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A Nonlocal, Ordinary, State-Based Plasticity Model for Peridynamics

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Abstract

An implicit time integration algorithm for a non-local, state-based, peridynamics plasticity model is developed. The flow rule was proposed in [3] without an integration strategy or yield criterion. This report addresses both of these issues and thus establishes the first ordinary, state-based peridynamics plasticity model. Integration of the flow rule follows along the lines of the classical theories of rate independent J_2 plasticity [6]. It uses elastic force state relations, an additive decomposition of the deformation state, an elastic force state domain, a flow rule, loading/un-loading conditions, and a consistency condition. Just as in local theories of plasticity (LTP), state variables are required. It is shown that the resulting constitutive model does not violate the 2nd law of thermodynamics. The report also develops a useful non-local yield criterion that depends upon the yield stress and horizon for the material. The modulus state for both the ordinary elastic material [3] and aforementioned plasticity model is also developed and presented.

Acknowledgements

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1 Introduction and Approach

This report focuses on the recent development of a perfect plasticity model for peridynamics. It builds upon the state-based peridynamics model [3] and linearization framework [5] by Silling.

Section 2 provides a introduction to the kinematics associated with peridynamics. Nearly all variables relevant to ordinary state-based materials are presented. Fréchet derivative calculations relevant to subsequent sections are developed. Peridynamics treatment of the momentum equation is briefly presented with an emphasis towards its relationship with linearization and the kinematics of nonlinear solution methods. Section 2.4 introduces the modulus state, which represents the linearization of an ordinary state-based constitutive model. Section 3 shows that the linearization of all ordinary materials consists of two pieces: a geometric/material-independent piece and a material-dependent piece. This linearization applies to both the elastic material (Section 4) and the plastic material introduced in Section 5. In Section 7, it is shown that the plasticity model developed in Section 5 does not violate the 2nd law of thermodynamics. The plasticity model is linearized in Section 9. Of practical significance to the plasticity model is the yield condition, which is developed in Section 6. A summary is presented in Section 10.

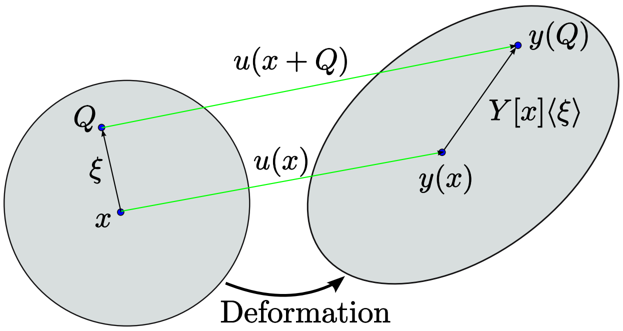

Figure 1. Peridynamics: bonds and states

2 Notation and Preliminary Development

In this Section, a review of the kinematics associated with peridynamics is given. Some quantities of use later will also be derived. The notation used closely follows [3, 5]. Figure 1 may be helpful for following the discussion.

2.1 Definitions

In the undeformed body, a bond vector ξ is defined between an arbitrary point $x \in \Re^3$ in the body and a second point $Q \in \Re^3$ within the body and neighborhood H_x of x, where $H_x = \{x' \mid \|x - x'\| \leq \delta\}$.

$$\text{Bond:} \qquad \xi = Q - x \tag{1}$$

States are functions that act on bonds. There are scalar, vector and modulus states used and developed in this report. These correspond with functions that act on bonds to produce scalars, vectors and 2nd order tensors respectively.

The deformed configuration of a body is described using the following states. Note that these are vector states unless specifically identified as a scalar state.

$$\text{Scalar reference state:} \qquad \begin{aligned} \underline{x} &= |X|\langle\xi\rangle \\ &= |Q - x| \\ &= |\xi| \end{aligned} \tag{2}$$

$$\text{Deformation state:} \qquad \begin{aligned} Y[x]\langle\xi\rangle &= y(x+\xi) - y(x) \\ &= y(Q) - y(x), \end{aligned} \tag{3}$$

where x is a point and should not to be confused with the scalar reference state $\underline{x}$ and $y(x)$ is the coordinate of the point x in the deformed configuration. The angle brackets $\langle\cdot\rangle$ are used to indicate the action of a state on a particular bond ξ while the $[\cdot]$ notation is used to indicate that the associated state depends upon the quantity within the square brackets.

Scalar deformation state: $$|Y|\langle\xi\rangle = [Y[x]\langle\xi\rangle \cdot Y[x]\langle\xi\rangle]^{\frac{1}{2}} \tag{4}$$

Deformed direction state: $$M(Y) = \frac{Y}{|Y|} \tag{5}$$

Note that $M(Y)$ is a unit vector directed along Y.

Displacement state: $$\begin{aligned} U[x]\langle\xi\rangle &= u(x+\xi) - u(x) \\ &= u(Q) - u(x) \end{aligned} \tag{6}$$

Scalar extension state: $$e(Y) = |Y| - |X| \tag{7}$$

The following definitions require the use of a dot product between two states. The formal definition for the dot product between two states is given in [3] and is denoted here as $a \bullet b$ for states a and b. The value of the dot product depends upon the point $x \in \Re^3$, its neighborhood H_x, and the order and values of the states a and b. To evaluate the dot product, the states a and b are evaluated as x. The dot product consists of an integral over all bonds ξ in the neighborhood H_x. The integrand for this integral is given by contracting $a[x]\langle\xi\rangle$ and $b[x]\langle\xi\rangle$. For scalar states the contraction is multiplication. For vector states, this contraction is the usual Euclidean dot product of vectors. These two cases cover all dot products between states in this report.

The following two scalars are used in conjunction with a constitutive model and the scalar extension state e to evaluate internal forces.

