

$$\left.\begin{aligned} I_{\boldsymbol{\sigma}}^2 &= (\sigma_1+\sigma_2+\sigma_3)^2 \\ &= \sigma_1^2+\sigma_2^2+\sigma_3^2+2\underbrace{(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)}_{II_{\boldsymbol{\sigma}}} \end{aligned}\right\} \quad \Rightarrow \quad \sigma_1^2+\sigma_2^2+\sigma_3^2 = I_{\boldsymbol{\sigma}}^2 - 2II_{\boldsymbol{\sigma}} \qquad (7.121)$$

Now, if we also consider the equation of the second invariant of the deviatoric stress tensor $II_{\boldsymbol{\sigma}^{dev}} = -\mathrm{J}_2$ which is given by $II_{\boldsymbol{\sigma}^{dev}} = II_{\boldsymbol{\sigma}} - \dfrac{I_{\boldsymbol{\sigma}}^2}{3} \Rightarrow II_{\boldsymbol{\sigma}} = II_{\boldsymbol{\sigma}^{dev}} + \dfrac{I_{\boldsymbol{\sigma}}^2}{3}$ (see Chapter 1), the equation in (7.121) becomes $\sigma_1^2+\sigma_2^2+\sigma_3^2 = I_{\boldsymbol{\sigma}}^2 - 2\left(-\mathrm{J}_2 + \dfrac{I_{\boldsymbol{\sigma}}^2}{3}\right) = \dfrac{I_{\boldsymbol{\sigma}}^2}{3} + 2\mathrm{J}_2 = \boldsymbol{\sigma}:\boldsymbol{\sigma}$, with which the strain energy density (7.120) can also be expressed as:

$$\Psi^e = \frac{-\lambda}{4\mu(3\lambda+2\mu)} I_{\boldsymbol{\sigma}}^2 + \frac{1}{4\mu}\left(\frac{I_{\boldsymbol{\sigma}}^2}{3} + 2\mathrm{J}_2\right) \quad \Rightarrow \quad \boxed{\Psi^e = \underbrace{\frac{1}{6(3\lambda+2\mu)} I_{\boldsymbol{\sigma}}^2}_{\textit{purely volumetric energy}} + \underbrace{\frac{1}{2\mu}\mathrm{J}_2}_{\textit{purely distortional energy}}} \qquad (7.122)$$

Problem 7.1: Given an isotropic linear elastic material whose elastic properties are $E = 71\,GPa$, $G = 26.6\,GPa$, find the strain tensor components and the strain energy density at the point in which the stress state, in Cartesian basis, is represented by:

$$\sigma_{ij} = \begin{bmatrix} 20 & -4 & 5 \\ -4 & 0 & 10 \\ 5 & 10 & 15 \end{bmatrix} MPa$$

Solution: Poisson's ratio can be obtained by means of the equation: $G = \dfrac{E}{2(1+\nu)}$

$$\Rightarrow \nu = \frac{E}{2G} - 1 = 0.335$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22}+\sigma_{33}\right)\right] = \frac{1}{71\times10^9}\left[20 - 0.335\left(0+15\right)\right]10^6 = 211\times10^{-6}$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11}+\sigma_{33}\right)\right] = \frac{1}{71\times10^9}\left[0 - 0.335\left(20+15\right)\right]10^6 = -165\times10^{-6}$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11}+\sigma_{22}\right)\right] = \frac{1}{71\times10^9}\left[15 - 0.335\left(20+0\right)\right]10^6 = 117\times10^{-6}$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} = \frac{1+0.335}{71\times10^9}(-4\times10^6) = 75\times10^{-6}$$

$$\varepsilon_{13} = \frac{1+\nu}{E}\sigma_{13} = \frac{1+0.335}{71\times10^9}(5\times10^6) = 94\times10^{-6}$$

$$\varepsilon_{23} = \frac{1+\nu}{E}\sigma_{23} = \frac{1+0.335}{71\times10^9}(10\times10^6) = 188\times10^{-6}$$

thus:

$$\varepsilon_{ij} = \begin{bmatrix} 211 & -75 & 94 \\ -75 & -165 & 188 \\ 94 & 188 & 117 \end{bmatrix} \times10^{-6}$$

Then, the strain energy density for an elastic material is obtained by the equation:

$$\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}^e:\boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} \xrightarrow{\textit{indicial}} \Psi^e = \frac{1}{2}\varepsilon_{ij}\sigma_{ij}$$

Next, by considering the symmetry of the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, the strain energy density can be calculated as follows:

$$\Psi^e = \frac{1}{2}[\varepsilon_{11}\sigma_{11} + \varepsilon_{22}\sigma_{22} + \varepsilon_{33}\sigma_{33} + 2\varepsilon_{12}\sigma_{12} + 2\varepsilon_{23}\sigma_{23} + 2\varepsilon_{13}\sigma_{13}]$$

$$= \frac{1}{2}[(211)(20) + (-165)(0) + (117)(15) + 2(-75)(-4) + 2(188)(10) + 2(94)(5)] = 5637.5\, J/m^3$$

We can also obtain the strain energy density by using the equation in (7.122), *i.e.*:

$$\Psi^e = \frac{1}{6(3\lambda + 2\mu)} I_{\boldsymbol{\sigma}}^2 - \frac{1}{2\mu} II_{\boldsymbol{\sigma}^{dev}} = \frac{1}{6(3\lambda + 2\mu)} I_{\boldsymbol{\sigma}}^2 + \frac{1}{2\mu} J_2$$

and if we consider that $I_{\boldsymbol{\sigma}} = 3.5\times10^7$; $II_{\boldsymbol{\sigma}} = -2.4933\times10^{14}$; $\lambda \approx 5.3804\times10^{10}\, Pa$; $\mu = G$, we can obtain $\Psi^e \approx 5638.03\, J/m^3$. Note that any discrepancies in the numerical results of Ψ^e are due to numerical approximations.

Problem 7.2: Find the strain energy density in terms of the principal invariants of $\boldsymbol{\varepsilon}$.

Solution:

$$\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}:[\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}] = \frac{\lambda \mathrm{Tr}(\boldsymbol{\varepsilon})}{2}\underbrace{\boldsymbol{\varepsilon}:\mathbf{1}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}$$

$$= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^T)$$

$$= \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}) = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}^2)$$

We can add and subtract the term $\mu[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2$ without altering the above outcome:

$$\Psi^e = \frac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 + \mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}^2) - \mu[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 = \frac{1}{2}(\lambda + 2\mu)[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mu\left\{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\}$$

Finally, if we consider that the principal invariants of the strain tensor $\boldsymbol{\varepsilon}$ are $I_{\boldsymbol{\varepsilon}} = \mathrm{Tr}(\boldsymbol{\varepsilon})$, $II_{\boldsymbol{\varepsilon}} = \frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2 - \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\}$, we can obtain: $\boxed{\Psi^e = \frac{1}{2}(\lambda + 2\mu)I_{\boldsymbol{\varepsilon}}^2 - 2\mu II_{\boldsymbol{\varepsilon}}}$

7.7 The Constitutive Law for Orthotropic Material

For orthotropic material the stress-strain relationship is given by the following equation:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \qquad (7.123)$$

in which there are 12 constants: E_1; E_2; E_3; ν_{12}; ν_{13}; ν_{23}; ν_{21}; ν_{31}; ν_{32}; G_{12}; G_{23}; G_{13}, but only 9 independents, (see equation (7.40)), since

$$\frac{\nu_{21}}{E_2}=\frac{\nu_{12}}{E_1} \quad ; \quad \frac{\nu_{31}}{E_3}=\frac{\nu_{13}}{E_1} \quad ; \quad \frac{\nu_{32}}{E_3}=\frac{\nu_{23}}{E_2} \tag{7.124}$$

Next, the reciprocal of (7.123) provides the generalized Hooke's law for orthotropic material:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}=\begin{bmatrix}\frac{E_1(\nu_{32}\nu_{23}-1)}{\chi} & \frac{-E_1(\nu_{21}+\nu_{23}\nu_{31})}{\chi} & \frac{-E_1(\nu_{31}+\nu_{32}\nu_{21})}{\chi} & 0 & 0 & 0\\ \frac{-E_1(\nu_{21}+\nu_{23}\nu_{31})}{\chi} & \frac{E_2(\nu_{13}\nu_{31}-1)}{\chi} & \frac{-E_2(\nu_{32}+\nu_{12}\nu_{31})}{\chi} & 0 & 0 & 0\\ \frac{-E_1(\nu_{31}+\nu_{32}\nu_{21})}{\chi} & \frac{-E_2(\nu_{32}+\nu_{12}\nu_{31})}{\chi} & \frac{E_3(\nu_{21}\nu_{12}-1)}{\chi} & 0 & 0 & 0\\ 0 & 0 & 0 & G_{12} & 0 & 0\\ 0 & 0 & 0 & 0 & G_{23} & 0\\ 0 & 0 & 0 & 0 & 0 & G_{13}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix} \tag{7.125}$$

where: $\chi=\nu_{32}\nu_{23}+\nu_{31}\nu_{13}+\nu_{21}\nu_{12}+2\nu_{21}\nu_{13}\nu_{32}-1$.

Note that when $E_1=E_2=E_3$; $\nu_{12}=\nu_{13}=\nu_{23}=\nu_{21}=\nu_{31}=\nu_{23}$; $G_{12}=G_{23}=G_{13}$ are satisfied, we revert to the isotropic case and thereby obtain the equations in (7.85) and (7.86).

7.8 Transversely Isotropic Materials

We can represent the elasticity matrix for a transversely isotropic material as follows:

$$[\mathcal{C}]=\begin{bmatrix}\lambda+2\mu & \lambda & \lambda_0 & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda_0 & 0 & 0 & 0\\ \lambda_0 & \lambda_0 & \lambda_0+2\mu_0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu_0 & 0\\ 0 & 0 & 0 & 0 & 0 & \mu_0\end{bmatrix} \tag{7.126}$$

Now, by decoupling the elasticity matrix into an isotropic and anisotropic part we obtain:

$$[\mathcal{C}]=\begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}+\begin{bmatrix}0 & 0 & \Delta\lambda & 0 & 0 & 0\\ 0 & 0 & \Delta\lambda & 0 & 0 & 0\\ \Delta\lambda & \Delta\lambda & \Delta\lambda+2\Delta\mu & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \Delta\mu & 0\\ 0 & 0 & 0 & 0 & 0 & \Delta\mu\end{bmatrix} \tag{7.127}$$

where we have taken into account that $\Delta\lambda=-\lambda+\lambda_0$ and $\Delta\mu=-\mu+\mu_0$. Moreover, the strain energy density, Ψ^e, can also be split into an isotropic and anisotropic part:

$$\Psi^e = \Psi^e_{iso} + \Psi^e_{Ani} \tag{7.128}$$

The isotropic part of the strain energy density, Ψ^e_{iso}, is the same as that seen previously for isotropic materials. The stress-strain relationship for the anisotropic part is considered as follows:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \Delta\lambda & 0 & 0 & 0 \\ 0 & 0 & \Delta\lambda & 0 & 0 & 0 \\ \Delta\lambda & \Delta\lambda & \Delta\lambda + 2\Delta\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} \tag{7.129}$$

Then, the anisotropic part of the strain energy density is given by:

$$\Psi^e_{Ani} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{23}\varepsilon_{23} + 2\sigma_{13}\varepsilon_{13}\right) \tag{7.130}$$

and by substituting the stresses given in (7.129) we obtain:

$$\begin{aligned} \Psi^e_{Ani} = \frac{1}{2}\{&\Delta\lambda\varepsilon_{33}\varepsilon_{11} + \Delta\lambda\varepsilon_{33}\varepsilon_{22} + [\Delta\lambda\varepsilon_{11} + \Delta\lambda\varepsilon_{22} + (\Delta\lambda + 2\Delta\mu)\varepsilon_{33}]\varepsilon_{33} \\ &+ 0 + 4\Delta\mu\sigma_{23}\varepsilon_{23} + 4\Delta\mu\sigma_{13}\varepsilon_{13}\} \end{aligned} \tag{7.131}$$

Then, if we simplify the above equation we obtain:

$$\Psi^e_{Ani} = \Delta\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]\varepsilon_{33} + \left(\Delta\mu - \frac{\Delta\lambda}{2}\right)\varepsilon^2_{33} + 2\Delta\mu\left(\varepsilon^2_{13} + \varepsilon^2_{23}\right) \tag{7.132}$$

7.9 The Saint-Venant's and Superposition Principles

If we consider two equivalent systems of forces as indicated in Figure 7.10, the Saint-Venant's principle states that when a point is far enough (damped area) from the point of disturbance, the two force systems are equivalent, *i.e.* they have the same outcome.

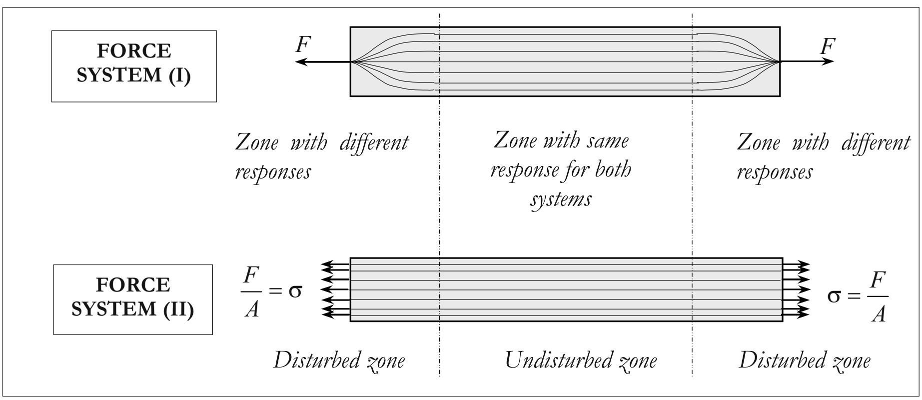
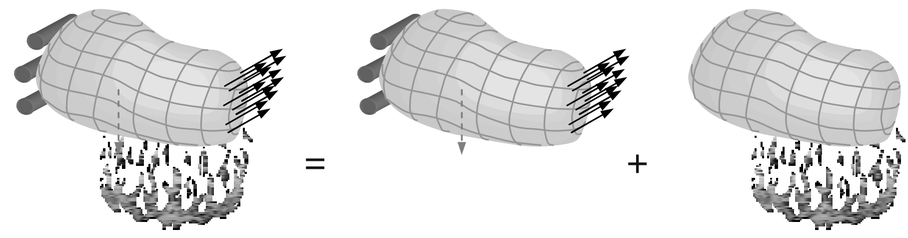


Figure 7.10: Saint-Venant's principle.

The superposition principle states that the balance of a system in which several actions take place is equal to the sum of all independent actions, (see Figure 7.11). This principle is valid because the governing equations have been linearized. As example, we can decouple the thermo-mechanical process, *i.e.* we can treat the different parts independently.



$$\begin{cases}\vec{\mathbf{u}}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\sigma}\end{cases} = \begin{cases}\vec{\mathbf{u}}(\boldsymbol{\varepsilon})\\ \boldsymbol{\varepsilon}(\vec{\mathbf{u}})\\ \boldsymbol{\sigma}(\boldsymbol{\varepsilon})\end{cases} + \begin{cases}\vec{\mathbf{u}}(\boldsymbol{\varepsilon}(T))\\ \boldsymbol{\varepsilon}(T)\\ \boldsymbol{\sigma}(\boldsymbol{\varepsilon}(T))\end{cases}$$

Figure 7.11: Superposition principle.

Problem 7.3: Let us consider a bar to which at one end we apply a force equal to $6000N$ as shown in Figure 7.12. Find $\varepsilon_x, \varepsilon_y, \varepsilon_z$, and the length change of the bar. Let us consider that the bar is made up of a material whose properties are: Young's modulus: $E = 10^7\,Pa$; Poisson's ratio: $\nu = 0.3$.

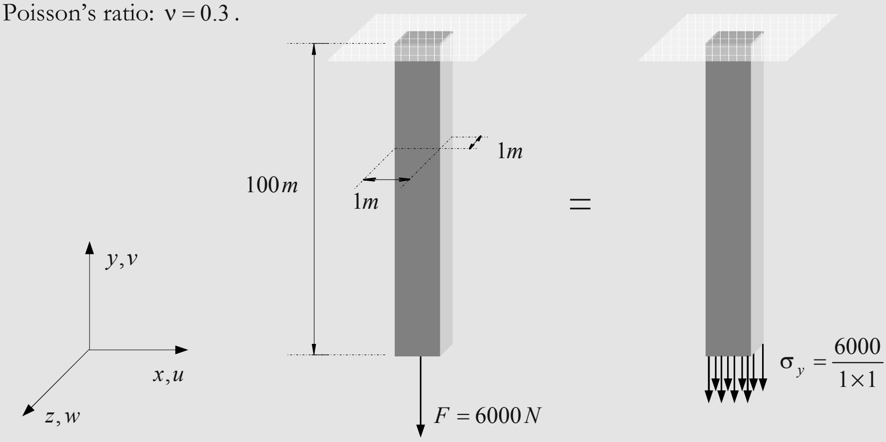


Figure 7.12

Solution: Using the normal strain expressions we can obtain:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = -\frac{\nu}{E}\sigma_y = -\frac{(0.3)(6000)}{10^7} = -0.00018$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] = \frac{\sigma_y}{E} = \frac{6000}{10^7} = 0.0006$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = -\frac{\nu}{E}\sigma_y = -0.00018$$

The total change in cross-sectional dimensions is $u = w = -0.00018 \times 1 = -1.8 \times 10^{-4}\,m$, and the total change in length is $v = 0.0006 \times 100 = 6.0 \times 10^{-2}\,m$.

7.10 Initial Stress/Strain

Some physical phenomena can be directly added to the constitutive equations because of the superposition principle. The effect of these phenomena can be represented by means of stress or strain.

7.10.1 Thermal Deformation

When temperature changes, there is an increase in internal energy, so, the atoms/molecules vibrate more intensely. This vibration causes the ligaments among the molecules to stretch, thereby causing the body volume to increase, (see Figure 7.13).

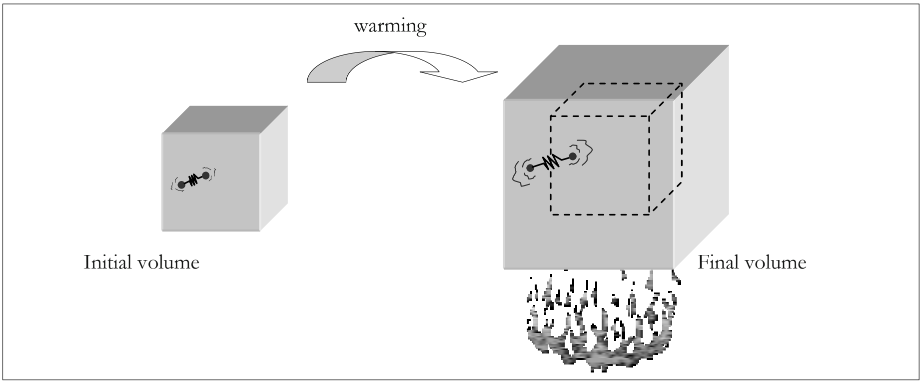


Figure 7.13: Body under a temperature change.

NOTE: In general, material properties change with temperature, *i.e.* material properties are temperature dependent. In this chapter, they are considered to be constant with regard to temperature, since the temperature variation range under consideration is not large enough. ■

In decoupled thermomechanical problems, it is possible to apply the superposition principle, *i.e.* we can obtain the strain field because of the mechanical problem $\varepsilon_{ij}(\vec{\mathbf{u}})$ (considering the isothermal process) and we can add the strain field due to the thermal effect $\varepsilon_{ij}(\Delta T)$, *i.e.*:

$$\varepsilon_{ij} = \varepsilon_{ij}(\vec{\mathbf{u}}) + \varepsilon_{ij}(\Delta T) \tag{7.133}$$

where $\varepsilon_{ij}(\vec{\mathbf{u}})$ shows the mechanical strain in terms of the displacement field ($\vec{\mathbf{u}}$) and $\varepsilon_{ij}(\Delta T)$ is the thermal strain in terms of the temperature variation (ΔT).

To obtain the temperature variation it is necessary to find the temperature distribution within the body by means of the heat flux equation, (see Chapter 5). For isotropic materials, the thermal strain caused by temperature variation is just represented by its normal components:

$$\boxed{\varepsilon_{ij}(T) = \alpha(T - T_0)\delta_{ij}} \tag{7.134}$$

where T_0 is the initial temperature; T is the final temperature, and α is the coefficient of thermal expansion. For further details about thermomechanical problems, see Chapter 10.

Next, we will show the coefficients of thermal expansion for some materials, namely: $\alpha_{steel} = 12\times10^{-6}\,\frac{1}{^{\circ}C}$, $\alpha_{aluminum} = 23\times10^{-6}\,\frac{1}{^{\circ}C}$, $\alpha_{copper} = 17\times10^{-6}\,\frac{1}{^{\circ}C}$.

Then, by using the equations in (7.133) and (7.89) we can obtain:

$$\varepsilon_{ij} = \frac{-\lambda}{\mu(3\lambda+2\mu)}\sigma_{kk}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} + \alpha(T-T_0)\delta_{ij} = \frac{-\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} + \alpha(T-T_0)\delta_{ij} \qquad (7.135)$$

The Hooke's law for isotropic materials including the thermal effect is given by the reciprocal of the equation in (7.135), the result of which is:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - \alpha(3\lambda+2\mu)(T-T_0)\delta_{ij}$$
$$\sigma_{ij} = \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij} + \frac{E}{(1+\nu)}\varepsilon_{ij} - \frac{E}{(1-2\nu)}\alpha(T-T_0)\delta_{ij} \qquad (7.136)$$

Problem 7.4: Let us consider a length rod equal to $L = 7.5m$, whose diameter is equal to $0.1m$, which is made up of a material whose properties are: $E = 2.0\times10^{11}\,Pa$ and $\alpha = 20\times10^{-6}\,\frac{1}{^{\circ}C}$. Initially the rod has a temperature equal to $15^{\circ}C$ which later rises to $50^{\circ}C$.

1) Considering that the rod can expand freely, calculate the total elongation of the rod, ΔL;
2) Now assume that the rod can not expand freely because concrete blocks have been placed at its ends, (see Figure 7.14(b)). Find the stress in the rod.
Hint: Consider the problem in one dimension.

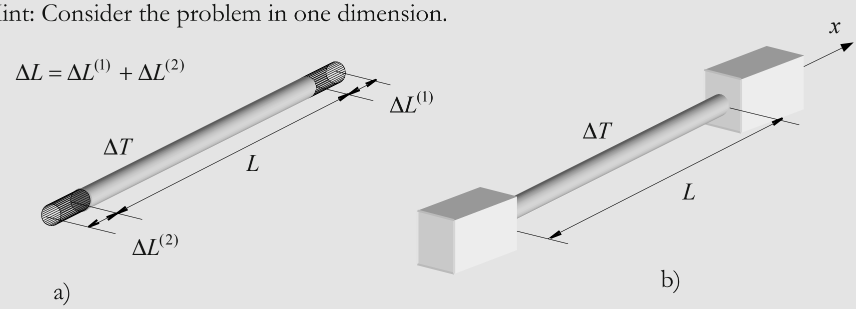


Figure 7.14: Rod under thermal effect.

Solution: 1) To obtain the elongation, we pre-calculate the thermal strain according to the rod axis direction $\varepsilon_{ij} = \alpha(\Delta T)\delta_{ij}$. Since this is a one-dimensional case, we need only consider the normal strain component according to the x-direction, $\varepsilon_{11} = \varepsilon_x$, then:

$$\varepsilon_{11} = \varepsilon_x = 20\times10^{-6}(50-15) = 7\times10^{-4}$$

Then, the total elongation, $\Delta L = \Delta L^{(1)} + \Delta L^{(2)}$, is obtained by solving the integral:

$$\Delta L = \int_0^L \varepsilon_x dx = \varepsilon_x L = 7\times10^{-4}\times7.5 = 5.25\times10^{-3}\,m$$

Note that as the rod can expand freely, it is stress-free.

2) If the ends can not move, there will be a homogeneous stress field equal to:

$$\sigma_x = -E\alpha(\Delta T)\delta_{ij} = -E\,"\varepsilon_x" = -2.0\times10^{11}\times7\times10^{-4} = -1.4\times10^{8}\,Pa$$

Note that in the case 2) there is no strain, since $\Delta L = 0$. Moreover, it is the same as when the initial length is equal to $L + \Delta L$ in which we apply compression stress in order to obtain a final length equal to L.

7.11 The Navier-Lamé Equations

By means of the constitutive law, (see the equations in (7.5) and (7.65)), we can calculate the Cauchy stress tensor divergence as follows:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \quad \Rightarrow \quad \sigma_{ij,j} = \lambda \varepsilon_{kk,j} \delta_{ij} + 2\mu \varepsilon_{ij,j} \tag{7.137}$$

Furthermore, by using the kinematic equations in (7.6) we can obtain the term $\varepsilon_{ij,j}$ and therefore $\varepsilon_{kk,j}$, *i.e.*:

$$\varepsilon_{ij} = \frac{1}{2}\left(\mathrm{u}_{i,j} + \mathrm{u}_{j,i}\right) \quad \Rightarrow \quad 2\varepsilon_{ij,j} = \left(\mathrm{u}_{i,jj} + \mathrm{u}_{j,ij}\right) \quad \Rightarrow \quad \varepsilon_{kk,j} = \mathrm{u}_{k,kj} \tag{7.138}$$

Then, by combining the equations in (7.138) and (7.137) we can obtain:

$$\sigma_{ij,j} = \lambda \mathrm{u}_{k,kj} \delta_{ij} + \mu\left(\mathrm{u}_{i,jj} + \mathrm{u}_{j,ij}\right) = (\lambda + \mu)\,\mathrm{u}_{j,ji} + \mu \mathrm{u}_{i,jj} \tag{7.139}$$

Finally, by substituting the equation (7.139) into the equations of motion given in (7.4), $\sigma_{ij,j} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i$, we can obtain:

$$\boxed{\begin{aligned} &(\lambda + \mu)\mathrm{u}_{j,ji} + \mu \mathrm{u}_{i,jj} + \rho \mathrm{b}_i = \rho \ddot{\mathrm{u}}_i \\ &(\lambda + \mu)\nabla\left(\nabla \cdot \vec{\mathbf{u}}\right) + \mu \nabla^2 \vec{\mathbf{u}} + \rho \vec{\mathbf{b}} = \rho \frac{\partial^2 \vec{\mathbf{u}}(\vec{x},t)}{\partial t^2} \end{aligned}} \quad \textit{The Navier-Lamé equations} \tag{7.140}$$

which are known as the Navier-Lamé equations. With that we have reduced the number of equations as well as the number of unknowns. Note that the only remaining unknowns are the displacement components. Finally, as for addressing specific problems, this equation can be used to obtain an analytical solution of the linear elasticity problem.

7.12 Two-Dimensional Elasticity

Occasionally, three-dimensional structures have certain geometrical and load features that enable us to treat them as two-dimensional problems (2D) which simplifies the problem immensely in two aspects: when solving the problem and when interpreting the results. Fundamentally, there are two kinds of simplifications:

1) *Simplification on a conceptual level*

Within this class of simplification there are two types of approach:

- The state of plane stress;
- The state of plane strain.

It should be stressed that such simplification are mere approximations of the real problem. Nevertheless, in many cases they turn out to be quite satisfactory, *i.e.* the error made when using them are insignificant.

2) *Simplification on a mathematical level*

We use these simplifications in structures that have radial symmetry. Such structures are known as:

- Solids of revolution (or Axisymmetric solids).

The results obtained by using this simplification are exactly the same as considering the problem from a three-dimensional point of view.

7.12.1 The State of Plane Stress

In this type of approach, one of the dimensions of the structural elements is very small when compared to the other two, (see Figure 7.15), and the load is perpendicular to the direction of smallest dimension. As a result of this the stress tensor components found in this direction are equal to zero. The deep beam is an example where we can apply this approach, (see Figure 7.16).

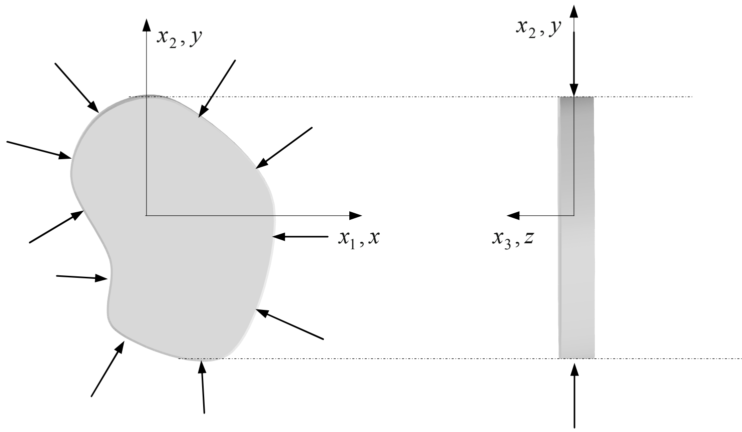


Figure 7.15: Plate

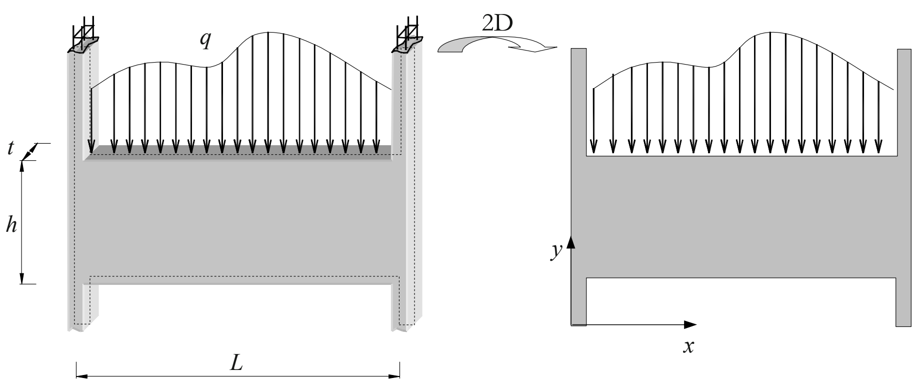


Figure 7.16: Deep beam.

The state of plane stress field, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2)$, is characterized by the absence of stress in one direction which we will show as $x_3 \equiv z$. Then the stress tensor components can be characterized by:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7.141}$$

Then, if we consider the above, the equation in (7.85) becomes:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \to 0 \\ \tau_{xy} \\ \tau_{yz} \to 0 \\ \tau_{zx} \to 0 \end{bmatrix} \tag{7.142}$$

Then, if we remove the columns and rows associated with the zero stresses, the stress-strain relationship for the plane stress case is given by:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \xrightarrow{G=\frac{E}{2(1+\nu)}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{7.143}$$

The reciprocal of the above equation will result in Hooke's law for the state of plane stress:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] = \frac{1}{E}\left[-\nu\left(\sigma_x + \sigma_y\right)\right] \tag{7.144}$$

Note that the normal strain ε_z is not equal to zero, since ε_z is not just dependant on the normal stress σ_z. Then, the strain tensor components are represented as follows:

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{7.145}$$

7.12.1.1 The Initial Strain

We can incorporate the initial strain ($\boldsymbol{\varepsilon}_0$) into the constitutive equation, such as those that appear from thermal phenomena:

$$\boldsymbol{\varepsilon} = \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma} + \boldsymbol{\varepsilon}_0 \quad ; \quad \boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon} - \mathbb{C}^e : \boldsymbol{\varepsilon}_0 \tag{7.146}$$

If we are considering thermal effects, the equations for the state of plane stress become:

- Strain:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{7.147}$$

$$\varepsilon_z = \frac{-\nu}{E}\left(\sigma_x + \sigma_y\right) + \alpha\Delta T$$

- Stress:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} - \frac{E\alpha\Delta T}{1-\nu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{7.148}$$

7.12.2 The State of Plane Strain

Let us now consider a structural element with prismatic features, in which the dimension that corresponds to the direction of the prismatic axis is much larger than the other dimensions. Additionally, the loads applied are normal to the prismatic axis, (see Figure 7.17). Under these conditions the strain components: ε_{13}, ε_{23} and ε_{33} are zero. This state is called the *plane strain*, examples of which include: retaining walls, cylinders under pressure (see Figure 7.17), dams (see Figure 7.18), tunnels (see Figure 7.19) and spread footing foundations.

It must be stressed that, in order to consider a state of plane strain the variables involved (load, section, material) must be constant along the prismatic axis. Otherwise, significant errors can occur.

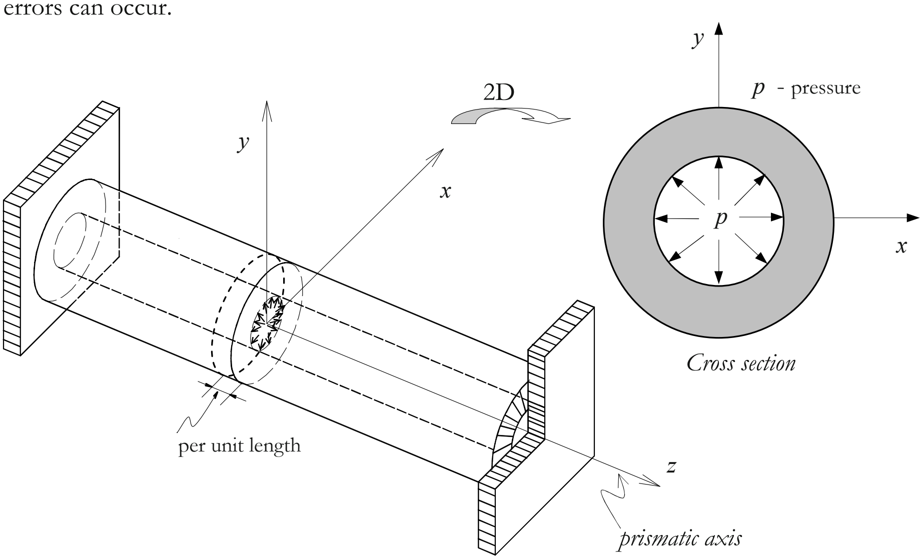


Figure 7.17: Cylinder under pressure.

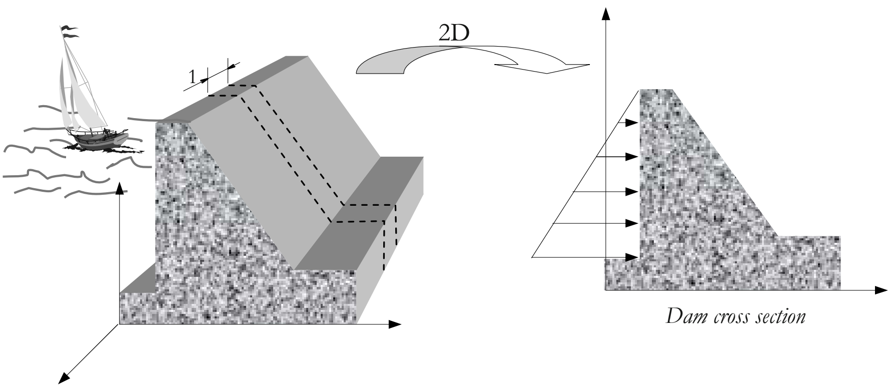


Figure 7.18: Dam.

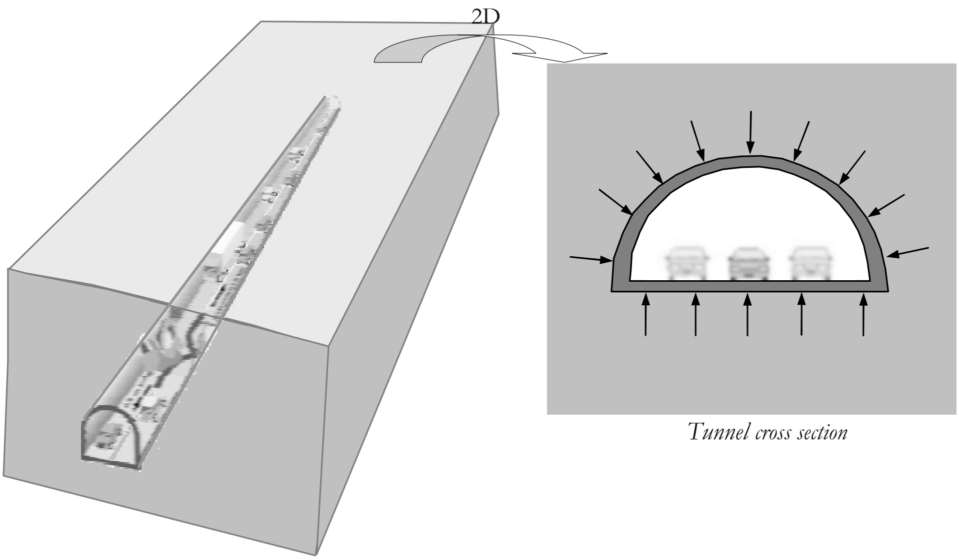


Figure 7.19: Tunnel.

If we start from the generalized Hooke's law (7.86) and by deleting the columns and rows associated with the zero strain, *i.e.*:

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \cancelto{0}{\varepsilon_z} \\ \gamma_{xy} \\ \cancelto{0}{\gamma_{yz}} \\ \cancelto{0}{\gamma_{zx}} \end{bmatrix} \tag{7.149}
$$

we obtain:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{7.150}$$

Then, the stress according to the direction z is given by:

$$\sigma_z = \frac{E\nu}{(1+\nu)(1-2\nu)} \left(\varepsilon_x + \varepsilon_y \right) \tag{7.151}$$

Additionally, the reciprocal of (7.150) is:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{7.152}$$

Afterwards, we can write the constitutive law for the state of stress and strain in a single equation as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{\overline{E}}{1-\overline{\nu}^2} \begin{bmatrix} 1 & \overline{\nu} & 0 \\ \overline{\nu} & 1 & 0 \\ 0 & 0 & \frac{1-\overline{\nu}}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{7.153}$$

where the values of $\overline{E}, \overline{\nu}$ assume the following values:

State of Plane Stress	State of Plane Strain
$\overline{E} = E \quad ; \quad \overline{\nu} = \nu$	$\overline{E} = \dfrac{E}{1-\nu^2} \quad ; \quad \overline{\nu} = \dfrac{\nu}{1-\nu}$

(7.154)

7.12.2.1 Thermal Strain

If we take into account the thermal effect, the stress tensor components can be obtained by means of the following equation:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{7.155}$$

Note that the above equation is the same as that given in (7.136) when we are dealing with two-dimensional cases, *i.e.* when $i, j = 1,2$. Our goal now is to obtain the strain field and to do so, we will restructure the above equation as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{7.156}$$

Then if we multiply the above equation by the matrix given in (7.152) we can obtain:

$$\frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \frac{E\alpha\Delta T}{1-2\nu}\frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{7.157}$$

with which the strain field becomes:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + (1+\nu)\alpha\Delta T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{7.158}$$

Problem 7.5: A *strain gauge* (or strain gage) is a device used to obtain the strain in only one direction. Consider a *strain rosette* that contains three strain gauges where there are 45º internal angles, (see Figure 7.20). At one point we have calculated the following strain values:

$$\varepsilon_x = 0.33\times 10^{-3} \quad ; \quad \varepsilon'_x = 0.22\times 10^{-3} \quad ; \quad \varepsilon_y = -0.05\times 10^{-3}$$

Find the maximum shear stress at the point in question.
Then consider an isotropic linear elastic material with the following mechanical properties: $E = 29000\,Pa$ (Young's modulus); $\nu = 0.3$ (Poisson's ratio). Find:

a) the eigenvalues (principal strains) and eigenvectors (principal directions) of the strain tensor;
b) the eigenvalues (principal stresses) and eigenvectors (principal directions) of the stress tensor.

Hint: Consider the state of plane strain.

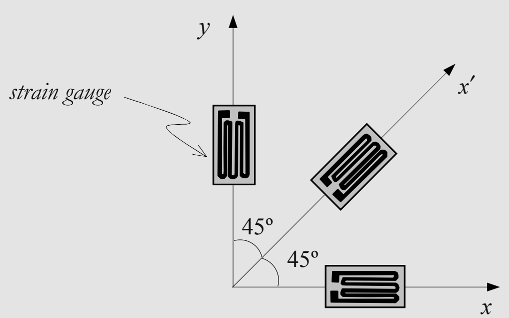


Figure 7.20: Strain rosette.

Solution:
Firstly, we have to obtain the strain tensor components in the system x, y, z and to do so we will use the coordinate transformation law in order to obtain the component $\gamma_{xy} = 2\varepsilon_{12}$.
Remember that in two-dimensional cases, the normal component in a new system is given by (see **Problem 1.40** in Chapter 1):

$$\varepsilon'_{11} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \frac{\varepsilon_{11}-\varepsilon_{22}}{2}\cos(2\theta) + \varepsilon_{12}\sin(2\theta)$$

The above equation was obtained by means of the transformation law, (see Chapter 1), which in engineering notation becomes:

$$\varepsilon'_x = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos(2\theta) + \frac{\gamma_{xy}}{2}\sin(2\theta)$$

Then, γ_{xy} can be obtained as follows:

$$\gamma_{xy} = \frac{2}{\sin(2\theta)}\left(\varepsilon'_x - \frac{(\varepsilon_x + \varepsilon_y)}{2} - \frac{(\varepsilon_x - \varepsilon_y)}{2}\cos(2\theta)\right) = 0.16\times 10^{-3}$$

thus

$$\varepsilon_{ij} = \begin{bmatrix} 0.33 & 0.08 & 0 \\ 0.08 & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

Then, the stress components can be evaluated as follows:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu\varepsilon_y\right] = 12.0462\,Pa$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_y + \nu\varepsilon_x\right] = 3.5692\,Pa$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = 1.7846\,Pa \;\; ; \;\; \sigma_z = \frac{E\nu}{(1+\nu)(1-2\nu)}\left[\varepsilon_x + \varepsilon_y\right] = 4.684\,Pa$$

Additionally, the maximum shear stress is given by:

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 4.5988\,Pa$$

a) The characteristic equation for the strain tensor (2D) is:

$$\varepsilon^2 - 0.28\,\varepsilon - 2.29\times 10^{-2} = 0 \qquad (\times 10^{-3})$$

Then, by solving the above equation we can find the eigenvalues (principal strains) given by:

$$\varepsilon_1 = 0.346155\times 10^{-3} \quad ; \quad \varepsilon_2 = -0.06615528\times 10^{-3}$$

Then, the eigenvectors of the strain tensor are:

$$\begin{array}{l} \varepsilon_1 \Rightarrow \\ \varepsilon_2 \Rightarrow \\ \varepsilon_3 \Rightarrow \end{array} \begin{bmatrix} 0.9802 & 0.1979 & 0 \\ -0.1979 & 0.9802 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Given the stress tensor components, we have:

$$\sigma_{ij} = \begin{bmatrix} 12.0462 & 1.7846 & 0 \\ 1.7846 & 3.5692 & 0 \\ 0 & 0 & 4.684 \end{bmatrix} Pa$$

We now obtain the characteristic determinant and in turn the eigenvalues (principal stresses) $\sigma_1 = 12.40654$, $\sigma_2 = 3.208843$. Additionally, the eigenvectors of the stress tensor are:

$$\begin{array}{l} \sigma_1 \Rightarrow \\ \sigma_2 \Rightarrow \\ \sigma_3 \Rightarrow \end{array} \begin{bmatrix} 0.9802 & 0.1979 & 0 \\ -0.1979 & 0.9802 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As expected, the eigenvectors of stress and strain are the same; since we are working with isotropic linear elastic material.

7.12.3 Axisymmetric Solids

In solids of revolution we use the cylindrical coordinate system to express the strain field:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad ; \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad ; \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{7.159}$$

where ε_r is the radial strain, ε_z is the axial strain, and γ_{rz} is the shear strain.

We can then introduce the strain in the circumferential direction ε_θ as:

$$\varepsilon_\theta = \frac{2\pi(r_P + u) - 2\pi r_P}{2\pi r_P} = \frac{u}{r_P} \tag{7.160}$$

Then, if we regroup the strain tensor components we can obtain:

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{u}{r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{bmatrix} \tag{7.161}$$

Next, the generalized Hooke's law for a solid of revolution is given by:

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{bmatrix} \tag{7.162}$$

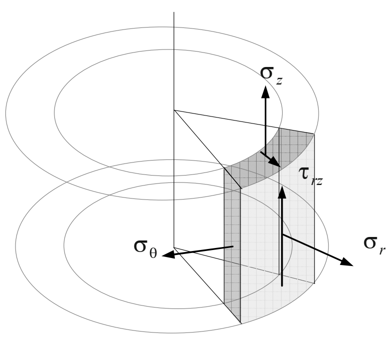


Figure 7.21: Stress components – Axisymmetric solid.

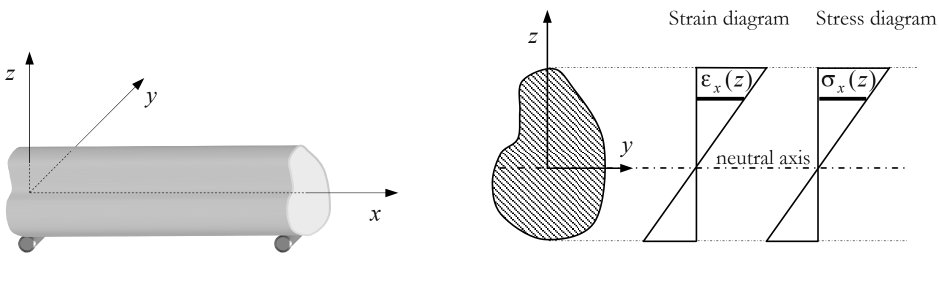
7.13 The Unidimensional Approach

7.13.1 Beam Structural Elements

Structural elements in which one dimension is much larger than the other two are subject to a particular stress/strain field and if we use this particular feature the problem can be greatly simplified. That is, a problem which is three dimensional by nature can be treated as if it were one-dimensional. A few of the structures that exemplify this problem type are: beams, trusses, arches, frames.

As two dimensions are smaller than the third one, if we also consider the linear elastic material and the small deformation regime, a planar cross section of the beam after deformation remains planar. Consequentially, the strain and stress fields at the beam cross section are defined by planes, (see Figure 7.22).

It must be pointed out that in the deep beam case, (see Figure 7.16), the approach adopted in this subsection is invalidated, since the beam cross section does not remain planar after deformation (bending).



a) beam b) beam cross section

Figure 7.22: Beam.

If we now make a cut in a cross section according to the orientation of the plane Π, in general, the stress state at a point in this cross section is given as shown in Figure 7.23. The intensity (or even the nonexistence) of stress depends on the load type (external force) and on the beam cross section.

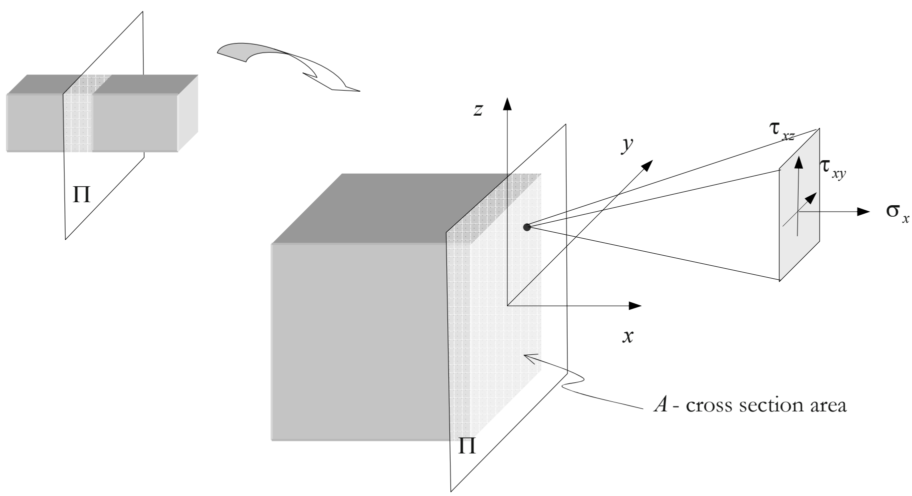


Figure 7.23: Stress at a point in the beam cross section.

As we know how the stress is distributed in the cross section, we can define some resultant internal forces caused by stress components, by integrating the cross-sectional area, namely: N - internal normal force; M - bending moment; Q - shear force; M_T - torsional moment. Next, we will outline how to obtain these internal resultant forces.

7.13.1.1 The Internal Normal Force and the Bending Moments

As we mentioned above, the stress distribution in the cross section is defined by a planar surface, (see Figure 7.24). The normal stress σ_x can be broken down as shown in Figure 7.24. Then, if we consider the normal stress σ_x by itself, it is possible to obtain the internal normal force (N) and the bending moments (M_y, M_z).

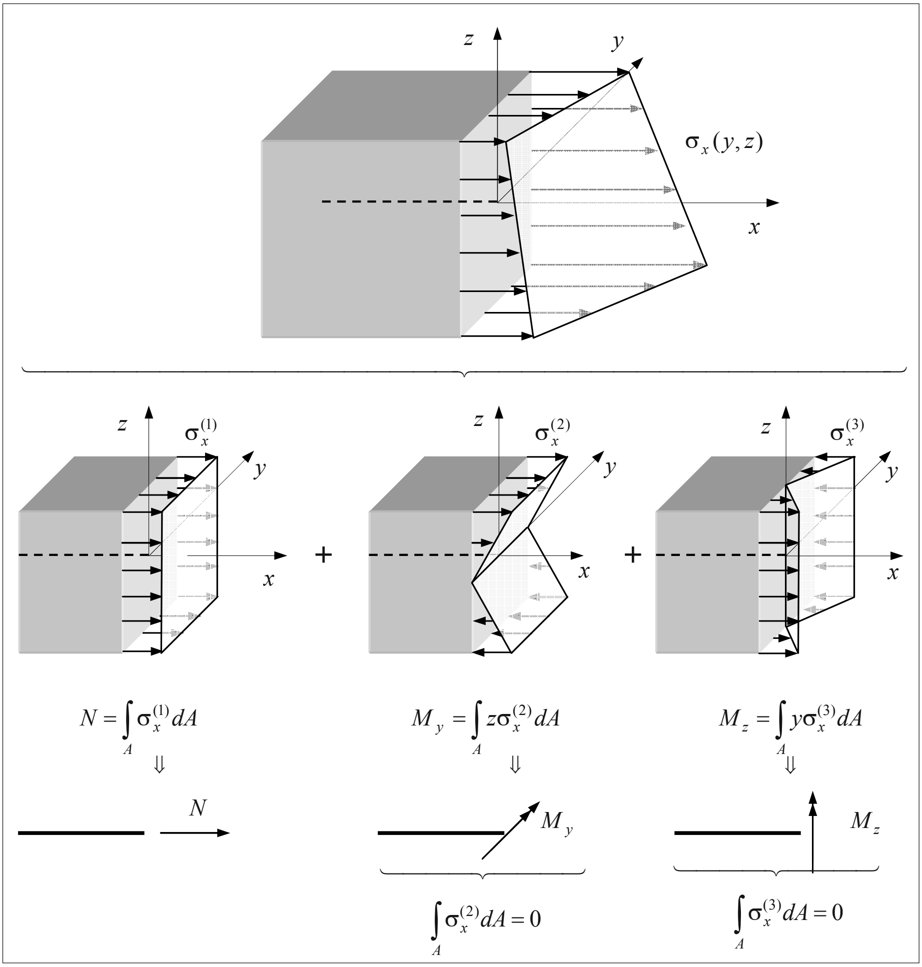


Figure 7.24: The internal normal force and the bending moments.

Then, by considering Figure 7.25, the bending moment M_y is defined as follows:

$$M_y = \int_A \sigma_x z dA = \int_A \frac{\sigma_S z}{c} z dA = \frac{\sigma_S}{c} \int_A z^2 dA = \frac{\sigma_S}{c} I_y \tag{7.163}$$

where I_y is the moment of inertia of the cross section about the y-axis. Then, if we observe that $\frac{\sigma_S}{c} = \frac{\sigma_x}{z}$, we can obtain:

$$\sigma_x(z) = \frac{M_y}{I_y} z \tag{7.164}$$

Similarly, we can obtain:

$$\sigma_x(y) = \frac{M_z}{I_z} y \tag{7.165}$$

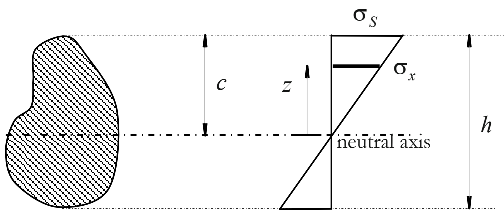


Figure 7.25: Normal stress distribution in the beam cross section.

7.13.1.2 The Shear Forces and the Torsional Moment

Due to shear stresses, the shear forces Q_y and Q_z appear, (see Figure 7.26) as well as the torsional moment (M_T), (see Figure 7.27):

$$M_T = \int_A (\tau_{xz} y - \tau_{yz} z) dA \tag{7.166}$$

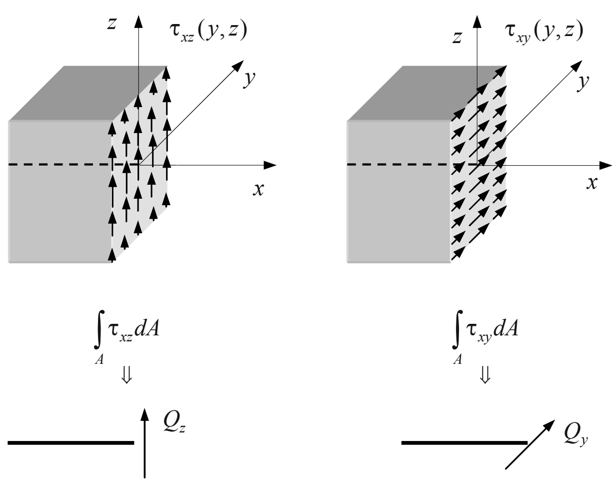


Figure 7.26: The shear stresses – Shear forces.

Warping is a phenomenon that comes about because shear stress is increasing at one point whilst decreasing at another, (see Figure 7.28(a)). In the circular cross-section there is no warping phenomenon, since the shear stress is uniform for given radius (r), (see Figure 7.28(b)).

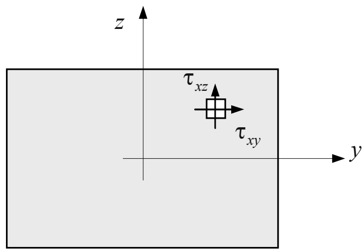


Figure 7.27: The shear stress (torsional moment).

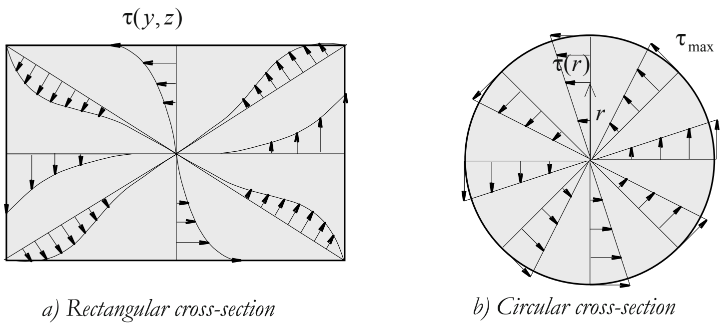


Figure 7.28: Distribution of the shear stress.

7.13.1.3 The Strain Energy

The strain energy associated with the normal stress, $\sigma_x^{(1)} = E\varepsilon_x^{(1)}$ (see Figure 7.24), can be expressed in terms of internal normal force as follows:

$$U = \frac{1}{2}\int_V \sigma_x^{(1)}\varepsilon_x^{(1)} dV = \frac{1}{2}\int_V \frac{\sigma_x^{(1)^2}}{E} dV = \frac{1}{2}\int_0^L \frac{N^2}{EA^2}\int_A dA dx = \frac{1}{2}\int_0^L \frac{N^2}{EA} dx \tag{7.167}$$

Likewise, we can obtain the strain energy associated with the normal stress $\sigma_x^{(2)} = E\varepsilon_x^{(2)}$ as:

$$U = \frac{1}{2}\int_V \sigma_x^{(2)}\varepsilon_x^{(2)} dV = \frac{1}{2}\int_0^L \int_A \frac{M_y}{I_y} z \frac{M_y}{EI_y} z dA dx = \frac{1}{2}\int_0^L \frac{M_y^2}{EI_y^2}\int_A z^2 dA dx = \frac{1}{2}\int_0^L \frac{M_y^2}{EI_y} dx \tag{7.168}$$

In a similar fashion, if we consider the component $\sigma_x^{(3)}$, we can obtain:

$$U = \frac{1}{2}\int_0^L \frac{M_z^2}{EI_z} dx \tag{7.169}$$

Then, if we follow the same procedure for the other stress components, we can obtain the strain energy of a bar in function of the internal forces as:

$$U = \frac{1}{2}\int_0^L \left(\frac{N^2}{EA} + \frac{M_y^2}{EI_y} + \frac{M_z^2}{EI_z} + \frac{\varsigma Q_y^2}{GA} + \frac{\varsigma Q_z^2}{GA} + \frac{M_T^2}{EJ_T} \right) dx \tag{7.170}$$

where ς is the correction factor for the cross-section, and J_T is the effective polar moment.

8 Hyperelasticity

8.1 Introduction

Some materials such as elastomers, polymers, rubber and biological matter (arteries, muscles, skin, etc.) may be subject to large deformations without there being any internal energy dissipation (which is typical en elastic process). These materials are classified as being hyperelastic and purely hyperelastic materials have no memory of motion history, *i.e.* they are only dependent on the current values of the state variables.

Physically speaking, elastic materials (linear elasticity, hyperelasticity) return to their initial state once their load disappears, (see Figure 8.1). In other words, the work done during the loading process is recovered during the unloading process, *i.e.* there is no internal energy dissipation (a reversible process).

Our goal in this chapter is to establish the constitutive equations for materials that behave according to the *hyperelasticity* theory, also known as *Green* or *nonlinear elasticity*. Moreover, we will limit our analysis to purely mechanical theories, so we have eliminated thermodynamic variables such as temperature and entropy.

Among the researchers who have used the hyperelastic constitutive equations to model rubberlike materials we can mention: Alexander (1968), Treloar (1975), Ogden (1984), Morman (1986) and Holzapfel (2000).

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_10,

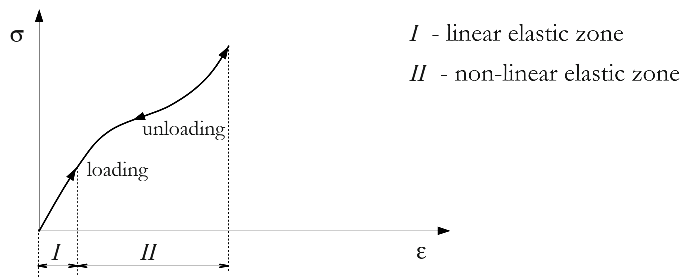


Figure 8.1: The stress-strain curve for elastic materials.

8.2 Constitutive Equations

A *hyperelastic material* supposes the existence of a function which is denoted by the Helmholtz free energy per unit reference volume (Ψ). The energy Ψ is also known as *strain energy density* or the *strain energy function*, or *elastic potential*. In hyperelastic materials, the strain energy function Ψ is only dependent on the deformation gradient $(\boldsymbol{F})$, *i.e.*, $\Psi = \Psi(\boldsymbol{F})$.

In pure deformation processes, which do not involve changes caused by entropy or temperature, internal energy dissipation is equal to zero ($\mathcal{D}_{int} = 0$), which thereby describes reversible processes. In this way, the Clausius-Planck inequality, (see Chapter 5), for reversible processes, turns into the following equations:

$$\begin{aligned}
&\mathcal{D}_{int} = \boldsymbol{\sigma} : \mathbf{D} - \dot{\Psi} = 0 \quad \Rightarrow \quad \dot{\Psi} = \boldsymbol{\sigma} : \mathbf{D} \quad \textit{(Current configuration)} \\
&\mathcal{D}_{int} = \mathbf{P} : \dot{\boldsymbol{F}} - \dot{\Psi} = 0 \quad \Rightarrow \quad \dot{\Psi} = \mathbf{P} : \dot{\boldsymbol{F}} \\
&\qquad = \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} - \dot{\Psi} = 0 \quad \Rightarrow \quad \dot{\Psi} = \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} \quad \textit{(Reference configuration)}
\end{aligned} \tag{8.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{D}$ is the rate-of-deformation tensor, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, and $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor. Then, taking into account the conjugate relations obtained in Chapter 5 we have:

$$\int_V \boldsymbol{\sigma} : \mathbf{D} dV = \int_{V_0} \underbrace{J\boldsymbol{\sigma}}_{\boldsymbol{\tau}} : \mathbf{D}\, dV_0 = \int_{V_0} \boldsymbol{\tau} : \mathbf{D} dV_0 = \int_{V_0} \mathbf{S} : \dot{\boldsymbol{E}} dV_0 = \int_{V_0} \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} dV_0 = \int_{V_0} \mathbf{P} : \dot{\boldsymbol{F}} dV_0 = \int_V \frac{\rho}{\rho_0}\mathbf{P} : \dot{\boldsymbol{F}}\, dV \tag{8.2}$$

Then, in summary we can state that the rate of change of the strain energy density can also be expressed as follows:

$$\dot{\Psi} = \underbrace{\mathbf{P} : \dot{\boldsymbol{F}} = \mathbf{S} : \dot{\boldsymbol{E}} = \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} = \boldsymbol{\tau} : \mathbf{D}}_{\textit{Stress Power}} \tag{8.3}$$

where $\boldsymbol{E}$ is the Green-Lagrange strain tensor, and $\boldsymbol{\tau}$ is the Kirchhoff stress tensor with which we can state that a material is considered to be hyperelastic if and only if the rate of change of the strain energy is equal to the stress power.

Then, by evaluating the rate of change of the strain energy $\Psi(\boldsymbol{F})$ we obtain:

$$\dot{\Psi}(\boldsymbol{F}) = \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} : \frac{\partial \boldsymbol{F}}{\partial t} = \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} \tag{8.4}$$

Next, by substituting the above equation into the internal energy dissipation given by the equation in (8.1) we obtain:

$$\mathbf{P} : \dot{\boldsymbol{F}} - \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} = 0 \quad \Rightarrow \quad \mathbf{P} : \dot{\boldsymbol{F}} = \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} : \dot{\boldsymbol{F}} \tag{8.5}$$

Thus we can draw the conclusion that:

$$\mathbf{P} = \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} \tag{8.6}$$

NOTE: The tensor $\mathbf{P}$ is a function of $\vec{X}$ and $\boldsymbol{F}$, *i.e.* $\mathbf{P} = \mathbf{P}(\boldsymbol{F}(\vec{X}), \vec{X}, t)$, and as $\mathbf{P}$ is directly dependent of $\vec{X}$ we can study non-homogeneous materials. Here, for the sake of simplicity, we will omit the material coordinate $\vec{X}$. ■

As discussed in Chapter 6, strain energy has to be objective, *i.e.* $\Psi(\boldsymbol{F}) = \Psi(\boldsymbol{F}^*) = \Psi(\mathbf{Q} \cdot \boldsymbol{F})$, where $\mathbf{Q}$ is an orthogonal tensor, (see Figure 8.2). Then, by applying the polar decomposition ($\boldsymbol{F} = \mathbf{R} \cdot \mathbf{U}$) and by adopting $\mathbf{Q} = \mathbf{R}^T$, we can obtain:

$$\begin{aligned} \Psi(\boldsymbol{F}) &= \Psi(\mathbf{Q} \cdot \boldsymbol{F}) \\ &= \Psi(\mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{U}) \\ &= \Psi(\mathbf{U}) \quad \Rightarrow \quad \overline{\Psi} = \Psi(\boldsymbol{C}) \quad \Rightarrow \quad \hat{\Psi} = \Psi(\boldsymbol{E}) \end{aligned} \tag{8.7}$$

where $\mathbf{U}$ is the right stretch tensor and $\mathbf{R}$ is the polar decomposition rotation tensor, (see Chapter 2). Since the tensors $\boldsymbol{C}$, $\mathbf{U}$, and $\boldsymbol{E}$ are directly linked by $\boldsymbol{C} = \mathbf{U}^2$, and $2\boldsymbol{E} = \boldsymbol{C} - \mathbf{1}$, the strain energy density can also be expressed in terms of the Green-Lagrange strain tensor ($\boldsymbol{E}$). Then, similarly to (8.6), it is possible to express the constitutive equation for stress in the material description. Next, if we take the rate of change of the energy $\Psi(\boldsymbol{C})$ we can obtain:

$$\dot{\Psi}(\boldsymbol{C}) = \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} : \frac{\partial \boldsymbol{C}}{\partial t} = \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}} \tag{8.8}$$

By substituting (8.8) into the internal energy dissipation (8.1) we obtain:

$$\begin{aligned} \mathcal{D}_{int} &= \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} - \dot{\Psi}(\boldsymbol{C}) = 0 \\ &= \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} - \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}} = 0 \\ &= \left(\frac{1}{2}\mathbf{S} - \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \right) : \dot{\boldsymbol{C}} = 0 \end{aligned} \tag{8.9}$$

Note that, the condition in (8.9) must hold for any thermodynamic process. Now, in a mechanical process with $\dot{\boldsymbol{C}} \neq \mathbf{0}$, the only scenario in which the condition (8.9) remains valid is when:

$$\frac{1}{2}\mathbf{S} - \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 0 \quad \Rightarrow \quad \mathbf{S} = 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \tag{8.10}$$

Likewise, we can show that:

$$\mathbf{S} = 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = \frac{\partial \Psi(\boldsymbol{E})}{\partial \boldsymbol{E}} \tag{8.11}$$

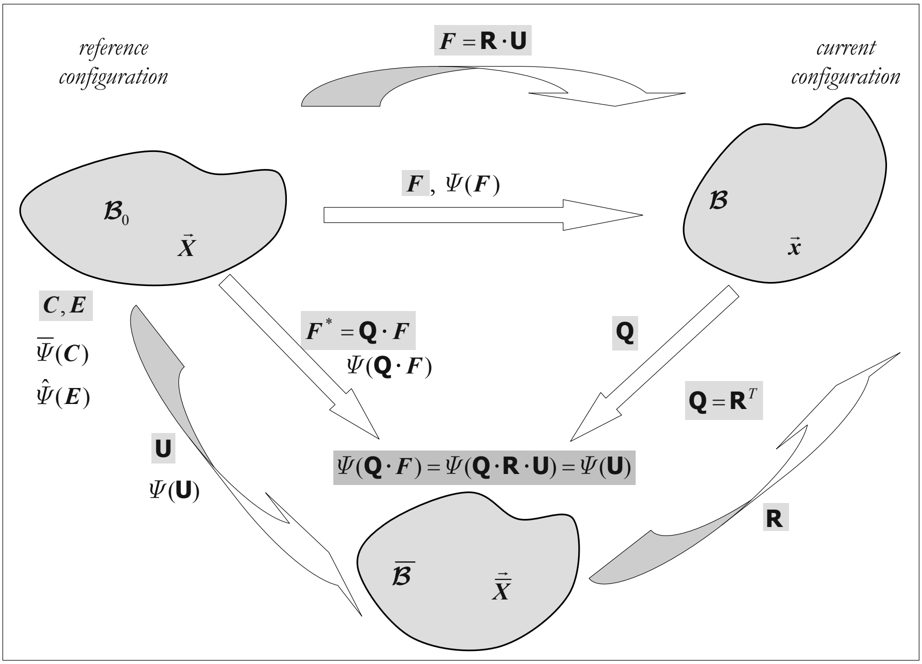


Figure 8.2: Objectivity of strain energy density.

Then, if we take into account the relationships between the stress tensors, (see Chapter 3), we can still express the constitutive equation for stress as follows:

- The Kirchhoff stress tensor ($\boldsymbol{\tau}$):

$$\boldsymbol{\tau} = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \frac{\partial \Psi(\boldsymbol{E})}{\partial \boldsymbol{E}} \cdot \boldsymbol{F}^T = 2\boldsymbol{F} \cdot \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T \tag{8.12}$$

- The Cauchy stress tensor ($\boldsymbol{\sigma}$):

$$\begin{aligned} \boldsymbol{\tau} = J\boldsymbol{\sigma} &= \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T \\ &= 2\boldsymbol{F} \cdot \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^T \end{aligned} \quad \Bigg| \quad \begin{aligned} \boldsymbol{\sigma} &= J\,\mathbf{P} \cdot \boldsymbol{F}^{-T} \\ &= J\frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^{-T} \end{aligned} \tag{8.13}$$

- The first Piola-Kirchhoff stress tensor ($\mathbf{P}$):

$$\mathbf{P} = \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F}^{-T} = \boldsymbol{F} \cdot \mathbf{S} = \boldsymbol{F} \cdot 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \tag{8.14}$$

- The Mandel stress tensor ($\mathbf{M}$):

$$\begin{aligned} \mathbf{M} &= \boldsymbol{C} \cdot \mathbf{S} &&= 2\boldsymbol{C} \cdot \frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \\ &= \boldsymbol{F}^T \cdot \mathbf{P} &&= \boldsymbol{F}^T \cdot \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} \end{aligned} \tag{8.15}$$

Hence, we can sum up the different ways of expressing the stress constitutive equations for hyperelastic materials as:

$$\boxed{\begin{array}{lll} \mathbf{P}=\dfrac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}=\boldsymbol{F}\cdot 2\dfrac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} & ; & \boldsymbol{\tau}=J\boldsymbol{\sigma}=2\boldsymbol{F}\cdot\dfrac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}}\cdot\boldsymbol{F}^T \\ \mathbf{S}=\dfrac{\partial\Psi(\boldsymbol{E})}{\partial\boldsymbol{E}}=2\dfrac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} & ; & \mathbf{M}=2\boldsymbol{C}\cdot\dfrac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}}=\boldsymbol{F}^T\cdot\dfrac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}} \end{array}} \qquad (8.16)$$

Stress constitutive equations for hyperelastic materials

The strain energy function (Ψ) has to satisfy the following:

- The normalization condition: $\Psi(\mathbf{1})=0$, *i.e.* the strain energy function vanishes when the material has been completely unloaded, *i.e.* when $\boldsymbol{F}=\mathbf{1}$;
- $\Psi(\boldsymbol{F})\geq 0$. The strain energy must increase monotonically with deformation.

In a reversible process (without internal energy dissipation) the following must be satisfied:

- The work done is independent of the path.

If we consider a particle which is deformed according to the path Γ at a given time interval, said deformation is defined by the deformation gradient $\boldsymbol{F}:[t_1,t_2]$. Then, the internal work done associated with this path is:

$$\int_{t_1}^{t_2}\mathbf{P}:\dot{\boldsymbol{F}}\,dt=\int_{t_1}^{t_2}\mathbf{S}:\dot{\boldsymbol{E}}\,dt=\int_{t_1}^{t_2}\frac{1}{2}\mathbf{S}:\dot{\boldsymbol{C}}\,dt \qquad (8.17)$$

Now, for an elastic material (reversible) the internal work done is independent of the path, so, the following must be met:

$$\int_{\Gamma}\mathbf{P}:d\boldsymbol{F}=\int_{\Gamma'}\mathbf{P}:d\boldsymbol{F} \qquad (8.18)$$

for any deformation path Γ, Γ'.

- For any closed cycle of deformation the work done is equal to zero:

$$\oint\mathbf{P}:d\boldsymbol{F}=0 \quad or \quad \oint\mathbf{S}:d\boldsymbol{E}=0 \qquad (8.19)$$

8.2.1 Elastic Tangent Stiffness Tensors

8.2.1.1 The Material Elastic Tangent Stiffness Tensor

The rate of change of the constitutive equation in (8.16), $\mathbf{S}(\boldsymbol{E})$, can be expressed as:

$$\dot{\mathbf{S}}=\frac{\partial^2\Psi(\boldsymbol{E})}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}=\mathbb{C}^{tan}:\dot{\boldsymbol{E}} \qquad \dot{S}_{ij}=\frac{\partial^2\Psi(\boldsymbol{E})}{\partial E_{ij}\,\partial E_{kl}}\dot{E}_{kl}=\mathbb{C}^{tan}_{ijkl}\dot{E}_{kl} \qquad (8.20)$$

where $\dot{\mathbf{S}}$, $\dot{\boldsymbol{E}}$ are objective rates, and $\mathbb{C}^{tan}$ is a fourth-order tensor known as the *material elastic tangent stiffness tensor* also called the *material tangent elasticity tensor*. Remember that the tensors $\boldsymbol{E}$ and $\boldsymbol{C}$ are related to each other by the equation $2\boldsymbol{E}=(\boldsymbol{C}-\mathbf{1})\Rightarrow 2\dot{\boldsymbol{E}}=\dot{\boldsymbol{C}}$, thus:

$$\dot{\mathbf{S}}=2\frac{\partial^2\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}\otimes\partial\boldsymbol{C}}:\dot{\boldsymbol{C}} \quad\Rightarrow\quad \dot{\mathbf{S}}=\underbrace{4\frac{\partial^2\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}\otimes\partial\boldsymbol{C}}}_{=\mathbb{C}^{tan}}:\dot{\boldsymbol{E}} \qquad (8.21)$$

Then, taking into account the equations in (8.11), (8.20) and (8.21) we can conclude that:

$$\boxed{\begin{aligned}\mathbb{C}^{tan} &= \frac{\partial^2 \Psi(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} = \frac{\partial \mathbf{S}}{\partial \boldsymbol{E}} \\ &= 4\frac{\partial^2 \Psi(\boldsymbol{C})}{\partial \boldsymbol{C} \otimes \partial \boldsymbol{C}} = 2\frac{\partial \mathbf{S}}{\partial \boldsymbol{C}}\end{aligned}} \quad \begin{array}{l}\textit{The material elastic tangent stiffness tensor} \\ \text{(Reference configuration)}\end{array} \tag{8.22}$$

Note that the tensors $\dot{\mathbf{S}}$ and $\dot{\boldsymbol{E}}$ are symmetrical, *i.e.* $\dot{\mathsf{S}}_{ij} = \dot{\mathsf{S}}_{ji}$, $\dot{E}_{ij} = \dot{E}_{ji}$, so, the fourth-order tensor, $\mathbb{C}^{tan}$, must feature at least minor symmetry, *i.e.*:

$$\mathbb{C}^{tan}_{ijkl} = \mathbb{C}^{tan}_{jikl} = \mathbb{C}^{tan}_{ijlk} = \mathbb{C}^{tan}_{jilk} \tag{8.23}$$

Then, taking into account the equation in (8.22) we can conclude that the tensor $\mathbb{C}^{tan}$ also has major symmetry:

$$\mathbb{C}^{tan}_{ijkl} = \frac{\partial^2 \Psi}{\partial E_{ij}\, \partial E_{kl}} = \frac{\partial}{\partial E_{ij}}\left(\frac{\partial \Psi}{\partial E_{kl}}\right) = \frac{\partial}{\partial E_{kl}}\left(\frac{\partial \Psi}{\partial E_{ij}}\right) = \mathbb{C}^{tan}_{klij} \tag{8.24}$$

Therefore, we can conclude that the tensor $\mathbb{C}^{tan}$ is symmetric. In the general case $\mathbb{C}^{tan}$ is anisotropic and has 21 independent components. For further details regarding symmetry types see Chapter 7.

8.2.1.2 The Spatial Elastic Tangent Stiffness Tensor

The rate of change of the second Piola-Kirchhoff stress tensor can be obtained by means of the equation $\mathbf{S} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T}$, *i.e.*:

$$\begin{aligned}\dot{\mathbf{S}} &= \dot{\boldsymbol{F}}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} + \boldsymbol{F}^{-1} \cdot \dot{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-T} + \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \dot{\boldsymbol{F}}^{-T} \\ &= -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} + \boldsymbol{F}^{-1} \cdot \dot{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-T} - \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{\ell}^{T} \cdot \boldsymbol{F}^{-T} \\ &= \boldsymbol{F}^{-1} \cdot \left(\dot{\boldsymbol{\tau}} - \boldsymbol{\ell} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\ell}^{T}\right) \cdot \boldsymbol{F}^{-T}\end{aligned} \tag{8.25}$$

where we have considered the equation $\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell}$ which was obtained in Chapter 2. Remember in Chapter 4 the Oldroyd rate of the Kirchhoff stress tensor was given by:

$$\overset{\square}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \boldsymbol{\ell} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\ell}^{T} \tag{8.26}$$

Remember also that $\dot{\boldsymbol{\tau}}$ is not objective, but $\overset{\square}{\boldsymbol{\tau}}$ is. Then, by substituting (8.26) into (8.25) we obtain:

$$\dot{\mathbf{S}} = \boldsymbol{F}^{-1} \cdot \overset{\square}{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-T} \tag{8.27}$$

Then, given the relationship between the rate of change of the Green-Lagrange strain tensor and the rate-of-deformation tensor we have:

$$\dot{\boldsymbol{E}} = \boldsymbol{F}^{T} \cdot \mathbf{D} \cdot \boldsymbol{F} \tag{8.28}$$

Additionally, by substituting the equations (8.27) and (8.28) into the constitutive equation (8.20) we obtain:

$$\begin{aligned}\dot{\mathbf{S}} &= \mathbb{C}^{tan} : \dot{\boldsymbol{E}} \\ \boldsymbol{F}^{-1} \cdot \overset{\square}{\boldsymbol{\tau}} \cdot \boldsymbol{F}^{-T} &= \mathbb{C}^{tan} : \boldsymbol{F}^{T} \cdot \mathbf{D} \cdot \boldsymbol{F}\end{aligned} \quad \Bigg| \quad \begin{aligned}\dot{\mathsf{S}}_{mn} &= \mathbb{C}^{tan}_{mnpq}\, \dot{E}_{pq} \\ F^{-1}_{ms}\, \overset{\square}{\tau}_{st}\, F^{-1}_{nt} &= \mathbb{C}^{tan}_{mnpq} F_{kp}\, \mathsf{D}_{kl} F_{lq}\end{aligned} \tag{8.29}$$

Then, taking into account the symmetry of the Oldroyd rate of the Kirchhoff stress tensor ($\overset{\square}{\boldsymbol{\tau}}$) and $\mathbf{D}$, we can obtain:

$$\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\overset{\square}{\boldsymbol{\tau}}\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{T}=\boldsymbol{F}\cdot\mathbb{C}^{tan}:\boldsymbol{F}^{T}\cdot\mathbf{D}\cdot\boldsymbol{F}\cdot\boldsymbol{F}^{T}$$

$$\overset{\square}{\boldsymbol{\tau}}=\left(\boldsymbol{F}\overline{\otimes}\boldsymbol{F}:\mathbb{C}^{tan}:\boldsymbol{F}^{T}\underline{\otimes}\boldsymbol{F}^{T}\right):\mathbf{D}$$

$$F_{im}F_{ms}^{-1}\overset{\square}{\tau}_{st}F_{jn}F_{nt}^{-1}=F_{jn}F_{im}\mathbb{C}_{mnpq}^{tan}F_{kp}F_{lq}\mathsf{D}_{kl}$$
$$\delta_{is}\overset{\square}{\tau}_{st}\delta_{jt}=F_{jn}F_{im}\mathbb{C}_{mnpq}^{tan}F_{kp}F_{lq}\mathsf{D}_{kl}$$
$$\overset{\square}{\tau}_{ij}=\underbrace{F_{jn}F_{im}\mathbb{C}_{mnpq}^{tan}F_{kp}F_{lq}}_{\mathbb{L}_{ijkl}}\mathsf{D}_{kl} \tag{8.30}$$

$$\overset{\square}{\boldsymbol{\tau}}=\mathbb{L}:\mathbf{D} \tag{8.31}$$

where we have introduced the spatial elastic tangent stiffness tensor, also called the spatial tangent elasticity tensor, in the current configuration, which is given by:

$$\mathbb{L}=(\boldsymbol{F}\overline{\otimes}\boldsymbol{F}):\mathbb{C}^{tan}:(\boldsymbol{F}^{T}\underline{\otimes}\boldsymbol{F}^{T}) \qquad \mathbb{L}_{ijkl}=F_{im}F_{jn}\mathbb{C}_{mnpq}^{tan}F_{kp}F_{lq} \tag{8.32}$$

Then, from the above equation we can obtain the inverse relationship:

$$\begin{aligned}F_{ai}^{-1}F_{bj}^{-1}\mathbb{L}_{ijkl}F_{ck}^{-1}F_{dl}^{-1}&=F_{ai}^{-1}F_{im}F_{bj}^{-1}F_{jn}\mathbb{C}_{mnpq}^{tan}F_{ck}^{-1}F_{kp}F_{dl}^{-1}F_{lq}=\delta_{am}\delta_{bn}\mathbb{C}_{mnpq}^{tan}\delta_{cp}\delta_{dq}\\&=\mathbb{C}_{abcd}^{tan}\end{aligned} \tag{8.33}$$

Thus

$$\mathbb{C}_{abcd}^{tan}=F_{ai}^{-1}F_{bj}^{-1}\mathbb{L}_{ijkl}F_{ck}^{-1}F_{dl}^{-1} \qquad \mathbb{C}^{tan}=\left(\boldsymbol{F}^{-1}\overline{\otimes}\boldsymbol{F}^{-1}\right):\mathbb{L}:\left(\boldsymbol{F}^{-T}\underline{\otimes}\boldsymbol{F}^{-T}\right) \tag{8.34}$$

In Chapter 4 we obtained the relationship between the Jaumann-Zaremba rate ($\overset{\circ}{\boldsymbol{\tau}}$) and the Oldroyd rate of the Kirchhoff stress tensor ($\overset{\square}{\boldsymbol{\tau}}$), *i.e.*:

$$\overset{\circ}{\boldsymbol{\tau}}=\overset{\square}{\boldsymbol{\tau}}+\mathbf{D}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\mathbf{D} \quad\Rightarrow\quad \overset{\square}{\boldsymbol{\tau}}=\overset{\circ}{\boldsymbol{\tau}}-\mathbf{D}\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\mathbf{D} \tag{8.35}$$

Next, by combining the above rate $\overset{\square}{\boldsymbol{\tau}}$ with the constitutive equation in (8.31) we can obtain:

$$\begin{aligned}&\overset{\circ}{\boldsymbol{\tau}}-\mathbf{D}\cdot\boldsymbol{\tau}-\boldsymbol{\tau}\cdot\mathbf{D}=\mathbb{L}:\mathbf{D}\\&\Rightarrow\overset{\circ}{\boldsymbol{\tau}}=\mathbb{L}:\mathbf{D}+\mathbf{D}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\mathbf{D}\end{aligned} \qquad \overset{\circ}{\tau}_{ij}=\mathbb{L}_{ijkl}\mathsf{D}_{kl}+\mathsf{D}_{ip}\tau_{pj}+\tau_{ip}\mathsf{D}_{pj} \tag{8.36}$$

Notice that the tensor $\mathbf{D}$ is symmetric, so the double dot product between the symmetric fourth-order unit tensor, $\mathbb{I}^{sym}$, and a symmetric second-order tensor turns out to be the same tensor, so,

$$\mathsf{D}_{ip}=\mathbb{I}_{ipkl}^{sym}\mathsf{D}_{kl}=\frac{1}{2}(\delta_{ik}\delta_{pl}+\delta_{il}\delta_{pk})\mathsf{D}_{kl} \qquad \mathsf{D}_{pj}=\mathbb{I}_{pjkl}^{sym}\mathsf{D}_{kl}=\frac{1}{2}(\delta_{pk}\delta_{jl}+\delta_{pl}\delta_{jk})\mathsf{D}_{kl} \tag{8.37}$$

Then, by substituting (8.37) into (8.36) we obtain:

$$\begin{aligned}\overset{\circ}{\tau}_{ij}&=\mathbb{L}_{ijkl}\mathsf{D}_{kl}+\frac{1}{2}(\delta_{ik}\delta_{pl}+\delta_{il}\delta_{pk})\tau_{pj}\mathsf{D}_{kl}+\frac{1}{2}\tau_{ip}(\delta_{pk}\delta_{jl}+\delta_{pl}\delta_{jk})\mathsf{D}_{kl}\\&=\left[\mathbb{L}_{ijkl}+\frac{1}{2}(\tau_{pj}\delta_{ik}\delta_{pl}+\tau_{pj}\delta_{il}\delta_{pk})+\frac{1}{2}(\tau_{ip}\delta_{pk}\delta_{jl}+\tau_{ip}\delta_{pl}\delta_{jk})\right]\mathsf{D}_{kl}\\&=\left[\mathbb{L}_{ijkl}+\frac{1}{2}(\tau_{lj}\delta_{ik}+\tau_{kj}\delta_{il})+\frac{1}{2}(\tau_{ik}\delta_{jl}+\tau_{il}\delta_{jk})\right]\mathsf{D}_{kl}\\&=\left[\mathbb{L}_{ijkl}+2\frac{1}{4}(\tau_{lj}\delta_{ik}+\tau_{kj}\delta_{il}+\tau_{ik}\delta_{jl}+\tau_{il}\delta_{jk})\right]\mathsf{D}_{kl}=\left[\mathbb{L}_{ijkl}+2\mathbb{H}_{ijkl}\right]\mathsf{D}_{kl}\end{aligned} \tag{8.38}$$

Therefore, the rate of change of the constitutive equation in terms of Jaumann-Zaremba rate of the Kirchhoff stress tensor becomes:

$$\overset{\circ}{\boldsymbol{\tau}} = \hat{\mathbb{L}} : \mathbf{D} \tag{8.39}$$

where $\hat{\mathbb{L}}$ is a fourth-order tensor and is defined by:

$$\hat{\mathbb{L}} = \mathbb{L} + 2\mathbb{H} \tag{8.40}$$

with

$$\mathbb{H}_{ijkl} = \frac{1}{4}(\tau_{lj}\delta_{ik} + \tau_{kj}\delta_{il} + \tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk}) \tag{8.41}$$

We can now summarize the relationships between the rate of change of the stress and the rate-of-deformation tensor as:

$$\boxed{\begin{cases} \overset{\square}{\boldsymbol{\tau}} = \mathbb{L} : \mathbf{D} \\ \overset{\square}{\tau}_{ij} = \mathbb{L}_{ijkl}\mathsf{D}_{kl} \end{cases} \quad ; \quad \begin{cases} \overset{\circ}{\boldsymbol{\tau}} = \hat{\mathbb{L}} : \mathbf{D} \\ \overset{\circ}{\tau}_{ij} = \left[\mathbb{L}_{ijkl} + 2\mathbb{H}_{ijkl}\right]\mathsf{D}_{kl} \end{cases}}$$

where

$$\boxed{\begin{aligned} \mathbb{L}_{ijkl} &= F_{jn}F_{im}\mathbb{C}^{tan}_{mnpq}F_{kp}F_{lq} \\ \hat{\mathbb{L}}_{ijkl} &= \mathbb{L}_{ijkl} + 2\mathbb{H}_{ijkl} \\ \mathbb{H}_{ijkl} &= \frac{1}{4}(\tau_{lj}\delta_{ik} + \tau_{kj}\delta_{il} + \tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk}) \end{aligned}} \tag{8.42}$$

The spatial elastic tangent stiffness tensor (Current configuration)

8.2.1.3 The Instantaneous Elastic Tangent Stiffness Tensor

The relationship between the Cauchy stress tensor and the second Piola-Kirchhoff stress tensors is given by $\mathbf{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$ whose rate of change becomes:

$$\dot{\mathbf{S}} = \dot{J}\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} + J\dot{\boldsymbol{F}}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} + J\boldsymbol{F}^{-1} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{F}^{-T} + J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{F}}^{-T} \tag{8.43}$$

where $\dot{J} = J\,\mathrm{Tr}(\mathbf{D})$ and $\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell}$, (see Chapter 2), which if substituted into the above equation yields the following result:

$$\dot{\mathbf{S}} = J\boldsymbol{F}^{-1} \cdot \left(\dot{\boldsymbol{\sigma}} - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^{-T} + \mathrm{Tr}(\mathbf{D})\boldsymbol{\sigma}\right) \cdot \boldsymbol{F}^{-T} \tag{8.44}$$

Now, remember in Chapter 4 that the Truesdell stress rate, which is objective, is given by $\overset{\mathrm{T}}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\ell} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\ell}^{T} + \boldsymbol{\sigma}\,\mathrm{Tr}(\mathbf{D})$ so, we can state that:

$$\dot{\mathbf{S}} = J\boldsymbol{F}^{-1} \cdot \overset{\mathrm{T}}{\boldsymbol{\sigma}} \cdot \boldsymbol{F}^{-T} \tag{8.45}$$

Then, by substituting the equation (8.45) into the constitutive equation in (8.20) and if we know that $\dot{\boldsymbol{E}} = \boldsymbol{F}^{T} \cdot \mathbf{D} \cdot \boldsymbol{F}$, we can obtain:

$$\begin{aligned} \dot{\mathbf{S}} &= \mathbb{C}^{tan} : \dot{\boldsymbol{E}} \\ J\boldsymbol{F}^{-1} \cdot \overset{\mathrm{T}}{\boldsymbol{\sigma}} \cdot \boldsymbol{F}^{-T} &= \mathbb{C}^{tan} : \boldsymbol{F}^{T} \cdot \mathbf{D} \cdot \boldsymbol{F} \\ \overset{\mathrm{T}}{\boldsymbol{\sigma}} &= \frac{1}{J}\left(\boldsymbol{F} \overline{\otimes} \boldsymbol{F} : \mathbb{C}^{tan} : \boldsymbol{F}^{T} \underline{\otimes} \boldsymbol{F}^{T}\right) : \mathbf{D} \end{aligned} \tag{8.46}$$

or

$$\overset{\mathrm{T}}{\boldsymbol{\sigma}} = \mathbb{A} : \mathbf{D} \tag{8.47}$$

where $\mathbb{A}$ is the *instantaneous elastic tangent stiffness tensor*, also called the instantaneous elastic moduli, (see Asaro&Lubarda(2006)), which is defined by:

$$\mathbb{A} = \frac{1}{J} \boldsymbol{F} \overline{\otimes} \boldsymbol{F} : \mathbb{C}^{tan} : \boldsymbol{F}^T \underline{\otimes} \boldsymbol{F}^T = \frac{1}{J} \mathbb{L} \qquad \Big| \qquad \mathbb{A}_{ijkl} = \frac{1}{J} F_{im} F_{jn} \mathbb{C}^{tan}_{mnpq} F_{kp} F_{lq} = \frac{1}{J} \mathbb{L}_{ijkl} \tag{8.48}$$

where $\mathbb{L}$ is the fourth-order tensor given in (8.32). So, in summary we have:

$$\boxed{\overset{\mathrm{T}}{\boldsymbol{\sigma}} = \mathbb{A} : \mathbf{D} \quad ; \quad \overset{\mathrm{T}}{\sigma}_{ij} = \mathbb{A}_{ijkl} \mathsf{D}_{kl}}$$

with (8.49)

$$\boxed{\mathbb{A}_{ijkl} = \frac{1}{J} F_{im} F_{jn} \mathbb{C}^{tan}_{mnpq} F_{kp} F_{lq} = \frac{1}{J} \mathbb{L}_{ijkl}} \quad \textit{The instantaneous elastic tangent stiffness tensor}$$

Then, by taking into account the relationship between the Truesdell stress rate ($\overset{\mathrm{T}}{\boldsymbol{\sigma}}$) and the Oldroyd rate of the Kirchhoff stress tensor $\overset{\square}{\boldsymbol{\tau}}$, *i.e.* $\overset{\square}{\boldsymbol{\tau}} = J \overset{\mathrm{T}}{\boldsymbol{\sigma}}$, the equation in (8.47) becomes:

$$\overset{\mathrm{T}}{\boldsymbol{\sigma}} = \mathbb{A} : \mathbf{D} \quad \Rightarrow \quad \frac{1}{J} \overset{\square}{\boldsymbol{\tau}} = \frac{1}{J} \mathbb{L} : \mathbf{D} \quad \Rightarrow \quad \overset{\square}{\boldsymbol{\tau}} = \mathbb{L} : \mathbf{D} \tag{8.50}$$

which is the same as that obtained in (8.31).

8.2.1.4 The Elastic Tangent Stiffness Pseudo-Tensor

The constitutive equation for stress that relates the first Piola-Kirchhoff stress tensor to the deformation gradient is:

$$\mathbf{P} = \frac{\partial \Psi(\boldsymbol{F})}{\partial \boldsymbol{F}} \qquad \Big| \qquad \mathsf{P}_{ij} = \frac{\partial \Psi(\boldsymbol{F})}{\partial F_{ij}} \tag{8.51}$$

Remember that $\boldsymbol{F}$ and $\mathbf{P}$ are two-point tensors (pseudo-tensors), *i.e.* they are not defined in any configuration. Then, the rate of change of the above constitutive equation is given by:

$$\begin{aligned} \dot{\mathbf{P}} &= \frac{\partial^2 \Psi(\boldsymbol{F})}{\partial \boldsymbol{F} \otimes \partial \boldsymbol{F}} : \dot{\boldsymbol{F}} \qquad \Big| \qquad \dot{\mathsf{P}}_{ij} = \frac{\partial}{\partial F_{ij}} \left(\frac{\partial \Psi(\boldsymbol{F})}{\partial F_{kl}} \right) \dot{F}_{kl} \\ \dot{\mathbf{P}} &= \mathbb{K} : \dot{\boldsymbol{F}} \qquad \Big| \qquad \dot{\mathsf{P}}_{ij} = \mathbb{K}_{ijkl} \dot{F}_{kl} \end{aligned} \tag{8.52}$$

We can now introduce the *elastic tangent stiffness pseudo-tensor* also called the elastic pseudomoduli, (see Lubarda&Benson (2001)), as follows:

$$\mathbb{K} = \frac{\partial^2 \Psi(\boldsymbol{F})}{\partial \boldsymbol{F} \otimes \partial \boldsymbol{F}} \qquad \Big| \qquad \mathbb{K}_{ijkl} = \frac{\partial^2 \Psi(\boldsymbol{F})}{\partial F_{ij} \partial F_{kl}} = \frac{\partial^2 \Psi(\boldsymbol{F})}{\partial F_{kl} \partial F_{ij}} = \mathbb{K}_{klij} \tag{8.53}$$

The elastic tangent stiffness pseudo-tensor is not a "real" moduli, because it is partially associated with the material spin tensor, (see Asaro&Lubarda (2006)).

Next, we can relate the tensors $\mathbb{K}$ and $\mathbb{C}^{tan}$. To do so, we need to evaluate the rate of change of $\mathsf{P}_{ij} = F_{ip} \mathsf{S}_{pj}$, (see Eq. (8.14)):

$$\dot{\mathsf{P}}_{ij} = \dot{\mathsf{S}}_{pj} F_{ip} + \dot{F}_{ip} \mathsf{S}_{pj} \tag{8.54}$$

Then, by substituting (8.21) and (8.52) into the above equation we obtain:

$$\mathbb{K}_{ijkl}\dot{F}_{kl} = \mathbb{C}_{pjkl}\dot{E}_{kl}F_{ip} + \dot{F}_{ip}\mathsf{S}_{pj} \tag{8.55}$$

and if we know that $\dot{E}_{kl} = \frac{1}{2}\left(\dot{F}_{qk}F_{ql} + F_{qk}\dot{F}_{ql}\right)$ the above equation becomes:

$$\begin{aligned}\mathbb{K}_{ijkl}\dot{F}_{kl} &= \frac{1}{2}\mathbb{C}^{tan}_{pjkl}\left(\dot{F}_{qk}F_{ql} + F_{qk}\dot{F}_{ql}\right)F_{ip} + \dot{F}_{ip}\mathsf{S}_{pj} \\ &= \frac{1}{2}\left(\mathbb{C}^{tan}_{pjkl}F_{ql}F_{ip}\dot{F}_{qk} + \mathbb{C}^{tan}_{pjkl}F_{qk}F_{ip}\dot{F}_{ql}\right) + \dot{F}_{ip}\mathsf{S}_{pj}\end{aligned} \tag{8.56}$$

Note that the dummy indices k and l from the expression $\mathbb{C}^{tan}_{pjkl}F_{ql}F_{ip}\dot{F}_{qk}$ can be exchanged without altering the result of the expression, and the dummy indices k and q from $\mathbb{C}^{tan}_{pjkl}F_{qk}F_{ip}\dot{F}_{ql}$ can also be exchanged, so:

$$\begin{aligned}\mathbb{K}_{ijkl}\dot{F}_{kl} &= \frac{1}{2}\left(\mathbb{C}^{tan}_{pjlk}F_{qk}F_{ip}\dot{F}_{ql} + \mathbb{C}^{tan}_{pjql}F_{kq}F_{ip}\dot{F}_{kl}\right) + \mathsf{S}_{pj}\dot{F}_{ip} \\ &= \frac{1}{2}\left(\mathbb{C}^{tan}_{pjlq}F_{kq}F_{ip}\dot{F}_{kl} + \mathbb{C}^{tan}_{pjql}F_{kq}F_{ip}\dot{F}_{kl}\right) + \mathsf{S}_{pj}\dot{F}_{ip}\end{aligned} \tag{8.57}$$

Then, if we make use of the minor symmetry $\mathbb{C}^{tan}_{pjlq} = \mathbb{C}^{tan}_{pjql}$, we can still state that:

$$\mathbb{K}_{ijkl}\dot{F}_{kl} = \left(\mathbb{C}^{tan}_{pjlq}F_{kq}F_{ip}\right)\dot{F}_{kl} + \dot{F}_{ip}\mathsf{S}_{pj} \tag{8.58}$$

Next, if we see that $\mathsf{S}_{pj}\dot{F}_{ip} = \mathsf{S}_{lj}\dot{F}_{il} = \mathsf{S}_{lj}\delta_{ik}\dot{F}_{kl}$ holds, we can conclude that:

$$\mathbb{K}_{ijkl}\dot{F}_{kl} = \left(\mathbb{C}^{tan}_{pjlq}F_{kq}F_{ip} + \mathsf{S}_{lj}\delta_{ik}\right)\dot{F}_{kl} \quad \Rightarrow \quad \mathbb{K}_{ijkl} = \mathbb{C}^{tan}_{pjlq}F_{kq}F_{ip} + \mathsf{S}_{lj}\delta_{ik} \tag{8.59}$$

Thus:

$$\boxed{\dot{\mathbf{P}} = \mathbb{K} : \dot{\boldsymbol{F}} \quad ; \quad \dot{\mathsf{P}}_{ij} = \mathbb{K}_{ijkl}\dot{F}_{kl}}$$

$$\boxed{\begin{aligned}\mathbb{K}_{ijkl} &= \frac{\partial^2 \Psi(\boldsymbol{F})}{\partial F_{ij}\partial F_{kl}} \\ \mathbb{K}_{ijkl} &= \mathbb{C}^{tan}_{pjlq}F_{kq}F_{ip} + \mathsf{S}_{lj}\delta_{ik}\end{aligned}} \quad \textit{The elastic tangent stiffness pseudo-tensor} \tag{8.60}$$

8.3 Isotropic Hyperelastic Materials

If the scalar-valued tensor function $\Psi(\boldsymbol{C})$ is isotropic it must satisfy the following:

$$\Psi(\boldsymbol{C}) = \Psi(\mathbf{Q}\cdot\boldsymbol{C}\cdot\mathbf{Q}^T) \tag{8.61}$$

for any orthogonal tensor $\mathbf{Q}$. Then if we use the polar decomposition rotation tensor, *i.e.* $\mathbf{Q} = \mathbf{R}$, and if we know that $\boldsymbol{C} = \boldsymbol{F}^T\cdot\boldsymbol{F}$, we can obtain:

$$\begin{aligned}\Psi(\boldsymbol{C}) &= \Psi(\mathbf{R}\cdot\boldsymbol{C}\cdot\mathbf{R}^T) = \Psi(\mathbf{R}\cdot\boldsymbol{F}^T\cdot\boldsymbol{F}\cdot\mathbf{R}^T) = \Psi(\mathbf{V}^T\cdot\mathbf{V}) = \Psi(\mathbf{V}^2) \\ &= \Psi(\boldsymbol{b})\end{aligned} \tag{8.62}$$

where $\mathbf{V} = \mathbf{V}^T$ is the left stretch tensor which is related to the left Cauchy-Green deformation tensor ($\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$) by means of $\boldsymbol{b} = \mathbf{V}^2$, where $\boldsymbol{F} = \mathbf{V} \cdot \mathbf{R} \Rightarrow \mathbf{V} = \boldsymbol{F} \cdot \mathbf{R}^T$ is satisfied, (see Chapter 2). Thus, in isotropic materials, the energy function Ψ can be expressed in terms of the left Cauchy-Green deformation tensor as follows:

$$\boxed{\Psi(\boldsymbol{C}, \vec{\boldsymbol{X}}) = \Psi(\boldsymbol{b}, \vec{\boldsymbol{x}})} \quad \textit{Energy function for isotropic hyperelastic materials} \tag{8.63}$$

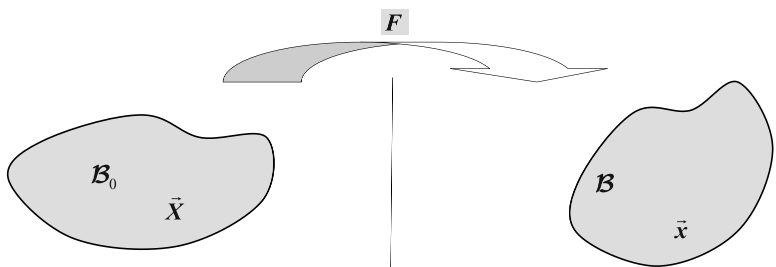
Reference configuration *Current configuration*

$$\dot{\mathsf{P}}_{ij} = \mathbb{K}_{ijkl} \dot{F}_{kl}$$

Elastic Tangent Stiffness Pseudo-Tensor

$$\mathbb{K}_{ijkl} = \frac{\partial^2 \Psi(\boldsymbol{F})}{\partial F_{ij} \partial F_{kl}}$$

$$\mathbb{K}_{ijkl} = \mathbb{C}^{tan}_{pjlq} F_{kq} F_{ip} + \mathsf{S}_{lj} \delta_{ik}$$



RATE-TYPE CONSTITUTIVE EQUATIONS

$$\dot{\mathbf{S}} = \mathbb{C}^{tan} : \dot{\boldsymbol{E}}$$

$$\overset{\square}{\tau}_{ij} = \mathbb{L}_{ijkl} \mathsf{D}_{kl} \qquad \overset{\circ}{\tau}_{ij} = \hat{\mathbb{L}}_{ijkl} \mathsf{D}_{kl} \qquad \overset{\mathrm{T}}{\sigma}_{ij} = \mathbb{A}_{ijkl} \mathsf{D}_{kl}$$

ELASTIC TANGENT STIFFNESS TENSORS

Material elastic tangent stiffness tensor

$$\mathbb{C}^{tan} = \frac{\partial^2 \Psi(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} = \frac{\partial \mathbf{S}}{\partial \boldsymbol{E}} = 4 \frac{\partial^2 \Psi(\boldsymbol{C})}{\partial \boldsymbol{C} \otimes \partial \boldsymbol{C}} = 2 \frac{\partial \mathbf{S}}{\partial \boldsymbol{C}}$$

Spatial elastic tangent stiffness tensor

$$\mathbb{L}_{ijkl} = F_{jn} F_{im} \mathbb{C}^{tan}_{mnpq} F_{kp} F_{lq}$$

$$\hat{\mathbb{L}}_{ijkl} = \mathbb{L}_{ijkl} + 2\mathbb{H}_{ijkl}$$

with

$$\mathbb{H}_{ijkl} = \frac{1}{4}(\tau_{lj}\delta_{ik} + \tau_{kj}\delta_{il} + \tau_{ik}\delta_{jl} + \tau_{il}\delta_{jk})$$

Instantaneous elastic tangent stiffness tensor

$$\mathbb{A}_{ijkl} = \frac{1}{J} F_{im} F_{jn} \mathbb{C}^{tan}_{mnpq} F_{kp} F_{lq} = \frac{1}{J} \mathbb{L}_{ijkl}$$

Figure 8.3: The elastic tangent stiffness tensors.

8.3.1 The Constitutive Equation in terms of Invariants

8.3.1.1 The Constitutive Equation in terms of $\boldsymbol{C}$ and $\boldsymbol{b}$

We showed in Chapter 1 that the scalar-valued isotropic tensor function, $\Psi(\boldsymbol{C})$, can be written in terms of the principal invariants of $\boldsymbol{C}$ ($I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}, I\!I\!I_{\boldsymbol{C}}$), or in terms of the invariants of $\boldsymbol{b}$, *i.e.*:

$$\Psi = \Psi(\boldsymbol{C}) = \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}, I\!I\!I_{\boldsymbol{C}}) = \Psi(I_{\boldsymbol{b}}, I\!I_{\boldsymbol{b}}, I\!I\!I_{\boldsymbol{b}}) \tag{8.64}$$

Then, it was verified in Chapter 2 that $I_{\boldsymbol{C}} = I_{\boldsymbol{b}}$, $I\!I_{\boldsymbol{C}} = I\!I_{\boldsymbol{b}}$ and $I\!I\!I_{\boldsymbol{C}} = I\!I\!I_{\boldsymbol{b}}$, which is obtained from:

$$I_{\boldsymbol{C}} = \mathrm{Tr}(\boldsymbol{C}), \quad \left| \; I\!I_{\boldsymbol{C}} = \frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{C})]^2 - \mathrm{Tr}(\boldsymbol{C}^2)\right\}, \quad \right| \; \begin{aligned} I\!I\!I_{\boldsymbol{C}} &= \det(\boldsymbol{C}) = J^2 \\ &= \frac{1}{3}\left\{\mathrm{Tr}(\boldsymbol{C}^3) - \frac{3}{2}\mathrm{Tr}(\boldsymbol{C})\mathrm{Tr}(\boldsymbol{C}^2) + \frac{1}{2}[\mathrm{Tr}(\boldsymbol{C})]^3\right\} \end{aligned}$$

Also in Chapter 1, we showed that for the scalar-valued tensor function, $\Psi(\boldsymbol{C}) = \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}, I\!I\!I_{\boldsymbol{C}})$, the following equations are valid:

$$\begin{aligned}
\frac{\partial \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}, I\!I\!I_{\boldsymbol{C}})}{\partial \boldsymbol{C}} &= \frac{\partial \Psi}{\partial I_{\boldsymbol{C}}}\frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\frac{\partial I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}}\frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} \\
&= \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1} \\
&= \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C}^2 \\
&= \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}}\right)\mathbf{1} + \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I\!I_{\boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-2}
\end{aligned} \tag{8.65}$$

where we held that:

$$\begin{aligned}
&\frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \mathbf{1}, \quad \frac{\partial I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C}^T = I_{\boldsymbol{C}}\mathbf{1} - \boldsymbol{C} = I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1} - I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-2}, \\
&\frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-T} = I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1} = \boldsymbol{C}^2 - I_{\boldsymbol{C}}\boldsymbol{C} + I\!I_{\boldsymbol{C}}\mathbf{1}
\end{aligned} \tag{8.66}$$

Now, taking into account the equations in (8.11) and (8.65) we can obtain the constitutive equation in terms of the principal invariants of $\boldsymbol{C}$ as follows:

$$\boxed{\begin{aligned}
\mathbf{S} &= 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} \\
&= 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}\right] \\
&= 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C}^2\right] \\
&= 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}}\right)\mathbf{1} + \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I\!I_{\boldsymbol{C}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-2}\right]
\end{aligned}} \tag{8.67}$$

In Chapter 1, in the subsection about the Tensor-Valued Tensor Function, it was shown that the following relationships are valid:

$$\Psi_{,\boldsymbol{b}} \cdot \boldsymbol{b} = \boldsymbol{F} \cdot \Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = \boldsymbol{b} \cdot \Psi_{,\boldsymbol{b}} \tag{8.68}$$

Now, if we consider the relationship between the Kirchhoff stress and the second Piola-Kirchhoff stress tensors, $\boldsymbol{\tau} = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T$, and the constitutive equation in the reference configuration (8.11), $\mathbf{S} = 2\Psi_{,\boldsymbol{C}}$, it is possible to obtain the constitutive equation in the current configuration as follows:

$$\boldsymbol{\tau} = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot 2\Psi_{,\boldsymbol{C}} \cdot \boldsymbol{F}^T = 2\Psi_{,\boldsymbol{b}} \cdot \boldsymbol{b} = 2\boldsymbol{b} \cdot \Psi_{,\boldsymbol{b}} \tag{8.69}$$

Next, by taking into account the equation $\boldsymbol{\tau} = J\,\boldsymbol{\sigma}$, we can also represent the constitutive equation for isotropic materials as:

$$\boldsymbol{\tau} = 2\frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = 2\boldsymbol{b} \cdot \frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \qquad \text{and} \qquad \boldsymbol{\sigma} = J^{-1} 2\frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = J^{-1} 2\boldsymbol{b} \cdot \frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \tag{8.70}$$

Then, in a similar fashion to (8.67), and by considering that $\boldsymbol{\tau} = J\,\boldsymbol{\sigma}$ we can obtain:

$$\boldsymbol{\tau} = J\,\boldsymbol{\sigma} = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T = 2\boldsymbol{F} \cdot \left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\boldsymbol{C} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\boldsymbol{C}^{-1}\right] \cdot \boldsymbol{F}^T \tag{8.71}$$

Now, if we consider that $\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F}$ and $\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^T$, the above equation becomes:

$$\boxed{\boldsymbol{\tau} = J\,\boldsymbol{\sigma} = 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{b}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} I_{\boldsymbol{b}}\right)\boldsymbol{b} - \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}}\boldsymbol{b}^2 + \frac{\partial \Psi}{\partial III_{\boldsymbol{b}}} III_{\boldsymbol{b}}\mathbf{1}\right]} \tag{8.72}$$

Alternatively, we can represent (8.72) by substituting the expression $\boldsymbol{b}^2$ obtained by means of the Cayley-Hamilton theorem $\boldsymbol{b}^3 - I_{\boldsymbol{b}}\boldsymbol{b}^2 + II_{\boldsymbol{b}}\boldsymbol{b} - III_{\boldsymbol{b}}\mathbf{1} = \mathbf{0}$, (see Chapter 1), *i.e.:*

$$\begin{aligned} &\Rightarrow \boldsymbol{b}^3 \cdot \boldsymbol{b}^{-1} - I_{\boldsymbol{b}}\boldsymbol{b}^2 \cdot \boldsymbol{b}^{-1} + II_{\boldsymbol{b}}\boldsymbol{b} \cdot \boldsymbol{b}^{-1} - III_{\boldsymbol{b}}\mathbf{1} \cdot \boldsymbol{b}^{-1} = \mathbf{0} \\ &\Rightarrow \boldsymbol{b}^2 - I_{\boldsymbol{b}}\boldsymbol{b} + II_{\boldsymbol{b}}\mathbf{1} - III_{\boldsymbol{b}}\boldsymbol{b}^{-1} = \mathbf{0} \\ &\Rightarrow \boldsymbol{b}^2 = I_{\boldsymbol{b}}\boldsymbol{b} - II_{\boldsymbol{b}}\mathbf{1} + III_{\boldsymbol{b}}\boldsymbol{b}^{-1} \end{aligned} \tag{8.73}$$

Then, the equation in (8.72) can still be written as:

$$\boxed{\boldsymbol{\tau} = J\,\boldsymbol{\sigma} = 2\left[\left(\frac{\partial \Psi}{\partial III_{\boldsymbol{b}}} III_{\boldsymbol{b}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} II_{\boldsymbol{b}}\right)\mathbf{1} + \left(\frac{\partial \Psi}{\partial I_{\boldsymbol{b}}}\right)\boldsymbol{b} + \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{b}}} III_{\boldsymbol{b}}\right)\boldsymbol{b}^{-1}\right]} \tag{8.74}$$

Then, by using equations between stress tensors, (see Chapter 3),

$$\mathbf{P} = \boldsymbol{\tau} \cdot \boldsymbol{F}^{-T} = \boldsymbol{F} \cdot \mathbf{S} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F}^{-T} = \boldsymbol{F} \cdot \mathbf{S} \tag{8.75}$$

the constitutive equation can also be written in terms of the first Piola-Kirchhoff stress tensor. To do so, let us consider $\mathbf{S}$ given in (8.67), and so we obtain:

$$\begin{aligned} \mathbf{P} &= 2\boldsymbol{F} \cdot \left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\boldsymbol{C} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\boldsymbol{C}^{-1}\right] \\ &= 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\boldsymbol{F} \cdot \mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\underbrace{\boldsymbol{F} \cdot \boldsymbol{F}^T}_{\boldsymbol{b}} \cdot \boldsymbol{F} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}} \cdot \underbrace{\boldsymbol{F} \cdot \boldsymbol{F}^{-1}}_{\mathbf{1}}\boldsymbol{F}^{-T}\right] \end{aligned} \tag{8.76}$$

Additionally, by considering that $\boldsymbol{F}^{-T} = \boldsymbol{b}^{-T} \cdot \boldsymbol{F}$ and the symmetry of $\boldsymbol{b}$, we obtain:

$$\boxed{\mathbf{P} = 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\boldsymbol{b} + \frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\boldsymbol{b}^{-1}\right] \cdot \boldsymbol{F} \quad \Rightarrow \quad \mathbf{P} = 2\frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{F}} \tag{8.77}$$

8.3.1.2 The Constitutive Equation in terms of $\boldsymbol{E}$

Energy can also be written in terms of the Green-Lagrange strain tensor $\boldsymbol{E}$, and if we are dealing with isotropic material the energy constitutive equation can be expressed in terms of the principal invariants of $\boldsymbol{E}$:

$$\Psi(\boldsymbol{E})=\Psi(I_{\boldsymbol{E}}, II_{\boldsymbol{E}}, III_{\boldsymbol{E}}) \tag{8.78}$$

where $I_{\boldsymbol{E}}=\mathrm{Tr}(\boldsymbol{E})$, $II_{\boldsymbol{E}}=\frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{E})]^2-\mathrm{Tr}(\boldsymbol{E}^2)\right\}$, $III_{\boldsymbol{E}}=\det(\boldsymbol{E})$. Then, if we consider the equation in (8.11), *i.e.* $\mathbf{S}=\frac{\partial\Psi(\boldsymbol{E})}{\partial\boldsymbol{E}}$, we can obtain another one analogous to that obtained in (8.65), *i.e.*:

$$\mathbf{S}=c_0\mathbf{1}-c_1\boldsymbol{E}+c_2\boldsymbol{E}^2 \tag{8.79}$$

where the parameters c_0, c_1, c_2 are given by:

$$c_0=\frac{\partial\Psi}{\partial I_{\boldsymbol{E}}}+\frac{\partial\Psi}{\partial II_{\boldsymbol{E}}}I_{\boldsymbol{E}}+\frac{\partial\Psi}{\partial III_{\boldsymbol{E}}}II_{\boldsymbol{E}} \quad ; \quad c_1=\frac{\partial\Psi}{\partial II_{\boldsymbol{E}}}+\frac{\partial\Psi}{\partial III_{\boldsymbol{E}}}I_{\boldsymbol{E}} \quad ; \quad c_2=\frac{\partial\Psi}{\partial III_{\boldsymbol{E}}} \tag{8.80}$$

8.3.2 Series Expansion of the Energy Function

Let us assume that $\Psi=\Psi(\boldsymbol{C})$ is a continuously differentiable function with respect to the $\boldsymbol{C}$-invariants. It is possible, then, to represent Ψ by means of infinite power series:

$$\Psi=\Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})=\sum_{p,q,r=0}^{\infty}c_{pqr}(I_{\boldsymbol{C}}-3)^p(II_{\boldsymbol{C}}-3)^q(III_{\boldsymbol{C}}-1)^r \tag{8.81}$$

where the coefficients c_{pqr} are independent of the deformation.

Here, we can notice that in an undeformed state we have $\boldsymbol{F}=\mathbf{1}\Rightarrow\boldsymbol{C}=\mathbf{1}$, then $I_{\boldsymbol{C}}=3$, $II_{\boldsymbol{C}}=3$, $III_{\boldsymbol{C}}=1$, which results in $\Psi=0$, as expected, since in said undeformed state strain energy is zero (normalization condition).

Then, taking into account that $\boldsymbol{C}=\mathbf{U}^2$, where $\mathbf{U}$ is the right stretch tensor, it is possible to express the tensor $\boldsymbol{C}$ in terms of the principal stretches (eigenvalues of $\mathbf{U}$) λ_1, λ_2, λ_3, (see Chapter 2), so we can use the following spectral representation:

$$\mathbf{U}=\sum_{a=1}^{3}\lambda_a\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} \quad\Rightarrow\quad \boldsymbol{C}=\mathbf{U}^2=\sum_{a=1}^{3}\lambda_a^2\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)} \tag{8.82}$$

Next, the principal invariants of $\boldsymbol{C}$ or $\boldsymbol{b}$ in terms of the principal stretches λ_i are given by:

$$C_{ij}=\mathsf{U}_{ij}^2=\begin{bmatrix}\lambda_1^2 & 0 & 0\\ 0 & \lambda_2^2 & 0\\ 0 & 0 & \lambda_3^2\end{bmatrix} \quad\Rightarrow\quad \begin{cases} I_{\boldsymbol{C}}=I_{\boldsymbol{b}}=\lambda_1^2+\lambda_2^2+\lambda_3^2\\ II_{\boldsymbol{C}}=II_{\boldsymbol{b}}=\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2\\ III_{\boldsymbol{C}}=III_{\boldsymbol{b}}=\lambda_1^2\lambda_2^2\lambda_3^2\end{cases} \tag{8.83}$$

Then, by substituting the values of (8.83) into the power series (8.81) and after some mathematical manipulations we obtain:

$$\Psi=\Psi(\lambda_1,\lambda_2,\lambda_3)=\sum_{p,q,r=0}^{\infty}a_{pqr}\left\{\left[\lambda_1^p\left(\lambda_2^q+\lambda_3^q\right)+\lambda_2^p\left(\lambda_3^q+\lambda_1^q\right)+\lambda_3^p\left(\lambda_1^q+\lambda_2^q\right)\right]\left(\lambda_1\lambda_2\lambda_3\right)^r-6\right\} \tag{8.84}$$

where the coefficients a_{pqr} are independent of the deformation.

In incompressible materials $I\!I\!I_{\boldsymbol{C}} = 1$ or $\lambda_1\lambda_2\lambda_3 = 1$ is satisfied and the equations in (8.81) and (8.84) becomes, respectively:

$$\Psi = \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}) = \sum_{p,q=0}^{\infty} c_{pq}\left(I_{\boldsymbol{C}} - 3\right)^p \left(I\!I_{\boldsymbol{C}} - 3\right)^q \tag{8.85}$$

$$\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{p,q=0}^{\infty} a_{pq}\left\{ \left[\lambda_1^p\left(\lambda_2^q + \lambda_3^q\right) + \lambda_2^p\left(\lambda_3^q + \lambda_1^q\right) + \lambda_3^p\left(\lambda_1^q + \lambda_2^q\right)\right] - 6\right\} \tag{8.86}$$

8.3.3 Constitutive Equations in terms of the Principal Stretches

As we have seen before, for isotropic materials, we can express the strain energy function in terms of the principal stretches λ_a, $a = 1,2,3$, *i.e.* $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3)$. Let us now suppose that $\hat{\mathbf{N}}^{(a)}$ and $\hat{\mathbf{n}}^{(a)}$ are the principal directions (eigenvectors) of the right stretch tensor ($\mathbf{U}$) and the left stretch tensor ($\mathbf{V}$), respectively, where the following holds:

$$\begin{aligned}
&\mathbf{U} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \quad ; \quad \boldsymbol{E} = \sum_{a=1}^{3} \frac{1}{2}(\lambda_a^2 - 1)\hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \\
&\mathbf{V} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} \quad ; \quad \boldsymbol{F} = \sum_{a=1}^{3} \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \quad ; \quad \boldsymbol{F}^{-1} = \sum_{a=1}^{3} \frac{1}{\lambda_a} \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}
\end{aligned} \tag{8.87}$$

To see how the above relationships are proven, see the Section on Polar Decomposition in Chapter 2.

Now, the second Piola-Kirchhoff stress tensor ($\mathbf{S}$) in terms of the principal stretches becomes:

$$\mathbf{S} = 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\frac{\partial \Psi}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial \boldsymbol{C}} = 2\left(\frac{\partial \Psi}{\partial \lambda_1}\frac{\partial \lambda_1}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial \lambda_2}\frac{\partial \lambda_2}{\partial \boldsymbol{C}} + \frac{\partial \Psi}{\partial \lambda_3}\frac{\partial \lambda_3}{\partial \boldsymbol{C}}\right) \tag{8.88}$$

Then by considering the spectral representation of $\boldsymbol{C}$ given in (8.82), the rate of change of $\boldsymbol{C}(\lambda_a)$ can be evaluated as follows:

$$\dot{\boldsymbol{C}} = \sum_{a=1}^{3}\left[2\lambda_a\dot{\lambda}_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \lambda_a^2 \dot{\hat{\mathbf{N}}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} + \lambda_a^2 \hat{\mathbf{N}}^{(a)} \otimes \dot{\hat{\mathbf{N}}}^{(a)}\right] \tag{8.89}$$

Now, if we apply the dot product of $\hat{\mathbf{N}}^{(a)}$ on the right and on the left of both sides of the equation we have:

$$\begin{aligned}
&\hat{\mathbf{N}}^{(a)} \cdot \dot{\boldsymbol{C}} \cdot \hat{\mathbf{N}}^{(a)} = \\
&\sum_{a=1}^{3}\left[2\lambda_a\dot{\lambda}_a \hat{\mathbf{N}}^{(a)} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \cdot \hat{\mathbf{N}}^{(a)} + \lambda_a^2\left(\hat{\mathbf{N}}^{(a)} \cdot \dot{\hat{\mathbf{N}}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \cdot \hat{\mathbf{N}}^{(a)} + \hat{\mathbf{N}}^{(a)} \cdot \hat{\mathbf{N}}^{(a)} \otimes \dot{\hat{\mathbf{N}}}^{(a)} \cdot \hat{\mathbf{N}}^{(a)}\right)\right]
\end{aligned} \tag{8.90}$$

Then, bearing in mind that $\hat{\mathbf{N}}^{(a)} \cdot \hat{\mathbf{N}}^{(a)} = 1$, and the fact that the rate of change of a vector with constant magnitude is always orthogonal to itself (Holzapfel (2000)), it follows that $\hat{\mathbf{N}}^{(a)} \cdot \dot{\hat{\mathbf{N}}}^{(a)} = 0$ and subsequently the above equation becomes:

$$\hat{\mathbf{N}}^{(a)} \cdot \dot{\boldsymbol{C}} \cdot \hat{\mathbf{N}}^{(a)} = 2\lambda_a\dot{\lambda}_a \tag{8.91}$$

The reader should be aware here that the index $a=1,2,3$ is not a dummy index, *i.e.* we are not dealing with indicial notation.

Next, using the property $\vec{\mathbf{a}}\cdot\mathbf{T}\cdot\vec{\mathbf{b}}=\mathbf{T}:(\vec{\mathbf{a}}\otimes\vec{\mathbf{b}})$ where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are vectors and $\mathbf{T}$ is a second-order tensor, the equation in (8.91) can be rewritten as $\dot{\boldsymbol{C}}:(\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)})=2\lambda_a\dot{\lambda}_a$, and if we also consider that $\dot{\boldsymbol{C}}(\lambda_a)=\dfrac{\partial\boldsymbol{C}}{\partial\lambda_a}\dot{\lambda}_a$ we can obtain:

$$
\begin{aligned}
&\frac{\partial\boldsymbol{C}}{\partial\lambda_a}\dot{\lambda}_a:\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}=2\lambda_a\dot{\lambda}_a\Rightarrow\frac{\partial\boldsymbol{C}}{\partial\lambda_a}:\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}=2\lambda_a\\
&\Rightarrow\left(\frac{\partial\boldsymbol{C}}{\partial\lambda_a}\right):\left(\frac{1}{2\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)=1\Rightarrow\left(\frac{\partial\boldsymbol{C}}{\partial\lambda_a}\right):\left(\frac{1}{2\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)=\underbrace{\left(\frac{\partial\boldsymbol{C}}{\partial\lambda_a}\right):\left(\frac{\partial\lambda_a}{\partial\boldsymbol{C}}\right)}_{=1}
\end{aligned}
\tag{8.92}
$$

which draws us to the conclusion that:

$$
\frac{1}{2\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}=\frac{\partial\lambda_a}{\partial\boldsymbol{C}}
\tag{8.93}
$$

Then, by using the second Piola-Kirchhoff stress tensor expression obtained in (8.88), *i.e.* $\mathbf{S}=2\dfrac{\partial\Psi(\lambda_a)}{\partial\lambda_k}\dfrac{\partial\lambda_k}{\partial\boldsymbol{C}}=2\sum_{a=1}^{3}\dfrac{\partial\Psi}{\partial\lambda_a}\dfrac{\partial\lambda_a}{\partial\boldsymbol{C}}$, and by considering the equation obtained in (8.93), we can express the tenor $\mathbf{S}$ as:

$$
\boxed{\mathbf{S}=\sum_{a=1}^{3}\frac{1}{\lambda_a}\frac{\partial\Psi}{\partial\lambda_a}\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}=\sum_{a=1}^{3}\mathsf{S}_a\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}}
\tag{8.94}
$$

where S_a are the second Piola-Kirchhoff stress tensor eigenvalues. Then, by comparing the equation in (8.94) with the spectral representation of the tensor $\boldsymbol{C}$, given in (8.82), we can conclude that in isotropic materials, $\boldsymbol{C}$ and $\mathbf{S}$ are coaxial tensors ($\boldsymbol{C}\cdot\mathbf{S}=\mathbf{S}\cdot\boldsymbol{C}$), *i.e.* they have the same principal directions.

Then, as regards the Cauchy stress tensor, we have:

$$
\begin{aligned}
\boldsymbol{\sigma}=J^{-1}\boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T&=\sum_{a=1}^{3}J^{-1}\frac{1}{\lambda_a}\frac{\partial\Psi}{\partial\lambda_a}\boldsymbol{F}\cdot\left(\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\right)\cdot\boldsymbol{F}^T\\
&=\sum_{a=1}^{3}J^{-1}\frac{1}{\lambda_a}\frac{\partial\Psi}{\partial\lambda_a}\left(\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(a)}\otimes\hat{\mathbf{N}}^{(a)}\cdot\boldsymbol{F}^T\right)=\sum_{a=1}^{3}J^{-1}\frac{1}{\lambda_a}\frac{\partial\Psi}{\partial\lambda_a}\left(\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(a)}\otimes\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(a)}\right)
\end{aligned}
\tag{8.95}
$$

Moreover, if we take into account the equation $\boldsymbol{F}\cdot\hat{\mathbf{N}}^{(a)}=\lambda_a\hat{\mathbf{n}}^{(a)}$, (see subsection 2.8 Polar Decomposition of $\boldsymbol{F}$ in Chapter 2), we can obtain:

$$
\boxed{\boldsymbol{\sigma}=\sum_{a=1}^{3}J^{-1}\lambda_a\frac{\partial\Psi}{\partial\lambda_a}\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}=\sum_{a=1}^{3}\sigma_a\hat{\mathbf{n}}^{(a)}\otimes\hat{\mathbf{n}}^{(a)}}
\tag{8.96}
$$

where σ_a are the Cauchy stress tensor eigenvalues.