Weighted volume: $$\begin{aligned} m &= (\omega \underline{x}) \bullet \underline{x} \\ &= \int_H \omega(|\xi|)|\xi|^2 \, dV_Q \end{aligned} \tag{8}$$

Dilatation: $$\begin{aligned} \theta(Y) &= \frac{3}{m}(\omega \underline{x}) \bullet e \\ &= \frac{3}{m}\int_H \omega |X|\langle\xi\rangle |Y|\langle\xi\rangle \, dV_Q - 3 \end{aligned} \tag{9}$$

A new state, ω, called the *influence function* is introduced in the proceeding two equations. A formal definition for ω is given in [3, Definition 2.1]. Note that the definitions for weighted volume and dilatation introduce the dot product between two states. Each state in the dot product is defined at a point and the integral associated with the dot product is over the neighborhood, H, of that point. Therefore the dot product produces a quantity (in these two cases a scalar) that depends upon the point. One application of ω is as a weight function in the above dot products.

$$\text{Deviatoric extension state:} \qquad \begin{aligned} e^d(Y) &= e(Y) - \frac{\theta(Y)|X|}{3} \\ &= |Y| - |X| - \frac{\theta(Y)|X|}{3} \end{aligned} \tag{10}$$

2.2 Selected Fréchet Derivative Evaluations

The Fréchet derivative calculations presented in this subsection are relevant to later sections of the report. These results are particularly relevant to developing analytical expressions for the Jacobian used in Newton-Raphson.

Deformation State

The Fréchet derivative [5] of the deformation state $|Y|$ is motivated by the need to linearize $|Y|$ about a configuration $|Y^0|$. The idea is that a small displacement state U is superposed on $|Y^0|$. With this in mind, the Fréchet derivative of the deformation state $|Y|$ can be calculated directly from the definition [5] as follows.

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}|Y^0 + \epsilon U|_{\epsilon=0} &= \frac{\partial}{\partial \epsilon}\left[(Y^0 + \epsilon U) \cdot (Y^0 + \epsilon U)\right]^{\frac{1}{2}}\Big|_{\epsilon=0} \\
&= \frac{Y^0}{|Y^0|} \cdot U \\
&= M(Y^0) \cdot U \qquad (11) \\
&= \int_H \Delta(\xi - \eta) M(Y^0)\langle\xi\rangle \cdot U\langle\xi\rangle \, dV_\eta \qquad (12) \\
&= \nabla Y \bullet U
\end{aligned}$$

Note that (11) above defines $M(Y^0)$ as a unit vector directed along the deformation state vector Y^0 and that (12) uses and includes the Dirac Δ function. This later step becomes crucial for identification of the modulus state when linearizing ordinary state-based constitutive models.

Dilatation and Deviatoric Extension State

The Fréchet derivative of the dilatation is required for linearizing ordinary (defined in Section 3), state-based constitutive models. This is computed directly using the definition of the dilatation (9) and the definition of the Fréchet derivative.

$$\begin{aligned} \nabla\theta \bullet U &= \frac{\partial}{\partial\epsilon}\theta(Y^0+\epsilon U)|_{\epsilon=0} \\ &= \frac{3}{m}\int \omega(|\xi|)|\xi| M(Y^0)\langle\xi\rangle \cdot U[x]\langle\xi\rangle \, dV_\xi \end{aligned} \tag{13}$$

The Fréchet derivative of the deviatoric extension state is also required when linearizing ordinary state-based constitutive models. Again, this is computed directly using the definition and properties for the Fréchet derivative as well the above Fréchet derivatives for the deformation state and dilatation.

$$\begin{aligned} \nabla e^d \bullet U &= \frac{\partial}{\partial\epsilon}\, e^d(Y^0+\epsilon U)|_{\epsilon=0} \\ &= \frac{\partial}{\partial\epsilon}|Y^0+\epsilon U|_{\epsilon=0} - \frac{1}{3}|X|(\nabla\theta \bullet U) \end{aligned} \tag{14}$$

Deformation Direction State

The Fréchet derivative of the deformation direction state $M(Y)$ is also required when linearizing ordinary state-based constitutive models. Using the definition of the Fréchet derivative and the definition of $M(Y)$, this is computed as follows.

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\, M(Y^0+\epsilon U)\big|_{\epsilon=0} &= \left[\frac{I - M(Y^0)\langle\xi\rangle \otimes M(Y^0)\langle\xi\rangle}{|Y^0|\langle\xi\rangle}\right] U \\ &= \int_H \Delta(\xi-\eta)\left[\frac{I - M(Y^0)\langle\xi\rangle \otimes M(Y^0)\langle\xi\rangle}{|Y^0|\langle\xi\rangle}\right] U \, dV_\eta \end{aligned} \tag{15}$$

where I is the usual 3×3 identity tensor.

2.3 Peridynamics Treatment of the Momentum Equation for Solids

Solution algorithms used for the momentum equation under both finite element and peridynamic discretizations are very similar. The internal force for peridynamics is identical to that used in nonlinear finite element codes in the sense that forces are summed at a point. In quasi-statics, the balance of forces at a point defines equilibrium, while in dynamics the sum of forces equals the product of density and acceleration at the point. In each case, it is necessary to evaluate the sum of forces at a point arising from deformations of the solid. The method for evaluation of this force at a point is what differentiates peridynamics [2, 3] from local continuum mechanics. This section is a brief introduction to the the internal force vector associated with peridynamics with an emphasis on its relation to constitutive models.

Using the state-based theory of peridynamics, the momentum equation is given as:

$$\begin{aligned} \rho(x)\ddot{u}(x,t) &= \int_{H(x)} \{T(Y[x])\langle p-x\rangle - T(Y[p])\langle x-p\rangle\}\, dV_p + b(x,t) \\ &= f(Y[x],t) + b(x,t) \end{aligned} \tag{16}$$

The above equation is the peridynamics representation of Newton's 2nd law for the Lagrangian point x and defines the internal force vector $f(Y[x],t)$. The function $b(x,t)$ denotes a body force per unit volume. $T(Y[x])$ is a vector force state function (to be defined later) and is somewhat analogous to the stress tensor in local continuum mechanics in the sense that it represents the material response due to deformations. For a specific material $T(Y[x])$ would be represented by a constitutive model. Note the notation $T(Y[x])$ is used to denote the force state dependency upon the deformation state Y at the point x. This dependency is used later to linearize the internal force vector about a deformation state $Y^0[x]$.

2.4 Kinematics and Linearization of the Peridynamics Internal Force

In this section, discussion of kinematics is limited to that required for constitutive model development and linearization. A detailed account of solution algorithms is not given here.

The primary solution variable is the displacement field $u(x,t)$. The solution algorithm estimates the displacement field at discrete times t_n for $n = 1, 2, 3...$. The phrase *load step* is used to refer to a time increment Δt between two successive time steps – say from t_n to t_{n+1}. For each load step, the solution algorithm iteratively determines an increment in the displacement.

The following notation is used to represent discrete values of the displacement for a particular time; superscripts denote function values at a particular time step n and point x based upon the following definitions:

$$\begin{aligned} u^n &= u(x,t_n) \\ u^{n+1} &= u(x,t_n+\Delta t) \end{aligned}$$

Because the problem is in general non-linear, the incremental solution strategy for a load step consists of finding an increment, Δu, to the displacement u^n. This is an iterative process; the following notation is used for iteration k of a load step:

$$\begin{aligned} y^{n+1}_{k+1} &= x + u^{n+1}_k + \Delta u_k \\ &= y^{n+1}_k + \Delta u_k \end{aligned} \tag{17}$$

where

$$y^{n+1}_k = x + u^{n+1}_k \tag{18}$$

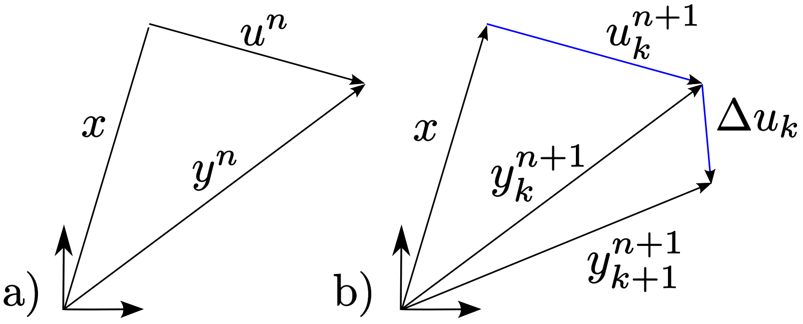

Figure 2. Incremental kinematics

In a typical Newton-Raphson strategy, u_k^{n+1} is known from the previous iteration. A graphical illustration of the kinematics for iteration k is shown in Figure 2(b).

To determine Δu_k, a Jacobian matrix associated with the internal force is required. Using (16) and (18), the internal force vector at step $n+1$ increment k is defined as:

$$\begin{aligned} f_k^{n+1} &= f(Y_k^{n+1}, t+\Delta t) \\ &= \int \left\{ T(Y_k^{n+1}[x]) \langle p-x \rangle - T(Y_k^{n+1}[p]) \langle x-p \rangle \right\} dV_p \end{aligned} \tag{19}$$

Then, the first order approximation to f_{k+1}^{n+1} is given as:

$$\begin{aligned} f_{k+1}^{n+1} &\approx f_k^{n+1} + Df(u_k^{n+1})[\Delta u_k] \qquad (20) \\ &= f_k^{n+1} + \int \left\{ (K[x] \bullet \Delta U_k[x]) \langle p-x \rangle - (K[p] \bullet \Delta U_k[p]) \langle x-p \rangle \right\} dV_p \end{aligned}$$

A special notation in (20) is used to indicate the directional derivative of f evaluated at u_k^{n+1} in the direction Δu_k. This is a linear operator with respect to Δu_k. The second part of the above equation expresses the directional derivative of the internal force vector f with respect to displacements u in terms of an increment in the displacement state ΔU, where K is the modulus state [3] of the material evaluated at Y_k^{n+1}. For implicit solution algorithms, evaluation of K is crucial for optimal convergence rates. The modulus state is a second order tensor that is material dependent. K is derived for an ordinary elastic and ordinary elastic-plastic materials in subsequent sections of this report.

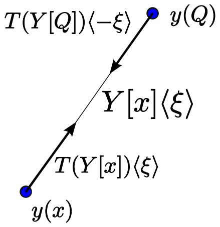

Figure 3. Ordinary material

3 Linearization of Ordinary Materials

As mentioned in the preceding section, the modulus state, K, is used to calculate the Jacobian associated with newton iterations. It represents a change in force at a point in the body for a given configuration y^0 when a small change to the configuration is superposed. The modulus state is given by linearizing the vector force state, $T(Y)$, about the deformation state Y^0 associated with configuration y^0. The vector force state for an ordinary material is defined as:

$$T(Y) = t(Y)\, M(Y) \tag{21}$$

where M is the direction deformation state defined in Eq. 5. An ordinary material is such that $T(Y)$ acts along the direction defined by $Y[x]\langle\xi\rangle$, with a magnitude given by the scalar force state $t(Y)$. A graphical representation of an ordinary material is shown if Figure 3.

To compute the modulus state, Eq. 21 is linearized using the Fréchet derivative. Note that U is the displacement state (see Eq. 6) associated with a small displacement field $u(x)$ superposed upon a deformed configuration y^0.

$$T(Y^0 + U) = T(Y^0) + \nabla T \bullet U \tag{22}$$

$$\begin{aligned}
\nabla T \bullet U &= \frac{\partial}{\partial\epsilon} T(Y^0 + \epsilon U)|_{\epsilon=0} \\
&= M(Y^0)\frac{\partial}{\partial\epsilon} t(Y^0 + \epsilon U)|_{\epsilon=0} + t(Y^0)\frac{\partial}{\partial\epsilon} M(Y^0 + \epsilon U)|_{\epsilon=0} \\
&= K \bullet U
\end{aligned} \tag{23}$$

The above relations define the modulus state for an ordinary material. Note that it consists of two parts: 1) a constitutive model piece defined by the Fréchet derivative of the scalar force state; 2) a geometric component that arises due to rotation of a force carrying bond.

4 Linearization of Isotropic Elastic Constitutive Model

In this section, results from Section 2.2 are used to derive the modulus state of Eq. 23 for an ordinary isostropic elastic constitutive model, where the scalar force state, $t(Y)$, is given as [3, Eq. 105]:

$$t = \frac{3k\theta}{m}\omega\underline{x} + \alpha\omega e^d \tag{24}$$

where k and α are properties that correspond with an isotropic Hookean material. k is the bulk modulus, α is a scalar multiple of the shear modulus, and ω is the influence function introduced in Section 2.1.