Since $\boldsymbol{C}$ and $\mathbf{S}$ are coaxial, we can obtain $\mathbf{S}$-eigenvalues by considering the principal directions of $\boldsymbol{C}$ by means of one of the equations in (8.67):

$$
\mathsf{S}_a=2\left[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\right)+\left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}II_{\boldsymbol{C}}+\frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}III_{\boldsymbol{C}}\right)\lambda_a^{-2}-\left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}III_{\boldsymbol{C}}\right)\lambda_a^{-4}\right]
\tag{8.97}
$$

The first Piola-Kirchhoff stress tensor is given by $\mathbf{P} = \boldsymbol{F} \cdot \mathbf{S}$, then it holds that:

$$
\begin{aligned}
\mathbf{P} = \boldsymbol{F} \cdot \mathbf{S} = \boldsymbol{F} \cdot \sum_{a=1}^{3} \mathsf{S}_a \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} &= \sum_{a=1}^{3} \mathsf{S}_a \boldsymbol{F} \cdot \hat{\mathbf{N}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \mathsf{S}_a \lambda_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} \\
&= \sum_{a=1}^{3} \mathsf{P}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}
\end{aligned}
\tag{8.98}
$$

Thus, we can express the components S_a as:

$$
\mathsf{P}_a = 2\lambda_a \left[\left(\frac{\partial \Psi}{\partial I_C} \right) + \left(\frac{\partial \Psi}{\partial II_C} II_C + \frac{\partial \Psi}{\partial III_C} III_C \right) \lambda_a^{-2} - \left(\frac{\partial \Psi}{\partial II_C} III_C \right) \lambda_a^{-4} \right]
\tag{8.99}
$$

We can also express these components in terms of P_a. To do so, let us consider that:

$$
\frac{\partial \Psi(I_C, II_C, III_C)}{\partial \lambda_a} = \frac{\partial \Psi}{\partial I_C} \frac{\partial I_C}{\partial \lambda_a} + \frac{\partial \Psi}{\partial II_C} \frac{\partial II_C}{\partial \lambda_a} + \frac{\partial \Psi}{\partial III_C} \frac{\partial III_C}{\partial \lambda_a}
\tag{8.100}
$$

Then, the derivatives of I_C with respect to λ_a are:

$$
\left.
\begin{aligned}
\frac{\partial I_C}{\partial \lambda_1} &= \frac{\partial}{\partial \lambda_1} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right) = 2\lambda_1 \\
\frac{\partial I_C}{\partial \lambda_2} &= \frac{\partial}{\partial \lambda_2} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right) = 2\lambda_2 \\
\frac{\partial I_C}{\partial \lambda_3} &= \frac{\partial}{\partial \lambda_3} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \right) = 2\lambda_3
\end{aligned}
\right\} \quad \Rightarrow \quad \frac{\partial I_C}{\partial \lambda_a} = 2\lambda_a
\tag{8.101}
$$

$$
\begin{aligned}
\frac{\partial II_C}{\partial \lambda_1} &= \frac{\partial}{\partial \lambda_1} \left(\lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_1^2 \lambda_3^2 \right) = 2\lambda_1 \lambda_2^2 + 2\lambda_1 \lambda_3^2 = 2\lambda_1 \left(\lambda_2^2 + \lambda_3^2 \right) \\
&= 2\lambda_1 \left[\left(\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 \right) \lambda_1^{-2} \right] = 2\lambda_1 \left[\left(\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \right) \lambda_1^{-2} - \lambda_2^2 \lambda_3^2 \lambda_1^{-2} \right] \\
&= 2\lambda_1 \left[\left(\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \right) \lambda_1^{-2} - \left(\lambda_2^2 \lambda_3^2 \lambda_1^2 \right) \lambda_1^{-4} \right] = 2\lambda_1 \left(II_C \lambda_1^{-2} - III_C \lambda_1^{-4} \right)
\end{aligned}
\tag{8.102}
$$

which is true for the other principal values, then

$$
\frac{\partial II_C}{\partial \lambda_a} = 2\lambda_a \left(II_C \lambda_a^{-2} - III_C \lambda_a^{-4} \right)
\tag{8.103}
$$

$$
\frac{\partial III_C}{\partial \lambda_1} = \frac{\partial III_C}{\partial \lambda_1} \left(\lambda_1^2 \lambda_2^2 \lambda_3^2 \right) = 2\lambda_1 \left(\lambda_2^2 \lambda_3^2 \right) = 2\lambda_1 \left[\left(\lambda_1^2 \lambda_2^2 \lambda_3^2 \right) \lambda_1^{-2} \right] = 2\lambda_1 \, III_C \lambda_1^{-2}
\tag{8.104}
$$

which is true for the other principal values, then

$$
\frac{\partial III_C}{\partial \lambda_a} = 2\lambda_a \, III_C \lambda_a^{-2}
\tag{8.105}
$$

Then, by substituting (8.101) into (8.100) we obtain:

$$
\frac{\partial \Psi(I_C, II_C, III_C)}{\partial \lambda_a} = 2\lambda_a \left[\left(\frac{\partial \Psi}{\partial I_C} \right) + \left(\frac{\partial \Psi}{\partial II_C} II_C + \frac{\partial \Psi}{\partial III_C} III_C \right) \lambda_a^{-2} - \left(\frac{\partial \Psi}{\partial II_C} III_C \right) \lambda_a^{-4} \right]
\tag{8.106}
$$

Additionally, if we compare (8.106) with (8.99) and with (8.97) we can draw the conclusion that:

$$
\mathsf{P}_a = \frac{\partial \Psi(I_C, II_C, III_C)}{\partial \lambda_a} = \lambda_a \mathsf{S}_a
\tag{8.107}
$$

Then, it holds that:

$$\boxed{\mathbf{P} = \sum_{a=1}^{3} \lambda_a \mathsf{S}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)} = \sum_{a=1}^{3} \mathsf{P}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{N}}^{(a)}} \tag{8.108}$$

Note that P_a are not the eigenvalues of $\mathbf{P}$. The Cauchy stress tensor is related to the first Piola-Kirchhoff stress tensor by means of $\boldsymbol{\sigma} = J^{-1} \mathbf{P} \cdot \boldsymbol{F}^T$, after which the eigenvalues of $\boldsymbol{\sigma}$ are given by:

$$\sigma_a = J^{-1} \lambda_a \frac{\partial \Psi}{\partial \lambda_a} \tag{8.109}$$

which is the same result as that obtained in (8.96). Note that index a does not indicate summation.

Then, in isotropic materials, the Kirchhoff stress tensor ($\boldsymbol{\tau}$) and the left stretch tensor ($\mathbf{V}$) have the same principal directions, and if we consider that $\boldsymbol{\tau} = J \boldsymbol{\sigma}$ we can obtain:

$$\boxed{\boldsymbol{\tau} = \sum_{a=1}^{3} \lambda_a^2 \mathsf{S}_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)} = \sum_{a=1}^{3} \tau_a \hat{\mathbf{n}}^{(a)} \otimes \hat{\mathbf{n}}^{(a)}} \tag{8.110}$$

Now, if we look back at the equations in (8.94), (8.96) and (8.110), we can conclude that the principal values of the tensors $\mathbf{S}$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, are interrelated by:

$$\boxed{\mathsf{S}_a = \frac{1}{\lambda_a} \frac{\partial \Psi}{\partial \lambda_a} = \frac{J}{\lambda_a^2} \sigma_a = \frac{1}{\lambda_a^2} \tau_a} \tag{8.111}$$

8.4 Compressible Materials

In compressible hyperelasticity materials (which go through a change in volume during the deformation process), it would be appropriate to separate the motion undergone into isochoric motion (volume-preserving) and another type characterized by dilatational transformation (purely volume-change). So, let us consider the multiplicative decomposition of the deformation gradient, (see Figure 8.4), as follows:

$$\boldsymbol{F} = \tilde{\boldsymbol{F}} \cdot \boldsymbol{F}^{vol} \tag{8.112}$$

where $\tilde{\boldsymbol{F}}$ show an isochoric transformation ($\tilde{\boldsymbol{F}} \equiv \boldsymbol{F}^{iso}$), and $\boldsymbol{F}^{vol}$ describes a dilatational transformation, (see Figure 8.4). Now let us look back at Chapter 2 subsection 2.13, where we obtained the following equations:

$$\tilde{\boldsymbol{F}} = J^{\frac{-1}{3}} \boldsymbol{F} \quad ; \quad \boldsymbol{F}^{vol} = J^{\frac{1}{3}} \mathbf{1} \quad \Rightarrow \quad \begin{cases} \tilde{\boldsymbol{C}} = J^{\frac{-2}{3}} \boldsymbol{C} & ; \quad \boldsymbol{C}^{vol} = J^{\frac{2}{3}} \mathbf{1} \\ \tilde{\boldsymbol{b}} = J^{\frac{-2}{3}} \boldsymbol{b} & ; \quad \boldsymbol{b}^{vol} = J^{\frac{2}{3}} \mathbf{1} \end{cases} \tag{8.113}$$

and

$$\tilde{J} = \left|\tilde{\boldsymbol{F}}\right| = \left|J^{\frac{-1}{3}} \boldsymbol{F}\right| = 1 \quad ; \quad J^{vol} = \left|\boldsymbol{F}^{vol}\right| = \left|J^{\frac{1}{3}} \mathbf{1}\right| = J \tag{8.114}$$

Moreover, in Chapter 1 it was proven that $\dfrac{\partial J}{\partial \boldsymbol{C}} = \dfrac{J}{2}\boldsymbol{C}^{-1}$, where $J^2 = I\!I\!I_{\boldsymbol{C}} = I\!I\!I_{\boldsymbol{b}}$. Likewise, we can obtain the following relationships:

$$
\begin{aligned}
\frac{\partial J^{\frac{-2}{3}}}{\partial \boldsymbol{C}} &= \frac{\partial\left[\left(I\!I\!I_{\boldsymbol{C}}\right)^{\frac{-1}{3}}\right]}{\partial \boldsymbol{C}} = -\frac{1}{3}\left(I\!I\!I_{\boldsymbol{C}}\right)^{\frac{-4}{3}} \frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = -\frac{1}{3}\left(I\!I\!I_{\boldsymbol{C}}\right)^{\frac{-4}{3}} I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-T} \\
&= -\frac{1}{3}\left(I\!I\!I_{\boldsymbol{C}}\right)^{\frac{-1}{3}}\boldsymbol{C}^{-1} = -\frac{1}{3}J^{\frac{-2}{3}}\boldsymbol{C}^{-1}
\end{aligned}
\tag{8.115}
$$

where we have used $\dfrac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-T} = I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1}$, (see Chapter 1). Additionally, we can obtain:

$$
\begin{aligned}
\frac{\partial \tilde{\boldsymbol{C}}}{\partial \boldsymbol{C}} &= \frac{\partial (J^{\frac{-2}{3}}\boldsymbol{C})}{\partial \boldsymbol{C}} \\
&= J^{\frac{-2}{3}}\frac{\partial(\boldsymbol{C})}{\partial \boldsymbol{C}} + \boldsymbol{C}\otimes\frac{\partial(J^{\frac{-2}{3}})}{\partial \boldsymbol{C}} \\
&= J^{\frac{-2}{3}}\mathbb{I} - \frac{1}{3}J^{\frac{-2}{3}}\boldsymbol{C}\otimes\boldsymbol{C}^{-1} \\
&= J^{\frac{-2}{3}}\left(\mathbb{I} - \frac{1}{3}\boldsymbol{C}\otimes\boldsymbol{C}^{-1}\right) \\
&= J^{\frac{-2}{3}}\mathbb{P}^T
\end{aligned}
\qquad
\begin{aligned}
\frac{\partial \tilde{C}_{ij}}{\partial C_{kl}} &= \frac{\partial (J^{\frac{-2}{3}}C_{ij})}{\partial C_{kl}} \\
&= J^{\frac{-2}{3}}\frac{\partial(C_{ij})}{\partial C_{kl}} + C_{ij}\frac{\partial(J^{\frac{-2}{3}})}{\partial C_{kl}} \\
&= J^{\frac{-2}{3}}\delta_{ik}\delta_{jl} - \frac{1}{3}J^{\frac{-2}{3}}C_{ij}C^{-1}_{kl} \\
&= J^{\frac{-2}{3}}\left(\mathbb{I}_{ijkl} - \frac{1}{3}C_{ij}C^{-1}_{kl}\right)
\end{aligned}
\tag{8.116}
$$

with which, we introduce the fourth-order tensor $\mathbb{P}$ known as the *projection tensor* with respect to the reference configuration, (see Holzapfel (2000)):

$$
\mathbb{P}^T = \mathbb{I} - \frac{1}{3}\boldsymbol{C}\otimes\boldsymbol{C}^{-1} \quad \Rightarrow \quad \mathbb{P} = \mathbb{I} - \frac{1}{3}\boldsymbol{C}^{-1}\otimes\boldsymbol{C}
\tag{8.117}
$$

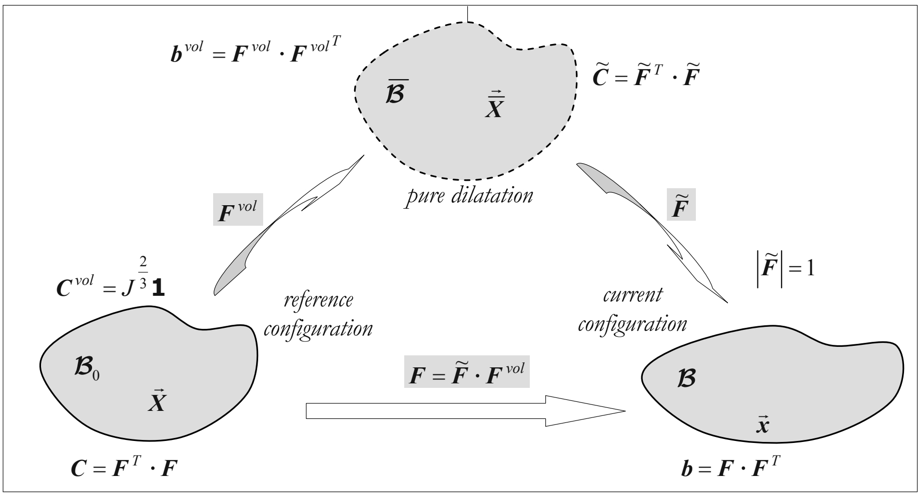


Figure 8.4: Multiplicative decomposition of deformation gradient – Kinematic tensors.

8.4.1 The Stress Tensors

Next, we will define the stress tensors in different configurations. To start off we will use the definitions of the Cauchy stress tensor (current configuration) $\boldsymbol{\sigma} = \frac{2}{J}\boldsymbol{F}\cdot\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}}\cdot\boldsymbol{F}^T$ and the second Piola-Kirchhoff stress tensor $\mathbf{S} = 2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}$ (reference configuration). Note that we can define a stress tensor, analogous to the Cauchy stress tensor, in the intermediate configuration ($\overline{\mathcal{B}}$) by means of the transformation $\boldsymbol{F}^{vol}$, *i.e.*:

$$\overline{\boldsymbol{\sigma}}^{vol} = \frac{2}{J^{vol}}\boldsymbol{F}^{vol}\cdot\frac{\partial\Psi(\boldsymbol{C}^{vol})}{\partial\boldsymbol{C}^{vol}}\cdot\boldsymbol{F}^{vol\,T} \tag{8.118}$$

Then, if we refer to $J^{vol} = J$, $\boldsymbol{F}^{vol} = J^{\frac{1}{3}}\mathbf{1}$, $\frac{\partial\Psi(\boldsymbol{C}^{vol})}{\partial\boldsymbol{C}^{vol}} = \frac{\partial\Psi(J^{vol})}{\partial J^{vol}}\frac{\partial J^{vol}}{\partial\boldsymbol{C}^{vol}}$, and $\frac{\partial J^{vol}}{\partial\boldsymbol{C}^{vol}} = \frac{J^{vol}}{2}\boldsymbol{C}^{vol\,-1} = \frac{J^{vol}}{2}J^{\frac{-2}{3}}\mathbf{1}$, the equation in (8.118) becomes:

$$\begin{aligned}
\overline{\boldsymbol{\sigma}}^{vol} &= \frac{2}{J^{vol}}\boldsymbol{F}^{vol}\cdot\frac{\partial\Psi(\boldsymbol{C}^{vol})}{\partial\boldsymbol{C}^{vol}}\cdot\boldsymbol{F}^{vol\,T} = \frac{2}{J^{vol}}J^{\frac{1}{3}}\mathbf{1}\cdot\left(\frac{\partial\Psi(J^{vol})}{\partial J^{vol}}\frac{\partial J^{vol}}{\partial\boldsymbol{C}^{vol}}\right)\cdot J^{\frac{1}{3}}\mathbf{1} \\
&= \frac{2}{J^{vol}}J^{\frac{1}{3}}\mathbf{1}\cdot\left(\frac{\partial\Psi(J^{vol})}{\partial J^{vol}}\frac{J^{vol}}{2}\boldsymbol{C}^{vol\,-1}\right)\cdot J^{\frac{1}{3}}\mathbf{1} \\
&= \frac{2}{J^{vol}}J^{\frac{1}{3}}\mathbf{1}\cdot\left(\frac{\partial\Psi(J^{vol})}{\partial J^{vol}}\frac{J^{vol}}{2}J^{\frac{-2}{3}}\mathbf{1}\right)\cdot J^{\frac{1}{3}}\mathbf{1} = \frac{\partial\Psi(J^{vol})}{\partial J^{vol}}\mathbf{1}
\end{aligned} \tag{8.119}$$

Thus,

$$\boxed{\overline{\boldsymbol{\sigma}}^{vol} = \frac{\partial\Psi(J)}{\partial J}\mathbf{1}} \tag{8.120}$$

We can also define a stress tensor in the intermediate configuration caused by the transformation $\widetilde{\boldsymbol{F}}$, (see Figure 8.5), as:

$$\widetilde{\widetilde{\mathbf{S}}} = 2\frac{\partial\widetilde{\Psi}(\widetilde{\boldsymbol{C}})}{\partial\widetilde{\boldsymbol{C}}} \tag{8.121}$$

We will now observe additive decomposition of the strain energy function in two parts, namely: isochoric and volumetric, *i.e.*:

$$\Psi(\boldsymbol{F}) = \widetilde{\Psi}(\widetilde{\boldsymbol{F}}) + \Psi^{vol}(\boldsymbol{F}^{vol}) \quad ; \quad \Psi(\boldsymbol{C}) = \widetilde{\Psi}(\widetilde{\boldsymbol{C}}) + \Psi^{vol}(\boldsymbol{C}^{vol}) \tag{8.122}$$

Then, if we take the chain rule of derivative of the strain energy function (8.122) we obtain:

$$\dot{\Psi}(\boldsymbol{C}) = \dot{\widetilde{\Psi}}(\widetilde{\boldsymbol{C}}) + \dot{\Psi}^{vol}(J) = \frac{\partial\widetilde{\Psi}(\widetilde{\boldsymbol{C}})}{\partial\widetilde{\boldsymbol{C}}}:\frac{\partial\widetilde{\boldsymbol{C}}}{\partial t} + \frac{d\Psi^{vol}(J)}{dJ}\frac{\partial J}{\partial t} = \frac{\partial\widetilde{\Psi}(\widetilde{\boldsymbol{C}})}{\partial\widetilde{\boldsymbol{C}}}:\dot{\widetilde{\boldsymbol{C}}} + \frac{d\Psi^{vol}(J)}{dJ}\dot{J} \tag{8.123}$$

Now, given that $\dot{J} = \frac{\partial J}{\partial\boldsymbol{C}}:\dot{\boldsymbol{C}}$ and $\frac{\partial J}{\partial\boldsymbol{C}} = \frac{J}{2}\boldsymbol{C}^{-1}$, we can obtain $\dot{J} = \frac{J}{2}\boldsymbol{C}^{-1}:\dot{\boldsymbol{C}}$ and the term $\dot{\widetilde{\boldsymbol{C}}}$ can be expressed as $\dot{\widetilde{\boldsymbol{C}}} = \frac{\partial\widetilde{\boldsymbol{C}}}{\partial\boldsymbol{C}}:\dot{\boldsymbol{C}} = J^{\frac{-2}{3}}\mathbb{P}^T:\dot{\boldsymbol{C}}$. So, the equation in (8.123) can also be expressed as follows:

$$\begin{aligned}\dot{\Psi}(\boldsymbol{C}) &= J^{\frac{-2}{3}} \frac{\partial\tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial\tilde{\boldsymbol{C}}} : \mathbb{P}^T : \dot{\boldsymbol{C}} + \frac{J}{2}\frac{d\Psi^{vol}(J)}{dJ}\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}} \\ &= J^{\frac{-2}{3}} \mathbb{P} : \frac{\partial\tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial\tilde{\boldsymbol{C}}} : \dot{\boldsymbol{C}} + \frac{J}{2}\frac{d\Psi^{vol}(J)}{dJ}\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}}\end{aligned} \tag{8.124}$$

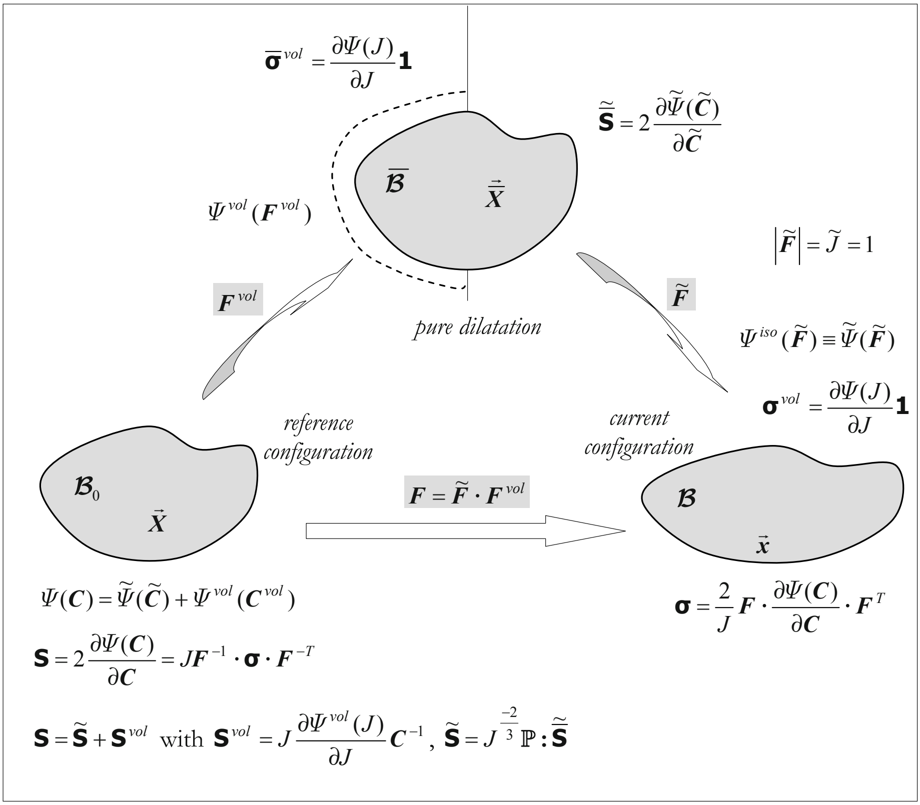


Figure 8.5: Multiplicative decomposition of deformation gradient – stress tensors.

In purely elastic materials, internal energy dissipation is zero. Remember that in Chapter 5 in a system with no entropy production, internal energy dissipation in the reference configuration is given by:

$$\mathcal{D}_{int} = \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} - \dot{\Psi}(\boldsymbol{C}) = 0 \tag{8.125}$$

Then, if we combine the equation in (8.124) with the one above we obtain:

$$\begin{aligned}\mathcal{D}_{int} &= \frac{1}{2}\mathbf{S} : \dot{\boldsymbol{C}} - J^{\frac{-2}{3}} \mathbb{P} : \frac{\partial\tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial\tilde{\boldsymbol{C}}} : \dot{\boldsymbol{C}} - \frac{d\Psi^{vol}(J)}{dJ}\frac{J}{2}\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}} = 0 \\ &= \left(\mathbf{S} - 2J^{\frac{-2}{3}} \mathbb{P} : \frac{\partial\tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial\tilde{\boldsymbol{C}}} - J\frac{d\Psi^{vol}(J)}{dJ}\boldsymbol{C}^{-1}\right) : \frac{\dot{\boldsymbol{C}}}{2} = 0\end{aligned} \tag{8.126}$$

Notice that the above must be satisfied for any admissible thermodynamic process. Let us now consider that $\dot{\boldsymbol{C}} \neq \boldsymbol{0}$, so, the only way for (8.126) to be satisfied is if:

$$\mathbf{S} = 2J^{\frac{-2}{3}} \mathbb{P} : \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial \tilde{\boldsymbol{C}}} + J \frac{d\Psi^{vol}(J)}{dJ} \boldsymbol{C}^{-1} \tag{8.127}$$

Next, if we take the definition of the tensor $\mathbf{S}$ given in (8.11), and use the definition of energy in (8.122), we can obtain:

$$\mathbf{S} = 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\frac{\partial}{\partial \boldsymbol{C}}\left[\tilde{\Psi}(\tilde{\boldsymbol{C}}) + \Psi^{vol}(\boldsymbol{C}^{vol})\right] = 2\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial \boldsymbol{C}} + 2\frac{\partial \Psi^{vol}(\boldsymbol{C}^{vol})}{\partial \boldsymbol{C}} = \tilde{\mathbf{S}} + \mathbf{S}^{vol} \tag{8.128}$$

where it holds that:

$$\mathbf{S}^{vol} = 2\frac{\partial \Psi^{vol}(\boldsymbol{C}^{vol})}{\partial \boldsymbol{C}} = 2\frac{\partial \Psi^{vol}(J)}{\partial J}\frac{\partial J}{\partial \boldsymbol{C}} = 2\frac{\partial \Psi^{vol}(J)}{\partial J}\frac{1}{2}J\boldsymbol{C}^{-1} = J\frac{\partial \Psi^{vol}(J)}{\partial J}\boldsymbol{C}^{-1} \tag{8.129}$$

and

$$\tilde{\mathbf{S}} = 2\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial \boldsymbol{C}} = 2\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial \tilde{\boldsymbol{C}}} : \frac{\partial \tilde{\boldsymbol{C}}}{\partial \boldsymbol{C}} = J^{\frac{-2}{3}} \mathbb{P} : \left(2\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial \tilde{\boldsymbol{C}}}\right) = J^{\frac{-2}{3}} \mathbb{P} : \tilde{\tilde{\mathbf{S}}} \tag{8.130}$$

where we have used the definition in (8.121). Additionally, we can verify that the tensor $\tilde{\tilde{\mathbf{S}}}$ is in the intermediate configuration, *i.e.* it is only characterized by a change of shape, (see Figure 8.5). Then, in summary we have:

$$\boxed{\mathbf{S} = \tilde{\mathbf{S}} + \mathbf{S}^{vol}} \tag{8.131}$$

where:

$$\boxed{\tilde{\mathbf{S}} = J^{\frac{-2}{3}} \mathbb{P} : 2\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial \tilde{\boldsymbol{C}}} = J^{\frac{-2}{3}} \mathbb{P} : \tilde{\tilde{\mathbf{S}}} \quad \textit{with} \quad \tilde{\tilde{\mathbf{S}}} = 2\frac{\partial \Psi^{iso}(\tilde{\boldsymbol{C}})}{\partial \tilde{\boldsymbol{C}}}} \tag{8.132}$$

$$\boxed{\mathbf{S}^{vol} = J\frac{d\Psi^{vol}(J)}{dJ}\boldsymbol{C}^{-1} = Jp\boldsymbol{C}^{-1}} \tag{8.133}$$

In addition, with the constitutive equation for hydrostatic pressure, Holzapfel (2000):

$$\boxed{p = \frac{D\Psi^{vol}(J)}{DJ}} \tag{8.134}$$

It is worth mentioning that the operator $\mathbb{P} = \mathbb{I} - \frac{1}{3}\boldsymbol{C}^{-1} \otimes \boldsymbol{C}$ given in (8.132) provides the correct deviatoric operator in the material (Lagrangian) description:

$$\left[\bullet(\vec{\boldsymbol{X}},t)\right]^{dev} = (\bullet) - \frac{1}{3}\left[(\bullet) : \boldsymbol{C}\right]\boldsymbol{C}^{-1} \tag{8.135}$$

Thus,

$$\tilde{\mathbf{S}} = J^{\frac{-2}{3}} \mathbb{P} : \tilde{\tilde{\mathbf{S}}} = J^{\frac{-2}{3}}\left[\mathbb{I} - \frac{1}{3}\boldsymbol{C}^{-1} \otimes \boldsymbol{C}\right] : \tilde{\tilde{\mathbf{S}}} = J^{\frac{-2}{3}}\left[\tilde{\tilde{\mathbf{S}}} - \frac{1}{3}\left(\tilde{\tilde{\mathbf{S}}} : \boldsymbol{C}\right)\boldsymbol{C}^{-1}\right] = J^{\frac{-2}{3}}\left[\tilde{\tilde{\mathbf{S}}}\right]^{dev} \tag{8.136}$$

Additionally, it holds that:

$$\left[\tilde{\tilde{\mathbf{S}}}\right]^{dev} : \boldsymbol{C} = 0 \tag{8.137}$$

8.4.2 Compressible Isotropic Materials

In compressible isotropic materials, the energy function decomposition can be given by:

$$\Psi(\boldsymbol{C}) = \Psi(\boldsymbol{b}) = \tilde{\Psi}(\tilde{\boldsymbol{b}}) + \Psi^{vol}(J) \tag{8.138}$$

Then, as $J = \sqrt{III_{\boldsymbol{b}}}$, we can show that if equations $\frac{\partial J}{\partial \boldsymbol{C}} = \frac{J}{2}\boldsymbol{C}^{-1}$ and (8.116) are valid, so are the following:

$$\frac{\partial J}{\partial \boldsymbol{b}} = \frac{J}{2}\boldsymbol{b}^{-1} \quad ; \quad \begin{cases} \dfrac{\partial \tilde{\boldsymbol{b}}}{\partial \boldsymbol{b}} = J^{\frac{-2}{3}}\left(\mathbb{I} - \dfrac{1}{3}\boldsymbol{b} \otimes \boldsymbol{b}^{-1}\right) \\ \dfrac{\partial \tilde{b}_{ij}}{\partial b_{kl}} = J^{\frac{-2}{3}}\left(\delta_{ik}\delta_{jl} - \dfrac{1}{3}b_{ij}b^{-1}_{kl}\right) \end{cases} \tag{8.139}$$

Then, if we refer to the constitutive equation for isotropic hyperelastic materials obtained in (8.70), $\boldsymbol{\sigma} = J^{-1}2\frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = J^{-1}2\boldsymbol{b} \cdot \frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}}$, and incorporate the definition of energy (8.138) into the stress equation, we can obtain:

$$\begin{aligned}\boldsymbol{\sigma} &= J^{-1}2\frac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = J^{-1}2\frac{\partial}{\partial \boldsymbol{b}}\left[\tilde{\Psi}(\tilde{\boldsymbol{b}}) + \Psi^{vol}(J)\right] \cdot \boldsymbol{b} = J^{-1}2\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} + J^{-1}2\frac{\partial \Psi^{vol}(J)}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} \\ &= \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{vol}\end{aligned} \tag{8.140}$$

Additionally, the volumetric contribution is:

$$\boldsymbol{\sigma}^{vol} = J^{-1}2\frac{\partial \Psi^{vol}(J)}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = J^{-1}2\frac{\partial \Psi^{vol}(J)}{\partial J}\frac{\partial J}{\partial \boldsymbol{b}} \cdot \boldsymbol{b} = J^{-1}2\frac{\partial \Psi^{vol}(J)}{\partial J}\frac{J}{2}\boldsymbol{b}^{-1} \cdot \boldsymbol{b} = \frac{\partial \Psi^{vol}(J)}{\partial J}\mathbf{1} \tag{8.141}$$

And, the isochoric contribution is:

$$\begin{aligned}\tilde{\boldsymbol{\sigma}} &= J^{-1}2\boldsymbol{b} \cdot \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \boldsymbol{b}} = J^{-1}2\boldsymbol{b} \cdot J^{\frac{-2}{3}}\left(\mathbb{I} - \frac{1}{3}\boldsymbol{b}^{-1} \otimes \boldsymbol{b}\right) : \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{\boldsymbol{b}}} \\ &= \boldsymbol{b} \cdot \left(J^{\frac{-2}{3}}\mathbb{I} - \frac{1}{3}\boldsymbol{b}^{-1} \otimes J^{\frac{-2}{3}}\boldsymbol{b}\right) \cdot \tilde{\boldsymbol{b}}^{-1} : 2J^{-1}\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{\boldsymbol{b}}} \cdot \tilde{\boldsymbol{b}} \\ &= \boldsymbol{b} \cdot \left(J^{\frac{-2}{3}}\mathbb{I} - \frac{1}{3}\boldsymbol{b}^{-1} \otimes \tilde{\boldsymbol{b}}\right) \cdot \tilde{\boldsymbol{b}}^{-1} : 2J^{-1}\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{\boldsymbol{b}}} \cdot \tilde{\boldsymbol{b}} = \left(\mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}\right) : \tilde{\boldsymbol{\sigma}} = [\tilde{\boldsymbol{\sigma}}]^{dev}\end{aligned} \tag{8.142}$$

where we have used the relationship $\boldsymbol{b} = J^{\frac{2}{3}}\tilde{\boldsymbol{b}}$, and it can be proven that if $\bullet$ is a second-order tensor, then it holds that: $\left(\mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}\right) : \bullet = \bullet^{dev}$, where $\bullet^{dev}$ represents the deviatoric part of the tensor $\bullet$, (see **Problem 1.26**).

Next, we will make the algebraic operations carried out in (8.142) using indicial notation:

$$
\begin{aligned}
\tilde{\sigma}_{ij} &= J^{-1} 2 b_{ik} \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial b_{kj}} = J^{-1} 2 b_{ik} \left(\frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{b}_{pq}} \frac{\partial \tilde{b}_{pq}}{\partial b_{kj}} \right) \\
&= J^{-1} 2 b_{ik} \left[J^{\frac{-2}{3}} \left(\delta_{pk} \delta_{qj} - \frac{1}{3} b_{pq} b_{kj}^{-1} \right) \right] \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{b}_{pq}} \\
&= J^{-1} 2 b_{ik} \left[J^{\frac{-2}{3}} \left(\delta_{pk} \delta_{qj} - \frac{1}{3} b_{pq} b_{kj}^{-1} \right) \right] \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{b}_{tq}} \delta_{tp} \\
&= J^{-1} 2 b_{ik} \left[J^{\frac{-2}{3}} \left(\delta_{pk} \delta_{qj} - \frac{1}{3} b_{pq} b_{kj}^{-1} \right) \right] \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{b}_{tq}} \underbrace{\tilde{b}_{ts} \tilde{b}_{sp}^{-1}}_{=\delta_{tp}} \\
&= J^{-1} 2 \left[\left(J^{\frac{-2}{3}} b_{ik} \delta_{pk} \delta_{qj} \tilde{b}_{sp}^{-1} - \frac{1}{3} \tilde{b}_{sp}^{-1} b_{pq} J^{\frac{-2}{3}} b_{ik} b_{kj}^{-1} \right) \right] \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{b}_{tq}} \tilde{b}_{ts} \\
&= J^{-1} 2 \left[\left(\tilde{b}_{ip} \tilde{b}_{sp}^{-1} \delta_{qj} - \frac{1}{3} \tilde{b}_{sp}^{-1} \tilde{b}_{pq} b_{ik} b_{kj}^{-1} \right) \right] \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{b}_{tq}} \tilde{b}_{ts} \\
&= \left[\left(\delta_{is} \delta_{qj} - \frac{1}{3} \delta_{ij} \delta_{sq} \right) \right] \left(J^{-1} 2 \frac{\partial \tilde{\Psi}(\tilde{\boldsymbol{b}})}{\partial \tilde{b}_{tq}} \tilde{b}_{ts} \right)
\end{aligned}
\tag{8.143}
$$

which is in accordance with (8.142).

In summary, in compressible isotropic materials, the following is satisfied:

$$
\boxed{\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^{vol}} \tag{8.144}
$$

where

$$
\boxed{\boldsymbol{\sigma}^{vol} = \frac{\partial \Psi^{vol}(J)}{\partial J} \mathbf{1} = p \mathbf{1}}, \quad \boxed{\tilde{\boldsymbol{\sigma}} = \left(\mathbb{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) : \tilde{\boldsymbol{\sigma}} = [\tilde{\boldsymbol{\sigma}}]^{dev}} \tag{8.145}
$$

in which the constitutive equation for hydrostatic pressure is:

$$
\boxed{p = \frac{D \Psi^{vol}(J)}{DJ}} \tag{8.146}
$$

8.4.2.1 Compressible Isotropic Material in terms of the Invariants

In isotropic materials, the strain energy function can be expressed in terms of the principal invariants as:

$$
\Psi = \tilde{\Psi}(I_{\tilde{C}}, II_{\tilde{C}}) + \Psi^{vol}(J) = \tilde{\Psi}(I_{\tilde{b}}, II_{\tilde{b}}) + \Psi^{vol}(J) \tag{8.147}
$$

where Ψ^{vol} is a function of the third invariant of the right Cauchy-Green deformation tensor ($III_{\boldsymbol{C}} = J^2$). This function, for an undeformed state, has to fulfill the following:

$$
\boldsymbol{C} = \mathbf{1} \longrightarrow III_{\boldsymbol{C}} = 1 \longrightarrow \left\{ \Psi^{vol} = 0 \quad ; \quad \frac{\partial \Psi^{vol}}{\partial \, III_{\boldsymbol{C}}} = 0 \right. \tag{8.148}
$$

In Chapter 2, subsection 2.13, we obtained the principal invariants of $\tilde{\boldsymbol{C}}$ in terms of those of $\boldsymbol{C}$, *i.e.*:

$$I_{\tilde{C}} = J^{\frac{-2}{3}} I_C = \frac{I_C}{\sqrt[3]{III_C}} = I_{\tilde{b}} \quad ; \quad II_{\tilde{C}} = J^{\frac{-4}{3}} II_C = \frac{II_C}{\sqrt[3]{III_C^2}} = II_{\tilde{b}} \quad ; \quad III_{\tilde{C}} = III_{\tilde{b}} = 1 \tag{8.149}$$

Note that the invariants $I_{\tilde{C}}$ and $II_{\tilde{C}}$ are independent of the volumetric deformation.

We can now express the constitutive equation in the material description by means of $\mathbf{S}$:

$$\mathbf{S} = \tilde{\mathbf{S}} + \mathbf{S}^{vol} \tag{8.150}$$

where $\mathbf{S}^{vol}$ is the volumetric part, and $\tilde{\mathbf{S}}$ is the isochoric part, both of which are given respectively by:

$$\mathbf{S}^{vol} = J \frac{d\Psi^{vol}(J)}{dJ} \boldsymbol{C}^{-1} = Jp\boldsymbol{C}^{-1} \quad ; \quad \tilde{\mathbf{S}} = J^{\frac{-2}{3}} \mathbb{P} : 2\frac{\partial\tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial\tilde{\boldsymbol{C}}} = J^{\frac{-2}{3}} \mathbb{P} : \tilde{\tilde{\mathbf{S}}} \tag{8.151}$$

where $\tilde{\mathbf{S}} = 2\dfrac{\partial\tilde{\Psi}(\tilde{\boldsymbol{C}})}{\partial\tilde{\boldsymbol{C}}}$ can be demonstrated by:

$$\tilde{\mathbf{S}} = 2\frac{\partial\tilde{\Psi}(I_{\tilde{C}}, II_{\tilde{C}})}{\partial\tilde{\boldsymbol{C}}} = 2\left[\left(\frac{\partial\tilde{\Psi}(I_{\tilde{C}}, II_{\tilde{C}})}{\partial I_{\tilde{C}}} + \frac{\partial\tilde{\Psi}(I_{\tilde{C}}, II_{\tilde{C}})}{\partial II_{\tilde{C}}} I_{\tilde{C}}\right)\mathbf{1} - \left(\frac{\partial\tilde{\Psi}(I_{\tilde{C}}, II_{\tilde{C}})}{\partial II_{\tilde{C}}}\right)\tilde{\boldsymbol{C}}\right] \tag{8.152}$$

where we have used one of the relationships obtained in (8.67).

Then, a very simple model for the volumetric part is $\Psi^{vol} = \Psi(\boldsymbol{F}^{vol}) = \Psi^{vol}(J)$, where $\Psi^{vol}(J)$ is given by:

$$\Psi^{vol}(J) = \frac{\kappa}{2}\left(J^2 - 1\right) = \frac{\kappa}{2}\left(III_C - 1\right) \tag{8.153}$$

and where κ is the bulk modulus. This model at the limit $J \to 0$ has no physical meaning since $\Psi^{vol}(J \to 0) = \dfrac{-\kappa}{2}$. Therefore, we can add the term $\Psi^{vol}(J) = \dfrac{\kappa}{2}(\log J)^2$ into the equation, *i.e.*:

$$\Psi^{vol}(J) = \frac{\kappa}{4}\left(J^2 - 1 - 2\log J\right) \tag{8.154}$$

which validated the following: when $J \to 0$, the energy function tends towards infinity, *i.e.* $\Psi^{vol}(J \to 0) = \infty$. Additionally, in a small deformation regime $J \approx 1$, the term $2\log J \to 0$.

8.5 Incompressible Materials

Many polymers can be subjected to large deformations without any volume change being observed, Holzapfel (2000). Hence, these materials can be considered to be incompressible, *i.e.* the continuum here is characterized by isochoric motion and the following is fulfilled:

$$\det(\boldsymbol{F}) = J = \lambda_1\lambda_2\lambda_3 = 1 \quad ; \quad J^2 = III_C = III_b = 1 \tag{8.155}$$

In incompressible materials, $(J = 1)$, the stress state is not completely determined by the strain state, because in an incompressible body we can add hydrostatic stress (pressure) to the current stress state without changing the strain state. Remember that, in isotropic materials, a hydrostatic state produces only volumetric deformations and because of this, the volumetric deformation in incompressible materials is equal to zero for any hydrostatic state. Note that here, even energy is not affected by the volumetric part, since: $\Psi(\boldsymbol{F}^{vol}) = \Psi(\mathbf{1}) = 0$.

According to the equation in (8.113), which is an incompressible case, it holds that $\tilde{\boldsymbol{F}} = \boldsymbol{F}$, $\boldsymbol{F}^{vol} = \mathbf{1}$, $\tilde{\boldsymbol{C}} = \boldsymbol{C}$, $\boldsymbol{C}^{vol} = \mathbf{1}$.

Then, the hydrostatic stress state is given by:

$$\boldsymbol{\sigma}^{hyd} = -p\mathbf{1} \qquad \Big| \qquad \sigma_{ij}^{hyd} = -p\delta_{ij} \tag{8.156}$$

where p denotes pressure. Then, if we refer to the relationship between the Cauchy stress and the second Piola-Kirchhoff stress tensors: $\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^{T}$, we can obtain the volumetric part of $\mathbf{S}$:

$$\mathbf{S}^{hyd} = J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{hyd}\cdot\boldsymbol{F}^{-T} = -J\,p\,\boldsymbol{F}^{-1}\cdot\mathbf{1}\cdot\boldsymbol{F}^{-T} = -J\,p\,\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T} = -J\,p\,\boldsymbol{B} = -J\,p\,\boldsymbol{C}^{-1} \tag{8.157}$$

where $\boldsymbol{B}$ is the Piola deformation tensor, given by $\boldsymbol{B} = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T} = \boldsymbol{C}^{-1}$, (see Chapter 2, subsection 2.6).

Then, for incompressible materials we have:

$$\mathbf{S} = 2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} - J\,p\,\boldsymbol{C}^{-1} \tag{8.158}$$

Next, to solve a problem with a constraint $(J = 1)$, we can introduce the Lagrange multiplier γ, which must satisfy the following:

$$\Psi = \Psi(\boldsymbol{C}) + \gamma\left(J - 1\right) \tag{8.159}$$

Now, the second Piola-Kirchhoff stress tensor can be obtained by taking the derivative of the above equation with respect to $\boldsymbol{C}$, *i.e.:*

$$\begin{aligned}\mathbf{S} &= 2\frac{\partial\Psi}{\partial\boldsymbol{C}} = 2\frac{\partial}{\partial\boldsymbol{C}}\left[\Psi(\boldsymbol{C}) + \gamma\left(J-1\right)\right] = 2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} + 2\gamma\frac{\partial J}{\partial\boldsymbol{C}} = 2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} + 2\gamma\frac{J}{2}\boldsymbol{C}^{-1} \\ &= 2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} + J\,\gamma\,\boldsymbol{C}^{-1}\end{aligned} \tag{8.160}$$

Now, if we compare (8.160) with (8.158) we can conclude that:

$$\gamma = -p = \frac{\sigma_{kk}}{3} \tag{8.161}$$

where p (pressure) is an unknown function to be determined by the incompressibility condition.

Now, let us consider the Cauchy stress tensor decomposition into a spherical (hydrostatic) and deviatoric part, *i.e.* $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{dev} + \sigma_m\mathbf{1}$, then, the second Piola-Kirchhoff stress tensor, defined as $\mathbf{S} = J\,\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T}$, can be split as follows:

$$\begin{aligned}\mathbf{S} &= J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} = J\boldsymbol{F}^{-1}\cdot(\boldsymbol{\sigma}^{dev} + \sigma_m\mathbf{1})\cdot\boldsymbol{F}^{-T} = J\boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}^{dev}\cdot\boldsymbol{F}^{-T} + J\boldsymbol{F}^{-1}\cdot\sigma_m\mathbf{1}\cdot\boldsymbol{F}^{-T} \\ &= \mathbf{S}^{dev} + J\sigma_m\boldsymbol{C}^{-1} = \mathbf{S}^{dev} + \mathbf{S}^{hyd}\end{aligned} \tag{8.162}$$

Then, by comparing the result above with (8.160) we can clearly identify the deviatoric part of the second Piola-Kirchhoff stress tensor.

After that, if we refer to the equation in (8.16) we can still write that in (8.160) as:

$$\boldsymbol{F}\cdot\mathbf{S}=\boldsymbol{F}\cdot 2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}}-p\,\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\quad\Rightarrow\quad\mathbf{P}=\frac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}-p\,\boldsymbol{F}^{-T}\tag{8.163}$$

and:

$$\begin{aligned}\mathbf{P}\cdot\boldsymbol{F}^{T}&=\frac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}\cdot\boldsymbol{F}^{T}-p\,\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{T}\\ \boldsymbol{\sigma}&=\frac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}\cdot\boldsymbol{F}^{T}-p\,\mathbf{1}=\boldsymbol{F}\cdot\left(\frac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}\right)^{T}-p\,\mathbf{1}\end{aligned}\tag{8.164}$$

where we have considered the incompressibility condition $J=1$. Then, in short, we can state that the constitutive equation for stress in an incompressible continuum can be given by:

$$\boxed{\begin{aligned}\mathbf{S}&=2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}}-p\,\boldsymbol{C}^{-1}\\ \mathbf{P}&=\frac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}-p\,\boldsymbol{F}^{-T}\\ \boldsymbol{\sigma}&=\frac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}\cdot\boldsymbol{F}^{T}-p\,\mathbf{1}=\boldsymbol{F}\cdot\left(\frac{\partial\Psi(\boldsymbol{F})}{\partial\boldsymbol{F}}\right)^{T}-p\,\mathbf{1}\end{aligned}}\tag{8.165}$$

The constitutive equation for hyperelastic incompressible materials

8.5.1 Geometrical Interpretation

In this section we will attempt to make a graphic interpretation of the results obtained previously, (see Bonet&Wood(1997)). To start with, let us consider the internal energy dissipation equation obtained in (8.9):

$$\mathcal{D}_{int}=\left(\frac{1}{2}\mathbf{S}-\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}}\right):\dot{\boldsymbol{C}}=0\tag{8.166}$$

As we saw before, in incompressible materials, the stress state is not completely defined by the strain state. Then, the term written in parentheses in the equation in (8.166) is not equal to zero, which indicates that $\dot{\boldsymbol{C}}$ is not arbitrary, *i.e.* it has restrictions. Remember that the rate of change of the Jacobian determinant is given by $\dot{J}=\frac{J}{2}\boldsymbol{C}^{-1}:\dot{\boldsymbol{C}}$, (see **Problem 2.12** in Chapter 2). Additionally, if we consider the incompressibility condition $J=1$, during motion $\dot{J}=0$, we obtain:

$$\dot{J}=\frac{J}{2}\boldsymbol{C}^{-1}:\dot{\boldsymbol{C}}=0\tag{8.167}$$

The above equation gives us the restrictions on $\dot{\boldsymbol{C}}$ which the equation in (8.166) has to satisfy, thereby implying that:

$$\frac{1}{2}\mathbf{S}-\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}}=\gamma\frac{J}{2}\boldsymbol{C}^{-1}\ \text{(unit of stress)}\tag{8.168}$$

where γ, as seen above, coincides with the hydrostatic pressure module.

Let us now consider an arbitrary plane defined by the normal $\hat{\boldsymbol{n}}$. Next, we will project the following tensors onto this plane, *i.e.*:

$$\dot{\boldsymbol{C}}\cdot\hat{\boldsymbol{n}}=\dot{\vec{\boldsymbol{d}}}^{(\hat{\boldsymbol{n}})}\quad;\quad\left(\frac{1}{2}\mathbf{S}-\frac{\partial\Psi}{\partial\boldsymbol{C}}\right)\cdot\hat{\boldsymbol{n}}=\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}\quad;\quad\frac{J}{2}\boldsymbol{C}^{-1}\cdot\hat{\boldsymbol{n}}=\frac{J}{2}\vec{\boldsymbol{d}}^{(\hat{\boldsymbol{n}})-1}\tag{8.169}$$

with which, the equations in (8.166) and (8.167) can be rewritten as:

$$\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})} \cdot \dot{\vec{\boldsymbol{d}}}^{(\hat{\boldsymbol{n}})} = 0 \quad ; \quad \frac{J}{2} \vec{\boldsymbol{d}}^{(\hat{\boldsymbol{n}})^{-1}} \cdot \dot{\vec{\boldsymbol{d}}}^{(\hat{\boldsymbol{n}})} = 0 \tag{8.170}$$

which indicates that $\vec{\mathbf{t}}^{(\hat{\boldsymbol{n}})}$ and $\vec{\boldsymbol{d}}^{(\hat{\boldsymbol{n}})^{-1}}$ are orthogonal to $\dot{\vec{\boldsymbol{d}}}^{(\hat{\boldsymbol{n}})}$, (see Figure 8.6).

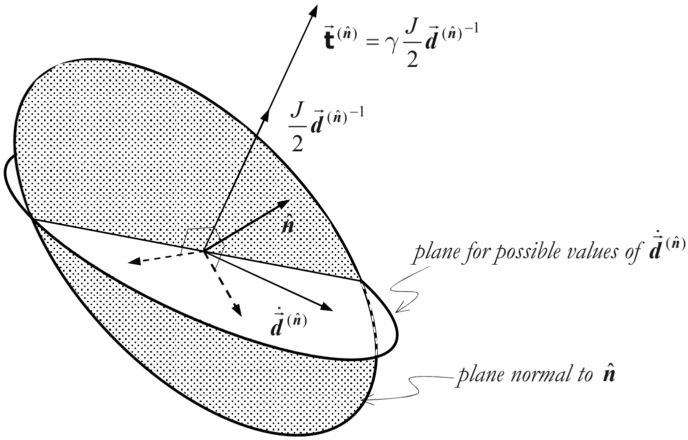


Figure 8.6: Incompressibility restriction.

8.5.2 Isotropic Incompressible Hyperelastic Materials

In isotropic incompressible hyperelastic materials, the scalar-valued tensor function $\Psi = \Psi(\boldsymbol{C}) = \Psi(\boldsymbol{b})$ can be expressed in terms of the invariants $I_{\boldsymbol{C}} = I_{\boldsymbol{b}}$ and $I\!I_{\boldsymbol{C}} = I\!I_{\boldsymbol{b}}$:

$$\Psi = \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}) = \Psi(I_{\boldsymbol{b}}, I\!I_{\boldsymbol{b}}) \tag{8.171}$$

or in terms of the principal invariants of the Green-Lagrange strain tensor:

$$\Psi = \Psi(I_{\boldsymbol{E}}, I\!I_{\boldsymbol{E}}) \tag{8.172}$$

Then, the constitutive equation for stress in hyperelastic materials becomes:

$$\mathbf{S} = \frac{\partial \Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}})}{\partial \boldsymbol{E}} - p \frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{E}} = \frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} \frac{\partial I_{\boldsymbol{C}}}{\partial \boldsymbol{E}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} \frac{\partial I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{E}} - p \frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{E}} \tag{8.173}$$

Next, the relationships between the principal invariants of the tensors $\boldsymbol{E}$ and $\boldsymbol{C}$ (obtained in **Problem 2.10** in Chapter 2) are interrelated by:

$$\begin{aligned}
I_{\boldsymbol{E}} &= \frac{1}{2}(I_{\boldsymbol{C}} - 3) & & & I_{\boldsymbol{C}} &= 2I_{\boldsymbol{E}} + 3 \\
I\!I_{\boldsymbol{E}} &= \frac{1}{4}(-2I_{\boldsymbol{C}} + I\!I_{\boldsymbol{C}} + 3) & &\xrightarrow{\textit{Reciprocal}} & I\!I_{\boldsymbol{C}} &= 4I\!I_{\boldsymbol{E}} + 4I_{\boldsymbol{E}} + 3 \\
I\!I\!I_{\boldsymbol{E}} &= \frac{1}{8}(I\!I\!I_{\boldsymbol{C}} - I\!I_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1) & & & I\!I\!I_{\boldsymbol{C}} &= 8\,I\!I\!I_{\boldsymbol{E}} + 4\,I\!I_{\boldsymbol{E}} + 2I_{\boldsymbol{E}} + 1
\end{aligned} \tag{8.174}$$

with which we can obtain the following derivatives:

$$\frac{\partial I_C}{\partial \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}}(2I_E + 3) = 2\mathbf{1}, \quad \frac{\partial II_C}{\partial \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}}(4\, II_E + 4I_E + 3) = 4(I_E\mathbf{1} - \boldsymbol{E}) + 4\mathbf{1}$$
$$\frac{\partial III_C}{\partial \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}}(8\, III_E + 4\, II_E + 2I_E + 1) = 8\, III_E \boldsymbol{E}^{-1} + 4(I_E\mathbf{1} - \boldsymbol{E}) + 2\mathbf{1} \tag{8.175}$$

Then the equation in (8.173) becomes:

$$\begin{aligned}\mathbf{S} &= \frac{\partial \Psi}{\partial I_C} 2\mathbf{1} + \frac{\partial \Psi}{\partial II_C}[4(I_E\mathbf{1} - \boldsymbol{E}) + 4\mathbf{1}] - p[8\, III_E \boldsymbol{E}^{-1} + 4(I_E\mathbf{1} - \boldsymbol{E}) + 2\mathbf{1}] \\ &= \frac{\partial \Psi}{\partial I_C} 2\mathbf{1} + \frac{\partial \Psi}{\partial II_C} 4[(I_E + 1)\mathbf{1} - \boldsymbol{E}] - 2p[(2I_E + 1)\mathbf{1} - 2\boldsymbol{E} + 4\, III_E \boldsymbol{E}^{-1}]\end{aligned} \tag{8.176}$$

Now, if we take into account that $III_C = 1$ into the equation in (8.174), we can then express III_E as a function of I_E and II_E, *i.e.*:

$$III_E = -\frac{1}{4}(2\, II_E + I_E) \tag{8.177}$$

Then, by using the equation in (8.72), the constitutive equation for isotropic incompressible materials becomes:

$$\boldsymbol{\sigma} = \left[2\left(\frac{\partial \Psi}{\partial I_b} + \frac{\partial \Psi}{\partial II_b} I_b\right)\boldsymbol{b} - 2\frac{\partial \Psi}{\partial II_b}\boldsymbol{b}^2\right] - p\mathbf{1} \tag{8.178}$$

Note that we can still express the constitutive equation for incompressible hyperelastic materials in terms of principal stretches, by means of the equation in (8.96):

$$\sigma_a = p + \lambda_a \frac{\partial \Psi}{\partial \lambda_a} \quad , \quad a = 1,2,3 \tag{8.179}$$

8.5.2.1 Series Expansion of the Energy Function for an Isotropic Incompressible Hyperelastic Materials

In incompressible materials it holds that $III_C = 1$, and the energy function $\Psi = \Psi(I_C, II_C)$ can be represented by means a power series, (see equation (8.81)), so:

$$\Psi = \Psi(I_C, II_C) = \sum_{p,q=0}^{\infty} c_{pq}(I_C - 3)^p (II_C - 3)^q \tag{8.180}$$

8.6 Examples of Hyperelastic Models

Several models have been developed to simulate the phenomenological behavior of hyperelastic materials. Here we mention some hyperelastic material models that can be found in the literature.

Remember that in elastic (or hyperelastic) materials, the only remaining constitutive equations are the one for energy and that for stress, one of which is redundant, that is, if we know the energy we can find the stress and vice versa, (see **Problem 6.1**).

8.6.1 The Neo-Hookean Material Model

In the Neo-Hookean material model, the strain energy function is given in terms of the isochoric part of $\boldsymbol{C}$ as follows:

$$\Psi = \frac{\mu}{2}\left(I_{\boldsymbol{C}} - 3\right) = c_1\left(I_{\boldsymbol{C}} - 3\right) \tag{8.181}$$

where $c_1 = \frac{\mu}{2}$. (The parameter μ was originally determined by statistical mechanics, Treloar (1944)), by $\mu = \frac{N}{2}\kappa T$, where N is the number of polymer chains per unit of the reference volume, κ is the Boltzmann constant and T is the temperature. Lastly, the parameter, $\mu = G$ can be determined experimentally and is known as the shear modulus.

Then, the stress constitutive equation for the Neo-Hookean material model becomes:

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2c_1\boldsymbol{b} \tag{8.182}$$

8.6.2 The Ogden Material Model

This model expresses the strain energy function in terms of the principal stretches and is given by:

$$\begin{aligned}\Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{M} a_p\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right) + \sum_{q=1}^{N} b_q\Big[(\lambda_1\lambda_2)^{\beta_q} + (\lambda_1\lambda_3)^{\beta_q} + \\ (\lambda_2\lambda_3)^{\beta_q} - 3\Big] + h(\lambda_1\lambda_2\lambda_3)\end{aligned} \tag{8.183}$$

where a_p, b_q are positive constants, $a_p \geq 1$, $b_q \geq 1$ and h is a one-variable convex function.

8.6.2.1 The Incompressible Ogden Material Model

This model expresses the strain energy function in terms of the principal stretches and is given by:

$$\Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right) \tag{8.184}$$

where N, μ_p, α_p are the material constants. In general, the shear modulus μ, in the reference configuration, becomes:

$$2\mu = \sum_{p=1}^{N} \mu_p \alpha_p \qquad \textit{with} \qquad \mu_p \alpha_p > 0 \tag{8.185}$$

In the literature, *e.g.* Holzapfel (2000), we can find the following values for the constants when $p = 3$:

$$\begin{aligned} &\alpha_1 = 1.3 \quad &;\quad &\alpha_2 = 5.0 \quad &;\quad &\alpha_3 = -2.0 \\ &\mu_1 = 6.3 \times 10^5\, N/m^2 \quad &;\quad &\mu_2 = 0.012 \times 10^5\, N/m^2 \quad &;\quad &\mu_3 = -0.1 \times 10^5\, N/m^2 \end{aligned} \tag{8.186}$$

Then, when $N = 1$ and $\alpha = 2$, the equation in (8.184) yields:

$$\Psi = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) = \frac{\mu}{2}\left(I_{\boldsymbol{C}} - 3\right) \tag{8.187}$$

which is the Neo-Hookean material model given in (8.181).

8.6.2.2 The Hadamard Material Model

This model is a simplified Ogden material model, where it holds that $M = N = 1$ and $\alpha_1 = \beta_1 = 2$ which reduces the equation in (8.183) to:

$$\Psi = \Psi(\boldsymbol{C}) = a_1(I_{\boldsymbol{C}} - 3) + b_1(I\!I_{\boldsymbol{C}} - 3) + h(J) \tag{8.188}$$

Then, taking into account the equations in (8.11) and (8.67), the constitutive equation becomes:

$$\mathbf{S} = 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\left[a_1 \mathbf{1} + b_1\left(I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1} - I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-2}\right) + h'(J)\frac{\partial J}{\partial \boldsymbol{C}}\right] \tag{8.189}$$

Afterwards, the derivative of the Jacobian determinant with respect to the tensor $\boldsymbol{C}$ can be evaluated as follows:

$$\frac{\partial J}{\partial \boldsymbol{C}} = \frac{\partial}{\partial \boldsymbol{C}}\left[(I\!I\!I_{\boldsymbol{C}})^{\frac{1}{2}}\right] = \frac{1}{2}(I\!I\!I_{\boldsymbol{C}})^{\frac{-1}{2}}\frac{\partial I\!I\!I_{\boldsymbol{C}}}{\partial \boldsymbol{C}} = \frac{1}{2}(I\!I\!I_{\boldsymbol{C}})^{\frac{-1}{2}} I\!I\!I_{\boldsymbol{C}}\boldsymbol{C}^{-1} = \frac{1}{2}J\boldsymbol{C}^{-1} \tag{8.190}$$

Finally, the equation in (8.189) may also be rewritten as:

$$\mathbf{S} = 2\left[a_1\mathbf{1} + \left(b_1 I\!I_{\boldsymbol{C}} + \frac{1}{2}h'(J)J\right)\boldsymbol{C}^{-1} - b_1 J^2\,\boldsymbol{C}^{-2}\right] \tag{8.191}$$

8.6.3 The Mooney-Rivlin Material Model

The Mooney-Rivlin material model was originally formulated to simulate rubber-like materials, today it is also used to simulate biological tissue-like materials.

8.6.3.1 Strain Energy Density

This model has the same energy expression as that provided by (8.184). Then, with the parameter values $N = 2$, $\alpha_1 = 2$, $\alpha_2 = -2$, and with the constraint $\lambda_1^2\lambda_2^2\lambda_3^2 = 1$ (incompressibility), the strain energy density given in (8.184) becomes:

$$\Psi(\lambda_1,\lambda_2,\lambda_3) = \frac{\mu_1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) - \frac{\mu_2}{2}\left(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3\right) \tag{8.192}$$

Note that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_{\tilde{\boldsymbol{C}}}$ and:

$$\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} = \frac{\lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_2^2}{\lambda_1^2\lambda_2^2\lambda_3^2} = \frac{I\!I_{\boldsymbol{C}}}{I\!I\!I_{\boldsymbol{C}}} = I\!I_{\boldsymbol{C}} \tag{8.193}$$

since we have the constraint $I\!I\!I_{\boldsymbol{C}} = 1$ we can summarize the strain energy density as follows:

$$\Psi(\boldsymbol{C}) = \frac{\mu_1}{2}(I_{\boldsymbol{C}} - 3) - \frac{\mu_2}{2}(I\!I_{\boldsymbol{C}} - 3) = c_1(I_{\boldsymbol{C}} - 3) + c_2(I\!I_{\boldsymbol{C}} - 3) \tag{8.194}$$

where $c_1 = \frac{\mu_1}{2}$ and $c_2 = -\frac{\mu_2}{2}$. Then, the terms $(I_{\boldsymbol{C}} - 3)$ and $(I\!I_{\boldsymbol{C}} - 3)$ ensure that the strain energy is zero when there is no deformation ($\boldsymbol{E} = \mathbf{0}$), since in this scenario and according to the equation in (8.172), we will obtain $(I_{\boldsymbol{C}} = 3)$ and $(I\!I_{\boldsymbol{C}} = 3)$. Note that in the particular case when $(c_2 = 0)$ we revert to the Neo-Hookean material model given in (8.181).

8.6.3.2 The Stress Tensor

The second Piola-Kirchhoff stress tensor for the Mooney-Rivlin material model becomes:

$$\begin{aligned}\mathbf{S} &= 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}\right] \\ &= 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\right)\boldsymbol{C}\right] = 2\left[(c_1 + c_2 I_{\boldsymbol{C}})\mathbf{1} - c_2 \boldsymbol{C}\right]\end{aligned} \tag{8.195}$$

and the Cauchy stress tensor can be obtained as follows:

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2c_1\boldsymbol{b} - 2c_2\boldsymbol{b}^{-1} \tag{8.196}$$

8.6.4 The Yeoh Material Model

8.6.4.1 Strain Energy Density

The Yeoh material model is used to simulate isotropic incompressible materials. Our starting point here is the series expansion of strain energy density:

$$\Psi = \Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}}) = \sum_{p,q,r=1}^{N} c_{pqr}(I_{\boldsymbol{C}} - 3)^p (II_{\boldsymbol{C}} - 3)^q (III_{\boldsymbol{C}} - 1)^r \tag{8.197}$$

Then, by considering the incompressible material, $III_{\boldsymbol{C}} = 1$, and also by discarding the second invariant we obtain:

$$\Psi = \Psi(I_{\boldsymbol{C}}) = \sum_{p=1}^{N=3} c_p (I_{\boldsymbol{C}} - 3)^p \tag{8.198}$$

with which we can obtain the strain energy density for the Yeoh material model as follows:

$$\Psi = c_1(I_{\boldsymbol{C}} - 3) + c_2(I_{\boldsymbol{C}} - 3)^2 + c_3(I_{\boldsymbol{C}} - 3)^3 \tag{8.199}$$

where c_1, c_2 and c_3 are material constants.

8.6.4.2 The Stress Tensor

In this model the second Piola-Kirchhoff stress tensor becomes:

$$\begin{aligned}\mathbf{S} &= 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial II_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial II_{\boldsymbol{C}}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}\right] \\ &= 2\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}}\mathbf{1} = 2\left[c_1 + 2c_2(I_{\boldsymbol{C}} - 3) + 3c_3(I_{\boldsymbol{C}} - 3)^2\right]\mathbf{1}\end{aligned} \tag{8.200}$$

8.6.5 The Arruda-Boyce Material Model

The Arruda-Boyce Material Model, also called the 8-chain model, takes into consideration that the shear modulus, μ, depends on the strain. This phenomenon is detected in some polymers.

Here, the strain energy density is given by:

$$\Psi = \Psi(\boldsymbol{C}) = \mu_0 \sum_{p=1}^{N} \frac{c_p}{\lambda_{lock}^{2p-2}} \left(I_{\boldsymbol{C}}^p - 3^p \right) \tag{8.201}$$

where μ_0 is the initial shear modulus, c_i are constants obtained by statistical theory and λ_{lock} and N are material constants. Then, if we consider that $N = 3$ we obtain:

$$\begin{aligned}
\Psi(\boldsymbol{C}) &= \mu_0 \left[c_1 \left(I_{\boldsymbol{C}} - 3 \right) + \frac{c_2}{\lambda_{lock}^2} \left(I_{\boldsymbol{C}}^2 - 9 \right) + \frac{c_3}{\lambda_{lock}^4} \left(I_{\boldsymbol{C}}^3 - 27 \right) \right] \\
&= \mu_0 \left[\frac{1}{2} \left(I_{\boldsymbol{C}} - 3 \right) + \frac{1}{20\lambda_{lock}^2} \left(I_{\boldsymbol{C}}^2 - 9 \right) + \frac{11}{1050\lambda_{lock}^4} \left(I_{\boldsymbol{C}}^3 - 27 \right) \right]
\end{aligned} \tag{8.202}$$

8.6.6 The Blatz-Ko Hyperelastic Model

Porous polymers should be considered as compressible materials. Blatz-Ko(1962) proposed the following strain energy density $\Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})$ based on experimental and numerical results, with isochoric and volumetric parts:

$$\Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}}) = f\left[\frac{\mu}{2}(I_{\boldsymbol{C}} - 3) - \frac{\mu}{2\beta}\left(III_{\boldsymbol{C}}^{-\beta} - 1 \right) \right] + (1-f)\frac{\mu}{2}\left[\left(\frac{II_{\boldsymbol{C}}}{III_{\boldsymbol{C}}} - 3 \right) + \frac{1}{\beta}(III_{\boldsymbol{C}}^{-\beta} - 1) \right] \tag{8.203}$$

where β is given in terms of the μ (shear modulus) and ν (Poisson's ratio) by $\beta = \dfrac{\nu}{1-2\nu}$, and $f \in [0,1]$ is an interpolation parameter. In the particular case in which $f = 1$, we obtain:

$$\Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}}) = \frac{\mu}{2}(I_{\boldsymbol{C}} - 3) - \frac{\mu}{2\beta}\left(III_{\boldsymbol{C}}^{-\beta} - 1 \right) \tag{8.204}$$

In the incompressibility case ($III_{\boldsymbol{C}} = 1$), the equation in (8.203) becomes the Mooney-Rivlin model, (see equation (8.194)). Then, in the restrictive case where $f = 1$, and with the incompressibility condition ($III_{\boldsymbol{C}} = 1$), we revert to the Neo-Hookean incompressible model given in (8.181).

8.6.7 The Saint Venant-Kirchhoff Model

8.6.7.1 Strain Energy Density

In the Saint Venant-Kirchhoff model, the strain energy density ($\Psi(I_{\boldsymbol{E}}, II_{\boldsymbol{E}})$) is given by:

$$\Psi(I_{\boldsymbol{E}}, II_{\boldsymbol{E}}) = \frac{1}{2}\left(\overline{\lambda} + 2\mu \right) I_{\boldsymbol{E}}^2 - 2\mu II_{\boldsymbol{E}} \tag{8.205}$$

where $\overline{\lambda}$ and μ are material constants.

We can express the strain energy density in terms of the $\boldsymbol{C}$-invariants. To do so, let us consider the relationships between the invariants of $\boldsymbol{E}$ and $\boldsymbol{C}$, (see the equations in (8.174)):

$$I_{\boldsymbol{E}} = \frac{1}{2}\left(I_{\boldsymbol{C}} - 3 \right) \quad ; \quad II_{\boldsymbol{E}} = \frac{1}{4}\left(-2I_{\boldsymbol{C}} + II_{\boldsymbol{C}} + 3 \right) \quad ; \quad III_{\boldsymbol{E}} = \frac{1}{8}\left(III_{\boldsymbol{C}} - II_{\boldsymbol{C}} + I_{\boldsymbol{C}} - 1 \right) \tag{8.206}$$

and by substituting them into the equation in (8.205) we obtain:

$$
\begin{aligned}
\Psi(I_{\boldsymbol{C}}, I\!I_{\boldsymbol{C}}) &= \frac{1}{2}\left(\overline{\lambda} + 2\mu\right)\left[\frac{1}{2}\left(I_{\boldsymbol{C}} - 3\right)\right]^2 - 2\mu\left[\frac{1}{4}\left(-2I_{\boldsymbol{C}} + I\!I_{\boldsymbol{C}} + 3\right)\right] \\
&= \frac{1}{8}\left(\overline{\lambda} + 2\mu\right)\left(I_{\boldsymbol{C}} - 3\right)^2 - \frac{\mu}{2}\left(-2I_{\boldsymbol{C}} + I\!I_{\boldsymbol{C}} + 3\right)
\end{aligned}
\tag{8.207}
$$

8.6.7.2 The Stress Tensor

The derivatives of the function (8.205) with respect to the invariants are:

$$
\frac{\partial \Psi}{\partial I_{\boldsymbol{E}}} = \left(\overline{\lambda} + 2\mu\right) I_{\boldsymbol{E}} \quad ; \quad \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{E}}} = -2\mu \tag{8.208}
$$

and by using the equation in (8.79), where the parameters are:

$$
\begin{aligned}
c_0 &= \frac{\partial \Psi}{\partial I_{\boldsymbol{E}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{E}}} I_{\boldsymbol{E}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{E}}} I\!I_{\boldsymbol{E}} = \left(\overline{\lambda} + 2\mu\right) I_{\boldsymbol{E}} - 2\mu I_{\boldsymbol{E}} = \overline{\lambda} I_{\boldsymbol{E}} \\
c_1 &= \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{E}}} + \frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{E}}} I_{\boldsymbol{E}} = -2\mu
\end{aligned}
\tag{8.209}
$$

and by substituting the above parameters into the equation in (8.79), we obtain:

$$
\mathbf{S} = \overline{\lambda} I_{\boldsymbol{E}} \mathbf{1} + 2\mu \boldsymbol{E} \tag{8.210}
$$

Then we can conclude that the Saint_Venant-Kirchhoff model describes geometric nonlinearity, but is also characterized by a material linearity, *i.e.* the stress-strain relationship is linear.

Then, taking into account that $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \mathbf{1}\right)$ and $I_{\boldsymbol{E}} = \frac{1}{2}\left(I_{\boldsymbol{C}} - 3\right)$, we can obtain the tensor $\mathbf{S}$ by means of $\boldsymbol{C}$ as follows:

$$
\mathbf{S} = \mathbf{S}(\boldsymbol{C}) = \overline{\lambda} I_{\boldsymbol{E}} \mathbf{1} + 2\mu \boldsymbol{E} = \overline{\lambda} \tfrac{1}{2}\left(I_{\boldsymbol{C}} - 3\right)\mathbf{1} + 2\mu \tfrac{1}{2}\left(\boldsymbol{C} - \mathbf{1}\right) = \left[\frac{\overline{\lambda}}{2}\left(I_{\boldsymbol{C}} - 3\right) - \mu\right]\mathbf{1} + \mu \boldsymbol{C} \tag{8.211}
$$

Note that the above equation could have been obtained by means of the constitutive equation in terms of $\mathbf{S}$ given in (8.67), *i.e.*:

$$
\begin{aligned}
\mathbf{S} &= 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C} + \left(\frac{\partial \Psi}{\partial I\!I\!I_{\boldsymbol{C}}} I\!I\!I_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}\right] \\
&= 2\left[\left(\frac{\partial \Psi}{\partial I_{\boldsymbol{C}}} + \frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}} I_{\boldsymbol{C}}\right)\mathbf{1} - \left(\frac{\partial \Psi}{\partial I\!I_{\boldsymbol{C}}}\right)\boldsymbol{C}\right] = 2\left\{\frac{2}{8}\left(\overline{\lambda} + 2\mu\right)\left(I_{\boldsymbol{C}} - 3\right) - \frac{\mu}{2} - \frac{\mu}{2}\left(I_{\boldsymbol{C}} - 3\right)\right\}\mathbf{1} + \frac{\mu}{2}\boldsymbol{C} \\
&= \left[\frac{\overline{\lambda}}{2}\left(I_{\boldsymbol{C}} - 3\right) - \mu\right]\mathbf{1} + \mu \boldsymbol{C}
\end{aligned}
\tag{8.212}
$$

8.6.7.3 The Elastic Tangent Stiffness Tensor

The elastic tangent stiffness tensor can be obtained as follows:

$$
\mathbb{C}^{tan} = \frac{\partial \mathbf{S}}{\partial \boldsymbol{E}} = \frac{\partial}{\partial \boldsymbol{E}}\left(\overline{\lambda} I_{\boldsymbol{E}} \mathbf{1} + 2\mu \boldsymbol{E}\right) \equiv \frac{\partial}{\partial \boldsymbol{E}}\left(\overline{\lambda} I_{\boldsymbol{E}} \otimes \mathbf{1} + 2\mu \otimes \boldsymbol{E}\right) = \overline{\lambda} \frac{\partial I_{\boldsymbol{E}}}{\partial \boldsymbol{E}} \otimes \mathbf{1} + 2\mu \otimes \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{E}} \tag{8.213}
$$

Note that $\boldsymbol{E}$ is a symmetric tensor, so the result of the operation $\boldsymbol{E}_{,\boldsymbol{E}}$ is also a symmetric tensor. Then, the equation in (8.213) in indicial notation becomes:

$$\mathbb{C}^{tan}_{ijkl} = \frac{\partial \mathsf{S}_{ij}}{\partial E_{kl}} = \overline{\lambda} \frac{\partial I_{\boldsymbol{E}}}{\partial E_{kl}} \delta_{ij} + 2\mu \frac{\partial E_{ij}}{\partial E_{kl}} = \overline{\lambda} \frac{\partial I_{\boldsymbol{E}}}{\partial E_{kl}} \delta_{ij} + 2\mu \frac{\partial \left[\frac{1}{2}\left(E_{ij} + E_{ji}\right)\right]}{\partial E_{kl}} \tag{8.214}$$

Thus,

$$\mathbb{C}^{tan}_{ijkl} = \overline{\lambda}\delta_{kl}\delta_{ij} + \mu\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right] \xrightarrow{\text{tensorial notation}} \mathbb{C}^{tan} = \overline{\lambda}\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I} \tag{8.215}$$

NOTE: Note that in a small deformation regime, the condition that all stress tensors are equal is satisfied, *i.e.* $\mathbf{S} \approx \boldsymbol{\sigma}$, and the same is true for the strain tensors, $\boldsymbol{E} \approx \boldsymbol{\varepsilon}$, after which the constitutive equation for stress in (8.210) becomes:

$$\boldsymbol{\sigma} = \overline{\lambda} I_{\boldsymbol{\varepsilon}} \mathbf{1} + 2\mu\boldsymbol{\varepsilon} \tag{8.216}$$

which is the same constitutive equation for isotropic linear elastic materials as that obtained in Chapter 7, (see also **Problem 6.1**). In addition, the elastic tangent stiffness tensor $\mathbb{C}^{tan}$ coincides with the elasticity tensor $\mathbb{C}^{e}$, (see Chapter 7). ■

8.6.8 The Compressible Neo-Hookean Material Model

8.6.8.1 Strain Energy Density

In the compressible Neo-Hookean material model, the Helmholtz free energy per unit reference volume (strain energy density), (see Bonet&Wood (1997)), is defined by:

$$\Psi(\boldsymbol{C}) = \underbrace{\frac{\overline{\lambda}}{2}(\ln J)^2 - \mu \ln J}_{\Psi(J)} + \underbrace{\frac{\mu}{2}(I_{\boldsymbol{C}} - 3)}_{\Psi(I_{\boldsymbol{C}})} = \Psi(J) + \Psi(I_{\boldsymbol{C}}) \tag{8.217}$$

where $\overline{\lambda}$ and μ are material parameters.

8.6.8.2 The Stress Tensor

The second Piola-Kirchhoff stress tensor, (see Eq. (8.67)), is given by:

$$\begin{aligned}\mathbf{S} &= 2\frac{\partial\Psi(\boldsymbol{C})}{\partial\boldsymbol{C}} = 2\left(\frac{\partial\Psi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})}{\partial\boldsymbol{C}} + \frac{\partial\Psi(J)}{\partial\boldsymbol{C}}\right)\\ &= 2\left[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\right)\mathbf{1} + \left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}} II_{\boldsymbol{C}} + \frac{\partial\Psi}{\partial III_{\boldsymbol{C}}} III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1} - \left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}} III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-2}\right] + 2\frac{\partial\Psi(J)}{\partial\boldsymbol{C}}\end{aligned} \tag{8.218}$$

where $\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}} = \frac{\mu}{2}$, $\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}} = 0$, $\frac{\partial\Psi}{\partial III_{\boldsymbol{C}}} = 0$ and $\frac{\partial\Psi(J)}{\partial\boldsymbol{C}} = \frac{\overline{\lambda}}{2} 2(\ln J)\frac{1}{J}\frac{\partial J}{\partial\boldsymbol{C}} - \mu\frac{1}{J}\frac{1}{2}J\boldsymbol{C}^{-1}$. Moreover, if we consider that $\frac{\partial J}{\partial\boldsymbol{C}} = \frac{J}{2}\boldsymbol{C}^{-1}$, we can conclude that:

$$\mathbf{S} = 2\left(\frac{\mu}{2}\mathbf{1}\right) + 2\left(\frac{\overline{\lambda}}{2}(\ln J)\boldsymbol{C}^{-1} - \frac{\mu}{2}\boldsymbol{C}^{-1}\right) = \mu\left(\mathbf{1} - \boldsymbol{C}^{-1}\right) + \overline{\lambda}(\ln J)\boldsymbol{C}^{-1} \tag{8.219}$$

Then, by considering the relationship between the Cauchy stress tensor and the second Piola-Kirchhoff stress tensor, $J\boldsymbol{\sigma} = \boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^{T}$, as well as $\boldsymbol{C}^{-1} = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}$, we can obtain:

$$J\boldsymbol{\sigma} = \boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T = \boldsymbol{F}\cdot\left[\mu\left(\mathbf{1}-\boldsymbol{C}^{-1}\right)+\overline{\lambda}(\ln J)\boldsymbol{C}^{-1}\right]\cdot\boldsymbol{F}^T$$
$$= \boldsymbol{F}\cdot\left[\mu\left(\mathbf{1}-\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\right)+\overline{\lambda}(\ln J)\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\right]\cdot\boldsymbol{F}^T \tag{8.220}$$
$$= \mu\boldsymbol{F}\cdot\boldsymbol{F}^T - \mu\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^T + \overline{\lambda}(\ln J)\boldsymbol{F}\cdot\boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^T$$

Thus,

$$\boldsymbol{\sigma} = \frac{1}{J}\left[\mu(\boldsymbol{b}-\mathbf{1})+\overline{\lambda}(\ln J)\mathbf{1}\right] \tag{8.221}$$

8.6.8.3 The Elastic Tangent Stiffness Tensor

The elastic tangent stiffness tensor, (see Figure 8.3), can be defined as:

$$\mathbb{C}^{tan} = 4\frac{\partial^2\Psi}{\partial\boldsymbol{C}\otimes\partial\boldsymbol{C}} = 2\frac{\partial\mathbf{S}}{\partial\boldsymbol{C}} \tag{8.222}$$

Then, given the second Piola-Kirchhoff stress tensor equation in (8.219), we obtain:

$$\mathbb{C}^{tan} = 2\frac{\partial}{\partial\boldsymbol{C}}\left[\mu\left(\mathbf{1}-\boldsymbol{C}^{-1}\right)+\overline{\lambda}(\ln J)\boldsymbol{C}^{-1}\right]$$
$$= 2\left[\mu\frac{\partial(\mathbf{1})}{\partial\boldsymbol{C}} - \mu\frac{\partial\boldsymbol{C}^{-1}}{\partial\boldsymbol{C}} + \overline{\lambda}\boldsymbol{C}^{-1}\otimes\frac{\partial(\ln J)}{\partial\boldsymbol{C}} + \overline{\lambda}(\ln J)\otimes\frac{\partial\boldsymbol{C}^{-1}}{\partial\boldsymbol{C}}\right] \tag{8.223}$$

In Chapter 1 we obtained $\dfrac{\partial C^{-1}_{ir}}{\partial C_{kl}} = -\dfrac{1}{2}\left[\left(C^{-1}_{ik}C^{-1}_{lr}+C^{-1}_{il}C^{-1}_{kr}\right)\right]$, after which the above equation, in indicial notation, becomes:

$$\mathbb{C}^{tan}_{irkl} = -2\mu\frac{\partial C^{-1}_{ir}}{\partial C_{kl}} + 2\overline{\lambda}C^{-1}_{ir}\frac{\partial(\ln J)}{\partial C_{kl}} + 2\overline{\lambda}(\ln J)\frac{\partial C^{-1}_{ir}}{\partial C_{kl}}$$
$$= \mu\left(C^{-1}_{ik}C^{-1}_{lr}+C^{-1}_{il}C^{-1}_{kr}\right) + 2\overline{\lambda}C^{-1}_{ir}\frac{1}{J}\frac{J}{2}C^{-1}_{kl} - \overline{\lambda}(\ln J)\left(C^{-1}_{ik}C^{-1}_{lr}+C^{-1}_{il}C^{-1}_{kr}\right) \tag{8.224}$$

and by simplifying we obtain:

$$\mathbb{C}^{tan}_{irkl} = \left[\mu-\overline{\lambda}(\ln J)\right]\left(C^{-1}_{ik}C^{-1}_{lr}+C^{-1}_{il}C^{-1}_{kr}\right)+\overline{\lambda}C^{-1}_{ir}C^{-1}_{kl}$$
$$\mathbb{C}^{tan} = \left[\mu-\overline{\lambda}(\ln J)\right]\left(\boldsymbol{C}^{-1}\overline{\otimes}\boldsymbol{C}^{-1}+\boldsymbol{C}^{-1}\underline{\otimes}\boldsymbol{C}^{-1}\right)+\overline{\lambda}\boldsymbol{C}^{-1}\otimes\boldsymbol{C}^{-1} \tag{8.225}$$

We can also obtain the tensor $\mathbb{L}$, (see Figure 8.3), which is related to the tensor $\mathbb{C}$ by the equation in (8.32), *i.e.*:

$$\mathbb{L}_{abcd} = F_{bq}F_{ap}\mathbb{C}^{tan}_{pqst}F_{ct}F_{ds}$$
$$= F_{bq}F_{ap}\left\{\left[\mu-\overline{\lambda}(\ln J)\right]\left(C^{-1}_{ps}C^{-1}_{tq}+C^{-1}_{pt}C^{-1}_{sq}\right)+\overline{\lambda}C^{-1}_{pq}C^{-1}_{st}\right\}F_{ct}F_{ds} \tag{8.226}$$

Note that: $F_{bq}F_{ap}C^{-1}_{ps}C^{-1}_{tq}F_{ct}F_{ds} = F_{bq}F_{ap}F^{-1}_{px}F^{-1}_{sx}F^{-1}_{tw}F^{-1}_{qw}F_{ct}F_{ds} = \delta_{ax}\delta_{bw}\delta_{xd}\delta_{wc} = \delta_{ad}\delta_{bc}$,

$F_{bq}F_{ap}C^{-1}_{pt}C^{-1}_{sq}F_{ct}F_{ds} = \delta_{ac}\delta_{bd}$, $F_{bq}F_{ap}C^{-1}_{pq}C^{-1}_{st}F_{ct}F_{ds} = \delta_{ab}\delta_{cd}$.

Then:

$$\mathbb{L}_{abcd} = \left[\mu-\overline{\lambda}(\ln J)\right]\left(\delta_{ad}\delta_{bc}+\delta_{ac}\delta_{bd}\right)+\overline{\lambda}\delta_{ab}\delta_{cd} \tag{8.227}$$

Note that $(\delta_{ad}\delta_{bc}+\delta_{ac}\delta_{bd}) = 2\mathbb{I}^{sym}_{abcd} = 2\mathbf{I}_{abcd}$, where $\mathbf{I}\equiv\mathbb{I}^{sym}$ is the symmetric fourth-order unit tensor, with which we can obtain:

$$\mathbb{L}_{abcd} = 2\left[\mu - \overline{\lambda}(\ln J)\right]\mathbb{I}^{sym}_{abcd} + \lambda\delta_{ab}\delta_{cd} \xrightarrow{\text{Tensorial notation}} \mathbb{L} = 2\left[\mu - \overline{\lambda}(\ln J)\right]\mathbf{I} + \overline{\lambda}\mathbf{1}\otimes\mathbf{1} \qquad (8.228)$$

Now, the tensor $\mathbb{A}$ (defined in (8.48)) becomes:

$$\mathbb{A} = \frac{1}{J}\mathbb{L} = 2\frac{\left[\mu - \overline{\lambda}(\ln J)\right]}{J}\mathbf{I} + \frac{\overline{\lambda}}{J}\mathbf{1}\otimes\mathbf{1} \qquad (8.229)$$

Then, if $\mu' = \dfrac{\left[\mu - \overline{\lambda}(\ln J)\right]}{J}$ and $\overline{\lambda}' = \dfrac{\overline{\lambda}}{J}$, where μ' and $\overline{\lambda}'$ are the equivalent Lamé constants, it follows that

$$\mathbb{A} = 2\mu'\mathbf{I} + \overline{\lambda}'\mathbf{1}\otimes\mathbf{1} \qquad (8.230)$$

NOTE: In a small deformation regime, we have $J \approx 1$, with which we obtain $\mu' \approx \mu$ and $\overline{\lambda}' \approx \overline{\lambda}$, and the following also holds: $\mathbb{C}^{tan} = \mathbb{L} = \mathbb{A} = 2\mu\mathbf{I} + \overline{\lambda}\mathbf{1}\otimes\mathbf{1} = \mathbb{C}^{e}$. ■

Problem 8.1: Consider a motion characterized by dilatation. The continuum is made up of an isotropic material resembling the compressible Neo-Hookean material model. Find the Cauchy stress tensor in terms of the Jacobian determinant.

Solution:

In isotropic materials dilation can be characterized by $dx_i = \lambda dX_i$, (see Figure 8.7).

$$\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\begin{bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{bmatrix}$$

Stretches $\lambda = \lambda_1 = \lambda_2 = \lambda_3$

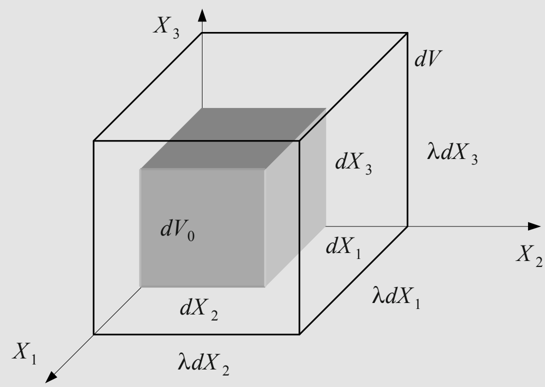


Figure 8.7: Dilatation.

In this scenario we have: $\boldsymbol{F} = \lambda\mathbf{1} \quad \Rightarrow \quad J \equiv \det(\boldsymbol{F}) = \lambda^3 \quad \Rightarrow \quad \lambda = J^{\frac{1}{3}}$

The Cauchy stress tensor can be obtained by means of the equation in (8.221), *i.e.*:

$$\boldsymbol{\sigma} = \frac{1}{J}\left[\mu(\boldsymbol{b} - \mathbf{1}) + \overline{\lambda}(\ln J)\mathbf{1}\right]$$

where the left Cauchy-Green deformation tensor can be evaluated as follows:

$$\boldsymbol{b} = \boldsymbol{F}\cdot\boldsymbol{F}^T = \lambda\mathbf{1}\cdot\lambda\mathbf{1} = \lambda^2\mathbf{1} = J^{\frac{2}{3}}\mathbf{1}$$

Therefore, the Cauchy stress tensor becomes:

$$\boldsymbol{\sigma} = \left[\frac{\mu}{J}\left(J^{\frac{2}{3}} - 1\right) + \frac{\overline{\lambda}}{J}(\ln J)\right]\mathbf{1}$$

8.6.9 The Gent Model

Strain energy density in the Gent model, Gent(1996), is characterized by the following logarithmic function:

$$\Psi(I_{\boldsymbol{C}}) = \frac{-\mu}{2} \mathrm{I}_m \ln\left(1 - \frac{I_{\boldsymbol{C}} - 3}{\mathrm{I}_m}\right) \tag{8.231}$$

where μ and I_m are material constants, and I_m is the constant that measures the limit value of $(I_{\boldsymbol{C}} - 3)$.

8.6.10 The Statistical Model

We can summarize this model as follows.

Strain Energy Density

Let $\tilde{\Psi}$ be the isochoric part and Ψ^{vol} the volumetric part, then, the energy function is given by:

$$\boxed{\Psi = \tilde{\Psi}(I_{\tilde{\boldsymbol{C}}}) + \Psi^{vol}(\mathit{III}_{\boldsymbol{C}})} \tag{8.232}$$

where

$$\begin{aligned}\tilde{\Psi} = \mu\Bigg[\frac{1}{2}\left(I_{\tilde{\boldsymbol{C}}} - 3\right) + \frac{1}{20N}\left(I_{\tilde{\boldsymbol{C}}}^2 - 9\right) + \frac{11}{1050N^2}\left(I_{\tilde{\boldsymbol{C}}}^3 - 27\right) + \\ \frac{19}{7000N^3}\left(I_{\tilde{\boldsymbol{C}}}^4 - 81\right) + \frac{519}{673750N^4}\left(I_{\tilde{\boldsymbol{C}}}^5 - 243\right) + \cdots\Bigg]\end{aligned} \tag{8.233}$$

$$\Psi^{vol} = \frac{\kappa}{2}\left(\ln\sqrt{\mathit{III}_{\boldsymbol{C}}}\right)^2 \tag{8.234}$$

The Second Piola-Kirchhoff Stress Tensor

$$\boxed{\mathbf{S} = a_0 \mathbf{1} + a_2 \boldsymbol{C}^{-1}} \quad \textit{Stress Tensor} - \textit{Statistical Model} \tag{8.235}$$

where the coefficients a_0, a_1 and a_2 are given by:

$$a_0 = 2\mu\left(\frac{1}{2} + \frac{1}{10N} I_{\tilde{\boldsymbol{C}}} + \frac{11}{350N^2} I_{\tilde{\boldsymbol{C}}}^2 + \frac{19}{1750N^3} I_{\tilde{\boldsymbol{C}}}^3 + \frac{519}{134750N^4} I_{\tilde{\boldsymbol{C}}}^4 + \cdots\right)\frac{1}{\sqrt[3]{\mathit{III}_{\boldsymbol{C}}}} \tag{8.236}$$

$$\begin{aligned}a_2 = -\frac{2}{3}\mu\Bigg(\frac{1}{2} I_{\tilde{\boldsymbol{C}}} + \frac{1}{10N} I_{\tilde{\boldsymbol{C}}}^2 + \frac{11}{350N^2} I_{\tilde{\boldsymbol{C}}}^3 + \frac{19}{1750N^3} I_{\tilde{\boldsymbol{C}}}^4 + \\ \frac{519}{134750N^4} I_{\tilde{\boldsymbol{C}}}^5 + \cdots\Bigg) + \kappa \ln\sqrt{\mathit{III}_{\boldsymbol{C}}}\end{aligned} \tag{8.237}$$

The Elastic Tangent Stiffness Tensor

$$\boxed{\mathbb{C}_{ijkl}^{tan} = b_1 \delta_{ij}\delta_{kl} + b_2\left(\delta_{ij} C_{kl}^{-1} + C_{ij}^{-1}\delta_{kl}\right) + (b_4 + b_5) C_{ij}^{-1} C_{kl}^{-1} - \frac{a_2}{2}\left(C_{ik}^{-1} C_{lj}^{-1} + C_{il}^{-1} C_{kj}^{-1}\right)} \tag{8.238}$$

where the parameters are:

$$b_1 = 2\mu\left(\frac{1}{10N} + \frac{11}{175N^2} I_{\tilde{\boldsymbol{C}}} + \frac{57}{1750N^3} I_{\tilde{\boldsymbol{C}}}^2 + \frac{1038}{67375N^4} I_{\tilde{\boldsymbol{C}}}^3 + \cdots\right)\frac{1}{\sqrt[3]{\mathit{III}_{\boldsymbol{C}}}}\left(\frac{1}{\sqrt[3]{\mathit{III}_{\boldsymbol{C}}}}\right) \tag{8.239}$$

$$b_2 = -\frac{2}{3}\mu\left(\frac{1}{2} + \frac{1}{5N} I_{\tilde{C}} + \frac{33}{350N^2} I_{\tilde{C}}^2 + \frac{38}{875N^3} I_{\tilde{C}}^3 + \frac{519}{26950N^4} I_{\tilde{C}}^4 + \cdots\right)\frac{1}{\sqrt[3]{III_C}} \tag{8.240}$$

$$b_4 = \frac{2}{9}\mu\left(\frac{1}{2} I_{\tilde{C}} + \frac{1}{5N} I_{\tilde{C}}^2 + \frac{33}{350N^2} I_{\tilde{C}}^3 + \frac{38}{875N^3} I_{\tilde{C}}^4 + \frac{519}{26950N^4} I_{\tilde{C}}^5 + \cdots\right) \tag{8.241}$$

$$b_5 = \frac{\kappa}{2} \tag{8.242}$$

$$\begin{aligned} a_2 = -\frac{2}{3}\mu\Bigg(&\frac{1}{2} I_{\tilde{C}} + \frac{1}{10N} I_{\tilde{C}}^2 + \frac{11}{350N^2} I_{\tilde{C}}^3 + \frac{19}{1750N^3} I_{\tilde{C}}^4 + \\ &\frac{519}{134750N^4} I_{\tilde{C}}^5 + \cdots\Bigg) + \kappa \ln\sqrt{III_C} \end{aligned} \tag{8.243}$$

The demonstration of the statistical model can be found in Chaves (2009), (see also Sansour *et al.* (2003)).

8.6.11 The Eight-Parameter Model

We summarize this model as follows.

Strain Energy Density

$$\boxed{\Psi = \tilde{\Psi}(I_{\tilde{C}}, II_{\tilde{C}}) + \Psi^{vol}(III_C)} \tag{8.244}$$

where

$$\tilde{\Psi}(I_{\tilde{C}}, II_{\tilde{C}}) = \alpha_1 I_{\tilde{C}} + \alpha_2 II_{\tilde{C}} + \alpha_3 I_{\tilde{C}}^2 + \alpha_4 I_{\tilde{C}} II_{\tilde{C}} + \alpha_5 I_{\tilde{C}}^3 + \alpha_6 II_{\tilde{C}}^2 + \alpha_7 I_{\tilde{C}}^4 + \alpha_8 I_{\tilde{C}} II_{\tilde{C}}^2 \tag{8.245}$$

$$\Psi^{vol} = \alpha_9 \left(\ln\sqrt{III_C}\right)^2 \tag{8.246}$$

The parameters α_1, α_2, ..., α_8 are the eight material parameters whereas α_9 represents the bulk modulus. The values these parameters were determined by Sansour (1998) as:

$$\begin{array}{llll} \alpha_1 = 0.1796; & \alpha_2 = 0.0145; & \alpha_3 = -0.1684\times10^{-2}; & \alpha_4 = 0.3268\times10^{-3} \\ \alpha_5 = 0.3473\times10^{-4}; & \alpha_6 = -0.8439\times10^{-3}; & \alpha_7 = 0.432\times10^{-7}; & \alpha_8 = 0.5513\times10^{-5} \end{array} \tag{8.247}$$

The Second Piola-Kirchhoff Stress Tensor

$$\boxed{\mathbf{S} = a_0 \mathbf{1} - a_1 \boldsymbol{C} + a_2 \boldsymbol{C}^{-1}} \quad \textit{Stress tensor – Eight-parameter model} \tag{8.248}$$

where

$$\begin{aligned} a_0 = \frac{2}{\sqrt[3]{III_C}}\Big(&\alpha_1 + \alpha_2 I_{\tilde{C}} + 2\alpha_3 I_{\tilde{C}} + \alpha_4 II_{\tilde{C}} + \alpha_4 I_{\tilde{C}}^2 + 3\alpha_5 I_{\tilde{C}}^2 + \\ &+ 2\alpha_6 I_{\tilde{C}} II_{\tilde{C}} + 4\alpha_7 I_{\tilde{C}}^3 + \alpha_8 II_{\tilde{C}}^2 + 2\alpha_8 I_{\tilde{C}}^2 II_{\tilde{C}}\Big) \end{aligned} \tag{8.249}$$

$$a_1 = 2\left(\alpha_2 + \alpha_4 I_{\tilde{C}} + 2\alpha_6 II_{\tilde{C}} + 2\alpha_8 I_{\tilde{C}} II_{\tilde{C}}\right)\frac{1}{\sqrt[3]{III_C^2}} \tag{8.250}$$

$$\begin{aligned} a_2 = -\frac{2}{3}\Big(&\alpha_1 I_{\tilde{C}} + 2\alpha_3 I_{\tilde{C}}^2 + 3\alpha_4 I_{\tilde{C}} II_{\tilde{C}} + 3\alpha_5 I_{\tilde{C}}^3 + 4\alpha_7 I_{\tilde{C}}^4 + 5\alpha_8 I_{\tilde{C}} II_{\tilde{C}}^2 \\ &+ 2\alpha_2 II_{\tilde{C}} + 4\alpha_6 II_{\tilde{C}}^2 - 3\alpha_9 \ln(III_C)\Big) \end{aligned} \tag{8.251}$$

The Elastic Tangent Stiffness Tensor

$$\boxed{\begin{aligned}\mathbb{C}^{tan}_{ijkl} &= b_0\delta_{ij}\delta_{kl} - b_1\left(\delta_{ij}C_{kl} + C_{ij}\delta_{kl}\right) + b_2\left(\delta_{ij}C^{-1}_{kl} + C^{-1}_{ij}\delta_{kl}\right) + b_4 C_{ij}C_{kl} \\ &\quad - \frac{a_1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right) - b_5\left(C^{-1}_{ij}C_{kl} + C_{ij}C^{-1}_{kl}\right) + b_8 C^{-1}_{ij}C^{-1}_{kl} - \frac{a_2}{2}\left(C^{-1}_{ik}C^{-1}_{lj} + C^{-1}_{il}C^{-1}_{kj}\right)\end{aligned}} \tag{8.252}$$

where

$$\begin{aligned} b_0 = \frac{2}{\sqrt[3]{III^2_{C}}}\Big(&\alpha_2 + 2\alpha_3 + 3\alpha_4 I_{\tilde{C}} + 6\alpha_5 I_{\tilde{C}} + 2\alpha_6\, II_{\tilde{C}} + 2\alpha_6 I^2_{\tilde{C}} + \\ &12\alpha_7 I^2_{\tilde{C}} + 6\alpha_8 I_{\tilde{C}}\, II_{\tilde{C}} + 2\alpha_8 I^3_{\tilde{C}}\Big)\end{aligned} \tag{8.253}$$

$$b_1 = \frac{2}{III_C}\left(\alpha_4 + 2\alpha_6 I_{\tilde{C}} + 2\alpha_8\, II_{\tilde{C}} + 2\alpha_8 I^2_{\tilde{C}}\right) \tag{8.254}$$

$$\begin{aligned} b_2 = -\frac{2}{3}\frac{1}{\sqrt[3]{III_C}}\Big(&\alpha_1 + 2\alpha_2 I_{\tilde{C}} + 4\alpha_3 I_{\tilde{C}} + 3\alpha_4 I^2_{\tilde{C}} + 3\alpha_4\, II_{\tilde{C}} + 9\alpha_5 I^2_{\tilde{C}} + 8\alpha_6 I_{\tilde{C}}\, II_{\tilde{C}} + \\ &+16\alpha_7 I^3_{\tilde{C}} + 5\alpha_8\, II^2_{\tilde{C}} + 10\alpha_8 I^2_{\tilde{C}}\, II_{\tilde{C}}\Big)\end{aligned} \tag{8.255}$$

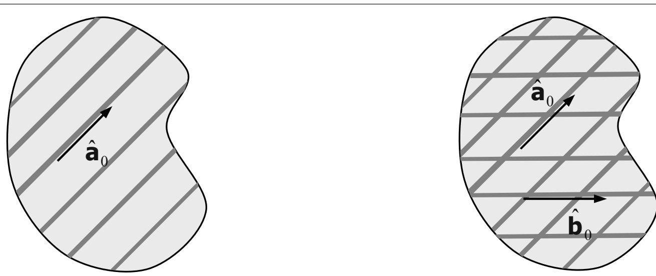
$$b_4 = \frac{2}{\sqrt[3]{III^4_C}}\left(2\alpha_6 + 2\alpha_8 I_{\tilde{C}}\right), \quad b_5 = -\frac{2}{3}\frac{1}{\sqrt[3]{III^2_C}}\left(2\alpha_2 + 3\alpha_4 I_{\tilde{C}} + 8\alpha_6\, II_{\tilde{C}} + 10\alpha_8 I_{\tilde{C}}\, II_{\tilde{C}}\right) \tag{8.256}$$

$$\begin{aligned} b_8 = \frac{2}{9}\Big(&\alpha_1 I_{\tilde{C}} + 4\alpha_2\, II_{\tilde{C}} + 4\alpha_3 I^2_{\tilde{C}} + 9\alpha_4 I_{\tilde{C}}\, II_{\tilde{C}} + 9\alpha_5 I^3_{\tilde{C}} + 16\alpha_6\, II^2_{\tilde{C}} + \\ &16\alpha_7 I^4_{\tilde{C}} + 25\alpha_8 I_{\tilde{C}}\, II^2_{\tilde{C}} + 9\alpha_9\Big)\end{aligned} \tag{8.257}$$

The demonstration of the eight-parameter model can be found in Chaves (2009), (see also Sansour *et al.* (2003)).

8.7 Anisotropic Hyperelasticity

Certain materials such as some biological tissues have fibers, and therefore lose their isotropy. When these fibers are arranged according to a preferential direction, $\hat{\mathbf{a}}_0$, we can approach this material by means of the transversely isotropic material. In other material such as heart tissue, the fibers are arranged according to two preferential directions, and are classified as tissue with two families of fibers, (see Figure 8.8). In this subsection we just describe the model for one family of fibers. Details about two families of fibers can be found in Holzapfel (2000).



b) One family of fibers *b) Two families of fibers*

Figure 8.8: Materials with fibers.

8.7.1 Transversely Isotropic Material

As discussed in Chapter 1, the scalar-valued tensor function $\varPsi = \varPsi(\boldsymbol{C})$ can be written in terms of the principal invariants of $\boldsymbol{C}$, *i.e.* $\varPsi = \varPsi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}})$. Now, if the $\varPsi$-arguments are $\boldsymbol{C}$ and the vector $\hat{\mathbf{a}}_0$, *i.e.* $\varPsi(\boldsymbol{C}, \hat{\mathbf{a}}_0)$, it can be shown that this function can be written in terms of the following invariants:

$$\varPsi(\boldsymbol{C}, \hat{\mathbf{a}}_0) = \varPsi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}}, \hat{\mathbf{a}}_0 \cdot \boldsymbol{C} \cdot \hat{\mathbf{a}}_0, \hat{\mathbf{a}}_0 \cdot \boldsymbol{C}^2 \cdot \hat{\mathbf{a}}_0) = \varPsi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}}, I_{\boldsymbol{C}}^{(4)}, I_{\boldsymbol{C}}^{(5)}) \tag{8.258}$$

where $I_{\boldsymbol{C}}^{(4)}$ and $I_{\boldsymbol{C}}^{(5)}$ are the pseudo-invariants of anisotropy. Moreover, if we consider that the energy, if independent of the sense of the vector $\hat{\mathbf{a}}_0$, fulfills that $\varPsi(\boldsymbol{C}, \hat{\mathbf{a}}_0) = \varPsi(\boldsymbol{C}, -\hat{\mathbf{a}}_0)$, we can then represent the strain energy function by:

$$\varPsi = \varPsi(\boldsymbol{C}, \hat{\mathbf{a}}_0 \otimes \hat{\mathbf{a}}_0) \tag{8.259}$$

We can also show that the previous function is objective as follows:

$$\varPsi(\boldsymbol{C}, \hat{\mathbf{a}}_0 \otimes \hat{\mathbf{a}}_0) = \varPsi(\mathbf{Q} \cdot \boldsymbol{C} \cdot \mathbf{Q}^T, \mathbf{Q} \cdot \hat{\mathbf{a}}_0 \otimes \hat{\mathbf{a}}_0 \cdot \mathbf{Q}^T) = \varPsi(\mathbf{Q} \cdot \boldsymbol{C} \cdot \mathbf{Q}^T, \mathbf{Q} \cdot \hat{\mathbf{a}}_0 \otimes \mathbf{Q} \cdot \hat{\mathbf{a}}_0) \tag{8.260}$$

As the rotated current configuration is defined by the transformation $\boldsymbol{F}^* = \mathbf{Q} \cdot \boldsymbol{F}$, (see Chapter 4), then $\boldsymbol{C}^* = \boldsymbol{F}^{*T} \cdot \boldsymbol{F}^* = \boldsymbol{C}$ is fulfilled, and a vector in the rotated current configuration is given by $\hat{\mathbf{a}}_0^* = \mathbf{Q} \cdot \hat{\mathbf{a}}_0$, thus:

$$\varPsi(\boldsymbol{C}, \hat{\mathbf{a}}_0 \otimes \hat{\mathbf{a}}_0) = \varPsi(\mathbf{Q} \cdot \boldsymbol{C} \cdot \mathbf{Q}^T, \mathbf{Q} \cdot \hat{\mathbf{a}}_0 \otimes \mathbf{Q} \cdot \hat{\mathbf{a}}_0) = \varPsi(\boldsymbol{C}^*, \hat{\mathbf{a}}_0^* \otimes \hat{\mathbf{a}}_0^*) \tag{8.261}$$

which proves objectivity.

Then, since the material has lost its isotropy, we can not express the energy function just in terms of the principal invariants. To use the energy function in terms of invariants, Spencer (1984) obtained two pseudo-invariants of anisotropy, which contribute to the energy function, (see Holzapfel (2000)), and are given by:

$$\begin{aligned} I_{\boldsymbol{C}}^{(4)} &= \hat{\mathbf{a}}_0 \cdot \boldsymbol{C} \cdot \hat{\mathbf{a}}_0 \quad &;\quad I_{\boldsymbol{C}}^{(5)} &= \hat{\mathbf{a}}_0 \cdot \boldsymbol{C}^2 \cdot \hat{\mathbf{a}}_0 \\ I_{\boldsymbol{C}}^{(4)} &= \hat{a}_{0i} C_{ij} \hat{a}_{0j} \quad &;\quad I_{\boldsymbol{C}}^{(5)} &= \hat{a}_{0i} C_{ip} C_{pj} \hat{a}_{0j} \end{aligned} \tag{8.262}$$

where $I_{\boldsymbol{C}}^{(4)} = \hat{\mathbf{a}}_0 \cdot \boldsymbol{C} \cdot \hat{\mathbf{a}}_0 = \lambda_{\hat{\mathbf{a}}_0}^2$, and $\lambda_{\hat{\mathbf{a}}_0}$ is the stretch according to the $\hat{\mathbf{a}}_0$-direction (fiber stretching).

So, the energy function can be expressed as follows:

$$\varPsi = \varPsi(I_{\boldsymbol{C}}, II_{\boldsymbol{C}}, III_{\boldsymbol{C}}, I_{\boldsymbol{C}}^{(4)}, I_{\boldsymbol{C}}^{(5)}) \tag{8.263}$$

Now, the derivatives of $I_{\boldsymbol{C}}^{(4)}$ and $I_{\boldsymbol{C}}^{(5)}$ with respect to the tensor $\boldsymbol{C}$ are given by:

$$\begin{aligned} \frac{\partial I_{\boldsymbol{C}}^{(4)}}{\partial C_{kl}} &= \frac{\partial}{\partial C_{kl}}\left(\hat{a}_{0i} C_{ij} \hat{a}_{0j}\right) = \hat{a}_{0i}\hat{a}_{0j}\frac{\partial C_{ij}}{\partial C_{kl}} = \hat{a}_{0i}\hat{a}_{0j}\delta_{ik}\delta_{jl} = \hat{a}_{0k}\hat{a}_{0l} \\ \frac{\partial I_{\boldsymbol{C}}^{(5)}}{\partial C_{kl}} &= \frac{\partial}{\partial C_{kl}}\left(\hat{a}_{0i} C_{ip} C_{pj} \hat{a}_{0j}\right) = \hat{a}_{0i}\hat{a}_{0j} C_{pj}\frac{\partial C_{ip}}{\partial C_{kl}} + \hat{a}_{0i} C_{ip}\hat{a}_{0j}\frac{\partial C_{pj}}{\partial C_{kl}} \\ &= \hat{a}_{0i}\hat{a}_{0j} C_{pj}\delta_{ik}\delta_{pl} + \hat{a}_{0i} C_{ip}\hat{a}_{0j}\delta_{pk}\delta_{jl} = \hat{a}_{0k}\hat{a}_{0j} C_{lj} + \hat{a}_{0i} C_{ik}\hat{a}_{0l} \end{aligned} \tag{8.264}$$

and such equations in tensorial notation become:

$$\frac{\partial I_{\boldsymbol{C}}^{(4)}}{\partial \boldsymbol{C}} = \hat{\mathbf{a}}_0 \otimes \hat{\mathbf{a}}_0 \quad ; \quad \frac{\partial I_{\boldsymbol{C}}^{(5)}}{\partial \boldsymbol{C}} = \hat{\mathbf{a}}_0 \otimes \left(\boldsymbol{C} \cdot \hat{\mathbf{a}}_0\right) + \left(\hat{\mathbf{a}}_0 \cdot \boldsymbol{C}\right) \otimes \hat{\mathbf{a}}_0 \tag{8.265}$$

Then, the constitutive equation for stress (reference configuration) can be represented by:

$$\mathbf{S}=2\frac{\partial\Psi(I_{\boldsymbol{C}},II_{\boldsymbol{C}},III_{\boldsymbol{C}},I_{\boldsymbol{C}}^{(4)},I_{\boldsymbol{C}}^{(5)})}{\partial\boldsymbol{C}}$$
$$=2\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\frac{\partial I_{\boldsymbol{C}}}{\partial\boldsymbol{C}}+\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}\frac{\partial II_{\boldsymbol{C}}}{\partial\boldsymbol{C}}+\frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}\frac{\partial III_{\boldsymbol{C}}}{\partial\boldsymbol{C}}+\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}^{(4)}}\frac{\partial I_{\boldsymbol{C}}^{(4)}}{\partial\boldsymbol{C}}+\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\frac{\partial I_{\boldsymbol{C}}^{(5)}}{\partial\boldsymbol{C}}\right) \tag{8.266}$$

Next, if we consider the derivatives in (8.264) and the equation in (8.67) we obtain:

$$\mathbf{S}=2\Bigg[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\right)\mathbf{1}+\left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}II_{\boldsymbol{C}}+\frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}-\left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-2}+$$
$$\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}^{(4)}}\hat{\mathbf{a}}_0\otimes\hat{\mathbf{a}}_0+\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}^{(5)}}\left(\hat{\mathbf{a}}_0\otimes(\boldsymbol{C}\cdot\hat{\mathbf{a}}_0)+(\hat{\mathbf{a}}_0\cdot\boldsymbol{C})\otimes\hat{\mathbf{a}}_0\right)\Bigg] \tag{8.267}$$

The second derivative of $I_{\boldsymbol{C}}^{(4)}$ is:

$$\frac{\partial^2 I_{\boldsymbol{C}}^{(4)}}{\partial\boldsymbol{C}\otimes\partial\boldsymbol{C}}=\mathbf{0}\qquad\qquad\frac{\partial^2 I_{\boldsymbol{C}}^{(4)}}{\partial C_{ij}\partial C_{kl}}=0_{ijkl} \tag{8.268}$$

Note that the second derivative of $I_{\boldsymbol{C}}^{(5)}$ becomes a fourth-order tensor that features both major and minor symmetry:

$$\frac{\partial^2 I_{\boldsymbol{C}}^{(5)}}{\partial\boldsymbol{C}\otimes\partial\boldsymbol{C}}=\left(\hat{\mathbf{a}}_0\otimes\hat{\mathbf{a}}_0\right)\otimes\mathbf{1}\qquad\qquad\frac{\partial^2 I_{\boldsymbol{C}}^{(5)}}{\partial C_{ij}\partial C_{kl}}=\hat{\mathrm{a}}_{0i}\hat{\mathrm{a}}_{0j}\delta_{kl} \tag{8.269}$$

If we now consider the directions of the fibers in the current configuration (deformed configuration), $\hat{\mathbf{a}}$, and using the expression of the Kirchhoff stress tensor, (8.71), we obtain:

$$\boldsymbol{\tau}=J\boldsymbol{\sigma}=\boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T$$
$$=2\boldsymbol{F}\cdot\Bigg[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}}\right)\mathbf{1}+\left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}II_{\boldsymbol{C}}+\frac{\partial\Psi}{\partial III_{\boldsymbol{C}}}III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-1}-\left(\frac{\partial\Psi}{\partial II_{\boldsymbol{C}}}III_{\boldsymbol{C}}\right)\boldsymbol{C}^{-2}+$$
$$+\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}^{(4)}}\hat{\mathbf{a}}_0\otimes\hat{\mathbf{a}}_0+\frac{\partial\Psi}{\partial I_{\boldsymbol{C}}^{(5)}}\left(\hat{\mathbf{a}}_0\otimes(\boldsymbol{C}\cdot\hat{\mathbf{a}}_0)+(\hat{\mathbf{a}}_0\cdot\boldsymbol{C})\otimes\hat{\mathbf{a}}_0\right)\Bigg]\cdot\boldsymbol{F}^T \tag{8.270}$$

Then, if we consider that $\boldsymbol{C}=\boldsymbol{F}^T\cdot\boldsymbol{F}$, $\boldsymbol{b}=\boldsymbol{F}\cdot\boldsymbol{F}^T$ and $\boldsymbol{F}\cdot\hat{\mathbf{a}}_0=\lambda_{\hat{\mathbf{a}}_0}\hat{\mathbf{a}}$, the above equation then becomes:

$$\boldsymbol{\tau}=J\boldsymbol{\sigma}=2\Bigg[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}}+\frac{\partial\Psi}{\partial II_{\boldsymbol{b}}}I_{\boldsymbol{b}}\right)\boldsymbol{b}-\frac{\partial\Psi}{\partial II_{\boldsymbol{b}}}\boldsymbol{b}^2+\frac{\partial\Psi}{\partial III_{\boldsymbol{b}}}III_{\boldsymbol{b}}\mathbf{1}+$$
$$+\lambda_{\hat{\mathbf{a}}_0}^2\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}^{(4)}}\hat{\mathbf{a}}\otimes\hat{\mathbf{a}}+\lambda_{\hat{\mathbf{a}}_0}^2\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}^{(5)}}\left(\hat{\mathbf{a}}\otimes(\boldsymbol{b}\cdot\hat{\mathbf{a}})+(\hat{\mathbf{a}}\cdot\boldsymbol{b})\otimes\hat{\mathbf{a}}\right)\Bigg] \tag{8.271}$$

Moreover, by considering that $I_{\boldsymbol{C}}^{(4)}=\hat{\mathbf{a}}_0\cdot\boldsymbol{C}\cdot\hat{\mathbf{a}}_0=\lambda_{\hat{\mathbf{a}}_0}^2$, we obtain:

$$\boldsymbol{\tau}=J\boldsymbol{\sigma}=2\Bigg[\left(\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}}+\frac{\partial\Psi}{\partial II_{\boldsymbol{b}}}I_{\boldsymbol{b}}\right)\boldsymbol{b}-\frac{\partial\Psi}{\partial II_{\boldsymbol{b}}}\boldsymbol{b}^2+\frac{\partial\Psi}{\partial III_{\boldsymbol{b}}}III_{\boldsymbol{b}}\mathbf{1}+$$
$$+I_{\boldsymbol{C}}^{(4)}\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}^{(4)}}\hat{\mathbf{a}}\otimes\hat{\mathbf{a}}+I_{\boldsymbol{C}}^{(4)}\frac{\partial\Psi}{\partial I_{\boldsymbol{b}}^{(5)}}\left(\hat{\mathbf{a}}\otimes(\boldsymbol{b}\cdot\hat{\mathbf{a}})+(\hat{\mathbf{a}}\cdot\boldsymbol{b})\otimes\hat{\mathbf{a}}\right)\Bigg] \tag{8.272}$$

9 Plasticity

9.1 Introduction

With elastic loading, atomic structures are not affected, which is typical of processes in which there is no internal energy dissipation. Then, once the load has been removed, the solid returns to its initial state. In certain types of materials, if we keep loading, a level will be reached in which the atoms begins to restructure (dislocation at the atomic level), so, in this way, we have internal energy dissipation (an irreversible process). Most of the dissipated energy will be used to increase the temperature (heat release), and as a result there will be an increase in system disorder, *i.e.* a rise in entropy. A rise in temperature also involves the dilation phenomenon. At the macroscopic level, in ductile materials, this atomic restructuring is characterized by permanent deformation (plastic strain). That is, if the material which has been internally restructured is completely unloaded, it can be observed that part of the total deformation is regained. This recoverable part is characterized by the *elastic strain*, and the permanent deformation by the *plastic strain*, (see Figure 9.1), and the constitutive models devised to represent this phenomenon are called *"plasticity models"* or *"elastoplastic models"*.

It can be complex to formulate a constitutive model that considers all possible phenomena that occur during the plasticity process. In general, a process which involves plastic strain typically involves a large deformation, heat production, and the loss of material isotropy in the plastic zone due to plastic fibers which are formed in this area. However, in certain types of materials, the effect of temperature can be discarded (*isothermal process*), and they can also be subjected to a deformation state in which the elastic strain is very small when compared with the plastic strain (*small deformation regime*). These simplifications have given rise to the classical Theory of Plasticity.

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_11,

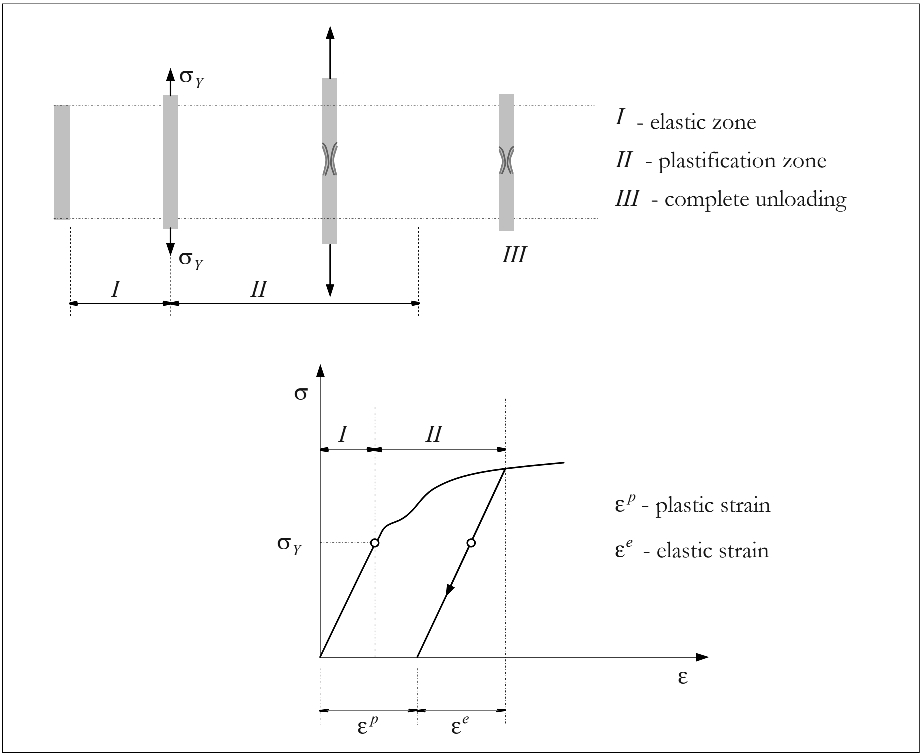


Figure 9.1: Tensile testing – Plastic behavior.

There were many researchers who endorsed this theory of plasticity, among whom we can cite: Rankine(1851), Tresca(1864), von Mises(1913), Prandtl(1924), Reuss(1930), Prager(1945), Hill(1950), Drucker(1950), Koiter(1953), Ziegler(1959), Naghdi(1960), Mroz(1967).

From a kinematics standpoint, the plasticity theory has been developed considering:

- Plasticity with small deformation (infinitesimal strain):
  - ✓ Without the effect of temperature (Classical theory of plasticity);
  - With the effect of temperature (Thermoplasticity in infinitesimal strain).
- Plasticity with large deformation (Finite strain):
  - ✓ Without the effect of temperature (Plasticity in finite strain);
  - With the effect of temperature (Thermoplasticity in finite strain).

In this chapter our approach will be to use plasticity models taking into account the small and large (finite) deformation regime, (see Figure 9.2), and we will omit the temperature effect (isothermal process).

However, before formulating these models, we will introduce some concepts that will be useful in the development of this chapter.

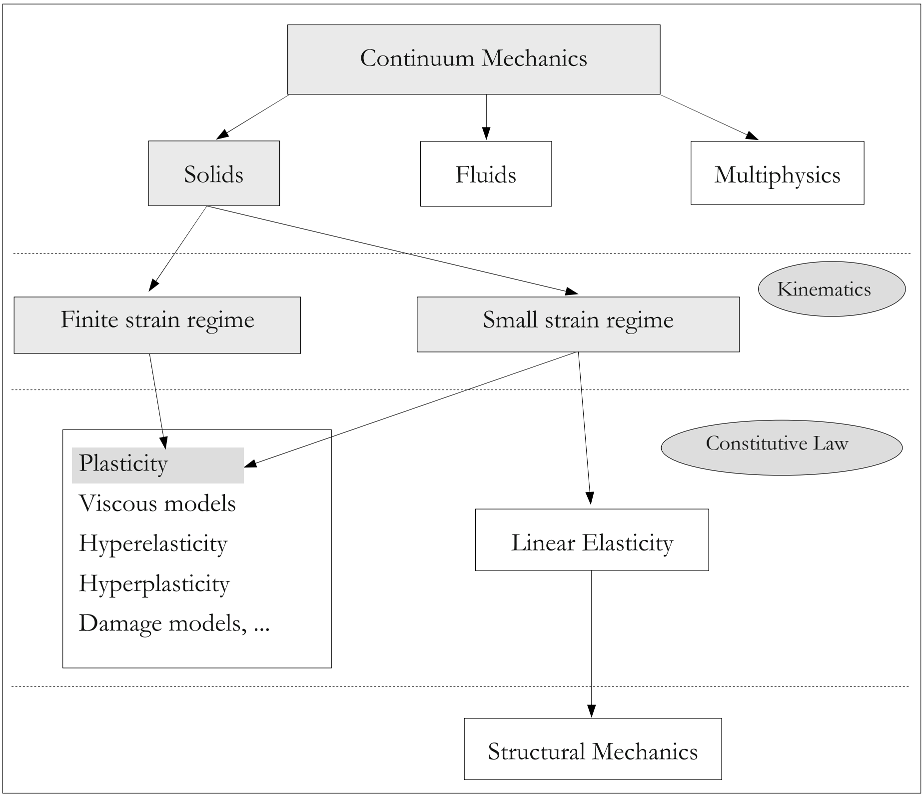


Figure 9.2: Overview of solids mechanics.

9.2 The Yield Criterion

An important concept in the classical plasticity theory (rate independent) is the concept of the yield surface, which defines a multiaxial stress state at the threshold of plastic strain. If the current stress state lies within the yield surface, the corresponding mechanical change is purely elastic. Plastic strain is only possible when the stress state is on the yield surface. Now, firstly, we will explain what this initial yield surface (yield criterion) is, and then we will establish how it evolves during plastification.

As discussed in subsection 6.5 in Chapter 6 related to the tensile testing, (see Figure 9.1), certain materials can exhibit two zones: an elastic zone, in which the upper limit stress is characterized by σ_Y ; and a plastic zone. Generally speaking in three dimensions cases, if we include the six independent components of the stress tensor as independent coordinate axes, the current stress state is defined by a point in the six-dimensional space (hyperspace). So, we take the infinite possibilities there are for the stress state to trigger plastification at a material point in order to define a hypersurface, which is known as the *yield hypersurface*. Fundamentally, we can state that the yield surface separates the elastic and plastic domains. Therefore, if the stress state is inside the region delimited by the yield surface, the corresponding strain change is purely elastic.

Metaphorically speaking, we can state that the constitutive equation reflects the "personality" of the material, *i.e.* each material (or kind of material) has its own yield surface and hence the concept of yield criterion appears. That is, the *yield criterion* establishes when certain materials start to plastify.

9.2.1 The Yield Surface for Anisotropic Materials

Let us consider a homogeneous material undergoing a purely mechanical process. Then, the yield surface, or the yield function, can be described mathematically as follows:

$$\mathcal{F}(\sigma_{ij}) = 0 \quad ; \quad \mathcal{F}(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13},) = 0 \tag{9.1}$$

which represent the hypersurface equation in the stress space (six dimensions), in which:

$$\begin{aligned} &\mathcal{F}(\sigma_{ij}) < 0 \qquad \text{elastic domain} \\ &\mathcal{F}(\sigma_{ij}) = 0 \qquad \text{plastic domain} \end{aligned} \tag{9.2}$$

Alternatively, the equation in (9.1) can be rewritten in terms of the eigenvalues (σ_a) and eigenvectors ($\hat{\boldsymbol{n}}^{(a)}$) of $\boldsymbol{\sigma}$, *i.e.*:

$$\mathcal{F}(\sigma_1, \sigma_2, \sigma_3, \hat{\boldsymbol{n}}^{(1)}, \hat{\boldsymbol{n}}^{(2)}, \hat{\boldsymbol{n}}^{(3)}) = 0 \tag{9.3}$$

As expected, determining the yield criterion for anisotropic material is rather difficult to achieve. Moreover, because of anisotropy, we need to establish 21 independent constants in the laboratory (mechanical properties) in order to fully describe elastic material behavior.

9.2.1.1 The Yield Surface Gradient

The yield surface gradient in the stress space is given as follows:

$$\mathbf{n} \equiv \nabla_{\boldsymbol{\sigma}} \mathcal{F} \equiv \frac{\partial \mathcal{F}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} \xrightarrow{\text{components}} \mathrm{n}_{ij} = \frac{\partial \mathcal{F}(\boldsymbol{\sigma})}{\partial \sigma_{ij}} \tag{9.4}$$

where $\mathbf{n}$ is a symmetric second-order tensor or the *plastic flow tensor*, with six independent components as well as being a coaxial tensor with $\boldsymbol{\sigma}$.

Then, the Frobenius norm of the tensor $\mathbf{n}$ is defined as follows:

$$\hat{\mathbf{n}} = \frac{\nabla_{\boldsymbol{\sigma}} \mathcal{F}}{\left\| \nabla_{\boldsymbol{\sigma}} \mathcal{F} \right\|} \qquad \textit{with} \qquad \left\| \nabla_{\boldsymbol{\sigma}} \mathcal{F} \right\| = \sqrt{\nabla_{\boldsymbol{\sigma}} \mathcal{F} : \nabla_{\boldsymbol{\sigma}} \mathcal{F}} = \sqrt{\mathbf{n} : \mathbf{n}} \tag{9.5}$$

which thus shows that the Frobenius norm ($\hat{\mathbf{n}}$) is unitary, *i.e.* $\left\| \hat{\mathbf{n}} \right\| = \sqrt{\hat{\mathbf{n}} : \hat{\mathbf{n}}} = 1$.