While the above relation is linear with respect to θ and e^d, it remains nonlinear with respect to displacements. To complete the modulus state calculation, Fréchet derivatives given in Eqs. 13 and 14 are used. The first step is to compute the Fréchet derivative of the scalar force state:

$$\frac{\partial}{\partial\epsilon}\, t(Y^0 + \epsilon U)\big|_{\epsilon=0} = \left(\frac{9k - \alpha\, m}{3m}\right)\omega\underline{x}\,\nabla\theta \bullet U + \alpha\,\omega M(Y^0)\cdot U \tag{25}$$

At this stage all the ingredients have been evaluated. It is only necessary to combine the above equation with the definition for the modulus state (Eq. 23) and the Fréchet derivative of the deformation direction state in Eq. 15. Also note that the modulus state is defined using the dot product $\bullet$ for states thus requiring the last form of Eq. 12 (for example). The modulus state for an ordinary isostropic elastic material is given as:

$$\begin{aligned} K\langle\zeta,\xi\rangle &= \left(\frac{9k - \alpha\, m}{m^2}\right)\omega(|\zeta|)\omega(|\xi|)|\zeta|\,|\xi| M(Y^0)\langle\zeta\rangle \otimes M(Y^0)\langle\xi\rangle \\ &+ \alpha\omega(|\zeta|) M(Y^0)\langle\zeta\rangle \otimes M(Y^0)\langle\xi\rangle\,\Delta(\zeta - \xi) \\ &+ t(Y^0)\langle\zeta\rangle\left(\frac{I - M(Y^0)\langle\xi\rangle \otimes M(Y^0)\langle\xi\rangle}{|Y^0|\langle\xi\rangle}\right)\Delta(\zeta-\xi) \end{aligned} \tag{26}$$

5 A Nonlocal, Ordinary, State-Based Plasticity Model

A non-local elastic-plastic constitutive model was proposed for Peridynamics in [3]. The peridynamics model follows along the lines of the classical theories of rate-independent J_2 plasticity [6] in that it uses:

- Additive decomposition of deviatoric extension state into elastic and plastic parts
- Elastic force state relations
- Elastic force states domain defined by a yield surface/function that depends upon the deviatoric force state
- Flow rule which describes rate of plastic deformation
- Loading/un-loading conditions (Kuhn-Tucker constraints)
- Consistency condition for both loading/un-loading conditions

Just as in local theories of plasticity (LTP), state variables are required – in this case a scalar deviatoric plastic deformation state. Due to the close similarities between LTP and the elastic-plastic constitutive model for peridynamics, much of the logic used to integrate the rate equation of LTP carries over to the constitutive model.

In this section, the above items are discussed in detail thus providing a return mapping algorithm for the rate-independent peridynamics perfect plasticity model given in [3].

5.1 Decomposition of the scalar deformation state and the isotropic elastic constitutive relation

A key ingredient in the peridynamics constitutive model for plasticity is the decomposition of the scalar extension state into dilatation and deviatoric parts, as well as the additive decomposition of deviatoric extension state into elastic $e^{de}$and plastic e^{dp} parts. Although the nature of the deviatoric extension state is fundamentally different from the local theory, conceptually both of these decompositions are used in LTP. The total deviatoric extension state given in Eq. 10 is additively decomposed into elastic and plastic parts as:

$$e^d \;=\; e^{de} + e^{dp} \tag{27}$$

Motivated by observations that for many ductile materials plastic deformations are independent of the pressure, the isotropic elastic constitutive model given in Eq. 24 is written using the additive decomposition as:

$$\begin{aligned} t &= -\frac{3p}{m}\omega \underline{x} + \alpha\omega(e^d - e^{dp}) \\ &= t^i + t^d, \end{aligned} \tag{28}$$

where $p = -k\theta$ is the the peridynamic pressure and k is the bulk modulus. A rate form of the above equation is given as:

$$\begin{aligned} \dot{t} &= \frac{3k\dot{\theta}}{m}\omega \underline{x} + \omega\alpha(\dot{e}^d - \dot{e}^{dp}) \\ &= \dot{t}^i + \dot{t}^d. \end{aligned} \tag{29}$$

where $\dot{\theta} = \theta(\dot{e})$. The above relations follow from the fact that there exists an elastic stored energy functional W (see [3], Eq 104) of the form

$$W(\theta, e^d, e^{dp}) = \frac{k\theta^2}{2} + \frac{\alpha}{2}(e^d - e^{dp}) \bullet \omega(e^d - e^{dp}), \tag{30}$$

where

$$t^i = \left(\frac{3\omega\underline{x}}{m}\right)\frac{\partial W}{\partial\theta}, \quad t^d = \frac{\partial W}{\partial e^d}. \tag{31}$$

5.2 Elastic domain and the plastic flow rule: perfect plasticity

The description in this section conceptually follows the classical approach to plasticity [6] while utilizing the results and development in [3].

In order to utilize the elastic constitutive relation for plasticity calculations, a scalar valued function, f, called the yield function is used to define a set of allowable scalar deviatoric force states E_{t^d} as follows:

$$E_{t^d} = \left\{t^d \epsilon S^d | f(t^d) = \psi(t^d) - \psi_0 \leq 0\right\} \tag{32}$$

where the ψ_0 is a positive constant and ψ is a function on S (space of scalar states) and S^d is the set defined [3] by:

$$S^d = \left\{t^d \epsilon S | t^d \bullet \underline{x} = 0\right\}. \tag{33}$$

Note that f does not include hardening. This form of f is analogous to local perfect-plasticity, where ψ_0 represents the yield (flow) point of the material.

The plastic flow rule is defined as:

$$\dot{e}^{dp} = \lambda \nabla^d \psi, \tag{34}$$

where $\nabla^d\psi$ is the constrained Fréchet derivative of $\psi\epsilon S$ on S^d and λ is the so-called consistency parameter. A formal definition for the constrained Fréchet derivative is given in [3]. The key idea is that $\nabla^d\psi$ produces functions that have no dilatation.

5.3 Loading/unloading and consistency

Just as in the local theory of plasticity, the evolution of peridynamic plastic deformations is governed by the consistency parameter λ (see Eq 34) and the yield function f.

When force states t^d are such that $f(t^d) < 0$, then it is assumed that there is no change in the plastic deformation and $\underline{\dot{e}}^{dp} = 0$. The scalar force state t remains in the interior of E_{t^d} (Eq 32) and the material is instantaneously elastic. On the other hand, force states t^d are inadmissible if $f(t^d) > 0$. Therefore, plastic deformations only occur when $f(t^d) = 0$.

The consistency parameter λ is a non-negative function that correlates with the rate of plastic deformation. When deformations are elastic, $\lambda = 0$, otherwise under plastic deformations $\lambda > 0$. When taken together, the consistence parameter and the yield function are assumed to satisfy the Kuhn-Tucker complementary conditions given as follows:

$$\lambda \geq 0, \quad f(t^d) \leq 0, \quad \lambda f(t^d) = 0 \tag{35}$$

The above conditions imply that when $f(t^d) < 0$, the consistency parameter must be zero $(\lambda = 0)$. Under this condition, deformations are elastic.

When $\lambda > 0$, admissability of scalar force states implies that $f(t^d) = 0$. Under these conditions, the force state is said to be on the yield surface. Furthermore, it must persist on the yield surface – therefore $\dot{f}(t^d) = 0$. This is a plastic loading scenario.

On the other hand, an unloading condition occurs if $f(t^d) = 0$ and $\dot{f}(t^d) < 0$. In this case, $\lambda = 0$.

Combining these conditions (both the loading and unloading conditions) gives the consistency condition

$$\lambda \dot{f}(t^d) = 0 \tag{36}$$

Using the above consistency condition, the flow rule (Eq. 34), and Eq. 29, a value for the consistency parameter can be computed.

$$\begin{aligned} \dot{f}(t^d) &= 0 \\ &= \frac{\partial f}{\partial t^d} \bullet \dot{t}^d \\ &= \nabla^d \psi \bullet (\alpha \underline{\omega}(\dot{e}^d - \dot{e}^{dp})) \end{aligned} \tag{37}$$

After substituting the flow rule (Eq. 34), the above equation can be solved for λ, which is given as:

$$\lambda = \frac{\nabla^d \psi \bullet \underline{\omega} \dot{e}^d}{\nabla^d \psi \bullet \underline{\omega} \nabla^d \psi}. \tag{38}$$

5.4 Implicit time integration of plasticity model

In this section, a backward Euler scheme is used to incrementally integrate the deviatoric portion of the scalar force state (Eq. 29). Subscripts n denote function values evaluated at time t_n and subscripts $n+1$ denote function values time $t+\Delta t$. Note that the notation here overloads the use of t_n (once for time at step n and once for the scalar force state at step n). The difference should be clear from context.

Given $\{e^d_{n+1}, e^d_n, e^{dp}_n\}$, the problem is to find e^{dp}_{n+1}. This approach to integration is extension state-driven and is analogous to the strain-driven approach used in local plasticity. Note that given a value for e_{n+1}, it is straight forward to calculate e^d_{n+1} using Eqs. 9 and 10.

At step n, corresponding to time t_n, the scalar deviatoric extension state e^d_n and the deviatoric plastic extension state e^{dp}_n (see Eq 27) are denoted as:

$$e^d_n = e^d(t_n), \quad e^{dp}_n = e^{dp}(t_n). \tag{39}$$

Using the constitutive model for t^d in Eq. 24, and the flow rule given in Eq. 34, a trial value for the deviatoric force state t^d_{trial} is defined and computed as:

$$\begin{aligned} t^d_{n+1} &= \alpha\underline{\omega}(e^d_{n+1} - e^{dp}_{n+1}) \\ &= \alpha\underline{\omega}(e^d_{n+1} - e^{dp}_n) - \alpha\underline{\omega}(e^{dp}_{n+1} - e^{dp}_n) \\ &= t^d_{trial} - \alpha\omega\Delta\lambda\nabla^d\psi, \end{aligned} \tag{40}$$

where $\Delta\lambda = \lambda\Delta t$. Given t^d_{trial}, the yield function $f(t^d_{trial})$ (see Eq. 32) is used to determine if the step is elastic or incrementally plastic.

- if $f(t^d_{trial}) \leq 0$, then the loading/unloading conditions are automatically satisfied with $\Delta\lambda = 0$. This leads to:

  $$\begin{aligned} t^d_{n+1} &= t^d_{trial} \\ e^{dp}_{n+1} &= e^{dp}_n \end{aligned} \tag{41}$$

- if $f(t^d_{trial}) > 0$, then the step is incrementally plastic and $\Delta\lambda > 0$. In this case, an algorithmic incremental form of the consistency condition is required. The approach here is to select a value for $\Delta\lambda > 0$ so that $f(t^d_{n+1}) = 0$. For the yield function f defined by $\psi(t^d) = \frac{\|t^d\|^2}{2}$ and $\omega = 1$, and using Eq. 40, the deviatoric force state at the end of the step is computed as:

  $$t^d_{n+1} = \frac{t^d_{trial}}{1+\alpha\Delta\lambda} \tag{42}$$

  Therefore, the value of the yield function at the end of the step is given by:

  $$f(t^d_{n+1}) = \frac{\|t^d_{trial}\|^2}{2}\frac{1}{(1+\alpha\Delta\lambda)^2} - \psi_0 \tag{43}$$

Setting the above equal to zero yields a value for $\Delta\lambda$ as:

$$\Delta\lambda = \frac{1}{\alpha}\left[\frac{\|t^d_{trial}\|}{\sqrt{2\psi_0}} - 1\right] \tag{44}$$

With this value of $\Delta\lambda$, the deviatoric force and plastic states are updated as:

$$t^d_{n+1} = \sqrt{2\psi_0}\,\frac{t^d_{trial}}{\|t^d_{trial}\|}, \quad e^{dp}_{n+1} = e^{dp}_n + \frac{1}{\alpha}\left[\frac{\|t^d_{trial}\|}{\sqrt{2\psi_0}} - 1\right] t^d_{n+1} \tag{45}$$

6 A Yield Condition for the Plasticity Model

In preceding sections, a flow rule for the perfect plasticity model was integrated under the assumption that the yield condition value denoted by ψ_0 in Eq 32 is given. The purpose of the present section is to develop a value for ψ_0.