In isotropic materials the yield surface is only dependent on the $\boldsymbol{\sigma}$-eigenvalues, hence, it can be shown in the space defined by the principal stresses (three dimensions), so the plastic flow becomes a vector, and because of this some researchers refer to $\mathbf{n}$ as the *flow plastic vector*.

9.2.2 The Yield Surface for Isotropic Materials

As we saw in Chapter 1, the scalar-valued isotropic tensor function, $\mathcal{F}(\boldsymbol{\sigma})$, can be represented only by the eigenvalues of the arguments, or in terms of the principal invariants of the argument $\boldsymbol{\sigma}$. From the above, we can draw the conclusion that for isotropic materials the beginning of plastification does not depend on the principal directions (eigenvectors) of the Cauchy stress tensor, thus:

$$\boxed{\begin{aligned}&\mathcal{F}(\sigma_1,\sigma_2,\sigma_3)=0\\&\mathcal{F}(I_{\boldsymbol{\sigma}},II_{\boldsymbol{\sigma}},III_{\boldsymbol{\sigma}})=0\end{aligned}} \quad \textit{Yield surface for isotropic materials} \tag{9.6}$$

where $I_{\boldsymbol{\sigma}}$, $II_{\boldsymbol{\sigma}}$, $III_{\boldsymbol{\sigma}}$ are the Cauchy stress tensor principal invariants, whose values are given by $I_{\boldsymbol{\sigma}}=\mathrm{Tr}(\boldsymbol{\sigma})$, $II_{\boldsymbol{\sigma}}=\frac{1}{2}\left\{[\mathrm{Tr}(\boldsymbol{\sigma})]^2-\mathrm{Tr}(\boldsymbol{\sigma}^2)\right\}$, $III_{\boldsymbol{\sigma}}=\det(\boldsymbol{\sigma})$. Note that the equations in (9.6) represent the yield surface in the space defined by the Cauchy stress tensor eigenvectors. As discussed in the chapter on stress, (see **Problem 3.3**), the principal invariants are related to the invariants of the deviatoric part of $\boldsymbol{\sigma}$ as follows:

$$\mathrm{J}_2=-\frac{1}{3}\left(3\,II_{\boldsymbol{\sigma}}-I_{\boldsymbol{\sigma}}^2\right) \quad ; \quad \mathrm{J}_3=\frac{1}{27}\left(2I_{\boldsymbol{\sigma}}^3-9I_{\boldsymbol{\sigma}}\,II_{\boldsymbol{\sigma}}+27\,III_{\boldsymbol{\sigma}}\right) \tag{9.7}$$

where $\mathrm{J}_1=I_{\boldsymbol{\sigma}^{dev}}=0$, $\mathrm{J}_2=-II_{\boldsymbol{\sigma}^{dev}}=\frac{1}{2}\boldsymbol{\sigma}^{dev}:\boldsymbol{\sigma}^{dev}$, $\mathrm{J}_3=III_{\boldsymbol{\sigma}^{dev}}=\det(\boldsymbol{\sigma}^{dev})$, with which the yield surface for isotropic materials can still be represented as:

$$\boxed{\mathcal{F}(I_{\boldsymbol{\sigma}},\mathrm{J}_2,\mathrm{J}_3)=0} \quad \textit{Yield surface for isotropic materials} \tag{9.8}$$

Now, in certain kinds of materials such as non-porous metals, experimental evidence suggests that hydrostatic pressure does not influence the initiation of plastification, that is, the yield surface is independent of the first principal invariant $I_{\boldsymbol{\sigma}}$, so it is a function of the deviatoric part of the stress tensor, $\boldsymbol{\sigma}^{dev}\equiv\mathbf{s}$:

$$\boxed{\mathcal{F}(\mathrm{J}_2,\mathrm{J}_3)=0} \quad \textit{Yield surface for isotropic materials independent of hydrostatic pressure} \tag{9.9}$$

In this case, the yield surface is represented by a prism in the principal stress space, (see Figure 9.3).

NOTE: Before starting to study the yield criteria, a review of subsection *A.4 Graphical Representation of the Spherical and Deviatoric Parts* in Appendix A is recommended. ■

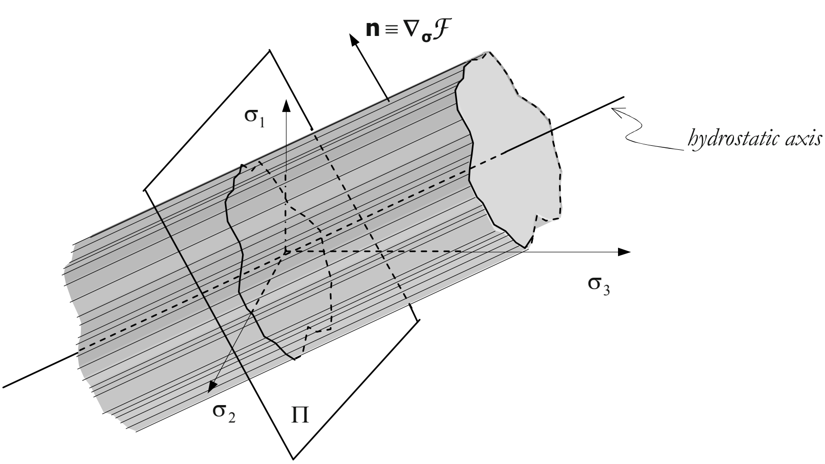


Figure 9.3: Yield surface for isotropic materials independent of pressure.

In Appendix A we saw that a stress state in the principal stress space (Haigh-Westergaard stress space) can be represented by means of three variables (p, q, θ) called the Haigh-Westergaard coordinates, (see Figure 9.4). Thus, the yield surface for isotropic materials can also be shown as:

$$\boxed{\mathcal{F}(p,q,\theta)=0} \quad \textit{Yield surface for isotropic materials} \tag{9.10}$$

and for materials independent of pressure as:

$$\boxed{\mathcal{F}(q,\theta)=0} \quad \textit{Yield surface for isotropic materials independent of pressure} \tag{9.11}$$

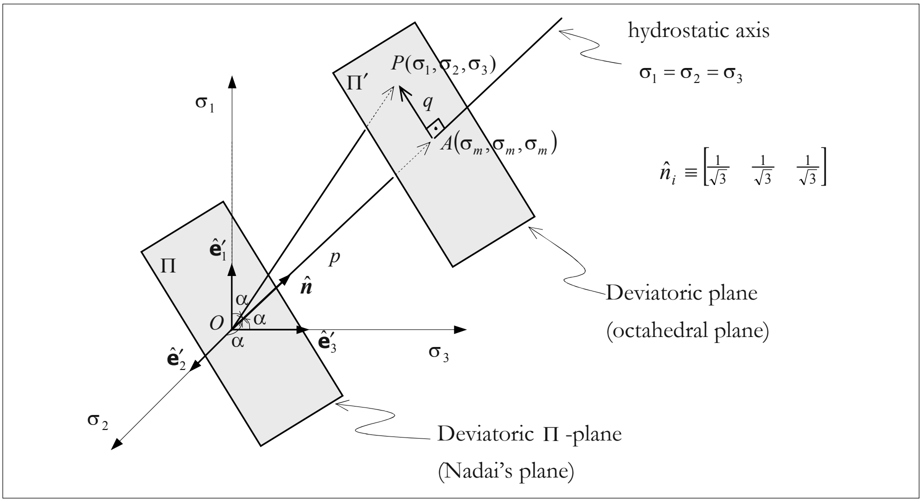


Figure 9.4: Haigh-Westergaard stress space.

By referring to (9.11), it is sufficient to represent the yield surface projected onto Π-plane (deviatoric plane that passes through the origin, also called Nadai's plane), (see Appendix A). The curve in the deviatoric plane is called the *yield curve*. As illustrated in Figure 9.5, the yield curve has triple symmetry and because of this one need only analyze the sector $0 \le \theta \le \frac{\pi}{3}$.

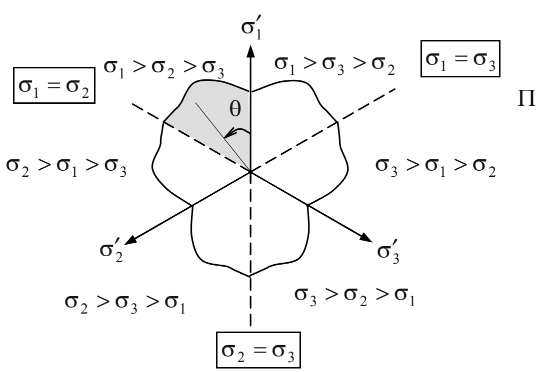


Figure 9.5: Yield surface projected onto a Π -plane.

9.2.3 The Yield Surface for Materials Independent of Pressure

In the previous section we saw that, generally speaking, for materials in which the yield surface is independent of the hydrostatic pressure, the former has a prismatic shape in which the prismatic and hydrostatic axes coincide. The yield surface cross section is then defined depending on the model adopted. In this subsection, we will introduce some models developed to represent material plastification where the hydrostatic pressure has no influence on plastification, (see Chen&Han(1988)).

9.2.3.1 The von Mises Yield Criterion

The yield criterion of von Mises (1913) assumes that plastification occurs when the second invariant of the deviatoric tensor, J_2, reaches a critical value k^2. We can define this plastification criterion as:

$$\begin{aligned} &\mathsf{J}_2 - \mathsf{k}^2 < 0 \qquad \text{elastic domain} \\ &\mathsf{J}_2 - \mathsf{k}^2 = 0 \qquad \text{plastic domain} \end{aligned} \tag{9.12}$$

where k is a material property (yield stress in pure shear). Remember that in Chapter 1 (see also **Problem 3.3**), the second invariant J_2 can be written in terms of the Cauchy stress tensor components. Thus, the yield surface for von Mises criterion can be written as:

$$\underbrace{\frac{1}{6}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{11}-\sigma_{33})^2\right]+\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2}_{\mathsf{J}_2}=\mathsf{k}^2 \tag{9.13}$$

or even in terms of the principal stresses:

$$\begin{aligned} &\frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]=\mathsf{k}^2 \\ &\frac{1}{3}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3\right)=\mathsf{k}^2 \end{aligned} \tag{9.14}$$

The von Mises surface is defined by a cylinder, (see Figure 9.6), which prismatic axis is parallel to hydrostatic axis, and does not depend on $I_{\boldsymbol{\sigma}}$ nor θ. We can obtain the cylinder radius by $r = q = \sqrt{2\mathsf{J}_2} = \sqrt{2}\,\mathsf{k}$.

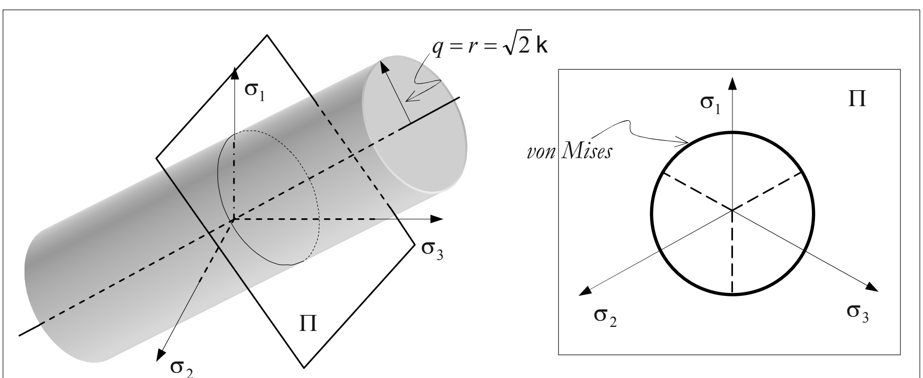


Figure 9.6: Yield surface for von Mises yield criterion (independent of pressure).

The parameter k is obtained by means of tensile testing, where it holds that $\sigma_1 = \sigma_Y$ and $\sigma_2 = \sigma_3 = 0$. Under these conditions the equation in (9.14) becomes:

$$\sigma_Y^2 = 3\mathsf{k}^2 \quad \Rightarrow \quad \mathsf{k} = \frac{\sigma_Y}{\sqrt{3}} \tag{9.15}$$

The strain energy density (the energy per unit volume) is given by:

$$\begin{aligned}
\Psi &= \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \\
&= \frac{1}{2}\left(\sigma_{ij}^{dev} + \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(\varepsilon_{ij}^{dev} + \frac{1}{3}\varepsilon_{pp}\delta_{ij}\right) \\
&= \frac{1}{2}\sigma_{ij}^{dev}\varepsilon_{ij}^{dev} + \frac{1}{6}\sigma_{ij}^{dev}\varepsilon_{pp}\delta_{ij} + \frac{1}{6}\sigma_{kk}\delta_{ij}\varepsilon_{ij}^{dev} + \frac{1}{18}\sigma_{kk}\delta_{ij}\varepsilon_{pp}\delta_{ij} \\
&= \frac{1}{2}\sigma_{ij}^{dev}\varepsilon_{ij}^{dev} + \frac{1}{6}\underbrace{\sigma_{ii}^{dev}}_{=0}\varepsilon_{pp} + \frac{1}{6}\sigma_{kk}\underbrace{\varepsilon_{ii}^{dev}}_{=0} + \frac{1}{18}\sigma_{kk}\underbrace{\delta_{ii}}_{=3}\varepsilon_{pp} \\
&= \frac{1}{2}\sigma_{ij}^{dev}\varepsilon_{ij}^{dev} + \frac{1}{6}\sigma_{kk}\varepsilon_{pp} = \frac{1}{2}\boldsymbol{\sigma}^{dev}:\boldsymbol{\varepsilon}^{dev} + \frac{1}{6}\mathrm{Tr}(\boldsymbol{\sigma})\mathrm{Tr}(\boldsymbol{\varepsilon}) = \Psi^{dev} + \Psi^{vol}
\end{aligned} \tag{9.16}$$

If we now consider the generalized Hooke's law, $\boldsymbol{\sigma} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$ (isotropic material), (see Chapter 7), and its deviatoric part, $\boldsymbol{\sigma}^{dev} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon}^{dev})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^{dev} = 2\mu\boldsymbol{\varepsilon}^{dev}$, ($\sigma_{ij}^{dev} = 2\mu\varepsilon_{ij}^{dev}$) (where it was considered that the trace of any deviatoric tensor is zero) then, the part of the strain energy density associated with the deviatoric part is given by:

$$\Psi^{dev} = \frac{1}{2}\sigma_{ij}^{dev}\varepsilon_{ij}^{dev} = \frac{1}{4\mu}\sigma_{ij}^{dev}\sigma_{ij}^{dev} = \frac{1}{2\mu}\mathsf{J}_2 \tag{9.17}$$

Thus, it is possible to interpret the von Mises yield criterion as: plastification begins when the energy Ψ^{dev} reaches the critical value $\dfrac{\mathsf{J}_2}{2\mu} = \dfrac{\mathsf{k}^2}{2\mu} = \dfrac{\sigma_Y^2}{6\mu}$.

Another interpretation of the von Mises yield criterion is related to the octahedral shear stress:

$$\left.\begin{aligned}
\tau_{oct}^2 &= \frac{1}{9}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right] \\
\mathsf{J}_2 &= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_1-\sigma_3)^2\right]
\end{aligned}\right\} \Rightarrow \tau_{oct}^2 = \frac{2}{3}\mathsf{J}_2 \tag{9.18}$$

whose equations were obtained in Appendix A. Then, by applying the von Mises yield criterion (9.12), $\mathsf{J}_2 - \mathsf{k}^2 = 0$, we obtain:

$$\tau_{oct}^2 = \frac{2}{3}\mathsf{J}_2 = \frac{2}{3}\mathsf{k}^2 \quad \Rightarrow \quad \tau_{oct} = \sqrt{\frac{2}{3}}\,\mathsf{k} \tag{9.19}$$

This criterion can also be interpreted as that in which material starts to plastify when the octahedral shear stress reaches the critical value $\sqrt{\dfrac{2}{3}}\,\mathsf{k} = \dfrac{\sigma_Y}{3}\sqrt{2}$.

Now, by considering the equations in (9.14) and (9.15) we can still write the von Mises yield criterion as:

$$(\sigma_1-\sigma_3)^2 - (\sigma_1-\sigma_3)(\sigma_2-\sigma_3) + (\sigma_2-\sigma_3)^2 = \sigma_Y^2 \tag{9.20}$$

which shows the ellipse equation in the coordinate system $(\sigma_1 - \sigma_3)$ - $(\sigma_2 - \sigma_3)$, whose ellipse axis form a 45° angle to the axes $(\sigma_1 - \sigma_3)$ and $(\sigma_2 - \sigma_3)$, (see Figure 9.7).

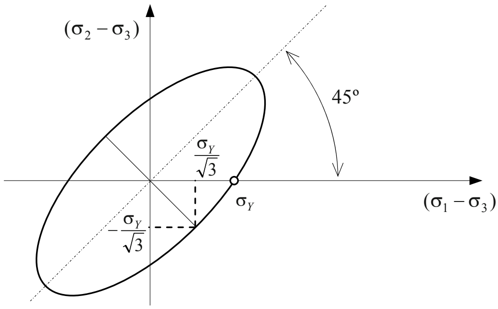


Figure 9.7: The von Mises yield criterion.

For the state of plane stress $(\sigma_3 = 0)$, the yield criterion in (9.20) becomes:

$$\sigma_1^2 - \sigma_1 \sigma_2 - \sigma_2^2 = 3\mathsf{k}^2 = \sigma_Y^2 \tag{9.21}$$

which represents an ellipse in the $\sigma_1 - \sigma_2$-space, (see Figure 9.8), and in uniaxial cases the yield surface is reduced to a point.

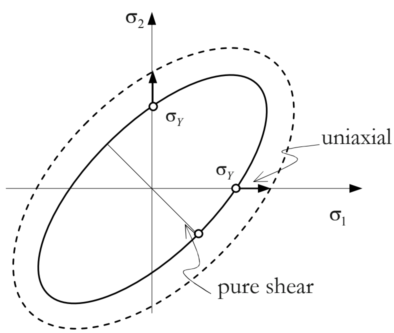


Figure 9.8: The von Mises yield criterion – the state of plane stress.

The von Mises yield criterion can also be expressed in terms of the Frobenius norm of the Cauchy deviatoric stress tensor ($\boldsymbol{\sigma}^{dev} \equiv \mathbf{s}$), which is given by:

$$\|\mathbf{s}\| = \sqrt{\mathbf{s}:\mathbf{s}} = \sqrt{\mathsf{s}_{ij}\mathsf{s}_{ij}} = \sqrt{\frac{2}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3\right)} = \sqrt{2\mathrm{J}_2} \tag{9.22}$$

Then, by considering the equations (9.14), (9.15) and (9.22), the von Mises yield criterion can be expressed as follows:

$$\sqrt{\frac{3}{2}}\sqrt{2\mathrm{J}_2} - \sigma_Y = 0 \quad \Rightarrow \quad \sqrt{\frac{3}{2}}\|\mathbf{s}\| - \sigma_Y = 0 \tag{9.23}$$

We can then summarize the different ways of expressing the yield surface for the von Mises yield criterion:

$$\boxed{\begin{aligned}&\mathcal{F}(\mathsf{J}_2,\mathsf{k})=\mathsf{J}_2-\mathsf{k}^2=0\\&\mathcal{F}(\tau_{oct},\mathsf{k})=\tau_{oct}-\sqrt{\frac{2}{3}}\mathsf{k}=0\\&\mathcal{F}(\mathsf{J}_2,\sigma_Y)=\sqrt{3\mathsf{J}_2}-\sigma_Y=0\\&\mathcal{F}(\tau_{oct},\sigma_Y)=\frac{3}{\sqrt{2}}\tau_{oct}-\sigma_Y=0\\&\mathcal{F}(\lVert\mathbf{s}\rVert,\sigma_Y)=\sqrt{\frac{3}{2}}\lVert\mathbf{s}\rVert-\sigma_Y=0\end{aligned}} \quad \textit{Yield surface for von Mises yield criterion} \tag{9.24}$$

Next, we will calculate the yield surface gradient for the von Mises criterion in the principal stress space. Starting with the definition $\mathbf{n}\equiv\nabla_{\boldsymbol{\sigma}}\mathcal{F}\equiv\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}$ and by using the function $\mathcal{F}=\sqrt{3\mathsf{J}_2}-\sigma_Y=0$, we obtain:

$$\mathsf{n}_{ij}=\frac{\partial\mathcal{F}}{\partial\sigma_{ij}}=\frac{\partial(\sqrt{3\mathsf{J}_2}-\sigma_Y)}{\partial\sigma_{ij}}=\frac{\partial(\sqrt{3\mathsf{J}_2})}{\partial\sigma_{ij}}=\frac{1}{2}(3\mathsf{J}_2)^{\frac{-1}{2}}\frac{\partial(3\mathsf{J}_2)}{\partial\sigma_{ij}} \tag{9.25}$$

Then, if we consider that $\mathsf{J}_2=\frac{1}{3}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_1\sigma_3\right)$, and because we are working in the principal stress space ($\sigma_{11}=\sigma_1$, $\sigma_{22}=\sigma_2$, $\sigma_{33}=\sigma_3$, $\sigma_{12}=\sigma_{13}=\sigma_{23}=0$) we can obtain:

$$\frac{\partial(\mathsf{J}_2)}{\partial\sigma_{ij}}=\begin{bmatrix}\frac{2\sigma_1-\sigma_2-\sigma_3}{3}&0&0\\0&\frac{2\sigma_2-\sigma_1-\sigma_3}{3}&0\\0&0&\frac{2\sigma_3-\sigma_1-\sigma_2}{3}\end{bmatrix}=\mathsf{s}_{ij} \tag{9.26}$$

Now, returning to the equation in (9.25), we can conclude that:

$$\mathsf{n}_{ij}=\frac{\partial\mathcal{F}}{\partial\sigma_{ij}}=\frac{1}{2}(3\mathsf{J}_2)^{\frac{-1}{2}}\frac{\partial(3\mathsf{J}_2)}{\partial\sigma_{ij}}=\frac{3}{2}\frac{1}{\sqrt{3\mathsf{J}_2}}\mathsf{s}_{ij}=\sqrt{\frac{3}{2}}\frac{\mathsf{s}_{ij}}{\sqrt{2\mathsf{J}_2}}=\sqrt{\frac{3}{2}}\frac{\mathsf{s}_{ij}}{\sqrt{\mathsf{s}_{pq}\mathsf{s}_{pq}}} \tag{9.27}$$

In tensorial notation the above equation becomes:

$$\mathbf{n}\equiv\nabla_{\boldsymbol{\sigma}}\mathcal{F}\equiv\frac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}=\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}}=\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\lVert\mathbf{s}\rVert}=\sqrt{\frac{3}{2}}\hat{\mathbf{s}} \tag{9.28}$$

where $\lVert\hat{\mathbf{s}}\rVert=\sqrt{\hat{\mathbf{s}}:\hat{\mathbf{s}}}=1$ holds. The same result as in (9.28) could have been obtained starting from the yield surface $\mathcal{F}(\lVert\mathbf{s}\rVert,\sigma_Y)=\sqrt{\frac{3}{2}}\lVert\mathbf{s}\rVert-\sigma_Y=0$, *i.e.*:

$$\begin{aligned}&\mathsf{n}_{ij}=\frac{\partial\mathcal{F}}{\partial\sigma_{ij}}=\frac{\partial\left(\sqrt{\frac{3}{2}}\lVert\mathbf{s}\rVert-\sigma_Y\right)}{\partial\sigma_{ij}}=\sqrt{\frac{3}{2}}\frac{\partial\left(\lVert\mathbf{s}\rVert\right)}{\partial\sigma_{ij}}=\sqrt{\frac{3}{2}}\frac{\partial\left(\lVert\mathbf{s}\rVert\right)}{\partial\mathsf{s}_{ij}}\frac{\partial\left(\mathsf{s}_{ij}\right)}{\partial\sigma_{kl}}=\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\lVert\mathbf{s}\rVert}\frac{\partial\left(\mathsf{s}_{ij}\right)}{\partial\sigma_{kl}}\\&\Rightarrow\mathbf{n}=\sqrt{\frac{3}{2}}\frac{\mathbf{s}}{\lVert\mathbf{s}\rVert}:\frac{\partial\mathbf{s}}{\partial\boldsymbol{\sigma}}\end{aligned} \tag{9.29}$$

It can be shown that $\mathbf{s} : \dfrac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}} = \mathbf{s}$, (see **Problem 1.39**), then it follows that $\mathbf{n} = \sqrt{\dfrac{3}{2}} \dfrac{\mathbf{s}}{\|\mathbf{s}\|}$ and the reader will be left to verify that the tensors $\boldsymbol{\sigma}$ and $\mathbf{n} \equiv \nabla_{\boldsymbol{\sigma}} \mathcal{F}$ are coaxial, *i.e.* $\boldsymbol{\sigma} \cdot \nabla_{\boldsymbol{\sigma}} \mathcal{F} = \nabla_{\boldsymbol{\sigma}} \mathcal{F} \cdot \boldsymbol{\sigma}$. In other words, they have the same principal directions (eigenvectors).

9.2.3.2 The Tresca Yield Criterion

In the *Tresca yield criterion* (1864) (also called the *criterion of maximum shear stress*) plastification of the material begins when the maximum shear stress reaches the critical value k_{T}. Mathematically, this criterion is represented by:

$$\begin{aligned} &\tau_{\max} < \mathsf{k}_{\mathrm{T}} && \text{elastic domain} \\ &\tau_{\max} = \mathsf{k}_{\mathrm{T}} && \text{plastic domain} \end{aligned} \tag{9.30}$$

or more explicitly by:

$$\max\left(\frac{\|\sigma_1 - \sigma_2\|}{2}, \frac{\|\sigma_2 - \sigma_3\|}{2}, \frac{\|\sigma_1 - \sigma_3\|}{2} \right) = \mathsf{k}_{\mathrm{T}} \tag{9.31}$$

Then, by considering the principal stresses such as $\sigma_I > \sigma_{II} > \sigma_{III}$, the Tresca yield criterion is given by:

$$\frac{1}{2}(\sigma_I - \sigma_{III}) = \mathsf{k}_{\mathrm{T}} \quad \Rightarrow \quad \mathcal{F}(\sigma_I, \sigma_{III}, \mathsf{k}_{\mathrm{T}}) = \frac{1}{2}(\sigma_I - \sigma_{III}) - \mathsf{k}_{\mathrm{T}} = 0 \tag{9.32}$$

where we have considered that $(\sigma_I - \sigma_{III}) > 0$. Note that the principal stress σ_{II} has no influence on the Tresca criterion. Then, the material constant k_{T} is evaluated by tensile testing, where it holds that $\sigma_I = \sigma_Y$, $\sigma_{III} = 0$, thus:

$$\sigma_1 = \sigma_Y \quad \Rightarrow \quad \mathsf{k}_{\mathrm{T}} = \frac{\sigma_Y}{2} \tag{9.33}$$

Figure 9.9 represents the stress state, at a material point, by means of a Mohr's circle in stress. In this figure we can appreciate the elastic stress state (before plastification), and we can verify the evolution of the stress sates from elastic to the initiation of plastification, (see Figure 9.9).

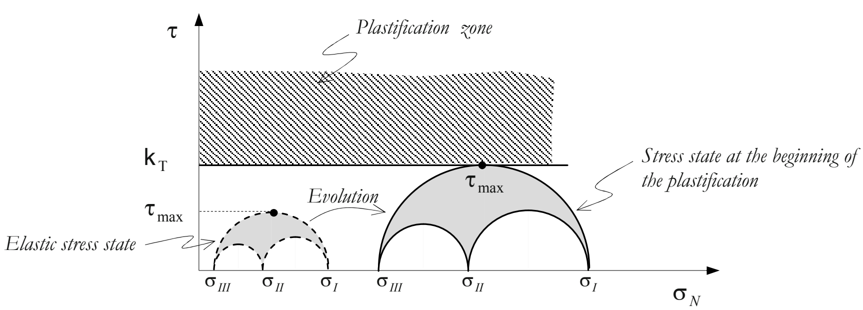


Figure 9.9: The Tresca yield criterion.

To obtain the Tresca yield surface shape, let us consider the three principal stresses (σ_1, σ_2, σ_3), and according to the Tresca criterion (9.31) we can define the following equations:

$$\begin{array}{lll} \sigma_1 - \sigma_2 = \sigma_Y & ; \quad \sigma_2 - \sigma_3 = \sigma_Y & ; \quad \sigma_3 - \sigma_1 = \sigma_Y \\ \sigma_1 - \sigma_2 = -\sigma_Y & ; \quad \sigma_2 - \sigma_3 = -\sigma_Y & ; \quad \sigma_3 - \sigma_1 = -\sigma_Y \end{array} \tag{9.34}$$

where we have considered the equation in (9.33). Note that each one represents a plane equation which is parallel to the hydrostatic axis. The surface generated by putting these planes together is a prism whose cross-section is defined by a regular hexagon, (see Figure 9.10). As we can verify, the cross section of the prism is the same in shape and size for any point on the hydrostatic axis.

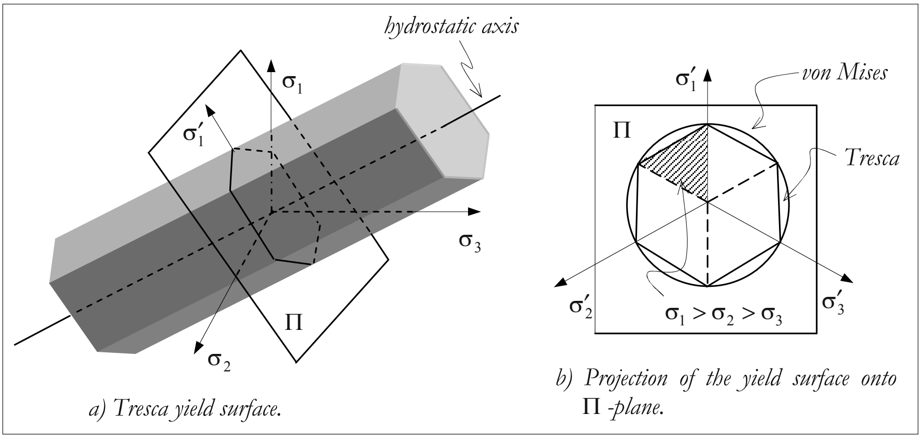


Figure 9.10: Tresca yield surface.

This cross section could also have been obtained by using the expressions of the principal stresses obtained in subsection A.4 in Appendix A:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \frac{2}{\sqrt{3}}\sqrt{J_2}\begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos(\theta - \frac{2\pi}{3}) & 0 \\ 0 & 0 & \cos(\theta + \frac{2\pi}{3}) \end{bmatrix} \tag{9.35}$$

Then, by substituting the stresses σ_1 and σ_3 from then above equation into the yield criterion in (9.32) we obtain:

$$\frac{1}{2}\left\| \left(\sigma_m + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta\right) - \left(\sigma_m + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta + \tfrac{2\pi}{3})\right) \right\| = k_T \tag{9.36}$$

$$\Rightarrow \sqrt{J_2}\,\frac{1}{\sqrt{3}}\left[\cos\theta - \cos(\theta + \tfrac{2\pi}{3})\right] = k_T \tag{9.37}$$

Then, if we consider the equation $\frac{1}{\sqrt{3}}\left[\cos\theta - \cos(\theta + \frac{2\pi}{3})\right] = \sin(\theta + \frac{\pi}{3})$, we can still state that:

$$\sqrt{J_2}\sin(\theta + \tfrac{\pi}{3}) = k_T \quad \Rightarrow \quad \frac{1}{\sqrt{2}}q\sin(\theta + \tfrac{\pi}{3}) = k_T \tag{9.38}$$

Let us also remember that in Appendix A the equation $q = \sqrt{2J_2}$ was obtained. Then, by varying the angle θ from 0° to 60°, we can obtain the yield curve in the deviatoric plane, (see Figure 9.11). Thus, we can obtain the yield surface in terms of the parameters q and θ:

$$\mathcal{F}(q,\theta) = \sqrt{2}q\sin(\theta + \tfrac{\pi}{3}) - 2\,k_T = 0 \qquad (0 \le \theta \le \tfrac{\pi}{3}) \tag{9.39}$$

As expected, we can see that the yield surface is not a function of the hydrostatic pressure p.

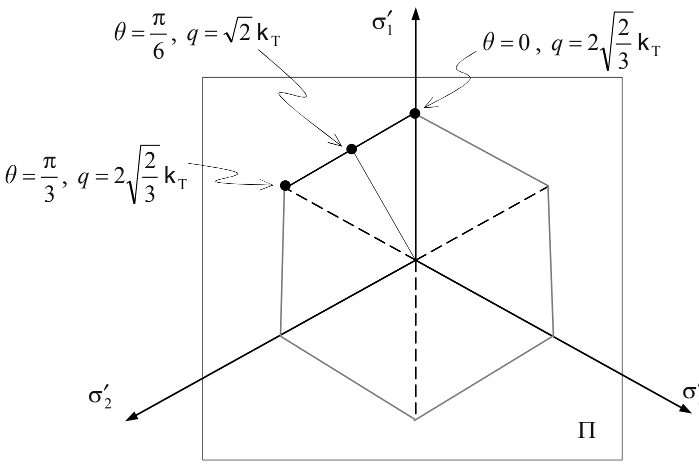


Figure 9.11: Tresca yield surface – Projection of the surface onto the deviatoric plane.

As for the state of plane stress, $(\sigma_3 = 0)$, the equations in (9.34) become:

$$\begin{matrix} \sigma_1 - \sigma_2 = \sigma_Y & ; & \sigma_2 = \sigma_Y & ; & -\sigma_1 = \sigma_Y \\ \sigma_1 - \sigma_2 = -\sigma_Y & ; & \sigma_2 = -\sigma_Y & ; & -\sigma_1 = -\sigma_Y \end{matrix} \tag{9.40}$$

which is represented by a hexagon in the principal stress space $\sigma_1 - \sigma_2$, (see Figure 9.12).

$$\sigma_I - \sigma_{II} = \pm 2\mathsf{k}_{\mathrm{T}} \quad \Rightarrow \quad \begin{cases} \sigma_I = \pm 2\mathsf{k}_{\mathrm{T}} \\ \sigma_{II} = \pm 2\mathsf{k}_{\mathrm{T}} \end{cases} \tag{9.41}$$

In Figure 9.13 we can see the von Mises and Tresca criteria for the state of plane stress ($\sigma_3 = 0$), which show that the Tresca hexagon is inscribed into the ellipse of von Mises.

The yield surface gradient for the Tresca yield criterion can be obtained by means of the definition $\mathbf{n} \equiv \nabla_{\boldsymbol{\sigma}} \mathcal{F} \equiv \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}}$. Then, we can use the definition of the Tresca yield surface given in (9.32), to obtain:

$$\frac{1}{2}(\sigma_I - \sigma_{III}) = \mathsf{k}_{\mathrm{T}} \quad \Rightarrow \quad \mathcal{F}(\sigma_I, \sigma_{III}, \mathsf{k}_{\mathrm{T}}) = \frac{1}{2}(\sigma_I - \sigma_{III}) - \mathsf{k}_{\mathrm{T}} = 0 \tag{9.42}$$

$$\mathsf{n}_{ij} = \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} = \frac{\partial \left[\frac{1}{2}(\sigma_I - \sigma_{III}) - \mathsf{k}_{\mathrm{T}} \right]}{\partial \sigma_{ij}} = \frac{1}{2} \frac{\partial [(\sigma_I - \sigma_{III})]}{\partial \sigma_{ij}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix} \tag{9.43}$$

9.2.4 The Yield Criteria for Pressure-Dependent Materials

Porous materials, *e.g.* soil, rock, concrete, and some porous metals, are affected by hydrostatic pressure, *i.e.* these materials are dependent on the first invariant $I_{\boldsymbol{\sigma}}$, (see Chen&Han(1988)). Here, can mention some criteria that take into account this phenomenon, namely: the Mohr-Coulomb criterion, the Drucker-Prager criterion and the Rankine criterion, among others.

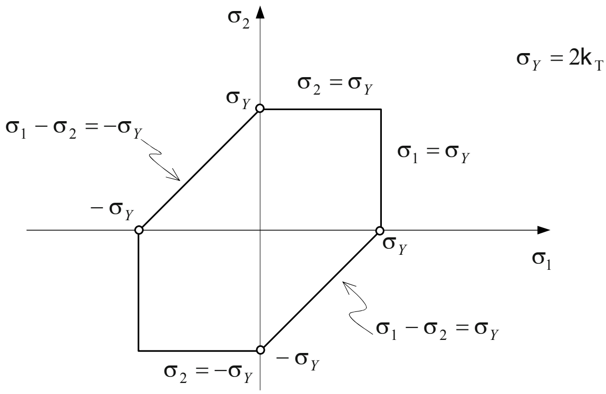


Figure 9.12: Tresca yield curve – Plane stress.

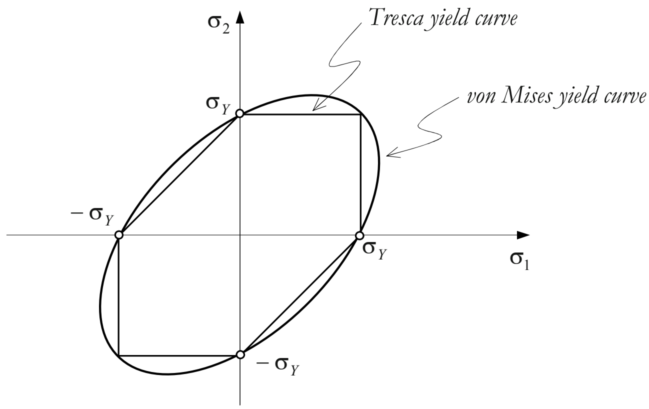


Figure 9.13: The yield curves for von Mises and de Tresca – Plane stress.

9.2.4.1 The Mohr-Coulomb Criterion

This criterion was formulated by Coulomb in 1773, and was enhanced by Mohr in 1882, Oller (2001). Mathematically, the Mohr-Coulomb criterion is given by:

$$\tau = \tau(\sigma_i, k_1, k_2, \cdots) \tag{9.44}$$

where $(k_1, k_2, \cdots)$ are material constants and the function $\tau(\sigma_i)$ is obtained by means of laboratory experiments; (see the *Triaxial Compression Test* in subsection 6.5.12 in Chapter 6). The function $\tau(\sigma_i)$ corresponds to an envelope curve of the Mohr's circles at the failure moment, where each Mohr's circle is obtained for a different hydrostatic pressure state. When the envelope is a straight line, (see Figure 9.14), the Mohr-Coulomb criterion becomes:

$$\tau = c - \sigma_N \operatorname{tg}\phi \tag{9.45}$$

where τ is the shear stress magnitude in the failure plane, σ_N is the normal stress in the failure plane, c is the cohesion (material property) and ϕ is the angle of internal friction (material property). In the particular case when $\phi = 0$ we revert to the Tresca yield criterion, with $\tau = c$, and where the cohesion is interpreted as $c = \mathsf{k}_T$.

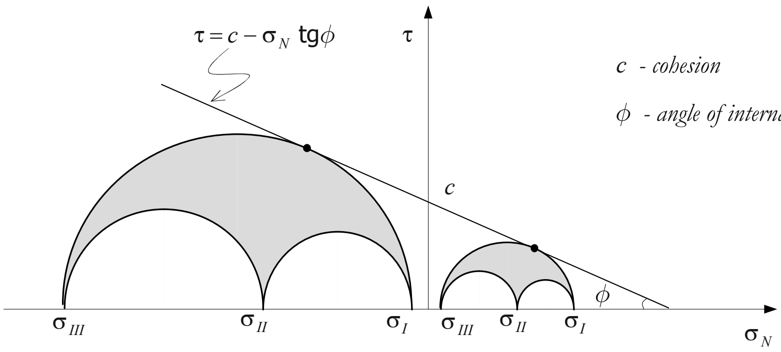


Figure 9.14: Varying the hydrostatic pressure so as to define the Mohr-Coulomb criterion.

Mohr proved by means of a graph that the equation in (9.45) represents a straight line which is tangent to the greatest circle defined by σ_I and σ_{III}, (see Figure 9.15). Moreover, we can also observe that this criterion is independent of the principal stress σ_{II}.

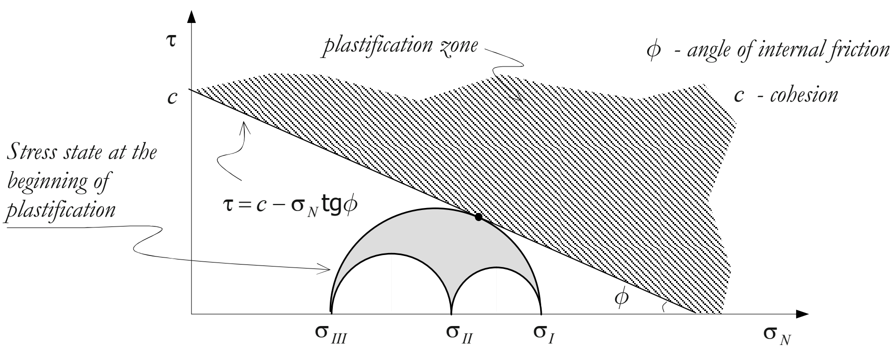


Figure 9.15: Mohr-Coulomb yield criterion.

Now, to obtain the mathematical equation that represents the Mohr-Coulomb yield surface, let us consider Figure 9.16 in which we can obtain:

$$\sigma_N^A = \frac{\sigma_I + \sigma_{III}}{2} - \frac{\sigma_I - \sigma_{III}}{2}\cos(2\alpha) \quad ; \quad \tau^A = \frac{\sigma_I - \sigma_{III}}{2}\sin(2\alpha) \tag{9.46}$$

Then, by substituting the equations in (9.46) into (9.45) we obtain:

$$\begin{aligned}
&\tau = c - \sigma_N \,\mathrm{tg}\phi \\
&\frac{\sigma_I - \sigma_{III}}{2}\sin(2\alpha) = c - \left[\frac{\sigma_I + \sigma_{III}}{2} - \frac{\sigma_I - \sigma_{III}}{2}\cos(2\alpha)\right]\mathrm{tg}\phi \\
&(\sigma_I - \sigma_{III})\sin(\frac{\pi}{2} - \phi) = 2c - (\sigma_I + \sigma_{III})\mathrm{tg}\phi + (\sigma_I - \sigma_{III})\cos(\frac{\pi}{2} - \phi)\mathrm{tg}\phi \\
&(\sigma_I - \sigma_{III})\cos\phi = 2c - (\sigma_I + \sigma_{III})\frac{\sin\phi}{\cos\phi} - (\sigma_I - \sigma_{III})\sin\phi\frac{\sin\phi}{\cos\phi} \\
&(\sigma_I - \sigma_{III})(\cos^2\phi + \sin^2\phi) = 2c\cos\phi - (\sigma_I + \sigma_{III})\sin\phi \\
&(\sigma_I - \sigma_{III}) = 2c\cos\phi - (\sigma_I + \sigma_{III})\sin\phi
\end{aligned} \tag{9.47}$$

where we have considered that $2\alpha + \dfrac{\pi}{2} + \phi = \pi \Rightarrow 2\alpha = \dfrac{\pi}{2} - \phi$.

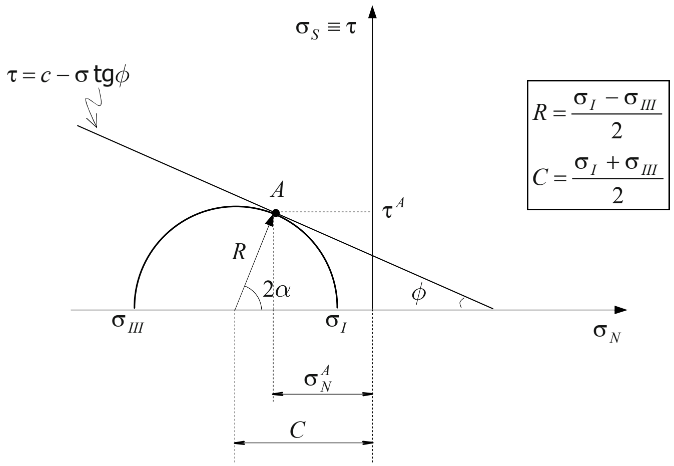


Figure 9.16: Mohr-Coulomb criterion.

Thus, the yield surface is defined as follows:

$$\boxed{\mathcal{F}(\boldsymbol{\sigma}, c, \phi) = (\sigma_I - \sigma_{III}) - 2c\cos\phi + (\sigma_I + \sigma_{III})\sin\phi = 0} \quad \textit{Mohr-Coulomb yield surface} \tag{9.48}$$

Then, if we consider the values of σ_I and σ_{III} given in (9.35), the equation in (9.47) becomes:

$$\sqrt{J_2}\,\frac{1}{\sqrt{3}}\left[\cos\theta - \cos\left(\theta + \tfrac{2\pi}{3}\right)\right] + \sigma_m \sin\phi + \frac{\sqrt{J_2}}{\sqrt{3}}\left[\cos\theta + \cos\left(\theta + \tfrac{2\pi}{3}\right)\right]\sin\phi = c\cos\phi \tag{9.49}$$

Next, by considering that $\dfrac{1}{\sqrt{3}}\left[\cos\theta - \cos\left(\theta + \tfrac{2\pi}{3}\right)\right] = \sin\left(\tfrac{\pi}{3} + \theta\right)$ and $\left[\cos\theta + \cos\left(\theta + \tfrac{2\pi}{3}\right)\right] = \cos\left(\tfrac{\pi}{3} + \theta\right)$, we can obtain:

$$\sqrt{J_2}\sin\left(\tfrac{\pi}{3} + \theta\right) + \sigma_m \sin\phi + \frac{\sqrt{J_2}}{\sqrt{3}}\cos\left(\tfrac{\pi}{3} + \theta\right)\sin\phi = c\cos\phi \tag{9.50}$$

Also by considering that $q = \sqrt{2J_2}$ and $p = \sqrt{3}\sigma_m$, the equation in (9.49) can still be expressed as follows:

$$q\sqrt{3}\sin\left(\tfrac{\pi}{3} + \theta\right) + \sqrt{2}\,p\sin\phi + q\cos\left(\tfrac{\pi}{3} + \theta\right)\sin\phi = \sqrt{6}\,c\cos\phi \tag{9.51}$$

thereby obtaining the yield surface on the deviatoric plane:

$$\boxed{\mathcal{F}(p, q, \theta, \phi, c) = q\sqrt{3}\sin\left(\theta + \tfrac{\pi}{3}\right) + \sqrt{2}\,p\sin\phi + q\cos\left(\theta + \tfrac{\pi}{3}\right)\sin\phi - \sqrt{6}\,c\cos\phi = 0} \tag{9.52}$$

From the above we can verify that for a constant angle θ the equation is linear with q and p. Then, when $q = 0$ we have:

$$\sqrt{2}\,p\sin\phi = \sqrt{6}\,c\cos\phi \quad \Rightarrow \quad p = \frac{\sqrt{6}}{\sqrt{2}}\frac{\cos\phi}{\sin\phi}c \quad \Rightarrow \quad p = \sqrt{3}\,c\,\mathrm{cotg}\phi \tag{9.53}$$

Note that when $\theta = 0^\circ$ we obtain:

$$
\begin{aligned}
& q\sqrt{3}\sin\left(\tfrac{\pi}{3}+\theta\right)+\sqrt{2}\,p\sin\phi+q\cos\left(\tfrac{\pi}{3}+\theta\right)\sin\phi=\sqrt{6}\,c\cos\phi \\
& q\sqrt{3}\sin\left(\tfrac{\pi}{3}\right)+\sqrt{2}\,p\sin\phi+q\cos\left(\tfrac{\pi}{3}\right)\sin\phi=\sqrt{6}\,c\cos\phi \\
& \Rightarrow q\sqrt{3}\,\frac{\sqrt{3}}{2}+\sqrt{2}\,p\sin\phi+q\,\frac{1}{2}\sin\phi=\sqrt{6}\,c\cos\phi
\end{aligned}
\tag{9.54}
$$

This line intercepts the q-axis with $p=0$, thus:

$$
q\sqrt{3}\,\frac{\sqrt{3}}{2}+q\,\frac{1}{2}\sin\phi=\sqrt{6}\,c\cos\phi \quad\Rightarrow\quad q=\frac{2\sqrt{6}\,c\cos\phi}{(3+\sin\phi)}
\tag{9.55}
$$

When $\theta=\frac{\pi}{3}$, the equation in (9.51) becomes:

$$
\begin{aligned}
& q\sqrt{3}\sin\left(\tfrac{2\pi}{3}\right)+\sqrt{2}\,p\sin\phi+q\cos\left(\tfrac{2\pi}{3}\right)\sin\phi=\sqrt{6}\,c\cos\phi \\
& \Rightarrow q\sqrt{3}\,\frac{\sqrt{3}}{2}+\sqrt{2}\,p\sin\phi+q\,\frac{-1}{2}\sin\phi=\sqrt{6}\,c\cos\phi
\end{aligned}
\tag{9.56}
$$

and this line intercepts the q-axis when $p=0$, thus:

$$
q=\frac{2\sqrt{6}\,c\cos\phi}{(3-\sin\phi)}
\tag{9.57}
$$

In this way we can draw a graph $p-q$, *i.e.* to show how pressure varies with the deviatoric part, (see Figure 9.17).

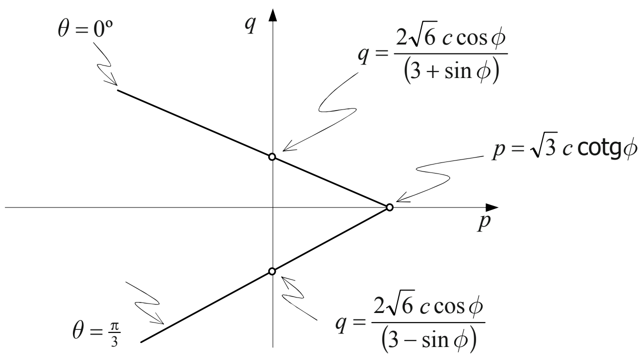


Figure 9.17: Pressure *vs.* deviatoric part – Mohr-Coulomb yield criterion.

In the principal stress space, the yield surface is represented by a conical surface whose cross section is shown by an irregular hexagon, (see Figure 9.18).

Next, we calculate the gradient of the Mohr-Coulomb yield surface in the principal stress space by means of the definition $\mathbf{n}\equiv\nabla_{\boldsymbol{\sigma}}\mathcal{F}\equiv\frac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}$. Then, if we consider the expression of $\mathcal{F}(\boldsymbol{\sigma},c,\phi)$ given in (9.48), and that $\frac{\partial\mathcal{F}}{\partial\sigma_I}=1+\sin\phi$ and $\frac{\partial\mathcal{F}}{\partial\sigma_{III}}=-1+\sin\phi$, we can obtain:

$$\mathsf{n}_{ij} = \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} = \begin{bmatrix} 1+\sin\phi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1+\sin\phi \end{bmatrix} \tag{9.58}$$

The norm of $\mathbf{n}$ is given by $\|\mathbf{n}\| = \sqrt{\mathbf{n}:\mathbf{n}} = \sqrt{(1+\sin\phi)^2 + (-1+\sin\phi)^2} = \sqrt{2(1+\sin^2\phi)}$, with which we can conclude that:

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|} \quad \Rightarrow \quad \hat{\mathsf{n}}_{ij} = \frac{\dfrac{\partial \mathcal{F}}{\partial \sigma_{ij}}}{\left\|\dfrac{\partial \mathcal{F}}{\partial \sigma_{ij}}\right\|} = \begin{bmatrix} \dfrac{1+\sin\phi}{\sqrt{2(1+\sin^2\phi)}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{-1+\sin\phi}{\sqrt{2(1+\sin^2\phi)}} \end{bmatrix} \tag{9.59}$$

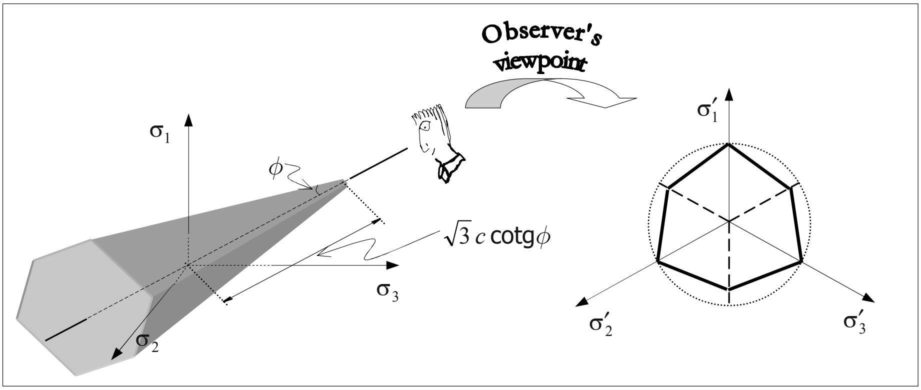


Figure 9.18: The Mohr-Coulomb yield surface.

9.2.4.2 The Drucker-Prager Yield Criterion

The Drucker-Prager yield criterion modifies that of von Mises, $\mathcal{F}(q,\mathsf{k}) = q - \sqrt{2}\,\mathsf{k} = 0$, by adding the effect of pressure. In order to do so, we will start from the result obtained in the Mohr-Coulomb graph made up of the axes $p-q$ when $\theta = 0^\circ$, (see Figure 9.17).

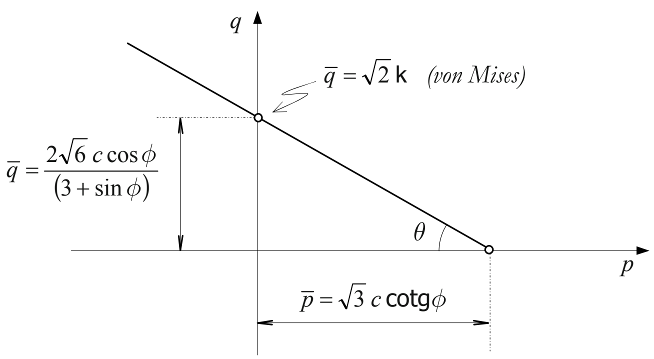


Figure 9.19: Pressure *vs.* deviatoric part – Drucker-Prager criterion.

The line equation depicted in Figure 9.19 is given by:

$$q=\sqrt{2}\,\mathsf{k}-p\,\mathrm{tg}\theta \quad \Rightarrow \quad q-\sqrt{2}\,\mathsf{k}+p\,\mathrm{tg}\theta=0 \tag{9.60}$$

where:

$$\mathrm{tg}\theta=\frac{\dfrac{2\sqrt{6}\,c\cos\phi}{(3+\sin\phi)}}{\sqrt{3}\,c\,\mathrm{cotg}}=\frac{2\sqrt{2}\sin\phi}{(3+\sin\phi)}=\underbrace{\frac{2\sin\phi}{\sqrt{3}(3+\sin\phi)}}_{=\alpha}\sqrt{6}=\alpha\sqrt{6} \tag{9.61}$$

From the graph with the $p-q$ axes described in Figure 9.19 we can still obtain the following equations:

$$\sqrt{2}\,\mathsf{k}=\frac{2\sqrt{6}\,c\cos\phi}{(3+\sin\phi)} \quad \Rightarrow \quad \begin{cases} \mathsf{k}=\dfrac{2\sqrt{3}\,c\cos\phi}{(3+\sin\phi)}=\dfrac{6\,c\cos\phi}{\sqrt{3}(3+\sin\phi)} \\ c=\dfrac{\sqrt{2}\,\mathsf{k}(3+\sin\phi)}{2\sqrt{6}\,\cos\phi} \end{cases} \tag{9.62}$$

Then, by substituting c given above into the equation of $\overline{p}$ in Figure 9.19 we can obtain:

$$\overline{p}=\sqrt{3}\,c\frac{\cos\phi}{\sin\phi}=\sqrt{3}\,\frac{\sqrt{2}\,\mathsf{k}(3+\sin\phi)}{2\sqrt{6}\,\cos\phi}\frac{\cos\phi}{\sin\phi}=\frac{\sqrt{3}(3+\sin\phi)}{2\sin\phi}\frac{\mathsf{k}}{\sqrt{3}}=\frac{\mathsf{k}}{\sqrt{3}\,\alpha} \tag{9.63}$$

Then, with the above results we can represent the Drucker-Prager yield criterion as:

$$\mathcal{F}(p,q,\mathsf{k},\alpha)=q-\sqrt{2}\,\mathsf{k}+\alpha\sqrt{6}\;p=0 \qquad \textit{with} \qquad \alpha=\frac{2\sin\phi}{\sqrt{3}(3+\sin\phi)} \tag{9.64}$$

Now, if we consider that $q=\sqrt{2\mathrm{J}_2}=\sqrt{3}\,\tau_{oct}$, $p=\dfrac{\sqrt{3}\,I_{\boldsymbol{\sigma}}}{3}=\sqrt{3}\,\sigma_{oct}$, (see Appendix A), the yield surface can still be represented as $\mathcal{F}(p,q,\mathsf{k},\phi)=\sqrt{2\mathrm{J}_2}-\sqrt{2}\,\mathsf{k}+\alpha\sqrt{6}\,\dfrac{\sqrt{3}\,I_{\boldsymbol{\sigma}}}{3}=0$, thus:

$$\mathcal{F}(\mathrm{J}_2,I_{\boldsymbol{\sigma}},\mathsf{k},\phi)=\sqrt{\mathrm{J}_2}-\mathsf{k}+\alpha\,I_{\boldsymbol{\sigma}}=0 \tag{9.65}$$

We can still express the Drucker-Prager yield surface in terms of the octahedral stresses, *i.e.* by means of $\sigma_{oct}=\dfrac{I_{\boldsymbol{\sigma}}}{3}$ (normal octahedral stress) and $\tau_{oct}=\sqrt{\dfrac{2}{3}}\sqrt{\mathrm{J}_2}$ (tangential octahedral stress), (see Appendix A). Thus, the equation in (9.65) can be rewritten as:

$$\mathcal{F}(\sigma_{oct},\tau_{oct},\mathsf{k},\phi)=\sqrt{\frac{3}{2}}\,\tau_{oct}-\mathsf{k}+3\alpha\,\sigma_{oct}=0 \tag{9.66}$$

Then, if we consider the expression of k given in (9.62), the Drucker-Prager criterion becomes:

$$\mathcal{F}(\mathrm{J}_2,I_{\boldsymbol{\sigma}},\phi)=\alpha I_{\boldsymbol{\sigma}}+\sqrt{\mathrm{J}_2}-\beta=0 \tag{9.67}$$

where $\alpha=\dfrac{2\sin\phi}{\sqrt{3}(3+\sin\phi)}$, $\beta=\dfrac{6c\cos\phi}{\sqrt{3}(3+\sin\phi)}$, $I_{\boldsymbol{\sigma}}=\sigma_1+\sigma_2+\sigma_3$, c is the cohesion, and ϕ is the angle of internal friction.

Note that on the yield curve the edge of the Mohr-Coulomb yield curve coincides with the Drucker-Prager yield curve when $\theta=0^{\circ}$. The yield surface is not longer a cylinder and becomes a cone, (see Figure 9.20).

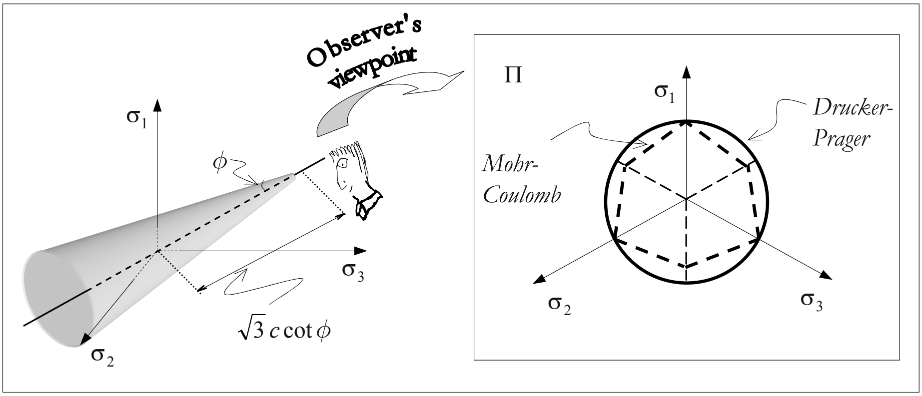


Figure 9.20: Drucker-Prager yield surface.

Then, in summary the different ways of expressing the Drucker-Prager yield surface are:

$$\boxed{\begin{aligned}&\mathcal{F}(p,q,\mathsf{k},\alpha)=q-\sqrt{2}\,\mathsf{k}+\alpha\sqrt{6}\;p=0\\&\mathcal{F}(\mathrm{J}_2,I_{\boldsymbol{\sigma}},\mathsf{k},\alpha)=\sqrt{\mathrm{J}_2}-\mathsf{k}+\alpha\,I_{\boldsymbol{\sigma}}=0\\&\mathcal{F}(\sigma_{oct},\tau_{oct},\mathsf{k},\alpha)=\sqrt{\frac{3}{2}}\,\tau_{oct}-\mathsf{k}+3\alpha\,\sigma_{oct}=0\\&\mathcal{F}(\mathrm{J}_2,I_{\boldsymbol{\sigma}},\phi)=\alpha I_{\boldsymbol{\sigma}}+\sqrt{\mathrm{J}_2}-\beta=0\end{aligned}}\quad \textit{Drucker-Prager yield surface} \tag{9.68}$$

The definition $\mathbf{n}\equiv\nabla_{\boldsymbol{\sigma}}\mathcal{F}\equiv\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}$ give us the yield surface gradient in the stress space, so, for the Drucker-Prager yield surface, $\mathcal{F}(\mathrm{J}_2,I_{\boldsymbol{\sigma}},\mathsf{k},\alpha)=\sqrt{\mathrm{J}_2}-\mathsf{k}+\alpha\,I_{\boldsymbol{\sigma}}=0$, $\mathbf{n}$ becomes:

$$\mathbf{n}=\frac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}=\frac{1}{2\sqrt{\mathrm{J}_2}}\mathbf{s}+\alpha\,\mathbf{1} \tag{9.69}$$

where we have considered the equations $\dfrac{\partial\left(\sqrt{\mathrm{J}_2}\right)}{\partial\boldsymbol{\sigma}}=\dfrac{1}{2\sqrt{\mathrm{J}_2}}\mathbf{s}$ and $\dfrac{\partial I_{\boldsymbol{\sigma}}}{\partial\boldsymbol{\sigma}}=\mathbf{1}$.

The Alternative Drucker-Prager Yield Criterion

Note that according to Figure 9.21 and by using the equations obtained previously, $\mathrm{tg}\theta=\alpha\sqrt{6}$, $q=\sqrt{2\mathrm{J}_2}=\sqrt{3}\,\tau_{oct}$, $p=\sqrt{3}\sigma_m=\dfrac{\sqrt{3}\,I_{\boldsymbol{\sigma}}}{3}=\sqrt{3}\,\sigma_{oct}$, we can obtain:

$$\mathrm{tg}\theta=\frac{\overline{q}-q}{p}=\frac{\sqrt{2}\,\mathsf{k}-\sqrt{3}\,\tau_{oct}}{\sqrt{3}\,\sigma_{oct}}=K \tag{9.70}$$

whose material parameters are θ and k. Then, given any stress state, if the point is on the line, then the material begins plastification. We can now use the definition in (9.70) to find the norm (measurement of distance in the stress space):

$$\overline{\overline{q}}(\boldsymbol{\sigma})=\sqrt{3}\left(K\sigma_{oct}+\tau_{oct}\right) \tag{9.71}$$

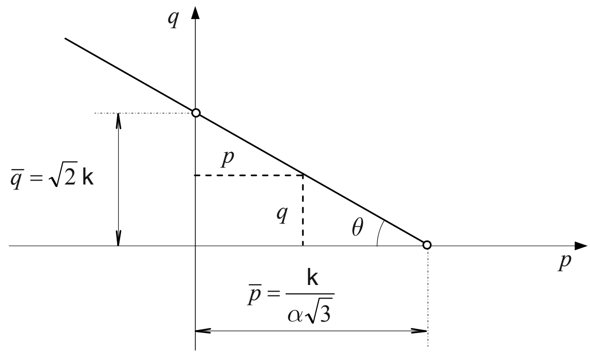


Figure 9.21: Pressure *vs.* deviatoric part – the Drucker-Prager criterion.

To obtain the parameters (θ, k) two different compression tests must be carried out, that is, a one-dimensional test 1D $(\sigma_1 = 0, \sigma_2 = 0, \sigma_3)$, and a two-dimensional one 2D $(\sigma_1 = 0, \sigma_2 = \sigma_3)$. Both of these must reach the nonlinearity limit. Then, if $f^-_{0_{1D}}$ and $f^-_{0_{2D}}$ represent the maximum values of the elastic stress (σ_3) for 1D and 2D respectively, we obtain:

- Test 1D

$$\sigma_{oct}^{1D} = \frac{1}{3}\mathsf{Tr}(\boldsymbol{\sigma}) = \frac{1}{3} f^-_{0_{1D}} \quad ; \quad \tau_{oct}^{1D} = \sqrt{\frac{2}{3}\mathsf{J}_2} = -\frac{\sqrt{2}}{3} f^-_{0_{1D}} \tag{9.72}$$

- Test 2D

$$\sigma_{oct}^{2D} = \frac{2}{3}\mathsf{Tr}(\boldsymbol{\sigma}) = \frac{1}{3} f^-_{0_{2D}} \quad ; \quad \tau_{oct}^{2D} = -\frac{\sqrt{2}}{3} f^-_{0_{2D}} \tag{9.73}$$

Then, by using the equation in (9.70) we can obtain:

$$\mathsf{tg}\theta = \frac{\sqrt{2}\,\mathsf{k} - \sqrt{3}\,\tau_{oct}^{1D}}{\sqrt{3}\,\sigma_{oct}^{1D}} = \frac{\sqrt{2}\,\mathsf{k} - \sqrt{3}\,\tau_{oct}^{2D}}{\sqrt{3}\,\sigma_{oct}^{2D}} \tag{9.74}$$

Moreover, by rearranging the above we have:

$$\sqrt{3}\,\sigma_{oct}^{2D}\left(\sqrt{2}\,\mathsf{k} - \sqrt{3}\,\tau_{oct}^{1D}\right) = \sqrt{3}\,\sigma_{oct}^{1D}\left(\sqrt{2}\,\mathsf{k} - \sqrt{3}\,\tau_{oct}^{2D}\right) \quad \Rightarrow \quad \mathsf{k} = \sqrt{\frac{2}{3}}\,\frac{\tau_{oct}^{1D}\sigma_{oct}^{2D} - \tau_{oct}^{2D}\sigma_{oct}^{1D}}{\sigma_{oct}^{2D} - \sigma_{oct}^{1D}} \tag{9.75}$$

and by substituting the equations (9.72) and (9.73) into the above we obtain:

$$\mathsf{k} = \frac{1}{\sqrt{3}}\,\frac{f^-_{0_{1D}} f^-_{0_{2D}}}{f^-_{0_{1D}} - 2 f^-_{0_{2D}}} \tag{9.76}$$

The constant K is then defined as follows:

$$K = \frac{\sqrt{2}\,\mathsf{k} - \sqrt{3}\,\tau_{oct}^{1D}}{\sqrt{3}\,\sigma_{oct}^{1D}} = \frac{\sqrt{\frac{2}{3}}\left(\frac{f^-_{0_{1D}} f^-_{0_{2D}}}{f^-_{0_{1D}} - 2 f^-_{0_{2D}}} + f^-_{0_{1D}}\right)}{\frac{1}{\sqrt{3}} f^-_{0_{1D}}} \tag{9.77}$$

which after simplifying becomes:

$$K = \sqrt{2}\,\frac{f_{0_{1D}}^{-} f_{0_{2D}}^{-} + f_{0_{1D}}^{-} f_{0_{1D}}^{-} - 2 f_{0_{1D}}^{-} f_{0_{2D}}^{-}}{\left(f_{0_{1D}}^{-} - 2 f_{0_{2D}}^{-}\right) f_{0_{1D}}^{-}} = \sqrt{2}\,\frac{f_{0_{1D}}^{-} + f_{0_{2D}}^{-}}{f_{0_{1D}}^{-} - 2 f_{0_{2D}}^{-}} \tag{9.78}$$

We can now introduce an auxiliary variable such as:

$$R_0 = \frac{f_{0_{2D}}^{-}}{f_{0_{1D}}^{-}} \tag{9.79}$$

Then, we can rewrite the equation of the parameter K by:

$$K = \frac{\sqrt{2}\,\mathsf{k} - \sqrt{3}\,\tau_{oct}^{1D}}{\sqrt{3}\,\sigma_{oct}^{1D}} = \frac{\sqrt{\frac{2}{3}}\left(\frac{f_{0_{1D}}^{-} f_{0_{2D}}^{-}}{f_{0_{1D}}^{-} - 2 f_{0_{2D}}^{-}} + f_{0_{1D}}^{-}\right)}{\frac{1}{\sqrt{3}} f_{0_{1D}}^{-}} = \sqrt{2}\,\frac{f_{0_{1D}}^{-} f_{0_{2D}}^{-} + f_{0_{1D}}^{-} f_{0_{1D}}^{-} - 2 f_{0_{1D}}^{-} f_{0_{2D}}^{-}}{\left(f_{0_{1D}}^{-} - 2 f_{0_{2D}}^{-}\right) f_{0_{1D}}^{-}}$$

$$= \sqrt{2}\,\frac{1 - R_0}{1 - 2R_0} \tag{9.80}$$

9.2.4.3 The Rankine Yield Criterion

The Rankine yield criterion also known as the *maximum-tensile-stress criterion*, (see Chen&Han (1988)), was formulated by Rankine in 1876. This criterion established that material fails when the maximum principal stress, σ_I, reaches the critical value $\sigma_T^{\max}$. Mathematically, this criterion is represented by:

$$\sigma_I = \sigma_T^{\max} \tag{9.81}$$

where $\sigma_I > \sigma_{II} > \sigma_{III}$ and $\sigma_T^{\max} > 0$ are satisfied. As a result, this criterion is used for materials that fail only due to traction ($\sigma_I > 0$). Thus, we can show the Rankine yield surface as follows:

$$\mathcal{F}(\sigma_I, \sigma_T^{\max}) = \sigma_I - \sigma_T^{\max} = 0 \tag{9.82}$$

If we now consider the three principal stresses σ_1, σ_2, σ_3, the Rankine yield surface is represented in the principal stress space as shown in Figure 9.22.

Then, by substituting the value of the principal stress σ_I given by the equation (9.35) into the Rankine criterion equation (9.82) we obtain:

$$\sigma_I - \sigma_T^{\max} = 0 \quad \Rightarrow \quad \sigma_m + \frac{2}{\sqrt{3}}\sqrt{\mathrm{J}_2}\cos\theta - \sigma_T^{\max} = \frac{I_{\boldsymbol{\sigma}}}{3} + \frac{2}{\sqrt{3}}\sqrt{\mathrm{J}_2}\cos\theta - \sigma_T^{\max} = 0 \tag{9.83}$$

Thus,

$$\mathcal{F}(I_{\boldsymbol{\sigma}}, \mathrm{J}_2, \theta) = I_{\boldsymbol{\sigma}} + 2\sqrt{3\,\mathrm{J}_2}\cos\theta - 3\sigma_T^{\max} = 0 \tag{9.84}$$

Then, if we consider the equations $q = \sqrt{2\mathrm{J}_2}$ and $p = \sqrt{3}\sigma_m$, the yield surface can now be rewritten in terms of the Haigh-Westergaard coordinates, *i.e.*:

$$\mathcal{F}(p, q, \theta) = p + \sqrt{2}\,q\cos\theta - \sqrt{3}\,\sigma_T^{\max} = 0 \tag{9.85}$$

Note that the yield curve shape on the deviatoric plane Π (Nadai's plane) can be obtained by using the equation in (9.31) with $p = 0$, thus:

$$\sqrt{2}\, q \cos\theta - \sqrt{3}\, \sigma_T^{\max} = 0 \quad \Rightarrow \quad q = \frac{\sqrt{3}\, \sigma_T^{\max}}{\sqrt{2}\cos\theta} \tag{9.86}$$

Note that the projection of the deviatoric vector q onto the axis σ_1' is $q\cos\theta = \frac{\sqrt{3}\,\sigma_T^{\max}}{\sqrt{2}}$, which is a constant value. When $\theta = 0^{\circ} \Rightarrow q = \frac{\sqrt{3}\,\sigma_T^{\max}}{\sqrt{2}}$ and $\theta = 60^{\circ} \Rightarrow q = \sqrt{6}\,\sigma_T^{\max}$.

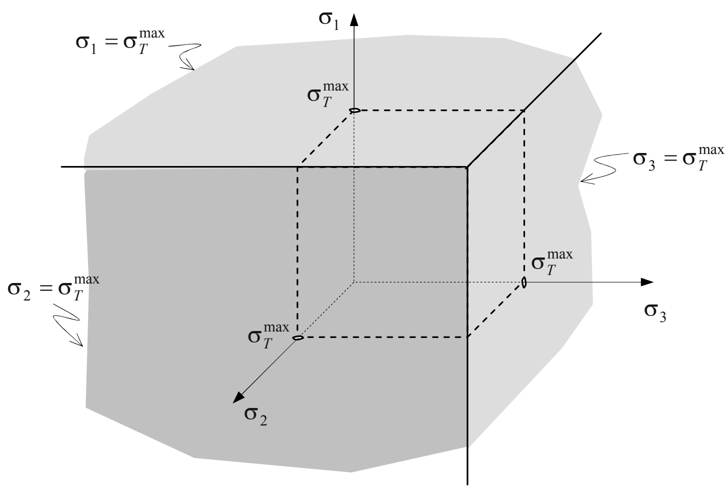


Figure 9.22: The Rankine criterion.

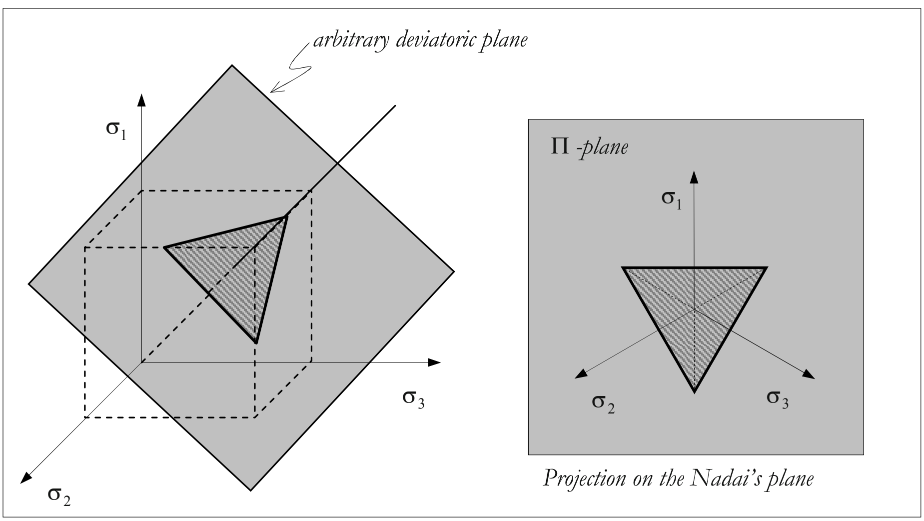


Figure 9.23: The Rankine yield criterion.

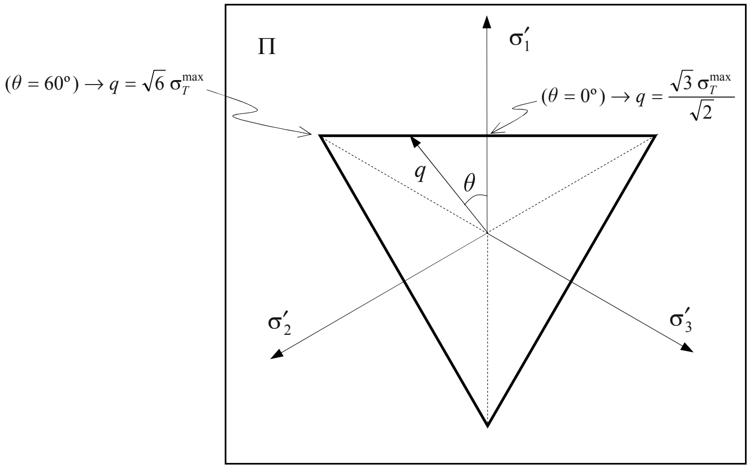


Figure 9.24: The Rankine yield criterion – Nadai's plane.

We can now verify that with a constant angle θ the equation (9.85) is linear with p and q. Then, when $q=0$ we obtain:

$$p=\sqrt{3}\,\sigma_T^{\max} \tag{9.87}$$

and when $p=0$ we find:

$$q=\frac{\sqrt{3}\,\sigma_T^{\max}}{\sqrt{2}\cos\theta} \tag{9.88}$$

and in the particular case when $\theta=0^{\text{o}}$ we obtain $q=\dfrac{\sqrt{3}\,\sigma_T^{\max}}{\sqrt{2}}$, and when $\theta=60^{\text{o}}$ we have $q=\sqrt{6}\,\sigma_T^{\max}$.

Then, in summary, the different ways of expressing the Rankine yield surface are:

$$\boxed{\begin{aligned}&\mathcal{F}(\sigma_I,\sigma_T^{\max})=\sigma_I-\sigma_T^{\max}=0\\&\mathcal{F}(I_{\boldsymbol{\sigma}},\mathrm{J}_2,\theta)=I_{\boldsymbol{\sigma}}+2\sqrt{3\,\mathrm{J}_2}\cos\theta-3\sigma_T^{\max}=0\\&\mathcal{F}(p,q,\theta)=p+\sqrt{2}\,q\cos\theta-\sqrt{3}\,\sigma_T^{\max}=0\end{aligned}}\quad \textit{Rankine yield surface} \tag{9.89}$$

Then, given the definition $\mathbf{n}\equiv\nabla_{\boldsymbol{\sigma}}\mathcal{F}\equiv\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}$, the gradient of the Rankine yield surface given in (9.82), $\mathcal{F}(\sigma_I,\sigma_T^{\max})=\sigma_I-\sigma_T^{\max}=0$, can be obtained as follows:

$$\mathcal{F}(\sigma_I,\sigma_T^{\max})=\sigma_I-\sigma_T^{\max}=0\quad\Rightarrow\quad \mathsf{n}_{ij}=\frac{\partial\mathcal{F}}{\partial\sigma_{ij}}=\frac{\partial\sigma_I}{\partial\sigma_{ij}}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \tag{9.90}$$

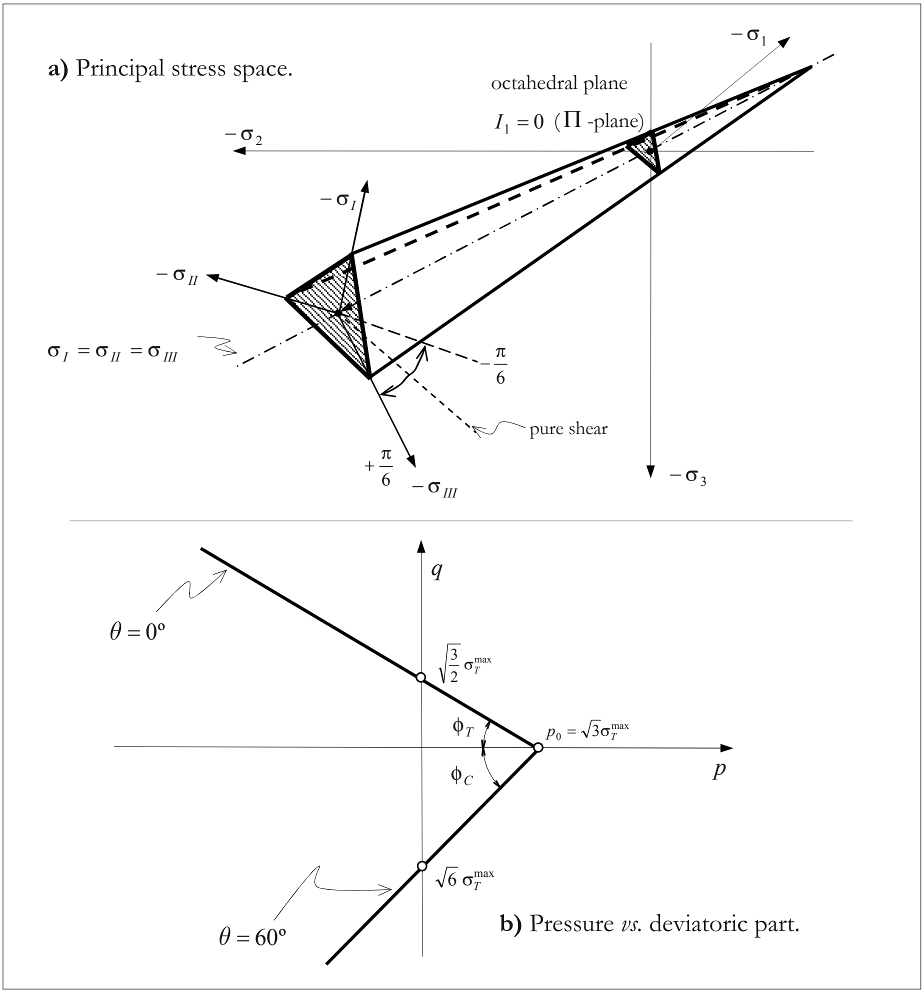


Figure 9.25: The Rankine yield surface.

9.2.5 Evolution of the Yield Surface

In order to fully describe material behavior, we must not just look at the onset of plastification, but rather at its evolution too, *i.e.* at how such material behavior evolves when undergoing loading/unloading/loading. From a material point of view, the yield surface can be altered after the start of plastification, but such changes depend on the type of material being analyzed. Next, we will discuss some ways of how the yield surface evolves, Chen&Han (1988).

The simplest model to characterize material behavior during plastification is the so-called *elastic-perfectly plasticity model*, which is characterized by a uniaxial stress-strain curve (as shown in Figure 9.28). In this scenario the yield surface remains unchanged during plastification. That is, the yield stress value is not affected, and the yield surface remains unaltered as plastification evolves.

Another idealized model used to show how plasticity evolves is known as hardening plasticity where we can emphasize two basic models: the isotropic and the kinematic. In uniaxial cases, the *isotropic hardening plasticity* model is represented in Figure 9.32. As we can see in said figure, as plastification evolves the elastic range develops symmetrically. In three-dimensional cases and (if we dealing with the Drucker-Prager yield surface) the evolution of this yield surface is as shown in Figure 9.26 in which we can observe that it does not change its shape but, rather, expands symmetrically during plastification.

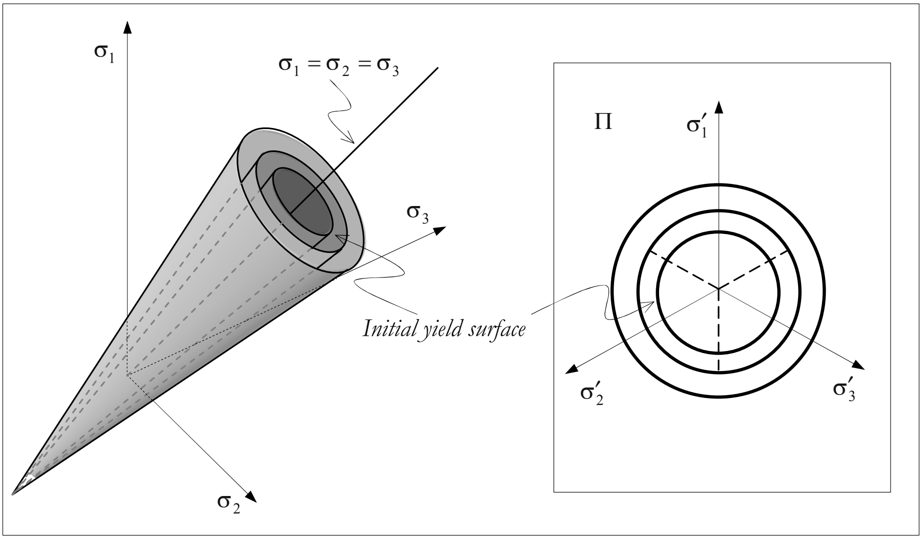


Figure 9.26: Evolution of the yield surface in the Drucker-Prager model with isotropic hardening.

The *kinematic hardening plasticity* model is characterized by the fact that the size and the shape of the elastic range do not change whilst plasticity evolves, but the elastic range is able to move. In Figure 9.37 we show a uniaxial stress-strain curve whose yield surface evolution neither changes its shape nor expands, but rather changes its position. Then, in a bidimensional case example, in which the initial yield curve is represented by the circumference in the principal stress space, if we dealing with the kinematic hardening model the yield curve only moves in this space, (see Figure 9.27).

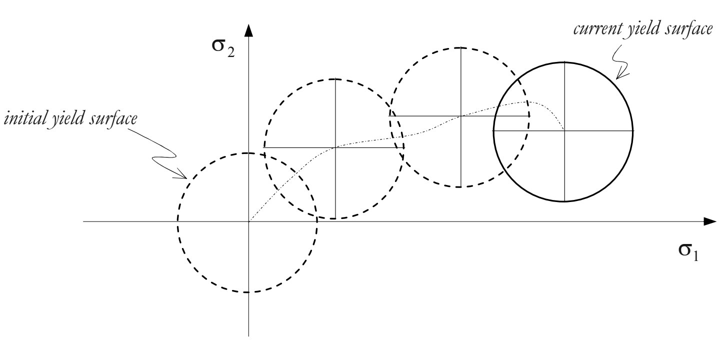


Figure 9.27: Evolution of the yield curve – kinematic hardening behavior.

It is possible to formulate more complex models by combining basic models, such as the *isotropic-kinematic hardening plasticity* model which considers isotropic and the kinematic behaviors simultaneously, *i.e.* the yield surface expands and can also move.

Then, we summarize some criterion to describe how plasticity evolves:

$$\text{Law of evolution of the Yield Surface}\begin{cases}\bullet\ \textit{Perfect Platicity} \\ \quad (\text{The surface does not evolve}) \\ \bullet\ \textit{Isotropic Hardening Plasticity} \\ \quad (\text{The surface evolves symmetrically}) \\ \bullet\ \textit{Kinematic Hardening Plasticity} \\ \quad (\text{The surface does not change its shape, just moves}) \\ \bullet\ \textit{Isotropic - Kinematic Hardening Plasticity} \\ \quad (\text{The surface expands and moves})\end{cases}$$

The following section will deal with the mathematical formulations that govern the models described above. For simplicity, we will use the one-dimensional case to describe and formulate the mathematical expressions that characterize each of these models, and then we will extend these models to the three-dimensional case (3D), (see Simo&Hughes (1998)).

9.3 Plasticity Models in Small Deformation Regime (Uniaxial Cases)

9.3.1 Rate-Independent Plasticity Models (Uniaxial Case)

Next, a few one-dimensional plasticity models will be described. However in some cases these simple models do not describe how most materials really act. Nevertheless, they are useful to gain and understanding of the plasticity mechanism, and can also be used to establish more complex models.

9.3.1.1 Perfect Elastoplastic Behavior

Let us assume that at a hypothetical material point, a material undergoing loading/unloading shows a stress-strain curve as described in Figure 9.28. Such behavior is known as *perfect plasticity*.

The loading and unloading steps represented in Figure 9.28 are indicated by the numbers: 1, 2, 3, 4, 5, 6, 7 and we can verify that after unloading (step 4) there is a residual strain, called the plastic strain denoted by $\varepsilon^{p^{(1)}}$, and the corresponding recoverable strain is called the elastic strain, $\varepsilon^{e^{(1)}}$.

The rheological model (a device used to interpret physical phenomena) used to represent the perfect elastoplastic behavior is made up of a linear spring and a Coulomb friction device in series, (see Figure 9.30). The behavior of each device in isolation can be appreciated in Figure 9.29. Note that until stress σ_Y is reached, the friction device does not suffer permanent deformation, and when it reaches the stress limit (which is not permitted to exceed σ_Y), it starts to deform without increasing stress. Note also that, when subjected to unloading, it maintains the strain value at the start.

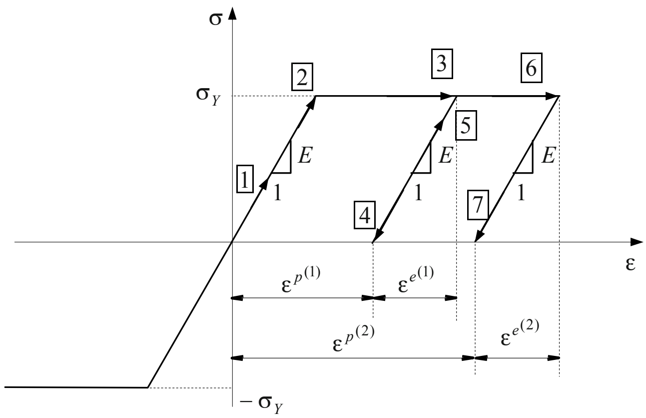


Figure 9.28: One-dimensional perfect plasticity.

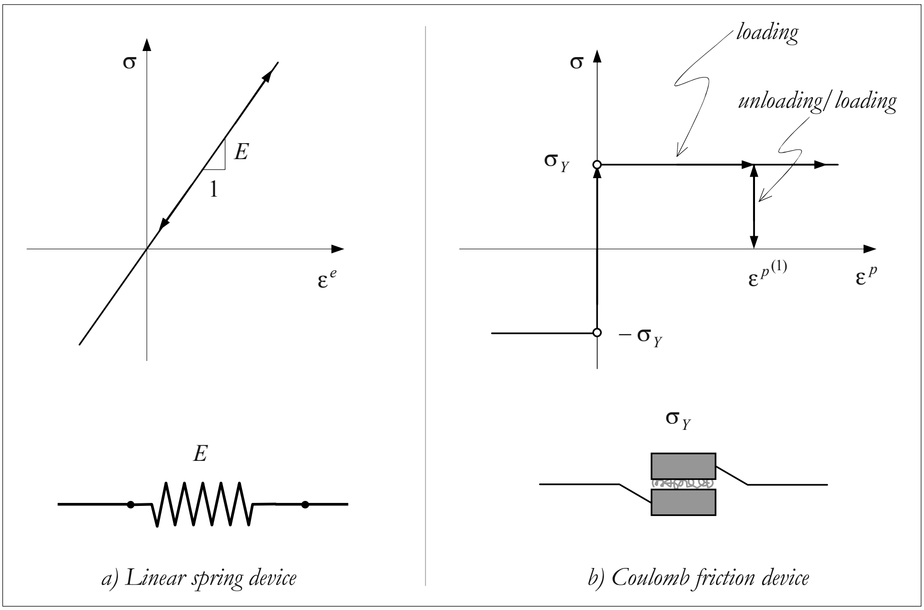


Figure 9.29: Behavior of some devices.

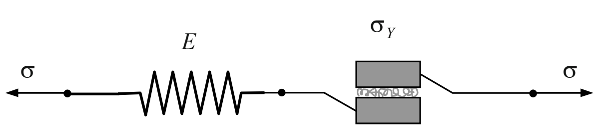


Figure 9.30: Rheological model for the perfect elastoplastic behavior.

A physical interpretation of the rheological model described in Figure 9.30 follows: in the beginning all the stress is absorbed by the spring device until the threshold σ_Y is reached. Afterwards, all the additional strain is absorbed by the Coulomb friction device, without there being any increase in stress (see Figure 9.28, steps 2-3-6). If there is any unloading the strain undergone by the Coulomb friction device can not be recovered, and the elastic device starts to recover its elastic strain, (see Figure 9.28 at the branch 4 or 7).

Next, we will establish the mathematical model that characterizes this type of behavior.

We can verify that the following holds:

- Additive decomposition of the strain into elastic and plastic parts:

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad \Rightarrow \qquad \varepsilon^e = \varepsilon - \varepsilon^p \tag{9.91}$$

- Constitutive equation for stress:

$$\sigma = E\varepsilon^e = E\left(\varepsilon - \varepsilon^p\right) \tag{9.92}$$

The Criterion of Plastification

If by means of the material behavior described in Figure 9.28, we can check that the magnitude can never be greater than σ_Y, then we can define the admissible stress space, $\mathcal{E}_{\boldsymbol{\sigma}}$, such as:

$$\mathcal{E}_{\boldsymbol{\sigma}} = \left\{\sigma \in \mathcal{R} \quad \middle| \quad \mathcal{F}(\sigma) = \left\|\sigma\right\| - \sigma_Y \leq 0\right\} \tag{9.93}$$

If we can verify that when the stress state satisfies $\mathcal{F}(\sigma) = \left\|\sigma\right\| - \sigma_Y < 0$ it means that there is no production of plastic strain, *i.e.* $\dot{\varepsilon}^p = 0$, then:

$$\mathcal{F}(\sigma) = \left\|\sigma\right\| - \sigma_Y < 0 \qquad \Rightarrow \qquad \dot{\varepsilon}^p = 0 \tag{9.94}$$

We can define the interior of that admissible stress space ($\operatorname{int}(\mathcal{E}_{\boldsymbol{\sigma}})$) as:

$$\operatorname{int}(\mathcal{E}_{\boldsymbol{\sigma}}) = \left\{\sigma \in \mathcal{R} \quad \middle| \quad \mathcal{F}(\sigma) = \left\|\sigma\right\| - \sigma_Y < 0\right\} \tag{9.95}$$

We can also define the yield surface as follows:

$$\partial\mathcal{E}_{\boldsymbol{\sigma}} = \left\{\sigma \in \mathcal{R} \quad \middle| \quad \mathcal{F}(\sigma) = \left\|\sigma\right\| - \sigma_Y = 0\right\} \tag{9.96}$$

where it meets the plastification criteria, and also holds that:

$$\mathcal{E}_{\boldsymbol{\sigma}} = \partial\mathcal{E}_{\boldsymbol{\sigma}} \cup \operatorname{int}(\mathcal{E}_{\boldsymbol{\sigma}}) \quad ; \quad \partial\mathcal{E}_{\boldsymbol{\sigma}} \cap \operatorname{int}(\mathcal{E}_{\boldsymbol{\sigma}}) = \varnothing \tag{9.97}$$

For one-dimensional case, the yield surface is limited by two points, and does not change during a loading/unloading/loading process, (see Figure 9.31).

Note that, since the stress space does not support the stress value $\left\|\sigma\right\| \not> \sigma_Y$ thus $\mathcal{F}(\sigma) \not> 0$, and also when $\mathcal{F}(\sigma) = \left\|\sigma\right\| - \sigma_Y < 0$ it implies that $\dot{\varepsilon}^p = 0$ whereas when $\dot{\varepsilon}^p \neq 0$ the implication is that $\mathcal{F}(\sigma) = \left\|\sigma\right\| - \sigma_Y = 0$, *i.e.* the material undergoes plastification when the stress state is on the yield surface.

We can now analyze what happens when the stress has negative values ($\sigma < 0$). To do so, we will adopt the scalar $\dot{\gamma} \geq 0$ known as the *plastic multiplier*. Then, it follows that:

$$\begin{aligned} \dot{\varepsilon}^p &= \dot{\gamma} \geq 0 \quad \textit{iff} \quad \sigma = \sigma_Y > 0 \\ \dot{\varepsilon}^p &= -\dot{\gamma} \leq 0 \quad \textit{iff} \quad \sigma = -\sigma_Y < 0 \end{aligned} \tag{9.98}$$

Then we can define the flow rule as:

$$\dot{\varepsilon}^p = \dot{\gamma}\,\operatorname{sign}(\sigma) \qquad \textit{Flow rule for perfect plasticity} \tag{9.99}$$

where we have introduced the *sign function* defined as:

$$\boxed{\operatorname{sign}(\sigma)=\begin{cases}+1 & if \quad \sigma>0\\ -1 & if \quad \sigma<0\end{cases}} \quad \textit{Sign function} \tag{9.100}$$

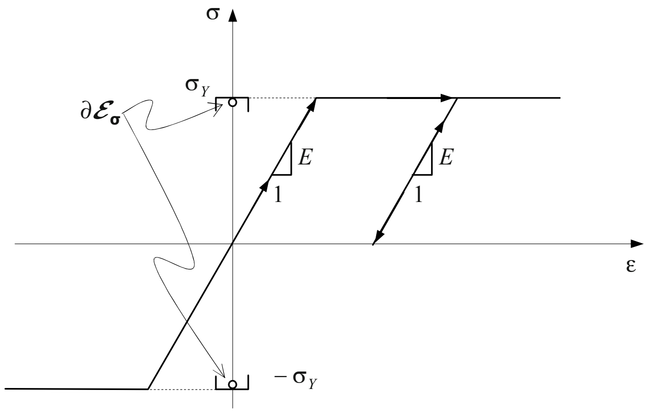


Figure 9.31: Perfect elastoplastic behavior.

The flow rule defines the rate of change of the plastic strain, and, in general, is given by:

$$\dot{\varepsilon}_{ij}^{p}=\dot{\gamma}\frac{\partial\Phi}{\partial\sigma_{ij}} \tag{9.101}$$

where Φ is a potential. In the particular case when Φ is equal to the yield surface, *i.e.* $\Phi=\mathcal{F}$, the flow rule is said to be *associated*, which is what happens in the case under consideration:

$$\dot{\varepsilon}^{p}=\dot{\gamma}\frac{\partial\mathcal{F}}{\partial\sigma}=\dot{\gamma}\frac{\partial\left(\|\sigma\|-\sigma_Y\right)}{\partial\sigma}=\dot{\gamma}\frac{\partial\left(\|\sigma\|\right)}{\partial\sigma}=\dot{\gamma}\operatorname{sign}(\sigma) \tag{9.102}$$

Then, in summary we have:

$$\begin{array}{lll} \dot{\gamma}\geq 0 \quad ; \quad \mathcal{F}(\sigma)\leq 0 & & \\ if \quad \mathcal{F}(\sigma)=\|\sigma\|-\sigma_Y<0 & \Rightarrow & \dot{\gamma}=0 \\ if \quad \mathcal{F}(\sigma)=\|\sigma\|-\sigma_Y=0 & \Rightarrow & \dot{\gamma}\neq 0 \end{array} \tag{9.103}$$

Now, all the conditions above can be unified into one single condition $\dot{\gamma}\mathcal{F}(\sigma)=0$. Then, we can introduce the *Kuhn-Tucker conditions*:

$$\boxed{\dot{\gamma}\geq 0 \quad ; \quad \mathcal{F}(\sigma)\leq 0 \quad ; \quad \dot{\gamma}\mathcal{F}(\sigma)=0} \quad \textit{The Kuhn-Tucker conditions} \tag{9.104}$$

Notice that, with respect to $\dot{\gamma}$, we have not yet fully characterized the mathematical model. Let us now consider a general scenario in which: at a certain instant in time t the variables $\varepsilon(t)$, $\varepsilon^{p}(t)$ and $\sigma(t)$ are known, and $\mathcal{F}(\sigma)=0$ is fulfilled, and by the fact that the point is in the plastification process we have $\dot{\gamma}>0$. The rate of change of $\dot{\mathcal{F}}(\sigma)$ can now be approached by time discretization by means of the intervals Δt in such a way that $\Delta t\dot{\mathcal{F}}(\sigma)=\mathcal{F}_{t+\Delta t}(\sigma)-\mathcal{F}_{t}(\sigma)=0$ is satisfied. Note that, for $\dot{\mathcal{F}}(\sigma)>0$ it implies that $\mathcal{F}_{t+\Delta t}(\sigma)>0$, whose result is inadmissible. That is, for any given stress state, if the point is outside the yield surface, the only way in which the condition $\mathcal{F}_{t+\Delta t}(\sigma)=0$ is satisfied occurs if the yield surface evolves, $\dot{\mathcal{F}}(\sigma)=0$. In perfect plasticity, this point cannot be outside the yield surface, the only possibility is by moving on the yield surface. Then, mathematically, we can summarize the previous comment as follows:

$$\left.\begin{matrix} \dot{\gamma}>0 & \textit{iff} & \dot{\mathcal{F}}=0 \\ \dot{\gamma}=0 & \textit{iff} & \dot{\mathcal{F}}<0 \end{matrix}\right\} \quad \Rightarrow \quad \dot{\gamma}\dot{\mathcal{F}}=0 \tag{9.105}$$

The latter condition is known as the consistency (or persistency) condition:

$$\boxed{\dot{\gamma}\dot{\mathcal{F}}=0} \quad \textit{The consistency (or persistency) condition} \tag{9.106}$$

Then, we can summarize perfect elastoplastic behavior, (see Simo&Hughes (1998)), by:

$$\boxed{\begin{array}{ll} \textbf{i.} & \text{Elastic stress - strain relationship} \\ & \qquad \sigma = E\varepsilon^{e} = E\left(\varepsilon-\varepsilon^{p}\right) \\ \textbf{ii.} & \text{Flow rule} \\ & \qquad \dot{\varepsilon}^{p} = \dot{\gamma}\,\mathsf{sign}(\sigma) \\ \textbf{iii.} & \text{Yield condition} \\ & \qquad \mathcal{F}(\sigma)=\|\sigma\|-\sigma_{Y}\leq 0 \\ \textbf{iv.} & \text{Kuhn - Tucker complementary condition} \\ & \qquad \dot{\gamma}\geq 0 \quad ; \quad \mathcal{F}(\sigma)\leq 0 \quad ; \quad \dot{\gamma}\mathcal{F}(\sigma)=0 \\ \textbf{v.} & \text{Consistency condition} \\ & \qquad \dot{\gamma}\dot{\mathcal{F}}=0 \end{array}} \quad \begin{array}{c}\textit{The perfect} \\ \textit{elastoplastic model}\end{array} \tag{9.107}$$

9.3.1.2 Isotropic Hardening Elastoplastic Behavior

Let us suppose now that a material point (particle) in a hypothetical material is undergoing the action of loading/unloading/loading in such a way that the stress-strain curve is that indicated in Figure 9.32. We can describe this behavior as *Isotropic Hardening Elastoplasticity*.

With regards to this figure, the loading and unloading steps are indicated by the numbers: 1, 2, 3, 4, 5, 6. We will check, once again, that after unloading step 4 there is a residual strain $\varepsilon^{p^{(1)}}$ (plastic strain) and a corresponding recoverable strain $\varepsilon^{e^{(1)}}$ (elastic strain). The difference between this and perfect elastoplastic behavior is: as the plastic strain evolves the elastic limits does so also, but symmetrically, *i.e.* the yield surface evolves symmetrically. As seen in Figure 9.32, before plastification begins the elastic limit in the material is defined by $[\sigma_Y,-\sigma_Y]$. When the stress state reaches point 3 unloading is applied, and we can then verify we have a new elastic limit defined by $[\sigma_Y^*,-\sigma_Y^*]$. Then, because the yield surface expands symmetrically this model is known as the isotropic hardening elastoplastic model.

The rheological model that shows isotropic hardening elastoplastic behavior is made up of a linear spring, a Coulomb friction device in parallel, and another linear spring device in series as shown in Figure 9.33.

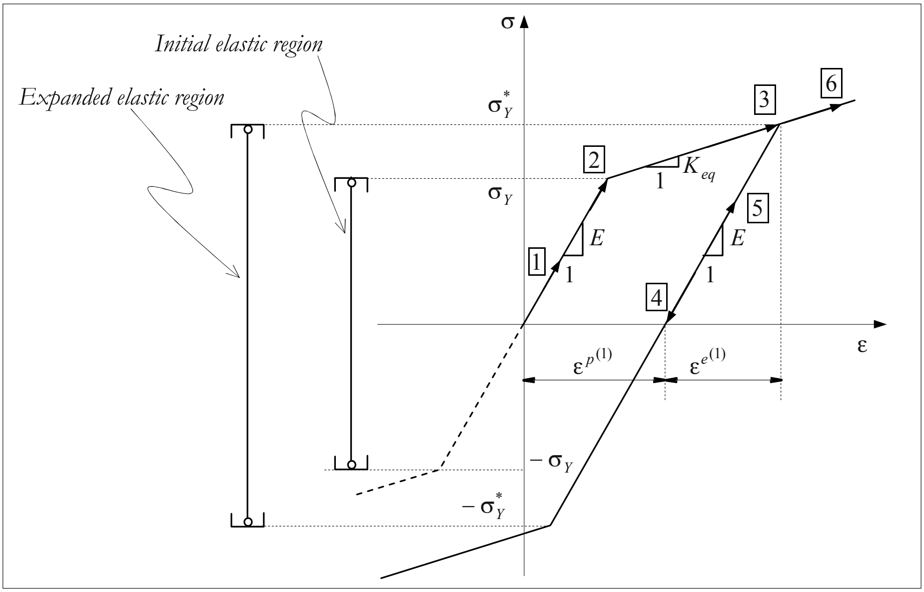


Figure 9.32: Isotropic hardening elastoplastic behavior.

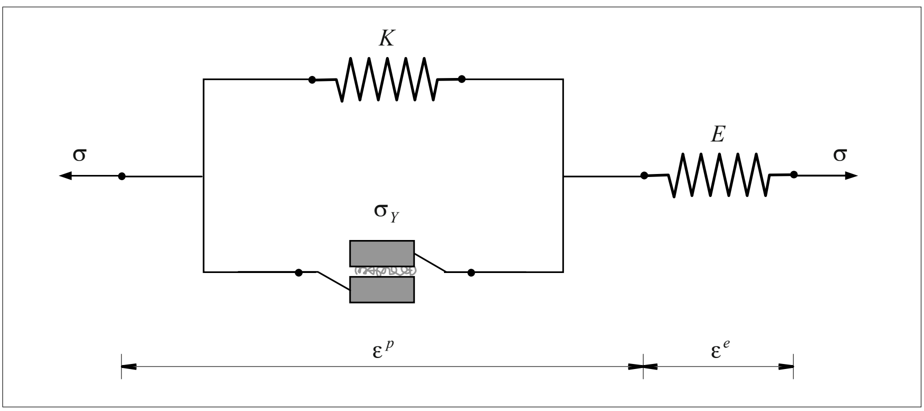


Figure 9.33: The rheological model for the isotropic hardening elastoplastic model.

Physically speaking, the rheological model shown in Figure 9.33 can be interpreted as follows. Initially, we just have the elastic strain, ε^e, which is caused by the spring with the constant E (since the Coulomb friction device does not deform for stress values less than σ_Y and because of this the spring with the constant K does not undergo deformation either). This load stage is typical of the initial elastic domain, and is shown by the branch $\boxed{1}$ in Figure 9.32. Note that here $\sigma = E\varepsilon^e \Rightarrow \dot{\sigma} = E\dot{\varepsilon}^e$ holds, *i.e.* the secant modulus and the tangent modulus are the same, *i.e.* $E^S = E^T = E$. Then, when the stress level reaches the value σ_Y, the Coulomb friction device starts to deform almost freely, *i.e.* the Coulomb friction device strain is controlled by the spring of constant K, and at this stage all the stress is absorbed by the springs. This stage is represented by the branch $\boxed{2-3}$ in Figure 9.32. Note that, in this model, the parameter K is constant during the deformation process, so, $\sigma = \sigma_Y + K\varepsilon^p \Rightarrow \dot{\sigma} = K\dot{\varepsilon}^p$ holds. Here, with $\boxed{2-3}$, we can state that the equivalent spring is that which results from those two in series, as shown in Figure 9.34. The constant of the equivalent spring, K_{eq}, is the tangent of the branch slope $\boxed{2-3-6}$.

$$\varepsilon = \varepsilon^p + \varepsilon^e$$

$$\Rightarrow \dot{\varepsilon} = \dot{\varepsilon}^p + \dot{\varepsilon}^e$$

$$\Rightarrow \dot{\varepsilon} = \frac{\dot{\sigma}}{K} + \frac{\dot{\sigma}}{E} = \frac{(K+E)}{KE}\dot{\sigma}$$

$$\Rightarrow \dot{\sigma} = \frac{KE}{(K+E)}\dot{\varepsilon}$$

$$\Rightarrow \dot{\sigma} = K_{eq}\dot{\varepsilon}$$

where

$$K_{eq} = \frac{KE}{(K+E)}$$

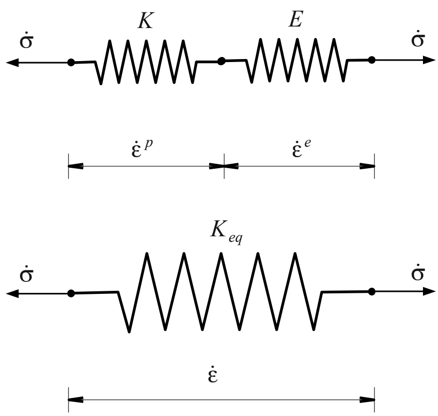


Figure 9.34: Linear springs disposed in series.

Once surpassed the threshold σ_Y, and then at the point $\boxed{3}$ of Figure 9.32 a decrease in stress (unloading) occurs, the plastic strain in the Coulomb friction device, $\varepsilon^{p^{(1)}}$, is maintained and consequently the strain in the spring K is also, which causes a new elastic limit in the material to appear, which is shown here by $\sigma_Y^* = \sigma_Y + K\varepsilon^{p^{(1)}}$. At this unloading stage, (see Figure 9.32, branch $\boxed{4}$), there is only strain recovery via the spring device E, *i.e.* an elastic process.

Next, we can establish the mathematical model that characterizes the behavior described above. We can verify that the following relations are still valid:

- Additive decomposition of the strain into elastic and plastic parts:

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad \Rightarrow \qquad \varepsilon^e = \varepsilon - \varepsilon^p \tag{9.108}$$

- Constitutive relation and its rate of change:

$$\sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p) \xrightarrow{\textit{rate of change}} \dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) \tag{9.109}$$

The Elastic domain and Yield surface

For isotropic hardening elastoplastic behavior the yield criterion is given by:

$$\mathcal{F}(\sigma, K\alpha) = \|\sigma\| - (\sigma_Y + K\alpha) \le 0 \tag{9.110}$$

where $\sigma_Y > 0$ and $K > 0$ are material constants and where K is known as the *plastic modulus*, (see Simo&Hughes(1998)). Note that the term $K\alpha$ has unit of stress, K has the same unit as E (unit of stress), and $\alpha \ge 0$ plays the role of plastic strain which is dimensionless, (see Figure 9.35). Then, by adopting the hardening hypothesis $\dot{\alpha} = \|\dot{\varepsilon}^p\|$ and by considering that $\dot{\varepsilon}^p = \dot{\gamma}\,\mathrm{sign}(\sigma)$, we can obtain $\dot{\alpha} = \dot{\gamma}$.

We can define the admissible stress space, (see Figure 9.35(a)), as:

$$\mathcal{E}_{\boldsymbol{\sigma}} = \{\sigma \in \mathcal{R} \quad | \quad \mathcal{F}(\sigma, K\alpha) = \|\sigma\| - (\sigma_Y + K\alpha) \leq 0\} \tag{9.111}$$

the elastic domain as:

$$\operatorname{int}(\mathcal{E}_{\boldsymbol{\sigma}}) = \{\sigma \in \mathcal{R} \quad | \quad \mathcal{F}(\sigma, K\alpha) = \|\sigma\| - (\sigma_Y + K\alpha) < 0\} \tag{9.112}$$

and the yield surface as:

$$\partial\mathcal{E}_{\boldsymbol{\sigma}} = \{\sigma \in \mathcal{R} \quad | \quad \mathcal{F}(\sigma, K\alpha) = \|\sigma\| - (\sigma_Y + K\alpha) = 0\} \tag{9.113}$$

The elastic domain, $\operatorname{int}(\mathcal{E}_{\boldsymbol{\sigma}})$, together with its boundary, $\partial\mathcal{E}_{\boldsymbol{\sigma}}$, describes the admissible stress space $\mathcal{E}_{\boldsymbol{\sigma}}$, *i.e.*:

$$\mathcal{E}_{\boldsymbol{\sigma}} = \operatorname{int}(\mathcal{E}_{\boldsymbol{\sigma}}) \cup \partial\mathcal{E}_{\boldsymbol{\sigma}} \tag{9.114}$$

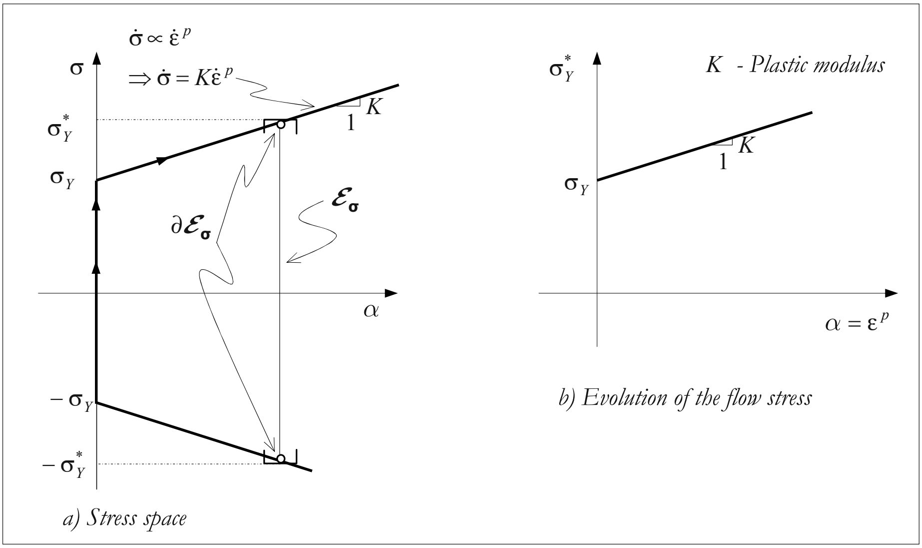


Figure 9.35: Isotropic hardening elastoplastic behavior.

The Kuhn-Tucker conditions are still valid:

$$\boxed{\dot{\gamma} \geq 0 \quad ; \quad \mathcal{F}(\sigma, K\alpha) \leq 0 \quad ; \quad \dot{\gamma}\,\mathcal{F}(\sigma, K\alpha) = 0} \quad \textit{The Kuhn-Tucker conditions} \tag{9.115}$$

as is the consistency condition:

$$\boxed{\dot{\gamma}\,\dot{\mathcal{F}} = 0} \quad \textit{The consistency condition} \tag{9.116}$$

The consistency condition allows us to obtain the explicit relationship for $\dot{\gamma}$. That is, when $\dot{\gamma} > 0$, we must satisfy $\dot{\mathcal{F}} = \dot{\mathcal{F}}(\sigma, K\alpha) = 0$:

$$\dot{\mathcal{F}}(\sigma, K\alpha) = \frac{\partial \mathcal{F}}{\partial \sigma}\dot{\sigma} + \frac{\partial \mathcal{F}}{\partial K\alpha}\frac{D(K\alpha)}{Dt} = \frac{\partial \mathcal{F}}{\partial \sigma}\dot{\sigma} + (-1)(\dot{K}\alpha + K\dot{\alpha}) = 0 \tag{9.117}$$

Then, according to the equation in (9.110) we can verify that $\dfrac{\partial \mathcal{F}}{\partial \sigma} = \operatorname{sign}(\sigma)$. In addition, if K is a constant variable we have $\dot{K} = 0$, and by considering $\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$ and $\dot{\alpha} = \dot{\gamma}$, the equation in (9.117) becomes:

$$\dot{\mathcal{F}}(\sigma, K\alpha) = \text{sign}(\sigma)\, E\left(\dot{\varepsilon} - \dot{\varepsilon}^p\right) - K\,\dot{\gamma} = \text{sign}(\sigma)\, E\,\dot{\varepsilon} - \text{sign}(\sigma)\, E\,\dot{\varepsilon}^p - K\,\dot{\gamma}$$
$$= \text{sign}(\sigma)\, E\,\dot{\varepsilon} - \text{sign}(\sigma)\, E\,\dot{\gamma}\,\text{sign}(\sigma) - K\,\dot{\gamma} \tag{9.118}$$

Note that $[\text{sign}(\sigma)]^2 = 1$, so the above equation becomes:

$$\dot{\mathcal{F}}(\sigma, \alpha) = \text{sign}(\sigma)\, E\,\dot{\varepsilon} - E\,\dot{\gamma} - K\,\dot{\gamma} = 0 \quad \Rightarrow \quad \dot{\gamma} = \frac{\text{sign}(\sigma)\, E\,\dot{\varepsilon}}{(E+K)} \tag{9.119}$$

Then, considering the equation in (9.102) we can obtain:

$$\dot{\varepsilon}^p = \dot{\gamma}\,\text{sign}(\sigma) = \frac{\text{sign}(\sigma)\, E\,\dot{\varepsilon}}{(E+K)}\,\text{sign}(\sigma) = \frac{E\,\dot{\varepsilon}}{(E+K)} \tag{9.120}$$

The Elastoplastic Tangent Stiffness Modulus

The rate of change of the stress-strain relationship can be obtained as follows:

$$\dot{\sigma} = E\left(\dot{\varepsilon} - \dot{\varepsilon}^p\right) = E\left(\dot{\varepsilon} - \frac{E\,\dot{\varepsilon}}{(E+K)}\right) = \frac{E\,K}{(E+K)}\dot{\varepsilon} = E^{ep}\,\dot{\varepsilon} \tag{9.121}$$

where E^{ep} is the *elastoplastic tangent stiffness modulus* which is the same as the constant of the equivalent spring obtained in Figure 9.34.

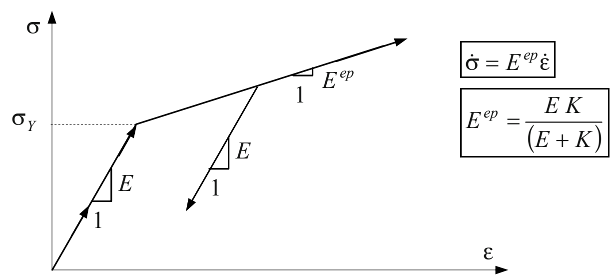


Figure 9.36: Elastoplastic tangent stiffness modulus.

Let us consider the equation in (9.120) for when $\dot{\gamma} > 0$, then we obtain:

$$\dot{\varepsilon}^p = \frac{E\,\dot{\varepsilon}}{(E+K)} \quad \Rightarrow \quad \dot{\varepsilon} = \frac{(E+K)}{E}\dot{\varepsilon}^p \tag{9.122}$$

and by substituting this into the equation in (9.121), we obtain:

$$\dot{\sigma} = \frac{E\,K}{(E+K)}\dot{\varepsilon} = \frac{E\,K}{(E+K)}\frac{(E+K)}{E}\dot{\varepsilon}^p \quad \Rightarrow \quad \dot{\sigma} = K\,\dot{\varepsilon}^p \tag{9.123}$$

Note that K is the line slope of the graph $\sigma \times \varepsilon^p$ described in Figure 9.35(b).

The constitutive relations are now defined as follows:

- Elastic regime (loading/unloading)

$$\mathcal{F}(\sigma, K\alpha) < 0 \quad ; \quad \sigma \in \text{int}\left(\mathcal{E}_{\boldsymbol{\sigma}}\right)\} \quad \Rightarrow \quad \dot{\sigma} = E\,\dot{\varepsilon} \tag{9.124}$$

- Elastoplastic regime when unloading

$$\dot{\mathcal{F}}(\sigma, K\alpha) < 0 \quad ; \quad \sigma \in \text{int}\left(\mathcal{E}_{\boldsymbol{\sigma}}\right)\} \quad \Rightarrow \quad \dot{\sigma} = E\,\dot{\varepsilon} \tag{9.125}$$

- Elastoplastic regime when loading

$$\left.\dot{\mathcal{F}}(\sigma, K\alpha) = 0 \quad ; \quad \sigma \in \partial \mathcal{E}_{\boldsymbol{\sigma}}\right\} \quad \Rightarrow \quad \dot{\sigma} = \frac{E\,K}{(E+K)}\dot{\varepsilon} = E^{ep}\,\dot{\varepsilon} \tag{9.126}$$

Then, we can summarize the mathematical model, (see Simo&Hughes(1998)), as:

The isotropic hardening elastoplastic model (9.127)

i. Elastic stress - strain relationship

$$\sigma = E\varepsilon^{e} = E\left(\varepsilon - \varepsilon^{p}\right)$$

ii. Flow rule

$$\dot{\varepsilon}^{p} = \dot{\gamma}\,\mathsf{sign}(\sigma)$$

iii. Isotropic hardening law

$$\dot{\alpha} = \dot{\gamma}$$

iv. Yield condition

$$\mathcal{F}(\sigma, K\alpha) = \|\sigma\| - (\sigma_{Y} + K\alpha) \leq 0$$

v. The Kuhn - Tucker complementary conditions

$$\dot{\gamma} \geq 0 \quad ; \quad \mathcal{F}(\sigma, K\alpha) \leq 0 \quad ; \quad \dot{\gamma}\,\mathcal{F}(\sigma, K\alpha) = 0$$

vi. The consistency condition

$$\dot{\gamma}\,\dot{\mathcal{F}} = 0$$

9.3.1.3 Kinematic Hardening Elastoplastic Behavior

Kinematic hardening elastoplastic behavior was first described by Prager (1955). In this model the shape of the yield surface, in the principal stress space, is not altered, however it can move. In uniaxial cases the yield surface is made up of two points, and additionally, in kinematic hardening the distance between these does not change during plastification, (see Figure 9.37).

Now, the yield criterion for kinematic hardening elastoplastic behavior is given by:

$$\mathcal{F}(\sigma, q) = \|\sigma - q\| - \sigma_{Y} = 0 \tag{9.128}$$

The internal variable q is associated with the new position of the yield surface center in the principal stress space, (see Figure 9.37) as well as being a function of the plastic strain, for example we can adopt a linear relationship, *e.g.*:

$$\dot{q} = H\,\dot{\varepsilon}^{p} \tag{9.129}$$

where H is known as the kinematic hardening modulus (material property) and generally speaking it is a tangent modulus of the curve $q \times \varepsilon^{p}$, (see Figure 9.37). In the case under consideration we will consider H to be a constant variable.

In this model the rate of change of the plastic strain is defined as follows:

$$\left.\begin{aligned} \dot{\varepsilon}^{p} &= \dot{\gamma} > 0 \quad \textit{if} \quad \sigma - q = \sigma_{Y} > 0 \\ \dot{\varepsilon}^{p} &= -\dot{\gamma} < 0 \quad \textit{if} \quad \sigma - q = -\sigma_{Y} < 0 \end{aligned}\right\} \quad \Rightarrow \quad \dot{\varepsilon}^{p} = \dot{\gamma}\,\mathsf{sign}(\sigma - q) \tag{9.130}$$

where

$$\mathsf{sign}(\sigma - q) = \begin{cases} +1 & \textit{if} \quad \sigma - q > 0 \\ -1 & \textit{if} \quad \sigma - q < 0 \end{cases} \tag{9.131}$$

Then, if we consider the equations in (9.129) and (9.130), we obtain:

$$\dot{q} = H\,\dot{\varepsilon}^{p} = H\,\dot{\gamma}\,\mathsf{sign}(\sigma - q) \tag{9.132}$$

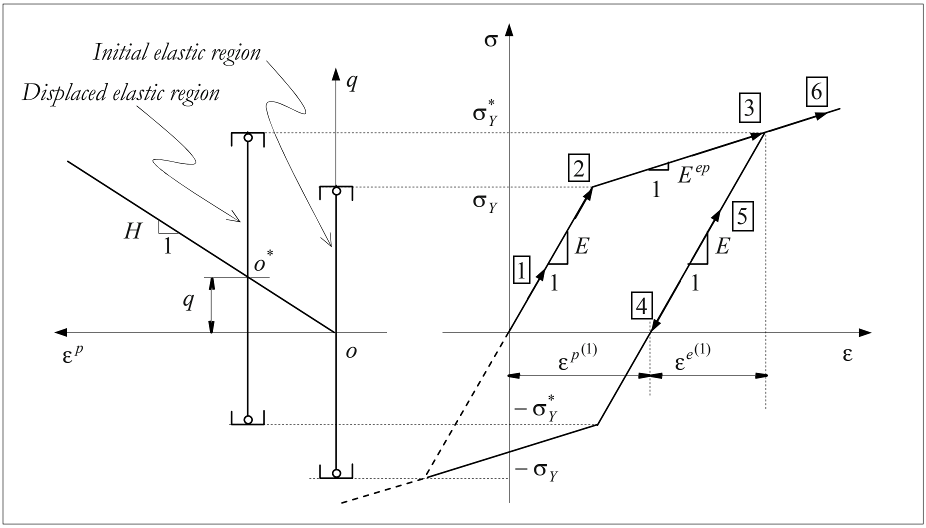


Figure 9.37: Kinematic hardening elastoplastic behavior.

Then, we can summarize the mathematical model, (see Simo&Hughes(1998)), as:

i. Elastic stress - strain relationship

$$\sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p)$$

ii. Flow rule

$$\dot{\varepsilon}^p = \dot{\gamma}\,\mathsf{sign}(\sigma - q)$$

iii. Kinematic hardening law

$$\dot{q} = H\,\dot{\varepsilon}^p$$

iv. Yield condition

$$\mathcal{F}(\sigma, q) = \|\sigma - q\| - \sigma_Y \le 0$$

v. Kuhn - Tucker complementary condition

$$\dot{\gamma} \ge 0 \quad ; \quad \mathcal{F}(\sigma, q) \le 0 \quad ; \quad \dot{\gamma}\,\mathcal{F}(\sigma, q) = 0$$

vi. Consistency condition

$$\dot{\gamma}\,\dot{\mathcal{F}} = 0$$

The kinematic hardening elastoplastic model (9.133)

The Elastoplastic Tangent Stiffness Modulus

As we saw before, the consistency condition, $\dot{\gamma}\,\dot{\mathcal{F}} = 0$, allows us to obtain the explicit relation for $\dot{\gamma} > 0$ after which we take the rate of change of $\mathcal{F} = \mathcal{F}(\sigma, q)$:

$$\dot{\mathcal{F}}(\sigma, q) = \frac{\partial \mathcal{F}}{\partial \sigma}\dot{\sigma} + \frac{\partial \mathcal{F}}{\partial q}\dot{q} = 0 \tag{9.134}$$

Then, according to the equations given in (9.133), we can verify that

$\dfrac{\partial \mathcal{F}}{\partial \sigma} = \dfrac{\partial \mathcal{F}}{\partial \|\sigma - q\|}\dfrac{\partial \|\sigma - q\|}{\partial \sigma} = \mathsf{sign}(\sigma - q)$, $\dfrac{\partial \mathcal{F}}{\partial q} = \dfrac{\partial \mathcal{F}}{\partial \|\sigma - q\|}\dfrac{\partial \|\sigma - q\|}{\partial q} = -\mathsf{sign}(\sigma - q)$, and also if we consider that $\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$ and $\dot{q} = \dot{\gamma}\,H\,\mathsf{sign}(\sigma - q)$, the equation in (9.134) becomes:

$$
\begin{aligned}
\dot{\mathcal{F}}(\sigma,q) &= \frac{\partial \mathcal{F}}{\partial \sigma}\dot{\sigma} + \frac{\partial \mathcal{F}}{\partial q}\dot{q} = 0 \\
&= \mathrm{sign}(\sigma-q)\,\dot{\sigma} - \mathrm{sign}(\sigma-q)\,\dot{q} = 0 \\
&= \mathrm{sign}(\sigma-q)E\left(\dot{\varepsilon}-\dot{\varepsilon}^{p}\right) - \mathrm{sign}(\sigma-q)\dot{\gamma}H\,\mathrm{sign}(\sigma-q) = 0 \\
&= \mathrm{sign}(\sigma-q)E\dot{\varepsilon} - \mathrm{sign}(\sigma-q)E\dot{\varepsilon}^{p} - \dot{\gamma}H = 0 \\
&= \mathrm{sign}(\sigma-q)E\dot{\varepsilon} - \mathrm{sign}(\sigma-q)E\dot{\gamma}\,\mathrm{sign}(\sigma-q) - \dot{\gamma}H = 0 \\
&= \mathrm{sign}(\sigma-q)E\dot{\varepsilon} - \dot{\gamma}[E+H] = 0
\end{aligned}
\tag{9.135}
$$

Thus

$$
\dot{\gamma} = \frac{\mathrm{sign}(\sigma-q)E}{[E+H]}\dot{\varepsilon} \quad \Rightarrow \quad \dot{\varepsilon} = \frac{[E+H]}{\mathrm{sign}(\sigma-q)E}\dot{\gamma} \tag{9.136}
$$

after which the rate of change of the plastic strain becomes:

$$
\dot{\varepsilon}^{p} = \dot{\gamma}\,\mathrm{sign}(\sigma-q) = \frac{\mathrm{sign}(\sigma-q)E}{[E+H]}\dot{\varepsilon}\,\mathrm{sign}(\sigma-q) = \frac{E}{[E+H]}\dot{\varepsilon} \tag{9.137}
$$

Then, by means of the rate of change of the stress-strain relationship we can obtain the elastoplastic tangent stiffness modulus E^{ep}, *i.e.*:

$$
\dot{\sigma} = E\left(\dot{\varepsilon}-\dot{\varepsilon}^{p}\right) = E\left(\dot{\varepsilon} - \frac{E}{[E+H]}\dot{\varepsilon}\right) = \frac{EH}{[E+H]}\dot{\varepsilon} = E^{ep}\,\dot{\varepsilon} \qquad \therefore \qquad E^{ep} = \frac{EH}{[E+H]} \tag{9.138}
$$

Thus, we summarize that:

$$
\dot{\sigma} = E\,\dot{\varepsilon} \quad \text{if} \quad \dot{\gamma} = 0 \qquad \text{and} \qquad \dot{\sigma} = \frac{EH}{[E+H]}\dot{\varepsilon} \quad \text{if} \quad \dot{\gamma} > 0 \tag{9.139}
$$

9.3.1.4 Isotropic-Kinematic Elastoplastic Behavior

This model is a combination of the models discussed previously. Here, the yield criterion can be defined as follows:

$$
\mathcal{F}(\sigma,q,K\alpha) = \|\sigma-q\| - (\sigma_Y + K\alpha) \le 0 \tag{9.140}
$$

We can summarize the mathematical model, (see Simo&Hughes(1998)), as:

i. Elastic stress - strain relationship

$$\sigma = E\varepsilon^{e} = E\left(\varepsilon - \varepsilon^{p}\right)$$

ii. Flow rule

$$\dot{\varepsilon}^{p} = \dot{\gamma}\,\mathrm{sign}(\sigma-q)$$

iii. Isotropic - Kinematic hardening law

$$\dot{q} = \dot{\gamma}\,H\,\mathrm{sign}(\sigma-q)$$

$$\dot{\alpha} = \dot{\gamma}$$

iv. Yield condition

$$\mathcal{F}(\sigma,q,K\alpha) = \|\sigma-q\| - (\sigma_Y + K\alpha) \le 0$$

v. Kuhn - Tucker complementary condition

$$\dot{\gamma} \ge 0 \quad ; \quad \mathcal{F}(\sigma,q,K\alpha) \le 0 \quad ; \quad \dot{\gamma}\mathcal{F}(\sigma,q,K\alpha) = 0$$

vi. Consistency condition

$$\dot{\gamma}\dot{\mathcal{F}} = 0$$

The isotropic-kinematic elastoplastic model (9.141)

The Elastoplastic Tangent Stiffness Modulus

Once again, we will use the consistency condition, $\dot{\mathcal{F}}\,\dot{\gamma}=0$, to obtain $\dot{\gamma}$. Firstly, we will calculate the rate of change of $\mathcal{F}(\sigma,q,\alpha)$ by means of the chain rule of the derivative, *i.e.:*

$$\dot{\mathcal{F}}(\sigma,q,K\alpha)=\frac{\partial\mathcal{F}}{\partial\sigma}\dot{\sigma}+\frac{\partial\mathcal{F}}{\partial q}\dot{q}+\frac{\partial\mathcal{F}}{\partial(K\alpha)}\frac{D(K\alpha)}{Dt}\le 0 \tag{9.142}$$

Then, according to the equation in (9.141), we will verify that $\frac{\partial\mathcal{F}}{\partial\sigma}=\frac{\partial\mathcal{F}}{\partial\|\sigma-q\|}\frac{\partial\|\sigma-q\|}{\partial\sigma}=\mathrm{sign}(\sigma-q)$, $\frac{\partial\mathcal{F}}{\partial q}=\frac{\partial\mathcal{F}}{\partial\|\sigma-q\|}\frac{\partial\|\sigma-q\|}{\partial q}=-\,\mathrm{sign}(\sigma-q)$, $\frac{\partial\mathcal{F}}{\partial(K\alpha)}\frac{D(K\alpha)}{Dt}=-K\dot{\alpha}$, and also if we consider that $\dot{\sigma}=E\left(\dot{\varepsilon}-\dot{\varepsilon}^{p}\right)$, $\dot{\alpha}=\dot{\gamma}$ and $\dot{q}=\dot{\gamma}\,H\,\mathrm{sign}(\sigma-q)$, the equation in (9.142) becomes:

$$\begin{aligned}\dot{\mathcal{F}}(\sigma,q,K\alpha)&=\frac{\partial\mathcal{F}}{\partial\sigma}\dot{\sigma}+\frac{\partial\mathcal{F}}{\partial q}\dot{q}+\frac{\partial\mathcal{F}}{\partial(K\alpha)}\frac{D(K\alpha)}{Dt}=0\\&=\mathrm{sign}(\sigma-q)\,\dot{\sigma}-\mathrm{sign}(\sigma-q)\,\dot{q}-K\,\dot{\alpha}=0\\&=\mathrm{sign}(\sigma-q)E\left(\dot{\varepsilon}-\dot{\varepsilon}^{p}\right)-\mathrm{sign}(\sigma-q)\dot{\gamma}\,H\,\mathrm{sign}(\sigma-q)-K\dot{\gamma}=0\\&=\mathrm{sign}(\sigma-q)E\dot{\varepsilon}-\mathrm{sign}(\sigma-q)E\dot{\varepsilon}^{p}-\dot{\gamma}[H+K]=0\\&=\mathrm{sign}(\sigma-q)E\dot{\varepsilon}-\mathrm{sign}(\sigma-q)E\dot{\gamma}\,\mathrm{sign}(\sigma-q)+\dot{\gamma}[H-K]=0\\&=\mathrm{sign}(\sigma-q)E\dot{\varepsilon}-\dot{\gamma}[E+H-K]=0\end{aligned} \tag{9.143}$$

Thus

$$\dot{\gamma}=\frac{\mathrm{sign}(\sigma-q)E}{[E+H+K]}\dot{\varepsilon}\qquad\Rightarrow\qquad\dot{\varepsilon}=\frac{[E+H+K]}{\mathrm{sign}(\sigma-q)E}\dot{\gamma} \tag{9.144}$$

after which we can obtain the rate of change of the plastic strain as follows:

$$\dot{\varepsilon}^{p}=\dot{\gamma}\,\mathrm{sign}(\sigma-q)=\frac{\mathrm{sign}(\sigma-q)E}{[E+H+K]}\dot{\varepsilon}\,\mathrm{sign}(\sigma-q)=\frac{E}{[E+H+K]}\dot{\varepsilon} \tag{9.145}$$

Then, by means of the rate of change of the stress-strain relationship we can obtain the elastoplastic tangent stiffness modulus E^{ep}, *i.e.*:

$$\dot{\sigma}=E\left(\dot{\varepsilon}-\dot{\varepsilon}^{p}\right)=E\left(\dot{\varepsilon}-\frac{E}{[E+H+K]}\dot{\varepsilon}\right)=\frac{E(H+K)}{[E+H+K]}\dot{\varepsilon}=E^{ep}\dot{\varepsilon}\quad\therefore\quad E^{ep}=\frac{E(H+K)}{[E+H+K]} \tag{9.146}$$

Thus, in summary we have:

$$\dot{\sigma}=E\,\dot{\varepsilon}\quad \mathit{if}\quad \dot{\gamma}=0\qquad \mathit{and}\qquad \dot{\sigma}=\frac{E(H+K)}{[E+H+K]}\dot{\varepsilon}=E^{ep}\dot{\varepsilon}\quad \mathit{if}\quad \dot{\gamma}>0 \tag{9.147}$$

9.4 Plasticity in Small Deformation Regime (The Classical Plasticity Theory)

In this section we generalize the theory of plasticity for three-dimensional cases. We will consider the small deformation regime, (see subsection 2.14 in Chapter 2), and, additionally, we will make the following assumptions:

- There is material isotropy. That is, during plastification the material does not lose its isotropy;
- The plastic strain produces no change in volume;
- The process is isothermal and adiabatic. That is, during plastification, the effect of temperature is not taken into account, *i.e.* the constitutive equation is independent of temperature.

The production of plastic strain is associated with internal energy dissipation, *i.e.* with irreversible processes. To put it another way: a process which involves plastic strain is path dependent, for example, in Figure 9.38 the points A and B are associated with the same stress but have different strain values, so, here we have a process that depends on the load history. Now, we will adopt the constitutive equation with internal variables, (see subsection 6.4.1 in Chapter 6), in order to describe the motion history, albeit indirectly.

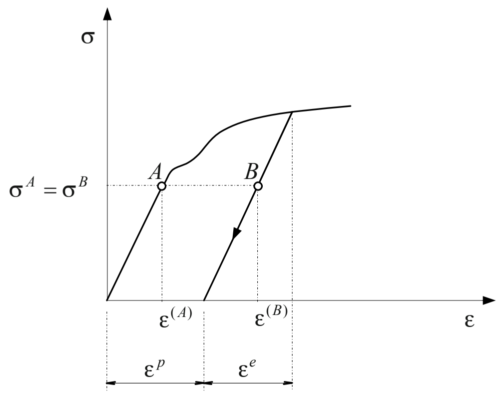


Figure 9.38: Plasticity – dependence of the load history.

9.4.1 The Infinitesimal Strain Tensor and Constitutive Equation

With small deformation, we use additive decomposition of the infinitesimal strain tensor into elastic and plastic parts, *i.e.*:

$$\boxed{\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p} \quad \textit{Additive decomposition of the infinitesimal strain tensor} \tag{9.148}$$

where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, whose components in the Cartesian basis are $\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p$, (see Chapter 2 in subsection 2.14). The assumption in (9.148) forms the basis of the classical theory of plasticity, which is valid in a small deformation regime.

Then, if we consider (9.148) the constitutive equation for stress becomes:

$$\boxed{\boldsymbol{\sigma} = \mathbb{C}^e : \boldsymbol{\varepsilon}^e = \mathbb{C}^e : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \quad ; \quad \boldsymbol{\varepsilon}^e = \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} \tag{9.149}$$

where $\mathbb{C}^e = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$ is the elasticity tensor for isotropic materials, and the elastic compliance tensor is given by $\mathbb{C}^{e^{-1}} = \dfrac{-\lambda}{2\mu(3\lambda + 2\mu)} \mathbf{1} \otimes \mathbf{1} + \dfrac{1}{2\mu} \mathbf{I}$, where λ, μ are the Lamé constants, (see Chapter 7).

9.4.2 Helmholtz Free Energy

We apply the constitutive equation with internal variables, (see Chapter 6), where, in general, the Helmholtz free energy, $\psi = \psi(\boldsymbol{\varepsilon}, T, \overline{\boldsymbol{\alpha}}_k)$, is described in terms of the infinitesimal strain tensor ($\boldsymbol{\varepsilon}$), temperature (T), and a set of internal variables ($\overline{\boldsymbol{\alpha}}_k$), which could be a scalar, vector, or higher order tensor. Then, if we consider the process to be isothermal, we have $\psi = \psi(\boldsymbol{\varepsilon}, \overline{\boldsymbol{\alpha}}_k)$. We can now reformulate this energy expression, in order to obtain the elastic part of $\boldsymbol{\varepsilon}$ as a free variable, *i.e.*: $\psi = \psi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}_k)$. Now the set of internal variables $\boldsymbol{\alpha}_k$ do not include the plastic component ($\boldsymbol{\varepsilon}^p$) of the strain tensor, because it is already included in the free variable $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$. Then, the following is satisfied:

$$\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^p} \tag{9.150}$$

Now, the rate of change of the Helmholtz free energy $\psi = \psi(\boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}_k)$ becomes:

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e + \frac{\partial \psi}{\partial \boldsymbol{\alpha}_k} \square \dot{\boldsymbol{\alpha}}_k \tag{9.151}$$

NOTE: The operator $\square$ is substituted by the number of contractions of the order of $\boldsymbol{\alpha}$. That is, if $\boldsymbol{\alpha}$ is a scalar (zeroth-order tensor), $\square$ has no contraction; if $\boldsymbol{\alpha}$ is a vector (first-order tensor), $\square = \cdot$ there is one contraction, *i.e.* the scalar (dot) product; if $\boldsymbol{\alpha}$ is second-order tensor, $\square =$**:** there are two contractions, *i.e.* the double scalar product; and so on. ■

9.4.3 Internal Energy Dissipation and the Evolution of the Internal Variables

As we discussed in Chapter 6, the entropy inequality is not an additional equation to the governing equations, but rather it is used to put restrictions on the variables of the problems. That is, the entropy inequality tells us how the internal variables must evolve during plastification. Then, by using the Clausius-Duhem inequality, (see Chapter 5), we obtain:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \mathbf{D} - \rho \eta \dot{T} - \rho \dot{\psi} - \frac{1}{T} \vec{\mathbf{q}} \cdot \nabla_{\vec{x}} T \geq 0 \quad \xrightarrow[\text{process}]{\text{isothermal}} \quad \mathcal{D}_{int} = \boldsymbol{\sigma} : \mathbf{D} - \rho \dot{\psi} \geq 0 \tag{9.152}$$

Note that, in isothermal processes the internal energy dissipation is purely mechanical. By substituting the rate of change of the Helmholtz free energy given in (9.151) into the Clausius-Duhem inequality, and also if we know that $\mathbf{D} = \dot{\boldsymbol{E}} \approx \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$ holds in a small deformation regime, we can obtain:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : (\dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p) - \rho \left[\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e + \frac{\partial \psi}{\partial \boldsymbol{\alpha}_k} \square \dot{\boldsymbol{\alpha}}_k \right] \geq 0 \tag{9.153}$$

$$\Rightarrow \mathcal{D}_{int} = \left(\boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} \right) : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \left(\boldsymbol{A}_k \square \dot{\boldsymbol{\alpha}}_k \right) \geq 0 \tag{9.154}$$

where we have denoted by $\boldsymbol{A}_k = -\rho \dfrac{\partial \psi}{\partial \boldsymbol{\alpha}_k}$ (the thermodynamic forces). Then, as the inequality in (9.154) must be valid for any admissible thermodynamic process, (see Chapter 6), so, we can adopt a process in which $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$ and $\dot{\boldsymbol{\alpha}}_k = \mathbf{0}$, then the following must hold:

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} \tag{9.155}$$

Note that $\boldsymbol{\sigma}$ is related to the gradient of ψ in the strain space, and $\boldsymbol{A}_k$ is related to the gradient of ψ in the space of $\boldsymbol{\alpha}_k$.

Then, by considering the constitutive equation in (9.155), the internal energy dissipation (9.154) becomes:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{A}_k \square \dot{\boldsymbol{\alpha}}_k \geq 0 \tag{9.156}$$

Next, we apply the maximum dissipation principle, which states that the dissipation in a material reaches a maximum during a change characterized by a dissipative process. Let us consider the current state $(\boldsymbol{\sigma}, \boldsymbol{A}_k)$, which is the current distribution of Cauchy stress tensor and thermodynamic forces in a body subjected to plastic strain. The principle of maximum energy dissipation requires that for a change of state, represented by $(\boldsymbol{\sigma}^*, \boldsymbol{A}_k^*)$, the following must be satisfied:

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \dot{\boldsymbol{\varepsilon}}^p + (\boldsymbol{A}_k - \boldsymbol{A}_k^*) \square \dot{\boldsymbol{\alpha}}_k \geq 0 \tag{9.157}$$

The inequality in (9.157) describes an optimization problem with constraint. We can also maximize the dissipation by minimizing the negative dissipation with the constraint $\Phi(\boldsymbol{\sigma}, \boldsymbol{A}_k) \leq 0$. To this end, we define the Lagrangian as:

$$\mathcal{L} = -\mathcal{D}_{int} + \gamma \Phi = -\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{A}_k \square \dot{\boldsymbol{\alpha}}_k + \dot{\gamma} \Phi \tag{9.158}$$

where $\dot{\gamma} \geq 0$ is the Lagrangian multiplier (plastic multiplier) that enforces $\Phi \leq 0$.

Then, the flow rule can be obtained by:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\sigma}} = \boldsymbol{0} \quad \Rightarrow \quad -\dot{\boldsymbol{\varepsilon}}^p + \dot{\gamma} \frac{\partial \Phi(\boldsymbol{\sigma}, \boldsymbol{A}_k)}{\partial \boldsymbol{\sigma}} = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \frac{\partial \Phi(\boldsymbol{\sigma}, \boldsymbol{A}_k)}{\partial \boldsymbol{\sigma}} \tag{9.159}$$

and the evolution of the internal variables is given by:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{A}_k} = \boldsymbol{0} \quad \Rightarrow \quad -\dot{\boldsymbol{\alpha}}_k + \dot{\gamma} \frac{\partial \Phi}{\partial \boldsymbol{A}_k} = \boldsymbol{0} \quad \Rightarrow \quad \dot{\boldsymbol{\alpha}}_k = \dot{\gamma} \frac{\partial \Phi}{\partial \boldsymbol{A}_k} \tag{9.160}$$

If we now make a change of variable such that $\Phi = \mathcal{G}$, where $\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{A}_k)$ is a plastic potential, we obtain:

$$\boxed{\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}}} \quad \textit{Plastic flow rule} \tag{9.161}$$

$$\boxed{\dot{\boldsymbol{\alpha}}_k = \dot{\gamma} \frac{\partial \mathcal{G}}{\partial \boldsymbol{A}_k}} \quad \textit{Evolution of the internal variables } \boldsymbol{\alpha}_k \tag{9.162}$$

Then to fully define the model we need to introduce the loading/unloading Kuhn-Tucker conditions:

$$\boxed{\dot{\gamma} \geq 0 \quad ; \quad \mathcal{F}(\boldsymbol{\sigma}, \boldsymbol{A}_k) \leq 0 \quad ; \quad \dot{\gamma} \mathcal{F}(\boldsymbol{\sigma}, \boldsymbol{A}_k) = 0} \quad \textit{The Kuhn-Tucker conditions} \tag{9.163}$$

and the persistency (or consistency) condition:

$$\boxed{\dot{\gamma} \dot{\mathcal{F}}(\boldsymbol{\sigma}, \boldsymbol{A}_k) = 0} \quad \textit{The persistency condition} \tag{9.164}$$

where $\mathcal{F}(\boldsymbol{\sigma}, \boldsymbol{A}_k)$ is the yield surface, and $\dot{\gamma}$ is the plastic multiplier.

In associated flow, the plastic potential $\mathcal{G}$ is equal to the yield surface $\mathcal{F}$, *i.e.*

$$\boxed{\mathcal{G}=\mathcal{F}} \quad \textit{Associated flow} \tag{9.165}$$

We can now summarize the elastoplastic model for isothermal processes under the small deformation regime as:

$$\boxed{\begin{array}{ll} \textbf{i.} & \text{Elastic stress - strain relationship} \\ & \quad \boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^e+\boldsymbol{\varepsilon}^p \\ \textbf{ii.} & \text{Constitutive equation for stress} \\ & \quad \boldsymbol{\sigma}=\dfrac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e} \\ \textbf{iii.} & \text{Plastic flow rule} \\ & \quad \dot{\boldsymbol{\varepsilon}}^p=\dot{\gamma}\dfrac{\partial\mathcal{G}}{\partial\boldsymbol{\sigma}} \\ \textbf{iv.} & \text{Evolution of the internal variables} \\ & \quad \dot{\boldsymbol{\alpha}}_k=\dot{\gamma}\dfrac{\partial\mathcal{G}}{\partial\boldsymbol{A}_k} \\ \textbf{v.} & \text{Kuhn - Tucker complementary condition} \\ & \dot{\gamma}\geq 0 \quad ; \quad \mathcal{F}(\boldsymbol{\sigma},\boldsymbol{A}_k)\leq 0 \quad ; \quad \dot{\gamma}\mathcal{F}(\boldsymbol{\sigma},\boldsymbol{A}_k)=0 \\ \textbf{vi.} & \text{Consistency condition} \\ & \quad \dot{\gamma}\dot{\mathcal{F}}(\boldsymbol{\sigma},\boldsymbol{A}_k)=0 \end{array}} \quad \textit{Elastoplastic model for isothermal small deformation regime} \tag{9.166}$$

9.4.4 The Elastoplastic Tangent Stiffness Tensor

In this section we will establish the rate of change of the stress-strain relationship, *i.e.*:

$$\dot{\boldsymbol{\sigma}}=\mathbb{C}^{tan\_ep}:\dot{\boldsymbol{\varepsilon}} \tag{9.167}$$

with which we can obtain the *elastoplastic tangent stiffness tensor*, $\mathbb{C}^{tan\_ep}$, starting off from the constitutive equation:

$$\boldsymbol{\sigma}=\mathbb{C}^e:\boldsymbol{\varepsilon}^e \tag{9.168}$$

where $\mathbb{C}^e$ is the elasticity tensor, which does not vary with time ($\dot{\mathbb{C}}^e=\mathbb{O}$), and $\boldsymbol{\varepsilon}^e$ is the elastic part of the infinitesimal strain tensor. Then, by means of the strain tensor additive decomposition, (see Eq. (9.148)), we can express the elastic part as $\boldsymbol{\varepsilon}^e=\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p$, and by substituting this into the equation in (9.168) we obtain:

$$\boldsymbol{\sigma}=\mathbb{C}^e:\boldsymbol{\varepsilon}^e=\mathbb{C}^e:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)\xrightarrow{\text{rate of change}}\dot{\boldsymbol{\sigma}}=\mathbb{C}^e:(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^p) \tag{9.169}$$

Next, using the flow rule defined in (9.161), *i.e.* $\dot{\boldsymbol{\varepsilon}}^p=\dot{\gamma}\dfrac{\partial\mathcal{G}}{\partial\boldsymbol{\sigma}}$, the above equation becomes:

$$\dot{\boldsymbol{\sigma}}=\mathbb{C}^e:(\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}^p)=\mathbb{C}^e:\left(\dot{\boldsymbol{\varepsilon}}-\dot{\gamma}\frac{\partial\mathcal{G}}{\partial\boldsymbol{\sigma}}\right) \tag{9.170}$$

According to the consistency condition when $\dot{\gamma}>0$ the implication is that $\dot{\mathcal{F}}(\boldsymbol{\sigma},\boldsymbol{A}_k)=0$. The rate of change of $\mathcal{F}=\mathcal{F}(\boldsymbol{\sigma},\boldsymbol{A}_k)$ can be evaluated as follows:

$$\dot{\mathcal{F}}(\boldsymbol{\sigma},\boldsymbol{A}_k)=\frac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}}+\sum_{a=1}^{n}\frac{\partial\mathcal{F}}{\partial\boldsymbol{A}_a}\dot{\boldsymbol{A}}_a=0 \tag{9.171}$$

where n is the number of internal variables. Then, by assuming that $\boldsymbol{A}_a$ is a function of plastic strain we can conclude that:

$$\dot{\boldsymbol{A}}_a = \frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p = \frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \dot{\gamma} = \dot{\gamma} \frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} = \dot{\gamma} \left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right) \tag{9.172}$$

Then, the equation in (9.171) becomes:

$$\dot{\mathcal{F}}(\boldsymbol{\sigma}, \boldsymbol{A}_k) = \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a} \dot{\boldsymbol{A}}_a = \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \dot{\gamma} \sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a} \left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right) = 0 \tag{9.173}$$

Next, by substituting the equation in (9.170) into the one above we obtain:

$$\begin{aligned}
\dot{\mathcal{F}}(\boldsymbol{\sigma}, \boldsymbol{A}_k) &= \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \dot{\gamma} \sum_{a=1}^{n} \mathcal{H}_a : \frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} = 0 \\
&= \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \left[\mathbb{C}^e : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} \right) \right] + \dot{\gamma} \sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a} \left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right) = 0 \\
&= \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} + \dot{\gamma} \sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a} \left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right) = 0
\end{aligned} \tag{9.174}$$

with which the plastic multiplier $\dot{\gamma}$ can be evaluated as follows:

$$\dot{\gamma} = \frac{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \dfrac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} - \displaystyle\sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a} \left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right)} \tag{9.175}$$

Then, drawing once more on the equation in (9.170) and if we consider the plastic multiplier obtained in (9.175), it follows that:

$$\begin{aligned}
\dot{\boldsymbol{\sigma}} &= \mathbb{C}^e : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} \right) = \mathbb{C}^e : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \otimes \frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} \right) \\
&= \mathbb{C}^e : \left(\dot{\boldsymbol{\varepsilon}} - \frac{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \dfrac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} - \displaystyle\sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a} \left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right)} \otimes \frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} \right)
\end{aligned} \tag{9.176}$$

For the sake of simplicity, we will change the variables such as $\mathbf{n} = \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}}$, $\mathbf{m} = \dfrac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}}$, where $\mathbf{n}$ and $\mathbf{m}$ are symmetric second-order tensors after which the equation in (9.176) can be rewritten as:

$$\dot{\boldsymbol{\sigma}} = \mathbb{C}^e : \left(\dot{\boldsymbol{\varepsilon}} - \frac{\mathbf{n} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}}{\mathbf{n} : \mathbb{C}^e : \mathbf{m} - \displaystyle\sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a} \left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right)} \otimes \mathbf{m} \right) \tag{9.177}$$

Note that the denominator is a scalar, and by denoting the inverse of the denominator by $\mathcal{X}$ we obtain:

$$\dot{\boldsymbol{\sigma}} = \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \mathcal{X}\, \mathbb{C}^e : \left(\mathbf{n} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} \otimes \mathbf{m} \right) \tag{9.178}$$

We will now continue by using indicial notation:

$$
\begin{aligned}
\dot{\sigma}_{ij} &= \mathbb{C}^e_{ijkl}\dot{\varepsilon}_{kl} - \mathcal{X}\,\mathbb{C}^e_{ijkl}\left(\mathsf{n}_{ab}\mathbb{C}^e_{abcd}\dot{\varepsilon}_{cd}\mathsf{m}_{kl}\right) \\
&= \mathbb{C}^e_{ijkl}\dot{\varepsilon}_{st}\delta_{sk}\delta_{tl} - \mathcal{X}\,\mathbb{C}^e_{ijkl}\left(\mathsf{n}_{ab}\mathbb{C}^e_{abcd}\dot{\varepsilon}_{st}\delta_{sc}\delta_{td}\mathsf{m}_{kl}\right) \\
&= \left(\mathbb{C}^e_{ijkl}\delta_{sk}\delta_{tl} - \mathcal{X}\,\mathbb{C}^e_{ijkl}\mathsf{n}_{ab}\mathbb{C}^e_{abcd}\delta_{sc}\delta_{td}\mathsf{m}_{kl}\right)\dot{\varepsilon}_{st} \\
&= \left(\mathbb{C}^e_{ijst} - \mathcal{X}\,\mathbb{C}^e_{ijkl}\mathsf{n}_{ab}\mathbb{C}^e_{abst}\mathsf{m}_{kl}\right)\dot{\varepsilon}_{st} \\
&= \left(\mathbb{C}^e_{ijst} - \mathcal{X}\,\mathbb{C}^e_{ijkl}\mathsf{m}_{kl}\mathsf{n}_{ab}\mathbb{C}^e_{abst}\right)\dot{\varepsilon}_{st}
\end{aligned}
\tag{9.179}
$$

Then, in tensorial notation the above equation becomes:

$$
\dot{\boldsymbol{\sigma}} = \left(\mathbb{C}^e - \mathcal{X}\,\mathbb{C}^e : \mathbf{m}\otimes\mathbf{n} : \mathbb{C}^e\right) : \dot{\boldsymbol{\varepsilon}} \tag{9.180}
$$

$$
\dot{\boldsymbol{\sigma}} = \left(\mathbb{C}^e - \frac{\mathbb{C}^e : \mathbf{m}\otimes\mathbf{n} : \mathbb{C}^e}{\mathbf{n} : \mathbb{C}^e : \mathbf{m} - \sum_{a=1}^{n}\frac{\partial\mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p} : \frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)}\right) : \dot{\boldsymbol{\varepsilon}} \quad\Rightarrow\quad \dot{\boldsymbol{\sigma}} = \mathbb{C}^{tan\_ep} : \dot{\boldsymbol{\varepsilon}} \tag{9.181}
$$

Then, we define the elastoplastic tangent stiffness tensor as follows:

$$
\boxed{\mathbb{C}^{tan\_ep} = \mathbb{C}^e - \frac{\mathbb{C}^e : \dfrac{\partial\mathcal{G}}{\partial\boldsymbol{\sigma}} \otimes \dfrac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}} : \mathbb{C}^e}{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}} : \mathbb{C}^e : \dfrac{\partial\mathcal{G}}{\partial\boldsymbol{\sigma}} - \sum_{a=1}^{n}\dfrac{\partial\mathcal{F}}{\partial A_a}\left(\dfrac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p} : \dfrac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)}} \quad \textit{Elastoplastic tangent stiffness tensor} \tag{9.182}
$$

Note that $\mathbb{C}^{tan\_ep}$ is a fourth-order (but, not necessarily symmetric) tensor, showing only minor symmetry due to that $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\varepsilon}}$. Then, in terms of $\mathbf{n}$ and $\mathbf{m}$ the elastoplastic tangent stiffness tensor becomes:

$$
\boxed{\mathbb{C}^{tan\_ep} = \mathbb{C}^e - \frac{\mathbb{C}^e : \mathbf{m}\otimes\mathbf{n} : \mathbb{C}^e}{\mathbf{n} : \mathbb{C}^e : \mathbf{m} - \sum_{a=1}^{n}\dfrac{\partial\mathcal{F}}{\partial A_a}\left(\dfrac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p} : \dfrac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)}} \quad \textit{Elastoplastic tangent stiffness tensor} \tag{9.183}
$$

In associated flow, *i.e.* $\mathcal{F} = \mathcal{G}$, the elastoplastic tangent stiffness tensor becomes:

$$
\boxed{\mathbb{C}^{tan\_ep} = \mathbb{C}^e - \frac{\mathbb{C}^e : \mathbf{n}\otimes\mathbf{n} : \mathbb{C}^e}{\mathbf{n} : \mathbb{C}^e : \mathbf{n} - \sum_{a=1}^{n}\dfrac{\partial\mathcal{F}}{\partial A_a}\left(\dfrac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p} : \dfrac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)}} \quad \textit{Elastoplastic tangent stiffness tensor for associated flow} \tag{9.184}
$$

In associated flow, the elastoplastic tangent stiffness tensor, $\mathbb{C}^{tan\_ep}$, is a symmetric tensor, *i.e.* it features both major and minor symmetry.

Problem 9.1: Consider a one-dimensional case, find the elastoplastic tangent stiffness tensor (elastoplastic tangent stiffness modulus) for the following: 1) perfect plasticity; 2) isotropic hardening plasticity, 3) kinematic hardening plasticity, and 4) isotropic-kinematic hardening plasticity.

Solution:

We start from the general definition of the elastoplastic tangent stiffness tensor given in (9.182), in which the associated flow $\mathcal{G} = \mathcal{F}$ takes place. Then:

$$\mathbb{C}^{tan\_ep} = \mathbb{C}^{e} - \frac{\mathbb{C}^{e} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^{e}}{\frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^{e} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} - \sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a}\left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right)}$$

1) Perfect plasticity

With perfect plasticity, the yield surface is given by:

$$\mathcal{F}(\sigma) = \|\sigma\| - \sigma_Y \qquad \Rightarrow \qquad \frac{\partial \mathcal{F}}{\partial \sigma} = \mathrm{sign}(\sigma)$$

and the plastic flow is:

$$\dot{\varepsilon}^p = \dot{\gamma}\,\mathrm{sign}(\sigma)$$

Note that, since there are no internal variables we have $\sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a}\left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right) = 0$.
Furthermore, as we are dealing with a one-dimensional case, the tensors can be represented by their only nonzero component, *i.e.* $\dot{\sigma}_{11} = \mathbb{C}^{tan\_ep}_{1111} \dot{\varepsilon}_{11} \Rightarrow \dot{\sigma} = E^{tan\_ep} \dot{\varepsilon}$, and

$$\mathbb{C}^{tan\_ep} \to E^{tan\_ep} \quad ; \quad \mathbb{C}^{e} \to E \quad ; \quad \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} \to \frac{\partial \mathcal{F}}{\partial \sigma}$$

Thus, as expected:

$$E^{tan\_ep} = E - \frac{[E\,\mathrm{sign}(\sigma)]\,[\mathrm{sign}(\sigma) E]}{\mathrm{sign}(\sigma) E\,\mathrm{sign}(\sigma) - 0} = E - E = 0$$

2) Isotropic hardening plasticity

In the case of isotropic hardening plasticity, the yield surface is given by:

$$\mathcal{F}(\sigma, \underbrace{K\alpha}_{A}) = \|\sigma\| - (\sigma_Y + K\alpha) \qquad \Rightarrow \qquad \begin{cases} \frac{\partial \mathcal{F}}{\partial \sigma} = \mathrm{sign}(\sigma) \\ \frac{\partial \mathcal{F}}{\partial A} = \frac{\partial \mathcal{F}}{\partial (K\alpha)} = -1 \end{cases}$$

where the internal variable in stress is the scalar $A = K\alpha$.
Then, the plastic flow is:

$$\dot{\varepsilon}^p = \dot{\gamma}\,\mathrm{sign}(\sigma) \quad ; \quad \dot{\alpha} = \dot{\gamma} \qquad \Rightarrow \qquad \begin{cases} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}} = \mathrm{sign}(\sigma) \\ \frac{\partial \alpha}{\partial \varepsilon^p} = \mathrm{sign}(\sigma) \end{cases}$$

The term $\sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a}\left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right)$ is evaluated as follows:

$$\sum_{a=1}^{1} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_1}\left(\frac{\partial \boldsymbol{A}_1}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}} \right) \quad \to \quad \sum_{a=1}^{1} \frac{\partial \mathcal{F}}{\partial K\alpha}\left(\frac{\partial K\alpha}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}} \right) = -1\big(K\,\mathrm{sign}(\sigma)\mathrm{sign}(\sigma)\big) = -K$$

Then, the elastoplastic tangent stiffness modulus becomes:

$$E^{tan\_ep} = E - \frac{E\,\mathrm{sign}(\sigma)\mathrm{sign}(\sigma) E}{\mathrm{sign}(\sigma) E\,\mathrm{sign}(\sigma) - (-K)} = E - \frac{E^2}{E+K} = \frac{EK}{E+K}$$

3) Kinematic hardening plasticity

In the case of kinematic hardening plasticity, the yield surface is given by:

$$\mathcal{F}(\sigma,\underbrace{q}_{A}) = \|\sigma - q\| - \sigma_Y \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \mathcal{F}}{\partial \sigma} = \mathrm{sign}(\sigma - q) \\ \dfrac{\partial \mathcal{F}}{\partial q} = -\mathrm{sign}(\sigma - q) \end{cases}$$

where the internal variable in stress is represented by the scalar $A = q$.
Then, the plastic flow is:

$$\begin{cases} \dot{\varepsilon}^p = \dot{\gamma}\,\mathrm{sign}(\sigma) \\ \dot{q} = H\,\dot{\varepsilon}^p \end{cases} \quad \Rightarrow \quad \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}} = \mathrm{sign}(\sigma)$$

The term $\displaystyle\sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_a}\left(\frac{\partial \boldsymbol{A}_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}}\right)$ is evaluated as follows:

$$\sum_{a=1}^{1} \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}_1}\left(\frac{\partial \boldsymbol{A}_1}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}}\right) \quad \rightarrow \quad \sum_{a=1}^{1} \frac{\partial \mathcal{F}}{\partial q}\left(\frac{\partial q}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) = -\mathrm{sign}(\sigma - q)\big(H\,\mathrm{sign}(\sigma - q)\big) = -H$$

Then, the elastoplastic tangent stiffness modulus becomes:

$$E^{tan\_ep} = E - \frac{E\mathrm{sign}(\sigma)\mathrm{sign}(\sigma)E}{\mathrm{sign}(\sigma)E\mathrm{sign}(\sigma) - (-H)} = E - \frac{E^2}{E+H} = \frac{EH}{E+H}$$

4*) Isotropic-kinematic hardening plasticity*

In the case of isotropic-kinematic hardening plasticity, the yield surface is given by:

$$\mathcal{F}(\sigma, A_1, A_2) = \mathcal{F}(\sigma, q, K\alpha) = \|\sigma - q\| - (\sigma_Y + K\alpha) \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \mathcal{F}}{\partial \sigma} = \mathrm{sign}(\sigma - q) \\ \dfrac{\partial \mathcal{F}}{\partial q} = -\mathrm{sign}(\sigma - q) \\ \dfrac{\partial \mathcal{F}}{\partial K\alpha} = -1 \end{cases}$$

Thus:

$$\sum_{a=1}^{2} \frac{\partial \mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) = \frac{\partial \mathcal{F}}{\partial A_1}\left(\frac{\partial A_1}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) + \frac{\partial \mathcal{F}}{\partial A_2}\left(\frac{\partial A_2}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right)$$

The internal variables are $A_1 = q$, $A_2 = K\alpha$, with which we obtain:

$$\begin{aligned} \sum_{a=1}^{2} \frac{\partial \mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) &= \frac{\partial \mathcal{F}}{\partial A_1}\left(\frac{\partial A_1}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) + \frac{\partial \mathcal{F}}{\partial A_2}\left(\frac{\partial A_2}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) \\ &= \frac{\partial \mathcal{F}}{\partial q}\left(\frac{\partial q}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) + \frac{\partial \mathcal{F}}{\partial K\alpha}\left(\frac{\partial K\alpha}{\partial \varepsilon^p} \frac{\partial \dot{\varepsilon}^p}{\partial \dot{\gamma}}\right) \\ &= -\mathrm{sign}(\sigma - q)\big(H\,\mathrm{sign}(\sigma - q)\big) + \big[-1\big(K\mathrm{sign}(\sigma - q)\mathrm{sign}(\sigma - q)\big)\big] \\ &= -H - K \end{aligned}$$

Then, the elastoplastic tangent stiffness modulus becomes:

$$E^{tan\_ep} = E - \frac{E^2}{E + (H+K)} = \frac{E(H+K)}{E+H+K}$$

9.4.5 The Classical J_2 Flow Theory

9.4.5.1 Perfect Plasticity

Let us consider the following assumptions, (see Simo&Hughes (1998)):

1. Isotropic linear elastic behavior;
2. The Huber-von Mises yield condition

$$\mathcal{F}(\boldsymbol{\sigma}) = \sqrt{\|\boldsymbol{\sigma}\|^2 - \tfrac{1}{3}[\mathrm{Tr}(\boldsymbol{\sigma})]^2} - R \qquad \textit{with} \qquad R = \sqrt{\tfrac{2}{3}}\sigma_Y \tag{9.185}$$

   where R is the radius of the yield surface, and σ_Y is the yield stress (elastic limit stress), and $\|\boldsymbol{\sigma}\|$ is the Frobenius norm of the Cauchy stress tensor, (see subsection 1.5.7 Norms of Tensors in Chapter 1).
3. There is Levy-Saint_Venant associated plastic flow
4. There is no hardening

Note that $\mathcal{F}(\boldsymbol{\sigma})$ can be expressed as follows:

$$\begin{aligned}\mathcal{F}(\boldsymbol{\sigma}) &= \sqrt{\|\boldsymbol{\sigma}\|^2 - \tfrac{1}{3}[\mathrm{Tr}(\boldsymbol{\sigma})]^2} - R = \sqrt{\boldsymbol{\sigma}:\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}:\boldsymbol{\sigma}} - R = \sqrt{\left(\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}\right):\boldsymbol{\sigma}} - R \\ &= \sqrt{\mathbf{s}:\boldsymbol{\sigma}} - R\end{aligned} \tag{9.186}$$

where $\mathbf{s} \equiv \boldsymbol{\sigma}^{dev}$, *i.e.* it is the deviatoric part of the Cauchy stress tensor. Then by substituting $\boldsymbol{\sigma} = \mathbf{s} - \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$ into the above equation we obtain:

$$\begin{aligned}\mathcal{F}(\boldsymbol{\sigma}) &= \sqrt{\mathbf{s}:\boldsymbol{\sigma}} - R = \sqrt{\mathbf{s}:\left(\mathbf{s} - \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}\right)} - R = \sqrt{\mathbf{s}:\mathbf{s}} - R \\ &= \|\mathbf{s}\| - R\end{aligned} \tag{9.187}$$

where we have taken into account that the trace of any deviatoric tensor is equal to zero, *i.e.* $\mathbf{s}:\mathbf{1} = \mathrm{Tr}(\mathbf{s}) = 0$.

We then obtain the gradient of $\mathcal{F}(\boldsymbol{\sigma})$ in the stress space, the result of which is the plastic flow tensor $\mathbf{n}$:

$$\mathbf{n} = \frac{\partial \mathcal{F}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \frac{\partial}{\partial \boldsymbol{\sigma}}\left[\sqrt{\mathbf{s}:\mathbf{s}} - R\right] = \frac{1}{2}(\mathbf{s}:\mathbf{s})^{\frac{-1}{2}}\left(\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}}:\mathbf{s} + \mathbf{s}:\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}}\right) = (\mathbf{s}:\mathbf{s})^{\frac{-1}{2}}\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}}:\mathbf{s} \tag{9.188}$$

Note that $\dfrac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}}:\mathbf{s} = \mathbf{s}$, (see **Problem 1.39**) with which we obtain:

$$\mathbf{n} = \frac{\partial \mathcal{F}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = (\mathbf{s}:\mathbf{s})^{\frac{-1}{2}}\frac{\partial \mathbf{s}}{\partial \boldsymbol{\sigma}}:\mathbf{s} = \frac{\mathbf{s}}{\sqrt{\mathbf{s}:\mathbf{s}}} = \frac{\mathbf{s}}{\|\mathbf{s}\|} \tag{9.189}$$

Notice also that the Frobenius norm of $\mathbf{n}$ is unitary, $\|\mathbf{n}\| = \sqrt{\mathbf{n}:\mathbf{n}} = 1$.

With the above, we can obtain the plastic multiplier such as that defined in (9.175) for the associated flow case:

$$\dot{\gamma} = \frac{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}}:\mathbb{C}^e:\dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}}:\mathbb{C}^e:\dfrac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} - \displaystyle\sum_{a=1}^{n}\frac{\partial \mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial \boldsymbol{\varepsilon}^p}:\frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \dot{\gamma}}\right)} = \frac{\mathbf{n}:\mathbb{C}^e:\dot{\boldsymbol{\varepsilon}}}{\mathbf{n}:\mathbb{C}^e:\mathbf{n} - 0} \tag{9.190}$$

where the elasticity tensor, (see Chapter 7), can be expressed as follows:

$$\mathbb{C}^e = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I} = \kappa \mathbf{1} \otimes \mathbf{1} + 2\mu \left[\mathbf{I} - \tfrac{1}{3} \mathbf{1} \otimes \mathbf{1}\right] \tag{9.191}$$

Thus

$$\dot{\gamma} = \frac{\mathbf{n} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}}{\mathbf{n} : \mathbb{C}^e : \mathbf{n} - 0} = \frac{\mathbf{n} : (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) : \dot{\boldsymbol{\varepsilon}}}{\mathbf{n} : (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) : \mathbf{n} - 0} = \frac{(\lambda \mathbf{n} : \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{n} : \mathbf{I}) : \dot{\boldsymbol{\varepsilon}}}{\lambda \mathbf{n} : \mathbf{1} \otimes \mathbf{1} : \mathbf{n} + 2\mu \mathbf{n} : \mathbf{I} : \mathbf{n}} \tag{9.192}$$

Then, if we consider that $\mathbf{n} : \mathbf{1} = \mathrm{Tr}(\mathbf{n}) = 0$ (trace of the deviatoric tensor), $\mathbf{n} : \mathbf{I} = \mathbf{n}^{sym} = \mathbf{n}$, we obtain:

$$\boxed{\dot{\gamma} = \frac{\mathbf{n} : \dot{\boldsymbol{\varepsilon}}}{\mathbf{n} : \mathbf{n}} = \mathbf{n} : \dot{\boldsymbol{\varepsilon}} \quad \textit{with} \quad \mathbf{n} = \frac{\mathbf{s}}{\|\mathbf{s}\|}} \tag{9.193}$$

We can also find the elastoplastic tangent stiffness tensor for the associated flow case, given in (9.184):

$$\begin{aligned} \mathbb{C}^{tan\_ep} &= \mathbb{C}^e - \frac{\mathbb{C}^e : \mathbf{n} \otimes \mathbf{n} : \mathbb{C}^e}{\mathbf{n} : \mathbb{C}^e : \mathbf{n} - \underbrace{\sum_{a=1}^{n} \frac{\partial \mathcal{F}}{\partial A_a} \left(\frac{\partial A_a}{\partial \boldsymbol{\varepsilon}^p} : \frac{\partial \dot{\boldsymbol{\varepsilon}}^p}{\partial \gamma} \right)}_{=0}} \\ &= (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) - \frac{(\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) : \mathbf{n} \otimes \mathbf{n} : (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I})}{\mathbf{n} : (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) : \mathbf{n}} \end{aligned} \tag{9.194}$$

Then, by simplifying the above equation we obtain:

$$\boxed{\begin{aligned} \mathbb{C}^{tan\_ep} &= \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu [\mathbf{I} - \mathbf{n} \otimes \mathbf{n}] \\ &= \kappa \mathbf{1} \otimes \mathbf{1} + 2\mu \left[\mathbf{I} - \tfrac{1}{3} \mathbf{1} \otimes \mathbf{1} - \mathbf{n} \otimes \mathbf{n}\right] \end{aligned}} \tag{9.195}$$

for when $\dot{\gamma} > 0$.

9.4.5.2 Isotropic-Kinematic Hardening Plasticity

This model has two internal variables, namely $\{\boldsymbol{A}_1 = \boldsymbol{q}; A_2 = \overline{\alpha}(K\alpha)\}$ where α (scalar) is the *equivalent plastic strain* which defines isotropic hardening behavior and $\boldsymbol{q}$ (a second-order tensor) defines the center of the von Mises yield surface in the deviatoric stress space. In this model we have the following hardening law and plastic flow rules:

$$\left. \begin{aligned} &\boldsymbol{\eta} = \mathbf{s} - \boldsymbol{q} \quad ; \quad \mathrm{Tr}(\boldsymbol{q}) = 0 \\ &\mathcal{F}(\boldsymbol{\sigma}, \boldsymbol{A}_1, A_2) = \|\boldsymbol{\eta}\| - \sqrt{\tfrac{2}{3}} K(\alpha) \end{aligned} \quad \right| \quad \begin{aligned} &\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|} \\ &\boldsymbol{q} = \dot{\gamma} \sqrt{\tfrac{2}{3}} H(\alpha) \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|} \quad ; \quad \dot{\alpha} = \dot{\gamma} \sqrt{\tfrac{2}{3}} \end{aligned} \tag{9.196}$$

where $\mathbf{s} \equiv \boldsymbol{\sigma}^{dev}$, $K(\alpha)$ is the isotropic hardening modulus, and $H(\alpha)$ is the kinematic hardening modulus. Then, given that $\|\dot{\boldsymbol{\varepsilon}}^p\| = \dot{\gamma}$ we can obtain the equivalent plastic strain as follows:

$$\alpha(t) = \int_0^t \sqrt{\frac{2}{3}} \|\dot{\boldsymbol{\varepsilon}}^p(\tau)\| d\tau \tag{9.197}$$

For the function $K(\alpha)$ we assume the linear variation:

$$K(\alpha) = \sigma_Y + \overline{K}\alpha \tag{9.198}$$

where $\overline{K}$ is a constant, and σ_Y is the yield stress.

The plastic multiplier can be found by means of the equation in (9.175) for the associated flow case, *i.e.*:

$$\begin{aligned}\dot{\gamma} &= \frac{\mathbf{n}:\mathbb{C}^e:\dot{\boldsymbol{\varepsilon}}}{\mathbf{n}:\mathbb{C}^e:\mathbf{n} - \sum\limits_{a=1}^{n}\frac{\partial\mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)} = \frac{2\mu\mathbf{n}:\dot{\boldsymbol{\varepsilon}}}{2\mu\mathbf{n}:\mathbf{n} - \sum\limits_{a=1}^{n}\frac{\partial\mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)}\\ &= \frac{2\mu\mathbf{n}:\dot{\boldsymbol{\varepsilon}}}{2\mu - \sum\limits_{a=1}^{n}\frac{\partial\mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)}\end{aligned} \tag{9.199}$$

where it holds that $\|\mathbf{n}\| = 1$. Furthermore, we can verify that the following holds:

$$\begin{aligned}\sum_{a=1}^{n}\frac{\partial\mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right) &= \frac{\partial\mathcal{F}}{\partial A_1}\left(\frac{\partial A_1}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right) + \frac{\partial\mathcal{F}}{\partial A_2}\left(\frac{\partial A_2}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)\\ &= \frac{\partial\mathcal{F}}{\partial\boldsymbol{q}}\left(\frac{\partial\boldsymbol{q}}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right) + \frac{\partial\mathcal{F}}{\partial(K\alpha)}\left(\frac{\partial(K\alpha)}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)\\ &= -\mathrm{sign}(\boldsymbol{\eta})\left(\frac{2}{3}H(\alpha)\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}:\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\mathrm{sign}(\boldsymbol{\eta})\right) - \sqrt{\frac{2}{3}}\left(K\sqrt{\frac{2}{3}}\mathrm{sign}(\boldsymbol{\eta})\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}:\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}\mathrm{sign}(\boldsymbol{\eta})\right)\\ &= -\frac{2}{3}H\mathbf{n}:\mathbf{n} - \frac{2}{3}K\mathbf{n}:\mathbf{n}\\ &= -\frac{2}{3}H - \frac{2}{3}K\end{aligned} \tag{9.200}$$

where $\mathbf{n} = \dfrac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}$ *with* $\|\mathbf{n}\| = 1$.

Then, the plastic multiplier becomes:

$$\dot{\gamma} = \frac{2\mu\mathbf{n}:\dot{\boldsymbol{\varepsilon}}}{2\mu - \sum\limits_{a=1}^{n}\frac{\partial\mathcal{F}}{\partial A_a}\left(\frac{\partial A_a}{\partial\boldsymbol{\varepsilon}^p}:\frac{\partial\dot{\boldsymbol{\varepsilon}}^p}{\partial\dot{\gamma}}\right)} = \frac{2\mu\mathbf{n}:\dot{\boldsymbol{\varepsilon}}}{2\mu - \left[-\frac{2}{3}H - \frac{2}{3}K\right]} \quad\Rightarrow\quad \boxed{\dot{\gamma} = \frac{\mathbf{n}:\dot{\boldsymbol{\varepsilon}}}{1 + \dfrac{H+K}{3\mu}}} \tag{9.201}$$

and the elastoplastic tangent stiffness tensor is given by:

$$\boxed{\mathbb{C}^{tan\_ep} = \kappa\mathbf{1}\otimes\mathbf{1} + 2\mu\left[\mathbf{I} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1} - \frac{\mathbf{n}\otimes\mathbf{n}}{1 + \dfrac{H+K}{3\mu}}\right]} \tag{9.202}$$

9.5 Plastic Potential Theory

Von Mises concluded that the strain tensor, $\boldsymbol{\varepsilon}$, was related to the stress tensor, $\boldsymbol{\sigma}$, by means of the elastic potential function (the complementary strain energy), U_c, as $\boldsymbol{\varepsilon} = \dfrac{\partial U_c}{\partial \boldsymbol{\sigma}}$. Similarly, von Mises suggested there was a plastic potential function, $\mathcal{G}(\boldsymbol{\sigma})$, with which the rate of change of the plastic strain is given by:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial \mathcal{G}}{\partial \boldsymbol{\sigma}} \qquad \dot{\varepsilon}^p_{ij} = \dot{\gamma}\frac{\partial \mathcal{G}}{\partial \sigma_{ij}} \tag{9.203}$$

where $\dot{\gamma}$ is the plastic multiplier- a positive scalar. This theory is known as the *plastic potential theory*. One possible approach we can take to this is to consider the plastic potential to be equal to the yield surface $\mathcal{G} = \mathcal{F}$, with which it is said that the flow is associated. Otherwise, *i.e.* when $\mathcal{G} \neq \mathcal{F}$, there is said to be non-associated flow rule. In the case of the former we obtain:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} \qquad \dot{\varepsilon}^p_{ij} = \dot{\gamma}\frac{\partial \mathcal{F}}{\partial \sigma_{ij}} \tag{9.204}$$

The Drucker's Stability postulates:

- The plastic work done by an external agency, during the application of additional stress, is positive.
- The total work done by an external agency during a cycle can not be negative.

If any of these criteria is not met, the material is said to be unstable.

The Normality

- The incremental plastic strain tensor is normal to the yield surface.

We will discuss these rules in the following example. First, let us consider a stress relationship as shown in Figure 9.39.

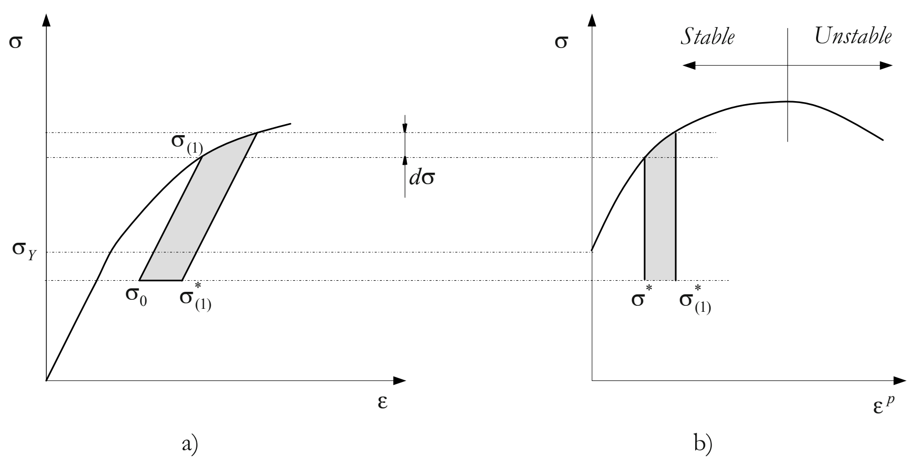


Figure 9.39: Plasticity, the load history dependency.

Let us consider the stress state σ_{ij} which is inside the initial yield surface at the initial instant of time (t_0). At time t_1, the point is found on the yield surface, so the process observed between t_0 and t_1 is purely elastic. From t_1 to $t_1+\delta t$, we apply a load increment described here by $d\sigma$, and then we apply unloading, as shown in Figure 9.40.

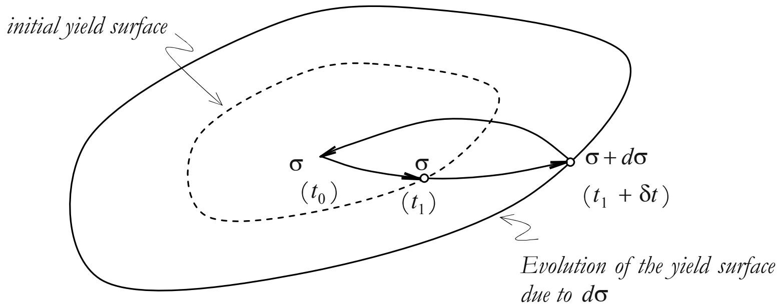


Figure 9.40: Evolution of the yield surface.

The total work done is given by $dW_n = dW_T - dW_0$, then that done in the process described in Figure 9.39 can be evaluated as follows:

$$dW_T = \int_0^{t_1} \sigma_{ij} d\varepsilon_{ij}^e dt + \int_{t_1}^{t_1+\delta t} \sigma_{ij}\left(d\varepsilon_{ij}^e + d\varepsilon_{ij}^p\right)dt + \int_{t_1+\delta t}^{t^*} \sigma_{ij} d\varepsilon_{ij}^e dt = \oint \sigma_{ij} d\varepsilon_{ij}^e dt + \int_{t_1}^{t_1+\delta t} \sigma_{ij} d\varepsilon_{ij}^p dt$$
$$= \int_{t_1}^{t_1+\delta t} \sigma_{ij} d\varepsilon_{ij}^p dt \tag{9.205}$$

The work done by σ_{ij}^*, is $dW_0 = \int_{t_1}^{t_1+\delta t} \sigma_{ij}^* d\varepsilon_{ij}^p dt$, and therefore,

$$dW_n = dW_T - dW_0 = \int_{t_1}^{t_1+\delta t} \sigma_{ij} d\varepsilon_{ij}^p dt - \int_{t_1}^{t_1+\delta t} \sigma_{ij}^* d\varepsilon_{ij}^p dt = \int_{t_1}^{t_1+\delta t} \left(\sigma_{ij} - \sigma_{ij}^*\right) d\varepsilon_{ij}^p dt \tag{9.206}$$

According to Drucker's stability criterion, the following must be satisfied

$$dW_n \geq 0 \quad \Rightarrow \quad \int_{t_1}^{t_1+\delta t} \left(\sigma_{ij} - \sigma_{ij}^*\right) d\varepsilon_{ij}^p dt > 0 \tag{9.207}$$

which represents Drucker's second postulate and because the above integrand is valid at any time, it holds that:

$$\left(\sigma_{ij} - \sigma_{ij}^*\right) d\varepsilon_{ij}^p \geq 0 \qquad \Big| \qquad \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\right) : \boldsymbol{\varepsilon}^p \geq 0 \tag{9.208}$$

Then, with $d\sigma_{ij} = \sigma_{ij} - \sigma_{ij}^*$ we obtain:

$$d\sigma_{ij} d\varepsilon_{ij}^p \geq 0 \qquad \Big| \qquad d\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p \geq 0 \tag{9.209}$$

Under these conditions the material is said to be plastically stable. For a material with hardening behavior we obtain:

$$d\sigma_{ij} d\varepsilon_{ij}^p > 0 \quad ; \quad d\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p > 0 \tag{9.210}$$

and for a material with perfect plasticity behavior we obtain:

$$d\sigma_{ij} d\varepsilon_{ij}^{p} = 0 \quad ; \quad d\boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^{p} = 0 \tag{9.211}$$

Let us now suppose that there is a scalar-valued tensor function $\mathcal{F} = \mathcal{F}(\boldsymbol{\sigma})$ called the plastic potential or yield function. In the elastic regime the following is satisfied:

$$\mathcal{F} < 0 \quad ; \quad d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \tag{9.212}$$

and in the plastic zone:

$$\mathcal{F} = 0 \quad ; \quad d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} d\sigma_{ij} \geq 0 \tag{9.213}$$

Then, from the condition in (9.209) the plastic energy becomes:

$$d\sigma_{ij} d\varepsilon_{ij}^{p} = 0 \tag{9.214}$$

with the restriction $d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} d\sigma_{ij} = 0$ on the yield surface.

Now, to solve a problem with a restriction we will introduce the Lagrange multiplier $d\lambda$:

$$d\sigma_{ij} d\varepsilon_{ij}^{p} - d\gamma \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \quad \Rightarrow \quad \left(d\varepsilon_{ij}^{p} - d\gamma \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} \right) d\sigma_{ij} = 0 \tag{9.215}$$

This condition must be fulfilled for any arbitrary value of $d\sigma_{ij}$, thus:

$$d\varepsilon_{ij}^{p} = d\gamma \frac{\partial \mathcal{F}}{\partial \sigma_{ij}} \quad ; \quad \boxed{d\boldsymbol{\varepsilon}^{p} = d\gamma \frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}}} \quad \textit{Prandtl-Reuss's flow rule} \tag{9.216}$$

where $d\gamma$ is a positive scalar. The equation (9.216) is called *Prandtl-Reuss's flow rule*.

Note that in an isotropic material the yield surface can be expressed in terms of the three principal stresses, *i.e.* $\mathcal{F} = \mathcal{F}(\sigma_1, \sigma_2, \sigma_3)$. In this case, $d\boldsymbol{\varepsilon}^{p}$ can be represented by a vector in the principal stress space, (see Figure 9.41).

Note also that $\frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}} \equiv \nabla_{\boldsymbol{\sigma}} \mathcal{F}$ is the gradient of $\mathcal{F}$ in the principal stress space and by definition is normal to the yield surface. Then, the plastic flow vector, $d\boldsymbol{\varepsilon}^{p}$, is also normal to the yield surface, so, together with the normality condition, we can conclude that the yield surface must be *convex*, since by the condition $d\sigma d\varepsilon^{p} \geq 0$ the angle formed by $d\sigma$ and $d\varepsilon^{p}$ cannot be obtuse.

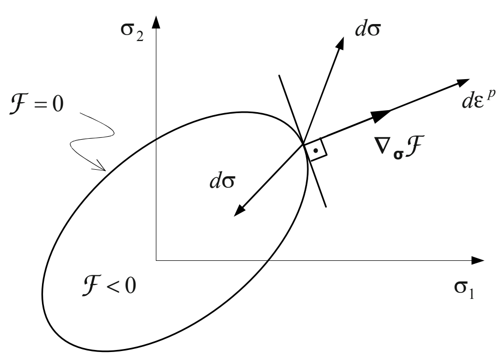


Figure 9.41: Normality condition – principal stress space.

9.6 Plasticity in Large Deformation Regime

Several theories have been developed for describing plasticity in the field of large deformation, among which we can mention:

- That based on the multiplicative decomposition of the deformation gradient proposed by Lee(1969) in the field of Solids Mechanics:

$$\boxed{\boldsymbol{F}(\vec{X},t)=\boldsymbol{F}^{e}(\vec{X},t)\cdot\boldsymbol{F}^{p}(\vec{X},t)} \quad \textit{Multiplicative decomposition of the deformation gradient} \tag{9.217}$$

- That based on the additive decomposition of the Green-Lagrange strain tensor, proposed by Green & Naghdi(1965):

$$\boxed{\boldsymbol{E}(\vec{X},t)=\boldsymbol{E}^{e}(\vec{X},t)+\boldsymbol{E}^{p}(\vec{X},t)} \quad \textit{Additive decomposition of the Green-Lagrange strain tensor} \tag{9.218}$$

- That based on the additive decomposition of the rate-of-deformation tensor, proposed by Nemat-Nasser(1982):

$$\boxed{\mathbf{D}(\vec{x},t)=\mathbf{D}^{e}(\vec{x},t)+\mathbf{D}^{p}(\vec{x},t)} \quad \textit{Additive decomposition of the rate-of-deformation tensor} \tag{9.219}$$

In the next subsection we will look at approaching large-deformation plasticity by means of the multiplicative decomposition of the deformation gradient caused by two transformations, namely, the elastic and the plastic transformations, Lee (1969), Simo (1992), Simo&Hughes (1998).

9.7 Large-Deformation Plasticity Based on the Multiplicative Decomposition of the Deformation Gradient

9.7.1 Kinematic Tensors

The multiplicative decomposition of the deformation gradient is given by:

$$\boxed{\boldsymbol{F}(\vec{X},t)=\boldsymbol{F}^{e}(\vec{X},t)\cdot\boldsymbol{F}^{p}(\vec{X},t)} \quad \textit{Multiplicative decomposition} \tag{9.220}$$

where $\boldsymbol{F}^{e}$ is the elastic transformation, and $\boldsymbol{F}^{p}$ is the plastic transformation, (see Figure 9.42). Then, according to Figure 9.42 the following is satisfied:

$$d\vec{x}=\boldsymbol{F}\cdot d\vec{X}=\boldsymbol{F}^{e}\cdot\boldsymbol{F}^{p}\cdot d\vec{X} \tag{9.221}$$

Note that, first we make the transformation related to $\boldsymbol{F}^{p}$, thereby defining a new configuration called the intermediate (or stress-free) configuration in which it holds that $d\vec{\bar{X}}=\boldsymbol{F}^{p}\cdot d\vec{X}$. Then, we make the transformation associated with $\boldsymbol{F}^{e}$, where $d\vec{x}=\boldsymbol{F}^{e}\cdot d\vec{\bar{X}}$ holds, (see Figure 9.42). Next, from the multiplicative decomposition we can obtain the following relationships:

$$\boldsymbol{F}=\boldsymbol{F}^{e}\cdot\boldsymbol{F}^{p} \quad\Rightarrow\quad \boldsymbol{F}^{-1}=\boldsymbol{F}^{p^{-1}}\cdot\boldsymbol{F}^{e^{-1}} \quad\Rightarrow\quad \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{e}=\boldsymbol{F}^{p^{-1}} \tag{9.222}$$

Next we will establish the kinematic variables in the intermediate configuration $\overline{\mathcal{B}}$, and show how these are related to those defined in the reference and current configurations.

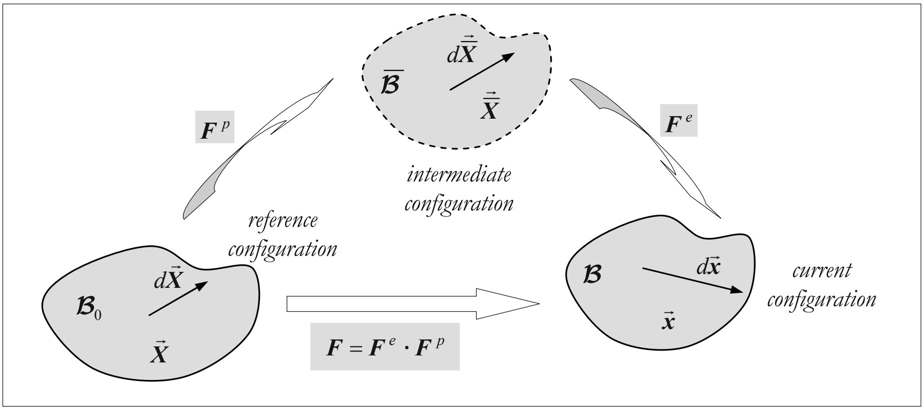


Figure 9.42: Multiplicative decomposition of the deformation gradient.

It is simple to show that the intermediate configuration is not unique, since here we can apply an orthogonal transformation (rotation) which remains in a stress-free state. In this scenario the deformation gradient can be represented by $\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^p = \hat{\boldsymbol{F}}^e \cdot \hat{\boldsymbol{F}}^p$, where $\hat{\boldsymbol{F}}^e = \boldsymbol{F}^e \cdot \mathbf{Q}^T$, $\hat{\boldsymbol{F}}^p = \mathbf{Q} \cdot \boldsymbol{F}^p$, (see Figure 9.43).

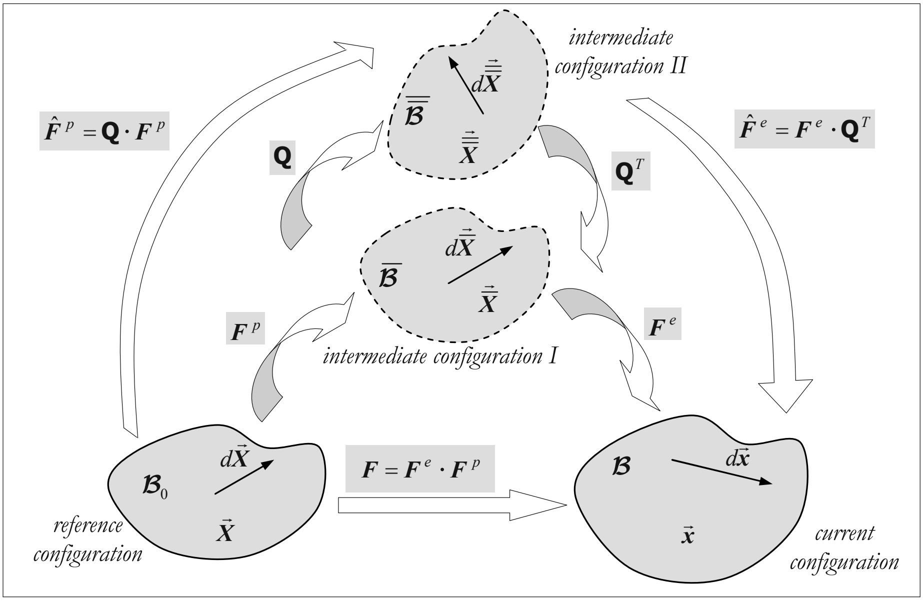


Figure 9.43: Non-uniqueness of the multiplicative decomposition of the deformation gradient.

9.7.1.1 Deformation and Strain Tensors

Now, remember from Chapter 2 that the right Cauchy-Green deformation tensor ($\boldsymbol{C}(\vec{X},t)$), the left Cauchy-Green deformation tensor ($\boldsymbol{b}(\vec{x},t)$), the Green-Lagrange strain tensor ($\boldsymbol{E}(\vec{X},t)$), the Cauchy deformation tensor ($\boldsymbol{c}(\vec{x},t)$), the Almansi strain tensor ($\boldsymbol{e}(\vec{x},t)$), the right stretch tensor ($\mathbf{U}(\vec{X},t)$), and the left stretch tensor ($\mathbf{V}(\vec{x},t)$) are related to each other as shown in Figure 9.44.

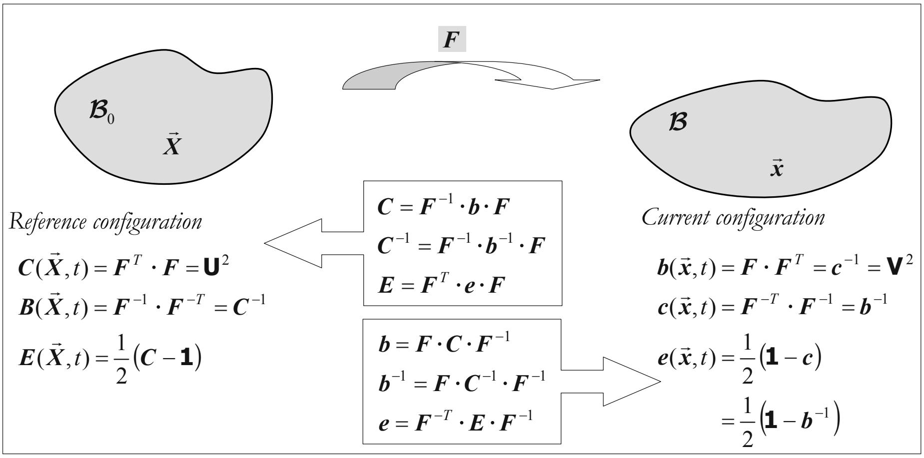


Figure 9.44: Kinematic tensors.

The right Cauchy-Green deformation tensor (reference configuration) is defined by $\boldsymbol{C}(\vec{X},t)=\boldsymbol{F}^T\cdot\boldsymbol{F}$, and the left Cauchy-Green deformation tensor by $\boldsymbol{b}(\vec{x},t)=\boldsymbol{F}\cdot\boldsymbol{F}^T=\mathbf{V}^2$ (current configuration). If we now consider the reference and intermediate configuration brought about by the transformation $\boldsymbol{F}^p$, (see Figure 9.42), we can define the following tensor:

$$\boxed{\boldsymbol{C}^p(\vec{X},t)=\boldsymbol{F}^{p^T}\cdot\boldsymbol{F}^p} \qquad \textit{Plastic part of the right Cauchy-Green deformation tensor} \text{ (reference configuration)} \tag{9.223}$$

and its inverse:

$$\boldsymbol{C}^{p^{-1}}(\vec{X},t)=\boldsymbol{F}^{p^{-1}}\cdot\boldsymbol{F}^{p^{-T}} \tag{9.224}$$

with which we can define the plastic part of the Green-Lagrange strain tensor in the reference configuration with:

$$\boxed{\boldsymbol{E}^p(\vec{X},t)=\frac{1}{2}\left(\boldsymbol{C}^p-\mathbf{1}\right)=\frac{1}{2}\left(\boldsymbol{F}^{p^T}\cdot\boldsymbol{F}^p-\mathbf{1}\right)} \qquad \textit{Plastic part of the Green-Lagrange strain tensor} \text{ (reference configuration)} \tag{9.225}$$

Then, we can also define:

$$\boxed{\bar{\boldsymbol{b}}^p(\vec{\bar{X}},t)=\boldsymbol{F}^p\cdot\boldsymbol{F}^{p^T}=\bar{\mathbf{V}}^{p^2}} \qquad \textit{Plastic part of the left Cauchy-Green deformation tensor} \text{ (intermediate configuration)} \tag{9.226}$$

Note that $\boldsymbol{C}^p(\vec{\boldsymbol{X}},t)$ and $\boldsymbol{E}^p(\vec{\boldsymbol{X}},t)$ are defined in the reference configuration while $\bar{\boldsymbol{b}}^p(\vec{\bar{\boldsymbol{X}}},t)$ is defined in the intermediate configuration, (see Figure 9.45). The Almansi strain tensor, defined in the intermediate configuration, is given by:

$$\boxed{\bar{\boldsymbol{e}}^p(\vec{\bar{\boldsymbol{X}}},t)=\frac{1}{2}\left(\mathbf{1}-\boldsymbol{b}^{p^{-1}}\right)=\frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}}\right)} \quad \text{\textit{Plastic part of the Almansi strain tensor} (intermediate configuration)} \tag{9.227}$$

In Chapter 2 we obtained the relationship between $\boldsymbol{E}$ and $\boldsymbol{e}$ given by $\boldsymbol{E}=\boldsymbol{F}^T\cdot\boldsymbol{e}\cdot\boldsymbol{F}$, (see Figure 9.44). Then, in comparison, the tensors $\boldsymbol{E}^p(\vec{\boldsymbol{X}},t)$ and $\bar{\boldsymbol{e}}^p(\vec{\bar{\boldsymbol{X}}},t)$ are interrelated, (see Figure 9.42), by:

$$\boxed{\boldsymbol{E}^p=\boldsymbol{F}^{p^T}\cdot\bar{\boldsymbol{e}}^p\cdot\boldsymbol{F}^p} \tag{9.228}$$

Proof of which follows:

$$\begin{aligned}
&\bar{\boldsymbol{e}}^p=\frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}}\right)\\
&\Rightarrow 2\bar{\boldsymbol{e}}^p=\mathbf{1}-\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}}\\
&\Rightarrow 2\boldsymbol{F}^{p^T}\cdot\bar{\boldsymbol{e}}^p\cdot\boldsymbol{F}^p=\boldsymbol{F}^{p^T}\cdot\left(\mathbf{1}-\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}}\right)\cdot\boldsymbol{F}^p\\
&\Rightarrow 2\boldsymbol{F}^{p^T}\cdot\bar{\boldsymbol{e}}^p\cdot\boldsymbol{F}^p=\boldsymbol{F}^{p^T}\cdot\mathbf{1}\cdot\boldsymbol{F}^p-\boldsymbol{F}^{p^T}\cdot\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}}\cdot\boldsymbol{F}^p\\
&\Rightarrow 2\boldsymbol{F}^{p^T}\cdot\bar{\boldsymbol{e}}^p\cdot\boldsymbol{F}^p=\boldsymbol{F}^{p^T}\cdot\boldsymbol{F}^p-\mathbf{1}\\
&\Rightarrow 2\boldsymbol{F}^{p^T}\cdot\bar{\boldsymbol{e}}^p\cdot\boldsymbol{F}^p=2\boldsymbol{E}^p\\
&\Rightarrow \boldsymbol{F}^{p^T}\cdot\bar{\boldsymbol{e}}^p\cdot\boldsymbol{F}^p=\boldsymbol{E}^p
\end{aligned} \tag{9.229}$$

If we now consider the transformation, $\boldsymbol{F}^e$, between the configurations $\bar{\mathcal{B}}$ (intermediate configuration) and $\mathcal{B}$ (current configuration), we can define the following tensors:

$$\boxed{\bar{\boldsymbol{C}}^e(\vec{\bar{\boldsymbol{X}}},t)=\boldsymbol{F}^{e^T}\cdot\boldsymbol{F}^e=\mathbf{U}^{e^2}} \quad \text{\textit{(intermediate configuration)}} \tag{9.230}$$

$$\boxed{\bar{\boldsymbol{E}}^e(\vec{\bar{\boldsymbol{X}}},t)=\frac{1}{2}\left(\bar{\boldsymbol{C}}^e-\mathbf{1}\right)=\frac{1}{2}\left(\boldsymbol{F}^{e^T}\cdot\boldsymbol{F}^e-\mathbf{1}\right)} \quad \text{\textit{(intermediate configuration)}} \tag{9.231}$$

$$\boxed{\boldsymbol{b}^e(\vec{\boldsymbol{x}},t)=\boldsymbol{F}^e\cdot\boldsymbol{F}^{e^T}} \quad \text{\textit{(current configuration)}} \tag{9.232}$$

$$\boxed{\boldsymbol{e}^e(\vec{\boldsymbol{x}},t)=\frac{1}{2}\left(\mathbf{1}-\boldsymbol{b}^{e^{-1}}\right)=\frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{e^{-T}}\cdot\boldsymbol{F}^{e^{-1}}\right)} \quad \text{\textit{(current configuration)}} \tag{9.233}$$

Then, by considering the Almansi strain tensor in the current configuration, $\boldsymbol{e}(\vec{\boldsymbol{x}},t)$, we can define a new tensor $\bar{\boldsymbol{E}}$ in the intermediate configuration as follows:

$$\bar{\boldsymbol{E}}=\boldsymbol{F}^{e^T}\cdot\boldsymbol{e}\cdot\boldsymbol{F}^e \tag{9.234}$$

We can now find the relationship between the tensors $\boldsymbol{E}$, $\boldsymbol{E}^p$ and $\bar{\boldsymbol{E}}^e$ starting from the definition of the Green-Lagrange strain tensor:

$$\boldsymbol{E}=\frac{1}{2}\left(\boldsymbol{F}^T\cdot\boldsymbol{F}-\mathbf{1}\right)\xrightarrow{\text{components}}E_{ij}=\frac{1}{2}\left(F_{ki}F_{kj}-\delta_{ij}\right) \tag{9.235}$$

and by considering the multiplicative decomposition, $F_{ij} = F^e_{ik} F^p_{kj}$, the Green-Lagrange strain tensor becomes:

$$E_{ij} = \frac{1}{2}\left(F^e_{ks} F^p_{si} F^e_{kt} F^p_{tj} - \delta_{ij}\right) \tag{9.236}$$

Then, from the equation in (9.231) we obtain $F^e_{ks} F^e_{kt} = 2\overline{E}^e_{st} + \delta_{st}$ which by substituting into the equation in (9.236) yields the following:

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left[\left(2\overline{E}^e_{st} + \delta_{st}\right) F^p_{si} F^p_{tj} - \delta_{ij}\right] = \frac{1}{2}\left[2\overline{E}^e_{st} F^p_{si} F^p_{tj} + \delta_{st} F^p_{si} F^p_{tj} - \delta_{ij}\right] \\ &= F^p_{si} \overline{E}^e_{st} F^p_{tj} + \frac{1}{2}\left[F^p_{si} F^p_{sj} - \delta_{ij}\right] = F^p_{si} \overline{E}^e_{st} F^p_{tj} + E^p_{ij} \end{aligned} \tag{9.237}$$

Then, the above equation in tensorial notation becomes:

$$\boxed{\begin{aligned} \boldsymbol{E} &= \boldsymbol{F}^{p^T} \cdot \overline{\boldsymbol{E}}^e \cdot \boldsymbol{F}^p + \boldsymbol{E}^p \\ &= \boldsymbol{E}^{(e\_p)} + \boldsymbol{E}^p \end{aligned}} \tag{9.238}$$

where we have defined a new tensor in the reference configuration:

$$\boxed{\boldsymbol{E}^{(e\_p)} = \boldsymbol{F}^{p^T} \cdot \overline{\boldsymbol{E}}^e \cdot \boldsymbol{F}^p} \tag{9.239}$$

The rate of change of (9.238) is given by:

$$\dot{\boldsymbol{E}} = \dot{\boldsymbol{E}}^{(e\_p)} + \dot{\boldsymbol{E}}^p \tag{9.240}$$

Let us see what we can obtain from the expression $\overline{\boldsymbol{E}}^e + \overline{\boldsymbol{e}}^p$. Now, if we use the relationships in (9.231) and (9.227) we obtain:

$$\overline{\boldsymbol{E}}^e + \overline{\boldsymbol{e}}^p = \frac{1}{2}\left(\boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e - \mathbf{1}\right) + \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{p^{-T}} \cdot \boldsymbol{F}^{p^{-1}}\right) = \frac{1}{2}\left(\boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e - \boldsymbol{F}^{p^{-T}} \cdot \boldsymbol{F}^{p^{-1}}\right) \tag{9.241}$$

Then, without altering the above outcome, we can apply the dot product of $\boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^{e^{-T}} = \mathbf{1}$ on the left, and the dot product of $\boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{F}^e = \mathbf{1}$ on the right, so, we obtain:

$$\begin{aligned} \overline{\boldsymbol{E}}^e + \overline{\boldsymbol{e}}^p &= \boldsymbol{F}^{e^T} \cdot \frac{1}{2}\left(\boldsymbol{F}^{e^{-T}} \cdot \boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e \cdot \boldsymbol{F}^{e^{-1}} - \boldsymbol{F}^{e^{-T}} \cdot \boldsymbol{F}^{p^{-T}} \cdot \boldsymbol{F}^{p^{-1}} \cdot \boldsymbol{F}^{e^{-1}}\right) \cdot \boldsymbol{F}^e \\ &= \boldsymbol{F}^{e^T} \cdot \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{-T} \cdot \boldsymbol{F}^{-1}\right) \cdot \boldsymbol{F}^e = \boldsymbol{F}^{e^T} \cdot \boldsymbol{e} \cdot \boldsymbol{F}^e \end{aligned} \tag{9.242}$$

Then, from the equations in (9.242) and (9.234) we can conclude that:

$$\boxed{\overline{\boldsymbol{E}} = \boldsymbol{F}^{e^T} \cdot \boldsymbol{e} \cdot \boldsymbol{F}^e = \overline{\boldsymbol{E}}^e + \overline{\boldsymbol{e}}^p} \tag{9.243}$$

If we now consider the equations in (9.232) and (9.224), we can obtain the relationship between $\boldsymbol{C}^{p^{-1}}$ and $\boldsymbol{b}^e$, *i.e.*:

$$\boldsymbol{b}^e = \boldsymbol{F}^e \cdot \boldsymbol{F}^{e^T} = \boldsymbol{F} \cdot \boldsymbol{F}^{p^{-1}} \cdot \left(\boldsymbol{F} \cdot \boldsymbol{F}^{p^{-1}}\right)^T = \boldsymbol{F} \cdot \boldsymbol{F}^{p^{-1}} \cdot \boldsymbol{F}^{p^{-T}} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \boldsymbol{C}^{p^{-1}} \cdot \boldsymbol{F}^T \tag{9.244}$$

$$\boxed{\boldsymbol{b}^e = \boldsymbol{F} \cdot \boldsymbol{C}^{p^{-1}} \cdot \boldsymbol{F}^T} \tag{9.245}$$

and the trace of $\boldsymbol{b}^e$ can be evaluated as follows:

$$\begin{aligned}\mathrm{Tr}(\boldsymbol{b}^e) &= \mathbf{1} : \left(\boldsymbol{F}\cdot\boldsymbol{C}^{p^{-1}}\cdot\boldsymbol{F}^T\right) = \delta_{ij}\left(F_{ik}C^{p^{-1}}_{kp}F_{jp}\right) = F_{jk}F_{jp}C^{p^{-1}}_{kp} = \left(\boldsymbol{F}^T\cdot\boldsymbol{F}\right) : \boldsymbol{C}^{p^{-1}} \\ &= \boldsymbol{C} : \boldsymbol{C}^{p^{-1}} = \mathrm{Tr}(\boldsymbol{C}\cdot\boldsymbol{C}^{p^{-1}})\end{aligned} \tag{9.246}$$

Next, we can obtain the relationship between the tensors $\overline{\boldsymbol{e}}^p(\vec{\overline{\boldsymbol{X}}},t)$ and $\boldsymbol{e}^e(\vec{\boldsymbol{x}},t)$:

$$\begin{aligned}\boldsymbol{e} &= \frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\right) = \frac{1}{2}\left[\mathbf{1}-\left(\boldsymbol{F}^e\cdot\boldsymbol{F}^p\right)^{-T}\cdot\left(\boldsymbol{F}^e\cdot\boldsymbol{F}^p\right)^{-1}\right] \\ &= \frac{1}{2}\left(\mathbf{1}-\left(\boldsymbol{F}^{p^T}\cdot\boldsymbol{F}^{e^T}\right)^{-1}\cdot\left(\boldsymbol{F}^e\cdot\boldsymbol{F}^p\right)^{-1}\right) = \frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{e^{-T}}\cdot\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}}\cdot\boldsymbol{F}^{e^{-1}}\right)\end{aligned} \tag{9.247}$$

Now, by considering that $\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}} = \left(\mathbf{1}-2\overline{\boldsymbol{e}}^p\right)$, (see equation (9.227)), we can obtain:

$$\begin{aligned}\boldsymbol{e} &= \frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{e^{-T}}\cdot\boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-1}}\cdot\boldsymbol{F}^{e^{-1}}\right) = \frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{e^{-T}}\cdot\left(\mathbf{1}-2\overline{\boldsymbol{e}}^p\right)\cdot\boldsymbol{F}^{e^{-1}}\right) \\ &= \frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{e^{-T}}\cdot\boldsymbol{F}^{e^{-1}}+\boldsymbol{F}^{e^{-T}}\cdot 2\overline{\boldsymbol{e}}^p\cdot\boldsymbol{F}^{e^{-1}}\right) \\ &= \boldsymbol{F}^{e^{-T}}\cdot\overline{\boldsymbol{e}}^p\cdot\boldsymbol{F}^{e^{-1}} + \underbrace{\frac{1}{2}\left(\mathbf{1}-\boldsymbol{F}^{e^{-T}}\cdot\boldsymbol{F}^{e^{-1}}\right)}_{=\boldsymbol{e}^e}\end{aligned} \tag{9.248}$$

Then, we can define a new tensor:

$$\boxed{\boldsymbol{e}^{(p\_e)} = \boldsymbol{F}^{e^{-T}}\cdot\overline{\boldsymbol{e}}^p\cdot\boldsymbol{F}^{e^{-1}}} \tag{9.249}$$

with which the Almansi strain tensor can also be defined as:

$$\boxed{\begin{aligned}\boldsymbol{e} &= \boldsymbol{F}^{e^{-T}}\cdot\overline{\boldsymbol{e}}^p\cdot\boldsymbol{F}^{e^{-1}}+\boldsymbol{e}^e \\ &= \boldsymbol{e}^{(p\_e)}+\boldsymbol{e}^e\end{aligned}} \tag{9.250}$$

Thus

$$\boldsymbol{e}^{(p\_e)} = \boldsymbol{e}-\boldsymbol{e}^e = \frac{1}{2}\left(\mathbf{1}-\boldsymbol{b}^{-1}\right)-\frac{1}{2}\left(\mathbf{1}-\boldsymbol{b}^{e^{-1}}\right) = \frac{1}{2}\left(\boldsymbol{b}^{e^{-1}}-\boldsymbol{b}^{-1}\right) \tag{9.251}$$

Then, from the equation in (9.249), and taking into account that $\boldsymbol{F}^{e^{-1}} = \boldsymbol{F}\cdot\boldsymbol{F}^{p^{-1}}$, we obtain:

$$\begin{aligned}&\boldsymbol{e}^{(p\_e)} = \boldsymbol{F}^{e^{-T}}\cdot\overline{\boldsymbol{e}}^p\cdot\boldsymbol{F}^{e^{-1}} \\ &\boldsymbol{e}^{(p\_e)} = \boldsymbol{F}^{p^{-T}}\cdot\boldsymbol{F}^T\cdot\overline{\boldsymbol{e}}^p\cdot\boldsymbol{F}\cdot\boldsymbol{F}^{p^{-1}} \\ &\Rightarrow \boldsymbol{F}^{p^T}\cdot\boldsymbol{e}^{(p\_e)}\cdot\boldsymbol{F}^p = \boldsymbol{F}^T\cdot\overline{\boldsymbol{e}}^p\cdot\boldsymbol{F}\end{aligned} \tag{9.252}$$

and by comparing the above equation with that in (9.228) we can conclude that:

$$\boxed{\boldsymbol{E}^p = \boldsymbol{F}^{p^T}\cdot\boldsymbol{e}^{(p\_e)}\cdot\boldsymbol{F}^p} \tag{9.253}$$

We can now appreciate all the relationships obtained above in Figure 9.45.

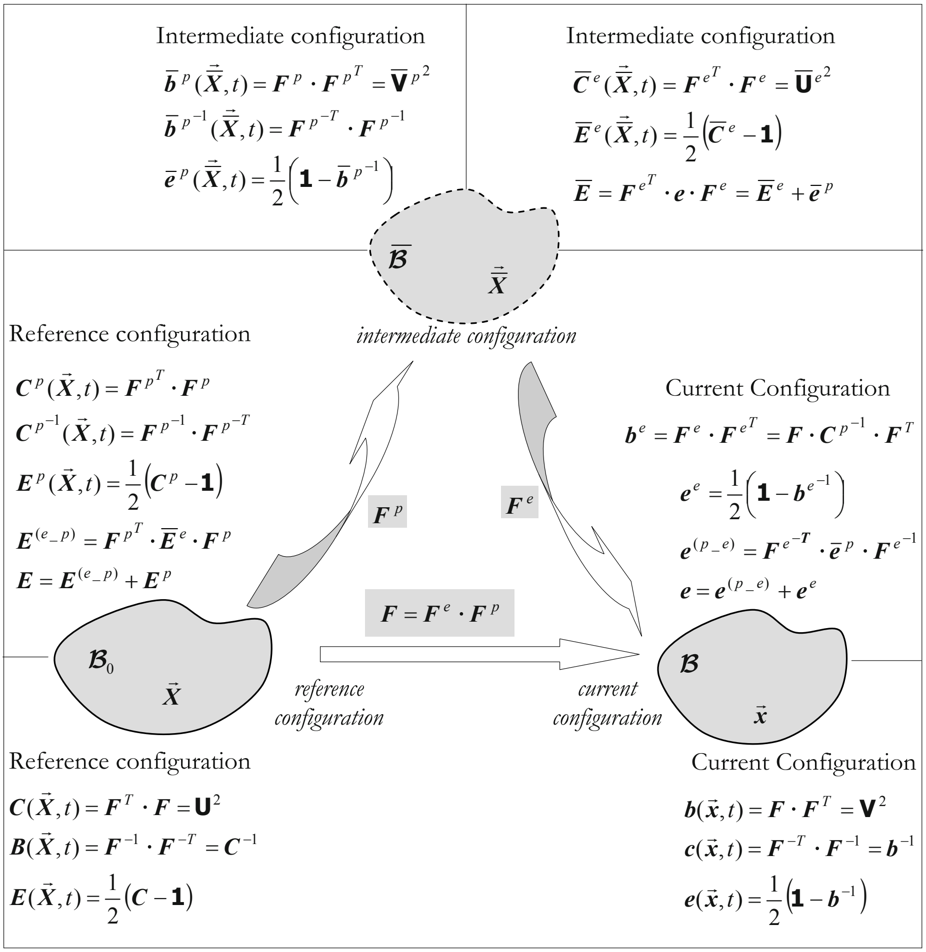


Figure 9.45: The kinematic tensors – Multiplicative decomposition.

9.7.1.2 Area and Volume Elements Deformation

With the definition of the Jacobian determinant and the multiplicative decomposition of the deformation gradient, we can obtain:

$$J=\det(\boldsymbol{F})=\det(\boldsymbol{F}^e\cdot\boldsymbol{F}^p)=\det(\boldsymbol{F}^e)\det(\boldsymbol{F}^p)=J^eJ^p \tag{9.254}$$

which thus defines the plastic Jacobian determinant J^p and the elastic Jacobian determinant J^e, respectively, as:

$$J^p=\det(\boldsymbol{F}^p)=\left[\det(\boldsymbol{C}^{p^{-1}})\right]^{\frac{1}{2}} \quad ; \quad J^e=\det(\boldsymbol{F}^e)=\left[\det(\boldsymbol{b}^e)\right]^{\frac{1}{2}} \tag{9.255}$$

Then, the differential volume elements in the respective configurations, (see Figure 9.46), are given by:

$$d\bar{V}(\vec{\bar{\boldsymbol{X}}},t)=J^p dV_0(\vec{\boldsymbol{X}},t) \quad ; \quad dV(\vec{\boldsymbol{x}},t)=J^e d\bar{V}(\vec{\bar{\boldsymbol{X}}},t) \tag{9.256}$$

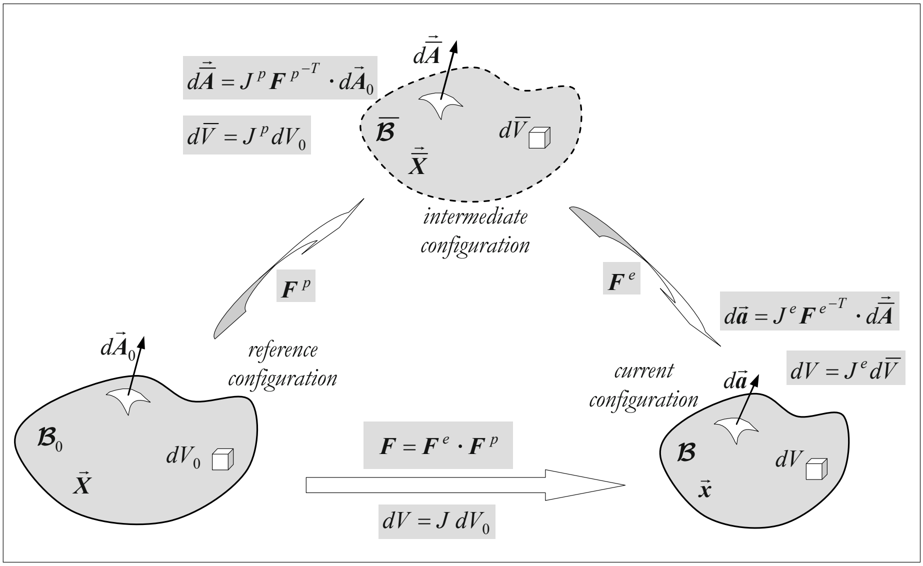


Figure 9.46: Deformation of the differential volume and area elements.

Let us consider now a differential area element in the reference configuration $d\vec{A}_0$, (see Figure 9.46), and by considering the multiplicative decomposition we obtain:

$$\boxed{d\vec{\bar{A}} = J^p \boldsymbol{F}^{p^{-T}} \cdot d\vec{A}_0} \quad \textit{The differential area element in the intermediate configuration} \tag{9.257}$$

$$\boxed{d\vec{a} = J^e \boldsymbol{F}^{e^{-T}} \cdot d\vec{\bar{A}}} \quad \textit{The differential area element in the current configuration} \tag{9.258}$$

Remember that the transformation between the $d\vec{A}_0$ and $d\vec{a}$ is given by Nanson's formula, $d\vec{a} = J\boldsymbol{F}^{-T} \cdot d\vec{A}_0$, (see Chapter 2), which can be validated by:

$$\begin{aligned} d\vec{a} &= J^e \boldsymbol{F}^{e^{-T}} \cdot d\vec{\bar{A}} = J^e \boldsymbol{F}^{e^{-T}} \cdot \left(J^p \boldsymbol{F}^{p^{-T}} \cdot d\vec{A}_0 \right) = J^e J^p \boldsymbol{F}^{e^{-T}} \cdot \boldsymbol{F}^{p^{-T}} \cdot d\vec{A}_0 \\ &= \left(J^e J^p\right)\left(\boldsymbol{F}^e \cdot \boldsymbol{F}^p\right)^{-T} \cdot d\vec{A}_0 = J\boldsymbol{F}^{-T} \cdot d\vec{A}_0 \end{aligned} \tag{9.259}$$

9.7.1.3 The Spatial Velocity Gradient

From the definition of the spatial velocity gradient, *i.e.* $\boldsymbol{\ell} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$, we can introduce the corresponding tensors that are brought about by the transformations $\boldsymbol{F}^p$ and $\boldsymbol{F}^e$, that is:

$$\boxed{\bar{\boldsymbol{\ell}}^p(\vec{\bar{X}},t) = \dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p^{-1}} \quad ; \quad \boldsymbol{\ell}^e(\vec{x},t) = \dot{\boldsymbol{F}}^e \cdot \boldsymbol{F}^{e^{-1}}} \tag{9.260}$$

Then, also based on the equation $\dot{\boldsymbol{F}}^{-1} = -\boldsymbol{F}^{-1} \cdot \boldsymbol{\ell}$, (see Chapter 2), we can introduce:

$$\boxed{\dot{\boldsymbol{F}}^{p^{-1}} = -\boldsymbol{F}^{p^{-1}} \cdot \bar{\boldsymbol{\ell}}^p \quad ; \quad \dot{\boldsymbol{F}}^{e^{-1}} = -\boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell}^e} \tag{9.261}$$

with which it is also possible to represent the spatial velocity gradient as:

$$
\begin{aligned}
\boldsymbol{\ell} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} &= \frac{D}{Dt}\left(\boldsymbol{F}^e \cdot \boldsymbol{F}^p\right) \cdot \left(\boldsymbol{F}^e \cdot \boldsymbol{F}^p\right)^{-1} = \left(\dot{\boldsymbol{F}}^e \cdot \boldsymbol{F}^p + \boldsymbol{F}^e \cdot \dot{\boldsymbol{F}}^p\right) \cdot \left(\boldsymbol{F}^{p^{-1}} \cdot \boldsymbol{F}^{e^{-1}}\right) \\
&= \dot{\boldsymbol{F}}^e \cdot \boldsymbol{F}^p \cdot \boldsymbol{F}^{p^{-1}} \cdot \boldsymbol{F}^{e^{-1}} + \boldsymbol{F}^e \cdot \dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p^{-1}} \cdot \boldsymbol{F}^{e^{-1}} = \dot{\boldsymbol{F}}^e \cdot \boldsymbol{F}^{e^{-1}} + \boldsymbol{F}^e \cdot \underbrace{\dot{\boldsymbol{F}}^p \cdot \boldsymbol{F}^{p^{-1}}}_{\bar{\boldsymbol{\ell}}^p} \cdot \boldsymbol{F}^{e^{-1}}
\end{aligned}
\tag{9.262}
$$

Thus, we can draw the conclusion that:

$$
\boxed{\boldsymbol{\ell} = \boldsymbol{\ell}^e + \boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^p \cdot \boldsymbol{F}^{e^{-1}} = \boldsymbol{\ell}^e + \boldsymbol{\ell}^{(p\_e)}}
\tag{9.263}
$$

Note that, $\boldsymbol{\ell}^e$ and $\boldsymbol{\ell}^{(p\_e)}$ are defined in the current configuration, whereas $\bar{\boldsymbol{\ell}}^p$ is in the intermediate configuration where the rate-of deformation and spin tensors are also established:

$$
\overline{\mathbf{D}}^p = \frac{1}{2}\left(\bar{\boldsymbol{\ell}}^p + \bar{\boldsymbol{\ell}}^{p^T}\right) \quad ; \quad \overline{\mathbf{W}}^p = \frac{1}{2}\left(\bar{\boldsymbol{\ell}}^p - \bar{\boldsymbol{\ell}}^{p^T}\right)
\tag{9.264}
$$

Note that $\boldsymbol{\ell}^{(p\_e)}(\vec{x},t) = \boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^p(\vec{\bar{X}},t) \cdot \boldsymbol{F}^{e^{-1}}$ and its inverse $\bar{\boldsymbol{\ell}}^p = \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell}^{(p\_e)} \cdot \boldsymbol{F}^e$, with which we can establish the following relationships:

$$
\begin{aligned}
\bar{\boldsymbol{\ell}} = \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell} \cdot \boldsymbol{F}^e \quad ; \quad \bar{\boldsymbol{\ell}}^p = \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell}^{(p\_e)} \cdot \boldsymbol{F}^e \quad ; \quad \bar{\boldsymbol{\ell}}^{(e\_e)} &= \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell}^e \cdot \boldsymbol{F}^e \\
&= \boldsymbol{F}^{e^{-1}} \cdot \dot{\boldsymbol{F}}^e \cdot \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{F}^e \\
&= \boldsymbol{F}^{e^{-1}} \cdot \dot{\boldsymbol{F}}^e
\end{aligned}
\tag{9.265}
$$

$$
\boldsymbol{\ell} = \boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}} \cdot \boldsymbol{F}^{e^{-1}} \quad ; \quad \boldsymbol{\ell}^{(p\_e)} = \boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^p \cdot \boldsymbol{F}^{e^{-1}} \quad ; \quad \boldsymbol{\ell}^e = \boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^{(e\_e)} \cdot \boldsymbol{F}^{e^{-1}}
\tag{9.266}
$$

and

$$
\begin{aligned}
\bar{\boldsymbol{\ell}} &= \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell} \cdot \boldsymbol{F}^e = \boldsymbol{F}^{e^{-1}} \cdot \left(\boldsymbol{\ell}^e + \boldsymbol{\ell}^{(p\_e)}\right) \cdot \boldsymbol{F}^e = \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell}^e \cdot \boldsymbol{F}^e + \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{\ell}^{(p\_e)} \cdot \boldsymbol{F}^e \\
&= \boldsymbol{F}^{e^{-1}} \cdot \dot{\boldsymbol{F}}^e \cdot \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{F}^e + \boldsymbol{F}^{e^{-1}} \cdot \left(\boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^p \cdot \boldsymbol{F}^{e^{-1}}\right) \cdot \boldsymbol{F}^e = \boldsymbol{F}^{e^{-1}} \cdot \dot{\boldsymbol{F}}^e + \bar{\boldsymbol{\ell}}^p \\
&= \bar{\boldsymbol{\ell}}^{(e\_e)} + \bar{\boldsymbol{\ell}}^p
\end{aligned}
\tag{9.267}
$$

We can define $\mathbf{D}^p(\vec{x},t)$ in the current configuration as follows:

$$
\begin{aligned}
\mathbf{D}^p = \boldsymbol{\ell}^{(p\_e)sym} &= \frac{1}{2}\left(\boldsymbol{\ell}^{(p\_e)} + \boldsymbol{\ell}^{(p\_e)^T}\right) = \frac{1}{2}\left(\boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^p \cdot \boldsymbol{F}^{e^{-1}} + \boldsymbol{F}^{e^{-T}} \cdot \bar{\boldsymbol{\ell}}^{p^T} \cdot \boldsymbol{F}^{e^T}\right) \\
&= \boldsymbol{F}^{e^{-T}} \cdot \frac{1}{2}\left(\boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^p \cdot \boldsymbol{F}^{e^{-1}} \cdot \boldsymbol{F}^e + \boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^{e^{-T}} \cdot \bar{\boldsymbol{\ell}}^{p^T} \cdot \boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e\right) \cdot \boldsymbol{F}^{e^{-1}} \\
&= \boldsymbol{F}^{e^{-T}} \cdot \frac{1}{2}\left(\boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e \cdot \bar{\boldsymbol{\ell}}^p + \bar{\boldsymbol{\ell}}^{p^T} \cdot \boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e\right) \cdot \boldsymbol{F}^{e^{-1}} \\
&= \boldsymbol{F}^{e^{-T}} \cdot \frac{1}{2}\left(\overline{\boldsymbol{C}}^e \cdot \bar{\boldsymbol{\ell}}^p + \bar{\boldsymbol{\ell}}^{p^T} \cdot \overline{\boldsymbol{C}}^e\right) \cdot \boldsymbol{F}^{e^{-1}} = \boldsymbol{F}^{e^{-T}} \cdot \left(\overline{\boldsymbol{C}}^e \cdot \bar{\boldsymbol{\ell}}^p\right)^{sym} \cdot \boldsymbol{F}^{e^{-1}}
\end{aligned}
\tag{9.268}
$$

We can also verify the following relationship:

$$
\begin{aligned}
\mathbf{D} &= \frac{1}{2}\left(\boldsymbol{\ell} + \boldsymbol{\ell}^T\right) = \frac{1}{2}\left[\left(\boldsymbol{\ell}^e + \boldsymbol{\ell}^{(p\_e)}\right) + \left(\boldsymbol{\ell}^e + \boldsymbol{\ell}^{(p\_e)}\right)^T\right] \\
&= \frac{1}{2}\left(\boldsymbol{\ell}^e + \boldsymbol{\ell}^{e^T}\right) + \frac{1}{2}\left(\boldsymbol{\ell}^{(p\_e)} + \boldsymbol{\ell}^{(p\_e)^T}\right) \\
&= \mathbf{D}^e + \mathbf{D}^{(p\_e)}
\end{aligned}
\tag{9.269}
$$

where $\mathbf{D}^e$ can also be represented by:

$$
\begin{aligned}
\mathbf{D}^e &= \frac{1}{2}\left(\boldsymbol{\ell}^e + \boldsymbol{\ell}^{eT}\right) = \frac{1}{2}\left(\dot{\boldsymbol{F}}^e \cdot \boldsymbol{F}^{e^{-1}} + \boldsymbol{F}^{e^{-T}} \cdot \dot{\boldsymbol{F}}^{e^T}\right) \\
&= \frac{1}{2}\boldsymbol{F}^{e^{-T}} \cdot \left(\boldsymbol{F}^{e^T} \cdot \dot{\boldsymbol{F}}^e + \dot{\boldsymbol{F}}^{e^T} \cdot \boldsymbol{F}^e\right) \cdot \boldsymbol{F}^{e^{-1}}
\end{aligned}
\tag{9.270}
$$

If we consider that

$$
\dot{\boldsymbol{C}}^e = \dot{\boldsymbol{F}}^{e^T} \cdot \boldsymbol{F}^e + \boldsymbol{F}^{e^T} \cdot \dot{\boldsymbol{F}}^e \tag{9.271}
$$

we can conclude that:

$$
\mathbf{D}^e = \frac{1}{2}\boldsymbol{F}^{e^{-T}} \cdot \dot{\boldsymbol{C}}^e \cdot \boldsymbol{F}^{e^{-1}} \tag{9.272}
$$

We can now appreciate all the relationships obtained above in Figure 9.47.

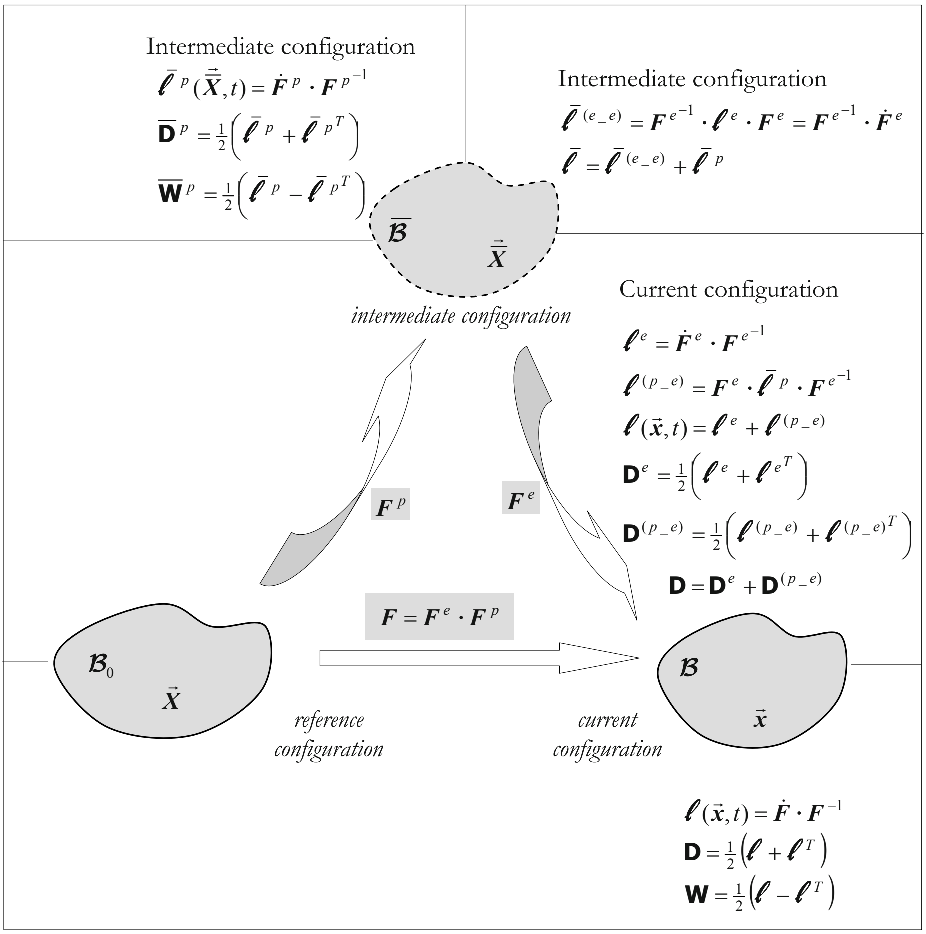


Figure 9.47: The rate of change of the deformation tensors – Multiplicative decomposition.

9.7.1.4 The Oldroyd Rate

By using the Oldroyd rate, $\overset{\square}{\mathbf{T}} = \dot{\mathbf{T}} - \boldsymbol{\ell} \cdot \mathbf{T} - \mathbf{T} \cdot \boldsymbol{\ell}^T$, (see Chapter 4), we can define the rate of change of an arbitrary second-order tensor $\overline{\mathbf{T}}$ in the intermediate configuration as:

$$\boxed{\overset{\square}{\overline{\mathbf{T}}} = \dot{\overline{\mathbf{T}}} - \overline{\boldsymbol{\ell}}^{\,p} \cdot \overline{\mathbf{T}} - \overline{\mathbf{T}} \cdot \overline{\boldsymbol{\ell}}^{\,pT}} \quad \textit{Oldroyd rate (intermediate configuration)} \tag{9.273}$$

Starting from said definition we can obtain the Oldroyd rate of the elastic part of the left Cauchy-Green deformation tensor as:

$$\boxed{\overset{\square}{\boldsymbol{b}}{}^e = \dot{\boldsymbol{b}}^e - \boldsymbol{\ell} \cdot \boldsymbol{b}^e - \boldsymbol{b}^e \cdot \boldsymbol{\ell}^T} \tag{9.274}$$

NOTE: In the literature, *e.g.* Marsden&Hughes (1983), we can find that $\overset{\square}{\boldsymbol{b}}{}^e$ is denoted by the *Lie derivative* of $\dot{\boldsymbol{b}}^e$. ■

If we now consider that $\boldsymbol{b}^e = \boldsymbol{F} \cdot \boldsymbol{C}^{p^{-1}} \cdot \boldsymbol{F}^T$, (see equation (9.245)), we can obtain the rate of change of $\boldsymbol{b}^e$ as follows:

$$\dot{\boldsymbol{b}}^e = \dot{\boldsymbol{F}} \cdot \boldsymbol{C}^{p^{-1}} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \dot{\boldsymbol{C}}^{p^{-1}} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \boldsymbol{C}^{p^{-1}} \cdot \dot{\boldsymbol{F}}^T \tag{9.275}$$

where $\dot{\boldsymbol{F}} = \boldsymbol{\ell} \cdot \boldsymbol{F}$, thus:

$$\begin{aligned} \dot{\boldsymbol{b}}^e &= \boldsymbol{\ell} \cdot \boldsymbol{F} \cdot \boldsymbol{C}^{p^{-1}} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \dot{\boldsymbol{C}}^{p^{-1}} \cdot \boldsymbol{F}^T + \boldsymbol{F} \cdot \boldsymbol{C}^{p^{-1}} \cdot \boldsymbol{F}^T \cdot \boldsymbol{\ell}^T \\ &= \boldsymbol{\ell} \cdot \boldsymbol{b}^e + \boldsymbol{F} \cdot \dot{\boldsymbol{C}}^{p^{-1}} \cdot \boldsymbol{F}^T + \boldsymbol{b}^e \cdot \boldsymbol{\ell}^T \end{aligned} \tag{9.276}$$

with which we can obtain:

$$\boldsymbol{F} \cdot \dot{\boldsymbol{C}}^{p^{-1}} \cdot \boldsymbol{F}^T = \dot{\boldsymbol{b}}^e - \boldsymbol{\ell} \cdot \boldsymbol{b}^e - \boldsymbol{b}^e \cdot \boldsymbol{\ell}^T \tag{9.277}$$

Then, by comparing the above equation with (9.274) we can conclude that the Oldroyd rate of $\boldsymbol{b}^e$ is given by:

$$\boxed{\overset{\square}{\boldsymbol{b}}{}^e = \boldsymbol{F} \cdot \dot{\boldsymbol{C}}^{p^{-1}} \cdot \boldsymbol{F}^T} \tag{9.278}$$

Note that if $\dot{\boldsymbol{b}} = \boldsymbol{\ell} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{\ell}^T$, it follows that the Oldroyd rate of $\boldsymbol{b}$ becomes the zero tensor, (see **Problem 4.1**), *i.e.:*

$$\overset{\square}{\boldsymbol{b}} = \dot{\boldsymbol{b}} - \boldsymbol{\ell} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{\ell}^T = \left(\boldsymbol{\ell} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{\ell}^T\right) - \boldsymbol{\ell} \cdot \boldsymbol{b} - \boldsymbol{b} \cdot \boldsymbol{\ell}^T = \mathbf{0} \tag{9.279}$$

Now, it was proven in Chapter 2 that $\mathbf{D} = \dot{\boldsymbol{e}} + \boldsymbol{e} \cdot \boldsymbol{\ell} + \boldsymbol{\ell}^T \cdot \boldsymbol{e}$, and if we now consider the definition of $\overset{\square}{\boldsymbol{e}}$, (see equation (9.273)), we can conclude that:

$$\begin{aligned} \overset{\square}{\boldsymbol{e}} &= \dot{\boldsymbol{e}} - \boldsymbol{\ell} \cdot \boldsymbol{e} - \boldsymbol{e} \cdot \boldsymbol{\ell}^T = \left(\mathbf{D} - \boldsymbol{e} \cdot \boldsymbol{\ell} - \boldsymbol{\ell}^T \cdot \boldsymbol{e}\right) - \boldsymbol{\ell} \cdot \boldsymbol{e} - \boldsymbol{e} \cdot \boldsymbol{\ell}^T \\ &= \left(\mathbf{D} + \boldsymbol{e} \cdot \boldsymbol{\ell}^T + \boldsymbol{\ell} \cdot \boldsymbol{e}\right) - \boldsymbol{\ell} \cdot \boldsymbol{e} - \boldsymbol{e} \cdot \boldsymbol{\ell}^T \\ &= \mathbf{D} \end{aligned} \tag{9.280}$$

The Oldroyd rate of $\boldsymbol{e}^e = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{b}^{e^{-1}}\right) = \frac{1}{2}\left(\mathbf{1} - \boldsymbol{F}^{e^{-T}} \cdot \boldsymbol{F}^{e^{-1}}\right)$ is given by:

$$\overset{\square}{\boldsymbol{e}}{}^e = \dot{\boldsymbol{e}}^e - \boldsymbol{\ell} \cdot \boldsymbol{e}^e - \boldsymbol{e}^e \cdot \boldsymbol{\ell}^T = \left(\mathbf{D}^e - \boldsymbol{e}^e \cdot \boldsymbol{\ell} - \boldsymbol{\ell}^T \cdot \boldsymbol{e}^e\right) - \boldsymbol{\ell}^e \cdot \boldsymbol{e}^e - \boldsymbol{e}^e \cdot \boldsymbol{\ell}^{eT} = \mathbf{D}^e \tag{9.281}$$

Thus, by starting from the equation $\boldsymbol{e} = \boldsymbol{e}^{(p\_e)} + \boldsymbol{e}^{e}$, (see equation (9.250)), we can obtain:

$$\overset{\square}{\boldsymbol{e}}{}^{(p\_e)} = \overset{\square}{\boldsymbol{e}} - \overset{\square}{\boldsymbol{e}}{}^{e} = \mathbf{D} - \mathbf{D}^{e} = \mathbf{D}^{(p\_e)} \tag{9.282}$$

Then, according to (9.251), *i.e.* $\boldsymbol{e}^{(p\_e)} = \frac{1}{2}\left(\boldsymbol{b}^{e^{-1}} - \boldsymbol{b}^{-1}\right)$, we can obtain:

$$\overset{\square}{\boldsymbol{e}}{}^{(p\_e)} = \frac{1}{2}\left(\overset{\square}{\boldsymbol{b}^{e^{-1}}}\right) - \underbrace{\left(\overset{\square}{\boldsymbol{b}^{-1}}\right)}_{=\mathbf{0}} = \frac{1}{2}\left(\overset{\square}{\boldsymbol{b}^{e^{-1}}}\right) \tag{9.283}$$

9.7.1.5 The Cotter-Rivlin Rate

By using the Cotter-Rivlin rate of a tensor, *i.e.* $\overset{\Delta}{\mathbf{T}} = \dot{\mathbf{T}} + \boldsymbol{\ell}^{T} \cdot \mathbf{T} + \mathbf{T} \cdot \boldsymbol{\ell}$, we can define said rate for $\overline{\mathbf{T}}(\vec{\overline{X}}, t)$ in the intermediate configuration as follows:

$$\boxed{\overset{\Delta}{\overline{\mathbf{T}}} = \dot{\overline{\mathbf{T}}} + \overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\mathbf{T}} + \overline{\mathbf{T}} \cdot \overline{\boldsymbol{\ell}}^{p}} \quad \textit{Cotter-Rivlin rate (intermediate configuration)} \tag{9.284}$$

Then, if we consider both the Cotter-Rivlin rate of the Almansi strain tensor $\overset{\Delta}{\boldsymbol{e}} = \dot{\boldsymbol{e}} + \boldsymbol{\ell}^{T} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \boldsymbol{\ell}$, and the relationship obtained in Chapter 2 $\mathbf{D} = \dot{\boldsymbol{e}} + \boldsymbol{\ell}^{T} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \boldsymbol{\ell}$ we can draw the conclusion that $\overset{\Delta}{\boldsymbol{e}} = \mathbf{D}$, (see **Problem 4.2**).

Another expression obtained in Chapter 2 is $\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}$, with which we define:

$$\overset{\Delta}{\overline{\boldsymbol{e}}}{}^{p} = \overline{\mathbf{D}}^{p} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{p} \cdot \boldsymbol{F}^{p^{-1}} \tag{9.285}$$

Then, by starting from the equation $\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}$ we obtain:

$$\begin{aligned} &\mathbf{D} = \boldsymbol{F}^{-T} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1} \\ &\Rightarrow \boldsymbol{F}^{T} \cdot \mathbf{D} \cdot \boldsymbol{F} = \dot{\boldsymbol{E}} \\ &\Rightarrow \boldsymbol{F}^{p^{-T}} \cdot \boldsymbol{F}^{T} \cdot \mathbf{D} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^{p^{-1}} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} \end{aligned} \tag{9.286}$$

Next, by considering the multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}^{e} \cdot \boldsymbol{F}^{p} \Rightarrow \boldsymbol{F}^{e} = \boldsymbol{F} \cdot \boldsymbol{F}^{p^{-1}}$, the above equation becomes:

$$\boldsymbol{F}^{e^{T}} \cdot \mathbf{D} \cdot \boldsymbol{F}^{e} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} \tag{9.287}$$

and if we consider that $\dot{\boldsymbol{E}} = \dot{\boldsymbol{E}}^{(e\_p)} + \dot{\boldsymbol{E}}^{p}$, (see equation (9.240)), into the above expression we obtain:

$$\begin{aligned} \boldsymbol{F}^{e^{T}} \cdot \mathbf{D} \cdot \boldsymbol{F}^{e} &= \boldsymbol{F}^{p^{-T}} \cdot \left(\dot{\boldsymbol{E}}^{(e\_p)} + \dot{\boldsymbol{E}}^{p}\right) \cdot \boldsymbol{F}^{p^{-1}} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{(e\_p)} \cdot \boldsymbol{F}^{p^{-1}} + \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{p} \cdot \boldsymbol{F}^{p^{-1}} \\ &= \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{(e\_p)} \cdot \boldsymbol{F}^{p^{-1}} + \overset{\Delta}{\overline{\boldsymbol{e}}}{}^{p} \end{aligned} \tag{9.288}$$

where we have applied the equation in (9.285). Then, the term $\dot{\boldsymbol{E}}^{(e\_p)}$ can be obtained by means of the expression of $\boldsymbol{E}^{(e\_p)}$ given in (9.239), *i.e.*: $\boldsymbol{E}^{(e\_p)} = \boldsymbol{F}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} \cdot \boldsymbol{F}^{p}$, thus:

$$\dot{\boldsymbol{E}}^{(e\_p)} = \dot{\boldsymbol{F}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} \cdot \boldsymbol{F}^{p} + \boldsymbol{F}^{p^{T}} \cdot \dot{\overline{\boldsymbol{E}}}{}^{e} \cdot \boldsymbol{F}^{p} + \boldsymbol{F}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} \cdot \dot{\boldsymbol{F}}^{p} \tag{9.289}$$

and

$$\boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{(e\_p)} \cdot \boldsymbol{F}^{p^{-1}} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{F}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} \cdot \boldsymbol{F}^{p} \cdot \boldsymbol{F}^{p^{-1}} + \boldsymbol{F}^{p^{-T}} \cdot \boldsymbol{F}^{p^{T}} \cdot \dot{\overline{\boldsymbol{E}}}^{e} \cdot \boldsymbol{F}^{p} \cdot \boldsymbol{F}^{p^{-1}} + \boldsymbol{F}^{p^{-T}} \cdot \boldsymbol{F}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} \cdot \dot{\boldsymbol{F}}^{p} \cdot \boldsymbol{F}^{p^{-1}} \tag{9.290}$$

Remember that $\overline{\boldsymbol{\ell}}^{p} = \dot{\boldsymbol{F}}^{p} \cdot \boldsymbol{F}^{p^{-1}}$, so the above equation becomes:

$$\boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{(e\_p)} \cdot \boldsymbol{F}^{p^{-1}} = \overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} + \dot{\overline{\boldsymbol{E}}}^{e} + \overline{\boldsymbol{E}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p} \tag{9.291}$$

Then, by comparing the above with (9.284) we can conclude that:

$$\boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{(e\_p)} \cdot \boldsymbol{F}^{p^{-1}} = \overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} + \dot{\overline{\boldsymbol{E}}}^{e} + \overline{\boldsymbol{E}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p} = \overset{\Delta}{\overline{\boldsymbol{E}}}{}^{e} \tag{9.292}$$

and by substituting the above into (9.288) we obtain:

$$\boldsymbol{F}^{e^{T}} \cdot \mathbf{D} \cdot \boldsymbol{F}^{e} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} = \overset{\Delta}{\overline{\boldsymbol{E}}}{}^{e} + \overset{\Delta}{\overline{\boldsymbol{e}}}{}^{p} \tag{9.293}$$

In the equation in (9.243) we obtained $\overline{\boldsymbol{E}} = \boldsymbol{F}^{e^{T}} \cdot \boldsymbol{e} \cdot \boldsymbol{F}^{e} = \overline{\boldsymbol{E}}^{e} + \overline{\boldsymbol{e}}^{p}$, thus we can conclude that:

$$\boxed{\boldsymbol{F}^{e^{T}} \cdot \mathbf{D} \cdot \boldsymbol{F}^{e} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} = \overset{\Delta}{\overline{\boldsymbol{E}}}{}^{e} + \overset{\Delta}{\overline{\boldsymbol{e}}}{}^{p} = \overset{\Delta}{\overline{\boldsymbol{E}}}} \tag{9.294}$$

Now, from (9.291) we can express $\dot{\overline{\boldsymbol{E}}}^{e}$ as follows:

$$\dot{\overline{\boldsymbol{E}}}^{e} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{(e\_p)} \cdot \boldsymbol{F}^{p^{-1}} - \overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} - \overline{\boldsymbol{E}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p} \tag{9.295}$$

and if we consider that $\dot{\boldsymbol{E}}^{(e\_p)} = \dot{\boldsymbol{E}} - \dot{\boldsymbol{E}}^{p}$, we obtain:

$$\dot{\overline{\boldsymbol{E}}}^{e} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} - \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{p} \cdot \boldsymbol{F}^{p^{-1}} - \overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} - \overline{\boldsymbol{E}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p} \tag{9.296}$$

Then, by starting from $\boldsymbol{E}^{p} = \frac{1}{2}\left(\boldsymbol{F}^{p^{T}} \cdot \boldsymbol{F}^{p} - \mathbf{1}\right)$ we can obtain its rate of change as follows $\dot{\boldsymbol{E}}^{p} = \frac{1}{2}\left(\dot{\boldsymbol{F}}^{p^{T}} \cdot \boldsymbol{F}^{p} + \boldsymbol{F}^{p^{T}} \cdot \dot{\boldsymbol{F}}^{p}\right)$, so, $\boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}}^{p} \cdot \boldsymbol{F}^{p^{-1}} = \frac{1}{2}\boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{F}}^{p^{T}} - \frac{1}{2}\dot{\boldsymbol{F}}^{p} \cdot \boldsymbol{F}^{p^{-1}}$. Then the equation in (9.296) becomes:

$$\begin{aligned}
\dot{\overline{\boldsymbol{E}}}^{e} &= \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} - \frac{1}{2}\left(\boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{F}}^{p^{T}} + \dot{\boldsymbol{F}}^{p} \cdot \boldsymbol{F}^{p^{-1}} + 2\overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} + 2\overline{\boldsymbol{E}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p}\right) \\
&= \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} - \frac{1}{2}\left(\overline{\boldsymbol{\ell}}^{p^{T}} + \overline{\boldsymbol{\ell}}^{p} + 2\overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\boldsymbol{E}}^{e} + 2\overline{\boldsymbol{E}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p}\right) \\
&= \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} - \frac{1}{2}\left[\overline{\boldsymbol{\ell}}^{p^{T}} \cdot \left(2\overline{\boldsymbol{E}}^{e} + \mathbf{1}\right) + \left(2\overline{\boldsymbol{E}}^{e} + \mathbf{1}\right) \cdot \overline{\boldsymbol{\ell}}^{p}\right] \\
&= \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} - \frac{1}{2}\left[\overline{\boldsymbol{\ell}}^{p^{T}} \cdot \overline{\boldsymbol{C}}^{e} + \overline{\boldsymbol{C}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p}\right]
\end{aligned} \tag{9.297}$$

Remember that $\overline{\boldsymbol{C}}^{e} = \overline{\boldsymbol{C}}^{e^{T}}$, so, the above becomes:

$$\boxed{\dot{\overline{\boldsymbol{E}}}^{e} = \boldsymbol{F}^{p^{-T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{p^{-1}} - \left(\overline{\boldsymbol{C}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p}\right)^{sym}} \tag{9.298}$$

We can also represent the equation in (9.298) as follows:

$$\dot{\overline{\boldsymbol{E}}}^{e} = \boldsymbol{F}^{p^{-T}} \cdot \left[\dot{\boldsymbol{E}} - \boldsymbol{F}^{p^{T}} \cdot \left(\overline{\boldsymbol{C}}^{e} \cdot \overline{\boldsymbol{\ell}}^{p}\right)^{sym} \cdot \boldsymbol{F}^{p}\right] \cdot \boldsymbol{F}^{p^{-1}} \tag{9.299}$$

9.7.2 The Stress Tensors

Remember from Chapter 3 that the Cauchy stress tensor $\boldsymbol{\sigma}(\vec{x},t)$, the Kirchhoff stress tensor $\boldsymbol{\tau}(\vec{x},t)$, the first Piola-Kirchhoff stress tensor $\mathbf{P}(\vec{X},t)$, the second Piola-Kirchhoff stress tensor $\mathbf{S}(\vec{X},t)$, the Biot stress tensor $\mathbf{T}(\vec{X},t)$, and the Mandel stress tensor $\mathbf{M}(\vec{X},t)$ are related to each other as indicated in Figure 9.48.

Remember that the tensors $\boldsymbol{F}$ and $\mathbf{P}$ are two-point tensors (pseudo-tensors), *i.e.* they are not defined in any configuration.