6.1 Pure shear deformation

The approach is to consider a state of pure shear at a point. This will be induced by a prescribed infinitesimal displacement field. The advantage of using a state of pure shear is that the dilatation is zero. Given the known displacement field, the scalar deformation state, scalar extension state, and the deviatoric extension state can be explicitly written down and subsequently linearized by Taylor series expansion. Note that linearization is required for the approach taken here, which involves difficult integrals that perhaps cannot be evaluated otherwise. However, this linearization is not a practical limitation – it is only used to derive a useful formula for yield. Subsquently, the deviatoric force state can be directly evaluated and substituted into the yield function. This produces a meaningful value for ψ_0 that only requires the readily available yield stress for a material and the horizon radius δ.

Pure shear kinematics

A displacement field that induces pure shear is given by:

$$u(\xi) = \gamma\, b\, \hat{i} \tag{46}$$

where γ is the shear strain, and $\hat{i}$ denotes the coordinate axis unit vector in the a-direction. A point in the undeformed configuration is located by the triplet (a, b, c) which is relative to the center of a sphere – see Figure 4. Ultimately, integrals will be performed on a sphere of radius δ.

With this value for the displacement field, the scalar deformation state (4) is written as a function of γ:

$$Y(\gamma) = \sqrt{b^2 + c^2 + (a + b\gamma)^2}. \tag{47}$$

After expanding the scalar deformation state in a Taylor series and keeping only terms up through linear, the scalar extension state becomes:

$$\begin{aligned} e &= |Y| - |\xi| \\ &\approx |\xi|\gamma \cos\phi \sin\phi \sin^2\theta \end{aligned} \tag{48}$$

Note that (ξ,ϕ,θ) are the spherical coordinates used to designate the location of the point in question. θ should not be confused with the dilatation. Using the above scalar extension state, it can be shown that the dilatation is zero. Furthermore, since the dilatation is zero, the scalar extension state and the deviatoric extension state are identical.

As a lead in to the evaluation of the yield condition, the quantity $\|e^d\|^2$ must be evaluated. This is given by integrating over a sphere of radius δ, where δ is the horizon of the point at the center of the sphere. This value is given by:

$$\|e^d\|^2 = \frac{4}{75}\pi\gamma^2\delta^5 \tag{49}$$

Given the deviatoric extension state e^d in (48), the associated scalar force state is deviatoric and given by:

$$\begin{aligned} t^d &= \alpha e^d \\ &= \frac{15\mu}{m}|\xi|\gamma\,\cos\phi\,\sin\phi\,\sin^2\theta \end{aligned} \tag{50}$$

This value is substituted into the yield functional:

$$\begin{aligned} \psi_0 &= \frac{\|t^d\|^2}{2} \\ &= \frac{1}{2}\left[\frac{15\mu}{m}\right]^2\|e^d\|^2 \\ &= \frac{75}{8\pi}\frac{(\mu\gamma_y)^2}{\delta^5} \\ &= \frac{75}{8\pi}\frac{E_y^2}{\delta^5}, \end{aligned} \tag{51}$$

where γ_y denotes the shear strain at material yield and a value of $m = \frac{4\pi\delta^5}{5}$ was used for the weighted volume. A yield stress corresponding to a state of pure shear is identified and define as:

$$E_y = \mu\gamma_y \tag{52}$$

Yield Condition and its Relation to Measured Yield Stress

The plasticity model developed in this report is the peridynamic equivalent of the local Von Mises plasticity model. The model developed here uses the yield stress E_y (see Eq 52) corresponding to a state of pure shear. To most appropriately use this yield condition, it is necessary to recognize that there is a difference between the value measured for E_y under conditions of pure shear and the value for E_y under conditions of uniaxial tension. To understand this relationship, consider the Von Mises effective stress σ_e. The effective stress is expressed using the 2nd invariant J_2 of the deviatoric stress tensor but can be written in terms of the stress tensor components σ_{ij} and its principal values (σ_1, σ_2, σ_3). The Von

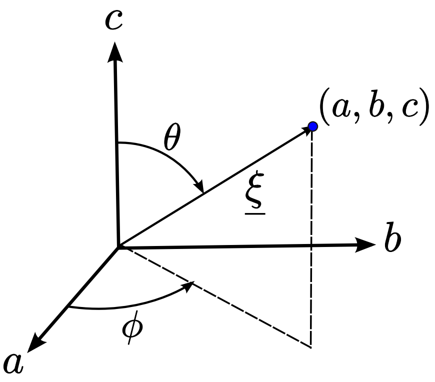

Figure 4. Coordinate system and neighborhood

Mises effective stress σ_e is defined as:

$$\begin{aligned} \sigma_e^2 &= 3\,J_2 \\ &= \frac{1}{6}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\right]+\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2 \qquad (53) \\ &= \frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]. \qquad (54) \end{aligned}$$

At material yield, the Von Mises yield function $f(\sigma_e, E_y) = 0$ and is given by:

$$f(\sigma_e, E_y) = \sigma_e - E_y = 0 \qquad (55)$$

In practical terms, it is important to know what stress conditions were used to evaluate the yield stress.

Yield stress measured under uniaxial stress: E_y can be evaluated using Eq 54 above. In this case, $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = 0$.

$$f(\sigma_e, E_y) = \frac{\sigma_1}{\sqrt{3}} - E_y = 0 \;\Rightarrow\; E_y = \frac{\sigma_1}{\sqrt{3}} \qquad (56)$$

Yield stress measured under pure shear: E_y is evaluated using Eq 53 above. In this case, all components of the stress tensor are zero except for one shear component – take that component to be σ_{12}. Under these conditions of pure shear, E_y is given by:

$$f(\sigma_e, E_y) = \sigma_{12} - E_y = 0 \;\Rightarrow\; E_y = \sigma_{12} \qquad (57)$$

7 Thermodynamic Considerations

The purpose of this section is to demonstrate that the proposed plasticity model does not violate the 2nd law of thermodynamics. This is shown by demonstrating that the rate of entropy production is positive. No attempt is made to justify this approach although it is widely used and accepted in the literature. It is also observed that the power due to plastic deformations is always positive thus satisfying the so-called *dissipation inequality* [1].