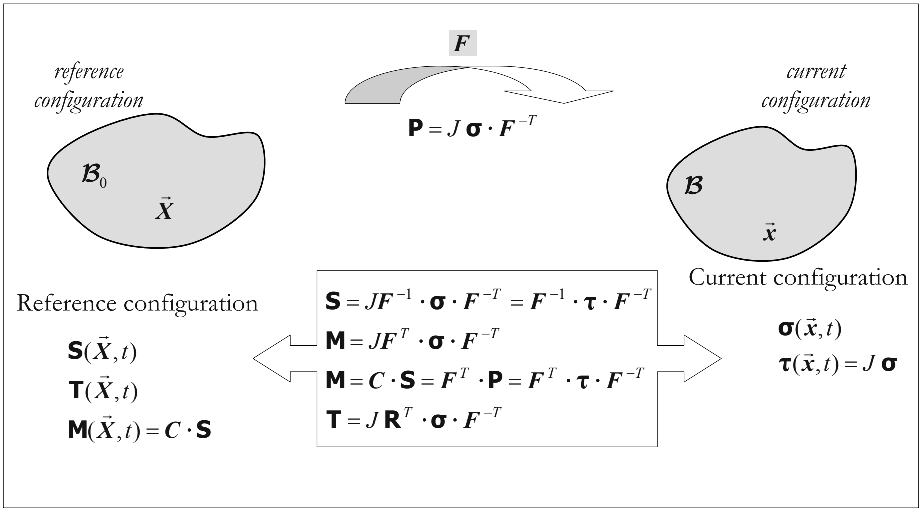


Figure 9.48: The stress tensors.

We can define the second Piola-Kirchhoff stress tensor in the intermediate configuration as:

$$\bar{\mathbf{S}}(\vec{\bar{X}},t) = \boldsymbol{F}^{e^{-1}}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{e^{-T}} \tag{9.300}$$

Then, from the above equation we can obtain:

$$\boldsymbol{F}^{e}\cdot\bar{\mathbf{S}}\cdot\boldsymbol{F}^{e^{T}} = \boldsymbol{\tau} \quad \Rightarrow \quad \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{e}\cdot\bar{\mathbf{S}}\cdot\boldsymbol{F}^{e^{T}}\cdot\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{-T} \tag{9.301}$$

Then, if we consider the multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}^{e}\cdot\boldsymbol{F}^{p}$, we can obtain the following equations $\Rightarrow \boldsymbol{F}^{-1} = \boldsymbol{F}^{p^{-1}}\cdot\boldsymbol{F}^{e^{-1}} \Rightarrow \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{e} = \boldsymbol{F}^{p^{-1}}$, thus we can rewrite (9.301) as follows:

$$\boxed{\begin{aligned}&\boldsymbol{F}^{p^{-1}}\cdot\bar{\mathbf{S}}\cdot\boldsymbol{F}^{p^{-T}} = \boldsymbol{F}^{-1}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{-T} = \mathbf{S}\\ &\bar{\mathbf{S}} = \boldsymbol{F}^{p}\cdot\mathbf{S}\cdot\boldsymbol{F}^{p^{T}}\end{aligned}} \tag{9.302}$$

Then, if we consider the Mandel stress tensor, $\mathbf{M} = \boldsymbol{C}\cdot\mathbf{S} = \boldsymbol{F}^{T}\cdot\mathbf{P} = \boldsymbol{F}^{T}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{-T}$, (see Figure 9.48), and the equation in (9.302), we can define the former in the intermediate configuration as:

$$\bar{\mathbf{M}} = \boldsymbol{F}^{e^{T}}\cdot\boldsymbol{\tau}\cdot\boldsymbol{F}^{e^{-T}} = \boldsymbol{F}^{e^{T}}\cdot\left(\boldsymbol{F}^{e}\cdot\bar{\mathbf{S}}\cdot\boldsymbol{F}^{e^{T}}\right)\cdot\boldsymbol{F}^{e^{-T}} = \boldsymbol{F}^{e^{T}}\cdot\boldsymbol{F}^{e}\cdot\bar{\mathbf{S}} = \bar{\boldsymbol{C}}^{e}\cdot\bar{\mathbf{S}} \tag{9.303}$$

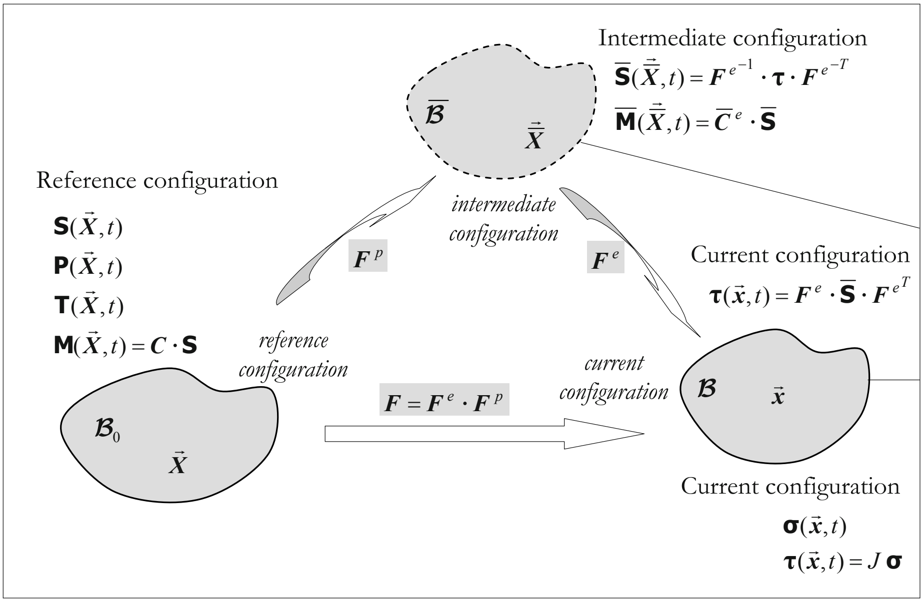


Figure 9.49: The stress tensors – Multiplicative decomposition.

9.7.2.1 Stress Tensor Rates

Starting from the equation in (9.302), *i.e.* $\bar{\mathbf{S}}=\boldsymbol{F}^{p}\cdot\mathbf{S}\cdot\boldsymbol{F}^{p^{T}}$, we can obtain the rate of change of $\bar{\mathbf{S}}$ as follows:

$$
\begin{aligned}
\dot{\bar{\mathbf{S}}} &= \dot{\boldsymbol{F}}^{p}\cdot\mathbf{S}\cdot\boldsymbol{F}^{p^{T}}+\boldsymbol{F}^{p}\cdot\dot{\mathbf{S}}\cdot\boldsymbol{F}^{p^{T}}+\boldsymbol{F}^{p}\cdot\mathbf{S}\cdot\dot{\boldsymbol{F}}^{p^{T}}\\
&= \boldsymbol{F}^{p}\cdot\left(\boldsymbol{F}^{p^{-1}}\cdot\dot{\boldsymbol{F}}^{p}\cdot\mathbf{S}+\dot{\mathbf{S}}+\mathbf{S}\cdot\dot{\boldsymbol{F}}^{p^{T}}\cdot\boldsymbol{F}^{p^{-T}}\right)\cdot\boldsymbol{F}^{p^{T}}\\
&= \boldsymbol{F}^{p}\cdot\left(\boldsymbol{F}^{p^{-1}}\cdot\bar{\boldsymbol{\ell}}^{p}\cdot\boldsymbol{F}^{p}\cdot\mathbf{S}+\dot{\mathbf{S}}+\mathbf{S}\cdot\boldsymbol{F}^{p^{T}}\bar{\boldsymbol{\ell}}^{p^{T}}\cdot\boldsymbol{F}^{p^{-T}}\right)\cdot\boldsymbol{F}^{p^{T}}
\end{aligned}
\tag{9.304}
$$

where we have considered that $\dot{\boldsymbol{F}}^{p}=\bar{\boldsymbol{\ell}}^{p}\cdot\boldsymbol{F}^{p}$. In addition, we have:

$$
\mathbf{Z}^{p}=\boldsymbol{F}^{p^{-1}}\cdot\bar{\boldsymbol{\ell}}^{p}\cdot\boldsymbol{F}^{p}=\boldsymbol{F}^{p^{-1}}\cdot\left(\dot{\boldsymbol{F}}^{p}\cdot\boldsymbol{F}^{p^{-1}}\right)\cdot\boldsymbol{F}^{p}
\tag{9.305}
$$

the equation in (9.304) can be rewritten as follows:

$$
\dot{\bar{\mathbf{S}}}=\boldsymbol{F}^{p}\cdot\left(\dot{\mathbf{S}}+\mathbf{Z}^{p}\cdot\mathbf{S}+\mathbf{S}\cdot\mathbf{Z}^{p^{T}}\right)\cdot\boldsymbol{F}^{p^{T}}
\tag{9.306}
$$

Now, by starting from the relation $\mathbf{S}=\boldsymbol{F}^{p^{-1}}\cdot\bar{\mathbf{S}}\cdot\boldsymbol{F}^{p^{-T}}$ we can evaluate $\dot{\mathbf{S}}$ as follows:

$$
\dot{\mathbf{S}}=\dot{\boldsymbol{F}}^{p^{-1}}\cdot\bar{\mathbf{S}}\cdot\boldsymbol{F}^{p^{-T}}+\boldsymbol{F}^{p^{-1}}\cdot\dot{\bar{\mathbf{S}}}\cdot\boldsymbol{F}^{p^{-T}}+\boldsymbol{F}^{p^{-1}}\cdot\bar{\mathbf{S}}\cdot\dot{\boldsymbol{F}}^{p^{-T}}
\tag{9.307}
$$

Then, if we consider (9.261), *i.e.* $\dot{\boldsymbol{F}}^{p^{-1}}=-\boldsymbol{F}^{p^{-1}}\cdot\bar{\boldsymbol{\ell}}^{p}$, and substitute this into the equation in (9.307), we can obtain:

$$
\begin{aligned}
\dot{\mathbf{S}} &= -\boldsymbol{F}^{p^{-1}}\cdot\bar{\boldsymbol{\ell}}^{p}\cdot\bar{\mathbf{S}}\cdot\boldsymbol{F}^{p^{-T}}+\boldsymbol{F}^{p^{-1}}\cdot\dot{\bar{\mathbf{S}}}\cdot\boldsymbol{F}^{p^{-T}}-\boldsymbol{F}^{p^{-1}}\cdot\bar{\mathbf{S}}\cdot\bar{\boldsymbol{\ell}}^{p^{-T}}\cdot\boldsymbol{F}^{p^{-T}}\\
&= \boldsymbol{F}^{p^{-1}}\cdot\left(\dot{\bar{\mathbf{S}}}-\bar{\boldsymbol{\ell}}^{p}\cdot\bar{\mathbf{S}}-\bar{\mathbf{S}}\cdot\bar{\boldsymbol{\ell}}^{p^{-T}}\right)\cdot\boldsymbol{F}^{p^{-T}}
\end{aligned}
\tag{9.308}
$$

Now, if we remember the Oldroyd rate of $\bar{\mathbf{S}}$, *i.e.* $\overset{\square}{\bar{\mathbf{S}}} = \dot{\bar{\mathbf{S}}} - \bar{\boldsymbol{\ell}}^p \cdot \bar{\mathbf{S}} - \bar{\mathbf{S}} \cdot \bar{\boldsymbol{\ell}}^{pT}$, the above expression becomes:

$$\boxed{\dot{\mathbf{S}} = \boldsymbol{F}^{p^{-1}} \cdot \overset{\square}{\bar{\mathbf{S}}} \cdot \boldsymbol{F}^{p^{-T}}} \tag{9.309}$$

9.7.3 The Helmholtz Free Energy

The Helmholtz free energy (per unit volume) is given by:

$$\psi = \psi(\boldsymbol{F}, T, \boldsymbol{\alpha}) \tag{9.310}$$

For temperature-independent processes the above expression is reduced to:

$$\psi = \psi(\boldsymbol{F}, \boldsymbol{\alpha}) \tag{9.311}$$

Then, if we consider the multiplicative decomposition of the deformation gradient we can adopt the intermediate configuration to define the energy expression. Note that the intermediate configuration is a stress-free one, *i.e.* it is elastically unloaded. Then, the energy function can be defined as:

$$\psi = \psi(\boldsymbol{F}, \boldsymbol{F}^p, \boldsymbol{\alpha}) \tag{9.312}$$

Considering that $\boldsymbol{F}^e = \boldsymbol{F}^e(\boldsymbol{F}, \boldsymbol{F}^p)$, the energy can be written in terms of

$$\Psi = \Psi(\boldsymbol{F}^e, \boldsymbol{\alpha}) \tag{9.313}$$

9.7.3.1 Decoupling the Helmholtz Free Energy

The Helmholtz free energy can be approached additively by two parts. One part is caused by the effect of $\boldsymbol{F}^e$ and the other part is caused by the effect of $\boldsymbol{\alpha}$, *i.e.*:

$$\Psi = \Psi^e(\boldsymbol{F}^e) + \Psi^p(\boldsymbol{\alpha}) \tag{9.314}$$

One advantage of this decoupling is that we can treat the elastic part of the energy, $\Psi^e(\boldsymbol{F}^e)$, in the same way as when we considered hyperelastic material which was discussed in Chapter 8.

9.7.3.2 The Objectivity Principle for the Helmholtz Free Energy

As we discussed in Chapter 6, the constitutive equation must satisfy certain principles including the principle of objectivity and as the energy is a scalar, it satisfies this principle, *i.e.* $\Psi = \Psi^*$. However, we can also use this principle to express the energy $\Psi = \Psi(\boldsymbol{F}^e, \boldsymbol{\alpha})$ in terms of other parameters and according to it, (see Chapter 4), the following holds:

$$\Psi = \Psi^*(\boldsymbol{F}^{e^*}, \boldsymbol{\alpha}^*) \tag{9.315}$$

The free variables $\boldsymbol{\alpha}^*$ can be scalars, vectors, or second-order tensors which fulfills the following law of transformation $\alpha^* = \alpha$ (scalar), $\vec{\boldsymbol{\alpha}}^* = \mathbf{Q} \cdot \vec{\boldsymbol{\alpha}}$ (vector), and $\boldsymbol{\alpha}^* = \mathbf{Q} \cdot \boldsymbol{\alpha} \cdot \mathbf{Q}^T$ (Eulerian second-order tensor). For the sake of simplicity, let us consider that $\boldsymbol{\alpha}^*$ is a scalar $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}$. Then, with respect to $\boldsymbol{F}^{e^*}$, remember that the deformation gradient does not obey the second-order tensor transformation law, and is given by $\boldsymbol{F}^{e^*} = \mathbf{Q} \cdot \boldsymbol{F}^e$. In addition, if we consider the polar decomposition of $\boldsymbol{F}^e$, (see Figure 9.50), then $\boldsymbol{F}^e = \mathbf{R}^e \cdot \overline{\mathbf{U}}^e = \overline{\mathbf{V}}^e \cdot \mathbf{R}^e$ is valid, thus:

$$\Psi = \Psi(\boldsymbol{F}^{e^*}, \boldsymbol{\alpha}^*) = \Psi(\mathbf{Q} \cdot \boldsymbol{F}^e, \boldsymbol{\alpha}) = \Psi(\mathbf{Q} \cdot \mathbf{R}^e \cdot \overline{\mathbf{U}}^e, \boldsymbol{\alpha}) \tag{9.316}$$

As the tensor $\mathbf{Q}$ can be any orthogonal tensor, we can adopt $\mathbf{Q} = \mathbf{R}^{e^T}$ with which we obtain:

$$\Psi = \Psi(\mathbf{R}^{e^T} \cdot \mathbf{R}^e \cdot \overline{\mathbf{U}}^e, \boldsymbol{\alpha}) = \Psi(\overline{\mathbf{U}}^e, \boldsymbol{\alpha}) \tag{9.317}$$

That is, the energy function can also be expressed in terms of the right stretch tensor of the intermediate configuration, (see Figure 9.50). Remember also that the right Cauchy-Green deformation tensor $\overline{\boldsymbol{C}}^e$ is related to $\overline{\mathbf{U}}^e$ by the following equation $\overline{\boldsymbol{C}}^e = \overline{\mathbf{U}}^{e^2}$, so the energy can also be expressed in terms of tensor $\overline{\boldsymbol{C}}^e$:

$$\Psi = \Psi(\overline{\boldsymbol{C}}^e, \boldsymbol{\alpha}) \tag{9.318}$$

Moreover, by considering that the Green-Lagrange strain tensor $\overline{\boldsymbol{E}}^e$ is related to $\overline{\boldsymbol{C}}^e$ by $2\overline{\boldsymbol{E}}^e = (\overline{\boldsymbol{C}}^e - \mathbf{1})$, we can still express the energy as:

$$\Psi = \Psi(\overline{\boldsymbol{E}}^e, \boldsymbol{\alpha}) \tag{9.319}$$

9.7.3.3 The Isotropic Helmholtz Free Energy

Remember that a scalar-valued isotropic tensor function can be written in terms of the principal values, *i.e.*:

$$\Psi = \Psi(\overline{\boldsymbol{C}}^e, \boldsymbol{\alpha}) = \Psi(\overline{\lambda}_1^{e^2}, \overline{\lambda}_2^{e^2}, \overline{\lambda}_3^{e^2}, \boldsymbol{\alpha}) \tag{9.320}$$

where $\overline{\lambda}_1^e$, $\overline{\lambda}_2^e$, $\overline{\lambda}_3^e$ are the principal values (eigenvalues) of $\overline{\mathbf{U}}^e$ (right stretch tensor). Then, as the tensors $\overline{\boldsymbol{C}}^e$ and $\boldsymbol{b}^e$ have the same eigenvalues, (see Chapter 2), we can obtain:

$$\Psi = \Psi(\overline{\boldsymbol{C}}^e, \boldsymbol{\alpha}) = \Psi(\overline{\lambda}_1^{e^2}, \overline{\lambda}_2^{e^2}, \overline{\lambda}_3^{e^2}, \boldsymbol{\alpha}) = \Psi(\boldsymbol{b}^e, \boldsymbol{\alpha}) \tag{9.321}$$

9.7.3.4 The Rate of Change of the Isotropic Helmholtz Free Energy

Let us consider the isotropic Helmholtz energy:

$$\boxed{\Psi = \Psi(\boldsymbol{b}^e, \boldsymbol{\alpha}) = \Psi(\overline{\lambda}_1^{e^2}, \overline{\lambda}_2^{e^2}, \overline{\lambda}_3^{e^2}, \boldsymbol{\alpha})} \quad \textit{Isotropic Helmholtz free energy} \tag{9.322}$$

The rate of change of (9.322) is evaluated as follows:

$$\dot{\Psi}(\boldsymbol{b}^e, \boldsymbol{\alpha}) = \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \dot{\boldsymbol{b}}^e + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \tag{9.323}$$

NOTE: The operator $\square$ is replaced by the number of contractions of the order of $\boldsymbol{\alpha}$. That is, if $\boldsymbol{\alpha}$ is an scalar (zeroth-order tensor), $\square$ does not contract; if $\boldsymbol{\alpha}$ is a vector (first order tensor), $\square = \cdot$ contracts once, *i.e.* the scalar (dot) product; if $\boldsymbol{\alpha}$ is second-order tensor, $\square =$ **:** contracts twice, *i.e.* the double scalar product; and so on. ■

Then, by substituting $\dot{\boldsymbol{b}}^e$ given in (9.274) into the equation (9.323) we obtain:

$$
\begin{aligned}
\dot{\Psi} &= \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \left[\overset{\square}{\boldsymbol{b}}{}^e + \boldsymbol{\ell} \cdot \boldsymbol{b}^e + \boldsymbol{b}^e \cdot \boldsymbol{\ell}^T \right] + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \\
&= \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \overset{\square}{\boldsymbol{b}}{}^e + \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \left(\boldsymbol{\ell} \cdot \boldsymbol{b}^e \right) + \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \left(\boldsymbol{b}^e \cdot \boldsymbol{\ell}^T \right) + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}}
\end{aligned}
\tag{9.324}
$$

It was proved in Chapter 1 that if Ψ is a scalar-valued isotropic tensor function, and if $\boldsymbol{b}^e$ is a symmetric second-order tensor, then $\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e = \boldsymbol{b}^e \cdot \frac{\partial \Psi}{\partial \boldsymbol{b}^e}$ holds, *i.e.* $\frac{\partial \Psi}{\partial \boldsymbol{b}^e}$ and $\boldsymbol{b}^e$ are coaxial tensosr. Note also that $\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e = \boldsymbol{b}^e \cdot \frac{\partial \Psi}{\partial \boldsymbol{b}^e}$ results in a symmetric second-order tensor, since $\frac{\partial \Psi}{\partial \boldsymbol{b}^e}$ and $\boldsymbol{b}^e$ are symmetric and coaxial tensors (see subsection 1.5.9 in Chapter 1).

Note that the following equations are also satisfied:

$$
\frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \left(\boldsymbol{\ell} \cdot \boldsymbol{b}^e \right) = \frac{\partial \Psi}{\partial b^e_{ij}} \ell_{ik} b^e_{kj} = \frac{\partial \Psi}{\partial b^e_{ij}} b^e_{jk} \ell_{ik} = \underbrace{\left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right)}_{\text{symmetric}} : \boldsymbol{\ell}
\tag{9.325}
$$

$$
\frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \left(\boldsymbol{b}^e \cdot \boldsymbol{\ell}^T \right) = \frac{\partial \Psi}{\partial b^e_{ij}} b^e_{ik} \ell_{jk} = \frac{\partial \Psi}{\partial b^e_{ji}} b^e_{ik} \ell_{jk} = \underbrace{\left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right)}_{\text{symmetric}} : \boldsymbol{\ell}
\tag{9.326}
$$

In Chapter 1, (see **Problem 1.16**), it was proven that: if $\mathbf{A}$ and $\mathbf{B}$ are arbitrary second-order tensors, the following is satisfied:

$$
\mathbf{A} : \mathbf{B} = \mathbf{A}^{sym} : \mathbf{B}^{sym} + \mathbf{A}^{skew} : \mathbf{B}^{skew}
\tag{9.327}
$$

Thus, we can conclude that:

$$
\left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \boldsymbol{\ell} = \left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right)^{sym} : \underbrace{\boldsymbol{\ell}^{sym}}_{\equiv \mathbf{D}} + \underbrace{\left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right)^{skew}}_{= \mathbf{0}} : \boldsymbol{\ell}^{skew} \quad = \left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \mathbf{D}
\tag{9.328}
$$

Then, going back to the equation in (9.324), we can conclude that:

$$
\begin{aligned}
\dot{\Psi} &= \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \overset{\square}{\boldsymbol{b}}{}^e + \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \left(\boldsymbol{\ell} \cdot \boldsymbol{b}^e \right) + \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \left(\boldsymbol{b}^e \cdot \boldsymbol{\ell}^T \right) + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \\
&= \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \overset{\square}{\boldsymbol{b}}{}^e \cdot \mathbf{1} + \left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \boldsymbol{\ell} + \left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \boldsymbol{\ell} + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \\
&= \frac{\partial \Psi}{\partial \boldsymbol{b}^e} : \overset{\square}{\boldsymbol{b}}{}^e \cdot \boldsymbol{b}^{e^{-1}} \cdot \boldsymbol{b}^e + \left(2 \frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \boldsymbol{\ell} + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \\
&= \left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \left(\overset{\square}{\boldsymbol{b}}{}^e \cdot \boldsymbol{b}^{e^{-1}} \right) + \left(2 \frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \boldsymbol{\ell} + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \\
&= \left(\frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \left(\overset{\square}{\boldsymbol{b}}{}^e \cdot \boldsymbol{b}^{e^{-1}} \right) + \left(2 \frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \mathbf{D} + \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}}
\end{aligned}
\tag{9.329}
$$

Then, the rate of change of the isotropic Helmholtz free energy becomes:

$$
\boxed{\dot{\Psi}(\boldsymbol{b}^e, \boldsymbol{\alpha}) = \left(2 \frac{\partial \Psi(\boldsymbol{b}^e, \boldsymbol{\alpha})}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \left[\frac{1}{2} \overset{\square}{\boldsymbol{b}}{}^e \cdot \boldsymbol{b}^{e^{-1}} + \mathbf{D} \right] + \frac{\partial \Psi(\boldsymbol{b}^e, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}}}
\tag{9.330}
$$

$$\boxed{\dot{\Psi}(\boldsymbol{b}^e,\boldsymbol{\alpha}) = \left(2\frac{\partial\Psi(\boldsymbol{b}^e,\boldsymbol{\alpha})}{\partial\boldsymbol{b}^e}\cdot\boldsymbol{b}^e\right):\left(\frac{1}{2}\overset{\square}{\boldsymbol{b}^e}\cdot\boldsymbol{b}^{e^{-1}}+\boldsymbol{\ell}\right)+\frac{\partial\Psi(\boldsymbol{b}^e,\boldsymbol{\alpha})}{\partial\boldsymbol{\alpha}}\square\dot{\boldsymbol{\alpha}}} \tag{9.331}$$

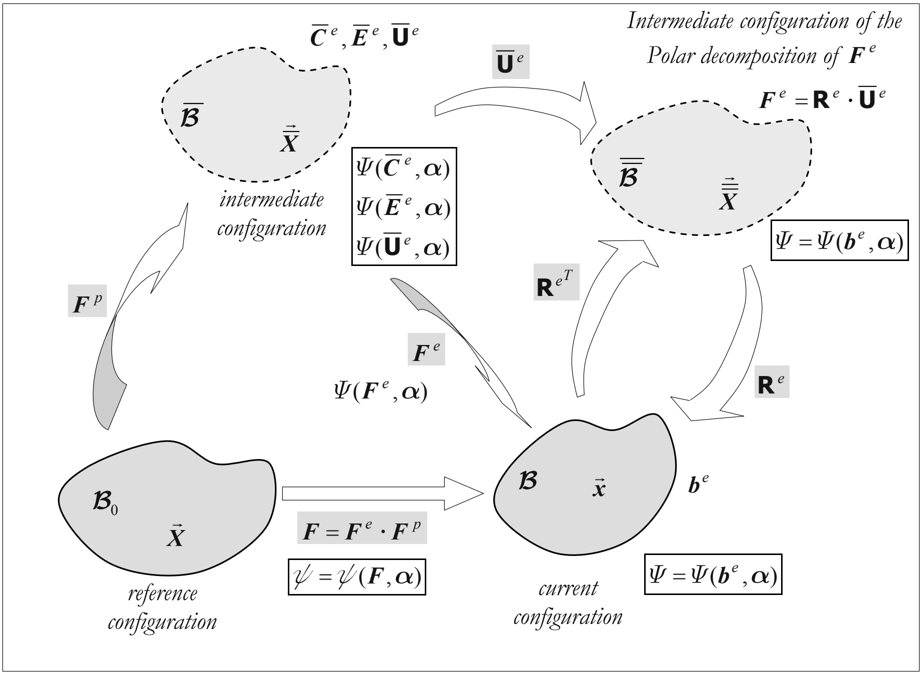


Figure 9.50: Helmholtz free energy.

9.7.4 The Plastic Potential and the Yield Criterion

The plastic potential in the strain space is given by:

$$\mathcal{G} = \mathcal{G}(\boldsymbol{F}^e,\alpha) \tag{9.332}$$

As we can see, it is a function that depends on the same parameters as the free energy.

The yield surface is defined analogously as follows:

$$\mathcal{F} = \mathcal{F}(\boldsymbol{F}^e,\alpha) \tag{9.333}$$

The plastic potential and the yield surface have to fulfill the principle of objectivity, so the following holds:

$$\mathcal{G} = \mathcal{G}(\overline{\boldsymbol{E}}^e,\alpha) \quad ; \quad \mathcal{F} = \mathcal{F}(\overline{\boldsymbol{E}}^e,\alpha) \tag{9.334}$$

We can also express these functions in the stress space. Then, if we consider that, in the intermediate configuration, $\overline{\mathbf{S}} = \overline{\mathbf{S}}(\overline{\boldsymbol{E}}^e)$, we then have:

$$\mathcal{G} = \overline{\mathcal{G}}(\overline{\mathbf{S}}(\overline{\boldsymbol{E}}^e),\overline{\alpha}) \quad ; \quad \mathcal{F} = \overline{\mathcal{F}}(\overline{\mathbf{S}}(\overline{\boldsymbol{E}}^e),\overline{\alpha}) \tag{9.335}$$

It is noteworthy that $\mathcal{G}$ and $\mathcal{F}$ are hypersurfaces as regards the six independent components of $\overline{\mathbf{S}}$. When we are dealing with isotropic materials, these functions can be represented in terms of the three eigenvalues of $\overline{\mathbf{S}}$.

9.7.5 The Dissipation and the Constitutive Equation

The internal dissipation $\mathcal{D}_{int}$ in the reference configuration, (see Chapter 5), is given by:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0 \left[\eta \dot{T} + \dot{\psi} \right] \geq 0 \tag{9.336}$$

where ψ is the Helmholtz free energy per unit mass, and $\Psi = \rho_0 \psi$ is the energy per unit volume. In the isothermal process, $\dot{T} = 0$, this dissipation becomes:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \dot{\Psi} = \boldsymbol{\tau} : \mathbf{D} - \dot{\Psi} \geq 0 \tag{9.337}$$

Then, if we consider the equation in (9.301), the above inequality becomes:

$$\begin{aligned} \mathcal{D}_{int} &= \boldsymbol{\tau} : \mathbf{D} - \dot{\Psi} = \left(\boldsymbol{F}^e \cdot \overline{\mathbf{S}} \cdot \boldsymbol{F}^{e^T} \right) : \mathbf{D} - \dot{\Psi} \geq 0 \\ &= F^e_{ik} \overline{\mathsf{S}}_{kp} F^e_{jp} \mathsf{D}_{ij} - \dot{\Psi} = \overline{\mathsf{S}}_{kp} F^e_{ik} \mathsf{D}_{ij} F^e_{jp} - \dot{\Psi} \geq 0 \\ &= \overline{\mathbf{S}} : \left(\boldsymbol{F}^{e^T} \cdot \mathbf{D} \cdot \boldsymbol{F}^e \right) - \dot{\Psi} \geq 0 \end{aligned} \tag{9.338}$$

Then, by considering the equation in (9.294), the dissipation becomes:

$$\boxed{\begin{aligned} \mathcal{D}_{int} &= \boldsymbol{\tau} : \mathbf{D} - \dot{\Psi} \geq 0 \\ &= \overline{\mathbf{S}} : \overset{\Delta}{\overline{\boldsymbol{E}}} - \dot{\Psi} \geq 0 \end{aligned}} \tag{9.339}$$

Next, by substituting (9.330) into the dissipation (9.339) we can obtain:

$$\begin{aligned} \mathcal{D}_{int} &= \boldsymbol{\tau} : \mathbf{D} - \dot{\Psi}(\boldsymbol{b}^e, \boldsymbol{\alpha}) \geq 0 \\ &= \boldsymbol{\tau} : \mathbf{D} - \left(2 \frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \left[\frac{1}{2} \overset{\square}{\boldsymbol{b}^e} \cdot \boldsymbol{b}^{e^{-1}} + \mathbf{D} \right] - \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \geq 0 \\ &= \left(\boldsymbol{\tau} - 2 \frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \mathbf{D} + \left(2 \frac{\partial \Psi}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e \right) : \left(-\frac{1}{2} \overset{\square}{\boldsymbol{b}^e} \cdot \boldsymbol{b}^{e^{-1}} \right) - \frac{\partial \Psi}{\partial \boldsymbol{\alpha}} \square \dot{\boldsymbol{\alpha}} \geq 0 \end{aligned} \tag{9.340}$$

As the above inequality must be satisfied for any thermodynamic process, we can deduce that:

$$\boxed{\boldsymbol{\tau} = 2 \frac{\partial \Psi(\boldsymbol{b}^e, \boldsymbol{\alpha})}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e} \quad \textit{Constitutive equation for stress} \tag{9.341}$$

In addition, by adopting $\boldsymbol{\beta} = -\dfrac{\partial \Psi}{\partial \boldsymbol{\alpha}}$ (thermodynamic forces), which represents the hardening internal forces, we obtain:

$$\boxed{\mathcal{D}_{int} = \boldsymbol{\tau} : \left(-\frac{1}{2} \overset{\square}{\boldsymbol{b}^e} \cdot \boldsymbol{b}^{e^{-1}} \right) + \boldsymbol{\beta} \square \dot{\boldsymbol{\alpha}} \geq 0} \tag{9.342}$$

Note that in a process that is purely elastic (hyperelasticity), we have $\dot{\boldsymbol{\alpha}} = 0$, and revert to the following scenario $\boldsymbol{b}^e = \boldsymbol{b}$, with which we find that the energy dissipation is equal to zero, *i.e.* $\mathcal{D}_{int} = 0$, (as we demonstrated in (9.279) that $\overset{\square}{\boldsymbol{b}} = \mathbf{0}$ with which the constitutive equation for stress (9.341) is reduced to $\boldsymbol{\tau} = 2 \dfrac{\partial \Psi(\boldsymbol{b})}{\partial \boldsymbol{b}} \cdot \boldsymbol{b}$ and which is the same expression obtained as that for an isotropic hyperelastic material, see subsection 8.3.1 in Chapter 8).

9.7.6 Evolution of the Internal Variables

In order to fully describe the constitutive model we have to establish how the internal variables evolve.

Firstly, we will define the elastic domain $\mathcal{E}_{\boldsymbol{\sigma}}$ in the stress space, and the yield criterion $\mathcal{F}$ in terms of the Kirchhoff stress tensor $\boldsymbol{\tau}$ (current configuration):

$$\mathcal{E}_{\boldsymbol{\sigma}} = \{(\boldsymbol{\tau}, \boldsymbol{\beta}) \in \mathcal{R}: \qquad \mathcal{F}(\boldsymbol{\tau}, \boldsymbol{\beta}) \leq 0\} \tag{9.343}$$

where $\boldsymbol{\beta}$ is the scalar-valued tensor function, denoted by the isotropic hardening function and $\mathcal{F}$ is assumed to be a convex function.

Next, we apply the maximum dissipation principle, which states that dissipation in the material reaches a maximum during a change characterized by a dissipative process. Let us consider the current state $(\boldsymbol{\tau}, \boldsymbol{\beta}) \in \mathcal{E}_{\boldsymbol{\sigma}}$ which represents the current distribution of the Kirchhoff stress tensor and thermodynamic forces in a body subjected to plastic strain. The maximum dissipation principle requires that for a change of state, show here by $(\boldsymbol{\tau}^*, \boldsymbol{\beta}^*) \in \mathcal{E}_{\boldsymbol{\sigma}}$, the following must be satisfied:

$$\left[\boldsymbol{\tau} - \boldsymbol{\tau}^*\right] : \left(-\frac{1}{2} \overset{\square}{\boldsymbol{b}}{}^e \cdot \boldsymbol{b}^{e^{-1}} \right) + \left[\boldsymbol{\beta} - \boldsymbol{\beta}^*\right] \square \dot{\boldsymbol{\alpha}} \geq 0 \tag{9.344}$$

The inequality in (9.344) describes an optimization problem with constraint. Here, we can maximize the dissipation by minimizing the negative dissipation under the constraint $\Phi(\boldsymbol{\tau}, \boldsymbol{\beta}) \leq 0$. To this end, we define the Lagrangian:

$$\mathcal{L} = -\mathcal{D}_{int} + \dot{\gamma}\, \Phi = -\boldsymbol{\tau} : \left(-\frac{1}{2} \overset{\square}{\boldsymbol{b}}{}^e \cdot \boldsymbol{b}^{e^{-1}} \right) - \boldsymbol{\beta} \square \dot{\boldsymbol{\alpha}} + \dot{\gamma}\, \Phi \tag{9.345}$$

where $\dot{\gamma}$ is the Lagrange multiplier that enforces $\Phi \leq 0$.

Then, the flow rule can be obtained as follows:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\tau}} = \mathbf{0} \quad \Rightarrow \quad \frac{1}{2} \overset{\square}{\boldsymbol{b}}{}^e \cdot \boldsymbol{b}^{e^{-1}} + \dot{\gamma} \frac{\partial \Phi}{\partial \boldsymbol{\tau}} = \mathbf{0} \quad \Rightarrow \quad -\overset{\square}{\boldsymbol{b}}{}^e = 2\dot{\gamma} \frac{\partial \Phi}{\partial \boldsymbol{\tau}} \cdot \boldsymbol{b}^e \tag{9.346}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\beta}} = 0 \quad \Rightarrow \quad -\dot{\boldsymbol{\alpha}} + \dot{\gamma} \frac{\partial \Phi}{\partial \boldsymbol{\beta}} = 0 \quad \Rightarrow \quad \dot{\boldsymbol{\alpha}} = \dot{\gamma} \frac{\partial \Phi}{\partial \boldsymbol{\beta}} \tag{9.347}$$

Next, we can summarize the evolution of the variables as:

$$\boxed{\overset{\square}{\boldsymbol{b}}{}^e = -2\dot{\gamma} \frac{\partial \Phi(\boldsymbol{\tau}, \boldsymbol{\beta})}{\partial \boldsymbol{\tau}} \cdot \boldsymbol{b}^e \quad ; \quad \dot{\boldsymbol{\alpha}} = \dot{\gamma} \frac{\partial \Phi(\boldsymbol{\tau}, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}}} \tag{9.348}$$

To fully define the model we introduce the loading/unloading Kuhn-Tucker conditions:

$$\dot{\gamma} \geq 0 \quad ; \quad \mathcal{F}(\boldsymbol{\tau}, \boldsymbol{\beta}) \leq 0 \quad ; \quad \dot{\gamma}\, \mathcal{F}(\boldsymbol{\tau}, \boldsymbol{\beta}) = 0 \tag{9.349}$$

and the persistency condition:

$$\dot{\gamma}\, \dot{\mathcal{F}}(\boldsymbol{\tau}, \boldsymbol{\beta}) = 0 \tag{9.350}$$

Then, by substituting the equation in (9.278), *i.e.* $\overset{\square}{\boldsymbol{b}}{}^e = \boldsymbol{F} \cdot \dot{\boldsymbol{C}}^{p^{-1}} \cdot \boldsymbol{F}^T$, into (9.346) we can obtain:

$$-\boldsymbol{F}\cdot\dot{\boldsymbol{C}}^{p^{-1}}\cdot\boldsymbol{F}^{T}=2\dot{\gamma}\frac{\partial\Phi}{\partial\boldsymbol{\tau}}\cdot\boldsymbol{b}^{e}\quad\Rightarrow\quad\dot{\boldsymbol{C}}^{p^{-1}}=-2\dot{\gamma}\,\boldsymbol{F}^{-1}\cdot\left(\frac{\partial\Phi}{\partial\boldsymbol{\tau}}\right)\cdot\boldsymbol{b}^{e}\cdot\boldsymbol{F}^{-T}\tag{9.351}$$

and if we consider that $\boldsymbol{b}^{e}=\boldsymbol{F}\cdot\boldsymbol{C}^{p^{-1}}\cdot\boldsymbol{F}^{T}$ we can conclude that:

$$\begin{aligned}\dot{\boldsymbol{C}}^{p^{-1}}&=-2\dot{\gamma}\,\boldsymbol{F}^{-1}\cdot\left(\frac{\partial\Phi}{\partial\boldsymbol{\tau}}\right)\cdot\boldsymbol{b}^{e}\cdot\boldsymbol{F}^{-T}=-2\dot{\gamma}\,\boldsymbol{F}^{-1}\cdot\left(\frac{\partial\Phi}{\partial\boldsymbol{\tau}}\right)\cdot\boldsymbol{F}\cdot\boldsymbol{C}^{p^{-1}}\cdot\boldsymbol{F}^{T}\cdot\boldsymbol{F}^{-T}\\&=-2\dot{\gamma}\left[\boldsymbol{F}^{-1}\cdot\left(\frac{\partial\Phi}{\partial\boldsymbol{\tau}}\right)\cdot\boldsymbol{F}\right]\cdot\boldsymbol{C}^{p^{-1}}\end{aligned}\tag{9.352}$$

Summary

Helmholtz Free Energy (per unit volume):	$\Psi=\Psi(\boldsymbol{b}^{e},\boldsymbol{\alpha})$	(9.353)
Measurement of elastic deformation:	$\boldsymbol{b}^{e}=\boldsymbol{F}^{e}\cdot\boldsymbol{F}^{e^{T}}=\boldsymbol{F}\cdot\boldsymbol{C}^{p^{-1}}\cdot\boldsymbol{F}^{T}$	(9.354)
Stress:	$\boldsymbol{\tau}=2\dfrac{\partial\Psi}{\partial\boldsymbol{b}^{e}}\cdot\boldsymbol{b}^{e}$	(9.355)
Isotropic hardening force:	$\boldsymbol{\beta}=-\dfrac{\partial\Psi}{\partial\dot{\boldsymbol{\alpha}}}$	(9.356)
Yield surface:	$\mathcal{F}=\mathcal{F}(\boldsymbol{\tau},\boldsymbol{\beta})$	(9.357)
Evolution equations:	$\dot{\boldsymbol{C}}^{p^{-1}}=-2\dot{\gamma}\left[\boldsymbol{F}^{-1}\cdot\left(\dfrac{\partial\Phi}{\partial\boldsymbol{\tau}}\right)\cdot\boldsymbol{F}\right]\cdot\boldsymbol{C}^{p^{-1}}$ $\dot{\boldsymbol{\alpha}}=\dot{\gamma}\dfrac{\partial\Phi}{\partial\boldsymbol{\beta}}$	(9.358)
Kuhn-Tucker condition (loading/unloading):	$\dot{\gamma}\geq 0\quad;\quad\mathcal{F}(\boldsymbol{\tau},\boldsymbol{\beta})\leq 0\quad;\quad\dot{\gamma}\,\mathcal{F}(\boldsymbol{\tau},\boldsymbol{\beta})=0$	(9.359)
Dissipation:	$\mathcal{D}_{int}=\boldsymbol{\tau}:\left(-\dfrac{1}{2}\overset{\square}{\boldsymbol{b}^{e}}\cdot\boldsymbol{b}^{e^{-1}}\right)+\boldsymbol{\beta}\square\dot{\boldsymbol{\alpha}}\geq 0$	(9.360)

9.7.7 The Elastoplastic Tangent Stiffness Tensors

Let us consider the energy function $\Psi^{e}=\Psi^{e}(\overline{\boldsymbol{E}}^{e})$, where $\overline{\boldsymbol{E}}^{e}=\boldsymbol{F}^{p^{-T}}\cdot\left(\boldsymbol{E}-\boldsymbol{E}^{p}\right)\cdot\boldsymbol{F}^{p^{-1}}$, (see equation (9.238)), and the second Piola-Kirchhoff stress tensor in the intermediate and in the reference configuration, respectively, are given by:

$$\overline{\mathbf{S}}=\frac{\partial\overline{\Psi}^{e}}{\partial\overline{\boldsymbol{E}}^{e}}\quad;\quad\mathbf{S}=\frac{\partial\Psi^{e}}{\partial\boldsymbol{E}}\tag{9.361}$$

where $\overline{\mathbf{S}}=\boldsymbol{F}^{p}\cdot\mathbf{S}\cdot\boldsymbol{F}^{p^{T}}$ is fulfilled (see equation (9.302)). The elastoplastic tangent stiffness fourth-order tensors $\overline{\mathbb{C}}^{e}$ and $\mathbb{C}$ are introduced, (see Chapter 8), as follows:

$$\overline{\mathbb{C}}^{e}=\frac{\partial^{2}\overline{\Psi}^{e}}{\partial\overline{\boldsymbol{E}}^{e}\otimes\partial\overline{\boldsymbol{E}}^{e}}\quad;\quad\mathbb{C}=\frac{\partial^{2}\Psi^{e}}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}\tag{9.362}$$

The tensors $\overline{\mathbb{C}}^{e}$ and $\mathbb{C}$ have minor and major symmetries and they are related to each other by:

$$\mathbb{C}_{ijkl}=F_{im}^{p^{-1}}F_{jn}^{p^{-1}}\overline{\mathbb{C}}^{e}_{mnpq}F_{kp}^{p^{-1}}F_{lq}^{p^{-1}}\tag{9.363}$$

The tensor $\overline{\mathbb{C}}^e$ appears in the following linear relationship:

$$\dot{\overline{\mathbf{S}}} = \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}}^e \tag{9.364}$$

Then, by substituting the expression of $\dot{\overline{\mathbf{S}}}$ given by (9.306), and $\dot{\overline{\boldsymbol{E}}}^e$ given by (9.299), into the equation in (9.364), we obtain:

$$\boldsymbol{F}^p \cdot \left(\dot{\mathbf{S}} + \mathbf{Z}^p \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{Z}^{p^T} \right) \cdot \boldsymbol{F}^{p^T} = \overline{\mathbb{C}}^e : \left\{ \boldsymbol{F}^{p^{-T}} \cdot \left[\dot{\boldsymbol{E}} - \boldsymbol{F}^{p^T} \cdot \left(\overline{\boldsymbol{C}}^e \cdot \overline{\boldsymbol{\ell}}^p \right)^{sym} \cdot \boldsymbol{F}^p \right] \cdot \boldsymbol{F}^{p^{-1}} \right\} \tag{9.365}$$

We then apply the dot product both between $\boldsymbol{F}^{p^{-1}}$ and (9.365), and also between the equation in (9.365) and $\boldsymbol{F}^{p^{-T}}$, with which we obtain:

$$\begin{aligned} &\dot{\mathbf{S}} + \mathbf{Z}^p \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{Z}^{p^T} = \boldsymbol{F}^{p^{-1}} \cdot \overline{\mathbb{C}}^e : \left\{ \boldsymbol{F}^{p^{-T}} \cdot \left[\dot{\boldsymbol{E}} - \boldsymbol{F}^{p^T} \cdot \left(\overline{\boldsymbol{C}}^e \cdot \overline{\boldsymbol{\ell}}^p \right)^{sym} \cdot \boldsymbol{F}^p \right] \cdot \boldsymbol{F}^{p^{-1}} \right\} \cdot \boldsymbol{F}^{p^{-T}} \\ &\Rightarrow \dot{\mathbf{S}} = \mathbb{C} : \left[\dot{\boldsymbol{E}} - \boldsymbol{F}^{p^T} \cdot \left(\overline{\boldsymbol{C}}^e \cdot \overline{\boldsymbol{\ell}}^p \right)^{sym} \cdot \boldsymbol{F}^p \right] - \left(\mathbf{Z}^p \cdot \mathbf{S} + \mathbf{S} \cdot \mathbf{Z}^{p^T} \right) \end{aligned} \tag{9.366}$$

9.7.7.1 The Elastoplastic Tangent Stiffness Tensor

In the intermediate configuration the following holds:

$$\dot{\overline{\mathbf{S}}} = \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}}^e = \overline{\mathbb{C}}^e : \left(\dot{\overline{\boldsymbol{E}}} - \dot{\overline{\boldsymbol{e}}}^p \right) \quad with \quad \overline{\mathbb{C}}^e = \frac{\partial^2 \overline{\Psi}^e}{\partial \overline{\boldsymbol{E}}^e \otimes \partial \overline{\boldsymbol{E}}^e} \tag{9.367}$$

We can express $\dot{\overline{\boldsymbol{e}}}^p$ as follows:

$$\dot{\overline{\boldsymbol{e}}}^p = \dot{\gamma} \frac{\partial \mathcal{G}(\overline{\mathbf{S}}, \boldsymbol{\alpha})}{\partial \overline{\mathbf{S}}} = \dot{\gamma} \nabla_{\overline{\mathbf{S}}} \mathcal{G} \tag{9.368}$$

where $\dot{\gamma}$ is the plastic multiplier. Then, by combining (9.368) with (9.367) we obtain:

$$\dot{\overline{\mathbf{S}}} = \overline{\mathbb{C}}^e : \left(\dot{\overline{\boldsymbol{E}}} - \dot{\overline{\boldsymbol{e}}}^p \right) = \overline{\mathbb{C}}^e : \left(\dot{\overline{\boldsymbol{E}}} - \dot{\gamma} \frac{\partial \mathcal{G}(\overline{\mathbf{S}}, \boldsymbol{\alpha})}{\partial \overline{\mathbf{S}}} \right) \tag{9.369}$$

Next, to obtain the parameter $\dot{\gamma}$ we use the consistency condition, *i.e.* any change in the intermediate configuration must allow the stress state to remain on the yield surface, thus for $\dot{\gamma} > 0 \Rightarrow \dot{\mathcal{F}} = 0$ we have:

$$\begin{aligned} &\dot{\mathcal{F}}(\overline{\mathbf{S}}, \alpha) = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \dot{\overline{\mathbf{S}}} + \sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_i} \dot{\alpha}_i = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \dot{\overline{\mathbf{S}}} + \sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_i} \dot{\gamma} H_i = 0 \\ &\Rightarrow \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \dot{\overline{\mathbf{S}}} + \dot{\gamma} \sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_i} H_i = 0 \end{aligned} \tag{9.370}$$

For the sake of simplicity we have considered that α_i are scalars. Then, by combining (9.369) with (9.370) we obtain:

$$\begin{aligned} &\frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \dot{\overline{\mathbf{S}}} + \dot{\gamma} \sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_i} H_i = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \left[\overline{\mathbb{C}}^e : \left(\dot{\overline{\boldsymbol{E}}} - \dot{\gamma} \frac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} \right) \right] + \dot{\gamma} \sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_i} H_i = 0 \\ &\Rightarrow \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}} - \dot{\gamma} \frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \frac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} + \dot{\gamma} \sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_i} H_i = 0 \end{aligned} \tag{9.371}$$

$$\Rightarrow \dot{\gamma} = \frac{\left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}}\right)}{\left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} - \sum_{i=1}^{n} \dfrac{\partial \mathcal{F}}{\partial \alpha_i} H_i\right)} \tag{9.372}$$

Next, by substituting the above expression of $\dot{\gamma}$ into the equation in (9.369), we find:

$$\dot{\overline{\mathbf{S}}} = \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}} - \dot{\gamma}\, \overline{\mathbb{C}}^e : \frac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} = \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}} - \frac{\left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}}\right)}{\left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} - \sum_{i=1}^{n} \dfrac{\partial \mathcal{F}}{\partial \alpha_i} H_i\right)} \overline{\mathbb{C}}^e : \frac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} \tag{9.373}$$

Denoting by:

$$\mathcal{K} = \frac{1}{\left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} - \sum_{i=1}^{n} \dfrac{\partial \mathcal{F}}{\partial \alpha_i} H_i\right)} \tag{9.374}$$

the equation in (9.373) becomes:

$$\dot{\overline{\mathbf{S}}} = \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}} - \mathcal{K}\left(\frac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dot{\overline{\boldsymbol{E}}}\right) \overline{\mathbb{C}}^e : \frac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} \quad \Bigg| \quad \dot{\overline{\mathsf{S}}}_{ij} = \overline{\mathbb{C}}^e{}_{ijkl} \dot{\overline{E}}_{kl} - \mathcal{K}\left(\frac{\partial \mathcal{F}}{\partial \overline{\mathsf{S}}_{pq}} \overline{\mathbb{C}}^e_{pqst} \dot{\overline{E}}_{st}\right) \overline{\mathbb{C}}^e{}_{ijkl} \frac{\partial \mathcal{G}}{\partial \overline{\mathsf{S}}_{kl}} \tag{9.375}$$

Then, if we consider the substitution operator property we obtain:

$$\begin{aligned}
\dot{\overline{\mathsf{S}}}_{ij} &= \overline{\mathbb{C}}^e{}_{ijkl} \dot{\overline{E}}_{ab} \delta_{ak} \delta_{bl} - \mathcal{K} \frac{\partial \mathcal{F}}{\partial \overline{\mathsf{S}}_{pq}} \overline{\mathbb{C}}^e_{pqst} \dot{\overline{E}}_{ab} \delta_{as} \delta_{bt} \overline{\mathbb{C}}^e{}_{ijkl} \frac{\partial \mathcal{G}}{\partial \overline{\mathsf{S}}_{kl}} \\
&= \left(\overline{\mathbb{C}}^e{}_{ijkl} \delta_{ak} \delta_{bl} - \mathcal{K} \frac{\partial \mathcal{F}}{\partial \overline{\mathsf{S}}_{pq}} \overline{\mathbb{C}}^e_{pqst} \delta_{as} \delta_{bt} \overline{\mathbb{C}}^e{}_{ijkl} \frac{\partial \mathcal{G}}{\partial \overline{\mathsf{S}}_{kl}}\right) \dot{\overline{E}}_{ab} \\
&= \left(\overline{\mathbb{C}}^e{}_{ijab} - \mathcal{K} \frac{\partial \mathcal{F}}{\partial \overline{\mathsf{S}}_{pq}} \overline{\mathbb{C}}^e_{pqab} \overline{\mathbb{C}}^e{}_{ijkl} \frac{\partial \mathcal{G}}{\partial \overline{\mathsf{S}}_{kl}}\right) \dot{\overline{E}}_{ab} = \left[\overline{\mathbb{C}}^e{}_{ijab} - \mathcal{K}\left(\overline{\mathbb{C}}^e{}_{ijkl} \frac{\partial \mathcal{G}}{\partial \overline{\mathsf{S}}_{kl}}\right)\left(\frac{\partial \mathcal{F}}{\partial \overline{\mathsf{S}}_{pq}} \overline{\mathbb{C}}^e_{pqab}\right)\right] \dot{\overline{E}}_{ab}
\end{aligned} \tag{9.376}$$

Thus:

$$\dot{\overline{\mathbf{S}}} = \left[\overline{\mathbb{C}}^e - \frac{\left(\overline{\mathbb{C}}^e : \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}}\right) \otimes \left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e\right)}{\left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} - \sum_{i=1}^{n} \dfrac{\partial \mathcal{F}}{\partial \alpha_i} H_i\right)}\right] : \dot{\overline{\boldsymbol{E}}} \quad \Rightarrow \quad \dot{\overline{\mathbf{S}}} = \overline{\mathbb{C}}^{tan\_ep} : \dot{\overline{\boldsymbol{E}}} \tag{9.377}$$

where we have introduced the elastoplastic tangent stiffness tensor as:

$$\boxed{\overline{\mathbb{C}}^{tan\_ep} = \overline{\mathbb{C}}^e - \frac{\left(\overline{\mathbb{C}}^e : \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}}\right) \otimes \left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e\right)}{\left(\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}} : \overline{\mathbb{C}}^e : \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}} - \sum_{i=1}^{n} \dfrac{\partial \mathcal{F}}{\partial \alpha_i} H_i\right)}} \quad \textit{Elastoplastic tangent stiffness tensor (intermediate configuration)} \tag{9.378}$$

If we consider that $\overline{\mathbf{m}} = \dfrac{\partial \mathcal{G}}{\partial \overline{\mathbf{S}}}$ and $\overline{\mathbf{n}} = \dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}}$, where both tensors are symmetric second-order tensors, then, the above equation can be rewritten as follows:

$$\overline{\mathbb{C}}^{tan\_ep} = \overline{\mathbb{C}}^{e} - \mathcal{K}\left(\overline{\mathbb{C}}^{e} : \overline{\mathbf{m}}\right) \otimes \left(\overline{\mathbf{n}} : \overline{\mathbb{C}}^{e}\right) \tag{9.379}$$

Note that $\overline{\mathbf{m}}$ is the gradient of $\mathcal{G}$ in the stress space, $\overline{\mathbf{n}}$ is the gradient of the yield surface in the same space, *i.e.* $\overline{\mathbf{n}}$ is normal to the yield surface. Then, in the case of associated flow, $\mathcal{F} = \mathcal{G}$, we obtain $\overline{\mathbf{m}} = \overline{\mathbf{n}}$, and the elastoplastic tangent stiffness tensor is symmetric (major and minor symmetry):

$$\boxed{\overline{\mathbb{C}}^{tan\_ep} = \overline{\mathbb{C}}^{e} - \frac{\left(\overline{\mathbb{C}}^{e} : \overline{\mathbf{n}}\right) \otimes \left(\overline{\mathbf{n}} : \overline{\mathbb{C}}^{e}\right)}{\left(\overline{\mathbf{n}} : \overline{\mathbb{C}}^{e} : \overline{\mathbf{n}} - \displaystyle\sum_{i=1}^{n} \frac{\partial \mathcal{F}}{\partial \alpha_i} H_i\right)}} \quad \textit{Elastoplastic tangent stiffness tensor for associated flow rule} \tag{9.380}$$

9.7.8 The Hyperelastoplastic Model with von Mises Yield Criterion

In this subsection we will formulate a model for finite strain plasticity considering the J_2 flow theory with the isotropic hardening law, (see Simo&Hughes (1998)).

9.7.8.1 The Helmholtz Free Energy

In this model, Simo&Hughes (1998), the Helmholtz free energy (per unit volume) is given by:

$$\boxed{\Psi = \frac{\kappa}{2}\left[\frac{1}{2}\left(J^2 - 1\right) - \ln(J)\right] + \frac{\mu}{2}\left[\mathrm{Tr}(\widetilde{\boldsymbol{b}}^{e}) - 3\right]} \tag{9.381}$$

where $J = \det(\boldsymbol{F})$, and $\widetilde{\boldsymbol{b}}^{e} = J^{-2/3}\boldsymbol{b}^{e}$ is the isochoric part of the elastic part of the left Cauchy-Green deformation tensor, κ is the bulk modulus, and $\mu = G$ is the shear modulus.

Then, starting from the intermediate configuration we apply a multiplicative decomposition by means of a volumetric transformation brought about by $\boldsymbol{F}^{e^{vol}} = J^{e^{\frac{1}{3}}}\mathbf{1}$, (see Figure 9.51), and another brought about by an isochoric transformation characterized by $\widetilde{\boldsymbol{F}}^{e} = J^{e^{\frac{-1}{3}}}\boldsymbol{F}^{e}$, (see also Chapter 2) which gives us the following tensors:

$$\boldsymbol{b}^{e^{vol}} = \boldsymbol{F}^{e^{vol}} \cdot \left(\boldsymbol{F}^{e^{vol}}\right)^T = J^{e^{\frac{1}{3}}}\mathbf{1} \cdot J^{e^{\frac{1}{3}}}\mathbf{1} = J^{e^{\frac{2}{3}}} \tag{9.382}$$

$$\widetilde{\boldsymbol{b}}^{e} = \widetilde{\boldsymbol{F}}^{e} \cdot \left(\widetilde{\boldsymbol{F}}^{e}\right)^T = J^{e^{\frac{-1}{3}}}\boldsymbol{F}^{e} \cdot \left(J^{e^{\frac{-1}{3}}}\boldsymbol{F}^{e}\right)^T = J^{e^{\frac{-2}{3}}}\boldsymbol{F}^{e} \cdot \boldsymbol{F}^{e^T} = J^{e^{\frac{-2}{3}}}\boldsymbol{b}^{e} \tag{9.383}$$

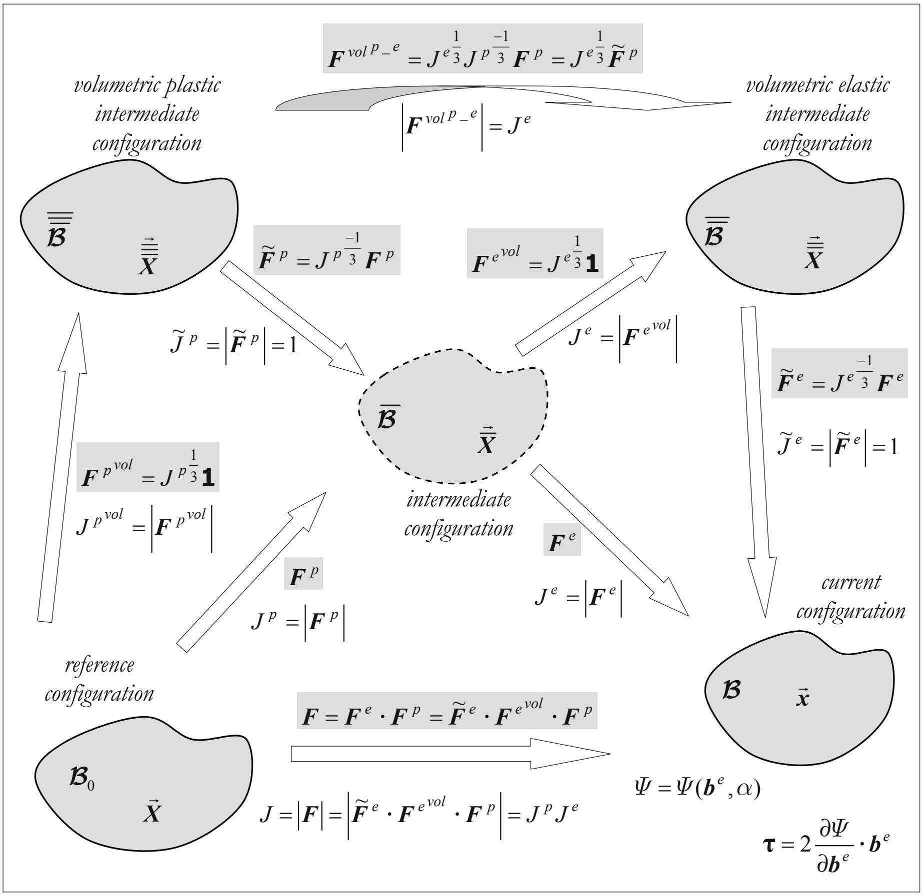


Figure 9.51: Volumetric and isochoric decomposition – multiplicative decomposition.

9.7.8.2 The Stress Tensor

The Kirchhoff stress tensor (9.341) becomes

$$
\begin{aligned}
\boldsymbol{\tau} &= 2\frac{\partial \Psi(\boldsymbol{b}^e, \boldsymbol{\alpha})}{\partial \boldsymbol{b}^e} \cdot \boldsymbol{b}^e = 2\frac{\partial}{\partial \boldsymbol{b}^e}\left\{\frac{\kappa}{2}\left[\frac{1}{2}\left(J^2 - 1\right) - \ln(J)\right] + \frac{\mu}{2}\left[\mathrm{Tr}(\tilde{\boldsymbol{b}}^e) - 3\right]\right\} \cdot \boldsymbol{b}^e \\
&= \left\{\kappa\left[\frac{1}{2}\frac{\partial}{\partial \boldsymbol{b}^e}\left(J^2 - 1\right) - \frac{\partial}{\partial \boldsymbol{b}^e}\left[\ln(J)\right]\right] + \mu\frac{\partial}{\partial \boldsymbol{b}^e}\left[J^{e\frac{-2}{3}}\mathrm{Tr}(\boldsymbol{b}^e)\right]\right\} \cdot \boldsymbol{b}^e
\end{aligned}
\tag{9.384}
$$

where $\mathrm{Tr}(\tilde{\boldsymbol{b}}^e) = \tilde{\boldsymbol{b}}^e : \mathbf{1} = J^{e\frac{-2}{3}} \boldsymbol{b}^e : \mathbf{1} = J^{e\frac{-2}{3}} \mathrm{Tr}(\boldsymbol{b}^e)$ holds and the derivatives of the invariants, (see Chapter 1), become:

$$\frac{\partial(J^e)}{\partial \boldsymbol{b}^e} = \frac{1}{2} J^e \boldsymbol{b}^{e^{-1}} , \quad \frac{\partial J}{\partial \boldsymbol{b}^e} = \frac{\partial(J^e J^p)}{\partial \boldsymbol{b}^e} = J^p \frac{\partial(J^e)}{\partial \boldsymbol{b}^e} = J^p \frac{1}{2} J^e \boldsymbol{b}^{e^{-1}} = J \frac{1}{2} \boldsymbol{b}^{e^{-1}} ,$$

$$\frac{\partial[\ln(J)]}{\partial \boldsymbol{b}^e} = \frac{1}{J}\frac{\partial(J)}{\partial \boldsymbol{b}^e} = \frac{1}{J} J \frac{1}{2} \boldsymbol{b}^{e^{-1}} = \frac{1}{2} \boldsymbol{b}^{e^{-1}} , \quad \frac{\partial[\mathrm{Tr}(\boldsymbol{b}^e)]}{\partial \boldsymbol{b}^e} = \mathbf{1} ,$$

$$\frac{\partial}{\partial \boldsymbol{b}^e}\left(J^2 - 1\right) = 2J \frac{\partial J}{\partial \boldsymbol{b}^e} = 2JJ\frac{1}{2}\boldsymbol{b}^{e^{-1}} = J^2 \boldsymbol{b}^{e^{-1}} ,$$

$$\frac{\partial}{\partial \boldsymbol{b}^e}\left[J^{e^{\frac{-2}{3}}} \mathrm{Tr}(\boldsymbol{b}^e)\right] = \frac{-2}{3} J^{e^{\frac{-5}{3}}} \frac{\partial J}{\partial \boldsymbol{b}^e} \mathrm{Tr}(\boldsymbol{b}^e) + J^{e^{\frac{-2}{3}}} \frac{\partial[\mathrm{Tr}(\boldsymbol{b}^e)]}{\partial \boldsymbol{b}^e} = \frac{-2}{3} J^{e^{\frac{-5}{3}}} J \frac{1}{2} \boldsymbol{b}^{e^{-1}} \mathrm{Tr}(\boldsymbol{b}^e) + J^{e^{\frac{-2}{3}}} \mathbf{1}$$

Then, by incorporating the above derivatives into the equation in (9.384) we obtain:

$$\begin{aligned}
\boldsymbol{\tau} &= \left\{ \kappa\left[\frac{1}{2}\frac{\partial}{\partial \boldsymbol{b}^e}\left(J^2-1\right) - \frac{\partial}{\partial \boldsymbol{b}^e}[\ln(J)]\right] + \mu \frac{\partial}{\partial \boldsymbol{b}^e}\left[J^{e^{\frac{-2}{3}}} \mathrm{Tr}(\boldsymbol{b}^e)\right]\right\} \cdot \boldsymbol{b}^e \\
&= \left\{ \kappa\left[\frac{1}{2} J^2 \boldsymbol{b}^{e^{-1}} - \frac{1}{2}\boldsymbol{b}^{e^{-1}}\right] + \mu\left[\frac{-2}{3} J^{e^{\frac{-5}{3}}} J \frac{1}{2} \boldsymbol{b}^{e^{-1}} \mathrm{Tr}(\boldsymbol{b}^e) + J^{e^{\frac{-2}{3}}} \mathbf{1}\right]\right\} \cdot \boldsymbol{b}^e \\
&= \left\{ \frac{\kappa}{2}\left[J^2 - 1\right]\boldsymbol{b}^{e^{-1}} + \mu\left[\frac{-1}{3} J^{e^{\frac{-2}{3}}} \boldsymbol{b}^{e^{-1}} \mathrm{Tr}(\boldsymbol{b}^e) + J^{e^{\frac{-2}{3}}} \mathbf{1}\right]\right\} \cdot \boldsymbol{b}^e \\
&= \left\{ \frac{\kappa}{2}\left[J^2 - 1\right]\boldsymbol{b}^{e^{-1}} + \mu\left[\frac{-1}{3} \mathrm{Tr}(\tilde{\boldsymbol{b}}^e) \boldsymbol{b}^{e^{-1}} + J^{e^{\frac{-2}{3}}} \mathbf{1}\right]\right\} \cdot \boldsymbol{b}^e \\
&= \frac{\kappa}{2}\left[J^2 - 1\right]\boldsymbol{b}^{e^{-1}} \cdot \boldsymbol{b}^e + \mu\left[\frac{-1}{3} \mathrm{Tr}(\tilde{\boldsymbol{b}}^e) \boldsymbol{b}^{e^{-1}} \cdot \boldsymbol{b}^e + J^{e^{\frac{-2}{3}}} \mathbf{1} \cdot \boldsymbol{b}^e\right] \\
&= \frac{\kappa}{2}\left[J^2 - 1\right]\mathbf{1} + \mu\left[\frac{-1}{3} \mathrm{Tr}(\tilde{\boldsymbol{b}}^e)\mathbf{1} + \tilde{\boldsymbol{b}}^e\right]
\end{aligned} \tag{9.385}$$

where $\frac{-1}{3}\mathrm{Tr}(\tilde{\boldsymbol{b}}^e)\mathbf{1} + \tilde{\boldsymbol{b}}^e = \left[\tilde{\boldsymbol{b}}^e\right]^{dev}$. Then, the constitutive equation for stress becomes:

$$\boxed{\boldsymbol{\tau} = \frac{\kappa}{2}\left[J^2 - 1\right]\mathbf{1} + \mu\left[\tilde{\boldsymbol{b}}^e\right]^{dev}} \quad \textit{The constitutive equation for stress} \tag{9.386}$$

9.7.8.3 Formulation Considering the Transformation $\boldsymbol{F}^p$ as an Isochoric Transformation

We will next consider the plastic deformation $\boldsymbol{F}^p$ to be purely isochoric:

$$\det(\tilde{\boldsymbol{F}}^p) = \det(\tilde{\boldsymbol{C}}^p) = 1 \quad \Rightarrow \quad J = \det(\boldsymbol{F}) = \det(\boldsymbol{F}^e) = J^e \tag{9.387}$$

With this simplification the energy and stress equations become:

$$\boxed{\begin{aligned}
\Psi &= \frac{\kappa}{2}\left[\frac{1}{2}\left(J^{e^2} - 1\right) - \ln(J^e)\right] + \frac{\mu}{2}\left[\mathrm{Tr}(\tilde{\boldsymbol{b}}^e) - 3\right] \\
\boldsymbol{\tau} &= \frac{\kappa}{2}\left[J^{e^2} - 1\right]\mathbf{1} + \mu\left[\tilde{\boldsymbol{b}}^e\right]^{dev}
\end{aligned}} \tag{9.388}$$

As seen in Chapter 2 – subsection 2.13, the following holds:

$$I_{\tilde{b}} = I_{\tilde{C}} = \frac{I_b}{\sqrt[3]{III_b}} \quad ; \quad II_{\tilde{b}} = II_{\tilde{C}} = \frac{II_b}{\sqrt[3]{III_b^2}} \quad ; \quad III_{\tilde{b}} = III_{\tilde{C}} = 1 \tag{9.389}$$

Then, considering the equation in (9.383), *i.e.* $\tilde{\boldsymbol{b}}^e = J^{e\frac{-2}{3}} \boldsymbol{b}^e$, we can show that:

$$III_{\tilde{b}^e} = \det(\tilde{\boldsymbol{b}}^e) = \det(J^{e\frac{-2}{3}} \boldsymbol{b}^e) = J^{e^{-2}} \det(\boldsymbol{b}^e) \tag{9.390}$$

Afterwards, if we consider that $\det(\boldsymbol{b}) = \det(\boldsymbol{C}) = J^2$, we obtain $\det(\boldsymbol{b}^e) = \det(\boldsymbol{C}^e) = J^{e^2}$, which give us:

$$III_{\tilde{b}^e} = \det(\tilde{\boldsymbol{b}}^e) = \det(J^{e\frac{-2}{3}} \boldsymbol{b}^e) = J^{e^{-2}} J^{e^2} = 1 \tag{9.391}$$

We can also show that, if the relationships in (9.389) hold, so do

$$I_{\tilde{b}^e} = I_{\tilde{C}^e} = \frac{I_{b^e}}{\sqrt[3]{III_{b^e}}} \quad ; \quad II_{\tilde{b}^e} = II_{\tilde{C}^e} = \frac{II_{b^e}}{\sqrt[3]{III_{b^e}^2}} \quad ; \quad III_{\tilde{b}^e} = III_{\tilde{C}^e} = 1 \tag{9.392}$$

9.7.8.4 The Rate of Change of the Helmholtz Free Energy

The rate of change of the Helmholtz free energy, (see equation (9.388)), is given by

$$\dot{\Psi} = \frac{D}{Dt}\left\{\frac{\kappa}{2}\left[\frac{1}{2}\left(J^{e2} - 1\right) - \ln(J^e)\right] + \frac{\mu}{2}\left[\tilde{\boldsymbol{b}}^e : \mathbf{1} - 3\right]\right\} = \frac{\kappa}{2}\left[\frac{1}{2} 2J^e \dot{J}^e - \frac{1}{J^e}\dot{J}^e\right] + \frac{\mu}{2}\left[\dot{\tilde{\boldsymbol{b}}}^e : \mathbf{1}\right] \tag{9.393}$$

Remember that $\dot{J} = J \,\mathrm{Tr}(\mathbf{D}) = J\, \boldsymbol{C}^{-1} : \dot{\boldsymbol{E}} = \frac{J}{2} \boldsymbol{C}^{-1} : \dot{\boldsymbol{C}} = J \boldsymbol{F}^{-T} : \dot{\boldsymbol{F}}$, (see **Problem 2.12**), and also that:

$$\dot{\tilde{\boldsymbol{b}}}^e = \frac{D}{Dt}\left\{J^{e\frac{-2}{3}} \boldsymbol{b}^e\right\} = -\frac{2}{3} J^{e\frac{-5}{3}} \dot{J}^e \boldsymbol{b}^e + J^{e\frac{-2}{3}} \dot{\boldsymbol{b}}^e = \frac{-2}{3} J^{e\frac{-5}{3}} \dot{J}^e \boldsymbol{b}^e + J^{e\frac{-2}{3}}\left(\boldsymbol{\ell}^e \cdot \boldsymbol{b}^e + \boldsymbol{b}^e \cdot \boldsymbol{\ell}^{eT}\right) \tag{9.394}$$

Note that $\dot{\boldsymbol{b}} = \boldsymbol{\ell} \cdot \boldsymbol{b} + \boldsymbol{b} \cdot \boldsymbol{\ell}^T$ holds, (see Chapter 2). Similarly, we can prove that the following relationship $\dot{\boldsymbol{b}}^e = \boldsymbol{\ell}^e \cdot \boldsymbol{b}^e + \boldsymbol{b}^e \cdot \boldsymbol{\ell}^{eT}$ is valid.

Then, by substituting (9.394) into (9.393) we obtain:

$$\begin{aligned}
\dot{\Psi} &= \frac{\kappa}{2}\left[\frac{1}{2} 2J^e \dot{J}^e - \frac{1}{J^e}\dot{J}^e\right] + \frac{\mu}{2}\left[\dot{\tilde{\boldsymbol{b}}}^e : \mathbf{1}\right] \\
&= \frac{\kappa}{2}\left[J^e - \frac{1}{J^e}\right]\dot{J}^e + \frac{\mu}{2}\left\{\left[-\frac{2}{3} J^{e\frac{-5}{3}} \dot{J}^e \boldsymbol{b}^e + J^{e\frac{-2}{3}}\left(\boldsymbol{\ell}^e \cdot \boldsymbol{b}^e + \boldsymbol{b}^e \cdot \boldsymbol{\ell}^{eT}\right)\right] : \mathbf{1}\right\} \\
&= \frac{\kappa}{2}\left[J^e - \frac{1}{J^e}\right] J^e \,\mathrm{Tr}(\mathbf{D}^e) + \frac{\mu}{2}\left\{\left[-\frac{2}{3} J^{e\frac{-5}{3}} J^e \,\mathrm{Tr}(\mathbf{D}^e)\mathrm{Tr}(\boldsymbol{b}^e) + J^{e\frac{-2}{3}}\left(\boldsymbol{\ell}^e \cdot \boldsymbol{b}^e + \boldsymbol{b}^e \cdot \boldsymbol{\ell}^{eT}\right) : \mathbf{1}\right]\right\}
\end{aligned} \tag{9.395}$$

Furthermore, if $\left(\boldsymbol{\ell}^e \cdot \boldsymbol{b}^e + \boldsymbol{b}^e \cdot \boldsymbol{\ell}^{eT}\right) : \mathbf{1} = \left(\ell^e_{ik} b^e_{kj} + b^e_{ik} \ell^e_{jk}\right)\delta_{ij} = 2\ell^e_{jk} b^e_{kj} = 2\mathrm{D}^e_{jk} b^e_{kj}$ holds.

Then:

$$\begin{aligned}
\dot{\Psi} &= \frac{\kappa}{2}\left[J^{e2}-1\right]\mathrm{Tr}(\mathbf{D}^e) + \frac{\mu}{2}\left[-\frac{2}{3}J^{e\frac{-2}{3}}\,\mathrm{Tr}(\mathbf{D}^e)\mathrm{Tr}(\boldsymbol{b}^e) + J^{e\frac{-2}{3}}2\boldsymbol{b}^e:\mathbf{D}^e\right] \\
&= \left[\frac{\kappa}{2}\left(J^{e2}-1\right)\mathbf{1} + \mu\left(-\frac{1}{3}\mathrm{Tr}(\tilde{\boldsymbol{b}}^e)\mathbf{1} + \tilde{\boldsymbol{b}}^e\right)\right]:\mathbf{D}^e = \left[\frac{\kappa}{2}\left(J^{e2}-1\right)\mathbf{1} + \mu\left[\tilde{\boldsymbol{b}}^e\right]^{dev}\right]:\mathbf{D}^e \\
&= \boldsymbol{\tau}:\mathbf{D}^e
\end{aligned} \tag{9.396}$$

9.7.8.5 Yield Criterion and Evolution of the Internal Variables

Let us consider the Mises-Huber's yield condition formulated in terms of the Kirchhoff stress tensor as:

$$\mathcal{F}(\boldsymbol{\tau},\boldsymbol{\beta}) = \left\|\boldsymbol{\tau}^{dev}\right\| - \sqrt{\frac{2}{3}}\left[\sigma_Y - K\alpha\right] \le 0 \tag{9.397}$$

where $\boldsymbol{\sigma}_Y$ is the yield stress, $K>0$ is the isotropic hardening modulus, and α is the hardening parameter.

The plastic flow rule in the current configuration is given by:

$$\left(\overset{\square}{\boldsymbol{b}^e}\right)^{dev} = -\frac{2}{3}\dot{\gamma}\,\mathrm{Tr}(\boldsymbol{b}^e)\frac{\boldsymbol{\tau}^{dev}}{\left\|\boldsymbol{\tau}^{dev}\right\|} \tag{9.398}$$

The Isotropic Hardening Law and the Loading/Unloading Conditions

We assume that the rate of change of the hardening is given by:

$$\dot{\alpha} = \sqrt{\frac{2}{3}}\,\dot{\gamma} \tag{9.399}$$

where $\dot{\gamma}$ is the consistency parameter subjected to Kuhn-Tucker (loading/unloading) conditions:

$$\dot{\gamma} \ge 0 \quad ; \quad \mathcal{F}(\boldsymbol{\tau},\boldsymbol{\beta}) \le 0 \quad ; \quad \dot{\gamma}\,\mathcal{F}(\boldsymbol{\tau},\boldsymbol{\beta}) = 0 \tag{9.400}$$

which together with the persistency condition:

$$\dot{\gamma}\,\dot{\mathcal{F}}(\boldsymbol{\tau},\boldsymbol{\beta}) = 0 \tag{9.401}$$

complete the model formulation.

10

Thermoelasticity

10.1 Thermodynamic Potentials

Remember from Chapter 5 that the Clausius-Duhem inequality can be expressed as follows:

$$\boxed{\rho\,\dot{\eta}(\vec{x},t)+\frac{1}{T}\boldsymbol{\sigma}:\mathbf{D}-\frac{1}{T}\rho\,\dot{u}-\frac{1}{T^2}\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\geq 0} \quad \textit{Clausius-Duhem inequality (current configuration)} \tag{10.1}$$

$$\boxed{\begin{array}{l}\rho_0\dot{\eta}+\frac{1}{T}\mathbf{S}:\dot{\boldsymbol{E}}-\frac{1}{T}\rho_0\dot{u}-\frac{1}{T^2}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\\ o\\ \rho_0\dot{\eta}+\frac{1}{T}\mathbf{P}:\dot{\boldsymbol{F}}-\frac{1}{T}\rho_0\dot{u}-\frac{1}{T^2}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0\end{array}} \quad \textit{Clausius-Duhem inequality (reference configuration)} \tag{10.2}$$

Note that $\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\leq 0$, since the sense of the heat flux vector ($\vec{\mathbf{q}}$) is always opposite to that of the temperature gradient ($\nabla_{\vec{x}}T$). Thus, we can formulate the heat conduction inequality, (see Chapter 5), as follows:

$$\textit{Heat conduction inequality} \quad \begin{array}{ll}\boxed{-\vec{\mathbf{q}}\cdot\nabla_{\vec{x}}T\geq 0} & \textit{(current configuration)}\\ \boxed{-\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T\geq 0} & \textit{(reference configuration)}\end{array} \tag{10.3}$$

Then, by imposing the restriction (10.3) into the Clausius-Duhem inequality (10.1) and (10.2) we are lead to the Clausius-Planck inequality:

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_12,

$$\boxed{\mathcal{D}_{int} = \rho\dot{\eta}(\vec{x},t) + \frac{1}{T}\boldsymbol{\sigma}:\mathbf{D} - \frac{1}{T}\rho\,\dot{u}(\vec{x},t) \geq 0}$$ *(current configuration)*

Clausius-Planck inequality

$$\mathcal{D}_{int} = \rho_0\dot{\eta} + \frac{1}{T}\mathbf{S}:\dot{\boldsymbol{E}} - \frac{1}{T}\rho_0\dot{u} \geq 0 \tag{10.4}$$

or *(reference configuration)*

$$\mathcal{D}_{int} = \rho_0\dot{\eta}(\vec{X},t) + \frac{1}{T}\mathbf{P}:\dot{\boldsymbol{F}} - \frac{1}{T}\rho_0\dot{u}(\vec{X},t) \geq 0$$

where $\mathcal{D}_{int}$ is the internal energy dissipation (or the local entropy production), which must be positive throughout the continuum at any point and time, *i.e.* $\mathcal{D}_{int} \geq 0$.

Then, in a reversible process we have $\mathcal{D}_{int} = 0$, with which we obtain:

$$\boxed{\mathcal{D}_{int} = \rho_0\dot{\eta} + \frac{1}{T}\mathbf{S}:\dot{\boldsymbol{E}} - \frac{1}{T}\rho_0\dot{u} = 0 \quad \Rightarrow \quad \dot{u} = \frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}} + T\dot{\eta}} \tag{10.5}$$

10.1.1 The Specific Internal Energy

The equation in (10.5) indicates that the *specific internal energy* (u) is a thermodynamic potential in terms of $\boldsymbol{E},\eta$ (independent state variables) when evaluating $\mathbf{S},T$ (the state function), (see Asaro&Lubarda (2006)). In fact, if $u(\boldsymbol{E},\eta)$ its rate of change becomes:

$$\dot{u}(\boldsymbol{E},\eta) = \frac{\partial u}{\partial \boldsymbol{E}}:\dot{\boldsymbol{E}} + \frac{\partial u}{\partial \eta}\dot{\eta} \tag{10.6}$$

Then, by comparing the equations (10.5) and (10.6) we can draw the conclusion that:

$$\boxed{\mathbf{S} = \rho_0\left(\frac{\partial u(\boldsymbol{E},\eta)}{\partial \boldsymbol{E}}\right)_{\dot{\eta}=0} \quad ; \quad T = \left(\frac{\partial u(\boldsymbol{E},\eta)}{\partial \eta}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}} \tag{10.7}$$

10.1.2 The Specific Helmholtz Free Energy

Now, another thermodynamic potential is the specific Helmholtz free energy denoted by ψ, which is defined as:

$$\boxed{\psi = u - T\eta} \quad \textit{Specific Helmholtz free energy} \quad \left[\frac{J}{kg}\right] \tag{10.8}$$

We now need to verify that we are working with specific energy, *i.e.* energy per unit mass: $[\psi] = [u] = [T\eta] = K\dfrac{J}{kgK} = \dfrac{J}{kg}$ and then the rate of change of (10.8) is given by:

$$\dot{\psi} = \dot{u} - \dot{T}\eta - T\dot{\eta} \tag{10.9}$$

Then, by substituting the rate of change of the specific internal energy given in (10.5) into the above equation we obtain:

$$\dot{\psi} = \dot{u} - \dot{T}\eta - T\dot{\eta} = \frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}} + T\dot{\eta} - \dot{T}\eta - T\dot{\eta} = \frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}} - \dot{T}\eta \tag{10.10}$$

where ψ is a thermodynamic potential in terms of $(\boldsymbol{E},T)$ (independent state variables) when evaluating $(\mathbf{S},\eta)$. Moreover, by calculating the rate of change of $\psi(\boldsymbol{E},T)$, we obtain:

$$\dot{\psi}(\boldsymbol{E},T)=\frac{\partial\psi}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial\psi}{\partial T}\dot{T} \tag{10.11}$$

Then, by comparing the equations (10.10) and (10.11) we can conclude that:

$$\boxed{\mathbf{S}(\boldsymbol{E},T)=\rho_0\left(\frac{\partial\psi(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0} \quad ; \quad \eta(\boldsymbol{E},T)=-\left(\frac{\partial\psi(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}} \tag{10.12}$$

Now, the rate of change of entropy $\eta(\boldsymbol{E},T)$ becomes:

$$\dot{\eta}(\boldsymbol{E},T)=\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}:\dot{\boldsymbol{E}}+\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}\dot{T} \tag{10.13}$$

Furthermore, let us imagine a process where $\boldsymbol{E}\to\boldsymbol{E}+d\boldsymbol{E}$, $T\to T+dT$, and $\eta\to\eta+d\eta$. It then holds that:

$$d\eta(\boldsymbol{E},T)=\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}:d\boldsymbol{E}+\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}dT \tag{10.14}$$

We can also express the equation (10.6) by means of differentials as follows:

$$du(\boldsymbol{E},\eta)=\frac{1}{\rho_0}\mathbf{S}:d\boldsymbol{E}+T\,d\eta \tag{10.15}$$

Then, by substituting (10.14) into (10.15) we obtain:

$$\begin{aligned} du&=\frac{1}{\rho_0}\mathbf{S}:d\boldsymbol{E}+T\left[\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}:d\boldsymbol{E}+\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}dT\right]\\ &=\left[\frac{1}{\rho_0}\mathbf{S}+T\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}\right]:d\boldsymbol{E}+T\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}dT \end{aligned} \tag{10.16}$$

The necessary and sufficient condition for du to be a total differential is guaranteed by:

$$\frac{\partial}{\partial T}\left[\frac{1}{\rho_0}\mathbf{S}+T\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}\right]=\frac{\partial}{\partial\boldsymbol{E}}\left[T\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}\right] \tag{10.17}$$

which give us the following equation:

$$\frac{1}{\rho_0}\frac{\partial\mathbf{S}}{\partial T}+\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}=\mathbf{0} \quad\Rightarrow\quad \boxed{\left(\frac{\partial\mathbf{S}}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}=-\rho_0\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}} \tag{10.18}$$

Then, we can introduce a new second-order tensor as follows:

$$\boxed{\boldsymbol{M}=\left(\frac{\partial\mathbf{S}}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}=-\rho_0\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}} \quad \textit{The thermal stress tensor} \quad \left[\frac{Pa}{K}\right] \tag{10.19}$$

10.1.3 The Specific Gibbs Free Energy

We will now introduce a new thermodynamic potential: the specific Gibbs free energy (G) which is a potential in terms of stress and temperature:

$$\boxed{\mathsf{G}(\mathbf{S},T)=\psi(\boldsymbol{E},T)-\frac{1}{\rho_0}\mathbf{S}:\boldsymbol{E}}\quad \textit{Specific Gibbs free energy}\quad \left[\frac{J}{kg}\right] \tag{10.20}$$

whose rate of change becomes:

$$\frac{\partial \mathsf{G}}{\partial \mathbf{S}}:\dot{\mathbf{S}}+\frac{\partial \mathsf{G}}{\partial T}\dot{T}=\dot{\psi}(\boldsymbol{E},T)-\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}-\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}} \tag{10.21}$$

Then, by substituting the rate of change of the Helmholtz free energy given in (10.10) into the above equation, we can obtain:

$$\begin{aligned}&\frac{\partial \mathsf{G}}{\partial \mathbf{S}}:\dot{\mathbf{S}}+\frac{\partial \mathsf{G}}{\partial T}\dot{T}=\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}-\dot{T}\eta-\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}-\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}\\ \Rightarrow &\frac{\partial \mathsf{G}}{\partial \mathbf{S}}:\dot{\mathbf{S}}+\frac{\partial \mathsf{G}}{\partial T}\dot{T}=-\dot{T}\eta-\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}\end{aligned} \tag{10.22}$$

with which we can draw the conclusion that:

$$\boxed{\boldsymbol{E}=\boldsymbol{E}(\mathbf{S},T)=-\rho_0\left(\frac{\partial \mathsf{G}(\mathbf{S},T)}{\partial \mathbf{S}}\right)_{\dot{T}=0}\quad ;\quad \eta=\eta(\mathbf{S},T)=-\left(\frac{\partial \mathsf{G}(\mathbf{S},T)}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}}} \tag{10.23}$$

and if we consider the relationship between the specific Helmholtz free energy and specific internal energy, $\psi(\boldsymbol{E},T)=u(\boldsymbol{E},\eta)-T\eta$, we can obtain the following equations:

$$\mathsf{G}(\mathbf{S},T)=\psi(\boldsymbol{E},T)-\frac{1}{\rho_0}\mathbf{S}:\boldsymbol{E}=u(\boldsymbol{E},\eta)-T\eta-\frac{1}{\rho_0}\mathbf{S}:\boldsymbol{E}\Rightarrow u(\boldsymbol{E},\eta)-\mathsf{G}(\mathbf{S},T)=\frac{1}{\rho_0}\mathbf{S}:\boldsymbol{E}+T\eta \tag{10.24}$$

10.1.4 The Specific Enthalpy

Next, we will introduce the specific enthalpy (H) which is a thermodynamic potential in terms of stress and entropy, such that:

$$\boxed{\mathsf{H}(\mathbf{S},\eta)=u(\boldsymbol{E},\eta)-\frac{1}{\rho_0}\mathbf{S}:\boldsymbol{E}=\mathsf{G}(\mathbf{S},T)+T\eta}\quad \textit{Specific Enthalpy}\quad \left[\frac{J}{kg}\right] \tag{10.25}$$

Now, by evaluating the rate of change of the above equation, we obtain:

$$\frac{\partial \mathsf{H}}{\partial \mathbf{S}}:\dot{\mathbf{S}}+\frac{\partial \mathsf{H}}{\partial \eta}\dot{\eta}=\dot{u}(\boldsymbol{E},\eta)-\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}-\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}} \tag{10.26}$$

Then, by substituting $\dot{u}=\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}+T\dot{\eta}$, (see equation (10.5)), into the above, we find:

$$\begin{aligned}&\frac{\partial \mathsf{H}}{\partial \mathbf{S}}:\dot{\mathbf{S}}+\frac{\partial \mathsf{H}}{\partial \eta}\dot{\eta}=\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}+T\dot{\eta}-\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}-\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}\\ \Rightarrow &\frac{\partial \mathsf{H}}{\partial \mathbf{S}}:\dot{\mathbf{S}}+\frac{\partial \mathsf{H}}{\partial \eta}\dot{\eta}=T\dot{\eta}-\frac{1}{\rho_0}\dot{\mathbf{S}}:\boldsymbol{E}\end{aligned} \tag{10.27}$$

with which we can draw the conclusion that:

$$\boxed{\boldsymbol{E}=-\rho_0\left(\frac{\partial \mathsf{H}(\mathbf{S},\eta)}{\partial \mathbf{S}}\right)_{\dot{\eta}=0}\quad ;\quad T=\left(\frac{\partial \mathsf{H}(\mathbf{S},\eta)}{\partial \eta}\right)_{\dot{\mathbf{S}}=\mathbf{0}}} \tag{10.28}$$

Now, let us suppose that we make a change in the system characterized by the following process $T \to T + dT$, $\mathbf{S} \to \mathbf{S} + d\mathbf{S}$, $\eta \to \eta + d\eta$, with which the equation in (10.27) can be expressed as follows:

$$d\mathsf{H} = Td\eta - \frac{1}{\rho_0}\boldsymbol{E} : d\mathbf{S} \tag{10.29}$$

Then, taking the differential of $\eta = \eta(\mathbf{S}, T)$, we obtain:

$$d\eta = \left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} : d\mathbf{S} + \left(\frac{\partial \eta}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}} : dT \tag{10.30}$$

and combining (10.30) with (10.29) gives us:

$$\begin{aligned} d\mathsf{H} &= T\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} : d\mathbf{S} + T\left(\frac{\partial \eta}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}} : dT - \frac{1}{\rho_0}\boldsymbol{E} : d\mathbf{S} \\ &= \left[T\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} - \frac{1}{\rho_0}\boldsymbol{E}\right] : d\mathbf{S} + T\left(\frac{\partial \eta}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}} : dT \end{aligned} \tag{10.31}$$

Then, the necessary and sufficient condition for $d\mathsf{H}$ to be a total differential is guaranteed by:

$$\frac{\partial}{\partial T}\left[T\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} - \frac{1}{\rho_0}\boldsymbol{E}\right] = \frac{\partial}{\partial \mathbf{S}}\left[T\left(\frac{\partial \eta}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}}\right] \tag{10.32}$$

the result of which is:

$$\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} - \frac{1}{\rho_0}\frac{\partial \boldsymbol{E}}{\partial T} = \mathbf{0} \quad \Rightarrow \quad \boxed{\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} = \frac{1}{\rho_0}\left(\frac{\partial \boldsymbol{E}}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}}} \tag{10.33}$$

Next, we can define a new tensor: the thermal expansion tensor, by:

$$\boxed{\boldsymbol{A} = \left(\frac{\partial \boldsymbol{E}}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}} = \rho_0\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0}} \quad \textit{The thermal expansion tensor} \quad \left[\frac{1}{K}\right] \tag{10.34}$$

Now, we can make a summary of all the thermodynamic potentials in Table 10.1 by means of which we can easily show that $(u + \mathsf{G}) - (\psi + \mathsf{H}) = 0$.

Table 10.1: Thermodynamic potentials.

Specific internal energy	*Specific Helmholtz free energy*	*Specific Gibbs free energy*	*Specific enthalpy*
$u(\boldsymbol{E}, \eta)$	$\psi(\boldsymbol{E}, T)$	$\mathsf{G}(\mathbf{S}, T)$	$\mathsf{H}(\mathbf{S}, \eta)$
$\mathbf{S}(\boldsymbol{E}, \eta) = \rho_0 \frac{\partial u}{\partial \boldsymbol{E}}$ $T(\boldsymbol{E}, \eta) = \frac{\partial u}{\partial \eta}$	$\mathbf{S}(\boldsymbol{E}, T) = \rho_0 \frac{\partial \psi}{\partial \boldsymbol{E}}$ $\eta(\boldsymbol{E}, T) = -\frac{\partial \psi}{\partial T}$	$\boldsymbol{E}(\mathbf{S}, T) = -\rho_0 \frac{\partial \mathsf{G}}{\partial \mathbf{S}}$ $\eta(\mathbf{S}, T) = -\frac{\partial \mathsf{G}}{\partial T}$	$\boldsymbol{E}(\mathbf{S}, \eta) = -\rho_0 \frac{\partial \mathsf{H}}{\partial \mathbf{S}}$ $T(\mathbf{S}, \eta) = \frac{\partial \mathsf{H}}{\partial \eta}$
$u = \mathsf{G} + T\eta + \frac{1}{\rho_0}\mathbf{S} : \boldsymbol{E}$	$\psi = u - T\eta$	$\mathsf{G} = \psi - \frac{1}{\rho_0}\mathbf{S} : \boldsymbol{E}$ $= \mathsf{H} - T\eta$	$\mathsf{H} = u - \frac{1}{\rho_0}\mathbf{S} : \boldsymbol{E}$ $= \mathsf{G} + T\eta$

10.2 Thermomechanical Parameters

10.2.1 Isothermal and Isentropic Processes

Isothermal processes are characterized by having a constant temperature ($\dot{T}=0$) during a system change. Good approximations of isothermal processes are those found in materials that are good heat conductors (*e.g.* metals) and which are subjected to quasi-static processes. We can describe *Isentropic processes* as those with constant entropy ($\dot{\eta}=0$) during a system change. A good approximation of an isentropic process is when the continuum is a poor heat conductor and quantities (velocity, stress, strain) vary rapidly.

Let us now return to some of the expressions obtained previously:

$$\dot{u}(\boldsymbol{E},\eta)=\frac{\partial u}{\partial \boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial u}{\partial \eta}\dot{\eta} \quad \Rightarrow \quad \begin{cases} \mathbf{S}_{ise}(\boldsymbol{E},\eta)=\rho_0\left(\dfrac{\partial u(\boldsymbol{E},\eta)}{\partial \boldsymbol{E}}\right)_{\dot{\eta}=0} \\ T(\boldsymbol{E},\eta)=\left(\dfrac{\partial u(\boldsymbol{E},\eta)}{\partial \eta}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} \end{cases} \tag{10.35}$$

and from the rate of change of the specific Helmholtz free energy we obtain:

$$\dot{\psi}(\boldsymbol{E},T)=\frac{\partial \psi}{\partial \boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial \psi}{\partial T}\dot{T} \quad \Rightarrow \quad \begin{cases} \mathbf{S}_{isoT}(\boldsymbol{E},T)=\rho_0\left(\dfrac{\partial \psi(\boldsymbol{E},T)}{\partial \boldsymbol{E}}\right)_{\dot{T}=0} \\ \eta(\boldsymbol{E},T)=-\left(\dfrac{\partial \psi(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} \end{cases} \tag{10.36}$$

which gives us two ways to obtain the stress tensor, namely:

$$\mathbf{S}_{ise}(\boldsymbol{E},\eta)=\rho_0\left(\frac{\partial u(\boldsymbol{E},\eta)}{\partial \boldsymbol{E}}\right)_{\dot{\eta}=0} \quad ; \quad \mathbf{S}_{isoT}(\boldsymbol{E},T)=\rho_0\left(\frac{\partial \psi(\boldsymbol{E},T)}{\partial \boldsymbol{E}}\right)_{\dot{T}=0} \tag{10.37}$$

Then, by calculating the rate of change of $\mathbf{S}(\boldsymbol{E},T)$, (see Table 10.1), we obtain:

$$\begin{aligned} \dot{\mathbf{S}}(\boldsymbol{E},T) &= \left(\frac{\partial \mathbf{S}}{\partial \boldsymbol{E}}\right)_{\dot{T}=0}:\dot{\boldsymbol{E}}+\left(\frac{\partial \mathbf{S}}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}\dot{T} \\ &= \rho_0\left(\frac{\partial^2 \psi}{\partial \boldsymbol{E}\otimes\partial \boldsymbol{E}}\right)_{\dot{T}=0}:\dot{\boldsymbol{E}}+\rho_0\left(\frac{\partial^2 \psi}{\partial T\partial \boldsymbol{E}}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}\dot{T}=\mathbb{C}^e_{isoT}:\dot{\boldsymbol{E}}+\boldsymbol{M}\dot{T} \end{aligned} \tag{10.38}$$

where we have introduced the symmetric fourth-order tensor:

- The isothermal elastic tangent stiffness tensor:

$$\boxed{\mathbb{C}^e_{isoT}=\left(\frac{\partial \mathbf{S}(\boldsymbol{E},T)}{\partial \boldsymbol{E}}\right)_{\dot{T}=0}=\rho_0\left(\frac{\partial^2 \psi(\boldsymbol{E},T)}{\partial \boldsymbol{E}\otimes\partial \boldsymbol{E}}\right)_{\dot{T}=0}} \quad \textit{The isothermal elastic tangent stiffness tensor} \quad [Pa] \tag{10.39}$$

Then, calculation of the rate of change of $\boldsymbol{E}(\mathbf{S},T)$ yields:

$$\begin{aligned} \dot{\boldsymbol{E}}(\mathbf{S},T) &= \left(\frac{\partial \boldsymbol{E}}{\partial \mathbf{S}}\right)_{\dot{T}=0}:\dot{\mathbf{S}}+\left(\frac{\partial \boldsymbol{E}}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}}\dot{T} \\ &= -\rho_0\left(\frac{\partial^2 \mathsf{G}}{\partial \mathbf{S}\otimes\partial \mathbf{S}}\right)_{\dot{T}=0}:\dot{\mathbf{S}}-\rho_0\left(\frac{\partial^2 \mathsf{G}}{\partial T\partial \mathbf{S}}\right)_{\dot{\mathbf{S}}=\mathbf{0}}\dot{T}=\mathbb{D}^e_{isoT}:\dot{\mathbf{S}}+\boldsymbol{A}\dot{T} \end{aligned} \tag{10.40}$$

where $\mathbb{D}^{e}_{isoT} = \mathbb{C}^{e\;-1}_{isoT}$ and $\boldsymbol{A} = -\mathbb{D}^{e\;-1}_{isoT} : \boldsymbol{M}$ holds.

Now, if we calculate the rate of change of $\mathbf{S}(\boldsymbol{E},\eta)$, (see Table 10.1), we can obtain:

$$\begin{aligned}\dot{\mathbf{S}}(\boldsymbol{E},\eta) &= \left(\frac{\partial \mathbf{S}}{\partial \boldsymbol{E}}\right)_{\dot{\eta}=0} : \dot{\boldsymbol{E}} + \left(\frac{\partial \mathbf{S}}{\partial \eta}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} \dot{\eta} \\ &= \rho_0 \left(\frac{\partial^2 u}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}}\right)_{\dot{\eta}=0} : \dot{\boldsymbol{E}} + \rho_0 \left(\frac{\partial^2 u}{\partial \eta \partial \boldsymbol{E}}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} \dot{\eta} = \mathbb{C}^{e}_{ise} : \dot{\boldsymbol{E}} + \boldsymbol{M}\dot{\eta}\end{aligned} \tag{10.41}$$

where we have introduced a new symmetric fourth-order tensor:

- The adiabatic elastic tangent stiffness tensor:

$$\boxed{\mathbb{C}^{e}_{ise} = \left(\frac{\partial \mathbf{S}(\boldsymbol{E},\eta)}{\partial \boldsymbol{E}}\right)_{\dot{\eta}=0} = \rho_0 \left(\frac{\partial^2 u(\boldsymbol{E},\eta)}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}}\right)_{\dot{\eta}=0}} \quad \textit{The adiabatic elastic tangent stiffness tensor} \quad [Pa] \tag{10.42}$$

Remember that in isotropic linear elastic materials the elasticity tensor is expressed in terms of the Lamé constants as follows: $\mathbb{C}^{e} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$. Likewise, we can define the adiabatic and isothermal elasticity tensors for isotropic linear elastic materials as:

$$\boxed{\mathbb{C}^{e}_{ise} = \lambda_{ise} \mathbf{1} \otimes \mathbf{1} + 2\mu_{ise} \mathbf{I}} \quad \textit{Adiabatic elasticity tensor for isotropic materials} \tag{10.43}$$

$$\boxed{\mathbb{C}^{e}_{isoT} = \lambda_{isoT} \mathbf{1} \otimes \mathbf{1} + 2\mu_{isoT} \mathbf{I}} \quad \textit{Isothermal elasticity tensor for isotropic materials} \tag{10.44}$$

where $(\lambda_{adi}, \mu_{adi})$, $(\lambda_{isoT}, \mu_{isoT})$ are the Lamé constants for isentropic and isothermal processes, respectively.

10.2.2 Specific Heats and Latent Heat Tensors

According to Asaro&Labarda(2006), "the ratio of the absorbed amount of heat and the temperature increase is called *heat capacity*". The heat capacity per unit mass is denoted by the specific heat capacity or simply the *specific heat*. To understand this concept, we can make an analogy. For example, we can take a dry sponge and put it under a tap (faucet) which we then open. We will observe that the sponge is able to retain a certain amount of water until it is fully saturated after which it will no longer be able to retain the water. We can also observe that the amount of water flowing out from the tip of the sponge varies over time. We will observe that, depending on the characteristics of the sponge, that is, if it has fewer of more holes, it will retain less or more water. The same is true with heat: materials have the ability to retain a certain amount of thermal energy (internal energy store).

By the fact that the increase of heat is not a perfect differential, the specific heat depends on the system change path, and then the specific heat can be measured under various conditions. We then set two types of transformations, one at a constant stress ("pressure"), and the other at a constant strain ("volume"), (see Asaro&Lubarda(2006)).

- *Specific heat at a constant strain:* it is a scalar that corresponds to the heat supplied to a unit of mass so as to achieve a unit temperature change whilst maintaining the strain constant, ($\dot{\boldsymbol{E}} = \mathbf{0}$):

$$\boxed{c_{\boldsymbol{E}} = T\left(\frac{\partial \eta}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} = \left(\frac{\partial u}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} = -T\left(\frac{\partial^2 \psi}{\partial T \partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}}$$ *Specific heat at a constant volume* $\left[\frac{J}{kgK}\right]$ (10.45)

Next, we can also check the SI unit: $[c_{\boldsymbol{E}}] = \left[T\left(\frac{\partial \eta}{\partial T}\right)\right] = [T]\frac{[\eta]}{[T]} = [\eta] = \frac{J}{kgK}$.

- *Specific heat at a constant stress:* it is a scalar that corresponds to the heat required for a unit temperature change while the stress is maintained constant ($\dot{\mathbf{P}} = \mathbf{0}$):

$$\boxed{c_{\mathbf{S}} = T\left(\frac{\partial \eta}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}} = \left(\frac{\partial \mathrm{H}}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}}}$$ *Specific heat at a constant stress* $\left[\frac{J}{kgK}\right]$ (10.46)

The entropy here can be expressed in terms of:

$$\eta = \overline{\eta}(\boldsymbol{E}, T) = \hat{\eta}(\mathbf{S}, T) \tag{10.47}$$

In Figure 10.1 we can appreciate the graph entropy *vs.* temperature for water, where we can verify that during the phase changes there is a jump in entropy without there being any temperature variation. In said graph we can also verify that $c_{\boldsymbol{E}} = c_{\boldsymbol{E}}(T)$ is temperature dependent.

We can define the *latent heat tensor of change of strain* as the heat that must be provided at the material point so as to achieve a unit strain change while the temperature is maintained constant:

$$\boldsymbol{L}_{\boldsymbol{E}} = T\left(\frac{\partial \eta}{\partial \boldsymbol{E}}\right)_{\dot{T}=0} \qquad \left[K\frac{J}{kgK} = \frac{J}{kg}\right] \tag{10.48}$$

Note that in Figure 10.1 in T_{sl} or T_{lg} there is an entropy jump, *i.e.*, we are providing heat with no temperature change, since at these points there is a phase change.

We can define the latent heat tensor of change of stress as the heat that must be provided at the material point so as to achieve a unit stress change while the temperature is maintained constant:

$$\boldsymbol{L}_{\mathbf{S}} = T\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} \qquad \left[K\frac{J}{kgK}\frac{m^2}{N} = \frac{m^3}{kg}\right] \tag{10.49}$$

Note that $\boldsymbol{L}_{\boldsymbol{E}}$ and $\boldsymbol{L}_{\mathbf{S}}$ are symmetric second-order tensors, since $\boldsymbol{E}$ and $\mathbf{S}$ are symmetric tensors too. Then, if we consider the following relationships, (see equations (10.19) and (10.34)) we have:

$$\rho_0\left(\frac{\partial \eta}{\partial \boldsymbol{E}}\right)_{\dot{T}=0} = -\left(\frac{\partial \mathbf{S}}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} = -\boldsymbol{M} \quad ; \quad \rho_0\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} = \left(\frac{\partial \boldsymbol{E}}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}} = \boldsymbol{A} \tag{10.50}$$

and the latent heat tensors can be rewritten as follows:

$$\boxed{\boldsymbol{L}_{\boldsymbol{E}} = T\left(\frac{\partial \eta}{\partial \boldsymbol{E}}\right)_{\dot{T}=0} = -\frac{T}{\rho_0}\left(\frac{\partial \mathbf{S}}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} = -\frac{T}{\rho_0}\boldsymbol{M}}$$ *Latent heat tensor of change of strain* (10.51)

$$\boxed{\boldsymbol{L}_{\mathbf{S}} = T\left(\frac{\partial \eta}{\partial \mathbf{S}}\right)_{\dot{T}=0} = \frac{T}{\rho_0}\left(\frac{\partial \boldsymbol{E}}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}} = \frac{T}{\rho_0}\boldsymbol{A}}$$ *Latent heat tensor of change of stress* (10.52)

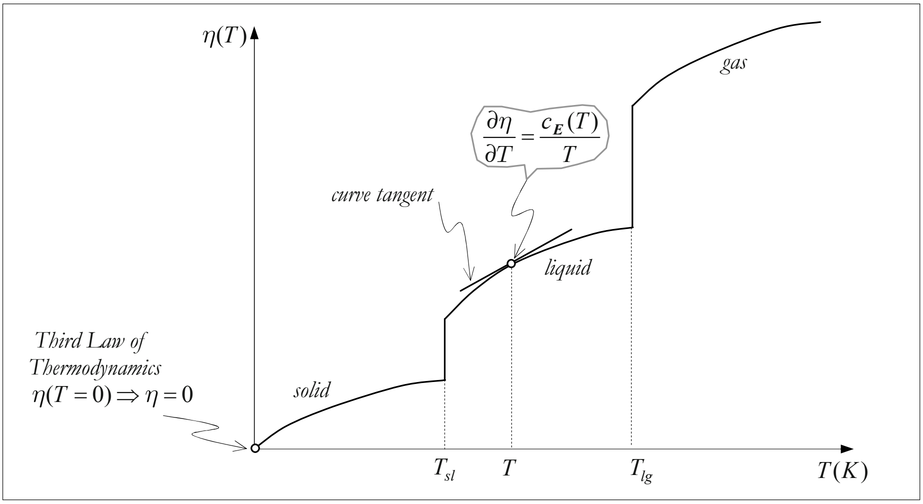


Figure 10.1: Entropy *vs.* temperature (water).

If we now take the derivative of $\eta=\overline{\eta}(\boldsymbol{E},T)=\hat{\eta}(\mathbf{S},T)$ with respect to temperature we obtain:

$$\frac{\partial\hat{\eta}(\mathbf{S},T)}{\partial T}=\frac{\partial\overline{\eta}(\boldsymbol{E},T)}{\partial T}+\frac{\partial\overline{\eta}(\boldsymbol{E},T)}{\partial\boldsymbol{E}}:\frac{\partial\boldsymbol{E}}{\partial T}\Rightarrow\underbrace{T\frac{\partial\hat{\eta}(\mathbf{S},T)}{\partial T}}_{c_{\mathbf{S}}}=\underbrace{T\frac{\partial\overline{\eta}(\boldsymbol{E},T)}{\partial T}}_{c_{\boldsymbol{E}}}+\frac{\partial\overline{\eta}(\boldsymbol{E},T)}{\partial\boldsymbol{E}}:\frac{\partial\boldsymbol{E}}{\partial T}T \tag{10.53}$$

Note that $T\left(\frac{\partial\boldsymbol{E}}{\partial T}\right)_{\dot{\mathbf{S}}=\mathbf{0}}=\rho_0\boldsymbol{L}_{\mathbf{S}}$, $\left(\frac{\partial\eta}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}=\frac{\boldsymbol{L}_{\boldsymbol{E}}}{T}$, thus:

$$\boxed{c_{\mathbf{S}}-c_{\boldsymbol{E}}=\frac{\rho_0}{T}\boldsymbol{L}_{\mathbf{S}}:\boldsymbol{L}_{\boldsymbol{E}}} \tag{10.54}$$

Additionally, the following holds:

$$\frac{\partial\hat{\eta}(\mathbf{S},T)}{\partial T}=\frac{\partial\overline{\eta}(\boldsymbol{E},T)}{\partial\boldsymbol{E}}:\frac{\partial\boldsymbol{E}}{\partial\mathbf{S}}=\frac{\partial\overline{\eta}(\boldsymbol{E},T)}{\partial\boldsymbol{E}}:{\mathbb{C}^{e}_{isoT}}^{-1}={\mathbb{C}^{e}_{isoT}}^{-1}:\frac{\partial\overline{\eta}(\boldsymbol{E},T)}{\partial\boldsymbol{E}} \tag{10.55}$$

or:

$$\frac{\boldsymbol{L}_{\mathbf{S}}}{T}=\frac{\boldsymbol{L}_{\boldsymbol{E}}}{T}:{\mathbb{C}^{e}_{isoT}}^{-1}\quad\Rightarrow\quad\boldsymbol{L}_{\mathbf{S}}=\boldsymbol{L}_{\boldsymbol{E}}:{\mathbb{C}^{e}_{isoT}}^{-1}\quad\Leftrightarrow\quad\boldsymbol{L}_{\boldsymbol{E}}=\boldsymbol{L}_{\mathbf{S}}:\mathbb{C}^{e}_{isoT} \tag{10.56}$$

$$\boldsymbol{L}_{\mathbf{S}}=\boldsymbol{L}_{\boldsymbol{E}}:{\mathbb{C}^{e}_{isoT}}^{-1}\quad\Rightarrow\quad\boldsymbol{L}_{\mathbf{S}}:\boldsymbol{L}_{\boldsymbol{E}}=\boldsymbol{L}_{\boldsymbol{E}}:{\mathbb{C}^{e}_{isoT}}^{-1}:\boldsymbol{L}_{\boldsymbol{E}}=\boldsymbol{L}_{\mathbf{S}}:\mathbb{C}^{e}_{isoT}:\boldsymbol{L}_{\mathbf{S}} \tag{10.57}$$

Then, by using the definition in (10.54) we can draw the conclusion that:

$$\boxed{c_{\mathbf{S}}-c_{\boldsymbol{E}}=\frac{\rho_0}{T}\left(\boldsymbol{L}_{\boldsymbol{E}}:{\mathbb{C}^{e}_{isoT}}^{-1}:\boldsymbol{L}_{\boldsymbol{E}}\right)=\frac{\rho_0}{T}\left(\boldsymbol{L}_{\mathbf{S}}:\mathbb{C}^{e}_{isoT}:\boldsymbol{L}_{\mathbf{S}}\right)} \tag{10.58}$$

and as ${\mathbb{C}^{e}_{isoT}}^{-1}$ is a positive definite tensor, *i.e.* $\boldsymbol{L}_{\boldsymbol{E}}:{\mathbb{C}^{e}_{isoT}}^{-1}:\boldsymbol{L}_{\boldsymbol{E}}>0$ is satisfied, so, the following must be met:

$$\boxed{c_{\mathbf{S}}>c_{\boldsymbol{E}}} \tag{10.59}$$

Furthermore, if we return to the equation in (10.13) we can also express this as follows:

$$\dot{\eta}(\boldsymbol{E},T)=\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}:\dot{\boldsymbol{E}}+\left(\frac{\partial\eta(\boldsymbol{E},T)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}\dot{T}=-\frac{1}{\rho_0}\boldsymbol{M}:\dot{\boldsymbol{E}}+\frac{c_{\boldsymbol{E}}}{T}\dot{T} \tag{10.60}$$

where we have used the equations in (10.50) and (10.45).

Next, we will obtain the relationship between $\mathbb{C}^e_{ise}$ and $\mathbb{C}^e_{isoT}$, (see Holzapfel(2000)). Note that in isentropic processes, $\dot{\eta}=0$, the equation in (10.60) becomes:

$$\dot{T}=\frac{T}{\rho_0 c_{\boldsymbol{E}}}\boldsymbol{M}:\dot{\boldsymbol{E}} \tag{10.61}$$

Then, if we take the rate of change of $\mathbf{S}(\boldsymbol{E},T)$, (see equation (10.12)), we are given:

$$\dot{\mathbf{S}}(\boldsymbol{E},T)=\left(\frac{\partial\mathbf{S}}{\partial\boldsymbol{E}}\right)_{\dot{T}=0}:\dot{\boldsymbol{E}}+\left(\frac{\partial\mathbf{S}}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}\dot{T}=\mathbb{C}^e_{isoT}:\dot{\boldsymbol{E}}+\boldsymbol{M}\,\dot{T} \tag{10.62}$$

Next, by substituting (10.61) into (10.62), we obtain:

$$\begin{aligned}\dot{\mathbf{S}}(\boldsymbol{E},T)&=\mathbb{C}^e_{isoT}:\dot{\boldsymbol{E}}+\boldsymbol{M}\otimes\dot{T}=\mathbb{C}^e_{isoT}:\dot{\boldsymbol{E}}+\boldsymbol{M}\otimes\left(\frac{T}{\rho_0 c_{\boldsymbol{E}}}\boldsymbol{M}:\dot{\boldsymbol{E}}\right)\\&=\left(\mathbb{C}^e_{isoT}+\frac{T}{\rho_0 c_{\boldsymbol{E}}}\boldsymbol{M}\otimes\boldsymbol{M}\right):\dot{\boldsymbol{E}}\end{aligned} \tag{10.63}$$

and by comparing this with the equation in (10.41) we can conclude that:

$$\boxed{\mathbb{C}^e_{ise}=\mathbb{C}^e_{isoT}+\frac{T}{\rho_0 c_{\boldsymbol{E}}}\boldsymbol{M}\otimes\boldsymbol{M}} \tag{10.64}$$

10.3 Linear Thermoelasticity

10.3.1 Linearization of the Constitutive Equations

As discussed in Chapter 6, the constitutive equations for simple thermoelastic materials can be expressed as follows:

$$\boxed{\begin{aligned}&\psi=\psi(\boldsymbol{E},T)\\&\mathbf{S}(\boldsymbol{E},T)=\rho_0\frac{\partial\psi(\boldsymbol{E},T)}{\partial\boldsymbol{E}}\\&\eta(\boldsymbol{E},T)=-\frac{\partial\psi}{\partial T}\\&\vec{\mathbf{q}}_0=\vec{\hat{\mathbf{q}}}_0(\boldsymbol{E},T,\nabla_{\vec{X}}T)\end{aligned}} \quad \text{The constitutive equations for simple thermoelastic materials (Reference configuration)} \tag{10.65}$$

where ψ is the specific Helmholtz free energy (per unit mass), $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, η is the specific entropy (per unit mass), $\vec{\mathbf{q}}_0$ is the heat flux vector, $\boldsymbol{E}$ is the Green-Lagrange strain tensor, T denotes temperature, $\nabla_{\vec{X}}T$ is the temperature gradient, and $\boldsymbol{F}$ is the deformation gradient.

To make the linearization of the constitutive equations, (Nowacki(1967), Šilhavý(1997), Pabst(2005)), we will use the Taylor series expansion, where the following condition holds: given a function $f(x)$, said function can be approximated by using the Taylor series:

$f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n f(a)}{\partial x^n}(x-a)^n$, applied at point a (the application point). We will now apply this same definition but applied to tensors, (see Chapter 1).

10.3.1.1 The Linearized Piola-Kirchhoff Stress Tensor

The Piola-Kirchhoff stress tensor can be represented by means of the Taylor series in which we will consider up to linear terms:

$$
\begin{aligned}
\mathbf{S}(\boldsymbol{E},T) &= \mathbf{S}(\boldsymbol{E}_O,T_O)+\frac{\partial \mathbf{S}(\boldsymbol{E}_O,T_O)}{\partial \boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_O)+\frac{\partial \mathbf{S}(\boldsymbol{E}_O,T_O)}{\partial T}(T-T_O)+\underbrace{\cdots}_{\substack{\text{Higher order}\\ \text{terms}}} \\
&\approx \mathbf{S}_O+\rho_0\frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial \boldsymbol{E}\otimes\partial \boldsymbol{E}}:(\boldsymbol{E}-\boldsymbol{E}_O)+\rho_0\frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial \boldsymbol{E}\partial T}(T-T_O)
\end{aligned}
\tag{10.66}
$$

where we have considered that $\mathbf{S}_O=\mathbf{S}(\boldsymbol{E}_O,T_O)=\rho_0\frac{\partial\psi(\boldsymbol{E}_O,T_O)}{\partial \boldsymbol{E}}$. Note that we have used the subscript O to indicate the variable value at the application point, so as to differentiate this from the subscript 0 which is used to identify variables in the reference configuration.

Note that the linearized constitutive equation for stress has a linear relationship with strain and temperature, but also considers large deformation kinematics.

Now, if we consider the equation in (10.66), we can identify these material properties:

- *The isothermal elasticity tensor* ($\mathbb{C}^e_{isoT}$) (reference configuration):

$$
\boxed{\mathbb{C}^e_{isoT}=\left(\frac{\partial \mathbf{S}(\boldsymbol{E}_O,T_O)}{\partial \boldsymbol{E}}\right)_{\dot{T}=0}=\rho_0\frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial \boldsymbol{E}\otimes\partial \boldsymbol{E}}}\quad \textit{The isothermal elasticity tensor} \tag{10.67}
$$

- *The thermal stress tensor* ($\boldsymbol{M}$):

$$
\boxed{\boldsymbol{M}=\left(\frac{\partial \mathbf{S}(\boldsymbol{E}_O,T_O)}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}=\rho_0\frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial T\otimes\partial \boldsymbol{E}}\equiv\rho_0\frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial T\partial \boldsymbol{E}}}\quad \textit{The thermal stress tensor} \tag{10.68}
$$

If we then consider the equations $\mathbf{P}=\boldsymbol{F}\cdot\mathbf{S}$, $\mathbf{P}(\boldsymbol{E},T)=\rho_0\boldsymbol{F}\cdot\frac{\partial\psi(\boldsymbol{E},T)}{\partial \boldsymbol{E}}$, $\mathbf{P}(\boldsymbol{F},T)=\rho_0\frac{\partial\psi(\boldsymbol{F},T)}{\partial \boldsymbol{F}}$, where $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, we can obtain:

$$
\boldsymbol{M}=\frac{\partial \mathbf{P}(\boldsymbol{F},T)}{\partial T}=\rho_0\frac{\partial^2\psi(\boldsymbol{F},T)}{\partial T\otimes\partial \boldsymbol{F}} \tag{10.69}
$$

Then, returning to the equation in (10.66) and by considering that $\mathbf{S}_O=\mathbf{0}$ and $\boldsymbol{E}_O=\mathbf{0}$, the linearized constitutive equation for stress becomes:

$$
\mathbf{S}=\mathbb{C}^e_{isoT}:\boldsymbol{E}+\boldsymbol{M}(T-T_O) \tag{10.70}
$$

Note that the following holds:

$$
\mathbf{S}=\boldsymbol{M}(T-T_O)\qquad \textit{with}\qquad \boldsymbol{E}=\mathbf{0} \tag{10.71}
$$

That is, the thermal stress tensor provides stress in the absence of strain ($\boldsymbol{E}=\mathbf{0}$).

We can also define:

- *The Latent heat tensor of change of strain* ($\boldsymbol{L}_{\boldsymbol{E}}$)

The thermal stress tensor is closely linked to the latent heat tensor $\boldsymbol{L}_{\boldsymbol{E}}$ (symmetric second-order tensor) which can be expressed as follows:

$$\boxed{\begin{aligned}\boldsymbol{L}_{\boldsymbol{E}} &= -\frac{T_O}{\rho_0}\frac{\partial \mathbf{S}(\boldsymbol{E}_O, T_O)}{\partial T} = -T_O\frac{\partial^2\psi(\boldsymbol{E}_O, T_O)}{\partial T\partial \boldsymbol{E}} \\ &= -\frac{T_O}{\rho_0}\boldsymbol{M}\cdot\boldsymbol{F}^T = -\frac{T_O}{\rho_0}\frac{\partial^2\psi}{\partial T\partial \boldsymbol{F}}\cdot\boldsymbol{F}^T\end{aligned}} \quad \textit{Latent heat tensor of change of strain} \tag{10.72}$$

Now, we can relate the latent heat tensor ($\boldsymbol{L}_{\boldsymbol{E}}$) to the thermal stress tensor ($\boldsymbol{M}$) by:

$$\boldsymbol{L}_{\boldsymbol{E}} = -T_O\frac{\partial^2\psi(\boldsymbol{E}_O, T_O)}{\partial T\partial \boldsymbol{E}} = -\frac{T_O}{\rho_0}\boldsymbol{M} \tag{10.73}$$

Next, we will define some material parameters which are related to deformation, *i.e.* those which are associated with the inverse of the equation in (10.70).

- *The thermal expansion tensor* ($\boldsymbol{A}$)

Now, based on the stress expression given in (10.70), $\mathbf{S} = \mathbb{C}^e_{isoT} : \boldsymbol{E} + \boldsymbol{M}(T - T_O)$, we can obtain the inverse relationship as follows:

$$\begin{aligned}&{\mathbb{C}^e_{isoT}}^{-1} : \mathbf{S} = \underbrace{{\mathbb{C}^e_{isoT}}^{-1} : \mathbb{C}^e_{isoT}}_{\mathbb{I}} : \boldsymbol{E} + {\mathbb{C}^e_{isoT}}^{-1} : \boldsymbol{M}(T - T_O) \\ &\Rightarrow \boldsymbol{E} = {\mathbb{C}^e_{isoT}}^{-1} : \mathbf{S} - {\mathbb{C}^e_{isoT}}^{-1} : \boldsymbol{M}(T - T_O)\end{aligned} \tag{10.74}$$

where $(\mathbb{C}^e_{isoT})^{-1} = \dfrac{\partial \boldsymbol{E}}{\partial \mathbf{S}}$ holds. Note that if the body can deform freely, the implication is that there is no stress $\mathbf{S} = \mathbf{0}$ and here the equation in (10.74) becomes:

$$\boldsymbol{E} = -{\mathbb{C}^e_{isoT}}^{-1} : \boldsymbol{M}(T - T_O) = \boldsymbol{A}(T - T_O) \tag{10.75}$$

Therefore, we can define the thermal expansion tensor, denoted by $\boldsymbol{A}$, as follows:

$$\boxed{\boldsymbol{A} = -{\mathbb{C}^e_{isoT}}^{-1} : \boldsymbol{M}} \quad \textit{The thermal expansion tensor} \tag{10.76}$$

Then, the equation in (10.74) can be rewritten as follows:

$$\boldsymbol{E} = {\mathbb{C}^e_{isoT}}^{-1} : \mathbf{S} + \boldsymbol{A}(T - T_O) \tag{10.77}$$

NOTE: Although we have defined the tensors $\boldsymbol{L}_{\boldsymbol{E}}$ and $\boldsymbol{A}$, in thermal stress analysis, we need only know the tensor $\boldsymbol{M}$. However, as regards practice in laboratory measurement, it is more convenient, from a practical standpoint, to obtain the thermal expansion tensor ($\boldsymbol{A}$) and then we can obtain the thermal stress tensor by means of the equation $\boldsymbol{M} = -\mathbb{C}^e_{isoT} : \boldsymbol{A}$. ■

10.3.1.2 The Linearized Heat Flux Vector

The linearization if the heat flux vector can be obtained as follows:

$$\vec{\mathbf{q}}_0(\nabla_{\vec{X}}T) = \vec{\mathbf{q}}_0(\boldsymbol{E}_O, T_O, \nabla_{\vec{X}}T) = \vec{\mathbf{q}}_{0O} + \frac{\partial \vec{\mathbf{q}}_0}{\partial \nabla_{\vec{X}}T}\cdot\nabla_{\vec{X}}T + \cdots \approx \frac{\partial \vec{\mathbf{q}}_0}{\partial \nabla_{\vec{X}}T}\cdot\nabla_{\vec{X}}T \tag{10.78}$$

where we have taken into account that $\vec{\mathbf{q}}_{0O} = \vec{\mathbf{q}}_0(\boldsymbol{E}_O, T_O, \nabla_{\vec{X}}T = \vec{\mathbf{0}}) = \vec{\mathbf{0}}$. Next, we can define the following thermal material properties:

- *The thermal conductivity tensor* ($\mathbf{K}$):

The thermal conductivity tensor is given by the following equation:

$$\boxed{\mathbf{K}_0 = -\frac{\partial \vec{\mathbf{q}}_0}{\partial \nabla_{\vec{X}} T}} \quad \textit{The thermal conductivity tensor (Reference configuration)} \quad \left[\frac{W}{mK}\right] \tag{10.79}$$

If we now consider the units: $[\vec{\mathbf{q}}_0] = \frac{J}{m^2 s}$, $[\nabla_{\vec{X}} T] = \frac{K}{m}$, it is easy to show that: $[\mathbf{K}_0] = \frac{[\vec{\mathbf{q}}_0]}{[\nabla_{\vec{X}} T]} = \frac{J}{m^2 s}\frac{m}{K} = \frac{W}{mK}$.

Then, we can rewrite the linearized heat flow vector as follows:

$$\vec{\mathbf{q}}_0(\nabla_{\vec{X}} T) = -\mathbf{K}_0 \cdot \nabla_{\vec{X}} T \tag{10.80}$$

Remember from Chapter 5 that the *heat conduction inequality* in the reference configuration is given by $-\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0$. Then, by substituting the linearized heat flow vector given in (10.80) into the heat conduction inequality we obtain:

$$-\vec{\mathbf{q}}_0 \cdot \nabla_{\vec{X}} T \geq 0 \quad \Rightarrow \quad -\left(-\mathbf{K}_0 \cdot \nabla_{\vec{X}} T\right) \cdot \nabla_{\vec{X}} T \geq 0 \tag{10.81}$$

which in indicial notation becomes:

$$-\left[-\mathsf{K}_{ij}\left(\nabla_{\vec{X}} T\right)_j\right]\left(\nabla_{\vec{X}} T\right)_i \geq 0 \quad \Rightarrow \quad \left(\nabla_{\vec{X}} T\right)_i \mathsf{K}_{ij}\left(\nabla_{\vec{X}} T\right)_j \geq 0 \tag{10.82}$$

or which is the same as:

$$\nabla_{\vec{X}} T \cdot \mathbf{K}_0 \cdot \nabla_{\vec{X}} T \geq 0 \tag{10.83}$$

Thus if we can conclude that the thermal conductivity tensor ($\mathbf{K}_0$) is a semi-positive definite tensor, then, the $\mathbf{K}_0$-eigenvalues are greater than or equal to zero, *i.e.* $\mathsf{K}_{0_1} \geq 0$, $\mathsf{K}_{0_2} \geq 0$, $\mathsf{K}_{0_3} \geq 0$.

Let us now consider a non-symmetric tensor $\mathbf{K}_0$ (anisotropic material) which we will then split into a symmetric ($\mathbf{K}_0^{sym}$) and antisymmetric ($\mathbf{K}_0^{skew}$) part, so that the inequality in (10.83) becomes:

$$\nabla_{\vec{X}} T \cdot \mathbf{K} \cdot \nabla_{\vec{X}} T = \nabla_{\vec{X}} T \cdot \left(\mathbf{K}_0^{sym} + \mathbf{K}_0^{skew}\right) \cdot \nabla_{\vec{X}} T = \nabla_{\vec{X}} T \cdot \mathbf{K}_0^{sym} \cdot \nabla_{\vec{X}} T + \nabla_{\vec{X}} T \cdot \mathbf{K}_0^{skew} \cdot \nabla_{\vec{X}} T \geq 0$$

Note that $\nabla_{\vec{X}} T \cdot \mathbf{K}_0^{skew} \cdot \nabla_{\vec{X}} T = \mathbf{K}_0^{skew} : \left(\nabla_{\vec{X}} T \otimes \nabla_{\vec{X}} T\right) = 0$, since the double scalar product between a symmetric and antisymmetric tensor is equal to zero. Then the above equation becomes:

$$\nabla_{\vec{X}} T \cdot \mathbf{K}_0^{sym} \cdot \nabla_{\vec{X}} T \geq 0 \tag{10.84}$$

That is, the antisymmetric part of the thermal conductivity tensor has no influence on entropy evolution or on the second law of thermodynamics, (Powers (2004)).

We will now take this opportunity to introduce the thermal diffusivity tensor as follows:

$$\boxed{\mathbf{D}_0 = \frac{1}{\rho_0 c_E}\mathbf{K}_0} \quad \textit{The thermal diffusivity tensor (Reference configuration)} \quad \left[\frac{m^2}{s}\right] \tag{10.85}$$

10.3.1.3 Linearized Entropy

Entropy linearization can be obtained as follows:

$$
\begin{aligned}
\eta(\boldsymbol{E},T) &= \eta_O + \frac{\partial\eta(\boldsymbol{E}_O,T_O)}{\partial\boldsymbol{E}} : (\boldsymbol{E}-\boldsymbol{E}_O) + \frac{\partial\eta(\boldsymbol{E}_O,T_O)}{\partial T}(T-T_O) + \cdots \\
&\approx \eta_O - \frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial\boldsymbol{E}\partial T} : (\boldsymbol{E}-\boldsymbol{E}_O) - \frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial T\partial T}(T-T_O)
\end{aligned}
\tag{10.86}
$$

We can the define the following material properties:

- *Specific heat* ($c^o_{\boldsymbol{E}}$) at a constant volume, (see equation (10.45)):

$$
\boxed{c^o_{\boldsymbol{E}} = T_O \frac{\partial\eta(\boldsymbol{E}_O,T_O)}{\partial T} = -T_O \frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial T\otimes\partial T}} \quad \textit{Specific heat} \tag{10.87}
$$

Then, by considering both $\boldsymbol{E}_O = \boldsymbol{0}$, $\Delta T = (T-T_O)$ and the material parameters in (10.87) and (10.68), the linearized entropy becomes:

$$
\eta(\boldsymbol{E},T) = \eta_O - \frac{1}{\rho_0}\boldsymbol{M}:\boldsymbol{E} + \frac{c^o_{\boldsymbol{E}}}{T_O}\Delta T \tag{10.88}
$$

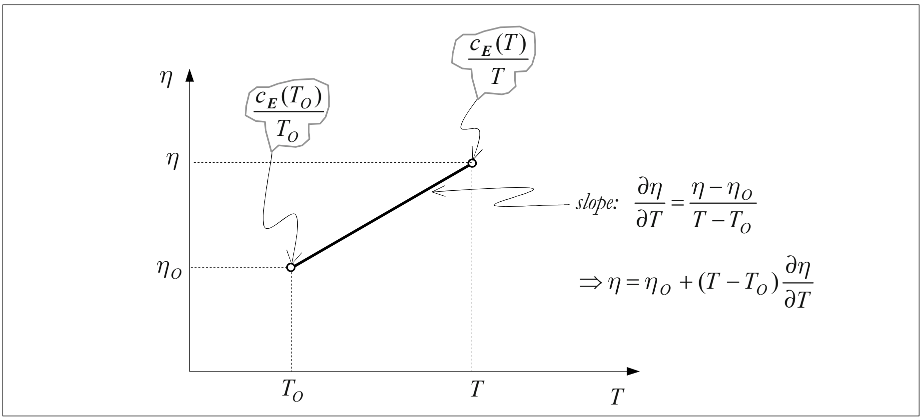


Figure 10.2: Curve entropy *vs.* temperature (linearization).

10.3.1.4 The Helmholtz Free Energy Approach

Note that to achieve consistency that $\mathbf{S}(\boldsymbol{E},T) = \rho_0 \frac{\partial\psi}{\partial\boldsymbol{E}}$ and $\eta(\boldsymbol{E},T) = -\frac{\partial\psi}{\partial T}$ are linear functions, the approximation of the Helmholtz free energy must present quadratic terms, so:

$$
\begin{aligned}
\psi(\boldsymbol{E},T) = \ & \psi(\boldsymbol{E}_O,T_O) + \frac{\partial\psi(\boldsymbol{E}_O,T_O)}{\partial\boldsymbol{E}} : (\boldsymbol{E}-\boldsymbol{E}_O) + \frac{\partial\psi(\boldsymbol{E}_O,T_O)}{\partial T}(T-T_O) + \\
& \frac{1}{2}(\boldsymbol{E}-\boldsymbol{E}_O) : \frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}} : (\boldsymbol{E}-\boldsymbol{E}_O) + (T-T_O)\frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial\boldsymbol{E}\otimes\partial T} : (\boldsymbol{E}-\boldsymbol{E}_O) + \\
& \frac{1}{2}\frac{\partial^2\psi(\boldsymbol{E}_O,T_O)}{\partial T\otimes\partial T}(T-T_O)^2 + \cdots
\end{aligned}
\tag{10.89}
$$

Then, by considering both the mechanical and thermal properties defined in the previous subsections as well as $\boldsymbol{E}_O = \mathbf{0}$ and $\Delta T = (T - T_O)$, we can express the approximation of the Helmholtz free energy as follows:

$$\psi(\boldsymbol{E},T) = \psi_O + \frac{1}{\rho_0}\underbrace{\mathbf{S}(\mathbf{0},T_O)}_{\mathbf{S}_O} : \boldsymbol{E} - \eta_O\,\Delta T + \frac{1}{2\rho_0}\boldsymbol{E} : \mathbb{C}^e_{isoT} : \boldsymbol{E} + \frac{\Delta T}{\rho_0}\boldsymbol{M} : \boldsymbol{E} - \frac{c^o_{\boldsymbol{E}}}{2T_O}\Delta T^2 \tag{10.90}$$

We must also consider that $\mathbf{S}_O = \mathbf{0}$ at the application point with which the above equation becomes:

$$\psi(\boldsymbol{E},T) = \psi_O - \eta_O\,\Delta T + \frac{1}{2\rho_0}\boldsymbol{E} : \mathbb{C}^e_{isoT} : \boldsymbol{E} + \frac{\Delta T}{\rho_0}\boldsymbol{M} : \boldsymbol{E} - \frac{c^o_{\boldsymbol{E}}}{2T_O}\Delta T^2 \tag{10.91}$$

10.3.1.5 Linearization of the Constitutive Equations

By considering the above in the previous subsections, we can sum up the linearized constitutive equations for simple thermoelastic materials as follows:

Linearized Constitutive Equations for Simple Thermoelastic Materials (Reference Configuration)		
Constitutive equation for energy	$\psi(\boldsymbol{E},T) = \psi_O - \eta_O\,\Delta T + \frac{1}{2\rho_0}\boldsymbol{E} : \mathbb{C}^e_{isoT} : \boldsymbol{E} + \frac{\Delta T}{\rho_0}\boldsymbol{M} : \boldsymbol{E} - \frac{c^o_{\boldsymbol{E}}}{2T_O}\Delta T^2$	(10.92)
Constitutive equations for stress	$\mathbf{S}(\boldsymbol{E},T) = \mathbb{C}^e_{isoT} : \boldsymbol{E} + \boldsymbol{M}\,\Delta T$	(10.93)
Constitutive equation for entropy	$\eta(\boldsymbol{E},T) = \eta_O - \frac{1}{\rho_0}\boldsymbol{M} : \boldsymbol{E} + \frac{c^o_{\boldsymbol{E}}}{T_O}\Delta T$	(10.94)
Constitutive equations for heat flux	$\vec{\mathbf{q}}_0(\nabla_{\vec{X}}T) = -\mathbf{K}_0 \cdot \nabla_{\vec{X}}T$	(10.95)

where $\mathbb{C}^e_{isoT}$, $\boldsymbol{M}$, $c^o_{\boldsymbol{E}}$, $\mathbf{K}_0$ are the thermo-mechanical material properties, which are obtained in the laboratory.

10.3.1.6 Linear Thermoelasticity in a Small Deformation Regime

When we are dealing with small deformation regime (infinitesimal strain), the displacement gradient is very small when compared with the unitary. Moreover, there is no distinction between the reference and current configurations (see Chapter 7). It is also true that the Green-Lagrange strain tensor $\boldsymbol{E}$ (reference configuration) and the Almansi strain tensor $\boldsymbol{e}$ (current configuration) collapse into the infinitesimal strain tensor, *i.e.* $\boldsymbol{\varepsilon} \approx \boldsymbol{E} \approx \boldsymbol{e}$, and the same happens to the stress tensors $\mathbf{P} \approx \mathbf{S} \approx \boldsymbol{\sigma}$. Here, we will define the following tensors: $\boldsymbol{M} \approx \boldsymbol{M}^L$ (the linear thermal stress tensor) and $\boldsymbol{A} \approx \boldsymbol{A}^L$ (the linear thermal expansion tensor). Note, in the small deformation regime, it is also true that $\vec{\mathbf{q}}_0 \to \vec{\mathbf{q}}$, $\rho \approx \rho_0$, $\nabla_{\vec{X}} \bullet \approx \nabla_{\vec{x}} \bullet \to \nabla \bullet$. Now, considering all the previous observations, the constitutive equation for stress given in (10.93) becomes:

$$\boxed{\boldsymbol{\sigma} = \mathbb{C}^e_{isoT} : \boldsymbol{\varepsilon} + \boldsymbol{M}^L (T - T_0)} \quad \textit{Duhamel-Neumann equations} \tag{10.96}$$

which is the generalized Hooke's law for thermoelastic material, which is also known as the Duhamel-Neumann equation.

The linear thermal stress tensor ($\boldsymbol{M}^L$) provides stress in absence of strain, *i.e.:*

$$\boldsymbol{\sigma} = \boldsymbol{M}^L (T - T_O) \qquad with \qquad (\boldsymbol{\varepsilon} = \boldsymbol{0}) \tag{10.97}$$

Now, the reciprocal of (10.96) is given by:

$$\begin{aligned} &\mathbb{C}^{e\;-1}_{isoT} : \boldsymbol{\sigma} = \underbrace{\mathbb{C}^{e\;-1}_{isoT} : \mathbb{C}^{e}_{isoT}}_{\mathbb{I}^{sym}} : \boldsymbol{\varepsilon} + \mathbb{C}^{e^{-1}} : \boldsymbol{M}^L (T - T_O) \\ &\Rightarrow \boldsymbol{\varepsilon} = \mathbb{C}^{e\;-1}_{isoT} : \boldsymbol{\sigma} - \mathbb{C}^{e\;-1}_{isoT} : \boldsymbol{M}^L (T - T_O) \end{aligned} \tag{10.98}$$

Then, considering that $\boldsymbol{A}^L = -\mathbb{C}^{e\;-1}_{isoT} : \boldsymbol{M}^L$, (see equation (10.76)) we obtain:

$$\boldsymbol{\varepsilon} = \mathbb{C}^{e\;-1}_{isoT} : \boldsymbol{\sigma} + \boldsymbol{A}^L (T - T_O) \tag{10.99}$$

Next, the linear thermal expansion tensor ($\boldsymbol{A}^L$) provides strain in the absence of stress, *i.e.:*

$$\boldsymbol{\varepsilon} = \boldsymbol{A}^L (T - T_O) \qquad with \qquad (\boldsymbol{\sigma} = \boldsymbol{0}) \tag{10.100}$$

Note that, here, the material point (particle) can expand freely.

Then, the constitutive equations for heat flux ($\vec{\mathbf{q}}_0 \approx \vec{\mathbf{q}}$) can be expressed by *Fourier's law of thermal conduction*, which is given by:

$$\vec{\mathbf{q}} = -\mathbf{K} \cdot \nabla T \tag{10.101}$$

where $\mathbf{K}$ is the thermal conductivity tensor.

Next, the constitutive equation for entropy becomes:

$$\eta = \eta_O + \frac{c_{\boldsymbol{E}}^o}{T_O}(T - T_O) - \frac{1}{\rho_0} \boldsymbol{M}^L : \boldsymbol{\varepsilon} \tag{10.102}$$

and the constitutive equation for energy becomes:

$$\psi = \psi_O - \eta_O \, \Delta T + \frac{1}{2\rho} \boldsymbol{\varepsilon} : \mathbb{C}^e_{isoT} : \boldsymbol{\varepsilon} + \frac{(T - T_O)}{\rho} \boldsymbol{M}^L : \boldsymbol{\varepsilon} - \frac{c_{\boldsymbol{E}}^o}{2T_O}(T - T_O)^2 \tag{10.103}$$

10.3.1.7 Linear Thermoelasticity in a Small Deformation Regime

In elastic materials, the fourth-order tensor $\mathbb{C}^e_{isoT}$, in general, is anisotropic and features 21 independent constants to be determined in the laboratory. For isotropy the number of constants reduces to 2 and the elasticity tensor can be expressed as follows $\mathbb{C}^e_{isoT} = \lambda_{isoT} \mathbf{1} \otimes \mathbf{1} + 2\mu_{isoT} \mathbf{I}$, where λ_{isoT} and μ_{isoT} are *isothermal Lamé constants*. Then, the constitutive equation for stress, which considers isotropic linear elastic materials and an isothermal processes, is given by

$$\boldsymbol{\sigma} = \lambda_{isoT} \mathrm{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} + 2\mu_{isoT} \boldsymbol{\varepsilon} \tag{10.104}$$

Remember that the isotropic second-order tensor is already spherical tensor and vice-versa. Then, in isotropic materials, the thermal tensors can be represented by the following:

$$\boldsymbol{M}^L = m\mathbf{1} \quad ; \quad \boldsymbol{A}^L = \alpha \mathbf{1} \quad ; \quad \mathbf{K} = \mathsf{K}\mathbf{1} \tag{10.105}$$

where K is the coefficient of thermal conductivity, m is the thermal stress coefficient and α is the coefficient of thermal expansion. We can now find the relationship between α and m by means of the definition in (10.76), thus:

$$\begin{aligned}\boldsymbol{M}^L &= -\mathbb{C}^e_{isoT} : \boldsymbol{A}^L \\ m\mathbf{1} &= -\left(\lambda_{isoT}\mathbf{1}\otimes\mathbf{1} + 2\mu_{isoT}\mathbf{I}\right):\alpha\mathbf{1} = -\alpha\left(\lambda_{isoT}\mathbf{1}\otimes\mathbf{1}:\mathbf{1} + 2\mu_{isoT}\mathbf{I}:\mathbf{1}\right) \\ &= -\alpha\left(3\lambda_{isoT}\mathbf{1} + 2\mu_{isoT}\mathbf{1}\right) = -\alpha(3\lambda_{isoT} + 2\mu_{isoT})\mathbf{1}\end{aligned} \tag{10.106}$$

with which we can draw the conclusion that:

$$m = -\alpha(3\lambda_{isoT} + 2\mu_{isoT}) = -3\alpha\kappa_{isoT} = -\frac{E_{isoT}}{(1-2\nu_{isoT})}\alpha \tag{10.107}$$

where we have introduced the isothermal bulk modulus: $\kappa_{isoT} = \dfrac{(3\lambda_{isoT} + 2\mu_{isoT})}{3}$ and where the following is also true $\kappa_{isoT} = \dfrac{E_{isoT}}{3(1-2\nu_{isoT})}$.

Then, the constitutive equations for stress, (see equation (10.96)), becomes:

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda_{isoT}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu_{isoT}\boldsymbol{\varepsilon} + m(T - T_O)\mathbf{1} \\ &= \lambda_{isoT}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu_{isoT}\boldsymbol{\varepsilon} - \alpha(3\lambda_{isoT} + 2\mu_{isoT})(T - T_O)\mathbf{1} \\ &= \lambda_{isoT}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu_{isoT}\boldsymbol{\varepsilon} - 3\alpha\,\kappa_{isoT}\Delta T\,\mathbf{1}\end{aligned} \tag{10.108}$$

When $\boldsymbol{\varepsilon} = \mathbf{0}$ we have:

$$\boldsymbol{\sigma} = m(T - T_O)\mathbf{1} = m\Delta T\,\mathbf{1} \tag{10.109}$$

which represents the spherical state brought about by the temperature change (pressure) in isotropic materials.

Then, the inverse of the constitutive equation for stress is given by (10.98) and by considering isotropic materials said equation becomes:

$$\boldsymbol{\varepsilon} = \mathbb{C}^{e\;-1}_{isoT} : \boldsymbol{\sigma} + \Delta T\,\boldsymbol{A}^L = \mathbb{C}^{e\;-1}_{isoT} : \boldsymbol{\sigma} + \alpha\Delta T\,\mathbf{1} \tag{10.110}$$

Then, considering that $\mathbb{C}^{e\;-1}_{isoT} = \dfrac{-\lambda_{isoT}}{2\mu_{isoT}(3\lambda_{isoT} + 2\mu_{isoT})}\mathbf{1}\otimes\mathbf{1} + \dfrac{1}{2\mu_{isoT}}\mathbf{I}$, the above becomes:

$$\boldsymbol{\varepsilon} = \frac{-\lambda_{isoT}}{2\mu_{isoT}(3\lambda_{isoT} + 2\mu_{isoT})}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \frac{1}{2\mu_{isoT}}\boldsymbol{\sigma} + \alpha\Delta T\,\mathbf{1} \tag{10.111}$$

and in stress-free cases we obtain:

$$\boldsymbol{\varepsilon} = \alpha\Delta T\,\mathbf{1} \qquad \textit{with} \qquad (\boldsymbol{\sigma} = \mathbf{0}) \tag{10.112}$$

which represents the uniform dilatation case.

Then, the constitutive equations for heat flux (Fourier's Law of thermal conduction) in isotropic materials become:

$$\vec{\mathbf{q}} = -\mathbf{K}\cdot\nabla T = -\mathsf{K}\mathbf{1}\cdot\nabla T = -\mathsf{K}\nabla T \tag{10.113}$$

Then, the constitutive equation for entropy (Biot's law), in isotropic materials, becomes:

$$\eta = \eta_O + \frac{c^o_E}{T_O}\Delta T - \frac{m}{\rho}\boldsymbol{M}^L : \boldsymbol{\varepsilon} = \eta_O + \frac{c^o_E}{T_O}\Delta T - \frac{1}{\rho}m\mathbf{1} : \boldsymbol{\varepsilon} = \eta_O + \frac{c^o_E}{T_O}\Delta T - \frac{m}{\rho}\mathrm{Tr}(\boldsymbol{\varepsilon}) \tag{10.114}$$

and, the constitutive equation for energy turns into:

$$\psi = \frac{1}{2\rho}\left\{\frac{\lambda_{isoT}[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu_{isoT}\ \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\} + m\frac{\Delta T}{\rho}\mathrm{Tr}(\boldsymbol{\varepsilon}) - \frac{c_{\boldsymbol{E}}^o}{2T_O}\Delta T^2 \tag{10.115}$$

Then, we can sum up the linearized constitutive equations obtained previously:

Linearized Constitutive Equations for Isotropic Thermoelastic Materials in a Small Deformation Regime		
Constitutive equation for energy	$\psi = \psi_O - \eta_O\,\Delta T + \frac{1}{2\rho}\left\{\frac{\lambda_{isoT}[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu_{isoT}\mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\} + m\frac{\Delta T}{\rho}\mathrm{Tr}(\boldsymbol{\varepsilon}) - \frac{c_{\boldsymbol{E}}^o}{2T_O}\Delta T^2$	(10.116)
Constitutive equations for stress	$\boldsymbol{\sigma} = \lambda_{isoT}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu_{isoT}\boldsymbol{\varepsilon} + m\Delta T\,\mathbf{1} = \lambda_{isoT}\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu_{isoT}\boldsymbol{\varepsilon} - \alpha(3\lambda_{isoT} + 2\mu_{isoT})\Delta T\,\mathbf{1}$	(10.117)
Constitutive equation for entropy	$\eta = \eta_O + \frac{c_{\boldsymbol{E}}^o}{T_O}\Delta T - \frac{m}{\rho}\mathrm{Tr}(\boldsymbol{\varepsilon})$	(10.118)
Constitutive equations for heat flux	$\vec{\mathbf{q}} = -\mathsf{K}\nabla T$	(10.119)

Next, we will obtain the relations between the isothermal and adiabatic Lamé constants. To do so, we will use the equations in (10.43), (10.44), and (10.64):

$$\begin{aligned}
&\mathbb{C}^e_{ise} = \mathbb{C}^e_{isoT} + \frac{T}{\rho_0 c_{\boldsymbol{E}}^o}\boldsymbol{M}^L \otimes \boldsymbol{M}^L \\
&\Rightarrow \lambda_{ise}\mathbf{1}\otimes\mathbf{1} + 2\mu_{ise}\mathbf{I} = \lambda_{isoT}\mathbf{1}\otimes\mathbf{1} + 2\mu_{isoT}\mathbf{I} + \frac{T}{\rho_0 c_{\boldsymbol{E}}^o}(m\mathbf{1})\otimes(m\mathbf{1}) \\
&\Rightarrow \lambda_{ise}\mathbf{1}\otimes\mathbf{1} + 2\mu_{ise}\mathbf{I} = \left(\lambda_{isoT} + \frac{m^2 T}{\rho_0 c_{\boldsymbol{E}}^o}\right)\mathbf{1}\otimes\mathbf{1} + 2\mu_{isoT}\mathbf{I}
\end{aligned} \tag{10.120}$$

with which we can conclude that:

$$\lambda_{ise} = \lambda_{isoT} + \frac{m^2 T}{\rho_0 c_{\boldsymbol{E}}^o} \qquad \text{and} \qquad \mu_{ise} = \mu_{isoT} \tag{10.121}$$

In real materials $c_{\boldsymbol{E}}^o > 0$ holds with which we can conclude that $\lambda_{ise} > \lambda_{isoT}$.

Now, if we start from the equations in (10.121), we can define other parameters such as the *adiabatic bulk modul*us:

$$\kappa_{ise} = \frac{(3\lambda_{ise} + 2\mu_{ise})}{3} = \frac{\left(3\lambda_{isoT} + \frac{m^2 T}{\rho_0 c_{\boldsymbol{E}}^o} + 2\mu_{isoT}\right)}{3} = \frac{(3\lambda_{isoT} + 2\mu_{isoT})}{3} + \frac{m^2 T}{3\rho_0 c_{\boldsymbol{E}}^o} = \kappa_{isoT} + \frac{m^2 T}{3\rho_0 c_{\boldsymbol{E}}^o} \tag{10.122}$$

The latent heat tensors, (of change of strain $\boldsymbol{L_E}$, and of change of stress $\boldsymbol{L_{\mathsf{S}}}$), (see equations (10.51) and (10.52)), in isotropic materials, are given by:

$$\boldsymbol{L_E} = -\frac{T}{\rho_0}\boldsymbol{M}^L = -\frac{T}{\rho_0}m\mathbf{1} \quad ; \quad \boldsymbol{L_{\mathsf{S}}} = \frac{T}{\rho_0}\boldsymbol{A}^L = \frac{T}{\rho_0}\alpha\,\mathbf{1} = -\frac{mT}{\rho_0(3\lambda_{isoT} + 2\mu_{isoT})}\mathbf{1} \tag{10.123}$$

where we have considered the equation $m = -\alpha(3\lambda_{isoT} + 2\mu_{isoT}) = -3\alpha\kappa_{isoT}$, (see Eq. (10.107)). Additionally, we can obtain:

$$
\begin{aligned}
c_{\mathbf{S}} - c_{\boldsymbol{E}} &= \frac{\rho_0}{T} \boldsymbol{L}_{\mathbf{S}} : \boldsymbol{L}_{\boldsymbol{E}} = \frac{\rho_0}{T}\left(-\frac{T}{\rho_0} m \mathbf{1}\right) : \left(-\frac{mT}{\rho_0(3\lambda_{isoT} + 2\mu_{isoT})}\mathbf{1}\right) \\
&= \frac{m^2 T}{\rho_0(3\lambda_{isoT} + 2\mu_{isoT})} \underbrace{\mathbf{1}:\mathbf{1}}_{=3} = \frac{3m^2 T}{\rho_0(3\lambda_{isoT} + 2\mu_{isoT})}
\end{aligned}
\tag{10.124}
$$

Then, if we start from (10.122), we can obtain:

$$
\frac{\kappa_{ise}}{\kappa_{isoT}} = \frac{\kappa_{isoT}}{\kappa_{isoT}} + \frac{m^2 T}{\kappa_{isoT} 3\rho_0 c_{\boldsymbol{E}}^o} \quad \Rightarrow \quad \frac{\kappa_{ise}}{\kappa_{isoT}} = 1 + \frac{m^2 T}{\kappa_{isoT} 3\rho_0 c_{\boldsymbol{E}}^o}
\tag{10.125}
$$

and by dividing that given in (10.124) by $c_{\boldsymbol{E}}$, we have:

$$
\begin{aligned}
&\frac{c_{\mathbf{S}}^o}{c_{\boldsymbol{E}}} - \frac{c_{\boldsymbol{E}}^o}{c_{\boldsymbol{E}}^o} = \frac{3m^2 T}{c_{\boldsymbol{E}}^o \rho_0 (3\lambda_{isoT} + 2\mu_{isoT})} \\
\Rightarrow \quad &\frac{c_{\mathbf{S}}^o}{c_{\boldsymbol{E}}^o} - 1 = \frac{3m^2 T}{c_{\boldsymbol{E}}^o \rho_0 (3\lambda_{isoT} + 2\mu_{isoT})} = \frac{m^2 T}{\kappa_{isoT} 3\rho_0 c_{\boldsymbol{E}}^o}
\end{aligned}
\tag{10.126}
$$

with which we can draw the conclusion that:

$$
\frac{c_{\mathbf{S}}^o}{c_{\boldsymbol{E}}^o} = \frac{\kappa_{ise}}{\kappa_{isoT}}
\tag{10.127}
$$

10.4 The Decoupled Thermo-Mechanical Problem in a Small Deformation Regime

In certain structures (solids) whose temperature variation is not sufficiently high (in the sense that their mechanical properties do not vary significantly) we can tackle this situation by means of the decoupled thermo-mechanical problem. That is, at any given time, we can carry out a thermal analysis without taking into consideration any deformation and then we can solve the mechanical problem considering initial deformations caused by temperature change, (see Figure 10.3, and Figure 10.4 and subsection 7.10 in Chapter 7).

As seen previously, the governing equations for simple thermoelastic materials are given by:

Basic Equations of Continuum Mechanics (Reference Configuration)		
The continuity equation	$\frac{D}{Dt}(J\rho) = 0$	(10.128)
The equations of motion	$\nabla_{\vec{X}} \cdot \mathbf{P} + \rho_0 \vec{\mathbf{b}}_0 = \rho_0 \dot{\vec{V}} = \rho_0 \dot{\vec{V}} = \rho_0 \ddot{\mathbf{u}}$ $\nabla_{\vec{X}} \cdot (\boldsymbol{F} \cdot \mathbf{S}) + \rho_0 \vec{\mathbf{b}}_0 = \rho_0 \dot{\vec{V}} = \rho_0 \ddot{\mathbf{u}}$	(10.129)
The second Piola-Kirchhoff stress tensor	$\mathbf{S} = \mathbf{S}^T$ or $\mathbf{P} \cdot \boldsymbol{F}^T = \boldsymbol{F} \cdot \mathbf{P}^T$	(10.130)
The energy equation	$\rho_0 \dot{u}(\vec{X}, t) = \mathbf{S} : \dot{\boldsymbol{E}} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r(\vec{X}, t)$ or $\rho_0 \dot{u}(\vec{X}, t) = \mathbf{P} : \dot{\boldsymbol{F}} - \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 + \rho_0 r(\vec{X}, t)$	(10.131)
The Clausius-Plack inequality	$\mathcal{D}_{int} = \rho_0 \dot{\eta} + \mathbf{S} : \dot{\boldsymbol{E}} - \frac{1}{T}\rho_0 \dot{u} \geq 0$	(10.132)

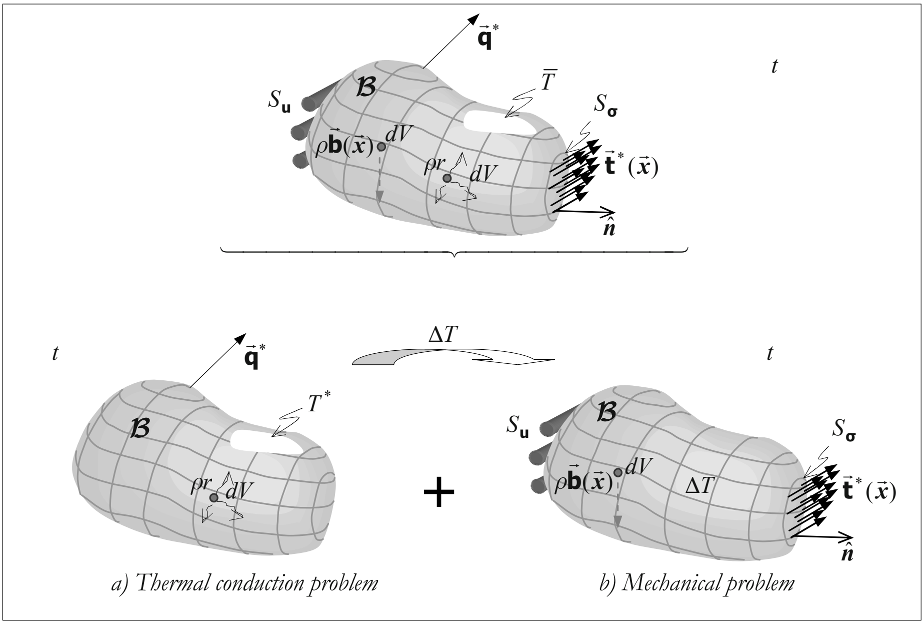


Figure 10.3: The decoupled thermo-mechanical problem.

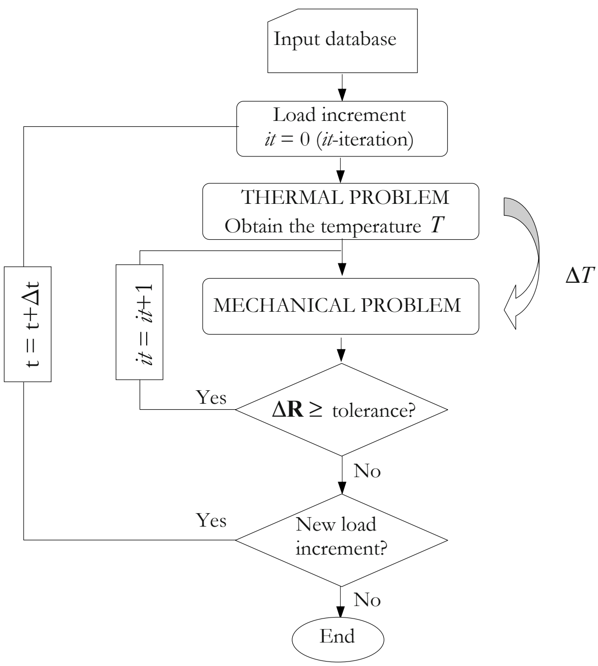


Figure 10.4: Flowchart of the decoupled thermo-mechanical problem.

Constitutive Equations for Simple Thermoelastic Materials (Reference Configuration)		
The constitutive equation for energy	$\psi = \psi(\boldsymbol{E}, T)$ (1 equation)	(10.133)
The constitutive equations for stress	$\mathbf{S} = \rho_0 \dfrac{\partial \psi(\boldsymbol{E}, T)}{\partial \boldsymbol{E}}$ (6 equations)	(10.134)
The constitutive equation for entropy	$\eta(\boldsymbol{E}, T) = -\dfrac{\partial \psi(\boldsymbol{E}, T)}{\partial T}$ (1 equation)	(10.135)
The constitutive equations for heat flux	$\vec{\mathbf{q}}_0 = \vec{\hat{\mathbf{q}}}_0(\boldsymbol{E}, T, \nabla_{\vec{X}} T)$ (3 equations)	(10.136)

Then, if we consider a small deformation regime we can make the following simplifications: $\mathbf{P} \approx \mathbf{S} \approx \boldsymbol{\sigma}$, $\boldsymbol{E} \approx \boldsymbol{e} \approx \boldsymbol{\varepsilon} = \nabla^{sym}\vec{\mathbf{u}}$, $\rho \approx \rho_0$, $\nabla_{\vec{X}} \approx \nabla_{\vec{x}} \approx \nabla$. Additionally, as the mass density is no longer an unknown the mass continuity equation plays no role. Then, we can summarize the basic equations for linear thermoelastic materials as follows:

Basic Equations for Linear Thermoelastic Solids		
The equations of motion	$\nabla \cdot \boldsymbol{\sigma} + \rho \vec{\mathbf{b}} = \rho \dot{\vec{V}} = \rho \dot{\vec{V}} = \rho \ddot{\vec{\mathbf{u}}}$	(10.137)
The energy equation	$\rho \dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \nabla \cdot \vec{\mathbf{q}} + \rho r$	(10.138)
The Clausius-Planck inequality	$\mathcal{D}_{int} = \rho \dot{\eta} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dfrac{1}{T} \rho \dot{u} \geq 0$	(10.139)

without forgetting the linearized constitutive equations obtained previously:

Linearized Constitutive Equations for Isotropic Thermoelastic Materials in a Small Deformation Regime		
The constitutive equation for energy	$\psi = \dfrac{1}{2\rho}\left\{\dfrac{\lambda[\mathrm{Tr}(\boldsymbol{\varepsilon})]^2}{2} + \mu\, \mathrm{Tr}(\boldsymbol{\varepsilon}^2)\right\} + m\dfrac{(T - T_0)}{\rho}\mathrm{Tr}(\boldsymbol{\varepsilon}) - \dfrac{c_{\boldsymbol{E}}}{2T_0}(T - T_0)^2$	(10.140)
The constitutive equations for stress	$\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} + m(T - T_0)\mathbf{1} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - \alpha(3\lambda + 2\mu)(T - T_0)\mathbf{1}$	(10.141)
The constitutive equation for entropy	$\eta = \dfrac{c_{\boldsymbol{E}}}{T_0}(T - T_0) - \dfrac{m}{\rho}\mathrm{Tr}(\boldsymbol{\varepsilon})$	(10.142)
The constitutive equations for heat flux	$\vec{\mathbf{q}} = -\mathbf{K} \cdot \nabla T$	(10.143)

10.4.1 The Purely Thermal Problem

Next, let us consider a purely thermal problem in which the equations of motion play no role. Then, by discarding the terms that are related to strain or stress we obtain:

Basic Equations for Linear Thermal Problems		
The energy equation	$\rho \dot{u} = -\nabla \cdot \vec{\mathbf{q}} + \rho r$	(10.144)
The Clausius-Planck equation (conservative process)	$\mathcal{D}_{int} = \rho \dot{\eta} - \dfrac{1}{T}\rho \dot{u} = 0 \quad \Rightarrow \quad \rho \dot{u} = T\rho\dot{\eta}$	(10.145)

Linearized Constitutive Equations for Thermal Problems in Small Deformation Regimes (isotropic material)		
The constitutive equation for energy	$\psi = -\frac{c_{\boldsymbol{E}}}{2T_0}(T - T_0)^2$	(10.146)
The constitutive equation for entropy	$\eta = \frac{c_{\boldsymbol{E}}}{T_0}(T - T_0)$	(10.147)
The constitutive equations for heat flux	$\vec{\mathbf{q}} = -\mathbf{K} \cdot \nabla T$	(10.148)

Then, by substituting the expression $(\rho\, \dot{u})$, given by the equation in (10.145) into (10.144), we can obtain the following:

$$T\rho\dot{\eta} = -\nabla \cdot \vec{\mathbf{q}} + \rho\, r \tag{10.149}$$

Next, the rate of change of the entropy $\eta = -\frac{\partial \psi(T)}{\partial T}$ can be obtained as follows:

$$\dot{\eta} = -\frac{\partial^2 \psi(T)}{\partial T \partial T}\dot{T} \tag{10.150}$$

and by substituting (10.150) into (10.149) we obtain:

$$-T\rho\frac{\partial^2 \psi(T)}{\partial T \partial T}\dot{T} = -\nabla \cdot \vec{\mathbf{q}} + \rho\, r \tag{10.151}$$

Additionally, by considering that $c_{\boldsymbol{E}} = -T\frac{\partial^2 \psi}{\partial T \partial T}$, we have:

$$\rho\, c_{\boldsymbol{E}}\dot{T} = -\nabla \cdot \vec{\mathbf{q}} + \rho\, r \tag{10.152}$$

Finally, by substituting the constitutive equations for heat flux given in (10.148) into (10.152) we obtain $\rho\, c_{\boldsymbol{E}}\dot{T} = -\nabla \cdot (-\mathbf{K} \cdot \nabla T) + \rho\, r$, or:

$$\boxed{\nabla \cdot (\mathbf{K} \cdot \nabla T) + \rho\, r = \rho\, c_{\boldsymbol{E}}\dot{T}} \quad \textit{The heat flux equation} \tag{10.153}$$

with which we have one equation and one unknown (temperature). Remember from Chapter 5 that the above equation is the same as that obtained when starting directly from the principle of conservation of energy, where we also defined the variable $Q = \rho\, r$. Now, to fully describe this problem (which was already discussed in subsection 5.12.1.4 in Chapter 5) we must add the boundary and initial conditions.

10.4.2 The Purely Mechanical Problem

The effect of temperature in mechanical problems can be addressed by means of the initial stress/strain as discussed in subsection 7.10. The mechanical problem statement was described in subsection 7.2, the only difference here lies in the constitutive equations for stress where we include the thermal effect by means of the initial stress, (see equation in (10.141)), *i.e.:*

The constitutive equations for stress (isotropic linear elastic materials):

$$\begin{aligned} \boldsymbol{\sigma} &= \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - \alpha(3\lambda + 2\mu)(T - T_0)\mathbf{1} \\ \sigma_{ij} &= \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - \alpha(3\lambda + 2\mu)(T - T_0)\delta_{ij} \end{aligned} \tag{10.154}$$

10.5 The Classical Theory of Thermoelasticity in Finite Strain (Large Deformation Regime)

The classical theory of thermoelasticity in finite strain considers two configurations, namely: the reference (or initial) configuration $\mathcal{B}_0$ and the current configuration $\mathcal{B}$, (see Vujošević&Lubarda (2002) as well as Figure 10.5). The hallmarks of the initial configuration denoted by $\mathcal{B}_0$ are a stress-free state and an initial temperature distribution $T_0(\vec{X})$. In the current configuration (deformed) the stress state is characterized by the Cauchy stress tensor $\boldsymbol{\sigma}(\vec{x},t)$ and by a temperature distribution denoted by $T(\vec{x},t)$.

We have defined the following tensors in the reference configuration: the Green-Lagrange strain tensor $\boldsymbol{E}(\vec{X},t)$, the second Piola-Kirchhoff stress tensor ($\mathbf{S}(\vec{X},t)$), and the heat flux vector $\vec{\mathbf{q}}_0(\vec{X},t)$. Then, in the current configuration we have defined: the Almansi strain tensor ($\boldsymbol{e}(\vec{x},t)$), the Cauchy stress tensor $\boldsymbol{\sigma}(\vec{x},t)$ and the heat flux vector $\vec{\mathbf{q}}(\vec{x},t)$, (see Figure 10.5).

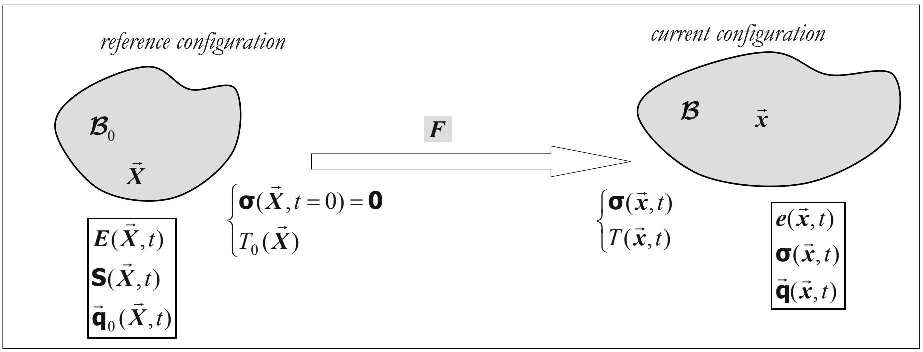


Figure 10.5: Reference and current configurations.

Remember from Chapter 5, (see Figure 5.9), that the heat flux vector (conduction) in the current configuration is related to the heat flux vector in the reference configuration by:

$$\vec{\mathbf{q}}_0 = J\,\vec{\mathbf{q}}\cdot\boldsymbol{F}^{-T} \qquad \Leftrightarrow \qquad \vec{\mathbf{q}} = J^{-1}\vec{\mathbf{q}}_0\cdot\boldsymbol{F}^{T} \tag{10.155}$$

Next, the principle of conservation of energy, by means of the Lagrangian variables, is given by:

$$\rho_0\dot{u}(\vec{X},t) = \mathbf{S}:\dot{\boldsymbol{E}} - \nabla_{\vec{X}}\cdot\vec{\mathbf{q}}_0 + \rho_0 r(\vec{X},t) \tag{10.156}$$

Then, by considering the continuum without any internal heat source, the energy equation becomes:

$$\rho_0\dot{u}(\vec{X},t) = \mathbf{S}:\dot{\boldsymbol{E}} - \nabla_{\vec{X}}\cdot\vec{\mathbf{q}}_0 \tag{10.157}$$

Next, by considering the Clausius-Duhem inequality we can obtain:

$$\rho_0\dot{\eta} - \rho_0\frac{r}{T} + \nabla_{\vec{X}}\cdot\left(\frac{\vec{\mathbf{q}}_0}{T}\right) = \rho_0\dot{\eta} - \underbrace{\rho_0\frac{r}{T}}_{=0} - \frac{1}{T}\nabla_{\vec{X}}\cdot\vec{\mathbf{q}}_0 + \underbrace{\frac{1}{T^2}\vec{\mathbf{q}}_0\cdot\nabla_{\vec{X}}T}_{>0} \geq 0 \tag{10.158}$$

Note that, here, the entropy production is only caused by the heat flux. Thus

$$\rho_0 \dot{\eta} \geq -\frac{1}{T} \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 \tag{10.159}$$

Therefore, in reversible processes, we have:

$$T\dot{\eta} = -\frac{1}{\rho_0} \nabla_{\vec{X}} \cdot \vec{\mathbf{q}}_0 \tag{10.160}$$

Remember that an alternative way to express the Clausius-Planck inequality is as follows:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0 \left[\eta \frac{DT}{Dt} + \frac{D\psi}{Dt} \right] \geq 0 \tag{10.161}$$

where ψ is the specific Helmholtz free energy (per unit mass).

Then, for processes with no internal energy dissipation $\mathcal{D}_{int} = 0$ (reversible), we can state that:

$$\dot{\psi} = \frac{1}{\rho_0} \mathbf{S} : \dot{\boldsymbol{E}} - \eta \dot{T} \tag{10.162}$$

where $\psi = \psi(\boldsymbol{E}, T)$ is a thermodynamic potential used to obtain the stress tensor ($\mathbf{S}$) and entropy (η):

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + \frac{\partial \psi}{\partial T} \dot{T} \tag{10.163}$$

Now, by comparing the equations (10.162) and (10.163), we can conclude, as expected, that:

$$\mathbf{S} = \rho_0 \frac{\partial \psi(\boldsymbol{E}, T)}{\partial \boldsymbol{E}} \quad ; \quad \eta = -\frac{\partial \psi(\boldsymbol{E}, T)}{\partial T} \tag{10.164}$$

10.5.1 The Coupled Heat Flux Equation

Let us now suppose that the heat flux vector is governed by Fourier's law of thermal conduction:

$$\vec{\mathbf{q}} = -\mathbf{K}(T) \cdot \nabla_{\vec{x}} T \quad ; \quad \vec{\mathbf{q}}_0 = -\mathbf{K}_0(\boldsymbol{F}, T) \cdot \nabla_{\vec{X}} T \tag{10.165}$$

where the thermal conductivity tensors $\mathbf{K}$ and $\mathbf{K}_0$ are interrelated by:

$$\mathbf{K}_0(\boldsymbol{F}, T) = \det(\boldsymbol{F})\, \boldsymbol{F}^{-1} \cdot \mathbf{K}(T) \cdot \boldsymbol{F}^{-T} \tag{10.166}$$

We can now prove the above equation is valid by starting from the following equation:

$$\vec{\mathbf{q}} \cdot d\vec{a} = \vec{\mathbf{q}}_0 \cdot d\vec{A} \tag{10.167}$$

where $d\vec{A}$ and $d\vec{a}$ are the differential area elements in the reference and current configuration, respectively, and which are related to each other by means of the Nanson's formula $d\vec{a} = J\,\boldsymbol{F}^{-T} \cdot d\vec{A}$ where $J = |\boldsymbol{F}|$. The following is then satisfied:

$$\begin{array}{c|c}
\vec{\mathbf{q}} \cdot d\vec{a} = \vec{\mathbf{q}}_0 \cdot d\vec{A} & \mathrm{q}_i da_i = \mathrm{q}_{0k} dA_k \\
J\,\vec{\mathbf{q}} \cdot \boldsymbol{F}^{-T} \cdot d\vec{A} = \vec{\mathbf{q}}_0 \cdot d\vec{A} & J\mathrm{q}_i F_{ki}^{-1} dA_k = \mathrm{q}_{0k} dA_k \\
J\,\vec{\mathbf{q}} \cdot \boldsymbol{F}^{-T} = \vec{\mathbf{q}}_0 & J\mathrm{q}_i F_{ki}^{-1} dA_k = \mathrm{q}_{0k} \\
\Rightarrow J\,\vec{\mathbf{q}} \cdot \boldsymbol{F}^{-T} = \vec{\mathbf{q}}_0 & \Rightarrow J\mathrm{q}_i F_{ki}^{-1} = \mathrm{q}_{0k}
\end{array} \tag{10.168}$$

Then, by substituting the equation in (10.165) into the above we obtain:

$$\begin{array}{c|c} \vec{\mathbf{q}}_0 = J\,\vec{\mathbf{q}}\cdot\boldsymbol{F}^{-T} & \mathsf{q}_{0I} = J\mathsf{q}_j F_{ij}^{-1} \\ -\mathbf{K}_0\cdot\nabla_{\vec{X}}T = -J\,\mathbf{K}\cdot\nabla_{\vec{x}}T\cdot\boldsymbol{F}^{-T} & -\mathsf{K}_{0IQ}T_{,Q} = -J\mathsf{K}_{jk}T_{,k}\,F_{ij}^{-1} \\ \mathbf{K}_0\cdot\nabla_{\vec{X}}T = J\,\boldsymbol{F}^{-1}\cdot\mathbf{K}\cdot\nabla_{\vec{x}}T & \mathsf{K}_{0IQ}T_{,Q} = JF_{ij}^{-1}\mathsf{K}_{jk}T_{,k} \end{array} \tag{10.169}$$

Note that the following is valid:

$$\begin{aligned} &\left(\nabla_{\vec{x}}T\right)_k = \frac{\partial T(\vec{x},t)}{\partial x_k} \equiv T_{,k} = \frac{\partial T(\vec{x}(\vec{X},t),t)}{\partial X_q}\underbrace{\frac{\partial X_q(\vec{X},t)}{\partial x_k}}_{=F_{qk}^{-1}} = \frac{\partial T(\vec{X},t)}{\partial X_q}F_{Qk}^{-1} = T_{,Q}\,F_{Qk}^{-1} \\ &\Rightarrow \nabla_{\vec{x}}T = \nabla_{\vec{X}}T\cdot\boldsymbol{F}^{-1} \end{aligned} \tag{10.170}$$

with which the equation in (10.169) becomes:

$$\begin{array}{l|l} \mathbf{K}_0\cdot\nabla_{\vec{X}}T = J\,\boldsymbol{F}^{-1}\cdot\mathbf{K}\cdot\nabla_{\vec{x}}T & \mathsf{K}_{0IQ}T_{,Q} = JF_{ij}^{-1}\mathsf{K}_{jk}T_{,k} \\ \mathbf{K}_0\cdot\nabla_{\vec{X}}T = J\,\boldsymbol{F}^{-1}\cdot\mathbf{K}\cdot(\nabla_{\vec{X}}T\cdot\boldsymbol{F}^{-1}) & \mathsf{K}_{0IQ}T_{,Q} = JF_{ij}^{-1}\mathsf{K}_{jk}T_{,Q}\,F_{Qk}^{-1} \\ \mathbf{K}_0\cdot\nabla_{\vec{X}}T = J\,\boldsymbol{F}^{-1}\cdot\mathbf{K}\cdot\boldsymbol{F}^{-T}\cdot\nabla_{\vec{X}}T & \mathsf{K}_{0IQ}T_{,Q} = (JF_{ij}^{-1}\mathsf{K}_{jk}F_{Qk}^{-1})T_{,Q} \end{array} \tag{10.171}$$

and:

$$\mathbf{K}_0 = J\,\boldsymbol{F}^{-1}\cdot\mathbf{K}\cdot\boldsymbol{F}^{-T} \tag{10.172}$$

which thus proves the equation in (10.166) is valid.

Then, by substituting Fourier's law (10.165) into the equation in (10.160) we obtain:

$$\dot{\eta}T = -\frac{1}{\rho_0}q_{0I,I} = -\frac{1}{\rho_0}\left[-\mathsf{K}_{0IQ}T_{,Q}\right]_{,I} = \frac{1}{\rho_0}\left[\mathsf{K}_{0IQ,I}T_{,Q} + \mathsf{K}_{0IQ}T_{,QI}\right] \tag{10.173}$$

where $T_{,QI} = \left(\dfrac{\partial^2 T}{\partial X_q \partial X_i}\right) = \left(\dfrac{\partial^2 T}{\partial X_i \partial X_q}\right) = T_{,IQ} = \left[\nabla_{\vec{X}}(\nabla_{\vec{X}}T)\right]_{iq}$ is satisfied and $\mathsf{K}_{0IQ,I}T_{,Q} = \dfrac{\partial \mathsf{K}_{0iq}}{\partial X_i}\dfrac{\partial T}{\partial X_q} = \dfrac{\partial \mathsf{K}_{0iq}}{\partial T}\dfrac{\partial T}{\partial X_i}\dfrac{\partial T}{\partial X_q} = \dfrac{\partial \mathbf{K}_0}{\partial T}:\left(\nabla_{\vec{X}}T\otimes\nabla_{\vec{X}}T\right)$ with which the equation in (10.173) becomes:

$$\dot{\eta}T = \frac{1}{\rho_0}\left\{\frac{\partial \mathbf{K}_0}{\partial T}:\left(\nabla_{\vec{X}}T\otimes\nabla_{\vec{X}}T\right) + \mathbf{K}_0:\left[\nabla_{\vec{X}}(\nabla_{\vec{X}}T)\right]\right\} \tag{10.174}$$

It can now be shown that the previous equation in the current configuration is given by:

$$\dot{\eta}T = \frac{1}{\rho}\left\{\frac{\partial \mathbf{K}}{\partial T}:\left(\nabla_{\vec{x}}T\otimes\nabla_{\vec{x}}T\right) + \mathbf{K}:\left[\nabla_{\vec{x}}(\nabla_{\vec{x}}T)\right]\right\} \tag{10.175}$$

and the rates of change of the equations in (10.164) are given by:

$$\begin{aligned} &\mathbf{S} = \rho_0\frac{\partial\psi(\boldsymbol{E},T)}{\partial\boldsymbol{E}} \xrightarrow{rate} \dot{\mathbf{S}} = \rho_0\left(\frac{\partial^2\psi(\boldsymbol{E},T)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}} + \frac{\partial^2\psi(\boldsymbol{E},T)}{\partial\boldsymbol{E}\otimes\partial T}\dot{T}\right) \\ &\eta = -\frac{\partial\psi(\boldsymbol{E},T)}{\partial T} \xrightarrow{rate} \dot{\eta} = -\left(\frac{\partial^2\psi(\boldsymbol{E},T)}{\partial\boldsymbol{E}\otimes\partial T}:\dot{\boldsymbol{E}} + \frac{\partial^2\psi(\boldsymbol{E},T)}{\partial T\otimes\partial T}\dot{T}\right) \end{aligned} \tag{10.176}$$

which yields:

$$T\dot{\eta} = -T\left(\frac{\partial^2\psi}{\partial \boldsymbol{E}\partial T}:\dot{\boldsymbol{E}} + \frac{\partial^2\psi}{\partial T^2}\dot{T}\right) = -T\frac{\partial}{\partial T}\left(\frac{\partial\psi}{\partial \boldsymbol{E}}\right):\dot{\boldsymbol{E}} - T\frac{\partial^2\psi}{\partial T^2}\dot{T} \tag{10.177}$$

Then, if we consider that $\mathbf{S} = \rho_0 \dfrac{\partial\psi(\boldsymbol{E},T)}{\partial \boldsymbol{E}}$, the above equation becomes:

$$T\dot{\eta} = -T\frac{\partial}{\partial T}\left(\frac{\partial\psi}{\partial \boldsymbol{E}}\right):\dot{\boldsymbol{E}} - T\frac{\partial^2\psi}{\partial T^2}\dot{T} = -\frac{T}{\rho_0}\frac{\partial}{\partial T}(\mathbf{S}):\dot{\boldsymbol{E}} - T\frac{\partial^2\psi}{\partial T^2}\dot{T} \tag{10.178}$$

We can now define a second-order tensor denoted by the latent heat tensor of change of strain, (see equation (10.51)), and given by:

$$\boxed{\boldsymbol{L}_{\boldsymbol{E}} = -\frac{T}{\rho_0}\left(\frac{\partial \mathbf{S}}{\partial T}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}} = -T\frac{\partial^2\psi}{\partial T\partial \boldsymbol{E}}} \quad \textit{Latent heat tensor of change of strain} \tag{10.179}$$

and the specific heat at a constant volume, (see subsection 10.2.2):

$$\boxed{c_{\boldsymbol{E}} = -T\left(\frac{\partial^2\psi}{\partial T^2}\right)_{\dot{\boldsymbol{E}}=\boldsymbol{0}}} \quad \textit{Specific heat at a constant volume} \tag{10.180}$$

with which we can rewrite the equation in (10.178) as follows:

$$T\dot{\eta} = \boldsymbol{L}_{\boldsymbol{E}}:\dot{\boldsymbol{E}} + c_{\boldsymbol{E}}\dot{T} \tag{10.181}$$

In Chapter 2 we obtained the relationship between the rate of change of the Green-Lagrange strain tensor ($\dot{\boldsymbol{E}}$) and the rate-of-deformation tensor ($\mathbf{D}$) as follows: $\dot{\boldsymbol{E}} = \boldsymbol{F}^T \cdot \mathbf{D} \cdot \boldsymbol{F}$, which is expressed in indicial notation as $\dot{E}_{ij} = F_{pi}\mathrm{D}_{pq}F_{qj}$ with which we can conclude that $L_{\boldsymbol{E}ij}\dot{E}_{ij} = L_{\boldsymbol{E}ij}F_{pi}\mathrm{D}_{pq}F_{qj} = F_{pi}L_{\boldsymbol{E}ij}F_{qj}\mathrm{D}_{pq} = \left(\boldsymbol{F}\cdot\boldsymbol{L}_{\boldsymbol{E}}\cdot\boldsymbol{F}^T\right):\mathbf{D}$. Additionally, by starting from the definition of $\boldsymbol{L}_{\boldsymbol{E}}$ given in (10.179) we can draw the conclusion that $\boldsymbol{F}\cdot\boldsymbol{L}_{\boldsymbol{E}}\cdot\boldsymbol{F}^T = \dfrac{-T}{\rho_0}\dfrac{\partial\left(\boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T\right)}{\partial T}$. Also, in Chapter 2 we obtained the relationship between the second Piola-Kirchhoff stress tensor ($\mathbf{S}$) and the Cauchy stress tensor ($\boldsymbol{\sigma}$) as follows: $\boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T = J\boldsymbol{\sigma}$, where $\rho J = \rho_0$ holds and by taking all of the above into consideration the equation in (10.181) can be rewritten as:

$$\begin{aligned} T\dot{\eta} &= \boldsymbol{L}_{\boldsymbol{E}}:\dot{\boldsymbol{E}} + c_{\boldsymbol{E}}\dot{T} = \left(\boldsymbol{F}\cdot\boldsymbol{L}_{\boldsymbol{E}}\cdot\boldsymbol{F}^T\right):\mathbf{D} + c_{\boldsymbol{E}}\dot{T} = \left(-\frac{T}{\rho_0}\frac{\partial\left(\boldsymbol{F}\cdot\mathbf{S}\cdot\boldsymbol{F}^T\right)}{\partial T}\right):\mathbf{D} + c_{\boldsymbol{E}}\dot{T} \\ &= \left(-\frac{JT}{\rho_0}\frac{\partial\boldsymbol{\sigma}}{\partial T}\right):\mathbf{D} + c_{\boldsymbol{E}}\dot{T} = \left(-\frac{T}{\rho}\frac{\partial\boldsymbol{\sigma}}{\partial T}\right):\mathbf{D} + c_{\boldsymbol{E}}\dot{T} \end{aligned} \tag{10.182}$$

10.5.2 The Specific Helmholtz Free Energy

Let us now assume that the stress tensor $\mathbf{S}$ varies linearly with the strain tensor $\boldsymbol{E}$, and also that the specific heat and the latent heat tensor vary linearly with temperature according to:

$$c_{\boldsymbol{E}} = c_{\boldsymbol{E}}^0 + c(T - T_0) \quad ; \quad \boldsymbol{L}_{\boldsymbol{E}} = \frac{T}{T_0}\boldsymbol{L}_{\boldsymbol{E}}^0 \tag{10.183}$$

where c is a constant while $c_{\boldsymbol{E}}^0$ and $\boldsymbol{L}_{\boldsymbol{E}}^0$ are the values of $c_{\boldsymbol{E}}$ and $\boldsymbol{L}_{\boldsymbol{E}}$, respectively, at $T = T_0$ and $\boldsymbol{E} = \boldsymbol{0}$. Then, we can represent the Helmholtz free energy (per unit mass):

$$\psi = \frac{1}{2\rho_0}\boldsymbol{E} : \mathbb{C}_0^e : \boldsymbol{E} - \left(\boldsymbol{L}_{\boldsymbol{E}}^0 : \boldsymbol{E}\right)\frac{T - T_0}{T_0} + \left(c_{\boldsymbol{E}}^0 - cT_0\right)\left[T - T_0 - T\ln\left(\frac{T}{T_0}\right)\right] - \frac{c}{2}(T - T_0)^2 \quad (10.184)$$

where $\mathbb{C}_0^e$ is the isothermal elasticity tensor (a symmetric fourth-order tensor).

The constitutive equations for stress are given by:

$$\mathbf{S} = \mathbb{C}_0^e : \boldsymbol{E} - \frac{\rho_0}{T_0}\frac{(T - T_0)}{T_0}\boldsymbol{L}_{\boldsymbol{E}}^0 \quad (10.185)$$

and the constitutive equation for entropy by:

$$\eta = \frac{1}{T_0}\boldsymbol{L}_{\boldsymbol{E}}^0 : \boldsymbol{E} + \left(c_{\boldsymbol{E}}^0 - cT_0\right)\ln\left(\frac{T}{T_0}\right) + c(T - T_0) \quad (10.186)$$

Then, if we consider:

$$\mathbb{C}_0^e : \boldsymbol{A}_0 = \frac{\rho_0}{T_0}\boldsymbol{L}_{\boldsymbol{E}}^0 \quad (10.187)$$

where $\boldsymbol{A}_0$ is the thermal expansion tensor defined in (10.76), we can obtain:

$$\boldsymbol{E} = \mathbb{C}_0^{e^{-1}} : \mathbf{S} + \boldsymbol{A}_0(T - T_0) \quad (10.188)$$

Next, in isotropic linear elastic materials, the following is satisfied:

$$\mathbb{C}_0^e = \lambda_0 \mathbf{1} \otimes \mathbf{1} + 2\mu_0 \mathbf{I} \quad ; \quad \boldsymbol{A}_0 = \alpha_0 \mathbf{1} \quad ; \quad \frac{\rho_0}{T_0}\boldsymbol{L}_{\boldsymbol{E}}^0 = 3\alpha_0 \kappa_0 \mathbf{1} \quad (10.189)$$

Then, the constitutive equations for stress and entropy become:

$$\begin{aligned} \mathbf{S} &= \lambda_0 \mathrm{Tr}(\boldsymbol{E})\mathbf{1} + 2\mu_0 \boldsymbol{E} - 3\alpha_0 (T - T_0)\kappa_0 \mathbf{1} \\ &= \lambda_0 \mathrm{Tr}(\boldsymbol{E})\mathbf{1} + 2\mu_0 \boldsymbol{E} - \alpha_0 (3\lambda_0 + 2\mu_0)(T - T_0)\mathbf{1} \end{aligned} \quad (10.190)$$

$$\eta = \frac{3}{\rho_0}\alpha_0 \kappa_0 \mathrm{Tr}(\boldsymbol{E}) + \left(c_{\boldsymbol{E}}^0 - cT_0\right)\ln\left(\frac{T}{T_0}\right) + c(T - T_0) \quad (10.191)$$

and the Green-Lagrange strain tensor is given by:

$$\boldsymbol{E} = \frac{1}{2\mu_0}\left[\mathbf{S} - \frac{\lambda_0}{3\kappa_0}\mathrm{Tr}(\mathbf{S})\mathbf{1}\right] + \alpha_0 (T - T_0)\mathbf{1} \quad (10.192)$$

10.6 Thermoelasticity based on the Multiplicative Decomposition of the Deformation Gradient

Now, we will tackle thermoelasticity in finite strain, in this subsection, by multiplicative decomposition of the deformation gradient into elastic ($\boldsymbol{F}^e$) and thermal parts ($\boldsymbol{F}^\theta$), Lubarda(2004), (see Figure 10.6). According to Vujošević&Lubarda(2002), this approach to describing thermal problems was first introduced by Stojanović. The first transformation is caused by $\boldsymbol{F}^\theta$, which then defines the intermediate configuration $\overline{\mathcal{B}}^\theta$, which is characterized by the absence of stress. After the transformation $\boldsymbol{F}^\theta$ takes place, we can carry out the transformation caused by $\boldsymbol{F}^e$. In this way, the deformation gradient can be represented by:

$$\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^\theta \tag{10.193}$$

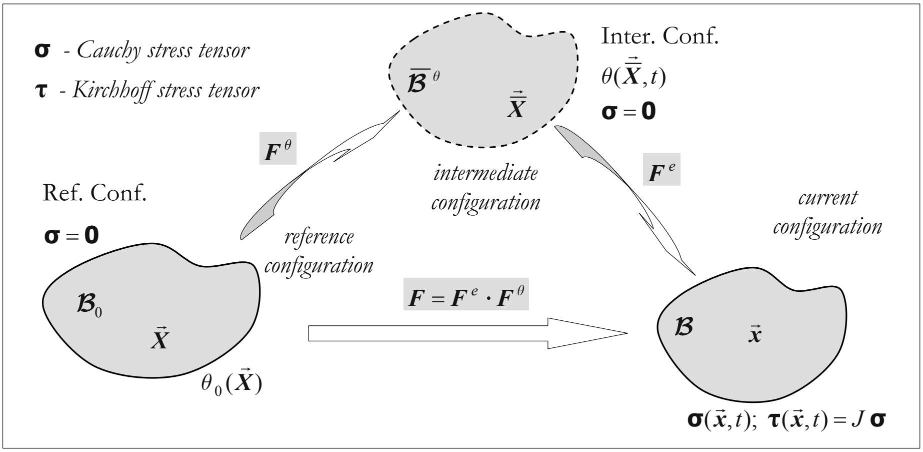


Figure 10.6: Multiplicative decomposition.

Thus, as proven in the chapter on plasticity, (see Chapter 9), multiplicative decomposition of the deformation gradient into either a plastic and elastic part ($\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^p$) or thermal and elastic parts is not unique.

OBS.: In this subsection we use the parameter θ instead of T to represent temperature so as to avoid confusion with the transpose $\boldsymbol{F}^T$.

10.6.1 Kinematic Tensors

Then, if we consider the transformations $\boldsymbol{F}^e$ and $\boldsymbol{F}^\theta$, we can define the elastic and thermal Lagrangian strain tensors as follows:

$$\boldsymbol{E}^e = \frac{1}{2}\left(\boldsymbol{F}^{e^T} \cdot \boldsymbol{F}^e - \mathbf{1}\right) \quad ; \quad \boldsymbol{E}^\theta = \frac{1}{2}\left(\boldsymbol{F}^{\theta^T} \cdot \boldsymbol{F}^\theta - \mathbf{1}\right) \tag{10.194}$$

thus, the rate of change of $\boldsymbol{E}^\theta$ can be evaluated as such:

$$\dot{\boldsymbol{E}}^\theta = \frac{1}{2}\left(\dot{\boldsymbol{F}}^{\theta^T} \cdot \boldsymbol{F}^\theta + \boldsymbol{F}^{\theta^T} \cdot \dot{\boldsymbol{F}}^\theta\right) \tag{10.195}$$

Then, by considering that $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}\right)$ we can obtain the following equation:

$$\boldsymbol{E} - \boldsymbol{E}^\theta = \frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \mathbf{1}\right) - \frac{1}{2}\left(\boldsymbol{F}^{\theta^T} \cdot \boldsymbol{F}^\theta - \mathbf{1}\right) = \frac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{F}^{\theta^T} \cdot \boldsymbol{F}^\theta\right) \tag{10.196}$$

and by using the multiplicative decomposition shown in (10.193), $\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^\theta$, the above can still be written as:

$$
\begin{aligned}
2\left(\boldsymbol{E}-\boldsymbol{E}^{\theta}\right) &= \boldsymbol{F}^{T}\cdot\boldsymbol{F}-\boldsymbol{F}^{\theta^{T}}\cdot\boldsymbol{F}^{\theta}=\left(\boldsymbol{F}^{e}\cdot\boldsymbol{F}^{\theta}\right)^{T}\cdot\left(\boldsymbol{F}^{e}\cdot\boldsymbol{F}^{\theta}\right)-\boldsymbol{F}^{\theta^{T}}\cdot\boldsymbol{F}^{\theta} \\
&= \boldsymbol{F}^{\theta^{T}}\cdot\boldsymbol{F}^{e^{T}}\cdot\boldsymbol{F}^{e}\cdot\boldsymbol{F}^{\theta}-\boldsymbol{F}^{\theta^{T}}\cdot\boldsymbol{F}^{\theta}=\boldsymbol{F}^{\theta^{T}}\cdot\left(\boldsymbol{F}^{e^{T}}\cdot\boldsymbol{F}^{e}-\mathbf{1}\right)\cdot\boldsymbol{F}^{\theta} \\
&= \boldsymbol{F}^{\theta^{T}}\cdot\left(2\boldsymbol{E}^{e}\right)\cdot\boldsymbol{F}^{\theta}
\end{aligned}
\tag{10.197}
$$

with which we can draw the conclusion that:

$$
\boxed{\boldsymbol{E}-\boldsymbol{E}^{\theta}=\boldsymbol{F}^{\theta^{T}}\cdot\boldsymbol{E}^{e}\cdot\boldsymbol{F}^{\theta}} \quad \Rightarrow \quad \boxed{\boldsymbol{E}^{e}=\boldsymbol{F}^{\theta^{-T}}\cdot\left(\boldsymbol{E}-\boldsymbol{E}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}}
\tag{10.198}
$$

Then, the rate of change of $\boldsymbol{E}^{e}$ can be evaluated as follows:

$$
\dot{\boldsymbol{E}}^{e}=\dot{\boldsymbol{F}}^{\theta^{-T}}\cdot\left(\boldsymbol{E}-\boldsymbol{E}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}+\boldsymbol{F}^{\theta^{-T}}\cdot\left(\dot{\boldsymbol{E}}-\dot{\boldsymbol{E}}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}+\boldsymbol{F}^{\theta^{-T}}\cdot\left(\boldsymbol{E}-\boldsymbol{E}^{\theta}\right)\cdot\dot{\boldsymbol{F}}^{\theta^{-1}}
\tag{10.199}
$$

Remember that the rate of change of the deformation gradient is given by $\dot{\boldsymbol{F}}=\boldsymbol{\ell}\cdot\boldsymbol{F}$, where $\boldsymbol{\ell}$ is the spatial velocity gradient, with which the equation $\dot{\boldsymbol{F}}^{-1}=-\boldsymbol{F}^{-1}\cdot\boldsymbol{\ell}$ holds. Likewise, we can define the following tensors as:

$$
\begin{cases}
\dot{\boldsymbol{F}}^{\theta}=\boldsymbol{\ell}^{\theta}\cdot\boldsymbol{F}^{\theta} \\
\Rightarrow \boldsymbol{\ell}^{\theta}=\dot{\boldsymbol{F}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}
\end{cases}
\qquad \textit{and} \qquad \dot{\boldsymbol{F}}^{\theta^{-1}}=-\boldsymbol{F}^{\theta^{-1}}\cdot\boldsymbol{\ell}^{\theta}
\tag{10.200}
$$

Now, returning to (10.199) and by taking into account the previous equations in (10.200), we can state that:

$$
\begin{aligned}
\dot{\boldsymbol{E}}^{e}=-\boldsymbol{\ell}^{\theta^{T}}\cdot\boldsymbol{F}^{\theta^{-T}}\cdot\left(\boldsymbol{E}-\boldsymbol{E}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}+\boldsymbol{F}^{\theta^{-T}}\cdot\left(\dot{\boldsymbol{E}}-\dot{\boldsymbol{E}}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}- \\
\boldsymbol{F}^{\theta^{-T}}\cdot\left(\boldsymbol{E}-\boldsymbol{E}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}\cdot\boldsymbol{\ell}^{\theta}
\end{aligned}
\tag{10.201}
$$

We can now substitute the term $\left(\boldsymbol{E}-\boldsymbol{E}^{\theta}\right)$ given by the equation in (10.198) into the above:

$$
\begin{aligned}
\dot{\boldsymbol{E}}^{e}=-\boldsymbol{\ell}^{\theta^{T}}\cdot\boldsymbol{F}^{\theta^{-T}}\cdot\boldsymbol{F}^{\theta^{T}}\cdot\boldsymbol{E}^{e}\cdot\boldsymbol{F}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}+\boldsymbol{F}^{\theta^{-T}}\cdot\left(\dot{\boldsymbol{E}}-\dot{\boldsymbol{E}}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}- \\
\boldsymbol{F}^{\theta^{-T}}\cdot\boldsymbol{F}^{\theta^{T}}\cdot\boldsymbol{E}^{e}\cdot\boldsymbol{F}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}\cdot\boldsymbol{\ell}^{\theta}
\end{aligned}
\tag{10.202}
$$

the result of which is:

$$
\begin{aligned}
\dot{\boldsymbol{E}}^{e} &= -\boldsymbol{\ell}^{\theta^{T}}\cdot\boldsymbol{E}^{e}+\boldsymbol{F}^{\theta^{-T}}\cdot\left(\dot{\boldsymbol{E}}-\dot{\boldsymbol{E}}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}}-\boldsymbol{E}^{e}\cdot\boldsymbol{\ell}^{\theta} \\
&= -\boldsymbol{\ell}^{\theta^{T}}\cdot\boldsymbol{E}^{e}+\boldsymbol{F}^{\theta^{-T}}\cdot\dot{\boldsymbol{E}}\cdot\boldsymbol{F}^{\theta^{-1}}-\boldsymbol{F}^{\theta^{-T}}\cdot\dot{\boldsymbol{E}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}-\boldsymbol{E}^{e}\cdot\boldsymbol{\ell}^{\theta}
\end{aligned}
\tag{10.203}
$$

Then, regarding the term $\boldsymbol{F}^{\theta^{-T}}\cdot\dot{\boldsymbol{E}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}$, we can substitute $\dot{\boldsymbol{E}}^{\theta}$ by means of the definition in (10.195), thus:

$$
\begin{aligned}
\boldsymbol{F}^{\theta^{-T}}\cdot\dot{\boldsymbol{E}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}} &= \boldsymbol{F}^{\theta^{-T}}\cdot\frac{1}{2}\left(\dot{\boldsymbol{F}}^{\theta^{T}}\cdot\boldsymbol{F}^{\theta}+\boldsymbol{F}^{\theta^{T}}\cdot\dot{\boldsymbol{F}}^{\theta}\right)\cdot\boldsymbol{F}^{\theta^{-1}} \\
&= \frac{1}{2}\left(\boldsymbol{F}^{\theta^{-T}}\cdot\dot{\boldsymbol{F}}^{\theta^{T}}\cdot\boldsymbol{F}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}+\boldsymbol{F}^{\theta^{-T}}\cdot\boldsymbol{F}^{\theta^{T}}\cdot\dot{\boldsymbol{F}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}\right) \\
&= \frac{1}{2}\left(\boldsymbol{F}^{\theta^{-T}}\cdot\dot{\boldsymbol{F}}^{\theta^{T}}+\dot{\boldsymbol{F}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}\right)=\frac{1}{2}\left[\left(\dot{\boldsymbol{F}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}\right)^{T}+\left(\dot{\boldsymbol{F}}^{\theta}\cdot\boldsymbol{F}^{\theta^{-1}}\right)\right] \\
&= \frac{1}{2}\left[\boldsymbol{\ell}^{\theta^{T}}+\boldsymbol{\ell}^{\theta}\right]=\boldsymbol{\ell}^{\theta^{sym}}\equiv\mathbf{D}^{\theta}
\end{aligned}
\tag{10.204}
$$

Hence, the equation in (10.203) becomes:

$$
\boxed{\dot{\boldsymbol{E}}^{e}=\boldsymbol{F}^{\theta^{-T}}\cdot\dot{\boldsymbol{E}}\cdot\boldsymbol{F}^{\theta^{-1}}-\boldsymbol{\ell}^{\theta^{T}}\cdot\boldsymbol{E}^{e}-\boldsymbol{E}^{e}\cdot\boldsymbol{\ell}^{\theta}-\boldsymbol{\ell}^{\theta^{sym}}}
\tag{10.205}
$$

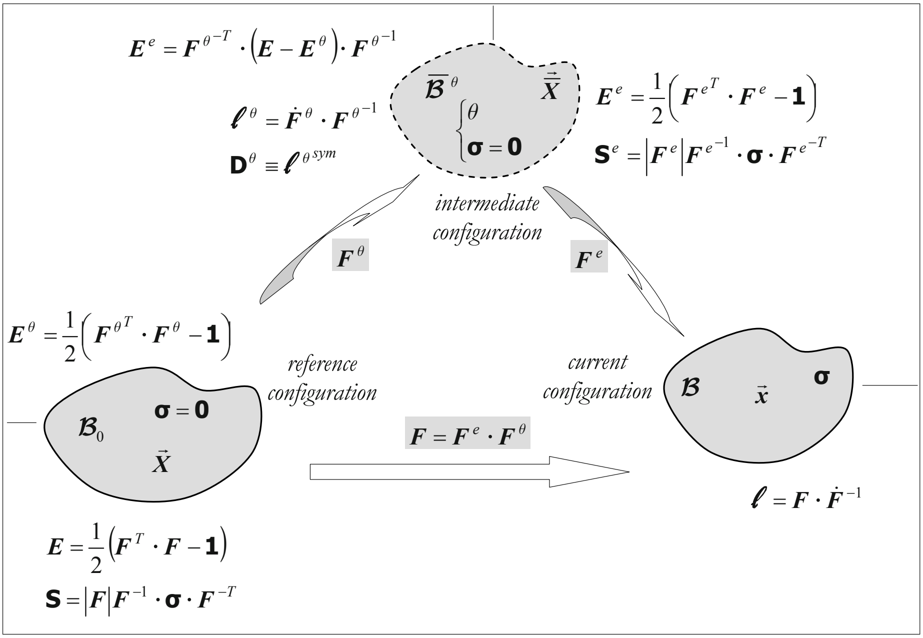


Figure 10.7: Kinematic and stress tensors in different configurations.

10.6.2 The Stress Tensor

Remember from Chapter 3 that the second Piola-Kirchhoff stress tensor is related to the Cauchy stress tensor by means of the equation $\mathbf{S} = \left|\boldsymbol{F}\right| \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$. Then, by using the multiplicative decomposition $\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^\theta$ we can obtain:

$$
\begin{aligned}
\mathbf{S} &= \left|\boldsymbol{F}\right| \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T} = \left|\boldsymbol{F}^e \cdot \boldsymbol{F}^\theta\right| \left(\boldsymbol{F}^e \cdot \boldsymbol{F}^\theta\right)^{-1} \cdot \boldsymbol{\sigma} \cdot \left(\boldsymbol{F}^e \cdot \boldsymbol{F}^\theta\right)^{-T} \\
&= \left|\boldsymbol{F}^e\right| \left|\boldsymbol{F}^\theta\right| \boldsymbol{F}^{\theta-1} \cdot \boldsymbol{F}^{e-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{e-T} \cdot \boldsymbol{F}^{\theta-T} = \left|\boldsymbol{F}^\theta\right| \boldsymbol{F}^{\theta-1} \cdot \mathbf{S}^e \cdot \boldsymbol{F}^{\theta-T}
\end{aligned}
\tag{10.206}
$$

where we have introduced:

$$
\mathbf{S}^e = \left|\boldsymbol{F}^e\right| \boldsymbol{F}^{e-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{e-T}
\tag{10.207}
$$

10.6.3 Area and Volume Elements

If we now consider the definition of the Jacobian determinant and the multiplicative decomposition of the deformation gradient, we can draw the conclusion that:

$$
J = \det(\boldsymbol{F}) = \det(\boldsymbol{F}^e \cdot \boldsymbol{F}^\theta) = \det(\boldsymbol{F}^e)\det(\boldsymbol{F}^\theta) = J^e J^\theta
\tag{10.208}
$$

Therefore, we can define the thermal Jacobian determinant J^θ and the elastic Jacobian determinant J^e, respectively, as follows:

$$
J^\theta = \det(\boldsymbol{F}^\theta) = \left[\det(\boldsymbol{C}^{\theta-1})\right]^{\frac{1}{2}} \quad ; \quad J^e = \det(\boldsymbol{F}^e) = \left[\det(\boldsymbol{b}^\theta)\right]^{\frac{1}{2}}
\tag{10.209}
$$

Then, the differential volume elements in the respective configurations, (see Figure 10.8), are given by:

$$dV^{\theta}(\vec{\bar{X}},t) = J^{\theta} dV_0(\vec{X},t) \quad ; \quad dV(\vec{x},t) = J^{e} d\bar{V}(\vec{\bar{X}},t) \tag{10.210}$$

Remember from Chapter 2 that the material time derivative of the volume element is given by $\frac{D(dV)}{Dt} = \mathrm{Tr}(\boldsymbol{\ell})dV = \mathrm{Tr}(\mathbf{D})dV$. Likewise, we can obtain the rate of change of dV^{θ} as:

$$\frac{D(dV^{\theta})}{Dt} = \mathrm{Tr}(\boldsymbol{\ell}^{\theta})dV^{\theta} = \mathrm{Tr}(\mathbf{D}^{\theta})dV^{\theta} \tag{10.211}$$

Let us now consider a differential area element in the reference configuration $d\vec{\boldsymbol{A}}_0$, (see Figure 10.8). Next, we will define the differential area elements in different configurations:

$$\boxed{d\vec{\boldsymbol{A}}^{\theta} = J^{\theta} \boldsymbol{F}^{\theta^{-T}} \cdot d\vec{\boldsymbol{A}}_0} \quad \textit{Differential area element in the intermediate configuration} \tag{10.212}$$

and

$$\boxed{d\vec{\boldsymbol{a}} = J^{e} \boldsymbol{F}^{e^{-T}} \cdot d\vec{\boldsymbol{A}}^{\theta}} \quad \textit{Differential area element in the current configuration} \tag{10.213}$$

Remember from Chapter 2 the transformation between $d\vec{\boldsymbol{A}}_0$ and $d\vec{\boldsymbol{a}}$ is given by the equation $d\vec{\boldsymbol{a}} = J\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}}_0$ (Nanson's formula). Hence, the following is valid:

$$\begin{aligned} d\vec{\boldsymbol{a}} &= J^{e} \boldsymbol{F}^{e^{-T}} \cdot d\vec{\boldsymbol{A}}^{\theta} = J^{e} \boldsymbol{F}^{e^{-T}} \cdot \left(J^{\theta} \boldsymbol{F}^{\theta^{-T}} \cdot d\vec{\boldsymbol{A}}_0 \right) = J^{e} J^{\theta} \boldsymbol{F}^{e^{-T}} \cdot \boldsymbol{F}^{\theta^{-T}} \cdot d\vec{\boldsymbol{A}}_0 \\ &= \left(J^{e} J^{\theta} \right) \left(\boldsymbol{F}^{e} \cdot \boldsymbol{F}^{\theta} \right)^{-T} \cdot d\vec{\boldsymbol{A}}_0 = J\boldsymbol{F}^{-T} \cdot d\vec{\boldsymbol{A}}_0 \end{aligned} \tag{10.214}$$

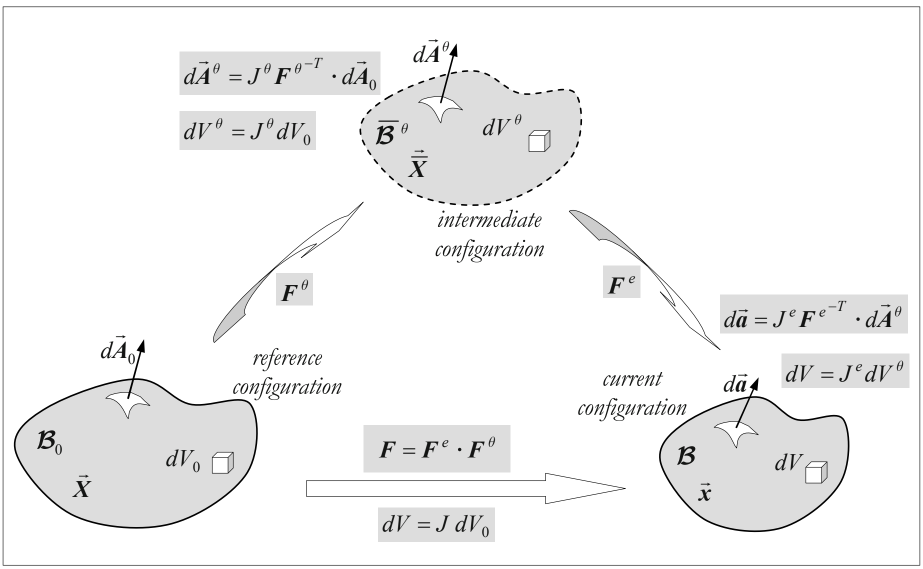


Figure 10.8: Area and volume elements deformations.

10.6.4 Isotropic Materials

Let us now consider an isotropic material in which the thermal deformation gradient is represented by:

$$\boldsymbol{F}^{\theta} = \upsilon_{\theta}\mathbf{1} \quad \Rightarrow \quad \begin{cases} \dot{\boldsymbol{F}}^{\theta} = \dfrac{d\upsilon_{\theta}}{d\theta}\dfrac{d\theta}{dt}\mathbf{1} = \dfrac{d\upsilon_{\theta}}{d\theta}\dot{\theta}\mathbf{1} \quad \text{(rate)} \\ \boldsymbol{F}^{\theta^{-1}} = \dfrac{1}{\upsilon_{\theta}}\mathbf{1} \quad \text{(inverse)} \end{cases} \tag{10.215}$$

where the scalar $\upsilon_{\theta} = \upsilon_{\theta}(\theta)$ is the *thermal stretch coefficient*. Note, an expression for υ_{θ} was proposed by Lu&Pister(1975) which is given by:

$$\upsilon_{\theta} = \exp^{\left(\int_{\theta_0}^{\theta}\alpha(\hat{\theta})d\hat{\theta}\right)} \tag{10.216}$$

where $\alpha_{\theta} = \alpha_{\theta}(\theta)$ is the linear thermal expansion coefficient, and where θ_0 and θ show the temperature in the configurations $\mathcal{B}_0$ and $\mathcal{B}$, respectively.

Now, if we consider the equation in (10.215) we can express $\boldsymbol{E}^{\theta}$ as follows:

$$\boldsymbol{E}^{\theta} = \frac{1}{2}\left(\boldsymbol{F}^{\theta^T}\cdot\boldsymbol{F}^{\theta} - \mathbf{1}\right) = \frac{1}{2}(\upsilon_{\theta}\mathbf{1}\cdot\upsilon_{\theta}\mathbf{1} - \mathbf{1}) = \frac{1}{2}(\upsilon_{\theta}^2 - 1)\mathbf{1} \tag{10.217}$$

and its rate of change by:

$$\dot{\boldsymbol{E}}^{\theta} = \frac{d}{dt}\left[\frac{1}{2}(\upsilon_{\theta}^2 - 1)\mathbf{1}\right] = \frac{1}{2}\left[2\upsilon_{\theta}\frac{d\upsilon_{\theta}}{d\theta}\frac{d\theta}{dt}\mathbf{1}\right] = \frac{d\upsilon_{\theta}}{d\theta}\upsilon_{\theta}\dot{\theta}\mathbf{1} \tag{10.218}$$

Then, if we consider the equation in (10.198) we are given:

$$\boldsymbol{E} - \boldsymbol{E}^{\theta} = \boldsymbol{F}^{\theta^T}\cdot\boldsymbol{E}^{e}\cdot\boldsymbol{F}^{\theta} = \upsilon_{\theta}\mathbf{1}\cdot\boldsymbol{E}^{e}\cdot\upsilon_{\theta}\mathbf{1} = \upsilon_{\theta}^2\boldsymbol{E}^{e} \tag{10.219}$$

or

$$\boldsymbol{E}^{e} = \frac{1}{\upsilon_{\theta}^2}\left(\boldsymbol{E} - \boldsymbol{E}^{\theta}\right) \tag{10.220}$$

Next, by substituting the expression of $\boldsymbol{E}^{\theta}$ given in (10.217) into the above equation we obtain:

$$\boldsymbol{E}^{e} = \frac{1}{\upsilon_{\theta}^2}\left(\boldsymbol{E} - \boldsymbol{E}^{\theta}\right) = \frac{1}{\upsilon_{\theta}^2}\left(\boldsymbol{E} - \frac{1}{2}(\upsilon_{\theta}^2 - 1)\mathbf{1}\right) \tag{10.221}$$

or

$$2\upsilon_{\theta}^2\boldsymbol{E}^{e} = 2\boldsymbol{E} - \upsilon_{\theta}^2\mathbf{1} + \mathbf{1} \quad \Rightarrow \quad \upsilon_{\theta}^2\left(\mathbf{1} + 2\boldsymbol{E}^{e}\right) = 2\boldsymbol{E} + \mathbf{1} \tag{10.222}$$

$$\mathbf{1} + 2\boldsymbol{E} = \upsilon_{\theta}^2\left(\mathbf{1} + 2\boldsymbol{E}^{e}\right) \tag{10.223}$$

Then, the relationship between υ_{θ} and the thermal expansion coefficient, α_{θ}, is given by:

$$\alpha_{\theta} = \alpha_{\theta}(\theta) = \frac{1}{\upsilon_{\theta}}\frac{d\upsilon_{\theta}}{d\theta} \tag{10.224}$$

Therefore, we can express the rate of change of the elastic strain tensor, $\dot{\boldsymbol{E}}^{e}$, by means of:

$$
\begin{aligned}
\dot{\boldsymbol{E}}^e &= \frac{d}{dt}\left[\frac{1}{\upsilon_\theta^2}\right]\left(\boldsymbol{E}-\boldsymbol{E}^\theta\right)+\frac{1}{\upsilon_\theta^2}\left(\dot{\boldsymbol{E}}-\dot{\boldsymbol{E}}^\theta\right)=-2\frac{1}{\upsilon_\theta^3}\frac{d\upsilon_\theta}{d\theta}\dot{\theta}\left(\boldsymbol{E}-\boldsymbol{E}^\theta\right)+\frac{1}{\upsilon_\theta^2}\left(\dot{\boldsymbol{E}}-\frac{d\upsilon_\theta}{d\theta}\upsilon_\theta\dot{\theta}\mathbf{1}\right)\\
&=\frac{1}{\upsilon_\theta^2}\left[-2\frac{1}{\upsilon_\theta}\upsilon_\theta\alpha_\theta\dot{\theta}\left(\boldsymbol{E}-\boldsymbol{E}^\theta\right)+\left(\dot{\boldsymbol{E}}-\upsilon_\theta^2\alpha_\theta\dot{\theta}\mathbf{1}\right)\right]\\
&=\frac{1}{\upsilon_\theta^2}\left[\dot{\boldsymbol{E}}-\alpha_\theta\left(2\left(\boldsymbol{E}-\boldsymbol{E}^\theta\right)+\upsilon_\theta^2\mathbf{1}\right)\dot{\theta}\right]=\frac{1}{\upsilon_\theta^2}\left[\dot{\boldsymbol{E}}-\alpha_\theta\left(2\upsilon_\theta^2\boldsymbol{E}^e+\upsilon_\theta^2\mathbf{1}\right)\dot{\theta}\right]\\
&=\frac{1}{\upsilon_\theta^2}\left[\dot{\boldsymbol{E}}-\alpha_\theta\upsilon_\theta^2\left(\mathbf{1}+2\boldsymbol{E}^e\right)\dot{\theta}\right]
\end{aligned}
\tag{10.225}
$$

and by using the equation in (10.223), we can obtain:

$$
\dot{\boldsymbol{E}}^e=\frac{1}{\upsilon_\theta^2}\left[\dot{\boldsymbol{E}}-\alpha_\theta\left(\mathbf{1}+2\boldsymbol{E}\right)\dot{\theta}\right]
\tag{10.226}
$$

Now, in isotropic materials, the spatial velocity gradient $\boldsymbol{\ell}^\theta$, (see equation (10.200)) becomes

$$
\boldsymbol{\ell}^\theta=\dot{\boldsymbol{F}}^\theta\cdot\boldsymbol{F}^{\theta^{-1}}=\left(\frac{d\upsilon_\theta}{d\theta}\dot{\theta}\mathbf{1}\right)\cdot\left(\frac{1}{\upsilon_\theta}\mathbf{1}\right)=\frac{1}{\upsilon_\theta}\frac{d\upsilon_\theta}{d\theta}\dot{\theta}\mathbf{1}
\tag{10.227}
$$

where we have used the expressions of $\dot{\boldsymbol{F}}^\theta$ and $\boldsymbol{F}^{\theta^{-1}}$ given in (10.215) with which we can obtain the rate of change of the differential volume element (dV^θ) as follows:

$$
\frac{D(dV^\theta)}{Dt}=\mathrm{Tr}(\boldsymbol{\ell}^\theta)dV^\theta=\frac{3}{\upsilon_\theta}\frac{d\upsilon_\theta}{d\theta}\dot{\theta}dV^\theta=3\alpha_\theta\dot{\theta}dV^\theta
\tag{10.228}
$$

10.6.5 The Constitutive Equations

10.6.5.1 The Constitutive Equation for Energy

Here, we will assume the specific Helmholtz free energy, ψ *(per unit mass)*, is as follows:

$$
\psi=\psi^e(\boldsymbol{E}^e,\theta)+\psi^\theta(\theta)
\tag{10.229}
$$

where ψ^e is given in terms of the strain tensor $\boldsymbol{E}^e$ and temperature θ, (see Figure 10.9), and ψ^θ can be adjusted according to the experimental results of specific heat, (see Lubarda(2004)). Next, we will evaluate the rate of change of the specific Helmholtz free energy:

$$
\dot{\psi}=\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}:\dot{\boldsymbol{E}}^e+\frac{\partial\psi^e}{\partial\theta}\dot{\theta}+\frac{d\psi^\theta}{d\theta}\dot{\theta}
\tag{10.230}
$$

Now, by substituting (10.226) into the above equation we obtain

$$
\dot{\psi}=\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}:\dot{\boldsymbol{E}}^e+\frac{\partial\psi^e}{\partial\theta}\dot{\theta}+\frac{d\psi^\theta}{d\theta}\dot{\theta}=\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}:\frac{1}{\upsilon_\theta^2}\left[\dot{\boldsymbol{E}}-\alpha_\theta\left(\mathbf{1}+2\boldsymbol{E}\right)\dot{\theta}\right]+\frac{\partial\psi^e}{\partial\theta}\dot{\theta}+\frac{d\psi^\theta}{d\theta}\dot{\theta}
\tag{10.231}
$$

and on simplifying we find:

$$
\dot{\psi}=\frac{1}{\upsilon_\theta^2}\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}:\dot{\boldsymbol{E}}-\left[\frac{\alpha_\theta}{\upsilon_\theta^2}\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}:\left(\mathbf{1}+2\boldsymbol{E}\right)-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}\right]\dot{\theta}
\tag{10.232}
$$

Then, starting from the Clausius-Planck inequality, we can express the rate of change of the energy as follows:

$$\begin{aligned}\mathbf{S}:\dot{\boldsymbol{E}}-\rho_0\eta\dot{\theta}-\rho_0\dot{\psi}\geq 0 \quad &\Rightarrow \quad -\rho_0\dot{\psi}\geq -\mathbf{S}:\dot{\boldsymbol{E}}+\rho_0\eta\dot{\theta}\\ &\Rightarrow \dot{\psi}\leq\frac{1}{\rho_0}\mathbf{S}:\dot{\boldsymbol{E}}-\eta\dot{\theta}\end{aligned} \tag{10.233}$$

and by comparing the equations (10.233) and (10.232), we obtain:

$$\boxed{\begin{aligned}&\mathbf{S}=\frac{\rho_0}{\upsilon_\theta^2}\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}\\ &\eta=\frac{\alpha_\theta}{\upsilon_\theta^2}\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}:(\mathbf{1}+2\boldsymbol{E})-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}\end{aligned}} \tag{10.234}$$

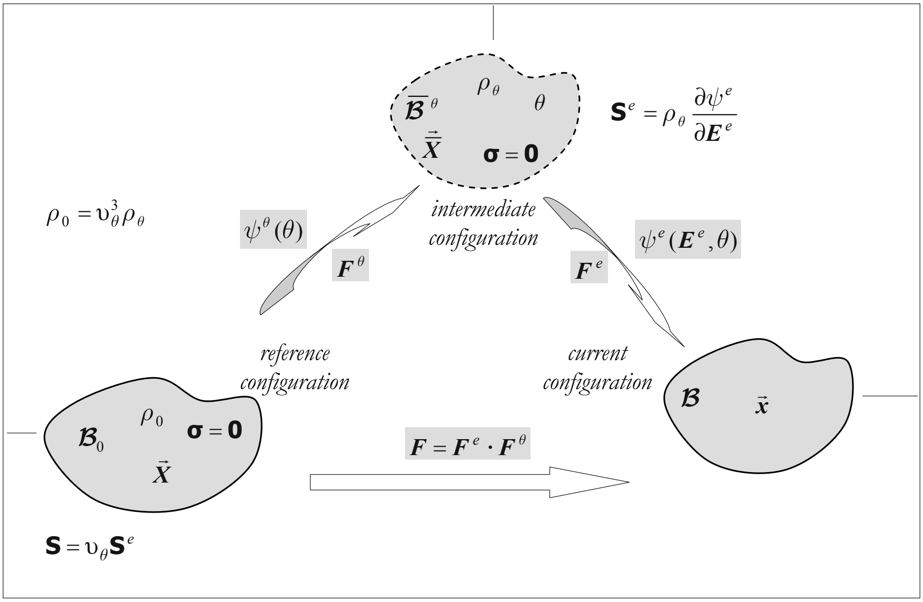


Figure 10.9: Energy, stress and mass density in different configurations.

10.6.5.2 The Constitutive Equations for Stress

If we now consider the relationship between mass density ρ_0 (reference configuration) and ρ_θ (intermediate configuration):

$$\begin{cases}\rho_0=J\rho\\ J=|\boldsymbol{F}|\end{cases} \quad\Rightarrow\quad \begin{cases}\rho_0=J^\theta\rho_\theta\\ J^\theta=|\boldsymbol{F}^\theta|=|\upsilon_\theta\mathbf{1}|=\upsilon_\theta^3>0\end{cases} \tag{10.235}$$

we can obtain

$$\rho_0=\upsilon_\theta^3\rho_\theta \tag{10.236}$$

Now, the constitutive equations for stress (10.234) can be rewritten as follows:

$$\mathbf{S} = \frac{\rho_0}{\upsilon_\theta^2}\frac{\partial \psi^e}{\partial \boldsymbol{E}^e} = \upsilon_\theta \underbrace{\rho_\theta \frac{\partial \psi^e}{\partial \boldsymbol{E}^e}}_{\mathbf{S}^e} \tag{10.237}$$

or:

$$\boxed{\mathbf{S} = \upsilon_\theta \mathbf{S}^e \qquad with \qquad \mathbf{S}^e = \rho_\theta \frac{\partial \psi^e}{\partial \boldsymbol{E}^e}} \tag{10.238}$$

Note that the equation $\mathbf{S} = \upsilon_\theta \mathbf{S}^e$ could have been directly obtained by means of that in (10.206), *i.e.:*

$$\mathbf{S} = \left|\boldsymbol{F}^\theta\right| \boldsymbol{F}^{\theta-1} \cdot \mathbf{S}^e \cdot \boldsymbol{F}^{\theta-T} = \left|\upsilon_\theta \mathbf{1}\right| (\upsilon_\theta \mathbf{1})^{-1} \cdot \mathbf{S}^e \cdot (\upsilon_\theta \mathbf{1})^{-T} = \upsilon_\theta \mathbf{S}^e \tag{10.239}$$

Let us now suppose that ψ^e is a quadratic function in terms of the elastic strain tensor as follows:

$$\rho_\theta \psi^e = \frac{1}{2}\lambda_\theta \left[\mathrm{Tr}(\boldsymbol{E}^e)\right]^2 + \mu_\theta \boldsymbol{E}^e : \boldsymbol{E}^e \tag{10.240}$$

where $\lambda_\theta = \lambda_\theta(\theta)$ and $\mu_\theta = \mu_\theta(\theta)$ are the Lamé constants which are dependent on temperature, and:

$$\mathbf{S}^e = \mathbb{C}^e_\theta : \boldsymbol{E}^e \qquad with \qquad \mathbb{C}^e_\theta = \lambda_\theta \mathbf{1} \otimes \mathbf{1} + 2\mu_\theta \mathbf{I} \tag{10.241}$$

where $\mathbb{C}^e_\theta = \mathbb{C}^e_\theta(\theta)$ is the thermal elasticity tensor, and $\mathbf{I}$ is the symmetric fourth-order unit tensor whose components are $\mathrm{I}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$.

Now, going back to the equation in (10.241) we can obtain:

$$\begin{aligned}
\mathbf{S}^e &= \mathbb{C}^e_\theta : \boldsymbol{E}^e = \left[\lambda_\theta \mathbf{1} \otimes \mathbf{1} + 2\mu_\theta \mathbf{I}\right] : \boldsymbol{E}^e = \lambda_\theta \mathbf{1} \otimes \underbrace{\mathbf{1} : \boldsymbol{E}^e}_{=\mathrm{Tr}(\boldsymbol{E}^e)} + 2\mu_\theta \underbrace{\mathbf{I} : \boldsymbol{E}^e}_{=\boldsymbol{E}^{e\,sym}} \\
&= \lambda_\theta \mathrm{Tr}(\boldsymbol{E}^e)\mathbf{1} + 2\mu_\theta \boldsymbol{E}^e
\end{aligned} \tag{10.242}$$

and if we then consider the expression of $\boldsymbol{E}^e$ given in (10.221) the following is valid:

$$\boldsymbol{E}^e = \frac{1}{\upsilon_\theta^2}\left(\boldsymbol{E} - \frac{1}{2}(\upsilon_\theta^2 - 1)\mathbf{1}\right) \xrightarrow{trace} \mathrm{Tr}(\boldsymbol{E}^e) = \frac{1}{\upsilon_\theta^2}\left[\mathrm{Tr}(\boldsymbol{E}) - \frac{3}{2}(\upsilon_\theta^2 - 1)\right] \tag{10.243}$$

Then, by substituting the above into the equation in (10.242) and by considering that $\mathbf{S} = \upsilon_\theta \mathbf{S}^e$ we obtain:

$$\begin{aligned}
\mathbf{S} &= \upsilon_\theta \mathbf{S}^e = \upsilon_\theta \left[\lambda_\theta \mathrm{Tr}(\boldsymbol{E}^e)\mathbf{1} + 2\mu_\theta \boldsymbol{E}^e\right] \\
&= \upsilon_\theta \left\{\lambda_\theta \frac{1}{\upsilon_\theta^2}\left[\mathrm{Tr}(\boldsymbol{E}) - \frac{3}{2}(\upsilon_\theta^2 - 1)\right]\mathbf{1} + 2\mu_\theta \frac{1}{\upsilon_\theta^2}\left[\boldsymbol{E} - \frac{1}{2}(\upsilon_\theta^2 - 1)\mathbf{1}\right]\right\} \\
&= \frac{1}{\upsilon_\theta}\left[\lambda_\theta \mathrm{Tr}(\boldsymbol{E})\mathbf{1} + 2\mu_\theta \boldsymbol{E}\right] - \frac{1}{2}\frac{(\upsilon_\theta^2 - 1)}{\upsilon_\theta}\left[3\lambda_\theta + 2\mu_\theta\right]\mathbf{1}
\end{aligned} \tag{10.244}$$

Now, if we consider that the bulk modulus is dependent on temperature ($\kappa_\theta = \kappa_\theta(\theta)$), which is given by $3\kappa_\theta = 3\lambda_\theta + 2\mu_\theta$, the above then becomes:

$$\mathbf{S}=\frac{1}{\upsilon_\theta}\left[\lambda_\theta \mathrm{Tr}(\boldsymbol{E})\mathbf{1}+2\mu_\theta \boldsymbol{E}\right]-\frac{3}{2}\frac{(\upsilon_\theta^2-1)}{\upsilon_\theta}\kappa_\theta\mathbf{1} \tag{10.245}$$

If we then take the following approach $\upsilon_\theta=\upsilon_\theta(\theta)\approx 1+\alpha_0(\theta-\theta_0)$ (where α_0 is the linear thermal expansion coefficient) and if in addition to that we assume that the Lamé constants are independent of temperature, the stress equation given in (10.245) becomes:

$$\begin{aligned}\mathbf{S}&=\lambda_0 \mathrm{Tr}(\boldsymbol{E})\mathbf{1}+2\mu_0\boldsymbol{E}-3\alpha_0(\theta-\theta_0)\kappa_0\mathbf{1}\\ &=\lambda_0 \mathrm{Tr}(\boldsymbol{E})\mathbf{1}+2\mu_0\boldsymbol{E}-\alpha_0(3\lambda_0+2\mu_0)(\theta-\theta_0)\mathbf{1}\end{aligned} \tag{10.246}$$

10.6.5.3 The Constitutive Equation for Entropy

According to (10.234) we can express the constitutive equation for entropy as:

$$\begin{aligned}\eta&=\frac{\alpha_\theta}{\upsilon_\theta^2}\frac{\partial\psi^e}{\partial\boldsymbol{E}^e}:(\mathbf{1}+2\boldsymbol{E})-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}=\frac{\alpha_\theta}{\rho_0}\mathbf{S}:(\mathbf{1}+2\boldsymbol{E})-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}\\ &=\frac{\alpha_\theta}{\rho_0}\mathbf{S}:\left[\upsilon_\theta^2(\mathbf{1}+2\boldsymbol{E}^e)\right]-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}=\frac{\upsilon_\theta^2\alpha_\theta}{\rho_0}\mathbf{S}:\mathbf{1}+\frac{2\upsilon_\theta^2\alpha_\theta}{\rho_0}\mathbf{S}:\boldsymbol{E}^e-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}\end{aligned} \tag{10.247}$$

where we have used the equation in (10.223): $\mathbf{1}+2\boldsymbol{E}=\upsilon_\theta^2(\mathbf{1}+2\boldsymbol{E}^e)$. Note also that the following holds:

$$\begin{aligned}\frac{1}{\upsilon_\theta}\mathbf{S}:\mathbf{1}=\mathbf{S}^e:\mathbf{1}=\mathrm{Tr}(\mathbf{S}^e)\quad&=\mathrm{Tr}\left[\lambda_\theta\mathrm{Tr}(\boldsymbol{E}^e)\mathbf{1}+2\mu_\theta\boldsymbol{E}^e\right]\\ &=3\lambda_\theta\mathrm{Tr}(\boldsymbol{E}^e)+2\mu_\theta\mathrm{Tr}(\boldsymbol{E}^e)\\ &=\left[3\lambda_\theta+2\mu_\theta\right]\mathrm{Tr}(\boldsymbol{E}^e)=3\kappa_\theta\mathrm{Tr}(\boldsymbol{E}^e)=3\kappa_\theta\mathbf{1}:\boldsymbol{E}^e\end{aligned} \tag{10.248}$$

Now, returning to the equation in (10.247) we obtain:

$$\begin{aligned}\eta&=\frac{\upsilon_\theta^2\alpha_\theta}{\rho_0}\mathbf{S}:\mathbf{1}+\frac{2\upsilon_\theta^2\alpha_\theta}{\rho_0}\mathbf{S}:\boldsymbol{E}^e-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}\\ &=\frac{3\upsilon_\theta^3\alpha_\theta\kappa_\theta}{\rho_0}\mathbf{1}:\boldsymbol{E}^e+\frac{2\upsilon_\theta^2\alpha_\theta}{\rho_0}\mathbf{S}:\boldsymbol{E}^e-\frac{\partial\psi^e}{\partial\theta}-\frac{d\psi^\theta}{d\theta}\end{aligned} \tag{10.249}$$

Note, the above term $\frac{\partial\psi^e}{\partial\theta}$ can be obtained by starting from the energy equation:

$$\rho_\theta\psi^e=\frac{1}{2}\mathbf{S}^e:\boldsymbol{E}^e\qquad\Rightarrow\qquad\rho_0\psi^e=\frac{\upsilon_\theta^3}{2}\mathbf{S}^e:\boldsymbol{E}^e \tag{10.250}$$

where we have used the equation in (10.236). Next, we will obtain the derivative of ψ^e with respect of the temperature θ:

$$\rho_0\frac{\partial\psi^e}{\partial\theta}=\frac{3}{2}\upsilon_\theta^2\frac{d\upsilon_\theta}{d\theta}\mathbf{S}^e:\boldsymbol{E}^e+\frac{\upsilon_\theta^3}{2}\frac{\partial\mathbf{S}^e}{\partial\theta}:\boldsymbol{E}^e \tag{10.251}$$

Now, if we consider the equation in (10.224) and $\mathbf{S}=\upsilon_\theta\mathbf{S}^e$, we can state that:

$$\rho_0\frac{\partial\psi^e}{\partial\theta}=\frac{3}{2}\upsilon_\theta^3\alpha_\theta\mathbf{S}^e:\boldsymbol{E}^e+\frac{\upsilon_\theta^3}{2}\frac{\partial\mathbf{S}^e}{\partial\theta}:\boldsymbol{E}^e=\frac{3}{2}\upsilon_\theta^2\alpha_\theta\mathbf{S}:\boldsymbol{E}^e+\frac{\upsilon_\theta^3}{2}\frac{\partial\mathbf{S}^e}{\partial\theta}:\boldsymbol{E}^e \tag{10.252}$$

and by using (10.221), the above equation becomes:

$$\rho_0 \frac{\partial \psi^e}{\partial \theta} = \frac{3}{2}\upsilon_\theta^2 \alpha_\theta \mathbf{S} : \left[\frac{1}{\upsilon_\theta^2}\left(\boldsymbol{E} - \frac{1}{2}(\upsilon_\theta^2 - 1)\mathbf{1} \right) \right] + \frac{\upsilon_\theta^3}{2} \frac{\partial \mathbf{S}^e}{\partial \theta} : \boldsymbol{E}^e$$
$$= \frac{3}{2}\alpha_\theta \left[\mathbf{S} : \boldsymbol{E} - \frac{1}{2}(\upsilon_\theta^2 - 1)\mathrm{Tr}(\mathbf{S}) \right] + \frac{\upsilon_\theta^3}{2} \frac{\partial \mathbf{S}^e}{\partial \theta} : \boldsymbol{E}^e \qquad (10.253)$$

Now, by substituting (10.252) into the entropy equation given in (10.249) we obtain:

$$\begin{aligned}
\eta &= \frac{3\upsilon_\theta^3 \alpha_\theta \kappa_\theta}{\rho_0} \mathbf{1} : \boldsymbol{E}^e + \frac{2\upsilon_\theta^2 \alpha_\theta}{\rho_0} \mathbf{S} : \boldsymbol{E}^e - \frac{\partial \psi^e}{\partial \theta} - \frac{d\psi^\theta}{d\theta} \\
&= \frac{3\upsilon_\theta^3 \alpha_\theta \kappa_\theta}{\rho_0} \mathbf{1} : \boldsymbol{E}^e + \frac{2\upsilon_\theta^2 \alpha_\theta}{\rho_0} \mathbf{S} : \boldsymbol{E}^e - \frac{1}{\rho_0}\left[\frac{3}{2}\upsilon_\theta^2 \alpha_\theta \mathbf{S} : \boldsymbol{E}^e + \frac{\upsilon_\theta^3}{2} \frac{\partial \mathbf{S}^e}{\partial \theta} : \boldsymbol{E}^e \right] - \frac{d\psi^\theta}{d\theta} \\
&= \left[\frac{3\upsilon_\theta^3 \alpha_\theta \kappa_\theta}{\rho_0} \mathbf{1} + \frac{2\upsilon_\theta^2 \alpha_\theta}{\rho_0} \mathbf{S} - \frac{3}{2\rho_0}\upsilon_\theta^2 \alpha_\theta \mathbf{S} - \frac{\upsilon_\theta^3}{2\rho_0} \frac{\partial \mathbf{S}^e}{\partial \theta} \right] : \boldsymbol{E}^e - \frac{d\psi^\theta}{d\theta} \\
&= \frac{1}{2\rho_0}\left[6\upsilon_\theta^3 \alpha_\theta \kappa_\theta \mathbf{1} + 4\upsilon_\theta^2 \alpha \mathbf{S} - 3\upsilon_\theta^2 \alpha_\theta \mathbf{S} - \upsilon_\theta^3 \frac{\partial \mathbf{S}^e}{\partial \theta} \right] : \boldsymbol{E}^e - \frac{d\psi^\theta}{d\theta} \\
&= \frac{1}{2\rho_0}\left[6\upsilon_\theta^3 \alpha_\theta \kappa_\theta \mathbf{1} + \upsilon_\theta^2 \alpha_\theta \mathbf{S} - \upsilon_\theta^3 \frac{\partial \mathbf{S}^e}{\partial \theta} \right] : \boldsymbol{E}^e - \frac{d\psi^\theta}{d\theta} \\
&= \frac{\upsilon_\theta^3}{2\rho_0}\left[6\alpha_\theta \kappa_\theta \mathbf{1} + \alpha_\theta \mathbf{S}^e - \frac{\partial \mathbf{S}^e}{\partial \theta} \right] : \boldsymbol{E}^e - \frac{d\psi^\theta}{d\theta}
\end{aligned} \qquad (10.254)$$

In addition, by considering that $\rho_0 = \upsilon_\theta^3 \rho_\theta$, the above equation becomes:

$$\eta = \frac{1}{2\rho_\theta}\left[6\alpha_\theta \kappa_\theta \mathbf{1} + \alpha_\theta \mathbf{S}^e - \frac{\partial \mathbf{S}^e}{\partial \theta} \right] : \boldsymbol{E}^e - \frac{d\psi^\theta}{d\theta} \qquad (10.255)$$

10.7 Thermoplasticity in a Small Deformation Regime

In this subsection we will extend the classical theory of plasticity, (see Chapter 9), in which temperature is included as a free variable.

10.7.1 The Specific Helmholtz Free Energy

In a small deformation regime, the specific Helmholtz free energy is a function of the following free variables: the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, the temperature T, and the internal variables $\overline{\boldsymbol{\alpha}}_k$, *i.e.:*

$$\psi = \psi(\boldsymbol{\varepsilon}, T, \overline{\boldsymbol{\alpha}}_i) \qquad (10.256)$$

As we have discussed in subsection 9.4.1, we can reformulate the energy equation in order to obtain the elastic part of strain $\boldsymbol{\varepsilon}^e$ as a free variable, *i.e.:*

$$\psi = \psi(\boldsymbol{\varepsilon}^e, T, \boldsymbol{\alpha}_i) \qquad (10.257)$$

Now the set of internal variables $\boldsymbol{\alpha}_i$ do not include the plastic part $\boldsymbol{\varepsilon}^p$ of the strain tensor, since this is already included in the free variable $\boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$. Then, the following is satisfied:

$$\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} - \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^p} \tag{10.258}$$

Finally, the rate of change of the free energy given in (10.257) becomes:

$$\dot{\psi} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e + \frac{\partial \psi}{\partial T} \dot{T} + \frac{\partial \psi}{\partial \boldsymbol{\alpha}_k} \square \dot{\boldsymbol{\alpha}}_k \tag{10.259}$$

10.7.2 Internal Energy Dissipation

Let us consider the Clausius-Duhem inequality:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \mathbf{D} - \rho \eta \dot{T} - \rho \dot{\psi} - \frac{1}{T} \vec{\mathbf{q}} \cdot \nabla T \geq 0 \tag{10.260}$$

Now, by substituting the rate of change of energy given in (10.259) into the above inequality and by considering that $\mathbf{D} = \dot{\boldsymbol{E}} \approx \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$ (in a small deformation regime), we obtain:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : (\dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p) - \rho \eta \dot{T} - \rho \left[\frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e + \frac{\partial \psi}{\partial T} \dot{T} + \frac{\partial \psi}{\partial \boldsymbol{\alpha}_k} \square \dot{\boldsymbol{\alpha}}_k \right] - \frac{1}{T} \vec{\mathbf{q}} \cdot \nabla T \geq 0 \tag{10.261}$$

which on simplifying yields:

$$\mathcal{D}_{int} = \left(\boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} \right) : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \rho \left(\eta + \frac{\partial \psi}{\partial T} \right) \dot{T} + \boldsymbol{A}_k \square \dot{\boldsymbol{\alpha}}_k - \frac{1}{T} \vec{\mathbf{q}} \cdot \nabla T \geq 0 \tag{10.262}$$

where $\boldsymbol{A}_k = -\rho \dfrac{\partial \psi}{\partial \boldsymbol{\alpha}_k}$ are the thermodynamic forces. As the above inequality must be valid for any thermodynamic process, (see Chapter 6), we obtain $\boldsymbol{\sigma} = \rho \dfrac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}$, $\eta = -\dfrac{\partial \psi}{\partial T}$. Then, we can summarize this model as follows:

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi(\boldsymbol{\varepsilon}^e, T, \boldsymbol{\alpha}_i)}{\partial \boldsymbol{\varepsilon}^e} \quad ; \quad \eta = \frac{\partial \psi(\boldsymbol{\varepsilon}^e, T, \boldsymbol{\alpha}_i)}{\partial T} \quad ; \quad \boldsymbol{A}_k = -\rho \frac{\partial \psi(\boldsymbol{\varepsilon}^e, T, \boldsymbol{\alpha}_i)}{\partial \boldsymbol{\alpha}_k} \tag{10.263}$$

and the internal energy dissipation becomes:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{A}_k \square \dot{\boldsymbol{\alpha}}_k - \frac{1}{T} \vec{\mathbf{q}} \cdot \nabla T \geq 0 \tag{10.264}$$

which we can the split into mechanical and thermal parts, *i.e.*:

$$\mathcal{D}_{int} = \mathcal{D}_{mechanical} + \mathcal{D}_{thermal} \tag{10.265}$$

where:

$$\mathcal{D}_{mechanical} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{A}_k \square \dot{\boldsymbol{\alpha}}_k \geq 0 \qquad \text{and} \qquad \mathcal{D}_{thermal} = -\frac{1}{T} \vec{\mathbf{q}} \cdot \nabla T \geq 0 \tag{10.266}$$

Now, by regrouping the variables we obtain:

$$\mathcal{D}_{mechanical} = \underbrace{\{\boldsymbol{\sigma} \quad \boldsymbol{A}_k\}}_{\boldsymbol{B}_\Theta}\underbrace{\{\dot{\boldsymbol{\varepsilon}}^p \quad \dot{\boldsymbol{\alpha}}_k\}^T}_{\dot{\boldsymbol{b}}_\Theta} \geq 0 \qquad \text{and} \qquad \mathcal{D}_{thermal} = \underbrace{\{\nabla T\}}_{B}\underbrace{\left\{-\frac{\vec{\mathbf{q}}}{T}\right\}}_{\dot{b}} \geq 0 \tag{10.267}$$

where $\boldsymbol{B}_\Theta$ and B include all forces, and $\dot{\boldsymbol{b}}_\Theta$, $\dot{b}$ include all variables related to flux. Then, using the dissipative potential $\varphi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}_k, \frac{\vec{\mathbf{q}}}{T})$, the complementary laws can be rewritten as follows:

$$\boldsymbol{\sigma} = \frac{\partial\varphi}{\partial\dot{\boldsymbol{\varepsilon}}^p} \quad ; \quad \boldsymbol{A}_k = \frac{\partial\varphi}{\partial\dot{\boldsymbol{\alpha}}_k} \quad ; \quad \nabla T = -\frac{\partial\varphi}{\partial\left(\dfrac{\vec{\mathbf{q}}}{T}\right)} \tag{10.268}$$

or by means of the flux variables as:

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\partial\varphi^*}{\partial\boldsymbol{\sigma}} \quad ; \quad \dot{\boldsymbol{\alpha}}_k = \frac{\partial\varphi^*}{\partial\boldsymbol{A}_k} \quad ; \quad \frac{\vec{\mathbf{q}}}{T} = -\frac{\partial\varphi^*}{\partial(\nabla T)} \tag{10.269}$$

where $\varphi^*(\boldsymbol{\sigma}, \boldsymbol{A}_k, \nabla T)$ is the dual of $\varphi(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}_k, \frac{\vec{\mathbf{q}}}{T})$ and is given by:

$$\varphi^*(\boldsymbol{\sigma}, \boldsymbol{A}_k, \nabla T) = \operatorname*{Sup}_{\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}_k, \frac{\vec{\mathbf{q}}}{T}} \left\{\left(\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{A}_k \square \dot{\boldsymbol{\alpha}}_k - \nabla T \cdot \frac{\vec{\mathbf{q}}}{T}\right) - \varphi\left(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}_k, \frac{\vec{\mathbf{q}}}{T}\right)\right\} \tag{10.270}$$

We will now introduce the potential Φ such that:

$$\dot{\boldsymbol{b}}_\Theta = \dot{\gamma}\frac{\partial\Phi}{\partial\boldsymbol{B}_\Theta} \quad ; \quad \dot{\gamma} \geq \mathbf{0} \tag{10.271}$$

and, according to the maximum plastic dissipation principle, the following is satisfied:

$$\boxed{\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial\mathcal{F}}{\partial\boldsymbol{\sigma}}} \quad \textit{Plastic flow rule} \tag{10.272}$$

where $\dot{\gamma}$ is the plastic multiplier, $\mathcal{F}$ is the yield surface (which is a convex function) and the plastic flow direction is normal to the surface $\mathcal{F}$, (see Chapter 9).

11

Damage Mechanics

11.1 Introduction

The term *Continuum Damage Mechanics* has been used to models materials which are characterized by loss of stiffness, *i.e.* by a decrease in their stiffness modulus. Damage models have also been used to simulate different materials (fragile and ductile), which are fundamentally characterized by irreversible material degradation. Physically speaking, we can describe the degradation of mechanical material properties as processes in which the initiation and growth (propagation) of micro-defects such as micro pores and microcracks take place.

In the pioneering work of Kachanov (1958) the concept of *effective stress* was introduced, and by using continuum damage he solved problems related to creep in metals. Rabotnov (1963) gave the problem physical meaning by suggesting we measure how the sectional area has reduced by means of the damage parameter. Nowadays, Continuum Damage Mechanics has become an important tool and is a consistent theory based on irreversible thermodynamic processes (the Clausius-Duhem inequality). Thermodynamic formalism was developed by Lemaitre&Chaboche (1985) and among important contributors to our knowledge about damage mechanics we can include: Mazars (1986), Mazars&Pijaudier-Cabot (1985), Chaboche (1979), Simo&Ju (1987 a,b), Ju(1989), Oliver *et al.* (1990) and Oller *et al.* (1990).

The continuum damage models, from a computational point of view, are very attractive since these present simple algorithms and are satisfactory for solving large problems.

In this chapter we will present some basic damage models used to study the failure mechanism after which we can develop more complex ones.

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11.2 The Isotropic Damage Model in a Small Deformation Regime

Continuum damage models have been widely accepted for simulating the behavior of materials whose mechanical properties are degrading due to the presence of small cracks that propagate during loading. To fully describe this phenomenon, we will first use a one-dimensional model (1D) which we will then extrapolate to three dimensional ones (3D). With regard to continuum kinematics, our study in this section will be carry out in a small deformation regime, and will be based on the lecture notes of Prof. Javier Oliver, Universitat Politècnica de Catalunya.

11.2.1 Description of the Isotropic Damage Model in Uniaxial Cases

Let us now suppose that a material point is subjected to the stress state as shown in Figure 11.1, whose apparent stress (σ) acts on the section s and due to the presence of faults (microcracks), only the undamaged region will be considered, *i.e.* the effective section ($\bar{s}$) on which the effective stress ($\bar{\sigma}$) acts.

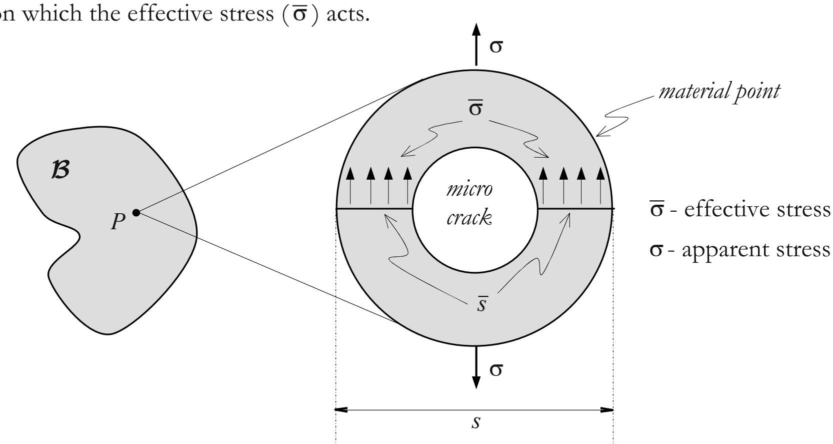


Figure 11.1: Continuum with microcracks.

Then, if we consider the force balance in Figure 11.1, we obtain:

$$s\sigma = \bar{s}\,\bar{\sigma} \tag{11.1}$$

The equation (11.1) can also be rewritten without altering its outcome as follows:

$$\sigma = \frac{\bar{s}}{s}\bar{\sigma} = \bar{\sigma}\left(\frac{\bar{s}-s}{s}+1\right) = \left(1-\frac{s-\bar{s}}{s}\right)\bar{\sigma} = \left(1-\frac{s_d}{s}\right)\bar{\sigma} \tag{11.2}$$

where s_d is the damaged section.

Note, the expression $\frac{s_d}{s}$ represents the amount of the original section which is corrupted, which in extreme cases, assumes the follows values:

- $s_d = 0 \Rightarrow \frac{s_d}{s} = 0 \Rightarrow \sigma = \overline{\sigma}$ - The section is not damaged;
- $s_d = s \Rightarrow \frac{s_d}{s} = 1 \Rightarrow \sigma = 0$ - The section is completely damaged.

The amount s_d depends on the stress state σ or indirectly on ε. The dimensionless ratio $\frac{s_d}{s}$ represents the *damage variable* and is denoted by $d = \frac{s_d}{s}$. Then, the equation in (11.2) can be written as:

$$\boxed{\sigma = (1-d)\overline{\sigma} \quad ; \quad 0 \le d \le 1} \tag{11.3}$$

where $\overline{\sigma}$ is the *effective stress*.

11.2.1.1 The Constitutive Equation

The effective stress $\overline{\sigma}$ and strain, in the undamaged area element, are interrelated by Hooke's law as:

$$\overline{\sigma} = E\,\varepsilon \tag{11.4}$$

where E is Young's modulus. Then, by substituting (11.3) into (11.4) we can obtain the constitutive equation for stress in the one-dimensional isotropic damage model:

$$\sigma = (1-d)E\,\varepsilon \qquad\qquad 0 \le d \le 1 \tag{11.5}$$

We can now verify that as the damage variable evolves, the state no longer returns to its original value. Physically speaking, we can interpret this as once the material has suffered damage this will be permanent. Hence, we can conclude that $\dot{d} \ge 0$, which characterizes an irreversible process. Now, the equation in (11.5) can still be written as:

$$\sigma = E^{sec\_d}\varepsilon \quad \text{with} \quad E^{sec\_d} = (1-d)E \tag{11.6}$$

where E^{sec_d} is the *damage secant stiffness modulus* with which we can observe that the damage variable can be interpreted as a measure of the loss of stiffness modulus of the material.

In general, materials have a yield stress that separates the elastic (reversible process) from the inelastic zone (irreversible process). In the strain space, we can represent the elastic limit by the variable ε_0, (see Figure 11.2), in which the damage process has not yet begun, *i.e.*:

$$d = 0 \qquad \text{if} \qquad \varepsilon < \varepsilon_0 \tag{11.7}$$

In the elastic region, the following is satisfied:

- $\dot{s}_d = 0 \quad \Rightarrow \quad \dot{d} = 0$;
- $\varepsilon \le \varepsilon_0$ where ε_0 is the threshold that defines the elastic region.

In a representative stress-strain curve, (see Figure 11.2), during an unloading process $(\dot{d} = 0)$, the secant modulus is given by $E^{sec\_d} = (1-d)E$ and after unloading is complete, there is no residual (permanent) strain, (see Figure 11.2), although the material has suffered some internal damaged.

We can now summarize the basic features of the one-dimensional damage model as follows:

$$\begin{aligned} \sigma &= (1-d)E\,\varepsilon \quad ; \quad (0 \le d \le 1) \\ d &= 0 \quad \text{if} \quad \varepsilon < \varepsilon_0 \end{aligned} \tag{11.8}$$

Now, by starting from the above equation we can obtain the energy equation in the system as follows:

$$\Psi = \frac{1}{2}\varepsilon\sigma = (1-d)\underbrace{\frac{1}{2}\varepsilon\,E\,\varepsilon}_{=\Psi^e} \quad \Rightarrow \quad \Psi = (1-d)\Psi^e \tag{11.9}$$

where Ψ^e is the elastic strain energy.