Following the procedure in [4], the 1st and 2nd laws of thermodynamics can be combined to produce the following condition

$$t^i \bullet \dot{e}^i + t^d \bullet \dot{e}^d - \dot{\theta}_T \eta - \dot{\phi} \geq 0, \tag{58}$$

where t^i and t^d are the scalar volumetric and deviatoric force states respectively, e^i and e^d are the scalar volumetric and deviatoric deformation states respectively, ϕ is the free energy, θ_T is the absolute temperature, and η is entropy. To proceed, a specific form for the free energy will be taken as follows:

$$\phi = W(\theta, e^d, e^{dp}) + h(\theta_T) \tag{59}$$

where W is the functional given in Eq. 30, θ is the dilatation, e^{dp} is the plastic portion of the deviatoric extension state e^d, and h is an arbitrary function of the temperature θ_T. With this functional, Eq. 58 is written as:

$$\left(t^i - \frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial e^i}\right) \bullet \dot{e}^i + \left(t^d - \frac{\partial W}{\partial e^d}\right) \bullet \dot{e}^d - \frac{\partial W}{\partial e^{dp}} \bullet \dot{e}^{dp} - \left(\eta + \frac{\partial h}{\partial \theta_T}\right)\dot{\theta}_T \geq 0 \tag{60}$$

From this equation, the following relations are established:

$$t^i = \frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial e^i}, \quad t^d = \frac{\partial W}{\partial e^d}, \quad \eta = -\frac{\partial h}{\partial \theta_T}, \quad -\frac{\partial W}{\partial e^{dp}} \bullet \dot{e}^{dp} \geq 0 \tag{61}$$

The first two of the above relations were used in Section 5.1 to derive the elastic material response (see Eq. 28). The last condition is of particular interest here because it is the power dissipation associated with plastic deformations. Using the plastic flow rule given in Eq. 34, and the yield function $\psi(t^d) = \frac{\|t^d\|^2}{2}$, the last condition above becomes:

$$\begin{aligned} -\frac{\partial W}{\partial e^{dp}} \bullet \dot{e}^{dp} &= \lambda\, t^d \bullet t^d \\ &\geq 0. \end{aligned} \tag{62}$$

By construction, the consistency parameter λ is a non-negative function (see Section 5.3) and therefore the thermodynamic condition on the constitutive model is satisfied.

8 Time Integration of Single Bond

As a simple demonstration of the time integration algorithm, a single bond is kinematically driven (back and forth) between the elastic condition and the plastic condition. The initial configuration of two points and the associated bond connecting the points is shown in the top image of Figure 5. The lower point is fixed. The motion of the upper point is prescribed: first to the right (loading), then to the left (unloading), and finally to the right (loading). The prescribed motion of the upper point is functionally represented in the lower left image. The load step parameter is somewhat analogous to time. As shown, γ is the prescribed shear and relates to the displacement of the upper point. Plasticity in the bond is governed by the applied shear. This scenario is very nearly identical to the conditions used to derive the yield condition (Section 6) with the exception that the dilatation is not zero because there is only one bond. For the purpose of this demonstration, the dilatation was set to zero. This has no impact on the relevance of the demonstration. In practice, yielding will occur independently of the dilatation.

Material properties (nominally 6061-T6 Aluminum) used for these calculations are given in Table 1. This is data that was collected using a tension test (see Eq 56).

There are three dots (red, green, yellow) on the lower two images in Figure 5. The red dot indicates the start of the simulation. Between the red and green dots, the motion of the upper point is prescribed and to the right. The green dot marks the change in direction of the applied motion – from right to left. Similarly, the yellow dot marks the change in direction of the applied motion – from left to right. As γ increases the upper point moves to the right and as it decreases the upper point moves to the left. In the lower right image, a positive force density corresponds with tension while a negative force density corresponds to compression. The force density magnitude refers to the signed magnitude of the force directed along the bond between the two points.

In the first loading phase the force increases linearly with γ – this is the elastic response. When γ exceeds approximately 1.5×10^{-3} (from the figure) plastic deformation begins and the force density remains constant with increasing γ. Subsequently, the bond is unloaded beginning at the green dot. The material response is instantaneously elastic and the force linearly decreases with γ. The elastic unloading line is parallel with the elastic loading line. As γ decreases, the force first decreases but then saturates with the onset of plasticity. Beginning with the yellow dot, the direction of the applied motion switches again and the material response is instantaneously elastic.