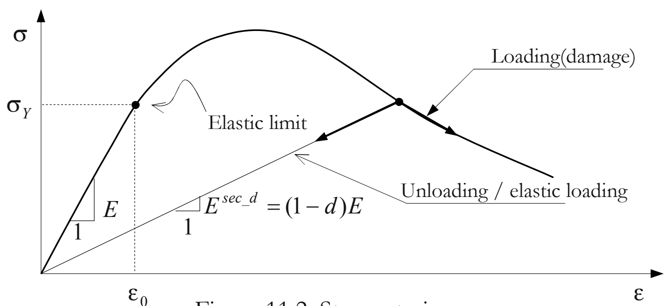


Figure 11.2: Stress-strain curve.

11.2.2 The Three-Dimensional Isotropic Damage Model

The basis of this damage model is to define a transformation between physical (real) and fictitious spaces (effective space) in which the material is undamaged, (see Figure 11.3).

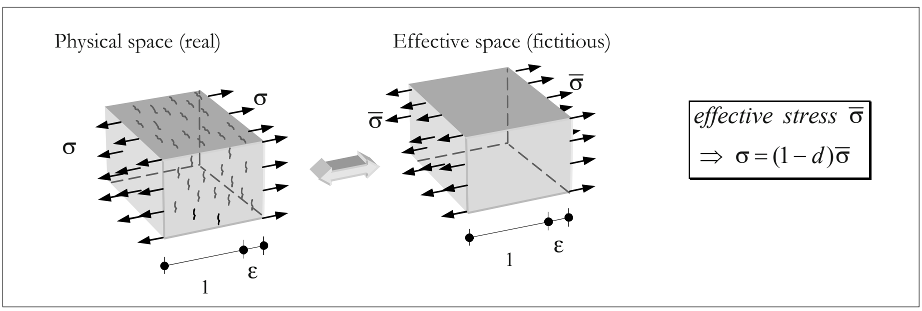


Figure 11.3: Real and fictitious spaces.

NOTE: As described above, this model depends on a single variable: the *damage parameter* d, which means that we are assuming a mechanical behavior in which the degradation is independent of the orientation, and because of this, this model is referred to as the *Isotropic Damage Model*. As a result of this, the fourth-order damaged elasticity tensor remains an isotropic tensor. ■

11.2.2.1 Helmholtz Free Energy

Let us now consider the Helmholtz free energy function $\Psi = \Psi(\boldsymbol{F}, T, \boldsymbol{\alpha}_k)$, or simply free energy, which is a function of the deformation gradient ($\boldsymbol{F}$), temperature (T) and the set of internal variables ($\boldsymbol{\alpha}_k$). Let us also consider there is a process independent of temperature, and the internal variable associated with the problem is characterized by the damage variable d. Furthermore, as seen in previous chapters, as the Helmholtz free energy must satisfy the principle of objectivity (see Chapter 6), we can express Ψ in terms of the Green-Lagrange strain tensor ($\boldsymbol{E}$), which in turn collapses with the infinitesimal strain tensor in a small deformation regime, *i.e.* $\boldsymbol{E} \approx \boldsymbol{\varepsilon}$. Then, if we consider all of the above, the Helmholtz free energy can be expressed in terms of:

$$\Psi = \Psi(\boldsymbol{\varepsilon}, d) \tag{11.10}$$

or explicitly as follows:

$$\boxed{\Psi = (1-d)\Psi^e = (1-d)\frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}} \quad \textit{Helmholtz free energy for isotropic damage model} \tag{11.11}$$

where $\Psi^e(\boldsymbol{\varepsilon})$ is the elastic strain energy density, which is a function of strain only, and $\mathbb{C}^e$ is the elasticity tensor (or elastic stiffness tensor).

11.2.2.2 Internal Energy Dissipation and the Constitutive Equations

The damage model has thermodynamic consistency, and so, entropy inequality is fulfilled. One way to express this entropy inequality is by means of the alternative form of the Clausius-Planck inequality, (see Chapter 5), which is expressed by:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \mathbf{D} - \left[\eta \dot{T} + \dot{\Psi}\right] \geq 0 \qquad \left[\frac{J}{m^3}\right] \tag{11.12}$$

Note that the terms $\boldsymbol{\sigma} : \mathbf{D}$, $\eta \dot{T}$, $\dot{\Psi}$ have the unit of energy per unit volume (density energy). In a small deformation regime $\mathbf{D} \approx \dot{\boldsymbol{\varepsilon}}$ holds, and by considering the isothermal process we have $\dot{T} = 0$, so, the equation in (11.12) becomes:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \geq 0 \tag{11.13}$$

Then, the rate of change of the free energy $\Psi = \Psi(\boldsymbol{\varepsilon}, d)$ can be evaluated as follows:

$$\dot{\Psi}(\boldsymbol{\varepsilon}, d) = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \Psi}{\partial d}\dot{d} \tag{11.14}$$

Next, by substituting (11.14) into the internal energy dissipation given in (11.13) we obtain:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi}(\boldsymbol{\varepsilon}, d) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Psi}{\partial d}\dot{d} = \left[\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}\right] : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Psi}{\partial d}\dot{d} \geq 0 \tag{11.15}$$

Note that the above inequality must hold for any admissible thermodynamic process, so, let us assume there is one where $\dot{d} = 0$. Here, we obtain $\mathcal{D}_{int} = \left[\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}\right] : \dot{\boldsymbol{\varepsilon}} \geq 0$, which in turn must also be true for any process. Additionally, if we have a process such that $\dot{\boldsymbol{\varepsilon}} \to -\dot{\boldsymbol{\varepsilon}}$, the only way for the entropy inequality to be satisfied is when $\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}$ holds with which we obtain the *constitutive equation for stress*. Thus, the entropy inequality becomes:

$$\mathcal{D}_{int} = \underbrace{\left[\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}\right]}_{=\mathbf{0}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Psi}{\partial d}\dot{d} = -\frac{\partial \Psi}{\partial d}\dot{d} \geq 0 \tag{11.16}$$

Now, if we consider the energy equation $\Psi = (1-d)\Psi^e$, we obtain $\dfrac{\partial \Psi}{\partial d} = -\Psi^e$, thus

$$\mathcal{D}_{int} = \Psi^e \dot{d} \ge 0 \tag{11.17}$$

where by definition $\Psi^e \ge 0$. Then, to satisfy the inequality (11.17), the rate of change of the damage parameter must satisfy:

$$\dot{d} \ge 0 \tag{11.18}$$

Then, by means of thermodynamic considerations we can draw the conclusion that:

$$\boxed{\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \quad ; \quad \dot{d} \ge 0} \tag{11.19}$$

We can also express the rate of change of the Helmholtz free energy by means of the equation in (11.11), *i.e.*:

$$\dot{\Psi} = \dot{\Psi}^e (1-d) - \dot{d}\,\Psi^e = (1-d)\boldsymbol{\varepsilon} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \dot{d}\,\Psi^e = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{d}\,\Psi^e \tag{11.20}$$

Next, the rate of change of the elastic strain energy, $\Psi^e = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}$, was obtained as follows:

$$\dot{\Psi}^e = \frac{1}{2}\left(\dot{\boldsymbol{\varepsilon}} : \mathbb{C}^e : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \dot{\mathbb{C}}^e : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}\right) \tag{11.21}$$

where $\dot{\mathbb{C}}^e = \boldsymbol{0}$, since $\mathbb{C}^e$ is constant, and as the elasticity tensor features major symmetry ($\mathbb{C}^e_{klij} = \mathbb{C}^e_{ijkl}$), the equation in (11.21) becomes:

$$\begin{aligned}\dot{\Psi}^e &= \frac{1}{2}\left(\dot{\varepsilon}_{ij}\mathbb{C}^e_{ijkl}\varepsilon_{kl} + \varepsilon_{ij}\mathbb{C}^e_{ijkl}\dot{\varepsilon}_{kl}\right) = \frac{1}{2}\left(\dot{\varepsilon}_{kl}\mathbb{C}^e_{klij}\varepsilon_{ij} + \varepsilon_{ij}\mathbb{C}^e_{ijkl}\dot{\varepsilon}_{kl}\right)\\ &= \varepsilon_{ij}\mathbb{C}^e_{ijkl}\dot{\varepsilon}_{kl} = \boldsymbol{\varepsilon} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} = \overline{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} = \frac{1}{(1-d)}\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\end{aligned} \tag{11.22}$$

Note that due to the major symmetry of $\mathbb{C}^e$, $\boldsymbol{\varepsilon} : \mathbb{C}^e = \mathbb{C}^e : \boldsymbol{\varepsilon}$ is fulfilled.

Then, starting from the equation in (11.11) we can obtain the stress by taking the derivative of the strain energy with respect to strain, *i.e.*:

$$\begin{aligned}\sigma_{ij} &= \frac{\partial \Psi(\boldsymbol{\varepsilon}, d)}{\partial \varepsilon_{ij}} = \frac{\partial}{\partial \varepsilon_{ij}}\left[(1-d)\frac{1}{2}\varepsilon_{pq}\mathbb{C}^e_{pqkl}\,\varepsilon_{kl}\right] = (1-d)\frac{1}{2}\mathbb{C}^e_{pqkl}\frac{\partial}{\partial \varepsilon_{ij}}\left[\varepsilon_{pq}\,\varepsilon_{kl}\right]\\ &= (1-d)\frac{1}{2}\mathbb{C}^e_{pqkl}\left(\varepsilon_{kl}\frac{\partial \varepsilon_{pq}}{\partial \varepsilon_{ij}} + \varepsilon_{pq}\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{ij}}\right)\\ &= (1-d)\frac{1}{2}\mathbb{C}^e_{pqkl}\left\{\varepsilon_{kl}\frac{\partial\left[\frac{1}{2}(\varepsilon_{pq} + \varepsilon_{qp})\right]}{\partial \varepsilon_{ij}} + \varepsilon_{pq}\frac{\partial\left[\frac{1}{2}(\varepsilon_{kl} + \varepsilon_{lk})\right]}{\partial \varepsilon_{ij}}\right\}\\ &= (1-d)\frac{1}{2}\mathbb{C}^e_{pqkl}\left\{\frac{1}{2}\varepsilon_{kl}\left(\delta_{pi}\delta_{qj} + \delta_{qi}\delta_{pj}\right) + \frac{1}{2}\varepsilon_{pq}\left(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}\right)\right\}\\ &= (1-d)\frac{1}{2}\left\{\frac{1}{2}\varepsilon_{kl}\left(\mathbb{C}^e_{ijkl} + \mathbb{C}^e_{jikl}\right) + \frac{1}{2}\varepsilon_{pq}\left(\mathbb{C}^e_{pqij} + \mathbb{C}^e_{pqji}\right)\right\} = (1-d)\frac{1}{2}\left\{\varepsilon_{kl}\mathbb{C}^e_{ijkl} + \varepsilon_{pq}\mathbb{C}^e_{pqij}\right\}\end{aligned} \tag{11.23}$$

where we have taken into account the minor symmetry of the elasticity tensor, *i.e.* $\mathbb{C}^e_{ijkl} = \mathbb{C}^e_{jikl}$, $\mathbb{C}^e_{pqij} = \mathbb{C}^e_{pqji}$. Note also that the indexes p, q are dummy indexes, so we can exchange them for k and l without altering the expression. Additionally, by taking into account the major symmetry of the elasticity tensor, $\mathbb{C}^e_{ijkl} = \mathbb{C}^e_{klij}$, we obtain:

$$\sigma_{ij} = (1-d)\mathbb{C}^e_{ijkl}\varepsilon_{kl} \tag{11.24}$$

which in tensorial notation becomes:

$$\boxed{\boldsymbol{\sigma} = \frac{\partial \Psi(\boldsymbol{\varepsilon}, d)}{\partial \boldsymbol{\varepsilon}} = (1-d)\mathbb{C}^e : \boldsymbol{\varepsilon} = (1-d)\bar{\boldsymbol{\sigma}}} \quad \textit{The constitutive equations for isotropic damage model} \tag{11.25}$$

where $\bar{\boldsymbol{\sigma}}$ is the *effective Cauchy stress tensor* and is defined as:

$$\boxed{\bar{\boldsymbol{\sigma}} = \mathbb{C}^e : \boldsymbol{\varepsilon}} \quad \textit{The effective Cauchy stress tensor} \tag{11.26}$$

and $\mathbb{C}^e$ is the elasticity tensor (fourth-order definite positive tensor) which contains the elastic mechanical properties. Remember that $\mathbb{C}^e$ can be represented in terms of the Lamé constants (λ, μ) as follows:

$$\mathbb{C}^e = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I} \quad ; \quad \mathbb{C}^e_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{11.27}$$

where $\mathbf{1}$ is the second-order unit tensor, and $\mathbf{I} \equiv \mathbb{I}^{sym}$ is the symmetric fourth-order unit tensor, whose components are expressed in terms of the Kronecker delta (δ_{ij}) as follows:

$$(\mathbf{1})_{ij} = \delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases} \quad ; \quad \mathbb{I}^{sym}_{ijkl} \equiv (\mathbf{I})_{ijkl} = \mathrm{I}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{11.28}$$

Then, by analyzing the constitutive equation in (11.25) we can put in evidence the following sentences:

- Since the damage parameter is a scalar, the stiffness degradation is isotropic;
- We can calculate the stress immediately once we know the current values of $\boldsymbol{\varepsilon}$ (strain) and d (internal variable);
- We can interpret the equation in (11.25) as the sum of elastic and inelastic parts, *i.e.*:

$$\boldsymbol{\sigma} = (1-d)\mathbb{C}^e : \boldsymbol{\varepsilon} = \underbrace{\mathbb{C}^e : \boldsymbol{\varepsilon}}_{elastic} - \underbrace{d\,\mathbb{C}^e : \boldsymbol{\varepsilon}}_{inelastic} = \boldsymbol{\sigma}^e - \boldsymbol{\sigma}^i \tag{11.29}$$

The Elastic-Damage Secant Stiffness Tensor

We can then define the elastic-damage secant stiffness tensor for the isotropic damage model as:

$$\boxed{\mathbb{C}^{sec\_d} = (1-d)\mathbb{C}^e} \quad \textit{The elastic-damage secant stiffness tensor} \tag{11.30}$$

Let us now consider a uniaxial case, (see Figure 11.4), where the material is loaded until the stress state reaches the point P represented in Figure 11.4, after which unloading occurs, with the unloading path being that indicated by the slope $E^{sec\_d} = (1-d)E$ defined in Figure 11.4.

11.2.2.3 "Ingredients" of the Damage Model

The damage constitutive model is completely determined when the damage variable d^t is known at each time step t of the loading/unloading process. Then, we can define the following elements of the constitutive equation:

- The energy norm of the stress (or strain) tensor;
- The damage surface and damage criterion. The damage surface defines the elastic limit, and the damage criterion establishes when the material is in a loading or in a elastic process, and;
- A set of evolution laws for internal variables.

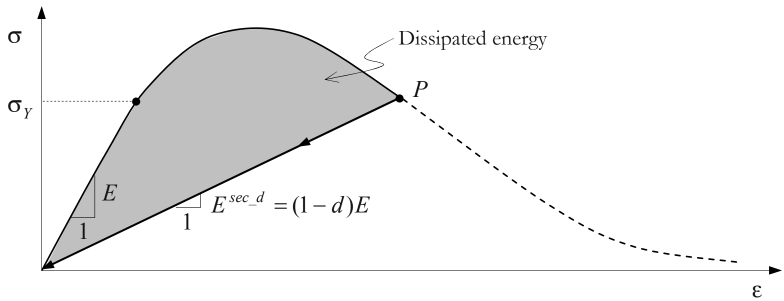


Figure 11.4: Stress-strain curve.

The Energy Norm in the Stress/Strain Space

The norm is a measure of distance and so is a scalar. Next, we will define a simple norm in the stress space denoted by $\mathcal{T}_{\boldsymbol{\sigma}}$ (equivalent stress), and in the strain space denoted by $\mathcal{T}_{\boldsymbol{\varepsilon}}$. The latter is also known as the *equivalent strain*:

$$\underbrace{\mathcal{T}_{\boldsymbol{\sigma}} = \| \boldsymbol{\sigma} \|_{\mathbb{C}^{e^{-1}}} = \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} \quad ; \quad \mathcal{T}_{\boldsymbol{\varepsilon}} = \| \boldsymbol{\varepsilon} \|_{\mathbb{C}^{e}} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^{e} : \boldsymbol{\varepsilon}} = \sqrt{2\Psi^{e}}}_{\Downarrow \atop \mathcal{T}_{\boldsymbol{\sigma}} = (1-d)\mathcal{T}_{\boldsymbol{\varepsilon}}} \tag{11.31}$$

Note that $\mathcal{T}_{\boldsymbol{\sigma}}$ and $\mathcal{T}_{\boldsymbol{\varepsilon}}$ are surface equations (ellipsoids) that characterize the stress state at the current point (see Figure 11.5). The proof of (11.31) now follows:

$$\left.\begin{aligned} \mathcal{T}_{\boldsymbol{\sigma}} &= \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} = \sqrt{(1-d)\boldsymbol{\sigma} : \boldsymbol{\varepsilon}} = \sqrt{(1-d)^2 \bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}} = (1-d)\sqrt{\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}} \\ \mathcal{T}_{\boldsymbol{\varepsilon}} &= \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^{e} : \boldsymbol{\varepsilon}} = \sqrt{\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}} \end{aligned}\right\} \Rightarrow \mathcal{T}_{\boldsymbol{\sigma}} = (1-d)\mathcal{T}_{\boldsymbol{\varepsilon}} \tag{11.32}$$

In order to better describe material behavior, others norms will be introduced (see subsection 11.2.4).

The Damage Criterion

Next we will define the damage criterion in the stress and strain space:

$$\underbrace{\mathcal{F}(\mathcal{T}_{\boldsymbol{\sigma}}, q) = \mathcal{T}_{\boldsymbol{\sigma}} - q(r) \le 0}_{stress \quad space} \quad \text{and} \quad \underbrace{\mathcal{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = \mathcal{T}_{\boldsymbol{\varepsilon}} - r \le 0}_{strain \quad space} \tag{11.33}$$

where r is an internal variable (current damage threshold), and q is a stress-like hardening/softening variable which is a function of r. Note that each material in its undamaged state is characterized by the initial value of r which is denoted by r_0 (the material parameter), which defines the initial yield in the strain space. Then, the material starts to fail (initial damage) when the energy norm exceeds the value r_0. Later we will relate the variables r and q to the damage variable.

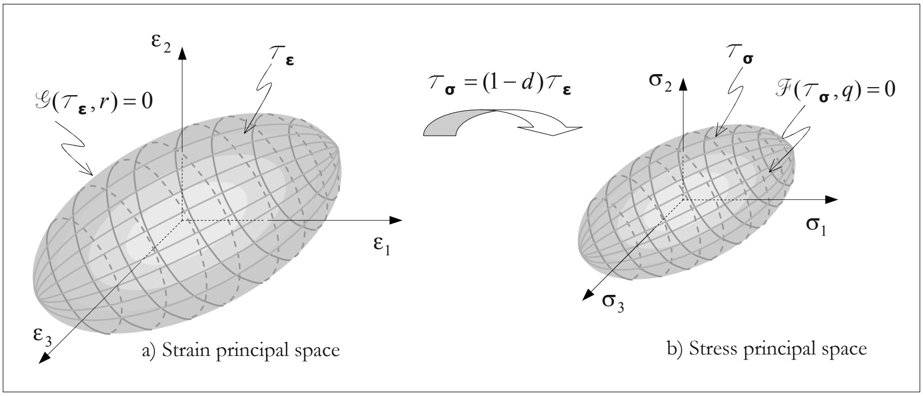


Figure 11.5: Strain and stress state in the principal space.

The damage criterion requires that the current stress state must be on or inside the damage surface. When the stress state lies inside said damage surface, the material shows elastic behavior, which can be elastic loading or unloading.

Then we can define the *admissible strain space* as follows:

$$\mathcal{E}_{\boldsymbol{\varepsilon}} := \left\{ \boldsymbol{\varepsilon} \;\middle|\; \mathcal{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) \le 0 \right\} \tag{11.34}$$

and the *admissible stress space* as:

$$\mathcal{E}_{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\sigma} \;\middle|\; \mathcal{F}(\mathcal{T}_{\boldsymbol{\sigma}}, q) \le 0 \right\} \tag{11.35}$$

When it holds that $\mathcal{F}(\mathcal{T}_{\boldsymbol{\sigma}}, q) = 0$, in the stress space, the stress state is on the surface as indicated in Figure 11.5(b).

The stress space ($\mathcal{E}_{\boldsymbol{\sigma}}$), (see Eq. (11.35)), can be decomposed into the inner domain $\mathrm{int}(\mathcal{E}_{\boldsymbol{\sigma}})$ (when the stress state is inside the surface), and other by the surface itself, $\partial\mathcal{E}_{\boldsymbol{\sigma}}$. We can define then the elastic region in strain and stress respectively as:

$$\mathrm{int}\mathcal{E}_{\boldsymbol{\varepsilon}} := \left\{ \boldsymbol{\varepsilon} \;\middle|\; \mathcal{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) < 0 \right\} \quad ; \quad \mathrm{int}\mathcal{E}_{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\sigma} \;\middle|\; \mathcal{F}(\mathcal{T}_{\boldsymbol{\sigma}}, q) \le 0 \right\} \tag{11.36}$$

and the elastic limit (damage surface):

$$\partial\mathcal{E}_{\boldsymbol{\varepsilon}} := \left\{ \boldsymbol{\varepsilon} \;\middle|\; \mathcal{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0 \right\} \quad ; \quad \partial\mathcal{E}_{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\sigma} \;\middle|\; \mathcal{F}(\mathcal{T}_{\boldsymbol{\sigma}}, q) = 0 \right\} \tag{11.37}$$

where it holds that:

$$\mathcal{E}_{\boldsymbol{\sigma}} = \mathrm{int}(\mathcal{E}_{\boldsymbol{\sigma}}) \cup \partial\mathcal{E}_{\boldsymbol{\sigma}} \quad ; \quad \mathrm{int}(\mathcal{E}_{\boldsymbol{\sigma}}) \cap \partial\mathcal{E}_{\boldsymbol{\sigma}} = \varnothing \tag{11.38}$$

Note that $\mathrm{int}(\mathcal{E}_{\boldsymbol{\sigma}})$ is the same as $\mathcal{F}(\mathcal{T}_{\boldsymbol{\sigma}}, q) < 0$ which describes the elastic region, and $\partial\mathcal{E}_{\boldsymbol{\sigma}}$ is the damage surface. Note that when the stress state is at a point inside of the space $\mathcal{E}_{\boldsymbol{\sigma}}$ it will also be inside the space $\mathcal{E}_{\boldsymbol{\varepsilon}}$, and when the stress state is on the surface $\partial\mathcal{E}_{\boldsymbol{\sigma}}$ it will also be on the surface $\partial\mathcal{E}_{\boldsymbol{\varepsilon}}$. Hence, we can use either the stress or strain space to describe how the damage evolves, proof of which follows:

$$\mathcal{T}_{\boldsymbol{\sigma}} - q(r) \le 0 \quad \Rightarrow \quad (1-d)\mathcal{T}_{\boldsymbol{\varepsilon}} - (1-d)r \le 0 \quad \Rightarrow \quad \mathcal{T}_{\boldsymbol{\varepsilon}} - r \le 0 \tag{11.39}$$

Said damage evolves when the norm $\mathcal{T}_{\boldsymbol{\varepsilon}}$ exceeds the maximum value reached by r. Then, considering (11.33) and (11.31) we can also conclude that:

$$q(r) = (1-d)r \tag{11.40}$$

In uniaxial cases, damage starts when $\mathcal{T}_{\boldsymbol{\varepsilon}}$ exceeds the first damage threshold value r_0. Then, from the equation in (11.31) and by means of Figure 11.2, we can obtain:

$$\begin{aligned} &\mathcal{T}_{\boldsymbol{\varepsilon}} = \sqrt{\boldsymbol{\varepsilon}:\mathbb{C}^e:\boldsymbol{\varepsilon}} \xrightarrow{\text{uniaxial}} \mathcal{T}_{\boldsymbol{\varepsilon}} = \sqrt{\varepsilon_0 E \varepsilon_0} = \varepsilon_0\sqrt{E} = \frac{\sigma_Y}{E}\sqrt{E} = \frac{\sigma_Y}{\sqrt{E}} \\ &\mathcal{T}_{\boldsymbol{\varepsilon}} - r_0 = 0 \quad \Rightarrow \quad r_0 = \frac{\sigma_Y}{\sqrt{E}} \end{aligned} \tag{11.41}$$

where σ_Y is the yield stress (obtained in the laboratory). Then, $r_0(\sigma_Y, E)$ can be interpreted as a material mechanical property also obtained in the laboratory.

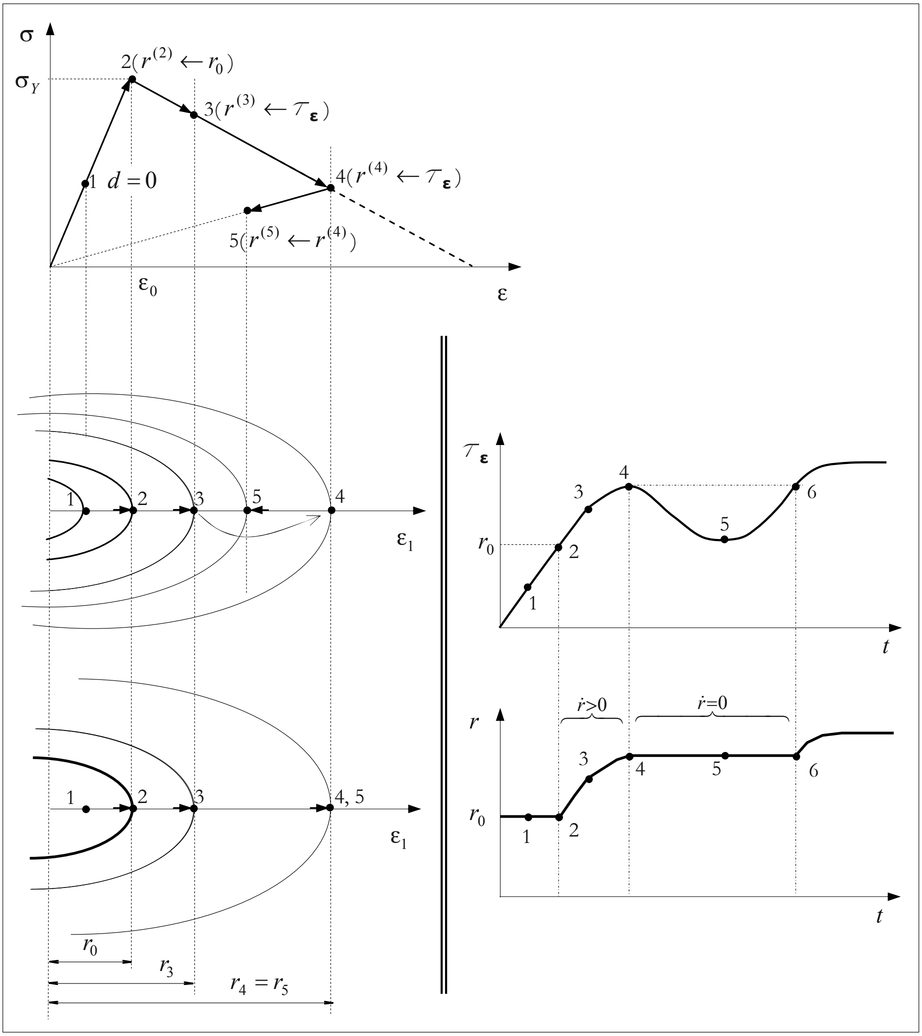


Figure 11.6: The evolution of $\mathcal{T}_{\boldsymbol{\varepsilon}}$ and r over time t.

The Internal Variable Evolution Law. The Kuhn-Tucker and Consistency Conditions

The constitutive equation described above uses three types of variables, namely: the free variable $\{\boldsymbol{\varepsilon}\}$; the internal variable $\{r\}$; the dependent variables $\{\Psi(\boldsymbol{\varepsilon}, r), \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d), d(r)\}$.

Now, to establish how the internal variable r evolves, let us take the example described in Figure 11.6. As we can observe, the discretized $\dot{r}$ between points 2-3 and 3-4 are positive and between points 1-2 and 4-5 are equal to zero, so we can conclude that r is a monotonically increasing function, *i.e.:*

$$\dot{r} \geq 0 \tag{11.42}$$

Graphically, we can see in Figure 11.6 how the variables r and $\mathcal{T}_{\boldsymbol{\varepsilon}}$ evolve. Furthermore, we can also verify that in the range between the points 4-6 $\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = \mathcal{T}_{\boldsymbol{\varepsilon}} - r < 0$ holds, *i.e.* there is an elastic regime.

Thus, we can establish that at time t, r^t is given by the following equation:

$$r^t = \max\left\{ r_0, \underbrace{\max}_{s \in (\infty, t]} \mathcal{T}_{\boldsymbol{\varepsilon}\, s} \right\} \tag{11.43}$$

As we saw, the Helmholtz free energy is a function of $\Psi = \Psi(\boldsymbol{\varepsilon}, d(r))$, where now the damage variable is in terms of the variable r (the internal variable), thus:

$$\Psi(\boldsymbol{\varepsilon}, d(r)) = [1 - d(r)]\Psi^e(\boldsymbol{\varepsilon}) \quad \Rightarrow \quad \dot{\Psi}(\boldsymbol{\varepsilon}, d(r)) \begin{aligned}[t] &= \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \Psi}{\partial d} \frac{\partial d(r)}{\partial r} \frac{\partial r}{\partial t} \\ &= \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \Psi^e \frac{\partial d(r)}{\partial r} \dot{r} \end{aligned} \tag{11.44}$$

where we have considered that $\dfrac{\partial \Psi}{\partial d} = -\Psi^e$. Then, by using the above equation, the internal energy dissipation becomes:

$$\begin{aligned} \mathcal{D}_{int} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi}(\boldsymbol{\varepsilon}, r) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \Psi^e \frac{\partial d(r)}{\partial r} \dot{r} \geq 0 \\ &= \underbrace{\left[\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \right]}_{=\boldsymbol{0}} : \dot{\boldsymbol{\varepsilon}} + \Psi^e \frac{\partial d(r)}{\partial r} \dot{r} = \Psi^e \frac{\partial d(r)}{\partial r} \dot{r} \geq 0 \end{aligned} \tag{11.45}$$

If we compare the above inequality with the one obtained in (11.17) we can conclude that:

$$\boxed{\dot{d} = \frac{\partial d(r)}{\partial r} \dot{r} = \mathcal{H}(\mathcal{T}_{\boldsymbol{\varepsilon}}, d)\, \dot{r}} \tag{11.46}$$

where $\mathcal{H}$ is the continuum hardening/softening modulus. The evolution laws for r and for d (damage variable) are then given by:

$$\dot{r} = \dot{\zeta}(\boldsymbol{\varepsilon}, r) \quad ; \quad \dot{d} = \dot{\zeta}\, \mathcal{H}(\mathcal{T}_{\boldsymbol{\varepsilon}}, d) \tag{11.47}$$

where $\dot{\zeta} \geq 0$ is the *damage parameter consistency* (damage multiplier).

Let us now consider a time t in which the process is characterized by $\boldsymbol{\varepsilon}^t$ and r. Then, we can observe the following: if at any time both $\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)^t = 0$ and $\dot{\mathscr{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) > 0$ hold, this implies that $\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)^{t+\Delta t} > 0$, which thereby violates the condition $\{\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)^t \leq 0 \quad \forall\, t\}$, so $\dot{\zeta} > 0 \Rightarrow \dot{\mathscr{G}} = 0$ must be satisfied. Another possible situation is when the current state is inside the damage surface, *i.e.* $\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)^t < 0$, and if in the next loading step $\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)^{t+\Delta t} < 0$ is satisfied, this implies that $\dot{\mathscr{G}} < 0 \Rightarrow \dot{r} = \dot{\zeta}(\boldsymbol{\varepsilon}, r) = 0$. We can gather these previous conditions by means of the *loading/unloading condition,* also called the *Kuhn-Tucker conditions*:

$$\boxed{\dot{\zeta} \geq 0 \quad ; \quad \mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) \leq 0 \quad ; \quad \dot{\zeta}\,\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0} \quad \textit{The Kuhn-Tucker conditions} \tag{11.48}$$

and by the consistency (persistency) condition:

$$\boxed{\dot{\zeta}\,\dot{\mathscr{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0} \quad \textit{The consistency condition} \tag{11.49}$$

If we are undergoing loading, this implies that $\dot{\zeta} > 0$, then by means of the Kuhn-Tucker conditions $\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0$ must be fulfilled. Here, the value of $\dot{\zeta}$ can be obtained by means of the consistency condition:

$$\mathscr{G}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = \dot{\mathscr{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0 \quad \Rightarrow \quad \dot{\mathcal{T}}_{\boldsymbol{\varepsilon}} = \dot{r} \tag{11.50}$$

Schematically, we can summarize the above loading/unloading states as follows:

$$\begin{cases} \mathscr{G} < 0 \quad \Rightarrow \quad \dot{\zeta} = 0 \quad \Rightarrow \quad \dot{d} = 0 \quad \Rightarrow \quad \textit{(elastic)} \\ \mathscr{G} = 0 \begin{cases} \dot{\mathscr{G}} < 0 \quad \Rightarrow \quad \dot{\zeta} = 0 \quad \Rightarrow \quad \dot{d} = 0 \quad \Rightarrow \quad \textit{(unloading)} \\ \dot{\mathscr{G}} = 0 \quad \Rightarrow \begin{cases} \dot{\zeta} = 0 \quad \Rightarrow \quad \dot{d} = 0 \quad \Rightarrow \quad \textit{(neutral loading)} \\ \dot{\zeta} > 0 \quad \Rightarrow \quad \dot{d} > 0 \quad \Rightarrow \quad \textit{(loading)} \end{cases} \end{cases} \end{cases} \tag{11.51}$$

NOTE: If the parameter $\boldsymbol{\mathcal{H}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, d)$, given in (11.46), is not a function of d, we can express it by means of $\boldsymbol{\mathcal{H}}(\mathcal{T}_{\boldsymbol{\varepsilon}}) = \dfrac{\partial G(\mathcal{T}_{\boldsymbol{\varepsilon}})}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}}$, where we have introduced the scalar function G which is a monotonically increasing function, which has proven to be a convenient way to express the damage criteria:

$$\begin{aligned} \overline{\mathscr{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) &= G(\mathcal{T}_{\boldsymbol{\varepsilon}}) - G(r) \leq 0 \quad ; \quad \forall\ t \geq 0 \\ \overline{\mathscr{F}}(\mathcal{T}_{\boldsymbol{\sigma}}, q) &= F(\mathcal{T}_{\boldsymbol{\sigma}}) - F(q) \leq 0 \quad ; \quad \forall\ t \geq 0 \end{aligned} \tag{11.52}$$

Here the loading/unloading condition becomes:

$$\dot{r} = \dot{\zeta}(\boldsymbol{\varepsilon}, r) \quad ; \quad \dot{d} = \dot{\zeta}\,\frac{\partial \overline{\mathscr{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)}{\partial r} \tag{11.53}$$

$$\boxed{\dot{\zeta} \geq 0 \quad ; \quad \overline{\mathscr{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) \leq 0 \quad ; \quad \dot{\zeta}\,\overline{\mathscr{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0} \quad \textit{The Kuhn-Tucker conditions} \tag{11.54}$$

$$\boxed{\dot{\zeta}\,\dot{\overline{\mathscr{G}}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0} \quad \textit{The consistency condition} \tag{11.55}$$

The Damage Variable

The parameter q is the stress-like hardening/softening parameter, and is defined in terms of r as follows:

$$q(r) = (1-d)r \quad \Rightarrow \quad d(r) = 1 - \frac{q(r)}{r} \tag{11.56}$$

Now, by using the equations in (11.56) and (11.25) we can obtain:

$$\boldsymbol{\sigma} = \frac{q(r)}{r}\bar{\boldsymbol{\sigma}} \tag{11.57}$$

in which the following holds:

$$0 \le d \le 1 \qquad r \in [r_0, \infty] \tag{11.58}$$

Note that with the new definition of the damage parameter given in (11.56), we can restructure the equation in (11.46) as follows:

$$d(r) = 1 - \frac{q(r)}{r} \tag{11.59}$$

$$\dot{d} = \frac{\partial d(r)}{\partial r}\dot{r} = \frac{\partial}{\partial r}\left[1 - \frac{q(r)}{r}\right]\dot{r} = \left[\frac{q(r) - \dfrac{\partial q(r)}{\partial r}}{r^2}\right]\dot{r} \quad \Rightarrow \quad \dot{d} = \left[\frac{q(r) - \mathcal{H}^d(r)}{r^2}\right]\dot{r} \tag{11.60}$$

where we have defined a new parameter $\mathcal{H}^d(r) = \dfrac{\partial q(r)}{\partial r}$, which is the hardening/softening parameter.

11.2.2.4 The Hardening/Softening Law

The expression $\dfrac{\partial q(r)}{\partial r}$ defines the hardening/softening parameter, thus:

$$\dot{q} = \mathcal{H}^d(r)\,\dot{r}\ ; \qquad r \in [r_0(d=0), \infty(d=1))\ ; \qquad q \in [r_0, a]\ ; \qquad q_0 = r_0 = \frac{\sigma_Y}{\sqrt{E}} \tag{11.61}$$

where $\mathcal{H}^d$ is the continuum hardening/softening parameter and which is characterized by:

$$\begin{array}{lll} \text{Damage with Hardening} & \Rightarrow & \mathcal{H}^d(r) > 0 \\ \text{Perfect Damage} & \Rightarrow & \mathcal{H}^d(r) = 0 \\ \text{Damage with Softening} & \Rightarrow & \mathcal{H}^d(r) < 0 \end{array} \tag{11.62}$$

Here, we will consider the relationship between q and r to be linear or exponential.

The Linear Hardening/Softening Law

Now, by assuming that q varies linearly with r, we have:

$$q(r) = \begin{cases} r_0 & r \le 0 \\ r_0 + \mathcal{H}^d(r - r_0) \not< 0 & r > r_0 \end{cases} \tag{11.63}$$

Then, taking into account the equation in (11.56) we can still state that:

$$d = 1 - \frac{q}{r} = \begin{cases} 0 & r \le r_0 \\ 1 - \frac{r_0}{r} - \mathcal{H}^d\left(1 - \frac{r_0}{r}\right) \not> 1 & r > r_0 \end{cases} \tag{11.64}$$

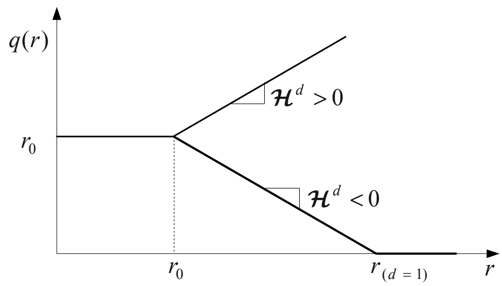


Figure 11.7: The linear hardening/softening law.

The Exponential Hardening/Softening Law

The exponential law is described by Figure 11.8. Then we can express $q(r)$ as follows:

$$q(r) = q_\infty - (q_\infty - r_0)\exp^{A\left(1-\frac{r}{r_0}\right)} \qquad with \qquad A > 0 \tag{11.65}$$

in addition to this we have:

$$\frac{\partial q(r)}{\partial r} = A\frac{(q_\infty - r_0)}{r_0}\exp^{A\left(1-\frac{r}{r_0}\right)} \tag{11.66}$$

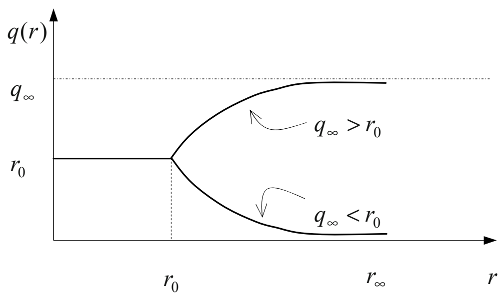


Figure 11.8: The exponential hardening/softening law.

Table 11.1: Summary of the Isotropic Damage Model in a small deformation regime described in the strain space.

ISOTROPIC DAMAGE MODEL IN A SMALL DEFORMATION REGIME		
Helmholtz free energy	$\Psi(\boldsymbol{\varepsilon}, r) = [1 - d(r)]\Psi^e \quad$ with $\quad \Psi^e = \frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon})$	(11.67)
Damage parameter	$d(r) = 1 - \frac{q}{r} \quad ; \quad q \in [r_0, a], a \neq \infty; \quad d \in [0,1]$	(11.68)
The constitutive equations	$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = (1-d)\bar{\boldsymbol{\sigma}} = (1-d)\mathbb{C}^e : \boldsymbol{\varepsilon}$	(11.69)
Evolution law	$\dot{r} = \dot{\zeta} \qquad \begin{cases} r \in [r_0, \infty) \\ r_0 = r\vert_{t=0} = \frac{\sigma_Y}{\sqrt{E}} \end{cases}$	(11.70)
Damage criterion	$\mathscr{G}(\boldsymbol{\varepsilon}, r) = \mathcal{T}_{\boldsymbol{\varepsilon}} - r = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}} - r$	(11.71)

Hardening Law	$\dot{q} = \mathcal{H}^d(r)\,\dot{r} \quad ; \quad \left(\mathcal{H}^d = q'(r) \le 0\right)$	(11.72)
Loading/unloading condition	$\mathscr{G} < 0 \quad ; \quad \dot{\zeta} \ge 0 \quad ; \quad \dot{\zeta}\,\mathscr{G} = 0$	(11.73)
Consistency condition	$\dot{\zeta}\,\dot{\mathscr{G}} = 0$	(11.74)

11.2.3 The Elastic-Damage Tangent Stiffness Tensor

Next, we will obtain the elastic-damage tangent stiffness tensor, which gives us an advantage, from a computational point of view, when we are dealing with the incremental-iterative solution procedures and as a result of this, convergence is improved considerably.

The relationship between $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\varepsilon}}$ give us this tensor denoted by $\mathbb{C}^{tan\_d}$, *i.e.* $\dot{\boldsymbol{\sigma}} = \mathbb{C}^{tan\_d} : \dot{\boldsymbol{\varepsilon}}$. Now, by considering the equation in (11.25), $\boldsymbol{\sigma} = (1-d)\mathbb{C}^e : \boldsymbol{\varepsilon}$, we can obtain the rate of change of the stress as follows:

$$\begin{aligned}\dot{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, d) &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \boldsymbol{\sigma}}{\partial d}\dot{d} = (1-d)\mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \mathbb{C}^e : \boldsymbol{\varepsilon}\dot{d} = (1-d)\mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}}\dot{d} \\ &= (1-d)\mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}} \otimes \dot{d}\end{aligned} \tag{11.75}$$

in which there is the following:

a) A process with elastic loading or unloading

$\zeta = 0 \Rightarrow \dot{d} = 0$, thus the equation in (11.75) becomes $\dot{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, d) = (1-d)\mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}$, with which the elastic-damage tangent stiffness tensor coincides with the elastic-damage secant stiffness tensor when we are dealing with elastic loading:

$$\mathbb{C}^{sec\_d} = \mathbb{C}^{tan\_d} = (1-d)\mathbb{C}^e = \xi\,\mathbb{C}^e \qquad where \qquad \xi = (1-d) \tag{11.76}$$

b) A process with damage loading

$\mathcal{T}_{\boldsymbol{\varepsilon}} = r \Rightarrow \dot{\mathcal{T}}_{\boldsymbol{\varepsilon}} = \dot{r}$, and the rate of change of the damage parameter $d = d(r)$ becomes:

$$\dot{d} = \frac{\partial d}{\partial r}\frac{\partial r}{\partial t} = \frac{\partial d}{\partial r}\dot{r} = \frac{\partial d}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}}\dot{\mathcal{T}}_{\boldsymbol{\varepsilon}} \tag{11.77}$$

where the rate of change of $\mathcal{T}_{\boldsymbol{\varepsilon}}$ can be evaluated as follows:

$$\begin{aligned}\mathcal{T}_{\boldsymbol{\varepsilon}} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}} \quad \longrightarrow \quad \dot{\mathcal{T}}_{\boldsymbol{\varepsilon}} &= \frac{1}{2}\left(\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}\right)^{\frac{-1}{2}}\left(\dot{\boldsymbol{\varepsilon}} : \mathbb{C}^e : \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}}\right) \\ &= \frac{1}{\sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}}}\boldsymbol{\varepsilon} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} = \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}}\boldsymbol{\varepsilon} : \mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} = \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}}\bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}\end{aligned} \tag{11.78}$$

Now, by substituting (11.78) into the equation in (11.77) we obtain:

$$\dot{d} = \frac{\partial d}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}}\frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}}\bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \tag{11.79}$$

Then, taking into account the equations (11.79) and (11.75), we can find the relationship between the rates of stress and strain change:

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= (1-d)\mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \bar{\boldsymbol{\sigma}} \otimes \dot{d} = (1-d)\mathbb{C}^e : \dot{\boldsymbol{\varepsilon}} - \frac{\partial d}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}}\frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}}\bar{\boldsymbol{\sigma}} \otimes \bar{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \\ &= \left[(1-d)\mathbb{C}^e - \frac{\partial d}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}}\frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}}\left(\bar{\boldsymbol{\sigma}} \otimes \bar{\boldsymbol{\sigma}}\right)\right] : \dot{\boldsymbol{\varepsilon}}\end{aligned} \tag{11.80}$$

which thus defines the *elastic-damage tangent stiffness tensor*:

$$\boxed{\mathbb{C}^{tan\_d} = \left[(1-d)\mathbb{C}^e - \frac{\partial d}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}}\frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}}(\overline{\boldsymbol{\sigma}} \otimes \overline{\boldsymbol{\sigma}})\right]} \tag{11.81}$$

and by considering that in a loading process $\mathcal{T}_{\varepsilon} = r$ holds we then find:

$$\frac{\partial d}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}}\frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}} = \frac{\partial d}{\partial r}\frac{1}{r} = \left(\frac{q(r) - \mathcal{H}^d r}{r^2}\right)\frac{1}{r} = \frac{q(r) - \mathcal{H}^d r}{r^3} \tag{11.82}$$

where we have taken into account the equation in (11.61), $\dfrac{\partial d}{\partial r} = \dfrac{q(r) - \mathcal{H}^d r}{r^2}$.

Then, by substituting the equation in (11.82) into that in (11.81) we can obtain $\mathbb{C}^{tan\_d}$ in terms of q and r:

$$\mathbb{C}^{tan\_d} = (1-d)\,\mathbb{C}^e - \left(\frac{q(r) - \mathcal{H}^d r}{r^3}\right)\left(\underbrace{\mathbb{C}^e : \boldsymbol{\varepsilon}}_{\overline{\boldsymbol{\sigma}}} \otimes \underbrace{\boldsymbol{\varepsilon} : \mathbb{C}^e}_{\overline{\boldsymbol{\sigma}}}\right) \tag{11.83}$$

Now, the general equation for the elastic-damage stiffness tensor $\mathbb{C}^{tan\_d}$ (symmetric fourth-order tensor) is given by:

$$\boxed{\mathbb{C}^{tan\_d} = \begin{cases} \mathbb{C}^e & \textit{elastic with } (d=0) \\ \xi\,\mathbb{C}^e - \mathcal{K}\left(\mathbb{C}^e : \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} : \mathbb{C}^e\right) & \textit{loading}\Big/\underbrace{\textit{unloading}}_{\dot{r}=0 \rightarrow \mathcal{K}=0} \end{cases}} \tag{11.84}$$

The elastic-damage stiffness tensor for isotropic damage model

where, $\mathcal{K} = \dfrac{q(r) - \mathcal{H}^d r}{r^3}$ and $\xi = (1-d)$.

11.2.4 The Energy Norms

Next, we will define some energy norms, which together with the damage criteria, play an important role in defining the yield damage surface.

In order to adequately represent the materials different norms will need to be defined so as to describe how these materials really behave. For example, in a simple model for concrete, if we only want to simulate the process of failure caused by tension, the *tension-only damage model* is used which means that it cannot capture the other type of failure caused by compression. Next, we will define some models used in the isotropic damage process.

11.2.4.1 The Symmetrical Damage Model (Tension-Compression) – Model I

This type of model shows when the material behave the same both with tension or and compression. The energy norm of this model is then represented by:

$$\mathcal{T}^I_{\boldsymbol{\sigma}} = \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} = (1-d)\sqrt{\overline{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}} \tag{11.85}$$

We can also define the energy norm of the strain tensor (also known as the equivalent strain), proposed by Simo&Ju(1987), (see equation (11.31)):

$$\mathcal{T}^I_{\boldsymbol{\varepsilon}} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}} = \sqrt{2\Psi^e} \tag{11.86}$$

To better illustrate this model, let us consider the state of plane stress ($\sigma_{i3}=0$). In this case, the yield surface is represented by an ellipse, (see Figure 11.9), where $\sigma_Y>0$ is the stress limit for tension and compression and the damage surface evolves symmetrically.

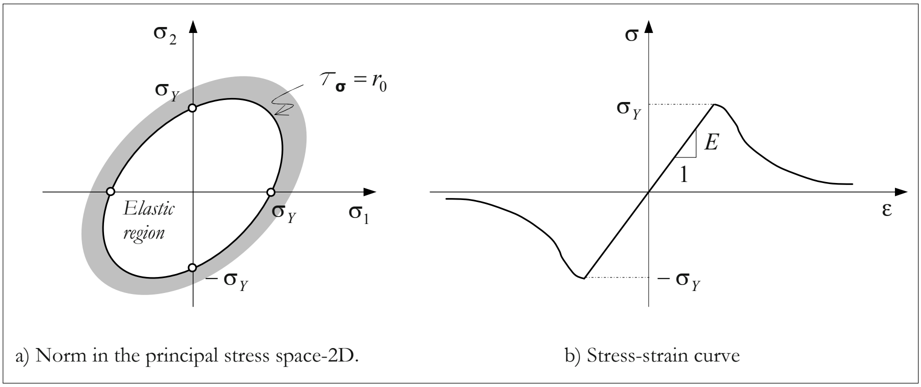


Figure 11.9: Damage surface in 2D and the uniaxial stress-strain curve for model I.

11.2.4.2 The Tension-Only Damage Model – Model II

The tension-only damage model does not take into account failure by compression, *i.e.* the material can only fail by tension and here we can define the following stress field:

$$\boldsymbol{\sigma}^+=\langle\boldsymbol{\sigma}\rangle=\frac{\boldsymbol{\sigma}+|\boldsymbol{\sigma}|}{2} \tag{11.87}$$

where $\langle\bullet\rangle\stackrel{def}{=}\dfrac{\bullet+|\bullet|}{2}$ is the Macaulay bracket whose graphical representation can be appreciated in Figure 11.10.

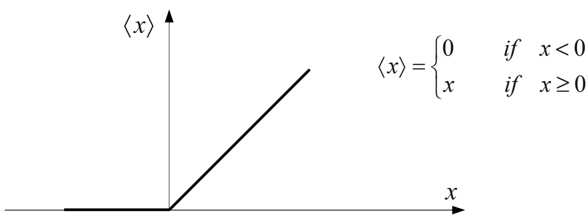


Figure 11.10: Ramp function.

Now, by means of spectral representation, we can represent the stress tensor in terms of eigenvalues (principal stresses) and eigenvectors as follows:

$$\boldsymbol{\sigma}=\sum_{a=1}^{3}\sigma_a\hat{\boldsymbol{n}}^{(a)}\otimes\hat{\boldsymbol{n}}^{(a)} \tag{11.88}$$

thus:

$$\boldsymbol{\sigma}^+=\sum_{a=1}^{3}\langle\sigma_a\rangle\hat{\boldsymbol{n}}^{(a)}\otimes\hat{\boldsymbol{n}}^{(a)} \tag{11.89}$$

Note, the relationship between the real and effective stress remains valid, *i.e.*:

$$\boldsymbol{\sigma}^+=(1-d)\overline{\boldsymbol{\sigma}}^+ \tag{11.90}$$

Then, the norm for the isotropic damage model defined previously becomes:

$$\mathcal{T}_{\boldsymbol{\varepsilon}} = \sqrt{2\Psi^e} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}} = \sqrt{\overline{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}} \tag{11.91}$$

Next, in the tension-only damage model $\boldsymbol{\sigma} \leftarrow \boldsymbol{\sigma}^+$, it follows that:

$$\mathcal{T}_{\boldsymbol{\varepsilon}}^{II} = \sqrt{\overline{\boldsymbol{\sigma}}^+ : \boldsymbol{\varepsilon}} = \sqrt{\overline{\boldsymbol{\sigma}}^+ : \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}}} = \sqrt{\frac{1}{(1-d)^2} \boldsymbol{\sigma}^+ : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} = \frac{1}{(1-d)} \sqrt{\boldsymbol{\sigma}^+ : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} \tag{11.92}$$

Then, if we consider the equation in (11.31), we can conclude that:

$$\mathcal{T}_{\boldsymbol{\sigma}}^{II} = \sqrt{\boldsymbol{\sigma}^+ : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} \tag{11.93}$$

Finally, in Figure 11.11 we can visualize the damage surface for two-dimensional cases (2D).

11.2.4.3 The Non-Symmetrical Damage Model – Model III

The non-symmetrical damage model is useful to simulate materials, such as concrete, whose tension domain differs with respect to compression. This model uses the following norm:

$$\mathcal{T}_{\boldsymbol{\sigma}}^{III} = \left(\theta + \frac{1-\theta}{n} \right) \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{e^{-1}} : \boldsymbol{\sigma}} \tag{11.94}$$

where the parameter θ is the weight factor dependant on the stress state $\boldsymbol{\sigma}$ which is given by:

$$\theta = \frac{\sum_{i=1}^{3} \langle \sigma_i \rangle}{\sum_{i=1}^{3} |\sigma_i|} \tag{11.95}$$

The parameter n is defined by means of the ratio of the compression elastic limit σ_Y^c to the tension elastic limit σ_Y^t, *i.e.*:

$$n = \frac{\sigma_Y^c}{\sigma_Y^t} \tag{11.96}$$

In the case of concrete n is approximately equal to $n \approx 10$.

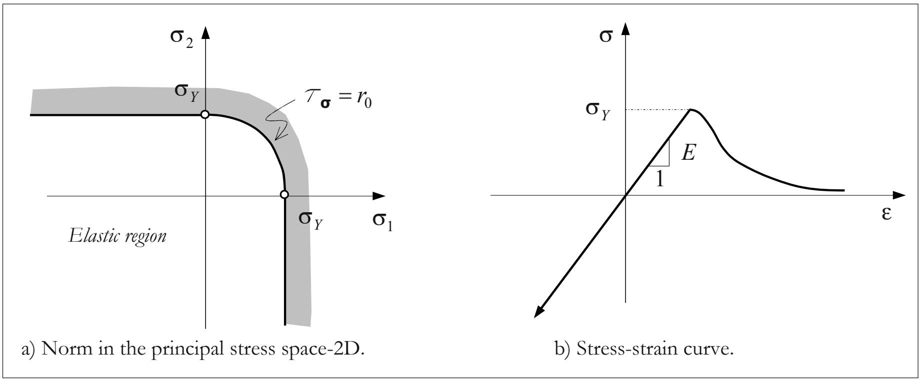


Figure 11.11: Damage surface in 2D and the uniaxial stress-strain curve for model II.

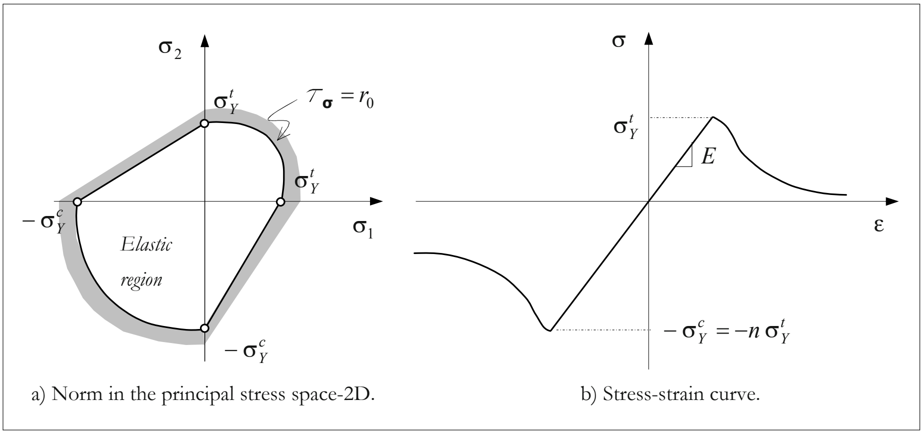


Figure 11.12: Damage surface in 2D and the uniaxial stress-strain curve for model III.

11.3 The Generalized Isotropic Damage Model

Note that the elasticity tensor $\mathbb{C}^e$ can be written in terms of the following sets of mechanical parameters (λ, μ), (E, ν), (κ, G):

$$\mathbb{C}^e = \lambda \mathbf{1} \otimes \mathbf{1} + 2\,\mu\,\mathbf{I} = \frac{\nu E}{(1+\nu)(1-2\nu)} \mathbf{1} \otimes \mathbf{1} + \frac{\nu E}{(1+\nu)} \mathbf{I} = \underbrace{\kappa \mathbf{1} \otimes \mathbf{1}}_{\substack{\text{volumetric} \\ \text{part}}} + \underbrace{2\,\mu \left[\mathbf{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right]}_{\text{isochoric part}} \tag{11.97}$$

where (E) =Young's modulus, (ν) =Poisson's ratio, (λ, μ) =Lamé constants, (κ) =bulk modulus, and $G = \mu$ is the shear modulus.

In the isotropic damage model the elastic-damage secant stiffness tensor can be represented as follows:

$$\mathbb{C}^{sec\_d} = (1-d)\mathbb{C}^e = \frac{\nu(1-d)E}{(1+\nu)(1-2\nu)} \mathbf{1} \otimes \mathbf{1} + \frac{\nu(1-d)E}{(1+\nu)} \mathbf{I} = \frac{\nu E^{sec\_d}}{(1+\nu)(1-2\nu)} \mathbf{1} \otimes \mathbf{1} + \frac{\nu E^{sec\_d}}{(1+\nu)} \mathbf{I}$$

Note that, in this model the damage variable affects only one of the mechanical parameters, namely, the Young's modulus. We can also verify that the same damage parameter equally affects both the spherical and deviatoric part:

$$\mathbb{C}^{sec\_d} = (1-d)\mathbb{C}^e = (1-d)\kappa \mathbf{1} \otimes \mathbf{1} + (1-d) 2\,\mu \left[\mathbf{I} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right] \tag{11.98}$$

Another model described by Carol *et al.* (1998) generalizes the isotropic damage model by considering independent degradation of the spherical and deviatoric parts and because of this the model requires two independent damage variables.

Now, the elasticity tensor components can be expressed by means of their spherical and deviatoric parts as follows:

$$\mathbb{C}^e_{ijkl} = \kappa \delta_{ij} \delta_{kl} + 2\,\mu \left[\frac{1}{2} \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) - \frac{1}{3} \delta_{ij} \delta_{kl} \right] \tag{11.99}$$

Then, with $\mathbb{P}^V_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$ and $\mathbb{P}^D_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \mathbb{P}^V_{ijkl}$, the above equation becomes:

$$\mathbb{C}^e_{ijkl} = 3\kappa\mathbb{P}^V_{ijkl} + 2\mu\,\mathbb{P}^D_{ijkl} \qquad \Big| \qquad \mathbb{C}^e = 3\kappa\mathbb{P}^V + 2\mu\,\mathbb{P}^D \tag{11.100}$$

Let us now consider that the material parameters κ and μ can be degraded by means of the variables d^V and d^D, respectively, and according to the following equations:

$$\kappa = (1 - d^V)\kappa_0 \quad ; \quad \mu = (1 - d^D)\mu_0 \tag{11.101}$$

with which the elastic-damage secant stiffness tensor becomes:

$$\mathbb{C}^{sec\_d}_{ijkl} = 3(1 - d^V)\kappa_0\mathbb{P}^V_{ijkl} + 2(1 - d^D)\mu_0\,\mathbb{P}^D_{ijkl} = (1 - d^V)\mathbb{C}^{e\_V}_{ijkl} + (1 - d^D)\mathbb{C}^{e\_D}_{ijkl} \tag{11.102}$$

where we have introduced:

$$\begin{cases} \mathbb{C}^{e\_V}_{ijkl} = 3\kappa_0\mathbb{P}^V_{ijkl} \\ \qquad\;\; = \kappa_0\delta_{ij}\delta_{kl} \end{cases} ; \quad \begin{cases} \mathbb{C}^{e\_D}_{ijkl} = 2\mu_0\,\mathbb{P}^D_{ijkl} \\ \qquad\;\; = 2\mu_0\left[\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{1}{3}\delta_{ij}\delta_{kl}\right] \end{cases} \tag{11.103}$$

11.3.1 The Strain Energy Function

Now, if we consider (11.100), the elastic strain energy function can be rewritten as follows:

$$\begin{aligned} \Psi^e &= \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} : \left(3\kappa\mathbb{P}^V + 2\mu\,\mathbb{P}^D\right) : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} : \left(3\kappa\mathbb{P}^V\right) : \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon} : \left(2\mu\,\mathbb{P}^D\right) : \boldsymbol{\varepsilon} \\ &= \Psi^{e\_vol} + \Psi^{e\_dev} \end{aligned} \tag{11.104}$$

where we have introduced:

$$\boxed{\left.\begin{aligned} \Psi^{e\_vol} &= \frac{1}{2}\boldsymbol{\varepsilon} : \left(3\kappa\mathbb{P}^V\right) : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{e\_V} : \boldsymbol{\varepsilon} \\ \Psi^{e\_vol} &= \frac{1}{2}\boldsymbol{\varepsilon} : \left(2\mu\,\mathbb{P}^D\right) : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{e\_D} : \boldsymbol{\varepsilon} \end{aligned}\right\} \Rightarrow \Psi^e(\boldsymbol{\varepsilon}) = \Psi^{e\_vol} + \Psi^{e\_dev}} \tag{11.105}$$

after which it becomes:

$$\begin{aligned} \Psi(\boldsymbol{\varepsilon}, d^V, d^D) &= \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{sec\_d} : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} : \left[(1 - d^V)\mathbb{C}^{e\_V} + (1 - d^D)\mathbb{C}^{e\_D}\right] : \boldsymbol{\varepsilon} \\ &= (1 - d^V)\frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{e\_V} : \boldsymbol{\varepsilon} + (1 - d^D)\frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{e\_D} : \boldsymbol{\varepsilon} \\ &= \underbrace{(1 - d^V)\Psi^{e\_vol}}_{=\Psi^{vol}} + \underbrace{(1 - d^D)\Psi^{e\_dev}}_{=\Psi^{dev}} = \Psi^{vol}(\boldsymbol{\varepsilon}, d^V) + \Psi^{dev}(\boldsymbol{\varepsilon}, d^D) \end{aligned} \tag{11.106}$$

Additionally, the following holds:

$$\begin{aligned} \Psi(\boldsymbol{\varepsilon}, d^V, d^D) &= (1 - d^V)\frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{e\_V} : \boldsymbol{\varepsilon} + (1 - d^D)\frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^{e\_D} : \boldsymbol{\varepsilon} \\ &= (1 - d^V)\frac{1}{2}\boldsymbol{\varepsilon}^{sph} : \mathbb{C}^{e\_V} : \boldsymbol{\varepsilon}^{sph} + (1 - d^D)\frac{1}{2}\boldsymbol{\varepsilon}^{dev} : \mathbb{C}^{e\_D} : \boldsymbol{\varepsilon}^{dev} \\ &= \underbrace{(1 - d^V)\Psi^{e\_vol}}_{=\Psi^{vol}} + \underbrace{(1 - d^D)\Psi^{e\_dev}}_{=\Psi^{dev}} = \Psi^{vol}(\boldsymbol{\varepsilon}^{sph}, d^V) + \Psi^{dev}(\boldsymbol{\varepsilon}^{dev}, d^D) \end{aligned} \tag{11.107}$$

11.3.2 Spherical and Deviatoric Effective Stress

Note that the following equations hold:

$$\boldsymbol{\sigma} = \mathbb{C}^{sec\_d} : \boldsymbol{\varepsilon} = (1-d^V)\mathbb{C}^{e\_V} : \boldsymbol{\varepsilon} + (1-d^D)\mathbb{C}^{e\_D} : \boldsymbol{\varepsilon} = (1-d^V)\bar{\boldsymbol{\sigma}}^{sph} + (1-d^D)\bar{\boldsymbol{\sigma}}^{dev} \tag{11.108}$$

where $\bar{\boldsymbol{\sigma}}^{sph}$, $\bar{\boldsymbol{\sigma}}^{dev}$ are the spherical and deviatoric effective stresses, respectively and where the following is valid:

$$\left.\begin{aligned} \boldsymbol{\sigma}^{sph} &= (1-d^V)\bar{\boldsymbol{\sigma}}^{sph} \\ \boldsymbol{\sigma}^{dev} &= (1-d^D)\bar{\boldsymbol{\sigma}}^{dev} \end{aligned}\right\} \Rightarrow \boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} \tag{11.109}$$

It is noteworthy that the following equations hold:

$$\begin{aligned} \boldsymbol{\sigma} &= (1-d^V)\mathbb{C}^{e\_V} : \boldsymbol{\varepsilon} + (1-d^D)\mathbb{C}^{e\_D} : \boldsymbol{\varepsilon} \\ &= (1-d^V)\mathbb{C}^{e\_V} : \left(\boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}\right) + (1-d^D)\mathbb{C}^{e\_D} : \left(\boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}\right) \\ &= (1-d^V)\mathbb{C}^{e\_V} : \boldsymbol{\varepsilon}^{sph} + (1-d^D)\mathbb{C}^{e\_D} : \boldsymbol{\varepsilon}^{dev} \end{aligned} \tag{11.110}$$

Then, the relationship between stress and strain in rate is given by:

$$\left.\begin{aligned} \dot{\boldsymbol{\sigma}} &= \mathbb{C}^{tan\_d} : \dot{\boldsymbol{\varepsilon}} \\ \dot{\boldsymbol{\sigma}}^{sph} + \dot{\boldsymbol{\sigma}}^{dev} &= \mathbb{C}^{tan\_d} : \left(\dot{\boldsymbol{\varepsilon}}^{sph} + \dot{\boldsymbol{\varepsilon}}^{dev}\right) \\ &= \mathbb{C}^{tan\_d} : \dot{\boldsymbol{\varepsilon}}^{sph} + \mathbb{C}^{tan\_d} : \dot{\boldsymbol{\varepsilon}}^{dev} \end{aligned}\right\} \Rightarrow \begin{cases} \dot{\boldsymbol{\sigma}}^{sph} = \mathbb{C}^{tan\_d} : \dot{\boldsymbol{\varepsilon}}^{sph} \\ \dot{\boldsymbol{\sigma}}^{dev} = \mathbb{C}^{tan\_d} : \dot{\boldsymbol{\varepsilon}}^{dev} \end{cases} \tag{11.111}$$

where $\mathbb{C}^{tan\_d}$ is the elastic-damage tangent stiffness tensor.

11.3.3 Thermodynamic Considerations

In a small deformation regime $\mathbf{D} \approx \dot{\boldsymbol{\varepsilon}}$ holds and in isothermal processes $\dot{T} = 0$ is satisfied, so, it then follows that the expression for internal energy dissipation given in (11.13) becomes:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \geq 0 \tag{11.112}$$

Then, by evaluating the rate of change of the strain energy function given in (11.106), $\Psi = (1-d^V)\Psi^{e\_V} + (1-d^D)\Psi^{e\_D}$, we can obtain:

$$\begin{aligned} \dot{\Psi} &= \dot{\Psi}^{e\_V}(1-d^V) - \Psi^{e\_V}\dot{d}^V + \dot{\Psi}^{e\_D}(1-d^D) - \Psi^{e\_D}\dot{d}^D \\ &= \dot{\Psi}^{e\_V}(1-d^V) + \dot{\Psi}^{e\_D}(1-d^D) - \Psi^{e\_V}\dot{d}^V - \Psi^{e\_D}\dot{d}^D \end{aligned} \tag{11.113}$$

and by using the stress equation given in (11.108) we have:

$$\begin{aligned} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} &= \left[(1-d^V)\mathbb{C}^{e\_V} : \boldsymbol{\varepsilon} + (1-d^D)\mathbb{C}^{e\_D} : \boldsymbol{\varepsilon}\right] : \dot{\boldsymbol{\varepsilon}} \\ &= (1-d^V)\boldsymbol{\varepsilon} : \mathbb{C}^{e\_V} : \dot{\boldsymbol{\varepsilon}} + (1-d^D)\boldsymbol{\varepsilon} : \mathbb{C}^{e\_D} : \dot{\boldsymbol{\varepsilon}} \end{aligned} \tag{11.114}$$

Note that $\dot{\Psi}^{e\_V} = \boldsymbol{\varepsilon} : \mathbb{C}^{e\_V} : \dot{\boldsymbol{\varepsilon}}$ and $\dot{\Psi}^{e\_D} = \boldsymbol{\varepsilon} : \mathbb{C}^{e\_D} : \dot{\boldsymbol{\varepsilon}}$, thus:

$$\begin{aligned} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} &= (1-d^V)\boldsymbol{\varepsilon} : \mathbb{C}^{e\_V} : \dot{\boldsymbol{\varepsilon}} + (1-d^D)\boldsymbol{\varepsilon} : \mathbb{C}^{e\_D} : \dot{\boldsymbol{\varepsilon}} \\ &= (1-d^V)\dot{\Psi}^{e\_V} + (1-d^D)\dot{\Psi}^{e\_D} \end{aligned} \tag{11.115}$$

Then, together the equations (11.115), (11.113), and the internal energy dissipation given in (11.112), yields:

$$
\begin{aligned}
\mathcal{D}_{int} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \geq 0 \\
&= (1-d^V)\dot{\Psi}^{e\_V} + (1-d^D)\dot{\Psi}^{e\_D} - \dot{\Psi}^{e\_V}(1-d^V) - \dot{\Psi}^{e\_D}(1-d^D) + \Psi^{e\_V}\dot{d}^V + \Psi^{e\_D}\dot{d}^D \geq 0 \\
&= \Psi^{e\_V}\dot{d}^V + \Psi^{e\_D}\dot{d}^D \geq 0
\end{aligned}
\tag{11.116}
$$

Since (11.116) must be satisfied for any admissible thermodynamic process, it follows that:

$$
\dot{d}^V \geq 0 \quad ; \quad \dot{d}^D \geq 0 \tag{11.117}
$$

where we have taken into account that $\Psi^{e\_V} \geq 0$ and $\Psi^{e\_D} \geq 0$.

11.3.4 The Elastic-Damage Tangent Stiffness Tensor

Initially we adopt the following norms:

$$
\mathcal{T}_{\boldsymbol{\varepsilon}}^V = \sqrt{2\Psi^{e\_V}} = \sqrt{\overline{\boldsymbol{\sigma}}^{sph} : \boldsymbol{\varepsilon}} = \sqrt{\overline{\boldsymbol{\sigma}}^{sph} : \boldsymbol{\varepsilon}^{sph}} = \sqrt{\boldsymbol{\varepsilon}^{sph} : \mathbb{C}^{e\_V} : \boldsymbol{\varepsilon}^{sph}} \tag{11.118}
$$

$$
\mathcal{T}_{\boldsymbol{\varepsilon}}^D = \sqrt{2\Psi^{e\_D}} = \sqrt{\overline{\boldsymbol{\sigma}}^{dev} : \boldsymbol{\varepsilon}} = \sqrt{\overline{\boldsymbol{\sigma}}^{dev} : \boldsymbol{\varepsilon}^{dev}} = \sqrt{\boldsymbol{\varepsilon}^{dev} : \mathbb{C}^{e\_D} : \boldsymbol{\varepsilon}^{dev}} \tag{11.119}
$$

where the following holds:

$$
\dot{\mathcal{T}}_{\boldsymbol{\varepsilon}}^V = \frac{1}{\sqrt{\boldsymbol{\varepsilon}^{sph} : \mathbb{C}^{e\_V} : \boldsymbol{\varepsilon}^{sph}}}\left(\boldsymbol{\varepsilon}^{sph} : \mathbb{C}^{e\_V} : \dot{\boldsymbol{\varepsilon}}^{sph}\right) = \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^V}\left(\overline{\boldsymbol{\sigma}}^{sph} : \dot{\boldsymbol{\varepsilon}}^{sph}\right) = \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^V}\left(\overline{\boldsymbol{\sigma}}^{sph} : \dot{\boldsymbol{\varepsilon}}\right) \tag{11.120}
$$

$$
\dot{\mathcal{T}}_{\boldsymbol{\varepsilon}}^D = \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^D}\left(\overline{\boldsymbol{\sigma}}^{dev} : \dot{\boldsymbol{\varepsilon}}\right) \tag{11.121}
$$

Next, we obtain the rate of change of the Cauchy stress tensor:

$$
\begin{aligned}
\dot{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, d^V, d^D) &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \boldsymbol{\sigma}}{\partial d^V}\dot{d}^V + \frac{\partial \boldsymbol{\sigma}}{\partial d^D}\dot{d}^D = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : (\dot{\boldsymbol{\varepsilon}}^{sph} + \dot{\boldsymbol{\varepsilon}}^{dev}) + \frac{\partial \boldsymbol{\sigma}}{\partial d^V}\dot{d}^V + \frac{\partial \boldsymbol{\sigma}}{\partial d^D}\dot{d}^D \\
&= \left(\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}^{sph} + \frac{\partial \boldsymbol{\sigma}}{\partial d^V}\dot{d}^V\right) + \left(\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}^{dev} + \frac{\partial \boldsymbol{\sigma}}{\partial d^D}\dot{d}^D\right) \\
&= \dot{\boldsymbol{\sigma}}^{sph} + \dot{\boldsymbol{\sigma}}^{dev}
\end{aligned}
\tag{11.122}
$$

where the following holds, (see equation (11.109)):

$$
\frac{\partial \boldsymbol{\sigma}}{\partial d^V} = -\overline{\boldsymbol{\sigma}}^{sph} \quad ; \quad \frac{\partial \boldsymbol{\sigma}}{\partial d^D} = -\overline{\boldsymbol{\sigma}}^{dev} \tag{11.123}
$$

and

$$
\dot{d}^V = \frac{\partial d^V}{\partial r^V}\frac{\partial r^V}{\partial t} = \frac{\partial d^V}{\partial r^V}\dot{r}^V = \frac{\partial d^V}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^V}\dot{\mathcal{T}}_{\boldsymbol{\varepsilon}}^V \quad ; \quad \dot{d}^D = \frac{\partial d^D}{\partial r^D}\frac{\partial r^D}{\partial t} = \frac{\partial d^D}{\partial r^D}\dot{r}^D = \frac{\partial d^D}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^D}\dot{\mathcal{T}}_{\boldsymbol{\varepsilon}}^D \tag{11.124}
$$

Then, we can express the rates of change $\dot{\boldsymbol{\sigma}}^{sph}$ and $\dot{\boldsymbol{\sigma}}^{dev}$ as follows:

$$\begin{aligned}\dot{\boldsymbol{\sigma}}^{sph} &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}^{sph} + \frac{\partial \boldsymbol{\sigma}}{\partial d^V} \dot{d}^V = (1-d^V)\mathbb{C}^{e\_V} : \dot{\boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^{sph} \frac{\partial d^V}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^V} \dot{\mathcal{T}}_{\boldsymbol{\varepsilon}}^V \\ &= (1-d^V)\mathbb{C}^{e\_V} : \dot{\boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^{sph} \frac{\partial d^V}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^V} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^V} \left(\overline{\boldsymbol{\sigma}}^{sph} : \dot{\boldsymbol{\varepsilon}} \right) \\ &= \left[(1-d^V)\mathbb{C}^{e\_V} - \frac{\partial d^V}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^V} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^V} \overline{\boldsymbol{\sigma}}^{sph} \otimes \overline{\boldsymbol{\sigma}}^{sph} \right] : \dot{\boldsymbol{\varepsilon}} \\ &= \left[(1-d^V)\mathbb{C}^{e\_V} - \frac{\partial d^V}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^V} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^V} \overline{\boldsymbol{\sigma}}^{sph} \otimes \overline{\boldsymbol{\sigma}}^{sph} \right] : \dot{\boldsymbol{\varepsilon}}^{sph}\end{aligned} \tag{11.125}$$

and

$$\begin{aligned}\dot{\boldsymbol{\sigma}}^{dev} &= \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}^{dev} + \frac{\partial \boldsymbol{\sigma}}{\partial d^D} \dot{d}^D = (1-d^D)\mathbb{C}^{e\_D} : \dot{\boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^{dev} \frac{\partial d^D}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^D} \dot{\mathcal{T}}_{\boldsymbol{\varepsilon}}^D \\ &= (1-d^D)\mathbb{C}^{e\_D} : \dot{\boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^{dev} \frac{\partial d^D}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^D} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^D} \left(\overline{\boldsymbol{\sigma}}^{dev} : \dot{\boldsymbol{\varepsilon}} \right) \\ &= \left[(1-d^D)\mathbb{C}^{e\_D} - \frac{\partial d^D}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^D} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^D} \overline{\boldsymbol{\sigma}}^{dev} \otimes \overline{\boldsymbol{\sigma}}^{dev} \right] : \dot{\boldsymbol{\varepsilon}} \\ &= \left[(1-d^D)\mathbb{C}^{e\_D} - \frac{\partial d^D}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^D} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^D} \overline{\boldsymbol{\sigma}}^{dev} \otimes \overline{\boldsymbol{\sigma}}^{dev} \right] : \dot{\boldsymbol{\varepsilon}}^{dev}\end{aligned} \tag{11.126}$$

with which we can define the following equation:

$$\dot{\boldsymbol{\sigma}} = \left[(1-d^D)\mathbb{C}^{e\_D} + (1-d^V)\mathbb{C}^{e\_V} - \frac{\partial d^D}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^D} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^D} \overline{\boldsymbol{\sigma}}^{dev} \otimes \overline{\boldsymbol{\sigma}}^{dev} - \frac{\partial d^V}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^V} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^V} \overline{\boldsymbol{\sigma}}^{sph} \otimes \overline{\boldsymbol{\sigma}}^{sph} \right] : \dot{\boldsymbol{\varepsilon}} \tag{11.127}$$

and by comparing the above with (11.111), we can conclude that:

$$\mathbb{C}^{tan\_d} = (1-d^D)\mathbb{C}^{e\_D} + (1-d^V)\mathbb{C}^{e\_V} - \frac{\partial d^D}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^D} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^D} \overline{\boldsymbol{\sigma}}^{dev} \otimes \overline{\boldsymbol{\sigma}}^{dev} - \frac{\partial d^V}{\partial \mathcal{T}_{\boldsymbol{\varepsilon}}^V} \frac{1}{\mathcal{T}_{\boldsymbol{\varepsilon}}^V} \overline{\boldsymbol{\sigma}}^{sph} \otimes \overline{\boldsymbol{\sigma}}^{sph} \tag{11.128}$$

11.4 The Elastoplastic-Damage Model in a Small Deformation Regime

The classical theory of damage has been modified and extended in order to include residual (plastic) strain. Among the researchers who worked in this area we can mention: Bazant&Kim (1979), Dragon&Mróz (1979), Ortiz (1985) and Simo&Ju (1987a,b).

Next, we will discuss the elasto-plastic damage model by considering there is an isothermal process under a small deformation regime.

Fundamentally, we can describe an elasto-plastic damage model as one that presents residual strain (plastic strain) and also where degradation of the secant stiffness tensor occurs, (see Figure 11.13).

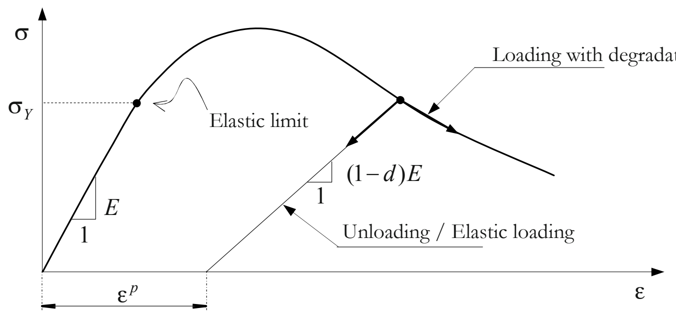


Figure 11.13: Stress-strain curve for elasto-plastic damage model.

Several elastoplastic-damage models have been developed, a few of which we mention below. We will start off from the strain energy function statement:

- One of the models: the elastoplastic-damage decoupled model, considers additive decomposition of the energy into elastic and plastic parts, where both types of energy are functions of the damage parameter:

$$\Psi = \Psi^e(\boldsymbol{\varepsilon}, d) + \Psi^p(\alpha^p, d) \tag{11.129}$$

- The next model considers the above plus an additional term which is only used in terms of the damage variable:

$$\Psi = \Psi^e(\boldsymbol{\varepsilon}, d) + \Psi^p(\alpha^p, d) + \Psi^d(\alpha^d) \tag{11.130}$$

- The next model considers energy decomposition into the elastic part $\Psi^e(\boldsymbol{\varepsilon}, d)$ and $\Psi^p(\alpha^p)$ which in turn is only used in terms of the plastic strain:

$$\Psi = \Psi^e(\boldsymbol{\varepsilon}, d) + \Psi^p(\alpha^p) \tag{11.131}$$

11.4.1 The Elasto-Plastic Damage Model by Simó&Ju (1987) in a Small Deformation Regime

Now, we will examine the isotropic damage model proposed by Simo&Ju (1987a,b) in which the following equation is still valid:

$$\boxed{\bar{\boldsymbol{\sigma}} = \frac{1}{(1-d)}\boldsymbol{\sigma}} \quad \textit{The effective stress tensor} \tag{11.132}$$

11.4.1.1 Helmholtz Free Energy

The free energy adopted for this model is given by:

$$\boxed{\Psi = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^p, \boldsymbol{q}, d) = (1-d)\Psi^0(\boldsymbol{\varepsilon}) - \boldsymbol{\varepsilon} : \boldsymbol{\sigma}^p + \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)} \quad \textit{Helmholtz free energy} \tag{11.133}$$

where d is the damage parameter, $\boldsymbol{q}$ is a set of plastic internal variables, $\boldsymbol{\sigma}^p$ is the plastic relaxation stress tensor, $\Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)$ is the plastic potential, and $\Psi^0(\boldsymbol{\varepsilon})$ is elastic strain function (energy density). In the particular case when the constitutive relationship is linear, we have $\Psi^0(\boldsymbol{\varepsilon}) = \Psi^e(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}$.

11.4.1.2 Internal Energy Dissipation. Constitutive Equations. Thermodynamic Considerations

Once again, let us consider the alternative form of the Clausius-Planck inequality, (see equation (11.12)) and if we consider the process to be purely mechanical we obtain:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^p, \boldsymbol{q}, d) \geq 0 \tag{11.134}$$

Then, taking into account the Helmholtz free energy we can obtain its rate of change as follows:

$$\dot{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^p, \boldsymbol{q}, d) = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \Psi}{\partial \boldsymbol{\sigma}^p} : \dot{\boldsymbol{\sigma}}^p + \frac{\partial \Psi}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + \frac{\partial \Psi}{\partial d} \dot{d} \tag{11.135}$$

and by substituting (11.135) into (11.134) we obtain:

$$\left(\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \left[\frac{\partial \Psi}{\partial \boldsymbol{\sigma}^p} : \dot{\boldsymbol{\sigma}}^p + \frac{\partial \Psi}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + \frac{\partial \Psi}{\partial d} \dot{d} \right] \geq 0 \tag{11.136}$$

Next, as the above inequality must be satisfied for any admissible thermodynamic process, we find:

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \tag{11.137}$$

and by considering the strain energy equation given in (11.133) we can obtain:

$$\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = (1-d)\frac{\partial \Psi^0}{\partial \boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^p = \boldsymbol{\sigma} \,, \quad \frac{\partial \Psi}{\partial \boldsymbol{\sigma}^p} = -\boldsymbol{\varepsilon} + \frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{\sigma}^p} \,, \quad \frac{\partial \Psi}{\partial \boldsymbol{q}} = \frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{q}} \,, \quad \frac{\partial \Psi}{\partial d} = -\Psi^0 \tag{11.138}$$

Now, substituting (11.138) in (11.136) yields:

$$\begin{aligned} &\left[\boldsymbol{\sigma} - \left((1-d)\frac{\partial \Psi^0}{\partial \boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^p \right) \right] : \dot{\boldsymbol{\varepsilon}} - \left[\left(-\boldsymbol{\varepsilon} + \frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{\sigma}^p} \right) : \dot{\boldsymbol{\sigma}}^p + \left(\frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{q}} \right) \dot{\boldsymbol{q}} - \Psi^0 \dot{d} \right] \geq 0 \\ &\Rightarrow \left(\boldsymbol{\sigma} - (1-d)\frac{\partial \Psi^0}{\partial \boldsymbol{\varepsilon}} + \boldsymbol{\sigma}^p \right) : \dot{\boldsymbol{\varepsilon}} - \left(\frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{\sigma}^p} - \boldsymbol{\varepsilon} \right) : \dot{\boldsymbol{\sigma}}^p - \left(\frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{q}} \right) \dot{\boldsymbol{q}} + \Psi^0 \dot{d} \geq 0 \end{aligned} \tag{11.139}$$

Note that the above inequality must hold for any admissible thermodynamic process, so, let us consider one in which we have $\dot{\boldsymbol{\sigma}}^p = \boldsymbol{0}$, $\dot{\boldsymbol{q}} = \boldsymbol{0}$, $\dot{d} = 0$, so the only way to fulfill the entropy inequality in (11.139) is when:

$$\boxed{\boldsymbol{\sigma} = (1-d)\frac{\partial \Psi^0}{\partial \boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^p = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = (1-d)\boldsymbol{\sigma}^0 - \boldsymbol{\sigma}^p} \tag{11.140}$$

or what is the same:

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^0 - \overline{\boldsymbol{\sigma}}^p \tag{11.141}$$

Then, by substituting the equation in (11.140) into the inequality (11.139) we find:

$$-\left(\frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{\sigma}^p} - \boldsymbol{\varepsilon} \right) : \dot{\boldsymbol{\sigma}}^p - \left(\frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{q}} \right) \dot{\boldsymbol{q}} + \Psi^0 \dot{d} \geq 0 \tag{11.142}$$

Now, we can assume a purely damage process, and another purely plastic process, with which we obtain the following restrictions:

$$\boxed{\Psi^0 \dot{d} \geq 0 \qquad and \qquad -\left(\frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{\sigma}^p} - \boldsymbol{\varepsilon}\right) : \dot{\boldsymbol{\sigma}}^p - \left(\frac{\partial \Xi(\boldsymbol{q}, \boldsymbol{\sigma}^p)}{\partial \boldsymbol{q}}\right) \dot{\boldsymbol{q}} \geq 0} \tag{11.143}$$

where $\Psi^0 \geq 0$ holds, then we can conclude that $\dot{d} \geq 0$.

11.4.1.3 Damage Characterization

Simo&Ju(1987) adopted the following norm:

$$\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}} = \sqrt{2\Psi^0(\boldsymbol{\varepsilon})} \xrightarrow{rate} \dot{\overline{\mathcal{T}}}_{\boldsymbol{\varepsilon}} = \frac{1}{2}\left(2\Psi^0\right)^{\frac{-1}{2}} \left(2\dot{\Psi}^0(\boldsymbol{\varepsilon})\right) = \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \frac{\partial \Psi^0}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}} \tag{11.144}$$

Note that we are dealing with a symmetrical norm, and also that when the constitutive equation is linear we obtain $\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}} = \mathcal{T}_{\boldsymbol{\varepsilon}} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^e : \boldsymbol{\varepsilon}} = \sqrt{2\Psi^e}$ which is the same as that outlined in the isotropic damage model, (see equation (11.31)).

We can describe the damage state in the material by means of the damage criteria in the strain space by:

$$\mathcal{G}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, r) = \overline{\mathcal{T}}_{\boldsymbol{\varepsilon}} - r \leq 0 \tag{11.145}$$

Next, we can define the evolution of the damage variable:

$$\dot{d} = \dot{\zeta} \frac{\partial \overline{\mathcal{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)}{\partial r} = \dot{\zeta}\, \mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d) \quad ; \quad \dot{r} = \dot{\zeta}(\boldsymbol{\varepsilon}, r) \tag{11.146}$$

where $\dot{\zeta}$ is the damage consistency parameter which defines the loading/unloading condition (the Kuhn-Tucker conditions):

$$\boxed{\dot{r} = \dot{\zeta} \geq 0 \quad ; \quad \overline{\mathcal{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) \leq 0 \quad ; \quad \dot{\zeta}\, \overline{\mathcal{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)) = 0} \quad \textit{Kuhn-Tucker condition} \tag{11.147}$$

and the consistency (persistency) condition:

$$\boxed{\dot{\zeta}\, \dot{\overline{\mathcal{G}}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r) = 0} \quad \textit{The consistency condition} \tag{11.148}$$

11.4.1.4 The Elastic-Damage Tangent Stiffness Tensor

The elastic-damage tangent stiffness tensor, with no plastic phenomena ($\dot{\boldsymbol{\sigma}}^p = \boldsymbol{0}$), can be obtained similarly to that obtained for the isotropic damage model (see equation (11.81)). The only difference is that in this situation we have strain energy density $\Psi^0(\boldsymbol{\varepsilon})$, not the elastic strain energy density $\Psi^e(\boldsymbol{\varepsilon})$.

Then, by evaluating the rate of change of $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d)$ we obtain:

$$\dot{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, d) = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \boldsymbol{\sigma}}{\partial d} \otimes \dot{d} \tag{11.149}$$

Remember that according to the equation in (11.140) we have $\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = (1-d) \frac{\partial^2 \Psi^0}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}}$, $\frac{\partial \boldsymbol{\sigma}}{\partial d} = -\boldsymbol{\sigma}^0$ and according to (11.146) we have $\dot{d} = \dot{\zeta}\, \mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)$. Additionally, when undergoing plastic loading we have $\dot{r} = \dot{\zeta}(\boldsymbol{\varepsilon}, r)$. So, here, the damage consistency parameter $\dot{\zeta}(\boldsymbol{\varepsilon}, r)$ can be determined by the persistency condition, $\dot{\zeta}(\boldsymbol{\varepsilon}, r) = \dot{\overline{\mathcal{T}}}_{\boldsymbol{\varepsilon}} = \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}}$, whereby:

$$\dot{d} = \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}} \tag{11.150}$$

Then, given all the above considerations, the equation in (11.149) can be rewritten as:

$$\begin{aligned}\dot{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, d) &= (1-d)\frac{\partial^2 \Psi^0}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\sigma}^0 \otimes \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}}\, \mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d) \\ &= \left[(1-d)\frac{\partial^2 \Psi^0}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 \otimes \boldsymbol{\sigma}^0 \right] : \dot{\boldsymbol{\varepsilon}}\end{aligned} \tag{11.151}$$

with which we have obtained the elastic-damage tangent stiffness tensor:

$$\boxed{\mathbb{C}^{tan\_d} = \left[(1-d)\frac{\partial^2 \Psi^0}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 \otimes \boldsymbol{\sigma}^0 \right]} \tag{11.152}$$

Note that the tensor $\mathbb{C}^{tan\_d}$ features major and minor symmetry.

11.4.1.5 Characterization of the Plastic Response. The Elastoplastic-Damage Tangent Stiffness Tensor

Characterization of the plastic response will be formulated in the effective stress space $\overline{\boldsymbol{\sigma}}$ and $\overline{\boldsymbol{\sigma}}^p$, then the following holds:

$$\overline{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \overline{\boldsymbol{\sigma}}^p) = \frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p \tag{11.153}$$

We can also postulate the yield function in the effective stress space, so that the elastic-damage domain is characterized by $\mathcal{F}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q}) \le 0$.

Then, by assuming there is an associated flow rule, the constitutive equations for plastic response are given by:

$$\begin{aligned} &\dot{\overline{\boldsymbol{\sigma}}}^p = \dot{\gamma} \frac{\partial \mathcal{F}\left(\frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p, \boldsymbol{q}\right)}{\partial \boldsymbol{\varepsilon}} && \textit{Plastic flow rule} \\ &\dot{\boldsymbol{q}} = \dot{\gamma}\, \boldsymbol{h}\left(\frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p, \boldsymbol{q}\right) && \textit{Plastic hardening law} \\ &\mathcal{F}\left(\frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p, \boldsymbol{q}\right) \le 0 && \textit{Yield criterion} \end{aligned} \tag{11.154}$$

where $\dot{\overline{\boldsymbol{\sigma}}}^p$ is the rate of change of the plastic relaxation stress tensor, $\dot{\gamma}$ is the plastic consistency parameter, and $\boldsymbol{h}$ is the hardening law and the loading/unloading conditions are given by:

$$\boxed{\mathcal{F}\left(\frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p, \boldsymbol{q}\right) \le 0; \quad \dot{\gamma} \ge 0; \quad \dot{\gamma}\mathcal{F}\left(\frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p, \boldsymbol{q}\right) = 0} \quad \textit{Kuhn-Tucker conditions} \tag{11.155}$$

Now, to obtain the value of the plastic consistency parameter $\dot{\gamma} > 0$ we turn to the consistency condition, with $\dot{\mathcal{F}}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q}) = 0$. Then, the rate of change of $\mathcal{F}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q})$ is given by:

$$\dot{\mathcal{F}}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q}) = \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dot{\overline{\boldsymbol{\sigma}}} + \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dot{\overline{\boldsymbol{\sigma}}} + \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} [\dot{\gamma}\, \boldsymbol{h}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q})] = 0 \tag{11.156}$$

Note that:

$$\dot{\overline{\boldsymbol{\sigma}}}^p = \dot{\gamma} \frac{\partial \mathcal{F}\left(\dfrac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p, \boldsymbol{q} \right)}{\partial \boldsymbol{\varepsilon}} = \dot{\gamma} \frac{\partial \mathcal{F}\left(\overline{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \overline{\boldsymbol{\sigma}}^p), \boldsymbol{q}\right)}{\partial \overline{\boldsymbol{\sigma}}} : \frac{\partial \overline{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}} = \dot{\gamma} \frac{\partial \mathcal{F}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q})}{\partial \overline{\boldsymbol{\sigma}}} : \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \tag{11.157}$$

The rate of change of $\dot{\overline{\boldsymbol{\sigma}}}$ is then evaluated as follows:

$$\overline{\boldsymbol{\sigma}} = \frac{\partial \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} - \overline{\boldsymbol{\sigma}}^p \quad \Rightarrow \quad \dot{\overline{\boldsymbol{\sigma}}} = \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \dot{\overline{\boldsymbol{\sigma}}}^p = \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} \right) \tag{11.158}$$

Next, substituting the above equation into that in (11.156) yields:

$$\begin{aligned}
\dot{\mathcal{F}}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q}) &= \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dot{\overline{\boldsymbol{\sigma}}} + \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = 0 \\
&= \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \left[\frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \left(\dot{\gamma} \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} \right) \right] + \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} [\dot{\gamma}\, \boldsymbol{h}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q})] = 0 \\
&= \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} + \dot{\gamma} \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \boldsymbol{h}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q}) = 0 \\
&= \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \left[\frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} - \frac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \boldsymbol{h}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q}) \right] = 0
\end{aligned} \tag{11.159}$$

Then, $\dot{\gamma}$ can be obtained as follows:

$$\dot{\gamma} = \frac{\dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} - \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \boldsymbol{h}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q})} \tag{11.160}$$

and by substituting (11.160) into (11.158), the result is:

$$\dot{\overline{\boldsymbol{\sigma}}} = \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\gamma} \frac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} \right) = \left[\frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} - \frac{\left(\dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} \right) \otimes \left(\dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \right)}{\dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} - \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \boldsymbol{h}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q})} \right] : \dot{\boldsymbol{\varepsilon}} \tag{11.161}$$

Note, the development of the equation (11.161) in indicial notation is similar to that seen in Chapter 9 in subsection *9.4.4 The Elastoplastic Tangent Tensor*.

Then, the effective elastoplastic tangent stiffness tensor is defined as follows:

$$\boxed{\overline{\mathbb{C}}^{tan\_p} = \frac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} - \frac{\left(\dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} \right) \otimes \left(\dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \right)}{\dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} : \dfrac{\partial^2 \Psi^0(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} : \dfrac{\partial \mathcal{F}}{\partial \overline{\boldsymbol{\sigma}}} - \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{q}} \boldsymbol{h}(\overline{\boldsymbol{\sigma}}, \boldsymbol{q})}} \tag{11.162}$$

Next, if we begin with the equation in (11.132), $\boldsymbol{\sigma} = (1-d)\overline{\boldsymbol{\sigma}}$, we can obtain the rate of change of the Cauchy stress tensor as:

$$\dot{\boldsymbol{\sigma}} = (1-d)\dot{\overline{\boldsymbol{\sigma}}} - \dot{d}\,\overline{\boldsymbol{\sigma}} = (1-d)\overline{\mathbb{C}}^{tan\_p} : \dot{\boldsymbol{\varepsilon}} - \dot{d}\,\overline{\boldsymbol{\sigma}} \tag{11.163}$$

Additionally, if we consider that $\dot{d} = \dfrac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}}$, (see equation (11.150)), we can obtain:

$$\dot{\boldsymbol{\sigma}} = (1-d)\overline{\mathbb{C}}^{tan\_p} : \dot{\boldsymbol{\varepsilon}} - \dot{d} \otimes \overline{\boldsymbol{\sigma}} = (1-d)\overline{\mathbb{C}}^{tan\_p} : \dot{\boldsymbol{\varepsilon}} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \boldsymbol{\sigma}^0 : \dot{\boldsymbol{\varepsilon}} \otimes \overline{\boldsymbol{\sigma}} \tag{11.164}$$

Then, by representing the above equation in indicial notation we can conclude that:

$$\dot{\sigma}_{ij} = (1-d)\overline{\mathbb{C}}^{tan\_p}_{ijkl} \dot{\varepsilon}_{kl} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \sigma^0_{kl} \dot{\varepsilon}_{kl} \overline{\sigma}_{ij} = \left[(1-d)\overline{\mathbb{C}}^{tan\_p}_{ijkl} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \overline{\sigma}_{ij} \sigma^0_{kl} \right] \dot{\varepsilon}_{kl} \tag{11.165}$$

which in tensorial notation becomes:

$$\dot{\boldsymbol{\sigma}} = \left[(1-d)\overline{\mathbb{C}}^{tan\_p} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \overline{\boldsymbol{\sigma}} \otimes \boldsymbol{\sigma}^0 \right] : \dot{\boldsymbol{\varepsilon}} = \mathbb{C}^{tan\_d\text{-}p} : \dot{\boldsymbol{\varepsilon}} \tag{11.166}$$

where we have introduced the elastoplastic-damage tangent stiffness tensor:

$$\boxed{\mathbb{C}^{tan\_d\text{-}p} = (1-d)\overline{\mathbb{C}}^{tan\_p} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{\varepsilon}}} \overline{\boldsymbol{\sigma}} \otimes \boldsymbol{\sigma}^0} \quad \textit{The elastoplastic-damage tangent stiffness tensor} \tag{11.167}$$

which is non-symmetric tensor, since $\mathbb{C}^{tan\_d\text{-}p}$ does not feature major symmetry.

11.5 The Tensile-Compressive Plastic-Damage Model

The next model has two independent internal variables used to describe material degradation caused by tension and compression. Next, we will describe this model according to Faria&Oliver (1993).

In this model we assume that the effective stress tensor can be decomposed as follows:

$$\overline{\boldsymbol{\sigma}} = \overline{\boldsymbol{\sigma}}^+ + \overline{\boldsymbol{\sigma}}^- \tag{11.168}$$

where $\overline{\boldsymbol{\sigma}}^+$ and $\overline{\boldsymbol{\sigma}}^-$ are the tensile and compressive effective stress tensors, respectively, and are defined by means of the spectral representations as follows:

$$\overline{\boldsymbol{\sigma}}^+ = \langle \overline{\boldsymbol{\sigma}} \rangle = \sum_{a=1}^{3} \langle \overline{\sigma}_a \rangle \hat{\boldsymbol{n}}^{(a)} \otimes \hat{\boldsymbol{n}}^{(a)} \quad ; \quad \overline{\boldsymbol{\sigma}}^- = \rangle \overline{\boldsymbol{\sigma}} \langle = \sum_{a=1}^{3} \rangle \overline{\sigma}_a \langle \, \hat{\boldsymbol{n}}^{(a)} \otimes \hat{\boldsymbol{n}}^{(a)} \tag{11.169}$$

where $\langle \bullet \rangle$ is the Macaulay brackets, and where $\langle \bullet \rangle + \rangle \bullet \langle = \bullet$ holds. For example, let us consider the following effective stresses $\overline{\sigma}_1 > 0$, $\overline{\sigma}_2 > 0$, $\overline{\sigma}_3 > 0$, and also let us suppose that the Cauchy stress tensor in the principal stress space is given by $(\overline{\sigma}_1, \overline{\sigma}_2, -\overline{\sigma}_3)$. We can now decompose this tensor as follows:

$$\bar{\sigma}'_{ij} = \begin{bmatrix} \bar{\sigma}_1 & 0 & 0 \\ 0 & \bar{\sigma}_2 & 0 \\ 0 & 0 & -\bar{\sigma}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{\sigma}_1 & 0 & 0 \\ 0 & \bar{\sigma}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=\bar{\sigma}^+_{ij}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\bar{\sigma}_3 \end{bmatrix}}_{=\bar{\sigma}^-_{ij}} \tag{11.170}$$

We can also verify that the following properties hold:

$$\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}^+ + \bar{\boldsymbol{\sigma}}^- \quad \Rightarrow \quad \bar{\boldsymbol{\sigma}} : \mathbf{1} = \mathrm{Tr}(\bar{\boldsymbol{\sigma}}) = \mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+ + \bar{\boldsymbol{\sigma}}^-) = \mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+) + \mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-) \tag{11.171}$$

$$\bar{\boldsymbol{\sigma}}^+ : \bar{\boldsymbol{\sigma}}^- = 0 \tag{11.172}$$

11.5.1 Helmholtz Free Energy

In this model the free energy is a function of the strain $\boldsymbol{\varepsilon}$, the plastic strain $\boldsymbol{\varepsilon}^p$, and the parameters d^+ and d^- and is expressed with:

$$\Psi = \Psi\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, d^+, d^-\right) = \left(1-d^+\right)\Psi^{e^+} + \left(1-d^-\right)\Psi^{e^-} \tag{11.173}$$

where

$$\Psi^{e^+} = \Psi^{e^+}\left(\bar{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)\right) = \frac{1}{2}\bar{\boldsymbol{\sigma}}^+ : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}} \quad ; \quad \Psi^{e^-} = \Psi^{e^-}\left(\bar{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)\right) = \frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}} \tag{11.174}$$

In Chapter 7 we verified $\mathbb{C}^{e^{-1}} \equiv \mathbb{D}^e$ to be the inverse of the elasticity tensor (symmetric fourth-order tensor), whose components are given by:

$$\begin{aligned} \mathbb{C}^{e^{-1}} &= \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{1}\otimes\mathbf{1} + \frac{1}{2\mu}\mathbf{I} = \frac{1}{E}\left[(1+\nu)\,\mathbf{I} - \nu\mathbf{1}\otimes\mathbf{1}\right] \\ \mathbb{C}^{e^{-1}}_{ijkl} \equiv \mathbb{D}^e_{ijkl} &= \frac{1}{E}\left\{(1+\nu)\left[\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right] - \nu\delta_{ij}\delta_{kl}\right\} \end{aligned} \tag{11.175}$$

We can now represent Ψ^{e^+} as:

$$\begin{aligned} \Psi^{e^+} &= \frac{1}{2}\bar{\boldsymbol{\sigma}}^+ : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}} = \frac{1}{2E}\bar{\boldsymbol{\sigma}}^+ : \left[(1+\nu)\,\mathbf{I} - \nu\mathbf{1}\otimes\mathbf{1}\right] : \bar{\boldsymbol{\sigma}} \\ &= \frac{(1+\nu)}{2E}\bar{\boldsymbol{\sigma}}^+ : \bar{\boldsymbol{\sigma}} - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}) \end{aligned} \tag{11.176}$$

Then, taking into account (11.171), the equation in (11.176) can be rewritten as follows:

$$\begin{aligned} \Psi^{e^+} &= \frac{(1+\nu)}{2E}\bar{\boldsymbol{\sigma}}^+ : \bar{\boldsymbol{\sigma}} - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}) \\ &= \frac{(1+\nu)}{2E}\bar{\boldsymbol{\sigma}}^+ : \left(\bar{\boldsymbol{\sigma}}^+ + \bar{\boldsymbol{\sigma}}^-\right) - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\left[\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+) + \mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-)\right] \\ &= \frac{(1+\nu)}{2E}\bar{\boldsymbol{\sigma}}^+ : \bar{\boldsymbol{\sigma}}^+ + \frac{(1+\nu)}{2E}\underbrace{\bar{\boldsymbol{\sigma}}^+ : \bar{\boldsymbol{\sigma}}^-}_{=0} - \frac{\nu}{2E}\left[\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\right]^2 - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-) \end{aligned} \tag{11.177}$$

Afterwards, Ψ^{e^+} can be expressed in the following ways:

$$\begin{aligned} \Psi^{e^+} &= \frac{(1+\nu)}{2E}\bar{\boldsymbol{\sigma}}^+ : \bar{\boldsymbol{\sigma}}^+ - \frac{\nu}{2E}\left[\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\right]^2 - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-) \\ &= \frac{(1+\nu)}{2E}\bar{\boldsymbol{\sigma}}^+ : \bar{\boldsymbol{\sigma}} - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}) = \frac{1}{2}\bar{\boldsymbol{\sigma}}^+ : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}}^+ - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-) \end{aligned} \tag{11.178}$$

Likewise, we can obtain Ψ^{e^-} as:

$$\begin{aligned}\Psi^{e^-} &= \frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}} = \frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \left(\bar{\boldsymbol{\sigma}}^+ + \bar{\boldsymbol{\sigma}}^-\right) = \frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}}^+ + \frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}}^- \\ &= \frac{1}{2E}\bar{\boldsymbol{\sigma}}^- : \left[(1+\nu)\,\mathbf{I} - \nu\mathbf{1}\otimes\mathbf{1}\right] : \bar{\boldsymbol{\sigma}}^+ + \frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}}^- \\ &= \frac{(1-\nu)}{2E}\underbrace{\bar{\boldsymbol{\sigma}}^- : \bar{\boldsymbol{\sigma}}^+}_{=0} - \frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-) + \frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}}^-\end{aligned} \tag{11.179}$$

Then, taking into account that $\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-) < 0$ and $\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+) > 0$ we can guarantee that:

$$\Psi^{e^-} = \underbrace{-\frac{\nu}{2E}\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^+)\mathrm{Tr}(\bar{\boldsymbol{\sigma}}^-)}_{\geq 0} + \underbrace{\frac{1}{2}\bar{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}}^-}_{\geq 0} \geq 0 \tag{11.180}$$

Now, the damage variable values lie between the following ranges:

$$0 \leq d^+ \leq 1 \quad ; \quad 0 \leq d^- \leq 1 \tag{11.181}$$

Next, by considering the Helmholtz free energy $\Psi = \Psi\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, d^+, d^-\right)$, its rate of change is evaluated as follows:

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^p} : \boldsymbol{\varepsilon}^p + \frac{\partial \Psi}{\partial d^+}\dot{d}^+ + \frac{\partial \Psi}{\partial d^-}\dot{d}^- \tag{11.182}$$

Remember from subsection *6.4.1 Constitutive Equations with Internal Variables* in Chapter 6, that the terms $-\dfrac{\partial \Psi}{\partial \boldsymbol{\alpha}_i} = \mathbf{A}_i$ were denoted by the thermodynamic forces, where $\boldsymbol{\alpha}_i$ are the set of internal variables. Therefore, with the denotations $\alpha_1 = d^+$, $\alpha_2 = d^-$, $\mathrm{A}_1 = \mathrm{A}^+$ and $\mathrm{A}_2 = \mathrm{A}^-$, the thermodynamic forces becomes:

$$\mathrm{A}^+ = -\frac{\partial \Psi}{\partial d^+} = \Psi^{e^+} \quad ; \quad \mathrm{A}^- = -\frac{\partial \Psi}{\partial d^-} = \Psi^{e^-} \tag{11.183}$$

where we have taken into account that:

$$\Psi = \left(1-d^+\right)\Psi^{e^+} + \left(1-d^-\right)\Psi^{e^-} \quad \Rightarrow \quad \begin{cases} \dfrac{\partial \Psi}{\partial d^+} = -\Psi^{e^+} \\ \dfrac{\partial \Psi}{\partial d^-} = -\Psi^{e^-} \end{cases} \tag{11.184}$$

11.5.2 Damage Characterization

In order to fully define this model we need to characterize the loading/unloading/loading process.

Next, we will define the norm in the tension stress space:

$$\bar{\tau}^+ = \sqrt{\bar{\boldsymbol{\sigma}}^+ : \mathbb{C}^{e^{-1}} : \bar{\boldsymbol{\sigma}}^+} \tag{11.185}$$

In compression, the norm (based on the Drucker-Prager model and which was obtained by Faria&Oliver (1993)) can be expressed in terms of the normal octahedral effective stress ($\bar{\sigma}^-_{oct}$) and octahedral tangential stress ($\bar{\tau}^-_{oct}$) and is given by:

$$\overline{\mathcal{T}}^{-}=\sqrt{\overline{\overline{q}}(\overline{\boldsymbol{\sigma}}^{-})}=\sqrt{\sqrt{3}\left(K\,\overline{\sigma}^{-}_{oct}+\overline{\tau}^{-}_{oct}\right)} \tag{11.186}$$

proof of which can be found in the chapter on Plasticity (the Drucker-Prager Criterion).

We will now introduce two damage criteria denoted by g^{+} and g^{-}:

$$g^{+}(\overline{\mathcal{T}}^{+},r^{+})=\overline{\mathcal{T}}^{+}-r^{+}\le 0 \quad ; \quad g^{-}(\overline{\mathcal{T}}^{-},r^{-})=\overline{\mathcal{T}}^{-}-r^{-}\le 0 \tag{11.187}$$

where r^{+} and r^{-} are the current damage thresholds which serve to "remind" us where the damage surface is during the loading/unloading/loading process and whose initial values are represented by r_0^{+} and r_0^{-}, respectively.

Then, by defining $f^{+}_{0_{1D}}$ and $f^{-}_{0_{1D}}$ as the stresses after a visible non-linearity in a uniaxial tensile and compression tests, we obtain $\boldsymbol{\sigma}^{1D}_{oct}=\frac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma})=\frac{1}{3}f^{-}_{0_{1D}}$, $\tau^{1D}_{oct}=\sqrt{\frac{2}{3}\mathrm{J}_2}=-\frac{\sqrt{2}}{3}f^{-}_{0_{1D}}$, (see Chapter 9 in the subsection: 9.2.4.2 The Drucker-Prager Yield Criterion), from which we define the following elastic thresholds:

$$r_0^{+}=\sqrt{f^{+}_{0_{1D}}\frac{1}{E}f^{+}_{0_{1D}}}=\frac{f^{+}_{0_{1D}}}{\sqrt{E}} \quad ; \quad r_0^{-}=\sqrt{\frac{\sqrt{3}}{3}(K-\sqrt{2})f^{-}_{0_{1D}}} \tag{11.188}$$

11.5.3 Evolution of the Damage Parameters

When observing how the damage parameters evolve the following equations in rates are considered:

- Tension: $$\dot{d}^{+}=\dot{\zeta}^{+}\frac{\partial G^{+}(\overline{\mathcal{T}}^{+})}{\partial\overline{\mathcal{T}}^{+}} \quad ; \quad \dot{\zeta}^{+}=\dot{r}^{+} \tag{11.189}$$
- Compression: $$\dot{d}^{-}=\dot{\zeta}^{-}\frac{\partial G^{-}(\overline{\mathcal{T}}^{-})}{\partial\overline{\mathcal{T}}^{-}} \quad ; \quad \dot{\zeta}^{-}=\dot{r}^{-} \tag{11.190}$$

where G^{+} and G^{-} are monotonically increasing functions (obtained from experimental observations), and $\dot{\zeta}^{+}$ and $\dot{\zeta}^{+}$ are the damage consistency parameters. Then, the Kuhn-Tucker conditions are:

Tension

$$\dot{\zeta}^{+}\ge 0 \quad ; \quad g^{+}\le 0 \quad ; \quad \dot{\zeta}^{+}g^{+}=0 \tag{11.191}$$

Compression

$$\dot{\zeta}^{-}\ge 0 \quad ; \quad g^{-}\le 0 \quad ; \quad \dot{\zeta}^{-}g^{-}=0 \tag{11.192}$$

We can verify that when $g^{+}<0$ damage ceases and from the Kuhn-Tucker conditions we obtain $\dot{\zeta}^{+}=0$. When $\dot{\zeta}^{+}>0$ there is damage evolution and here the Kuhn-Tucker conditions hold if $g^{+}=0$, that is, providing that the current state is on the damage surface. In these conditions it is also true that $\overline{\mathcal{T}}^{+}=\dot{r}^{+}$ and:

$$r^{+}=\max\left\{r_0^{+},\max\left(\overline{\mathcal{T}}^{+}\right)\right\} \tag{11.193}$$

likewise, for r^{-}:

$$r^- = \max\left\{ r_0^-, \max\left(\overline{\tau}^-\right) \right\} \tag{11.194}$$

Then, as the damage evolves, we can state that:

$$\dot{d}^+ = \dot{\overline{\tau}}^+ \frac{\partial G^+(\overline{\tau}^+)}{\partial \overline{\tau}^+} = \dot{G}^+ \geq 0 \quad ; \quad \dot{d}^- = \dot{\overline{\tau}}^- \frac{\partial G^-(\overline{\tau}^-)}{\partial \overline{\tau}^-} = \dot{G}^- \geq 0 \tag{11.195}$$

11.5.4 Evolution of the Plastic Strain Tensor

The plastic strain tensor evolution law, adopted by Faria&Oliver (1993), is defined as follows:

$$\boxed{\dot{\boldsymbol{\varepsilon}}^p = \beta E H(\dot{d}^-) \frac{\langle \overline{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \rangle}{\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}} \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}}} \tag{11.196}$$

where E is the Young's modulus and $\beta \geq 0$ is also a material parameter that will control the rate intensity of plastic strain. The value $\beta = 0$ is equivalent to the elastic damage case. $H(\dot{d}^-)$ is the Heaviside function of the compression damage rate, which was introduced to stop plastic evolution during compression unloading. Also, note that

$$\frac{\overline{\boldsymbol{\sigma}}}{\sqrt{\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}}} = \frac{\overline{\boldsymbol{\sigma}}}{\left\|\overline{\boldsymbol{\sigma}}\right\|} = \hat{\overline{\boldsymbol{\sigma}}} \quad \Rightarrow \quad \left\|\hat{\overline{\boldsymbol{\sigma}}}\right\| = 1 \tag{11.197}$$

with which the equation in (11.196) can be rewritten as:

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &= \beta E H(\dot{d}^-) \frac{\langle \overline{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \rangle}{\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}} \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} = \beta E H(\dot{d}^-) \frac{\langle \overline{\boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}} \rangle}{\sqrt{\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}}\sqrt{\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}}} \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} \\ &= \beta E H(\dot{d}^-) \langle \frac{\overline{\boldsymbol{\sigma}}}{\sqrt{\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}}} : \dot{\boldsymbol{\varepsilon}} \rangle \mathbb{C}^{e^{-1}} : \frac{\overline{\boldsymbol{\sigma}}}{\sqrt{\overline{\boldsymbol{\sigma}} : \overline{\boldsymbol{\sigma}}}} = \beta E H(\dot{d}^-) \langle \hat{\overline{\boldsymbol{\sigma}}} : \dot{\boldsymbol{\varepsilon}} \rangle \mathbb{C}^{e^{-1}} : \hat{\overline{\boldsymbol{\sigma}}} \end{aligned} \tag{11.198}$$

11.5.5 Internal Energy Dissipation

As how we have proceeded in previous models, we start from the Clausius-Planck inequality to put restrictions on the thermodynamic variables:

$$\mathcal{D}_{int} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^p, d^+, d^-) \geq 0 \tag{11.199}$$

Then, by evaluating the rate of change of $\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^p, d^+, d^-)$ we obtain:

$$\dot{\Psi}(\boldsymbol{\varepsilon}, \boldsymbol{\sigma}^p, d^+, d^-) = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p + \frac{\partial \Psi}{\partial d^+} \dot{d}^+ + \frac{\partial \Psi}{\partial d^-} \dot{d}^- \tag{11.200}$$

and by substituting this into the internal energy dissipation given in (11.199), we find:

$$\left(\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p - \frac{\partial \Psi}{\partial d^+} \dot{d}^+ - \frac{\partial \Psi}{\partial d^-} \dot{d}^- \geq 0 \tag{11.201}$$

Then, as the above inequality must be satisfied for any admissible thermodynamic process, we can draw the conclusion that:

$$\boxed{\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}} \tag{11.202}$$

which is the constitutive equation for stress.

In a small deformation regime, we can decompose additively the infinitesimal strain tensor into elastic ($\boldsymbol{\varepsilon}^e$) and plastic ($\boldsymbol{\varepsilon}^p$) parts, *i.e.*:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \qquad \Rightarrow \qquad \boldsymbol{\varepsilon}^e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p \tag{11.203}$$

In this way, we can replace two variables ($\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p$) with one $\boldsymbol{\varepsilon}^e$.

Then, the free energy defined in (11.173) can be expressed in terms of:

$$\Psi = \Psi\left(\boldsymbol{\varepsilon}^e, d^+, d^-\right) = \left(1 - d^+\right)\Psi^{e^+}(\boldsymbol{\varepsilon}^e) + \left(1 - d^-\right)\Psi^{e^-}(\boldsymbol{\varepsilon}^e) \tag{11.204}$$

whereby the stress is:

$$\begin{aligned}\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} &= \frac{\partial}{\partial \boldsymbol{\varepsilon}}\left[\left(1 - d^+\right)\Psi^{e^+}(\boldsymbol{\varepsilon}^e) + \left(1 - d^-\right)\Psi^{e^-}(\boldsymbol{\varepsilon}^e)\right] \\ &= \left(1 - d^+\right)\frac{\partial \Psi^{e^+}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} : \frac{\partial \boldsymbol{\varepsilon}^e}{\partial \boldsymbol{\varepsilon}} + \left(1 - d^-\right)\frac{\partial \Psi^{e^-}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} : \frac{\partial \boldsymbol{\varepsilon}^e}{\partial \boldsymbol{\varepsilon}}\end{aligned} \tag{11.205}$$

Then, taking into account that $\dfrac{\partial \boldsymbol{\varepsilon}^e}{\partial \boldsymbol{\varepsilon}} = \mathbf{I}$ (symmetric fourth-order unit tensor), the above equation becomes:

$$\boldsymbol{\sigma} = \left(1 - d^+\right)\frac{\partial \Psi^{e^+}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} + \left(1 - d^-\right)\frac{\partial \Psi^{e^-}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} = \left(1 - d^+\right)\overline{\boldsymbol{\sigma}}^+ + \left(1 - d^-\right)\overline{\boldsymbol{\sigma}}^- = \boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^- \tag{11.206}$$

thereby defining the following effective stresses:

$$\overline{\boldsymbol{\sigma}}^+ = \frac{\partial \Psi^{e^+}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} = \frac{1}{\left(1 - d^+\right)}\boldsymbol{\sigma}^+ \quad ; \quad \overline{\boldsymbol{\sigma}}^- = \frac{\partial \Psi^{e^-}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} = \frac{1}{\left(1 - d^-\right)}\boldsymbol{\sigma}^- \tag{11.207}$$

In addition, we can express the energy function as:

$$\begin{aligned}\Psi &= \left(1 - d^+\right)\Psi^{e^+} + \left(1 - d^-\right)\Psi^{e^-} = \left(1 - d^+\right)\frac{1}{2}\overline{\boldsymbol{\sigma}}^+ : \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} + \left(1 - d^-\right)\frac{1}{2}\overline{\boldsymbol{\sigma}}^- : \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} \\ &= \frac{1}{2}\boldsymbol{\sigma}^+ : \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} + \frac{1}{2}\boldsymbol{\sigma}^- : \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} = \frac{1}{2}\left(\boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^-\right) : \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} \\ &= \frac{1}{2}\boldsymbol{\sigma} : \mathbb{C}^{e^{-1}} : \overline{\boldsymbol{\sigma}} = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}\end{aligned} \tag{11.208}$$

where we have taken into account the expressions of Ψ^{e^+} and Ψ^{e^-} given in (11.174).

Then, if we consider the internal energy dissipation in (11.201) and the constitutive equation in (11.202), we can conclude that:

$$\mathcal{D}_{int} = -\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p - \frac{\partial \Psi}{\partial d^+}\dot{d}^+ - \frac{\partial \Psi}{\partial d^-}\dot{d}^- = -\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^p} : \dot{\boldsymbol{\varepsilon}}^p + \Psi^{e^+}\dot{d}^+ + \Psi^{e^-}\dot{d}^- \geq 0 \tag{11.209}$$

where $\Psi^{e^+} \geq 0$ and $\Psi^{e^-} \geq 0$ are positive by definition. Moreover if we consider there be a process with neither plastic evolution nor the evolution of the parameter d^-, we can conclude that $\dot{d}^+ \geq 0$. Likewise, we can obtain $\dot{d}^- \geq 0$. Thus:

$$\dot{d}^+ \geq 0 \quad ; \quad \dot{d}^- \geq 0 \tag{11.210}$$

If we now consider there to be a purely plastic process, the internal energy dissipation becomes:

$$-\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^p}:\dot{\boldsymbol{\varepsilon}}^p=-\frac{\partial\Psi(\boldsymbol{\varepsilon}^e)}{\partial\boldsymbol{\varepsilon}^e}:\frac{\partial\boldsymbol{\varepsilon}^e}{\partial\boldsymbol{\varepsilon}^p}:\dot{\boldsymbol{\varepsilon}}^p=-\frac{\partial\Psi(\boldsymbol{\varepsilon}^e)}{\partial\boldsymbol{\varepsilon}^e}:(-\mathbf{I}):\dot{\boldsymbol{\varepsilon}}^p=\frac{\partial\Psi(\boldsymbol{\varepsilon}^e)}{\partial\boldsymbol{\varepsilon}^e}:\dot{\boldsymbol{\varepsilon}}^p\geq 0 \tag{11.211}$$

Note, the above term $\frac{\partial\Psi(\boldsymbol{\varepsilon}^e)}{\partial\boldsymbol{\varepsilon}^e}$ can be obtained directly from the equation in (11.208):

$$\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^e}=\frac{\partial}{\partial\boldsymbol{\varepsilon}^e}\left[\frac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\varepsilon}\right]=\frac{1}{2}\left(\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{\varepsilon}^e}:\boldsymbol{\varepsilon}+\boldsymbol{\sigma}:\frac{\partial\boldsymbol{\varepsilon}}{\partial\boldsymbol{\varepsilon}^e}\right)=\frac{1}{2}\left(\mathbb{C}^e:\boldsymbol{\varepsilon}+\boldsymbol{\sigma}:\mathbf{I}\right)=\frac{1}{2}(\boldsymbol{\sigma}+\boldsymbol{\sigma})=\boldsymbol{\sigma} \tag{11.212}$$

Then, restructuring evolution law of $\dot{\boldsymbol{\varepsilon}}^p$ given in (11.196) we have:

$$\dot{\boldsymbol{\varepsilon}}^p=a\mathbb{C}^{e^{-1}}:\overline{\boldsymbol{\sigma}}= \qquad with \qquad a=\beta EH(\dot{d}^-)\frac{\langle\overline{\boldsymbol{\sigma}}:\dot{\boldsymbol{\varepsilon}}\rangle}{\overline{\boldsymbol{\sigma}}:\overline{\boldsymbol{\sigma}}}\geq 0 \tag{11.213}$$

and by incorporating the equations (11.212) and (11.213) into the internal energy dissipation in (11.211), we can obtain:

$$\frac{\partial\Psi(\boldsymbol{\varepsilon}^e)}{\partial\boldsymbol{\varepsilon}^e}:\dot{\boldsymbol{\varepsilon}}^p\geq 0 \quad ; \quad a\,\boldsymbol{\sigma}:\mathbb{C}^{e^{-1}}:\overline{\boldsymbol{\sigma}}\geq 0 \quad ; \quad 2a\Psi\geq 0 \tag{11.214}$$

after which the internal energy dissipation becomes:

$$\mathcal{D}_{int}=2a\Psi+\Psi^{e^+}\dot{d}^+ +\Psi^{e^-}\dot{d}^-\geq 0 \tag{11.215}$$

Further details about the numerical implementation of the tensile-compressive plastic-damage model are described in Faria&Oliver (1993).

11.6 Damage in a Large Deformation Regime

The classical hyperelastic models (large deformation regime) discussed in Chapter 8 are not capable of describing how certain polymers characterized by loss of stiffness behave. This dissipation phenomenon is known as the *Mullins effect* which was studied by several researchers, among whom we can cite: Bueche (1960), (1961), Mullins (1969) and Souza Neto *et al.* (1998). In the uniaxial cyclic test, the Mullins effect is phenomenologically characterized by degradation of the elastic properties, (see Figure 11.14). Let us now consider the stress-strain curve described in Figure 11.14. During loading (branch $[0-1]$), the path is A and unloading is done according to path B. Then, after the unloading is completed, the material fully recovers its initial state. The second loading will take place according to path B and follow on to path C. Note that according to the classical hyperelastic models, loading will take place according to path $A-C$ and unloading would occur along the same path $C-A$.

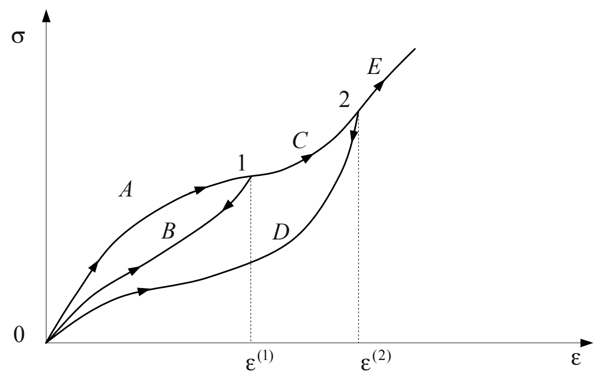


Figure 11.14: Mullins effects.

11.6.1 Gurtin & Francis' One-Dimensional Model

Gurtin&Francis(1981) proposed a simple one-dimensional model in which the current state of the damage variable is characterized by the maximum axial strain ε^m:

$$\varepsilon^m(t) = \max_{0\le s\le t}\{\varepsilon(s)\} \tag{11.216}$$

In this model Gurtin&Francis adopted as the constitutive equation for stress, σ, by means of the current strain state and the damage variable as follows:

$$\sigma = \bar{f}(\zeta)\,\bar{g}(\varepsilon^m) \tag{11.217}$$

where $\bar{g}(\varepsilon^m)$ is called the *virgin curve* and ζ is the relative strain:

$$\zeta = \frac{\varepsilon}{\varepsilon^m} \tag{11.218}$$

The function $\bar{f}(\zeta)$, called the *damage master curve*, defines the loss of stiffness and satisfies:

$$\bar{f}(1) = 1 \tag{11.219}$$

Then, when the maximum strain takes place in the current time $(\varepsilon^m = \varepsilon)$, the uniaxial stress is given by:

$$\sigma = \bar{g}(\varepsilon^m) \tag{11.220}$$

Then, the function $\bar{g}$ defines the uniaxial stress-strain curve obtained from a monotonically increasing/decreasing uniaxial test. In Figure 11.14, this function is defined according to path ACE.

Now, to fully describe the material parameters in this model, we need to determine both the virgin curve and the damage master curve $\bar{f}(\zeta)$. This latter curve is obtained from the uniaxial test.

11.6.2 The Rate Independent 3D Elastic-Damage Model

Based on the concepts of the Gurtin&Francis' model, Souza Neto *et al.* (1994), (1998) extended this model to the 3D model which will be explained below.