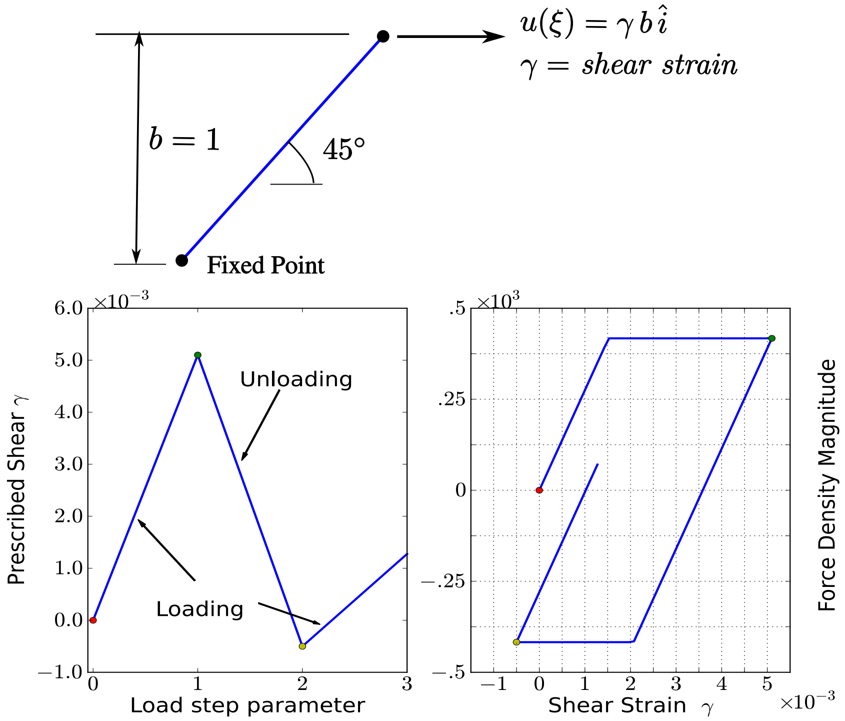

Figure 5. Time integration of plasticity model for single bond

Table 1. Material Properties

Property	Value	Units
Bulk Modulus	67.5×10^3	MPa
Shear Modulus	25.9×10^3	MPa
Yield Stress E_y	203.1	MPa
Density	2.71×10^{-3}	$\frac{g}{mm^3}$

9 Linearization of the Plasticity Model

To facilitate implicit solution methods for problems with the plasticity model, it is necessary to evaluate the modulus state. For every point, an important consideration is whether or not the load step implied by the configuration Y^0 is *elastic* or *plastic*. The choice is made using the same logic as was previously described for time integration of the constitutive model. The most likely scenario is one in which Y^0 is an un-converged iterate to a solution for step $n+1$ in a nonlinear solution scheme. The logic previously developed is immediately applicable and a trial force state t^d_{trial} can be evaluated using Eq. 40. If a point is found to be in the *elastic* condition, then the previously defined modulus state applies – see Eq. 26. If that is not the case, then the *plastic* condition applies and the elastic-plastic modulus state must be evaluated. Derivation of the modulus state for the plastic condition is the focus of the remainder of this section.

The modulus state for the elastic-plastic constitutive model requires linearization of the deviatoric force state at the end of a loadstep (perhaps an iterate representing the end of the load step). Because a closed-form solution for the deviatoric force state is given (Eq. 45), this derivative can be directly evaluated:

$$\nabla t^d_{n+1} \bullet U = \frac{\beta}{\|t^d_{trial}\|} \; \nabla t^d_{trial} \bullet U + \beta \, t^d_{trial} \frac{\partial}{\partial \epsilon} \left[\frac{1}{\|t^d_{trial}(Y^0 + \epsilon U)\|} \right]_{\epsilon = 0}, \tag{63}$$

where $\beta = \sqrt{2\psi_0}$. To evaluate the first term in the above expression, use the definition for t^d_{trial} (Eq. 40) and Eq. 14. The Fréchet derivative in the first term is given as:

$$\nabla t^d_{trial} \bullet U \;\; = \;\; \alpha\omega[M(Y^0) \cdot U - \frac{1}{3}|X|(\nabla\theta \bullet U)] \tag{64}$$

Noting that $\int t^d_{trial}\langle\xi\rangle |X| \, dV_\xi$ is zero (see Eq 114 [3]), the Fréchet derivative in the second term is given by:

$$\frac{\partial}{\partial \epsilon} \left[\frac{1}{\|t^d_{trial}(Y^0 + \epsilon U)\|} \right]_{\epsilon = 0} \;\; = \;\; -\frac{\alpha}{\|t^d_{trial}\|^3} \int t^d_{trial}\langle\xi\rangle \omega(|\xi|)[M(Y^0) \cdot U] \, dV_\xi$$

Combining these results, the modulus state of the elastic-plastic constitutive model is given by:

$$\begin{aligned} K\langle\zeta,\xi\rangle \;\; &= \;\; \left(\frac{3k}{m} - \frac{1}{3} \frac{\alpha\beta}{\|t^d_{trial}\|} \right) \omega(|\zeta|)\omega(|\xi|)|\zeta| \, |\xi| M(Y^0)\langle\zeta\rangle \otimes M(Y^0)\langle\xi\rangle \\ &+ \;\; \frac{\alpha\beta}{\|t^d_{trial}\|} \omega(|\zeta|) M(Y^0)\langle\zeta\rangle \otimes M(Y^0)\langle\xi\rangle \, \Delta(\zeta - \xi) \\ &- \;\; \frac{\alpha\beta}{\|t^d_{trial}\|^3} \omega(|\zeta|) \, t^d_{trial}\langle\zeta\rangle M(Y^0)\langle\zeta\rangle \otimes t^d_{trial}\langle\xi\rangle M(Y^0)\langle\xi\rangle \, \Delta(\zeta - \xi) \\ &+ \;\; t(Y^0)\langle\zeta\rangle \left(\frac{I - M(Y^0)\langle\xi\rangle \otimes M(Y^0)\langle\xi\rangle}{|Y^0|\langle\xi\rangle} \right) \Delta(\zeta - \xi) \end{aligned} \tag{65}$$

The similarities of this modulus state with the modulus state of the elastic material are notable. The first and second terms are very nearly identical to those of the elastic material. (Note that the ratio $\frac{\beta}{\|t^d_{trial}\|} < 1$ so that α is scaled downward.) The last term is the geometric contribution and is independent of the constitutive model. The third term is unique to the elastic-plastic constitutive model. The negative sign is apparently a softening effect and essentially works to eliminate stiffness associated with the deviatoric force state. A useful thought experiment is to observe what happens when there is only one bond. In that case, the magnitude of the third term is identical to that of the second term and so there is no change in the deviatoric force state and thus the change in force at a point arises solely because of changes in dilatation and geometry.

10 Summary and Path Forward

A non-local, state-based, ordinary, perfect plasticity model for peridynamics has been developed. All the necessary ingredients for computer implementation in both explicit and implicit peridynamics codes has been developed. This includes time integration of the plasticity model, development of a practical yield condition, and derivation of the modulus state which is crucial for implicit implementations.

While not shown here, large scale explicit calculations involving plastic deformations and damage have been conducted with this constitutive model. Results are very promising. Moving forward, additional calculations using both explicit and implicit time integration are needed to further verify and validate the model for practical calculations. These results and large deformation quasi-static calculations will be the subject of follow on reports.

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