Let us now consider a isotropic hyperelastic material, (see Chapter 8), which is governed by the free energy, Ψ^0, described in terms of the principal stretches (λ_1, λ_2, λ_3). Now, the Kirchhoff stress tensor eigenvalues, in terms of principal stretches, can be expressed as follows:

$$\tau_a = \lambda_a \frac{\partial \Psi^0}{\partial \lambda_a} =: g_a(\lambda_1, \lambda_2, \lambda_3) \tag{11.221}$$

The above equation is valid only when we are dealing with virgin material during loading. Then, the general form of (11.221) can be expressed as follows:

$$\tau_a = f(\xi) g_a(\lambda_1, \lambda_2, \lambda_3) \tag{11.222}$$

As when we looked at the 1D case, we will define a function dependent on the relative strain, ξ, in 3D, in which the following remains valid:

$$f(1) = 1 \tag{11.223}$$

11.6.3 The Damage Variable. Damage Evolution

We will now define the damage variable d, which records the level of damage suffered by the material during the loading history, as:

$$d(t) = \max_{0 \le s \le t} \left\{ \Psi^0(s) \right\} \tag{11.224}$$

We can now define the relative strain (ξ) as follows:

$$\xi := \frac{\Psi^0}{d} \tag{11.225}$$

If we draw an analogy with the yield surface from classical plasticity, we can define a *damage surface* in the principal stretch space as follows:

$$\Phi(\lambda_1, \lambda_2, \lambda_3, d) := \Psi^0(\lambda_1, \lambda_2, \lambda_3) - d = 0 \tag{11.226}$$

For a fixed value of d, the damage surface limits a region in the principal stretches space where the material behavior is purely elastic, *i.e.* where there is no damage evolution. As with plasticity, the damage variable evolution is characterized by the loading/unloading condition:

$$\Phi \le 0 \quad ; \quad \dot{d} \ge 0 \quad ; \quad \dot{d}\Phi = 0 \tag{11.227}$$

So, we can summarize this model as follows

$$\begin{array}{ll}
\textbf{i.} & \text{Damage Variable} \\
& \quad d(t) = \max\limits_{0 \le s \le t} \left\{ \Psi^0(s) \right\} \\
\textbf{ii.} & \text{The Constitutive Equation} \\
& \quad \tau_i = f(\xi) g_i(\lambda_1, \lambda_2, \lambda_3) \\
& \quad \xi := \dfrac{\Psi^0}{d} \\
\textbf{iii.} & \text{Damage Surface} \\
& \quad \Phi(\lambda_1, \lambda_2, \lambda_3, d) := \Psi^0(\lambda_1, \lambda_2, \lambda_3) - d = 0 \\
\textbf{iv.} & \text{Loading/Unloading Criterion} \\
& \quad \Phi \le 0 \qquad \dot{d} \ge 0 \qquad \dot{d}\Phi = 0
\end{array} \tag{11.228}$$

11.6.4 The Plastic-Damage Model by Simó & Ju (1989)

We will now discuss the plastic-damage model in a large deformation regime (finite strain) proposed by Simo&Ju (1989). Note that, the way in which the plastic-damage model by Simo&Ju (1987a,b) in a small deformation regime was described, the extension of this model to large deformation regime is almost trivial.

11.6.4.1 Specific Helmholtz Free Energy

The Helmholtz free energy (per unit mass) in the reference configuration is given by:

$$\begin{aligned}\psi = \psi(\boldsymbol{C}, \mathbf{S}^p, \boldsymbol{A}, d) &= (1-d)\psi^0(\boldsymbol{C}) - \frac{1}{\rho_0}\boldsymbol{E} : \mathbf{S}^p + \Xi(\boldsymbol{A}, \mathbf{S}^p) \\ &= (1-d)\psi^0(\boldsymbol{C}) - \frac{1}{\rho_0}\left[\frac{1}{2}(\boldsymbol{C} - \mathbf{1})\right] : \mathbf{S}^p + \Xi(\boldsymbol{A}, \mathbf{S}^p)\end{aligned} \tag{11.229}$$

where $\boldsymbol{C} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ is the left Cauchy-Green deformation tensor, $\boldsymbol{F}$ is the deformation gradient, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1})$ is the Green-Lagrange strain tensor, $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, $\boldsymbol{A}$ is the set of internal plastic variables, d is the damage variable, $\mathbf{S}^p$ is the plastic relaxation stress tensor, and $\Xi(\boldsymbol{A}, \mathbf{S}^p)$ is the plastic potential function. Moreover, it is noteworthy that ψ^0, $\frac{1}{\rho_0}\boldsymbol{E} : \mathbf{S}^p$ and $\Xi(\boldsymbol{A}, \mathbf{S}^p)$ have the unit of energy per unit mass.

11.6.4.2 Internal Energy Dissipation. Constitutive Equations. Thermodynamic Considerations

Remember that in Chapter 5 the alternative form of the Clausius-Planck inequality in the reference configuration is given by:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0\left[\eta\dot{T} + \dot{\psi}\right] \geq 0 \tag{11.230}$$

in which all variables are described in the reference configuration, and ρ_0 is the mass density, η is the specific entropy, T is temperature, and ψ is the specific Helmholtz free energy (*per mass unit*). In isothermal processes we have $\dot{T} = 0$, in which the Clausius-Planck inequality becomes:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \rho_0\dot{\psi} \geq 0 \tag{11.231}$$

Then, by evaluating the rate of change of the Helmholtz free energy, $\psi(\boldsymbol{C}, \mathbf{S}^p, \boldsymbol{A}, d)$, we obtain:

$$\dot{\psi}(\boldsymbol{C}, \mathbf{S}^p, \boldsymbol{A}, d) = \frac{\partial \psi}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}} + \frac{\partial \psi}{\partial \mathbf{S}^p} : \dot{\mathbf{S}}^p + \frac{\partial \psi}{\partial \boldsymbol{A}}\dot{\boldsymbol{A}} + \frac{\partial \psi}{\partial d}\dot{d} \tag{11.232}$$

Next, if we consider the expression of the Helmholtz free energy given in (11.229) we find:

$$\begin{aligned}&\frac{\partial \psi}{\partial \boldsymbol{C}} = (1-d)\frac{\partial \psi^0(\boldsymbol{C})}{\partial \boldsymbol{C}} - \frac{1}{2\rho_0}\mathbf{S}^p, \quad \frac{\partial \psi}{\partial \mathbf{S}^p} = -\frac{1}{\rho_0}\boldsymbol{E} + \frac{\partial \Xi(\boldsymbol{A}, \mathbf{S}^p)}{\partial \mathbf{S}^p}, \quad \frac{\partial \psi}{\partial \boldsymbol{A}} = \frac{\partial \Xi(\boldsymbol{A}, \mathbf{S}^p)}{\partial \boldsymbol{A}}, \\ &\frac{\partial \psi}{\partial d} = -\psi^0(\boldsymbol{C})\end{aligned} \tag{11.233}$$

After that, substituting (11.232) into the entropy inequality (11.231), and by considering the equations in (11.233), we have:

$$\mathbf{S}:\dot{\boldsymbol{E}}-\rho_0\left\{\left[(1-d)\frac{\partial\psi^0(\boldsymbol{C})}{\partial\boldsymbol{C}}-\frac{1}{2\rho_0}\mathbf{S}^p\right]:\dot{\boldsymbol{C}}+\left[\frac{\partial\Xi}{\partial\mathbf{S}^p}-\frac{1}{\rho_0}\boldsymbol{E}\right]:\dot{\mathbf{S}}^p+\frac{\partial\Xi}{\partial\boldsymbol{A}}\dot{\boldsymbol{A}}-\psi^0(\boldsymbol{C})\dot{d}\right\}\geq 0 \tag{11.234}$$

or:

$$\begin{aligned}&\mathbf{S}:\dot{\boldsymbol{E}}-\rho_0(1-d)\frac{\partial\psi^0(\boldsymbol{C})}{\partial\boldsymbol{C}}:\dot{\boldsymbol{C}}+\\&\qquad\frac{1}{2}\mathbf{S}^p:\dot{\boldsymbol{C}}-\rho_0\left[\frac{\partial\Xi}{\partial\mathbf{S}^p}-\frac{1}{\rho_0}\boldsymbol{E}\right]:\dot{\mathbf{S}}^p-\rho_0\frac{\partial\Xi}{\partial\boldsymbol{A}}\dot{\boldsymbol{A}}+\rho_0\psi^0(\boldsymbol{C})\dot{d}\geq 0\end{aligned} \tag{11.235}$$

Remember from Chapter 8 (Hyperelasticity) that $2\frac{\partial\psi^0(\boldsymbol{C})}{\partial\boldsymbol{C}}=\frac{\partial\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}}$, and in addition $\boldsymbol{E}=\frac{1}{2}(\boldsymbol{C}-\mathbf{1})\Rightarrow\dot{\boldsymbol{E}}=\frac{1}{2}\dot{\boldsymbol{C}}$ holds, so we can now rewrite the entropy inequality as:

$$\left[\mathbf{S}-\rho_0(1-d)\frac{\partial\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}}+\mathbf{S}^p\right]:\dot{\boldsymbol{E}}-\left[\rho_0\frac{\partial\Xi}{\partial\mathbf{S}^p}-\boldsymbol{E}\right]:\dot{\mathbf{S}}^p-\rho_0\frac{\partial\Xi}{\partial\boldsymbol{A}}\dot{\boldsymbol{A}}+\rho_0\psi^0(\boldsymbol{C})\dot{d}\geq 0 \tag{11.236}$$

which must be satisfied for any admissible thermodynamic process, so, by considering the following process $\dot{\mathbf{S}}^p=\mathbf{0}$, $\dot{\boldsymbol{A}}=\mathbf{0}$, $\dot{d}=0$, we obtain:

$$\left[\mathbf{S}-(1-d)\rho_0\frac{\partial\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}}+\mathbf{S}^p\right]:\dot{\boldsymbol{E}}\geq 0 \tag{11.237}$$

Furthermore, if we consider two processes where $\dot{\boldsymbol{E}}>\mathbf{0}$ and $\dot{\boldsymbol{E}}<\mathbf{0}$, the only way of enforcing the inequality is when the term within the brackets is equal to zero, *i.e.:*

$$\boxed{\mathbf{S}=(1-d)\rho_0\frac{\partial\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}}-\mathbf{S}^p=(1-d)\mathbf{S}^0-\mathbf{S}^p} \tag{11.238}$$

where $\mathbf{S}^0=\rho_0\frac{\partial\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}}$ is the non-damage stress tensor. Additionally, if we take into account that $\mathbf{S}=(1-d)\overline{\mathbf{S}}$ and $\mathbf{S}^p=(1-d)\overline{\mathbf{S}}^p$, where $\overline{\mathbf{S}}^p$ is the plastic relaxation effective stress tensor, we obtain:

$$\boxed{\begin{aligned}\mathbf{S}&=(1-d)\mathbf{S}^0-\mathbf{S}^p\\(1-d)\overline{\mathbf{S}}&=(1-d)\mathbf{S}^0-(1-d)\overline{\mathbf{S}}^p\\\overline{\mathbf{S}}&=\mathbf{S}^0-\overline{\mathbf{S}}^p\end{aligned}} \tag{11.239}$$

Then, substituting the constitutive equation in (11.238) into the inequality in (11.236) yields:

$$-\left[\rho_0\frac{\partial\Xi}{\partial\mathbf{S}^p}-\boldsymbol{E}\right]:\dot{\mathbf{S}}^p-\rho_0\frac{\partial\Xi}{\partial\boldsymbol{A}}\dot{\boldsymbol{A}}+\rho_0\psi^0(\boldsymbol{C})\dot{d}\geq 0 \tag{11.240}$$

Similarly, if there is a pure damage process and then another pure plastic process, we can obtain the following restrictions:

$$\boxed{\rho_0 \psi^0(\boldsymbol{C})\dot{d} \geq 0 \quad ; \quad -\left[\rho_0 \frac{\partial \Xi(\boldsymbol{A}, \mathbf{S}^p)}{\partial \mathbf{S}^p} - \boldsymbol{E}\right] : \dot{\mathbf{S}}^p - \rho_0 \frac{\partial \Xi(\boldsymbol{A}, \mathbf{S}^p)}{\partial \boldsymbol{A}} \dot{\boldsymbol{A}} \geq 0} \tag{11.241}$$

11.6.4.3 Damage Characterization

Here, we will adopt the following energy norm:

$$\overline{\mathcal{T}}_{\boldsymbol{E}} = \sqrt{2\Psi^0(\boldsymbol{E})} \xrightarrow{rate} \dot{\overline{\mathcal{T}}}_{\boldsymbol{E}} = \frac{1}{2}\left(2\Psi^0\right)^{\frac{-1}{2}}\left(2\dot{\Psi}^0(\boldsymbol{E})\right) = \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \frac{\partial \Psi^0}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} = \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \mathbf{S}^0 : \dot{\boldsymbol{E}} \tag{11.242}$$

We emphasize here that $\Psi^0(\boldsymbol{E}) = \rho_0 \psi^0(\boldsymbol{E})$ has the unit of energy per unit volume (energy density).

We can then characterize the state of damage in the material by means of the damage criterion defined in the principal strain space as:

$$\mathcal{G}(\overline{\mathcal{T}}_{\boldsymbol{E}}, r) = \overline{\mathcal{T}}_{\boldsymbol{E}} - r \leq 0 \tag{11.243}$$

Then, we can define the damage evolution law as follows:

$$\dot{d} = \dot{\zeta} \frac{\partial \overline{\mathcal{G}}(\mathcal{T}_{\boldsymbol{\varepsilon}}, r)}{\partial r} = \dot{\zeta}\, \mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d) \quad ; \quad \dot{r} = \dot{\zeta}(\boldsymbol{E}, r) \tag{11.244}$$

where $\dot{\zeta}$ is the damage consistency parameter which defines the loading/unloading Kuhn-Tucker conditions:

$$\boxed{\dot{r} = \dot{\zeta} \geq 0 \quad ; \quad \overline{\mathcal{G}}(\mathcal{T}_{\boldsymbol{E}}, r) \leq 0 \quad ; \quad \dot{\zeta}\, \overline{\mathcal{G}}(\mathcal{T}_{\boldsymbol{E}}, r) = 0} \quad \textit{Kuhn-Tucker conditions} \tag{11.245}$$

and the persistency (consistency) condition:

$$\boxed{\dot{\zeta}\, \dot{\overline{\mathcal{G}}}(\mathcal{T}_{\boldsymbol{E}}, r) = 0} \quad \textit{The consistency condition} \tag{11.246}$$

11.6.4.4 The Hyperelastic-Damage Tangent Stiffness Tensor

In the absence of plastic phenomena we have $\dot{\mathbf{S}}^p = \mathbf{0}$, which along with the rate of change of $\mathbf{S}(\boldsymbol{E}, d)$ gives us:

$$\dot{\mathbf{S}}(\boldsymbol{E}, d) = \frac{\partial \mathbf{S}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + \frac{\partial \mathbf{S}}{\partial d} \dot{d} \equiv \frac{\partial \mathbf{S}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + \frac{\partial \mathbf{S}}{\partial d} \otimes \dot{d} \tag{11.247}$$

Then, taking into account that $\frac{\partial \mathbf{S}}{\partial d} = -\mathbf{S}^0$, $\dot{\mathbf{S}}(\boldsymbol{E}, d) = \frac{\partial \mathbf{S}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + \frac{\partial \mathbf{S}}{\partial d} \dot{d}$, $\dot{d} = \dot{\zeta}\, \mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d)$, (see equation (11.244)), and also that in plastic loading $\dot{r} = \dot{\zeta}(\boldsymbol{E}, r)$ holds, in which the damage consistency parameter $\dot{\zeta}(\boldsymbol{E}, r)$ can be evaluated by means of the consistency condition $\dot{\zeta}(\boldsymbol{E}, r) = \dot{\overline{\mathcal{T}}}_{\boldsymbol{E}} = \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \mathbf{S}^0 : \dot{\boldsymbol{E}}$, the equation in (11.247) can be rewritten as follows:

$$\begin{aligned}
\dot{\mathbf{S}}(\boldsymbol{E}, d) &= (1-d)\rho_0 \frac{\partial^2 \psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + \frac{\partial \mathbf{S}}{\partial d} \otimes \dot{\zeta}\, \mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d) \\
&= (1-d)\rho_0 \frac{\partial^2 \psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : \dot{\boldsymbol{E}} - \mathbf{S}^0 \otimes \frac{1}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \mathbf{S}^0 : \dot{\boldsymbol{E}}\, \mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d)
\end{aligned}$$

$$\dot{\mathbf{S}}(\boldsymbol{E},d)=\left[(1-d)\rho_0\frac{\partial^2\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}-\frac{\mathcal{H}(\overline{\tau}_{\boldsymbol{E}},d)}{\overline{\tau}_{\boldsymbol{E}}}\mathbf{S}^0\otimes\mathbf{S}^0\right]:\dot{\boldsymbol{E}} \tag{11.248}$$

Thus, we can define the hyperelastic-damage tangent stiffness tensor in the reference configuration as:

$$\boxed{\mathbb{C}^{tan\_d}=(1-d)\rho_0\frac{\partial^2\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}-\frac{\mathcal{H}(\overline{\tau}_{\boldsymbol{E}},d)}{\overline{\tau}_{\boldsymbol{E}}}\mathbf{S}^0\otimes\mathbf{S}^0} \tag{11.249}$$

The tensor $\mathbb{C}^{tan\_d}$ features major and minor symmetry, which is, primarily, due to the fact that the norm adopted is symmetrical.

11.6.4.5 Characterization of the Plastic Response. The Effective Elastoplastic-Damage Tangent Stiffness Tensor

Characterization of the plastic response is formulated in the effective stress spaces $\overline{\mathbf{S}}$ and $\overline{\mathbf{S}}^p$ after which the following holds:

$$\overline{\mathbf{S}}=\rho_0\frac{\partial\psi^0(\boldsymbol{E})}{\partial\boldsymbol{E}}-\overline{\mathbf{S}}^p \tag{11.250}$$

We will now postulate the yield function in the effective stress space, such that the elastic-damage domain is characterized by $\mathcal{F}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})\le 0$.

Then, if we assume the associated flow rule holds, the constitutive equations for the plastic response are given by:

$$\dot{\overline{\mathbf{S}}}^p=\dot{\gamma}\frac{\partial\mathcal{F}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})}{\partial\boldsymbol{E}}\quad;\quad\dot{\boldsymbol{A}}=\dot{\gamma}\,\boldsymbol{H}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})\quad;\quad\mathcal{F}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})\le 0 \tag{11.251}$$

where $\dot{\overline{\mathbf{S}}}^p$ is the rate of change of the plastic relaxation effective stress tensor, $\dot{\gamma}$ is the plastic consistency parameter, and $\boldsymbol{H}$ is the hardening law.

Now, the loading/unloading condition can be expressed as follows:

$$\boxed{\mathcal{F}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})\le 0;\quad\dot{\gamma}\ge 0;\quad\dot{\gamma}\mathcal{F}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})=0}\quad\textit{Kuhn-Tucker conditions} \tag{11.252}$$

Then, to obtain the value of the plastic consistency parameter $\dot{\gamma}>0$ we turn to the consistency condition, which requires that $\dot{\mathcal{F}}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})=0$. Then the rate of change of $\mathcal{F}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})$ is given by:

$$\begin{aligned}\dot{\mathcal{F}}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})&=\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial\mathcal{F}}{\partial\overline{\mathbf{S}}^p}:\dot{\overline{\mathbf{S}}}^p+\frac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\dot{\boldsymbol{A}}=0\\&=\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial\mathcal{F}}{\partial\overline{\mathbf{S}}^p}:\left[\dot{\gamma}\frac{\partial\mathcal{F}(\boldsymbol{E},\overline{\mathbf{S}}^p,\boldsymbol{A})}{\partial\boldsymbol{E}}\right]+\frac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\left[\dot{\gamma}\,\boldsymbol{H}\left(\overline{\mathbf{S}},\boldsymbol{A}\right)\right]=0\end{aligned} \tag{11.253}$$

Next, $\dot{\gamma}$ can be obtained as follows:

$$\dot{\gamma}=\frac{-\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}}{\dfrac{\partial\mathcal{F}}{\partial\overline{\mathbf{S}}^p}:\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}+\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\boldsymbol{H}\left(\overline{\mathbf{S}},\boldsymbol{A}\right)} \tag{11.254}$$

The rate of change of $\overline{\mathbf{S}}$ is given by:

$$\overline{\mathbf{S}} = \frac{\partial \Psi^0(\boldsymbol{E})}{\partial \boldsymbol{E}} - \overline{\mathbf{S}}^p \quad \Rightarrow \quad \dot{\overline{\mathbf{S}}} = \frac{\partial^2 \Psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : \dot{\boldsymbol{E}} - \dot{\overline{\mathbf{S}}}^p \tag{11.255}$$

Then, substituting (11.254) into (11.255) yields:

$$\begin{aligned}
\dot{\overline{\mathbf{S}}} &= \frac{\partial^2 \Psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : \dot{\boldsymbol{E}} - \dot{\overline{\mathbf{S}}}^p = \frac{\partial^2 \Psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : \dot{\boldsymbol{E}} - \dot{\gamma} \otimes \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} \\
&= \frac{\partial^2 \Psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} : \dot{\boldsymbol{E}} - \left(\frac{-\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}}}{\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}^p} : \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}} + \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{A}} \boldsymbol{H}} \right) \otimes \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} \\
&= \left[\frac{\partial^2 \Psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} + \frac{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}} \otimes \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}}}{\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}^p} : \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}} + \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{A}} \boldsymbol{H}} \right] : \dot{\boldsymbol{E}}
\end{aligned} \tag{11.256}$$

Next, we will define the effective elastoplastic tangent stiffness tensor as follows:

$$\overline{\mathbb{C}}^{tan\_p} = \frac{\partial^2 \Psi^0(\boldsymbol{E})}{\partial \boldsymbol{E} \otimes \partial \boldsymbol{E}} + \frac{\dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}} \otimes \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}}}{\dfrac{\partial \mathcal{F}}{\partial \overline{\mathbf{S}}^p} : \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{E}} + \dfrac{\partial \mathcal{F}}{\partial \boldsymbol{A}} \boldsymbol{H}} \tag{11.257}$$

11.6.4.6 The Elastoplastic-Damage Tangent Stiffness Tensor

Starting from the equation $\mathbf{S} = (1-d)\overline{\mathbf{S}}$, the rate of change is evaluated as follows:

$$\dot{\mathbf{S}} = (1-d)\dot{\overline{\mathbf{S}}} - \dot{d}\,\overline{\mathbf{S}} = (1-d)\mathbb{C}^{tan\_p} : \dot{\boldsymbol{E}} - \dot{d}\,\overline{\mathbf{S}} \tag{11.258}$$

In addition, taking into account that $\dot{d} = \dfrac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \mathbf{S}^0 : \dot{\boldsymbol{E}}$, we obtain:

$$\begin{aligned}
\dot{\mathbf{S}} &= (1-d)\overline{\mathbb{C}}^{tan_p} : \dot{\boldsymbol{E}} - \dot{d} \otimes \overline{\mathbf{S}} = (1-d)\overline{\mathbb{C}}^{tan_p} : \dot{\boldsymbol{E}} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \mathbf{S}^0 : \dot{\boldsymbol{E}} \otimes \overline{\mathbf{S}} \\
&= \left[(1-d)\overline{\mathbb{C}}^{tan_p} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \overline{\mathbf{S}} \otimes \mathbf{S}^0 \right] : \dot{\boldsymbol{E}}
\end{aligned} \tag{11.259}$$

or:

$$\dot{\mathbf{S}} = \mathbb{C}^{tan\_d\text{-}p} : \dot{\boldsymbol{E}} \tag{11.260}$$

where we have introduced the elastoplastic-damage tangent stiffness tensor:

$$\boxed{\mathbb{C}^{tan\_d\text{-}p} = (1-d)\overline{\mathbb{C}}^{tan\_p} - \frac{\mathcal{H}(\overline{\mathcal{T}}_{\boldsymbol{E}}, d)}{\overline{\mathcal{T}}_{\boldsymbol{E}}} \overline{\mathbf{S}} \otimes \mathbf{S}^0} \quad \textit{The elastoplastic-damage tangent stiffness tensor} \tag{11.261}$$

which is a non symmetric tensor, due to the lack of major symmetry.

11.6.5 The Plastic-Damage Model by Ju(1989)

We will now discuss the formulation of damage based on strain coupled with the strain elastoplastic model described by Ju (1989). This formulation is based on the multiplicative decomposition of the deformation gradient into an elastic ($\boldsymbol{F}^e$) and plastic ($\boldsymbol{F}^p$) part, where $\boldsymbol{F} = \boldsymbol{F}^e \cdot \boldsymbol{F}^p$ holds. For further details on multiplicative decomposition see Chapter 9 in subsection 9.6 Large-Deformation Plasticity.

11.6.5.1 Helmholtz Free Energy

The Helmholtz free energy per unit volume in the reference configuration is given by:

$$\Psi = \Psi(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A},d) = (1-d)\Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A}) = (1-d)\left[\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p) + \Psi_0^p(\boldsymbol{A})\right] \tag{11.262}$$

where $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1})$ is the Green-Lagrange strain tensor, $\boldsymbol{C} = \boldsymbol{F} \cdot \boldsymbol{F}^T$ is the right Cauchy-Green deformation tensor, $\boldsymbol{F}$ is the deformation gradient, and $\boldsymbol{E}^p$ is the plastic part in the reference configuration.

11.6.5.2 Internal Energy Dissipation. Constitutive Equation. Thermodynamic Considerations

Remember that in Chapter 5 the alternative form of the Clausius-Planck inequality in the reference configuration is given by:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \left[\rho_0 \eta \dot{T} + \dot{\Psi}\right] \geq 0 \tag{11.263}$$

where ρ_0 is the mass density, η is the specific entropy (per unit mass), T is temperature, and Ψ is the Helmholtz free energy (per unit volume). In isothermal processes we have $\dot{T} = 0$, with which the Clausius-Planck inequality becomes:

$$\mathcal{D}_{int} = \mathbf{S} : \dot{\boldsymbol{E}} - \dot{\Psi} \geq 0 \tag{11.264}$$

Then, by taking the rate of change of $\Psi(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A},d)$, we obtain:

$$\begin{aligned}
&\dot{\Psi}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A},d) = \frac{\partial \Psi}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + \frac{\partial \Psi}{\partial \boldsymbol{E}^p} : \dot{\boldsymbol{E}}^p + \frac{\partial \Psi}{\partial \boldsymbol{A}} \dot{\boldsymbol{A}} + \frac{\partial \Psi}{\partial d} \dot{d} \\
&\Rightarrow \dot{\Psi} = (1-d)\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} + (1-d)\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}^p} : \dot{\boldsymbol{E}}^p + \frac{\partial \Psi_0^p(\boldsymbol{A})}{\partial \boldsymbol{A}} \dot{\boldsymbol{A}} - \Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})\dot{d}
\end{aligned} \tag{11.265}$$

Next, substituting the above equation into the internal energy dissipation given in (11.264), yields:

$$\begin{aligned}
&\mathbf{S} : \dot{\boldsymbol{E}} - (1-d)\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} - (1-d)\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}^p} : \dot{\boldsymbol{E}}^p - \frac{\partial \Psi_0^p(\boldsymbol{A})}{\partial \boldsymbol{A}} \dot{\boldsymbol{A}} + \frac{\partial \Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})}{\partial d} \dot{d} \geq 0 \\
&\left(\mathbf{S} - (1-d)\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}}\right) : \dot{\boldsymbol{E}} + (1-d)\left[-\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}^p} : \dot{\boldsymbol{E}}^p - \frac{\partial \Psi_0^p(\boldsymbol{A})}{\partial \boldsymbol{A}} \dot{\boldsymbol{A}}\right] + \Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})\dot{d} \geq 0
\end{aligned} \tag{11.266}$$

Then, as the above inequality must satisfy for any admissible thermodynamic process, we obtain:

$$\mathbf{S} = (1-d)\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}} \tag{11.267}$$

Thus the entropy inequality becomes:

$$-(1-d)\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}^p}:\dot{\boldsymbol{E}}^p-(1-d)\frac{\partial \Psi_0^p(\boldsymbol{A})}{\partial \boldsymbol{A}}\dot{\boldsymbol{A}}+\Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})\dot{d}\geq 0 \tag{11.268}$$

If only we have a process characterized by damage, we have:

$$\Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})\dot{d}\geq 0 \tag{11.269}$$

and if there is a purely plastic process, the following must be satisfied:

$$-\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}^p}:\dot{\boldsymbol{E}}^p-\frac{\partial \Psi_0^p(\boldsymbol{A})}{\partial \boldsymbol{A}}\dot{\boldsymbol{A}}\geq 0 \tag{11.270}$$

Then we can define the effective stress tensor as follows:

$$\bar{\mathbf{S}}=\frac{1}{(1-d)}\mathbf{S}=\frac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}} \tag{11.271}$$

11.6.5.3 Characterization of Damage. The Tangent Damage Hyperelasticity Tensor

The equation in (11.269) leads us to adopt the free energy $\Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})$ so as to characterize the loading/unloading conditions. That is, we will adopt the following variable:

$$\xi\equiv\Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A}) \tag{11.272}$$

Then, the damage criterion is represented by:

$$g(\xi^t,r^t)\equiv\xi^t-r^t\leq 0 \tag{11.273}$$

where (ξ^t,r^t) represent the current values of (ξ,r), at time t, where r^t is the current damage threshold, and by adopting r_0 as the material elastic threshold $r^t\geq r_0$ holds.

The evolution of the variables d and r are then given, respectively, by:

$$\dot{d}_t=\dot{\zeta}\,\Xi(d^t,\xi^t,s,a) \quad ; \quad \dot{r}^t=\dot{\zeta} \tag{11.274}$$

where the Kuhn-Tucker conditions hold:

$$\boxed{\dot{r}=\dot{\zeta}\geq 0 \quad ; \quad g(\xi^t,r^t)\leq 0 \quad ; \quad \dot{\zeta}\,g(\xi^t,r^t)=0} \quad \textit{Kuhn-Tucker conditions} \tag{11.275}$$

We can then obtain the parameter $\dot{\zeta}$ by imposing loading:

$$\dot{\zeta}>0\Rightarrow g(\xi^t,r^t)=\dot{g}(\xi^t,r^t)=0\Rightarrow\dot{\zeta}=\langle\dot{\xi}\rangle \tag{11.276}$$

where $\langle\bullet\rangle$ is the Macaulay brackets.

Then, the variable r^t is defined as follows:

$$\dot{\zeta}>0\Rightarrow g(\xi^t,r^t)=\dot{g}(\xi^t,r^t)=0\Rightarrow\dot{\zeta}=\langle\dot{\xi}\rangle \quad ; \quad r^t=\max\left\{r_0,\max_{s\in[0,t]}\xi^t\right\} \tag{11.277}$$

11.6.5.4 The Elastic-Damage Tangent Stiffness Tensor

In the absence of plastic phenomena we have $\dot{\boldsymbol{E}}^p=\mathbf{0}$, thus by evaluating the rate of change of the stress tensor given in (11.267), $\mathbf{S}=(1-d)\dfrac{\partial \Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial \boldsymbol{E}}$, we obtain:

$$\dot{\mathbf{S}}(\boldsymbol{E},d)=(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\frac{\partial\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}}\dot{d}$$

$$\begin{aligned}\dot{\mathbf{S}}(\boldsymbol{E},d)&=(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\overline{\mathbf{S}}\dot{d}\equiv(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\overline{\mathbf{S}}\otimes\dot{d}\\&=(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\Xi\overline{\mathbf{S}}\otimes\dot{\xi}\end{aligned}\tag{11.278}$$

Note that during loading $\dot{d}=\dot{r}\Xi=\dot{\xi}\Xi$ is valid, and also based on the expression $\xi\equiv\Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})$, the non-plastic process, we obtain:

$$\dot{\xi}=\frac{\partial\Psi_0}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}=\overline{\mathbf{S}}:\dot{\boldsymbol{E}}\tag{11.279}$$

Then

$$\begin{aligned}\dot{\mathbf{S}}(\boldsymbol{E},d)&=(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\Xi\overline{\mathbf{S}}\otimes\dot{\xi}=(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\Xi\overline{\mathbf{S}}\otimes\overline{\mathbf{S}}:\dot{\boldsymbol{E}}\\&=\left[(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}-\Xi\overline{\mathbf{S}}\otimes\overline{\mathbf{S}}\right]:\dot{\boldsymbol{E}}\end{aligned}\tag{11.280}$$

where we have introduced the elastic-damage tangent stiffness tensor:

$$\mathbb{C}^{tan\_d}=(1-d)\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}-\Xi\overline{\mathbf{S}}\otimes\overline{\mathbf{S}}\tag{11.281}$$

11.6.5.5 Characterization of Plastic Response. The elastoplastic Tangent Stiffness Tensor.

Now, by assuming there is an associated flow rule, the constitutive equation for the plastic response is given by $\dot{\boldsymbol{E}}^p=-\dot{\gamma}\,\mathbb{M}^{-1}:\dfrac{\partial\mathcal{F}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})}{\partial\boldsymbol{E}}$, where $\mathbb{M}=\dfrac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}^p}=\dfrac{\partial\overline{\mathbf{S}}}{\partial\boldsymbol{E}^p}$.

Now, in the stress space we have:

$$\dot{\overline{\mathbf{S}}}{}^p=-\mathbb{M}:\dot{\boldsymbol{E}}^p=\dot{\gamma}\frac{\partial\mathcal{F}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})}{\partial\boldsymbol{E}}\tag{11.282}$$

Then, the law of evolution for the variable $\boldsymbol{A}$ is given by:

$$\dot{\boldsymbol{A}}=\dot{\gamma}\,\boldsymbol{H}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})\tag{11.283}$$

where $\boldsymbol{H}$ is the generalized hardening law.

The loading/unloading conditions are then given by:

$$\boxed{\mathcal{F}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})\le 0;\quad \dot{\gamma}\ge 0;\quad \dot{\gamma}\mathcal{F}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})=0}\quad \textit{Kuhn-Tucker conditions}\tag{11.284}$$

Next, to obtain the plastic consistency parameter $\dot{\gamma}>0$ we turn to the consistency condition which requires that $\dot{\mathcal{F}}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})=0$. Then the rate of change of $\mathcal{F}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})$ is given by $\dot{\mathcal{F}}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})=\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}^p}:\dot{\boldsymbol{E}}^p+\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\dot{\boldsymbol{A}}=0$, which can be expressed as:

$$\dot{\mathcal{F}}(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})=\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}^p}:\left[-\dot{\gamma}\,\mathbb{M}^{-1}:\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}\right]+\frac{\partial\mathcal{F}}{\partial\boldsymbol{A}}[\dot{\gamma}\,\boldsymbol{H}]=0\tag{11.285}$$

Afterwards, $\dot{\gamma}$ can be obtained as follows:

$$\dot{\gamma}=\frac{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}}{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}^p}:\mathbb{M}^{-1}:\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}-\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\boldsymbol{H}} \tag{11.286}$$

and the rate of change of $\overline{\mathbf{S}}=\dfrac{\partial\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}}$ is evaluated as:

$$\begin{aligned}\dot{\overline{\mathbf{S}}}&=\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}^p}:\dot{\boldsymbol{E}}^p=\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\mathbb{M}:\dot{\boldsymbol{E}}^p\\&=\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\dot{\overline{\mathbf{S}}}^p=\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\dot{\gamma}\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}\end{aligned} \tag{11.287}$$

Then, if we consider that $\mathbb{C}_0^e=\dfrac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}$, and that the value of $\dot{\gamma}$ is given by the equation in (11.286), we obtain:

$$\begin{aligned}\dot{\overline{\mathbf{S}}}&=\frac{\partial^2\Psi_0^e(\boldsymbol{E},\boldsymbol{E}^p)}{\partial\boldsymbol{E}\otimes\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}-\dot{\gamma}\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}=\mathbb{C}_0^e:\dot{\boldsymbol{E}}-\dot{\gamma}\otimes\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}\\&=\mathbb{C}_0^e:\dot{\boldsymbol{E}}-\left(\frac{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}}{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}^p}:\mathbb{M}^{-1}:\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}-\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\boldsymbol{H}}\right)\otimes\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}=\left[\mathbb{C}_0^e-\frac{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}\otimes\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}}{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}^p}:\mathbb{M}^{-1}:\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}-\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\boldsymbol{H}}\right]:\dot{\boldsymbol{E}}\end{aligned} \tag{11.288}$$

thus we can define the effective elastoplastic tangent stiffness tensor ($\overline{\mathbb{C}}^{tan\_p}$) as:

$$\boxed{\overline{\mathbb{C}}^{tan\_p}=\mathbb{C}_0^e-\frac{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}\otimes\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}}{\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}^p}:\mathbb{M}^{-1}:\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{E}}-\dfrac{\partial\mathcal{F}}{\partial\boldsymbol{A}}\boldsymbol{H}}} \tag{11.289}$$

11.6.5.6 The Elastoplastic-Damage Tangent Stiffness Tensor

The elastoplastic-damage tangent stiffness tensor is defined according to the relationship $\dot{\mathbf{S}}=\mathbb{C}^{tan\_d\_p}:\dot{\boldsymbol{E}}$. Thus, we can start from the rate of change of $\mathbf{S}=(1-d)\overline{\mathbf{S}}$:

$$\dot{\mathbf{S}}=(1-d)\dot{\overline{\mathbf{S}}}-\dot{d}\,\overline{\mathbf{S}}=(1-d)\overline{\mathbb{C}}^{tan\_p}:\dot{\boldsymbol{E}}-\dot{\xi}\Xi\,\overline{\mathbf{S}} \tag{11.290}$$

Then, the rate of change of (11.272), $\xi\equiv\Psi_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})$, is given by:

$$\begin{aligned}\dot{\xi}\equiv\dot{\Psi}_0(\boldsymbol{E},\boldsymbol{E}^p,\boldsymbol{A})&=\frac{\partial\Psi_0}{\partial\boldsymbol{E}}:\dot{\boldsymbol{E}}+\frac{\partial\Psi_0}{\partial\boldsymbol{E}^p}:\dot{\boldsymbol{E}}^p+\frac{\partial\Psi_0}{\partial\boldsymbol{A}}:\dot{\boldsymbol{A}}\\&=\overline{\mathbf{S}}:\dot{\boldsymbol{E}}-\dot{\gamma}\left[\frac{\partial\Psi_0}{\partial\boldsymbol{E}^p}:\mathbb{M}^{-1}:\frac{\partial\mathcal{F}}{\partial\boldsymbol{E}}-\frac{\partial\Psi_0}{\partial\boldsymbol{A}}\boldsymbol{H}\right]\end{aligned} \tag{11.291}$$

or

$$\begin{aligned}
\dot{\xi} &= \overline{\mathsf{S}} : \dot{\boldsymbol{E}} - \dot{\gamma} \otimes \left[\frac{\partial \Psi_0}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \Psi_0}{\partial \boldsymbol{A}} \boldsymbol{H} \right] \\
&= \overline{\mathsf{S}} : \dot{\boldsymbol{E}} - \left(\frac{\frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}}}{\frac{\partial \mathcal{F}}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}} \boldsymbol{H}} \right) \otimes \left[\frac{\partial \Psi_0}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \Psi_0}{\partial \boldsymbol{A}} \boldsymbol{H} \right] \\
&= \overline{\mathsf{S}} : \dot{\boldsymbol{E}} - \left(\frac{\frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}}}{\mathcal{X}} \right) [\mathcal{Y}]
\end{aligned} \tag{11.292}$$

where we have introduced the following scalars $\mathcal{X} = \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}} \boldsymbol{H}$, $\mathcal{Y} = \frac{\partial \Psi_0}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \Psi_0}{\partial \boldsymbol{A}} \boldsymbol{H}$. Then, substituting (11.292) into (11.290) yields:

$$\begin{aligned}
\dot{\mathsf{S}} &= (1-d)\overline{\mathbb{C}}^{tan_p} : \dot{\boldsymbol{E}} - \Xi \dot{\xi} \otimes \overline{\mathsf{S}} = (1-d)\overline{\mathbb{C}}^{tan_p} : \dot{\boldsymbol{E}} - \Xi \left[\overline{\mathsf{S}} : \dot{\boldsymbol{E}} - \left(\frac{\frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}}}{\mathcal{X}} \right) [\mathcal{Y}] \right] \otimes \overline{\mathsf{S}} \\
&= (1-d)\overline{\mathbb{C}}^{tan_p} : \dot{\boldsymbol{E}} - \Xi \overline{\mathsf{S}} : \dot{\boldsymbol{E}} \otimes \overline{\mathsf{S}} + \Xi \left(\frac{\mathcal{Y}}{\mathcal{X}} \right) \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} : \dot{\boldsymbol{E}} \otimes \overline{\mathsf{S}}
\end{aligned} \tag{11.293}$$

in indicial notation:

$$\begin{aligned}
\dot{\mathsf{S}}_{ij} &= (1-d)\overline{\mathbb{C}}^{tan_p}_{ijkl} \dot{E}_{kl} - \Xi \overline{\mathsf{S}}_{kl} \dot{E}_{kl} \overline{\mathsf{S}}_{ij} + \Xi \left(\frac{\mathcal{Y}}{\mathcal{X}} \right) \frac{\partial \mathcal{F}}{\partial E_{kl}} \dot{E}_{kl} \overline{\mathsf{S}}_{ij} \\
&= \left[(1-d)\overline{\mathbb{C}}^{tan_p}_{ijkl} - \Xi \overline{\mathsf{S}}_{kl} \overline{\mathsf{S}}_{ij} + \Xi \left(\frac{\mathcal{Y}}{\mathcal{X}} \right) \frac{\partial \mathcal{F}}{\partial E_{kl}} \overline{\mathsf{S}}_{ij} \right] \dot{E}_{kl}
\end{aligned} \tag{11.294}$$

which in tensorial notation becomes:

$$\begin{aligned}
\dot{\mathsf{S}} &= \left[(1-d)\overline{\mathbb{C}}^{tan_p} - \Xi \overline{\mathsf{S}} \otimes \overline{\mathsf{S}} + \Xi \left(\frac{\mathcal{Y}}{\mathcal{X}} \right) \overline{\mathsf{S}} \otimes \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} \right] : \dot{\boldsymbol{E}} \\
&= \left[(1-d)\overline{\mathbb{C}}^{tan_p} - \Xi \overline{\mathsf{S}} \otimes \overline{\mathsf{S}} + \Xi \left(\frac{\frac{\partial \Psi_0}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \Psi_0}{\partial \boldsymbol{A}} \boldsymbol{H}}{\frac{\partial \mathcal{F}}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}} \boldsymbol{H}} \right) \overline{\mathsf{S}} \otimes \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} \right] : \dot{\boldsymbol{E}} \\
&= \mathbb{C}^{tan_d_p} : \dot{\boldsymbol{E}}
\end{aligned} \tag{11.295}$$

where we have introduced the elastoplastic-damage tangent stiffness tensor:

$$\boxed{\mathbb{C}^{tan\_d\_p} = (1-d)\overline{\mathbb{C}}^{tan\_p} - \Xi \overline{\mathsf{S}} \otimes \overline{\mathsf{S}} + \Xi \left(\frac{\frac{\partial \Psi_0}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \Psi_0}{\partial \boldsymbol{A}} \boldsymbol{H}}{\frac{\partial \mathcal{F}}{\partial \boldsymbol{E}^p} : \mathbb{M}^{-1} : \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}} - \frac{\partial \mathcal{F}}{\partial \boldsymbol{A}} \boldsymbol{H}} \right) \overline{\mathsf{S}} \otimes \frac{\partial \mathcal{F}}{\partial \boldsymbol{E}}} \tag{11.296}$$

12 Introduction to Fluids

12.1 Introduction

In this chapter, we will introduce an important branch of continuum mechanics: fluid mechanics with which we intend to study fluids in motion or at rest. These can be classified into:

$$\text{Fluids}\begin{cases}\text{Liquids}\\ \text{Gases}\end{cases}$$

There are several areas where fluids mechanics can be applied, *e.g.* meteorology, oceanography, aerodynamics, hydrodynamics and engineering, among others.

Fundamentally, we can state that solids can resist shear stress while liquids have very low (viscous fluids, *e.g.* oil) or no resistance to it (non-viscous fluids, *e.g.* water).

Both gases and liquids are materials consisting of molecules (an agglomeration of two or more atoms) colliding with each other. To treat fluids with assumption of continuum mechanics properties (*e.g.* mass density, pressure and velocity) are treated as continuous functions. Then, treating a system of molecules as a continuous medium is valid when comparing the mean free path of molecules (Λ) (average distance particles travel before colliding with each other) with the characteristic physical length scale (ℓ_C).For example, for solids and liquids we have $\Lambda \approx 10^{-7}\,cm$ and for gases $\Lambda \approx 10^{-6}\,cm$, Chung (1996). Then, the ratio $\dfrac{\Lambda}{\ell_C}$ is known as the *Knudsen number* (Kn). If this number is much smaller than unity, the domain can be treated as a continuum, otherwise we must use statistical mechanics to obtain the governing equations of the problem with which we can establish that:

E.W.V. Chaves, *Notes on Continuum Mechanics*, Lecture Notes on Numerical Methods in Engineering and Sciences 4, DOI 10.1007/978-94-007-5986-2_14,

$$
\begin{aligned}
Kn &= \frac{\Lambda}{\ell_C} << 1 \quad \Rightarrow \quad \text{macroscopic approach} \\
Kn &= \frac{\Lambda}{\ell_C} > 1 \quad \Rightarrow \quad \text{microscopic approach}
\end{aligned}
\tag{12.1}
$$

Let us now consider a fluid is between two surfaces separated by a distance d , (see Figure 12.1). The lower surface is fixed whilst the upper surface is moving at the constant velocity v_0 . We can then observe that the force required to maintain this motion is given by:

$$
\frac{F}{A} = \mu \frac{v_0}{d} \tag{12.2}
$$

where A is the surface area and μ is the fluid viscosity. The above equation indicates that the shear stress $\frac{F}{A}$ is proportional to how the velocity varies with distance (the velocity gradient).

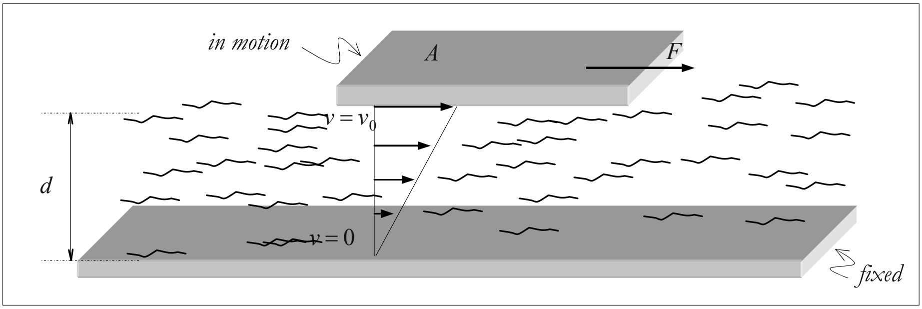


Figure 12.1: Motion of the plate.

12.2 Fluids at Rest and in Motion

12.2.1 Fluids at Rest

By means of experiments, it can be proven that a fluid at rest or with uniform flow is free of tangential stresses, *i.e.* the shear stress components are zero. Then, the traction vector on the surface element is only a function of pressure whereby we conclude that the traction vector $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}$ which acts on a surface is collinear with the normal $\hat{\mathbf{n}}$ and is given by:

Tensorial notation	components	
$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = -p_0 \hat{\mathbf{n}}$	$\mathrm{t}_i^{(\hat{\mathbf{n}})} = -p_0 \hat{\mathrm{n}}_i$	(12.3)

where p_0 is the *hydrostatic pressure*.

The traction vector can also be expressed in terms of the Cauchy stress tensor, (see Chapter 3), as $\vec{\mathbf{t}}^{(\hat{\mathbf{n}})} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = -\overline{p}\hat{\mathbf{n}}$, where $\overline{p}$ is given in terms of the *mean stress* as $\sigma_m = -\overline{p}$. Then, in the case of fluids at rest or with uniform motion we have:

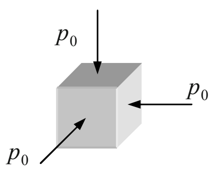
$$
p = -\overline{p} = \frac{\sigma_{kk}}{3} \tag{12.4}
$$

In this case, any direction is a principal direction and the stress tensor, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(p_0)$, is represented as follows:

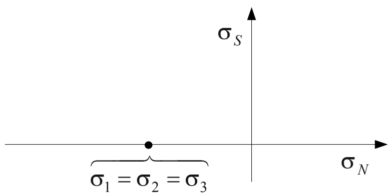
Tensorial notation	Components	
$\boldsymbol{\sigma} = -p_0 \mathbf{1}$	$\sigma_{ij} = -p_0 \delta_{ji}$	(12.5)

The constitutive law above was described by Bernoulli for a non-viscous fluid. Unfortunately this equation is not valid for any fluid, for instance a fluid in motion. The negative sign indicates a compressive stress with a positive pressure value.

Figure 12.2 shows the hydrostatic state by means of the Mohr's circle, here reduced to a single point. Note that for a fluid at rest the maximum tangential (shear) stress is zero and here the stress tensor is spherical.



a) Point under hydrostatic pressure



b) Mohr's circle

Figure 12.2: Hydrostatic pressure.

NOTE: In general, for a fluid in motion, the parameter ($\bar{p}$) and the hydrostatic pressure (p_0) do not match. However, as we saw previously, for a fluid at rest $p_0 = \bar{p}$ holds. ■

12.2.2 Fluids in Motion

For a fluid in motion, the shear stress components are, generally speaking, nonzero and the stress tensor, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(p, \boldsymbol{\tau})$, is usually decomposed into:

$$\boldsymbol{\sigma} = -p\mathbf{1} + \boldsymbol{\tau} \qquad \Big| \qquad \sigma_{ij} = -p\delta_{ji} + \tau_{ij} \tag{12.6}$$

where p is the *thermodynamic pressure* and $\boldsymbol{\tau}$ is the *viscous stress tensor*.

NOTE: The thermodynamic pressure (p) is a variable that is related to other thermodynamic variables such as mass density (ρ) and absolute temperature (T) by means of the equations of state $f(p, \rho, T) = 0$. ■

Note, in general, the thermodynamic pressure is different from the hydrostatic pressure, *i.e.* $p \neq p_0$.

12.3 Viscous and Non-Viscous Fluids

Real fluids are compressible and viscous, although, in many practical cases this viscosity or compressibility can be overlooked. So, traditionally, they have been classified into viscous and non-viscous fluids, *e.g.* water (an incompressible non-viscous fluid), air (a compressible non-viscous fluid) and oil (incompressible viscous fluid).

12.3.1 Non-Viscous Fluids (Perfect Fluids)

A non-viscous fluid is free of shear stress (negligible), so the viscous stress tensor is zero:

$$\boxed{\boldsymbol{\tau} \approx \mathbf{0}} \quad \textit{Non-viscous fluids} \tag{12.7}$$

In this case we have a perfect fluid and the equation in (12.3) holds.

12.3.2 Viscous Fluids

With viscous fluid in motion, resistance to tangential movement cannot be ignored, so, we have $\boldsymbol{\tau} \neq \mathbf{0}$. Then, by taking the equation in (12.6) and multiplying both sides of it by the Kronecker delta, δ_{ij}, *i.e.* by taking the trace of $\boldsymbol{\sigma}$, we obtain:

$$\begin{aligned}
&\sigma_{ij}\delta_{ij} = -p\delta_{ij}\delta_{ij} + \tau_{ij}\delta_{ij} \\
&\Rightarrow \sigma_{ii} = -3p + \tau_{ii} \\
&\Rightarrow \underbrace{\frac{1}{3}\sigma_{ii}}_{\sigma_m} = -p + \frac{1}{3}\tau_{ii}
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
&\boldsymbol{\sigma} : \mathbf{1} = -p\mathbf{1} : \mathbf{1} + \boldsymbol{\tau} : \mathbf{1} \\
&\mathrm{Tr}(\boldsymbol{\sigma}) = -3p + \mathrm{Tr}(\boldsymbol{\tau}) \\
&\Rightarrow \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3} = -p + \frac{\mathrm{Tr}(\boldsymbol{\tau})}{3}
\end{aligned} \tag{12.8}$$

Then, with regard to hydrostatic pressure we can state that:

- For viscous or non-viscous fluids at rest $\tau_{ij} = 0$, thus:

$$p = p_0 = \overline{p} \tag{12.9}$$

- For an incompressible fluids, p is an independent mechanical variable
- Compressible Fluids

  In general, pressure is a function of mass density and temperature which are related by means of the equation of state:

$$p = p(\rho, T) \quad \textit{Constitutive equation for pressure.} \tag{12.10}$$

where p is pressure, ρ is mass density, and T is absolute temperature. For example, the equation of state for ideal gases is $p = \rho RT$, where R is the gas constant.

When temperature is not included in the equation of state, $\rho = \rho(p)$, then the state change is called *barotropic*.

It can then be shown that in reversible adiabatic processes, an ideal gas is governed by the barotropic relationship:

$$\frac{p}{\rho^\gamma} = \text{constant} \quad ; \quad \gamma = \frac{c_p}{c_v} = 1 + \frac{R}{c_v} \tag{12.11}$$

where c_p is specific heat at constant pressure, c_v is the specific heat at a constant volume, R is the gas constant and a perfect incompressible fluid is governed y the barotropic equation:

$$\rho = \text{constant} \tag{12.12}$$

where we have $p = p_0$.

12.4 Laminar and Turbulent Flow

Fluid flow is considered to be laminar when the various fluids layers move in a parallel, uniform and regular fashion, (see Figure 12.3). As we will see later, the Navier-Stokes equations are only valid for laminar fluids. For such flows, it is well established that the shear stress ($\boldsymbol{\tau}$) is proportional to the velocity gradient. Therefore, the Navier-Poisson constitutive equations describe laminar flow behavior well.

A laminar flow is identified generally by so named *Reynolds number* ($\mathsf{R_e}$ -dimensionless), and is given by:

$$\mathsf{R_e} = \rho \frac{\vec{\boldsymbol{v}}\ \ell_C}{\mu^*} = \frac{\vec{\boldsymbol{v}}\ \ell_C}{\nu^*} \qquad (12.13)$$

where $\vec{\boldsymbol{v}}$ is the mean velocity of the object relative to the fluid; ℓ_C is the characteristic length; μ^* is the dynamic viscosity, and ν^* is the kinematic viscosity which is given by $\nu^* = \frac{\mu^*}{\rho}$. Then, the SI units of these are $[\nu^*] = m^2/s$ and $[\mu^*] = \frac{kg}{m^3}\frac{m^2}{s} = \frac{kg}{ms} = Pa \times s$.

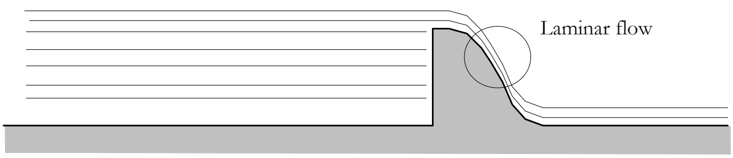


Figure 12.3: Laminar flow.

We state that a fluid flow is turbulent when stresses and velocities at each point randomly fluctuate over time, (see Figure 12.4).

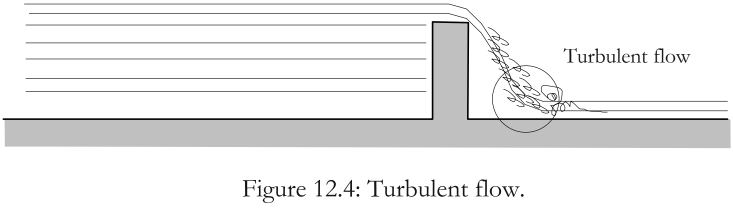


Figure 12.4: Turbulent flow.

12.5 Particular Cases

Reminder:

Remember that the material time derivative of velocity provides us with the acceleration, *i.e.*:

$$
\begin{aligned}
\vec{a}(\vec{x},t) &= \frac{D\vec{v}(\vec{x},t)}{Dt} \equiv \dot{\vec{v}}(\vec{x},t) = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\vec{v}\cdot\vec{v} \\
a_i(\vec{x},t) &= \frac{Dv_i}{Dt} \equiv \dot{v}_i = \frac{\partial v_i(\vec{x},t)}{\partial t} + v_k v_{i,k}
\end{aligned}
\tag{12.14}
$$

The spatial velocity gradient ($\nabla_{\vec{x}}\vec{v} \equiv \boldsymbol{\ell}$) can be split into a symmetric and an antisymmetric part as follows $\boldsymbol{\ell} = \mathbf{D} + \mathbf{W}$, where $\mathbf{D}$ is the rate-of deformation tensor (symmetric tensor) and $\mathbf{W}$ is the spin tensor (antisymmetric tensor). Then, the acceleration can also be expressed as:

$$
\begin{aligned}
\vec{a}(\vec{x},t) = \dot{\vec{v}}(\vec{x},t) &= \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\vec{v}\cdot\vec{v} \\
&= \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \boldsymbol{\ell}\cdot\vec{v} \\
&= \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \mathbf{D}\cdot\vec{v} + \mathbf{W}\cdot\vec{v}
\end{aligned}
\tag{12.15}
$$

Then, because of the antisymmetric tensor property, the equation $\mathbf{W}\cdot\vec{v} = \vec{w}\wedge\vec{v}$ holds, where $\vec{w}$ is the axial vector associated with $\mathbf{W}$. In addition, $\vec{\boldsymbol{\omega}} = 2\vec{w} = \mathrm{rot}(\vec{v}) = (\nabla_{\vec{x}}\wedge\vec{v})$ holds, where $\vec{\boldsymbol{\omega}}$ is the *vorticity vector*, $\mathbf{W}\cdot\vec{v} = \frac{1}{2}(\nabla_{\vec{x}}\wedge\vec{v})\wedge\vec{v}$. Next, the term $\mathbf{D}\cdot\vec{v}$ can be represented as follows:

$$
\mathbf{D}\cdot\vec{v} = \frac{1}{2}\left(\boldsymbol{\ell} + \boldsymbol{\ell}^T\right)\cdot\vec{v} = (\nabla^{sym}_{\vec{x}}\vec{v})\cdot\vec{v} \qquad \Bigg| \qquad \mathsf{D}_{ij}v_j = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)v_j
\tag{12.16}
$$

Then, the acceleration can still be represented as follows:

$$
\begin{aligned}
\vec{a}(\vec{x},t) &= \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \mathbf{D}\cdot\vec{v} + \mathbf{W}\cdot\vec{v} = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + (\nabla^{sym}_{\vec{x}}\vec{v})\cdot\vec{v} + \frac{1}{2}(\nabla_{\vec{x}}\wedge\vec{v})\wedge\vec{v} \\
&= \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + (\nabla^{sym}_{\vec{x}}\vec{v})\cdot\vec{v} + \frac{1}{2}\vec{\boldsymbol{\omega}}\wedge\vec{v}
\end{aligned}
\tag{12.17}
$$

Finally, remember in Chapter 2 it was shown that the following equation is valid:

$$
\vec{a}(\vec{x},t) = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \frac{1}{2}\nabla_{\vec{x}}(v^2) + \mathrm{rot}\,\vec{v}\wedge\vec{v} = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \frac{1}{2}\nabla_{\vec{x}}(v^2) + (\nabla_{\vec{x}}\wedge\vec{v})\wedge\vec{v}
\tag{12.18}
$$

12.5.1 Incompressible Fluids

As we saw in Chapter 2, an incompressible medium is characterized by isochoric motion, it then follows that $\nabla_{\vec{x}}\cdot\vec{v} = \mathsf{D}_{kk} = 0$ is satisfied for any incompressible fluid. Moreover, taking into account the mass continuity equation, $\dfrac{D\rho}{Dt} + \rho(\nabla_{\vec{x}}\cdot\vec{v}) = 0$, we can conclude that for any incompressible fluid the following is valid:

$$
\boxed{\nabla_{\vec{x}}\cdot\vec{v} = \mathrm{Tr}(\mathbf{D}) = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} = 0 \longrightarrow \rho = \rho_0} \quad \textit{Incompressible fluids}
\tag{12.19}
$$

12.5.2 Irrotational Flow

A flow is said to be irrotational when the spin tensor vanishes at any point in the fluid:

$$\mathsf{W}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = 0_{ij} \quad ; \quad \mathbf{W} = \frac{1}{2}\left[\nabla_{\vec{x}}\vec{v} - (\nabla_{\vec{x}}\vec{v})^T\right] = \mathbf{0}$$
$$\boxed{\mathbf{W} = \mathbf{0} \quad \Rightarrow \quad \nabla_{\vec{x}} \wedge \vec{v} = \vec{\mathbf{0}}} \qquad \textit{Irrotational flow} \qquad (12.20)$$

An incompressible irrotational flow is characterized by:

$$\boxed{\nabla_{\vec{x}} \cdot \vec{v} = 0 \quad and \quad \nabla_{\vec{x}} \wedge \vec{v} = \vec{\mathbf{0}}} \qquad \textit{Incompressible irrotational flow} \qquad (12.21)$$

When the flow is irrotational the acceleration given in (12.17) becomes:

$$\vec{a}(\vec{x},t) = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + (\nabla^{sym}_{\vec{x}}\vec{v}) \cdot \vec{v} + \underbrace{\frac{1}{2}\vec{\boldsymbol{\omega}} \wedge \vec{v}}_{=\vec{\mathbf{0}}} \qquad a_i(\vec{x},t) = \frac{\partial v_i(\vec{x},t)}{\partial t} + v_k\left(v_{i,k}\right)^{sym}$$
$$= \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + (\nabla^{sym}_{\vec{x}}\vec{v}) \cdot \vec{v} \qquad (12.22)$$

Note that $\left(v_k v_k\right)_{,i} = v_{k,i}v_k + v_{k,i}v_k$ holds and due to the symmetry of $v_{k,i} = v_{i,k}$, the following is valid $\left(2v_{k,i}v_k\right) = \left(v_k v_k\right)_{,i}$, *i.e.* $\boldsymbol{\ell} \cdot \vec{v} = \mathbf{D} \cdot \vec{v} = \frac{1}{2}\nabla_{\vec{x}}(\vec{v} \cdot \vec{v}) = \frac{1}{2}\nabla_{\vec{x}}(v^2)$, with which the acceleration for an irrotational flow becomes:

$$\boxed{\vec{a}(\vec{x},t) = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + (\nabla^{sym}_{\vec{x}}\vec{v}) \cdot \vec{v} = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \frac{1}{2}\nabla_{\vec{x}}(v^2)} \qquad \textit{Acceleration for an irrotational flow} \qquad (12.23)$$

12.5.3 Steady Flow

A steady flow, (see Chapter 2), is characterized by:

$$\boxed{\frac{\partial \vec{v}}{\partial t} = \vec{\mathbf{0}}} \qquad \textit{Steady flow} \qquad (12.24)$$

Remember that the material time derivative of the velocity is given by:

$$\frac{D\vec{v}}{Dt} \equiv \dot{\vec{v}} = \frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\vec{v} \cdot \vec{v} \qquad \frac{Dv_i}{Dt} \equiv \dot{v}_i = \frac{\partial v_i(\vec{x},t)}{\partial t} + v_{i,k}v_k \qquad (12.25)$$

and in steady flow the rate of change of the velocity becomes

$$\boxed{\dot{\vec{v}} = \nabla_{\vec{x}}\vec{v} \cdot \vec{v} \quad ; \quad \dot{v}_i = v_{i,k}v_k} \qquad \textit{Rate of change of velocity for steady flow} \qquad (12.26)$$

Problem 12.1: Demonstrate whether the following statements are true or false:

a) If the velocity field is steady, then the acceleration field is also;
b) If the velocity field is uniform, the acceleration field is always equal to zero;
c) If the velocity field is steady and the medium is incompressible, the acceleration is always zero.
Solution:

a) In a steady velocity field we have $\frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t} = \vec{\mathbf{0}}$ whereby the acceleration field becomes:

$$a_i = \dot{v}_i = \underbrace{\frac{\partial v_i(\vec{\boldsymbol{x}},t)}{\partial t}}_{=0_i} + v_{i,k} v_k = v_{i,k} v_k$$

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})}{\partial t} + \nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) = \underbrace{\nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})}_{\text{Independent of time}}$$

Then, assumption (a) is TRUE.

b) A uniform velocity field implies that $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{v}}(t)$, whereby:

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t} + \underbrace{\nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}_{=\vec{\mathbf{0}}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{\partial t}$$

Then, assumption (b) is FALSE.

c) A steady velocity field implies that $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})$ and an incompressible medium means that $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t) = 0$, so, we can conclude that:

$$\vec{\boldsymbol{a}} = \dot{\vec{\boldsymbol{v}}} = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})}{\partial t} + \nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) = \nabla_{\vec{x}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \cdot \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}})$$

Then, assumption (c) is FALSE.

12.6 Newtonian Fluids

The viscous stress tensor ($\boldsymbol{\tau}$) is associated with the internal energy dissipation brought about by the viscosity. Remember that in Chapter 6 in Subsection 6.4 in viscoelastic materials the dynamic part of the stress tensor is either a function of $\dot{\boldsymbol{F}}$ or a function of $\mathbf{D}$ (rate-of-deformation tensor), so:

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{D}) \quad \Rightarrow \quad \begin{cases} \text{If a linear relationship (Newtonian Fluid)} \\ \text{If a nonlinear relationship (Non - Newtonian Fluid)} \end{cases} \tag{12.27}$$

where $\mathbf{D}$ is the symmetric part of the spatial velocity gradient (see Chapter 2), whose components are given by:

$$\mathsf{D}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{12.28}$$

The equation in (12.27) is, general speaking, nonlinear, which is characteristic of the *Stokesian (non-Newtonian) fluid*. Blood, paints and sauces are all examples of this.

When the relationship in (12.27) is linear we have what we term *Newtonian fluids* which are described in the following format:

$$\tau_{ij} = \mathbb{K}_{ijkl} \mathsf{D}_{kl} \qquad \boxed{\boldsymbol{\tau} = \mathbb{K} : (\mathbf{D})} \quad \textit{Stress constitutive equation for Newtonian fluid} \tag{12.29}$$

where $\mathbb{K}$ is the tensor containing the viscosity coefficients.

Additionally, the Cauchy stress tensor is represented by:

$$\boxed{\boldsymbol{\sigma} = -p\mathbf{1} + \boldsymbol{\tau}} \qquad \boxed{\sigma_{ij} = -p\delta_{ji} + \tau_{ij}} \tag{12.30}$$

Then, to directly obtain an expression for $\boldsymbol{\tau}$, we can make an analogy with the stress constitutive equation for isotropic solid materials, (see Chapter 7), in which:

$$
\begin{array}{ccc}
\text{Isotropic solids} & & \text{Fluids} \\
\sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl} & \Rightarrow & \tau_{ij} = \mathbb{K}_{ijkl}\mathsf{D}_{kl} \\
\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} & \Rightarrow & \tau_{ij} = \lambda^*\delta_{ij}\mathsf{D}_{kk} + 2\mu^*\mathsf{D}_{ij}
\end{array}
\tag{12.31}
$$

where λ^* is the viscous dilatational coefficient and μ^* is the viscous tangential coefficient and, generally speaking, these variables are associated with other thermodynamics variables, *i.e.:*

$$
\lambda^* = \lambda^*(\rho, T) \quad ; \quad \mu^* = \mu^*(\rho, T) \tag{12.32}
$$

Now, by substituting the viscous stress tensor given in (12.31) into the equation in (12.30) we obtain:

$$
\boxed{\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + \lambda^*\mathsf{D}_{kk}\delta_{ij} + 2\mu^*\mathsf{D}_{ij} \\ \boldsymbol{\sigma} &= -p\mathbf{1} + \lambda^*\mathrm{Tr}(\mathbf{D})\mathbf{1} + 2\mu^*\mathbf{D} \end{aligned}} \quad \begin{array}{l} \textit{Navier-Poisson law} \\ \textit{(Newtonian fluid)} \end{array} \tag{12.33}
$$

These equations provide us with the *Navier-Poisson law* of a Newtonian fluid.

In incompressible fluids we have $\nabla_{\vec{x}} \cdot \vec{v} = \mathrm{Tr}(\mathbf{D}) = 0$, (see equation (12.19)), whereby the Navier-Poisson law becomes:

$$
\boxed{\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu^*\mathbf{D}} \quad \begin{array}{l} \textit{Navier-Poisson law} \\ \textit{(Incompressible Newtonian fluid)} \end{array} \tag{12.34}
$$

Then, by multiplying the equation in (12.33) by δ_{ij} we obtain:

$$
\begin{aligned}
\sigma_{ij}\delta_{ij} &= -p\delta_{ij}\delta_{ij} + \lambda^*\delta_{ij}\delta_{ij}\mathsf{D}_{kk} + 2\mu^*\delta_{ij}\mathsf{D}_{ij} \\
\sigma_{kk} &= -p\delta_{kk} + \lambda^*\delta_{kk}\mathsf{D}_{ii} + 2\mu^*\mathsf{D}_{kk} \\
\sigma_{kk} &= -3p + 3\lambda^*\mathsf{D}_{kk} + 2\mu^*\mathsf{D}_{kk} \\
\frac{1}{3}\sigma_{kk} &= -p + \left(\lambda^* + \frac{2}{3}\mu^*\right)\mathsf{D}_{kk} \\
\frac{1}{3}\sigma_{kk} &= -p + k^*\mathsf{D}_{kk} \quad \Rightarrow \quad \boxed{p - \bar{p} = k^*\mathsf{D}_{kk}}
\end{aligned}
\tag{12.35}
$$

which thus defines the *bulk viscosity coefficient* (also called the volume or second viscosity) (κ^*) as:

$$
\kappa^* = \lambda^* + \frac{2\mu^*}{3} \tag{12.36}
$$

It may be interesting to express the Navier-Poisson equations in terms of the deviatoric parts. To do so, we split the Cauchy stress tensor $\boldsymbol{\sigma}$ and the rate-of-deformation tensor $\mathbf{D}$ into a deviatoric and spherical part:

$$
\begin{cases}
\boldsymbol{\sigma} = \boldsymbol{\sigma}^{dev} + \dfrac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1} \\
\sigma_{ij} = \sigma_{ij}^{dev} + \dfrac{\sigma_{kk}}{3}\delta_{ij}
\end{cases}
\quad ; \quad
\begin{cases}
\mathbf{D} = \mathbf{D}^{dev} + \dfrac{\mathrm{Tr}(\mathbf{D})}{3}\mathbf{1} \\
\mathsf{D}_{ij} = \mathsf{D}_{ij}^{dev} + \dfrac{\mathsf{D}_{kk}}{3}\delta_{ij}
\end{cases}
\tag{12.37}
$$

which, substituting into the constitutive equation in (12.33), yields:

$$
\begin{aligned}
\sigma_{ij} &= -p\delta_{ij} + \lambda^*\delta_{ij}\mathsf{D}_{kk} + 2\mu^*\mathsf{D}_{ij} \\
\sigma_{ij}^{dev} + \underbrace{\frac{\sigma_{kk}}{3}}_{-\bar{p}}\delta_{ij} &= -p\delta_{ij} + \lambda^*\delta_{ij}\mathsf{D}_{kk} + 2\mu^*\left(\mathsf{D}_{ij}^{dev} + \frac{\mathsf{D}_{kk}}{3}\delta_{ij}\right) \\
&= \underbrace{-p\delta_{ij} + \left(\lambda^* + \frac{2\mu^*}{3}\right)\mathsf{D}_{kk}\delta_{ij}}_{\text{Related to volume change}} + \underbrace{2\mu^*\mathsf{D}_{ij}^{dev}}_{\substack{\text{Related to}\\ \text{shape change}}}
\end{aligned}
\tag{12.38}
$$

the result of which is:

$$
\boxed{
\begin{aligned}
\boldsymbol{\sigma}^{dev} &= (\bar{p} - p)\mathbf{1} + \kappa^*\mathrm{Tr}(\mathbf{D})\mathbf{1} + 2\mu^*\mathbf{D}^{dev} \\
\sigma_{ij}^{dev} &= (\bar{p} - p)\delta_{ij} + \kappa^*\,\mathsf{D}_{kk}\delta_{ij} + 2\mu^*\mathsf{D}_{ij}^{dev}
\end{aligned}
}
\tag{12.39}
$$

The above can be decomposed into two sets of equations. Then, we can consider the equations in (12.35) in which $(\bar{p} - p) = -\kappa^*\mathsf{D}_{kk}$ holds, which, substituted into the equation in (12.39), yields:

$$
\boldsymbol{\sigma}^{dev} = 2\mu^*\mathbf{D}^{dev} \quad ; \quad \sigma_{ij}^{dev} = 2\mu^*\mathsf{D}_{ij}^{dev}
\tag{12.40}
$$

Then, the equations in (12.39) can be replaced with the following set of equations:

$$
\boxed{
\begin{cases}
\boldsymbol{\sigma}^{dev} = 2\mu^*\mathbf{D}^{dev} \\
\bar{p} = p - \kappa^*\mathrm{Tr}(\mathbf{D})
\end{cases}
}
\tag{12.41}
$$

Now, let us remember that:

$$
\mathrm{Tr}(\boldsymbol{\ell}) = \mathrm{Tr}(\mathbf{D} + \mathbf{W}) = \mathrm{Tr}(\mathbf{D}) = \mathsf{D}_{kk}
\tag{12.42}
$$

where $\boldsymbol{\ell}$ is the spatial velocity gradient, and the trace of $\mathbf{D}$ can be expressed in terms of velocity divergence by:

$$
\mathsf{D}_{kk} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}
\tag{12.43}
$$

and by considering the mass continuity equation, (see Chapter 5), the following remains valid:

$$
\frac{D\rho}{Dt} + \rho\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = 0 \Rightarrow \nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \mathsf{D}_{kk}
\tag{12.44}
$$

Then, by substituting (12.44) into the equation $p_0 = \bar{p} = p - \kappa^*\mathrm{Tr}(\mathbf{D})$ we obtain:

$$
p_0 = p + \kappa^*\frac{1}{\rho}\frac{D\rho}{Dt}
\tag{12.45}
$$

The above equation indicates that the relationship $p_0 = p$ will only be fulfilled when:

1) The rate-of-deformation tensor trace is equal to zero (incompressible fluid, $\nabla_{\boldsymbol{x}} \cdot \vec{\boldsymbol{v}} = 0$):

$$
\mathsf{D}_{kk} = 0 \quad \textit{i.e.} \quad \frac{D\rho}{Dt} = 0 \quad \textit{(Incompressible fluid)}
\tag{12.46}
$$

2) The bulk viscosity coefficient is equal to zero:

$$
\kappa^* = 0 \quad \textit{(Stokes' condition)}
\tag{12.47}
$$

The latter condition is known as the Stokes' condition.

Then, in incompressible cases, ($\overline{p} = p - \kappa^* \mathrm{Tr}(\mathbf{D}) \Rightarrow \overline{p} = p$), the set of equations in (12.41) becomes:

$$\boxed{\boldsymbol{\sigma}^{dev} = 2\mu^* \mathbf{D}^{dev}} \quad \textit{Constitutive equations for incompressible Newtonian fluid} \tag{12.48}$$

12.6.1 The Stokes' Condition

The Stokes' condition is met when:

$$\boxed{\kappa^* = \lambda^* + \frac{2\mu^*}{3} = 0} \quad \textit{The Stokes' condition} \tag{12.49}$$

This condition ensures us that the pressure p is defined as the average of the normal stresses, *i.e.:*

$$\underbrace{\frac{1}{3}\sigma_{ii}}_{\sigma_m} = -p + \kappa^* \mathrm{Tr}(\mathbf{D}) \longrightarrow \sigma_m = -p \longrightarrow p = \overline{p} \tag{12.50}$$

12.7 Stress, Dissipated and Recoverable Powers

By considering the equation in (12.33), $\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} \mathsf{D}_{kk} + 2\mu^* \mathsf{D}_{ij}$, the mechanical (stress) power ($\boldsymbol{\sigma} : \mathbf{D}$) can be rewritten as follows:

$$\begin{aligned} \boldsymbol{\sigma} : \mathbf{D} &= -p\mathbf{1} : \mathbf{D} + \lambda^* \mathrm{Tr}(\mathbf{D})\mathbf{1} : \mathbf{D} + 2\mu^* \mathbf{D} : \mathbf{D} \\ &= -p\mathrm{Tr}(\mathbf{D}) + \lambda^* [\mathrm{Tr}(\mathbf{D})]^2 + 2\mu^* \mathbf{D} : \mathbf{D} \end{aligned} \qquad \begin{aligned} \sigma_{ij}\mathsf{D}_{ij} &= -p\delta_{ij}\mathsf{D}_{ij} + \lambda^* \delta_{ij}\mathsf{D}_{kk}\mathsf{D}_{ij} + 2\mu^* \mathsf{D}_{ij}\mathsf{D}_{ij} \\ &= -p\mathsf{D}_{ii} + \lambda^* \mathsf{D}_{kk}\mathsf{D}_{ii} + 2\mu^* \mathsf{D}_{ij}\mathsf{D}_{ij} \end{aligned} \tag{12.51}$$

Then, by splitting the rate-of-deformation tensor into a deviatoric and spherical part, ($\mathsf{D}_{ij} = \mathsf{D}^{dev}_{ij} + \frac{1}{3}\mathsf{D}_{kk}\delta_{ij}$), and by substituting them into the equation in (12.51) we can obtain:

$$\begin{aligned} \boldsymbol{\sigma} : \mathbf{D} &= -p\mathrm{Tr}(\mathbf{D}) + \lambda^* [\mathrm{Tr}(\mathbf{D})]^2 + 2\mu^* \left(\mathbf{D}^{dev} + \frac{\mathrm{Tr}(\mathbf{D})}{3}\mathbf{1} \right) : \left(\mathbf{D}^{dev} + \frac{\mathrm{Tr}(\mathbf{D})}{3}\mathbf{1} \right) \\ &= -p\mathrm{Tr}(\mathbf{D}) + \lambda^* [\mathrm{Tr}(\mathbf{D})]^2 + 2\mu^* \left(\frac{[\mathrm{Tr}(\mathbf{D})]^2}{3} + \mathbf{D}^{dev} : \mathbf{D}^{dev} \right) \\ &= -p\mathrm{Tr}(\mathbf{D}) + \left(\lambda^* + \frac{2\mu^*}{3} \right) [\mathrm{Tr}(\mathbf{D})]^2 + 2\mu^* \mathbf{D}^{dev} : \mathbf{D}^{dev} \end{aligned} \tag{12.52}$$

or in indicial notation:

$$\begin{aligned} \sigma_{ij}\mathsf{D}_{ij} &= -p\mathsf{D}_{ii} + \lambda^* \mathsf{D}_{kk}\mathsf{D}_{ii} + 2\mu^* \left(\mathsf{D}^{dev}_{ij} + \frac{\mathsf{D}_{kk}}{3}\delta_{ij} \right)\left(\mathsf{D}^{dev}_{ij} + \frac{\mathsf{D}_{pp}}{3}\delta_{ij} \right) \\ &= -p\mathsf{D}_{ii} + \lambda^* \mathsf{D}_{kk}\mathsf{D}_{ii} + 2\mu^* \left(\frac{\mathsf{D}_{kk}\mathsf{D}_{pp}}{3} + \mathsf{D}^{dev}_{ij}\mathsf{D}^{dev}_{ij} \right) \end{aligned} \tag{12.53}$$

where the deviatoric tensor trace is equal to zero, $\mathbf{D}^{dev} : \mathbf{1} = 0$. Then, by restructuring the above equation we obtain:

$$\begin{aligned}\sigma_{ij} \mathsf{D}_{ij} &= -p\mathsf{D}_{ii} + \left(\lambda^* + \frac{2\mu^*}{3} \right) \mathsf{D}_{kk} \mathsf{D}_{ii} + 2\mu^* \mathsf{D}_{ij}^{dev} \mathsf{D}_{ij}^{dev} \\ &= -p\mathsf{D}_{ii} + \kappa^* \mathsf{D}_{kk} \mathsf{D}_{ii} + 2\mu^* \mathsf{D}_{ij}^{dev} \mathsf{D}_{ij}^{dev}\end{aligned} \tag{12.54}$$

Next, the stress power can be expressed as follows:

$$\boxed{\boldsymbol{\sigma} : \mathbf{D} = -p\mathrm{Tr}(\mathbf{D}) + \underbrace{\kappa^* [\mathrm{Tr}(\mathbf{D})]^2 + 2\mu^* \mathbf{D}^{dev} : \mathbf{D}^{dev}}_{2W_D}} \quad \textit{Stress power} \tag{12.55}$$

The term $(-p\mathrm{Tr}(\mathbf{D}))$ is related to elastic energy, so, it is recoverable and because of this we have the definition:

$$\boxed{-p\mathrm{Tr}(\mathbf{D})} \quad \textit{Recoverable power} \tag{12.56}$$

which does not contribute to internal entropy generation in the system. We can also define the dissipated power per unit volume ($2W_D$), which is associated with internal energy dissipation:

$$\begin{cases} 2W_D = \kappa^* \mathsf{D}_{kk} \mathsf{D}_{ii} + 2\mu^* \mathsf{D}_{ij}^{dev} \mathsf{D}_{ij}^{dev} \\ 2W_D = \kappa^* [\mathrm{Tr}(\mathbf{D})]^2 + 2\mu^* \mathbf{D}^{dev} : \mathbf{D}^{dev} \end{cases} \quad \textit{Dissipated power} \tag{12.57}$$

NOTE: As we can verify by looking at the dissipated power, all dissipated energy is brought about by viscosity when the fluid is in relative motion between particles, (see Figure 12.5).

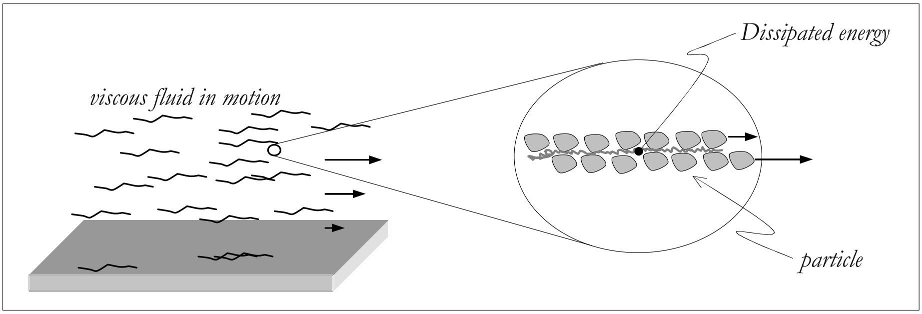


Figure 12.5: Viscous fluid in motion.

Now, by considering the second law of thermodynamics (nonnegative dissipation), we can conclude that:

$$(\mu^* > 0) \qquad \text{and} \qquad \kappa^* \geq 0 \Rightarrow \lambda^* \geq \frac{2\mu^*}{3} \tag{12.58}$$

For a fluid without viscosity, the dissipated power is zero, *i.e.* $2W_D = 0$, and the stress power becomes:

$$\boxed{\boldsymbol{\sigma} : \mathbf{D} = -p\mathrm{Tr}(\mathbf{D}) = -p\nabla_{\vec{x}} \cdot \vec{v}} \quad \textit{Stress power for non-viscous fluid} \tag{12.59}$$

12.8 The Fundamental Equations for Newtonian Fluids

The fundamental equations for Newtonian fluids (Current configuration)		
The mass continuity equation:	$\dfrac{D\rho}{Dt}+\rho(\nabla_{\vec{x}}\cdot\vec{v})=0 \quad ; \quad \dfrac{D\rho}{Dt}+\rho v_{i,i}=0$	(12.60)
The equations of motion:	$\nabla_{\vec{x}}\cdot\boldsymbol{\sigma}+\rho\vec{\mathbf{b}}=\rho\dot{\vec{v}} \quad ; \quad \sigma_{ij,j}+\rho\,\mathrm{b}_i=\rho\dot{v}_i$	(12.61)
The energy equation:	$\rho\,\dot{u}=\boldsymbol{\sigma}:\mathbf{D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r \quad ; \quad \dot{u}=\dfrac{1}{\rho}\sigma_{ij}\mathrm{D}_{ij}-\dfrac{1}{\rho}\mathrm{q}_{i,i}+r$ *or* $\rho\,\dot{u}=-p\mathrm{Tr}(\mathbf{D})+\underbrace{\kappa^*[\mathrm{Tr}(\mathbf{D})]^2+2\mu^*\mathbf{D}^{dev}:\mathbf{D}^{dev}}_{2W_D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r$	(12.62)

where u is the specific internal energy, r is the heat generated by internal sources and $\vec{\mathbf{q}}$ is the heat flux vector (non-convective).

The mass continuity equation, the equations of motion and the energy equation give us five equations in total. The unknowns are: velocity $\vec{v}$ (three components), temperature T, mass density ρ, the Cauchy stress tensor $\boldsymbol{\sigma}$ (six components), specific internal energy u, the heat flux vector $\vec{\mathbf{q}}$ (three components), entropy η, and pressure p, making a total of 17 unknowns.

For the problem to be well-posed 12 equations must be added to the system, as discussed in Chapter 6, these equations are the so-called constitutive equations:

The constitutive equations		
for stress:	$\boldsymbol{\sigma}=-p\mathbf{1}+\lambda^*\mathrm{Tr}(\mathbf{D})\mathbf{1}+2\mu^*\mathbf{D}$ $\sigma_{ij}=-p\delta_{ij}+\lambda^*\delta_{ij}\mathrm{D}_{kk}+2\mu^*\mathrm{D}_{ij}$	(12.63)
for heat conduction:	$\mathrm{q}_i=-\mathrm{K}_{ij}T_{,j} \quad ; \quad \vec{\mathbf{q}}=-\mathbf{K}\cdot\nabla_{\vec{x}}T$ (Fourier's law)	(12.64)
for entropy:	$\eta=\eta(\rho,T)$	(12.65)
The equations of state:	$p=p(\rho,T)$ $u=u(\rho,T)$	(12.66)

where $\mathbf{K}$ is the thermal conductivity tensor, (see Chapter 10). So, the problem results in a system of 17 equations with 17 unknowns.

For fluids in which is independent of temperature, the pressure can be expressed in term of mass density, $p=p(\rho)$ and the internal energy $u=u(\rho)$. So, the mechanical problem can be represented by the following equations:

The fundamental equations for barotropic Newtonian fluids		
The mass continuity equation	$\dfrac{D\rho}{Dt}+\rho(\nabla_{\vec{x}}\cdot\vec{v})=0$	(12.67)

The equations of motion:	$\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho\vec{\mathbf{b}} = \rho\dot{\vec{v}}$	(12.68)
	The constitutive equations	
for stress:	$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda^* \mathrm{Tr}(\mathbf{D})\mathbf{1} + 2\mu^*\mathbf{D}$	(12.69)
the equation of state:	$p = p(\rho)$	(12.70)

which results in a system with 11 equations and 11 unknown.

12.8.1 The Navier-Stokes-Duhem Equations of Motion

The Navier-Stokes-Duhem equations of motion are a combination of the equations of motion (12.61) and the constitutive equations (12.63). Then, by considering $\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}\mathsf{D}_{kk} + 2\mu^*\mathsf{D}_{ij}$ obtained in (12.33), the Cauchy stress tensor divergence ($\nabla_{\vec{x}} \cdot \boldsymbol{\sigma}$) can be evaluated as follows:

$$\sigma_{ij,j} = -p\delta_{ij,j} + \lambda^*\left(\delta_{ij}\mathsf{D}_{kk}\right)_{,j} + 2\mu^*\mathsf{D}_{ij,j} = \lambda^*\delta_{ij}\mathsf{D}_{kk,j} + 2\mu^*\mathsf{D}_{ij,j} \tag{12.71}$$

In addition, by considering $2\mathsf{D}_{ij} = v_{i,j} + v_{j,i}$ and $2\mathsf{D}_{kk} = v_{k,k} + v_{k,k} = 2v_{k,k}$, we obtain:

$$\begin{aligned} &\Rightarrow 2\mathsf{D}_{ij,j} = v_{i,jj} + v_{j,ij} = v_{i,jj} + v_{j,ji} \\ &\Rightarrow \mathsf{D}_{kk,j} = v_{k,kj} \end{aligned} \tag{12.72}$$

whereby the equation in (12.71) becomes:

$$\begin{aligned} \sigma_{ij,j} &= \lambda^*\delta_{ij}\mathsf{D}_{kk,j} + 2\mu^*\mathsf{D}_{ij,j} = \lambda^*\delta_{ij}v_{k,kj} + \mu^*\left(v_{i,jj} + v_{j,ji}\right) = \lambda^* v_{k,ki} + \mu^*\left(v_{i,jj} + v_{j,ji}\right) \\ &= \lambda^* v_{j,ji} + \mu^*\left(v_{i,jj} + v_{j,ji}\right) = \left(\lambda^* + \mu^*\right)v_{j,ji} + \mu^* v_{i,jj} \end{aligned} \tag{12.73}$$

Then, by substituting the equation in (12.73) into the equations of motion ($\sigma_{ij,j} + \rho\mathsf{b}_i = \rho\dot{v}_i$), (see equation (12.61)), we obtain:

$$\boxed{\begin{aligned} &\rho\dot{v}_i = \rho\mathsf{b}_i - p_{,i} + (\lambda^* + \mu^*)v_{j,ji} + \mu^* v_{i,jj} \\ &\rho\dot{\vec{v}} = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}}p + (\lambda^* + \mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) + \mu^*\nabla_{\vec{x}}^2\vec{v} \end{aligned}} \quad \textit{Navier-Stokes-Duhem equations of motion} \tag{12.74}$$

which are the Navier-Stokes-Duhem equations of motion. The terms on the right of the equation in (12.74) represents force terms, $\rho\vec{\mathbf{b}}$ represents force per unit volume and ($-\nabla_{\vec{x}}p$) is the pressure gradient and represents force per unit volume brought about by thermodynamic pressure. Finally, the remaining terms represent the viscous force per unit volume:

$$\boxed{\vec{f}_{vis} = (\lambda^* + \mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) + \mu^*\nabla_{\vec{x}}^2\vec{v}} \quad \textit{The viscous forces} \tag{12.75}$$

12.8.1.1 Alternative Form of the Fundamental Equations for Newtonian Fluids

With the above, the fundamental equations for Newtonian fluids can also be expressed as follows:

The fundamental equations for Newtonian fluids		
The mass continuity equation:	$\dfrac{D\rho}{Dt}+\rho(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})=0 \quad ; \quad \dfrac{D\rho}{Dt}+\rho v_{i,i}=0$	(12.76)
The Navier-Stokes-Duhem equations:	$\rho\dot{v}_i=\rho \mathsf{b}_i-p_{,i}+(\lambda^*+\mu^*)v_{j,ji}+\mu^* v_{i,jj}$ $\rho\dot{\vec{\boldsymbol{v}}}=\rho\vec{\mathbf{b}}-\nabla_{\vec{x}}p+(\lambda^*+\mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})+\mu^*\nabla_{\vec{x}}^2\vec{\boldsymbol{v}}$	(12.77)
The energy equation:	$\rho\,\dot{u}=-p\mathrm{Tr}(\mathbf{D})+\underbrace{\kappa^*[\mathrm{Tr}(\mathbf{D})]^2+2\mu^*\mathbf{D}^{dev}:\mathbf{D}^{dev}}_{2W_D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r$	(12.78)

The mass continuity equation, the Navier-Stokes-Duhem equations of motion and the energy equation give us five of these in total. The unknowns are: velocity $\vec{\boldsymbol{v}}$ (three components), temperature T, mass density ρ, specific internal energy u, the heat flux vector $\vec{\mathbf{q}}$ (three components), entropy η, and pressure p, which makes a total of 11 unknowns.

Then, for the problem to be well-posed, six equations must be added to the system, namely:

The constitutive equations		
for heat conduction:	$\mathsf{q}_i=-\mathsf{K}_{ij}T_{,j}$ $\vec{\mathbf{q}}=-\mathbf{K}\cdot\nabla_{\vec{x}}T$ (Fourier's law)	(12.79)
for entropy:	$\eta=\eta(\rho,T)$	(12.80)
The equations of state:	$p=p(\rho,T)$ $u=u(\rho,T)$	(12.81)

12.8.1.2 The Fundamental Equations for Incompressible Newtonian Fluid

With incompressible fluids $(\nabla_{\vec{x}}\cdot\vec{\boldsymbol{v}})=0$ holds, then:

The fundamental equations for incompressible Newtonian fluids		
The mass continuity equation:	$\dfrac{D\rho}{Dt}=0 \quad \Rightarrow \quad \rho=\rho_0$	(12.82)
The Navier-Stokes-Duhem equations of motion:	$\rho\dot{v}_i=\rho \mathsf{b}_i-p_{,i}+\mu^* v_{i,jj}$ $\rho\dot{\vec{\boldsymbol{v}}}=\rho\vec{\mathbf{b}}-\nabla_{\vec{x}}p+\mu^*\nabla_{\vec{x}}^2\vec{\boldsymbol{v}}$	(12.83)
The energy equation:	$\rho\,\dot{u}=-p\mathrm{Tr}(\mathbf{D})+\underbrace{\kappa^*[\mathrm{Tr}(\mathbf{D})]^2+2\mu^*\mathbf{D}^{dev}:\mathbf{D}^{dev}}_{2W_D}-\nabla_{\vec{x}}\cdot\vec{\mathbf{q}}+\rho r$	(12.84)

The constitutive equations		
for heat conduction:	$\mathsf{q}_i=-\mathsf{K}_{ij}T_{,j}$ $\vec{\mathbf{q}}=-\mathbf{K}\cdot\nabla_{\vec{x}}T$ (Fourier's law)	(12.85)
for entropy:	$\eta=\eta(\rho,T)$	(12.86)
The equations of state:	$p=p(\rho,T)$ $u=u(\rho,T)$	(12.87)

12.8.2 The Navier-Stokes Equations of Motion

If we have the Stokes' condition $\left(\lambda^* = -\frac{2}{3}\mu^*\right)$, the Navier-Stokes-Duhem equations of motion becomes the Navier-Stokes equations of motion. Then, by substituting $(\lambda^* + \mu^*) = \frac{1}{3}\mu^*$ into the equation in (12.77) we obtain:

$$\boxed{\begin{aligned} \rho\dot{v}_i &= \rho \mathrm{b}_i - p_{,i} + \frac{1}{3}\mu^* v_{j,ji} + \mu^* v_{i,jj} \\ \rho\dot{\vec{v}} &= \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p + \frac{1}{3}\mu^* \nabla_{\vec{x}}(\nabla_{\vec{x}} \cdot \vec{v}) + \mu^* \nabla_{\vec{x}}^2 \vec{v} \end{aligned}}$$ *The Navier-Stokes equations of motion (Compressible fluid)* (12.88)

when the fluid is incompressible ($\nabla_{\vec{x}} \cdot \vec{v} = v_{j,j} = 0$) the above equation becomes:

$$\boxed{\begin{aligned} \rho\dot{v}_i &= \rho \mathrm{b}_i - p_{,i} + \mu^* v_{i,jj} \\ \rho\dot{\vec{v}} &= \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p + \mu^* \nabla_{\vec{x}}^2 \vec{v} \end{aligned}}$$ *The Navier-Stokes equations of motion (Incompressible fluid)* (12.89)

Note that for in incompressible fluids the Navier-Stokes equations of motion (see (12.89)) and the Navier-Stokes-Duhem equations of motion (see (12.83)) coincide.

12.8.3 The Euler Equations of Motion

In a non-viscous fluid (perfect fluid) there is no viscous forces, *i.e.* $\vec{f}_{vis} = \vec{\mathbf{0}}$, so, the equations of motion become:

$$\boxed{\begin{aligned} \rho\dot{v}_i &= \rho \mathrm{b}_i - p_{,i} \\ \rho\dot{\vec{v}} &= \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p \end{aligned}}$$ *The Euler equations of motion (Non-viscous incompressible fluid)* (12.90)

which are known as the Euler equations of motion.

Then, by considering a perfect fluid and an isothermal and adiabatic process, we have:

The fundamental equations for perfect fluid (isothermal and adiabatic process)		
The mass continuity equation:	$\frac{D\rho}{Dt} + \rho(\nabla_{\vec{x}} \cdot \vec{v}) = 0$	(12.91)
The Navier-Stokes-Duhem equations of motion:	$\rho\dot{\vec{v}} = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p$ *or* $\frac{\partial \vec{v}(\vec{x},t)}{\partial t} + \nabla_{\vec{x}}\vec{v} \cdot \vec{v} = \vec{\mathbf{b}} - \frac{1}{\rho}\nabla_{\vec{x}} p$	(12.92)
The energy equation:	$\rho\dot{u} = -p\mathrm{Tr}(\mathbf{D}) = -p(\nabla_{\vec{x}} \cdot \vec{v})$	(12.93)

which results in a total of five equations and six unknowns, namely: $\vec{v}$, p, u, ρ. Then, to complete the set of equation we have to add:

The constitutive equation		
The equation of state	$p = p(\rho, u)$	(12.94)

Then, in the particular case when velocity is equal to zero, the equations of motion in (12.98) become:

$$\rho\vec{\mathbf{b}} = \nabla_{\vec{x}} p \qquad\qquad \rho \mathrm{b}_i = p_{,i} \tag{12.95}$$

which describes the hydrostatic equilibrium. Then, if we assume there is a barotropic condition, $\rho = \rho(p)$, it is possible to define the pressure function as:

$$P(p) = \int_{p_0}^{p} \frac{1}{\rho} dp \tag{12.96}$$

12.8.3.1 Non-Viscous and Incompressible Fluids

In incompressible fluids, $(\nabla_{\vec{x}} \cdot \vec{v}) = 0$, mass density is constant and is no longer an unknown. In addition to this, if we consider there is an isothermal and adiabatic process, we have:

The fundamental equations for incompressible perfect fluids		
The mass continuity equation:	$\nabla_{\vec{x}} \cdot \vec{v} = 0$	(12.97)
The Euler equations of motion:	$\rho \dot{v}_i = \rho \mathrm{b}_i - p_{,i} \quad ; \quad \rho\dot{\vec{v}} = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p$	(12.98)

which results in four unknowns and four equations. Here, we can verify that the Navier-Stokes-Duhem equations of motion become the Euler equations of motion (12.98). In addition to this, we can consider the body force field to be conservative, so, we can express it by means of the potential φ as follows $\vec{\mathbf{b}} = -\nabla_{\vec{x}}\varphi$ whereby the Euler equations of motion become:

$$\begin{aligned} \rho\dot{\vec{v}} &= \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p \\ &= -\rho\nabla_{\vec{x}}\varphi - \nabla_{\vec{x}} p \end{aligned} \qquad\qquad \begin{aligned} \rho\dot{v}_i &= \rho \mathrm{b}_i - p_{,i} \\ &= -\rho\varphi_{,i} - p_{,i} \end{aligned} \tag{12.99}$$

Then, by using the material time derivative, we can express the Eulerian velocity components as follows:

$$\dot{v}_i \equiv \frac{D v_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} v_j = \frac{\partial v_i}{\partial t} + v_{i,j} v_j \tag{12.100}$$

Note, the resulting components of the operation $v_{i,j} v_j$ are the same as those of $(\nabla_{\vec{x}}\vec{v}) \cdot \vec{v}$, (see Chapter 1) and it was shown that the following holds:

$$(\nabla_{\vec{x}}\vec{v}) \cdot \vec{v} = (\nabla_{\vec{x}} \wedge \vec{v}) \wedge \vec{v} + \frac{1}{2}\nabla_{\vec{x}}(\vec{v} \cdot \vec{v}) = (\nabla_{\vec{x}} \wedge \vec{v}) \wedge \vec{v} + \frac{1}{2}\nabla_{\vec{x}}(v^2) = \vec{\boldsymbol{\omega}} \wedge \vec{v} + \frac{1}{2}\nabla_{\vec{x}}(v^2) \tag{12.101}$$

where $v = \|\vec{v}\|$, and $\vec{\boldsymbol{\omega}} \equiv \mathrm{rot}(\vec{v}) \equiv (\nabla_{\vec{x}} \wedge \vec{v})$. Then, the velocity field can be expressed as follows:

$$\dot{\vec{v}} = \frac{\partial \vec{v}}{\partial t} + \vec{\boldsymbol{\omega}} \wedge \vec{v} + \frac{1}{2}\nabla_{\vec{x}}(v^2) \tag{12.102}$$

Then, returning to the equation in (12.99) we can affirm that:

$$\begin{aligned} &\rho\frac{\partial \vec{v}}{\partial t} + \rho\vec{\boldsymbol{\omega}} \wedge \vec{v} + \rho\frac{1}{2}\nabla_{\vec{x}}(v^2) = -\rho\nabla_{\vec{x}}\varphi - \nabla_{\vec{x}} p \\ \Rightarrow &\frac{\partial \vec{v}}{\partial t} + \vec{\boldsymbol{\omega}} \wedge \vec{v} + \frac{1}{2}\nabla_{\vec{x}}(v^2) + \nabla_{\vec{x}}\varphi + \frac{1}{\rho}\nabla_{\vec{x}} p = \vec{\mathbf{0}} \end{aligned} \tag{12.103}$$

Now, if we consider that the mass density field is homogenous, *i.e.* it does not very with the position vector, the following holds:

$$\nabla_{\vec{x}}\left(\frac{p}{\rho}\right)=\frac{1}{\rho}\nabla_{\vec{x}}(p)+p\underbrace{\nabla_{\vec{x}}\left(\frac{1}{\rho}\right)}_{=0} \tag{12.104}$$

whereby the equation in (12.103) can be rewritten as follows:

$$\frac{\partial\vec{\boldsymbol{v}}}{\partial t}+(\nabla_{\vec{x}}\wedge\vec{\boldsymbol{v}})\wedge\vec{\boldsymbol{v}}+\nabla_{\vec{x}}\left(\frac{1}{2}v^2+\varphi+\frac{p}{\rho}\right)=\vec{\mathbf{0}} \tag{12.105}$$

If we then consider the velocity field to be conservative, that means that the curl of the field is zero, $\vec{\boldsymbol{\omega}}=\nabla_{\vec{x}}\wedge\vec{\boldsymbol{v}}=\mathrm{rot}\,\vec{\boldsymbol{v}}=\vec{\mathbf{0}}$. Furthermore, a conservative field can be represented by a potential, thus:

$$\vec{\boldsymbol{v}}=\nabla_{\vec{x}}(\phi)\equiv\frac{\partial\phi}{\partial\vec{\boldsymbol{x}}}\qquad\Rightarrow\qquad\frac{\partial\vec{\boldsymbol{v}}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial\vec{\boldsymbol{x}}}\right)=\frac{\partial}{\partial\vec{\boldsymbol{x}}}\left(\frac{\partial\phi}{\partial t}\right) \tag{12.106}$$

whereupon the equation in (12.105) becomes:

$$\frac{\partial\vec{\boldsymbol{v}}}{\partial t}+\underbrace{(\nabla_{\vec{x}}\wedge\vec{\boldsymbol{v}})}_{\substack{=\vec{\mathbf{0}}\\ Irrotacional}}\wedge\vec{v}+\nabla_{\vec{x}}\left(\frac{1}{2}v^2+\phi+\frac{p}{\rho}\right)=\mathbf{0}_i\qquad\Rightarrow\qquad\nabla_{\vec{x}}\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}v^2+\varphi+\frac{p}{\rho}\right)=\mathbf{0}_i \tag{12.107}$$

Thus, we can conclude that:

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}v^2+\varphi+\frac{p}{\rho}=C(t) \tag{12.108}$$

where $C(t)$ is a constant and time-dependent.

12.8.3.2 Bernoulli's Equation

If the velocity field is stationary, $\frac{\partial\vec{\boldsymbol{v}}}{\partial t}=\vec{\mathbf{0}}$, and if the velocity field is irrotational,: $\nabla_{\vec{x}}\wedge\vec{\boldsymbol{v}}=\mathrm{rot}\,\vec{\boldsymbol{v}}=\vec{\mathbf{0}}$, the equation in (12.107) becomes:

$$\nabla_{\vec{x}}\left(\varphi+\frac{p}{\rho}+\frac{v^2}{2}\right)=\mathbf{0}_i\qquad\Rightarrow\qquad\varphi+\frac{p}{\rho}+\frac{v^2}{2}=constant \tag{12.109}$$

Then, by considering the potential $\varphi=gh$, where g is the acceleration of gravity, and h is the piezometric height, the equation in (12.109) becomes:

$$\boxed{gh+\frac{p}{\rho}+\frac{v^2}{2}=constant}\quad \textit{Bernoulli's equation} \tag{12.110}$$

Let us now check the SI units: $[gh]=\left[\frac{v^2}{2}\right]=\left[\frac{p}{\rho}\right]=\frac{N}{m^2}\frac{m^3}{kg}=\frac{Nm}{kg}=\frac{J}{kg}=\frac{m^2}{s^2}$, which is the unit of specific energy, *i.e.* energy per unit mass, (see Figure 12.6).

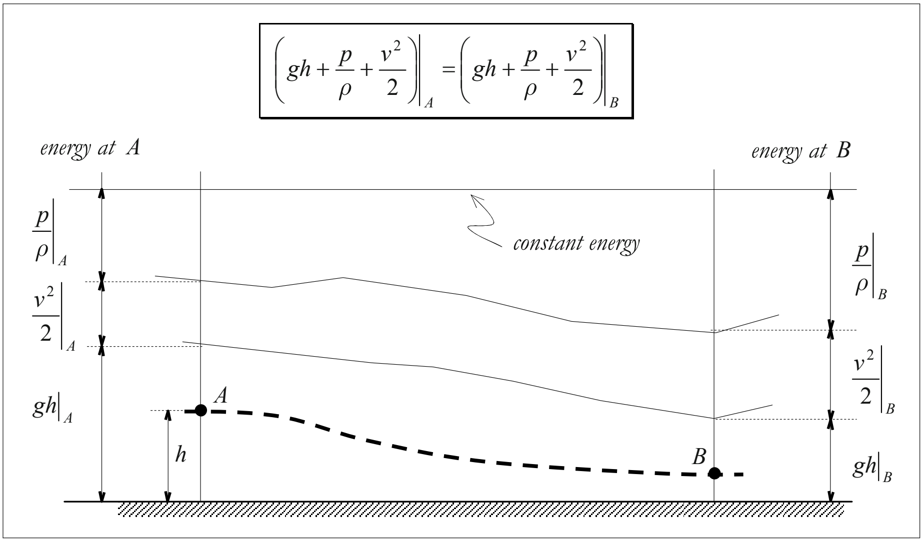


Figure 12.6: Energy *vs.* the Bernoulli' theorem.

12.8.4 The Equation of Vorticity

Next, we will establish the equation of vorticity. To do so, we will start from the Navier-Stokes-Duhem equations of motion given in (12.74):

$$\begin{aligned}\rho\dot{v}_i &= \rho \mathrm{b}_i - p_{,i} + (\lambda^* + \mu^*)v_{j,ji} + \mu^* v_{i,jj} \\ \rho\dot{\vec{v}} &= \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p + (\lambda^* + \mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v}) + \mu^*\nabla_{\vec{x}}^2\vec{v}\end{aligned} \tag{12.111}$$

Then, by taking into account the expression of velocity given in (12.102) the above equation becomes:

$$\begin{aligned}&\rho\dot{v}_i = \rho \mathrm{b}_i - p_{,i} + (\lambda^* + \mu^*)v_{j,ji} + \mu^* v_{i,jj} \\ &\rho\dot{\vec{v}} = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p + (\lambda^* + \mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v}) + \mu^*\nabla_{\vec{x}}^2\vec{v} \\ &\Rightarrow \rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{\boldsymbol{\omega}}\wedge\vec{v} + \frac{1}{2}\nabla_{\vec{x}}(v^2)\right) = \rho\vec{\mathbf{b}} - \nabla_{\vec{x}} p + (\lambda^* + \mu^*)\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v}) + \mu^*\nabla_{\vec{x}}^2\vec{v} \\ &\Rightarrow \frac{\partial\vec{v}}{\partial t} + \vec{\boldsymbol{\omega}}\wedge\vec{v} + \frac{1}{2}\nabla_{\vec{x}}(v^2) - \vec{\mathbf{b}} + \frac{1}{\rho}\nabla_{\vec{x}} p - \frac{(\lambda^* + \mu^*)}{\rho}\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v}) - \frac{\mu^*}{\rho}\nabla_{\vec{x}}^2\vec{v} = \vec{\mathbf{0}}\end{aligned} \tag{12.112}$$

Next, we can take the curl of the above equation, the result of which is:

$$\nabla_{\vec{x}}\wedge\left[\frac{\partial\vec{v}}{\partial t} + \vec{\boldsymbol{\omega}}\wedge\vec{v} + \frac{1}{2}\nabla_{\vec{x}}(v^2) - \vec{\mathbf{b}} + \frac{1}{\rho}\nabla_{\vec{x}} p - \frac{(\lambda^* + \mu^*)}{\rho}\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v}) - \frac{\mu^*}{\rho}\nabla_{\vec{x}}^2\vec{v}\right] = \vec{\mathbf{0}} \tag{12.113}$$

In Chapter 1 we proved that the following holds:

- $\nabla_{\vec{x}}\wedge\left[\nabla_{\vec{x}}(v^2)\right] = \vec{\mathbf{0}}$, $\nabla_{\vec{x}}\wedge\left[\nabla_{\vec{x}} p\right] = \vec{\mathbf{0}}$, $\nabla_{\vec{x}}\wedge\left[\nabla_{\vec{x}}(\nabla_{\vec{x}}\cdot\vec{v})\right] = \vec{\mathbf{0}}$;
- $\begin{aligned}&\nabla_{\vec{x}}\wedge\left[(\nabla_{\vec{x}}\wedge\vec{v})\wedge\vec{v}\right] = (\nabla_{\vec{x}}\cdot\vec{v})(\nabla_{\vec{x}}\wedge\vec{v}) + \left[\nabla_{\vec{x}}(\nabla_{\vec{x}}\wedge\vec{v})\right]\cdot\vec{v} - \nabla_{\vec{x}}\vec{v}\cdot(\nabla_{\vec{x}}\wedge\vec{v}) \\ &\Rightarrow \nabla_{\vec{x}}\wedge\left[\vec{\boldsymbol{\omega}}\wedge\vec{v}\right] = (\nabla_{\vec{x}}\cdot\vec{v})\vec{\boldsymbol{\omega}} + \left[\nabla_{\vec{x}}\vec{\boldsymbol{\omega}}\right]\cdot\vec{v} - \nabla_{\vec{x}}\vec{v}\cdot\vec{\boldsymbol{\omega}}\end{aligned}$;
- $\nabla_{\vec{x}}\wedge\left[\nabla_{\vec{x}}^2\vec{v}\right] = -\nabla_{\vec{x}}\wedge\left[\nabla_{\vec{x}}\wedge(\nabla_{\vec{x}}\wedge\vec{v})\right] = \nabla_{\vec{x}}^2\left[\nabla_{\vec{x}}\wedge\vec{v}\right] = \nabla_{\vec{x}}^2\,\vec{\boldsymbol{\omega}}$;

- $\nabla_{\vec{x}} \wedge \left[\dfrac{\partial \vec{v}}{\partial t}\right] = \dfrac{\partial}{\partial t}\left[\nabla_{\vec{x}} \wedge \vec{v}\right] = \dfrac{\partial \vec{\boldsymbol{\omega}}}{\partial t}$;
- If we consider that the field $\vec{\mathbf{b}}$ is conservative, and the curl of any conservative vector field is zero, we have $\nabla_{\vec{x}} \wedge \vec{\mathbf{b}} = \vec{\mathbf{0}}$.

Then, given the above, we can conclude that:

$$\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} + (\nabla_{\vec{x}} \vec{\boldsymbol{\omega}}) \cdot \vec{v} - \nabla_{\vec{x}} \vec{v} \cdot \vec{\boldsymbol{\omega}} - \frac{\mu^*}{\rho} \nabla_{\vec{x}}{}^2 \, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \tag{12.114}$$

Note that the following relationships hold:

$$\begin{aligned} (v_i \omega_j)_{,i} &= v_{i,i}\, \omega_j + v_i \omega_{j,i} \quad \Rightarrow \quad v_{i,i}\, \omega_j = (v_i \omega_j)_{,i} - v_i \omega_{j,i} \\ (v_i \omega_j)_{,j} &= v_{i,j}\, \omega_j + v_i \omega_{j,j} \quad \Rightarrow \quad v_{i,j}\, \omega_j = (v_i \omega_j)_{,j} - v_i \omega_{j,j} = (v_i \omega_j)_{,j} \end{aligned} \tag{12.115}$$

which is the same in tensorial notation:

$$\begin{aligned} (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} &= \nabla_{\vec{x}} \cdot \left[\vec{\boldsymbol{\omega}} \otimes \vec{v}\right] - (\nabla_{\vec{x}} \vec{\boldsymbol{\omega}}) \cdot \vec{v} \\ (\nabla_{\vec{x}} \vec{v}) \cdot \vec{\boldsymbol{\omega}} &= \nabla_{\vec{x}} \cdot \left[\vec{v} \otimes \vec{\boldsymbol{\omega}}\right] - (\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}})\vec{v} = \nabla_{\vec{x}} \cdot \left[\vec{v} \otimes \vec{\boldsymbol{\omega}}\right] \end{aligned} \tag{12.116}$$

where we have applied the definition that the divergence of the curl of a vector is zero, *i.e.* $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{\omega}} = \nabla_{\vec{x}} \cdot (\nabla_{\vec{x}} \wedge \vec{v}) = 0$. Then, by considering (12.116), the equation (12.114) becomes:

$$\begin{aligned} &\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + (\nabla_{\vec{x}} \cdot \vec{v})\vec{\boldsymbol{\omega}} + (\nabla_{\vec{x}} \vec{\boldsymbol{\omega}}) \cdot \vec{v} - \nabla_{\vec{x}} \vec{v} \cdot \vec{\boldsymbol{\omega}} - \frac{\mu^*}{\rho} \nabla_{\vec{x}}{}^2 \, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\ \Rightarrow &\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + \nabla_{\vec{x}} \cdot \left[\vec{\boldsymbol{\omega}} \otimes \vec{v}\right] - (\nabla_{\vec{x}} \vec{\boldsymbol{\omega}}) \cdot \vec{v} + (\nabla_{\vec{x}} \vec{\boldsymbol{\omega}}) \cdot \vec{v} - \nabla_{\vec{x}} \cdot \left[\vec{v} \otimes \vec{\boldsymbol{\omega}}\right] - \frac{\mu^*}{\rho} \nabla_{\vec{x}}{}^2 \, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\ \Rightarrow &\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + \nabla_{\vec{x}} \cdot \left[\vec{\boldsymbol{\omega}} \otimes \vec{v}\right] - \nabla_{\vec{x}} \cdot \left[\vec{v} \otimes \vec{\boldsymbol{\omega}}\right] - \frac{\mu^*}{\rho} \nabla_{\vec{x}}{}^2 \, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \\ \Rightarrow &\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + \nabla_{\vec{x}} \cdot \left[\vec{\boldsymbol{\omega}} \otimes \vec{v} - \vec{v} \otimes \vec{\boldsymbol{\omega}}\right] - \frac{\mu^*}{\rho} \nabla_{\vec{x}}{}^2 \, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \end{aligned} \tag{12.117}$$

$$\boxed{\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} + 2\nabla_{\vec{x}} \cdot \left[(\vec{\boldsymbol{\omega}} \otimes \vec{v})^{skew}\right] - \frac{\mu^*}{\rho} \nabla_{\vec{x}}{}^2 \, \vec{\boldsymbol{\omega}} = \vec{\mathbf{0}}} \quad \textit{The equation of vorticity} \tag{12.118}$$

Note that to obtain the equation in (12.118), the only simplification made was for the fluid to be Newtonian, and the $\vec{\mathbf{b}}$-field to be conservative.

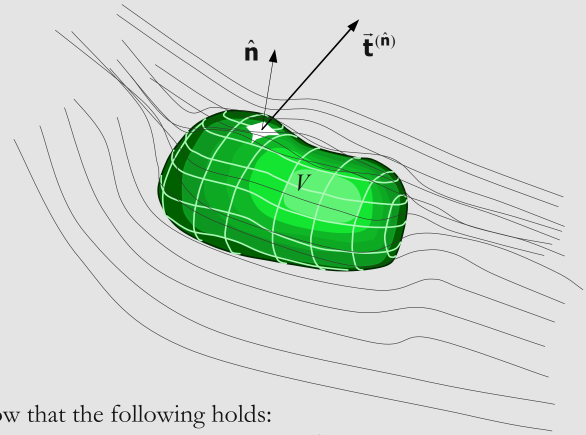
Problem 12.2: Prove that the Cauchy deviatoric stress tensor $\boldsymbol{\sigma}^{dev}$ is equal to $\boldsymbol{\tau}^{dev}$, where $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$.

Solution

If we consider that $\sigma_{kk} = -3p + \tau_{kk}$ we can obtain:

$$\sigma_{ij}^{dev} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij} = -p\delta_{ij} + \tau_{ij} - \frac{(-3p + \tau_{kk})}{3}\delta_{ij} = \tau_{ij} - \frac{\tau_{kk}}{3}\delta_{ij} = \tau_{ij}^{dev}$$

Problem 12.3: Let us consider a body immersed in a Newtonian fluid. Find the total traction force $\vec{\boldsymbol{E}}$ acting on the closed surface S which delimits the volume V. Consider that the bulk viscosity coefficient to be zero.



Solution: We know that the following holds:

$$dE_i = \mathsf{t}_i^{(\hat{\mathbf{n}})} dS$$

The total traction force is given by the following integral:

$$E_i = \int_S \mathsf{t}_i^{(\hat{\mathbf{n}})} dS = \int_S \sigma_{ij} \hat{\mathsf{n}}_j dS = \int_V \sigma_{ij,j} dV$$

where we have used the relationship $\sigma_{ij}\hat{\mathsf{n}}_j = \mathsf{t}_i^{(\hat{\mathbf{n}})}$.

Then, if the bulk viscosity coefficient is zero, we have: $\kappa^* = 0 \Rightarrow \lambda^* = -\frac{2}{3}\mu^*$ (Stokes' condition).

Next, by considering the stress constitutive equation for Newtonian fluids, we obtain:

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} \mathsf{D}_{kk} + 2\mu^* \mathsf{D}_{ij} = -p\delta_{ij} - \frac{2}{3}\mu^* \delta_{ij} \mathsf{D}_{kk} + 2\mu^* \mathsf{D}_{ij} = -p\delta_{ij} + 2\mu^* \underbrace{\left(\mathsf{D}_{ij} - \frac{\mathsf{D}_{kk}}{3}\delta_{ij} \right)}_{\mathsf{D}_{ij}^{dev}}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu^* \mathsf{D}_{ij}^{dev}$$

Then

$$E_i = \int_S (-p\delta_{ij} + 2\mu^* \mathsf{D}_{ij}^{dev}) \hat{\mathsf{n}}_j dS$$

and by applying the Gauss' theorem, we obtain:

$$E_i = \int_V \left(-p\delta_{ij} + 2\mu^* \mathsf{D}_{ij}^{dev} \right)_{,j} dV = \int_V \left(-p_{,j}\delta_{ij} + 2\mu^* \mathsf{D}_{ij,j}^{dev} \right) dV = \int_V (-p_{,i} + 2\mu^* \mathsf{D}_{ij,j}^{dev}) dV$$

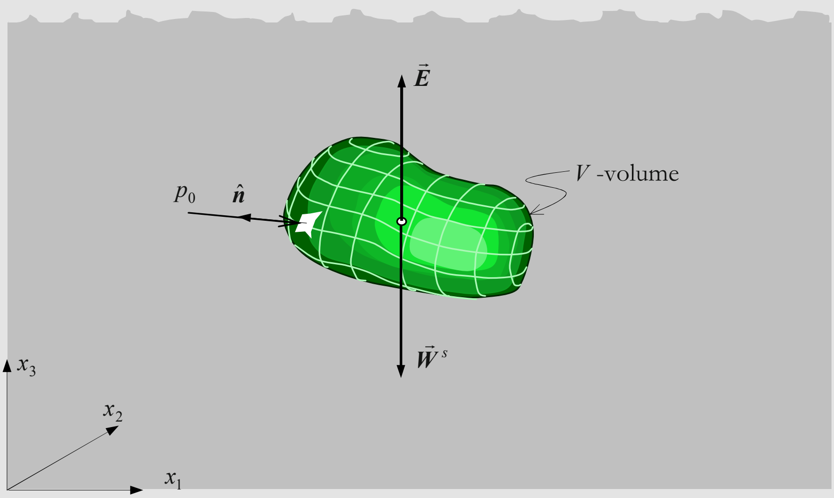
where we have considered that $\mu^*_{,j} = 0_j$, *i.e.* μ^* is a homogenous scalar field (homogenous material). Then, the above equation in tensorial notation becomes:

$$\vec{\boldsymbol{E}} = \int_V \left[-\nabla_{\vec{x}} p + 2\mu^* \nabla_{\vec{x}} \cdot (\mathbf{D}^{dev}) \right] dV \qquad (12.119)$$

Problem 12.4: Let us consider a fluid at rest which has the mass density ρ^f. Prove **Archimedes' Principle**: *"Any body immersed in a fluid at rest experiences an upward buoyant force equal to the weight of the volume fluid displaced by the body".*
If mass density in the body is equal to ρ^s and the body force (per unit mass) is given by $\mathsf{b}_i = -g\delta_{i3}$, obtain the resultant force and acceleration acting on the body.
Solution:

In **Problem 12.3** we showed that $\vec{\boldsymbol{E}} = \int \left[-\nabla_{\vec{x}} p + 2\mu^* \nabla_{\vec{x}} \cdot (\mathbf{D}^{dev})\right] dV$. If the fluid is at rest $\mathbf{D}^{dev} = \mathbf{0}$ holds, and the thermodynamic pressure is equal to the hydrostatic pressure, *i.e.* $p = p_0$ whereby we have:

$$\vec{\boldsymbol{E}} = \int_V \left[-\nabla_{\vec{x}} p_0\right] dV \tag{12.120}$$



The weight of the fluid volume displaced by the body is given by:

$$\vec{\boldsymbol{W}}^f = \int_V \rho^f \vec{\mathbf{b}} dV \tag{12.121}$$

Then, by applying the equilibrium equations we have:

$$\begin{array}{l|l}
\nabla_{\vec{x}} \cdot \boldsymbol{\sigma} + \rho^f \vec{\mathbf{b}} = \vec{\mathbf{0}} & \sigma_{ij,j} + \rho^f \mathsf{b}_i = 0_i \\
\Rightarrow \nabla_{\vec{x}} \cdot \boldsymbol{\sigma} = -\rho^f \vec{\mathbf{b}} & \Rightarrow \sigma_{ij,j} = -\rho^f \mathsf{b}_i \\
\Rightarrow \nabla_{\vec{x}} \cdot (-p_0 \mathbf{1}) = -\rho^f \vec{\mathbf{b}} & \Rightarrow (-p_0 \delta_{ij})_{,j} = -\rho^f \mathsf{b}_i \\
\Rightarrow \nabla_{\vec{x}} p_0 = \rho^f \vec{\mathbf{b}} & \Rightarrow p_{0,i} = \rho^f \mathsf{b}_i
\end{array} \tag{12.122}$$

Next, by considering both (12.120) and (12.121), we can conclude that:

$$\vec{\boldsymbol{W}}^f = \int_V \rho^f \vec{\mathbf{b}} dV = \int_V \nabla_{\vec{x}} p_0 dV = -\vec{\boldsymbol{E}} \tag{12.123}$$

which thereby proves Archimedes' principle.
Now, the body weight, with mass density ρ_s, can be obtained as follows:

$$\vec{\boldsymbol{W}}^s = \int_V \rho^s \vec{\mathbf{b}} dV$$

and the resultant force acting on the body is given by:

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{E}} + \vec{\boldsymbol{W}}_s = -\int_V \rho^f \vec{\mathbf{b}} dV + \int_V \rho^s \vec{\mathbf{b}} dV = \int_V (\rho^s - \rho^f) \vec{\mathbf{b}} dV$$

whose components are:

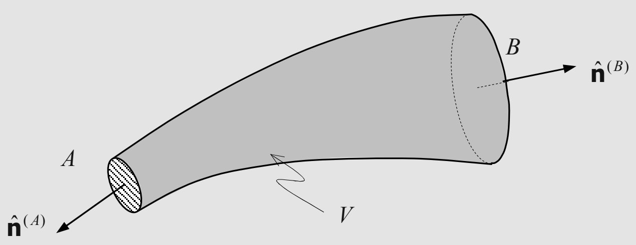
$$R_i = \int_V (\rho^s - \rho^f) \mathrm{b}_i dV = \int_V -g(\rho^s - \rho^f)\delta_{i3} dV = \begin{bmatrix} 0 \\ 0 \\ \int_V g(\rho^f - \rho^s) dV \end{bmatrix}$$

thereby verifying that: if the body has a mass density lower than fluid mass density, *e.g.* if the body is a gas, the body rises, *i.e.* $\rho^f > \rho^s \Rightarrow \vec{\boldsymbol{R}} > \vec{\mathbf{0}}$, and if not the body falls. Moreover, if we consider that $\vec{\boldsymbol{R}} = m^s \vec{\boldsymbol{a}}$, where m^s is the total mass of the submerged body, we can obtain the acceleration of the body as:

$$a_3 = \frac{R_3}{m^s} = \frac{\int_V g(\rho^f - \rho^s) dV}{m^s} = \frac{\int_V g(\rho^f - \rho^s)\frac{\rho^s}{\rho^s} dV}{m^s} = \frac{\frac{g(\rho^f - \rho^s)}{\rho^s}\int_V \rho^s dV}{m^s} = \frac{g(\rho^f - \rho^s)}{\rho^s}$$

NOTE: It is interesting to note that if the medium (f) is such that $\rho^f = 0$ we have $a_3 = -g$, *i.e.* the acceleration is independent of the mass. Here we have clearly seen, as did Galileo, by means of a simple experiment, that a freely falling body was independent of the mass. For example, on the moon where we can consider that the mass density of air is equal to zero, two bodies with different masses in free fall, *e.g.* a feather and a hammer, will have the same acceleration and will reach the moon surface at the same time.

Problem 12.5: Obtain the one-dimensional mass continuity equation for a non-viscous incompressible fluid flow through a pipeline. Then, consider the volume V between the two arbitrary cross sections A and B.



Solution:

In an incompressible medium, the mass density is independent of time $\frac{D\rho}{Dt} \equiv \dot{\rho} = 0$. Moreover, here we can consider the mass continuity equation $\dot{\rho} + \rho v_{k,k} = \rho(\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}}) = 0$, where $\nabla_{\vec{x}} \cdot \vec{\boldsymbol{v}} = 0$ or $v_{k,k} = 0$ holds. Then, by considering the entire volume we obtain:

$$\int_V \nabla_{\vec{x}} \cdot \vec{v}\, dV = 0 \quad ; \quad \int_V v_{k,k}\, dV = 0$$

and by applying the divergence theorem (Gauss' theorem) we obtain:

$$\int_S \vec{v} \cdot \hat{\mathbf{n}}\, dS = 0 \quad ; \quad \int_S v_k \hat{n}_k\, dS = 0$$

Thus:

$$\int_{S_A} \vec{v}_A \cdot \hat{\mathbf{n}}_A\, dS + \int_{S_B} \vec{v}_B \cdot \hat{\mathbf{n}}_B\, dS = 0$$

Next, the velocities at the cross sections S^A and S^B can be expressed as follows:

$$\vec{v}_A = -v_A \hat{\mathbf{n}}_A \quad ; \quad \vec{v}_B = v_B \hat{\mathbf{n}}_B$$

and by substituting the velocity into the integral we can obtain:

$$-v_A \int_{S_A} \hat{\mathbf{n}}_A \cdot \hat{\mathbf{n}}_A\, dS + v_B \int_{S_B} \hat{\mathbf{n}}_B \cdot \hat{\mathbf{n}}_B\, dS = 0$$

$$\boxed{v_A S_A = v_B S_B}$$

Problem 12.6: Determine the conditions needed for mean normal pressure $\bar{p} = -\frac{\sigma_{kk}}{3} = -\sigma_m$ to be equal to thermodynamic pressure p for a Newtonian fluid.

Solution:

It was deduced that:

$$\sigma_{ij}^{dev} = 2\mu^* \mathrm{D}_{ij}^{dev} \quad ; \quad \frac{1}{3}\sigma_{kk} = -p + k^* \mathrm{D}_{ii} \quad ; \quad \underbrace{\frac{\sigma_{kk}}{3}}_{-p} = -\bar{p} = -p + \kappa^* \mathrm{D}_{kk}$$

Thus, $\bar{p} = p$ holds when the following is satisfied:

$$\kappa^* = 0 \quad or \quad \begin{cases} \mathrm{D}_{ii} = 0 \\ \mathrm{Tr}(\mathbf{D}) = 0 \end{cases} \quad or \quad \lambda^* = -\frac{2}{3}\mu^*$$

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