

Peridynamics-Based Fracture Animation for Elastoplastic Solids: Supplementary Technical Document

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1 Introduction

This document presents the derivation of the constitutive model proposed in the paper. Section 2 introduces some preliminaries for the derivation and explains why the classical continuum elasticity theory could be regarded as a special case of peridynamics. Section 3 lists the common procedures to derive the peridynamics formulation of a general hyperelastic constitutive model in continuum mechanics. Finally, section 4 presents the derivation of the model introduced in the paper, which is a *nonlocal* extension of the linear elastic model in the *local* theory.

2 Preliminaries

2.1 Peridynamics equations of motion and the discretization

As is introduced in the paper, the peridynamics governing equation for any material point located at \mathbf{x} is formulated as below:

$$\rho \ddot{\mathbf{u}}(\mathbf{x}) = \int_{H_{\mathbf{x}}} [\mathbf{T} < \mathbf{x}' - \mathbf{x} > - \mathbf{T} < \mathbf{x} - \mathbf{x}' >] dV_{\mathbf{x}} + \mathbf{b}(\mathbf{x}). \quad (1)$$

The state of material point at \mathbf{x} is influenced by the possibly infinite number of material points \mathbf{x}' that belong to its family $H_{\mathbf{x}}$. When the continuum is discretized into particles, the integral in Equation 1 is replaced with the summations of particles within the horizon $\delta_{\mathbf{x}}$:

$$\rho \ddot{\mathbf{u}}(\mathbf{x}) = \sum_{\mathbf{x}', \mathbf{x}' \in H_{\mathbf{x}}} [\mathbf{T} < \mathbf{x}' - \mathbf{x} > - \mathbf{T} < \mathbf{x} - \mathbf{x}' >] V_{\mathbf{x}'} + \mathbf{b}(\mathbf{x}), \quad (2)$$

where $V_{\mathbf{x}'}$ is the volume of the discrete particle and its value depends on the distribution of particles.

2.2 Peridynamics for local interactions

In the limiting case where the horizon $\delta_{\mathbf{x}}$ approaches 0, the material point \mathbf{x} interacts only with its immediate neighbors. See Figure 1, the material point with label k interacts with the other six material points in the immediate vicinity denoted as $(k-l), (k+l), (k-m), (k+m), (k-n)$, and $(k+n)$. This conforms to the classical continuum mechanics, see the book [Bon08] for reference.

In the context of local interactions, Equation 2 for material point k is represented in the following form:

$$\rho_{(k)} \ddot{\mathbf{u}}_{(k)} = \sum_{j=k-l, k+l, k-m, k+m, k-n, k+n} (\mathbf{t}_{(k)(j)} - \mathbf{t}_{(j)(k)}) V_{(j)} + \mathbf{b}_{(k)}, \quad (3)$$

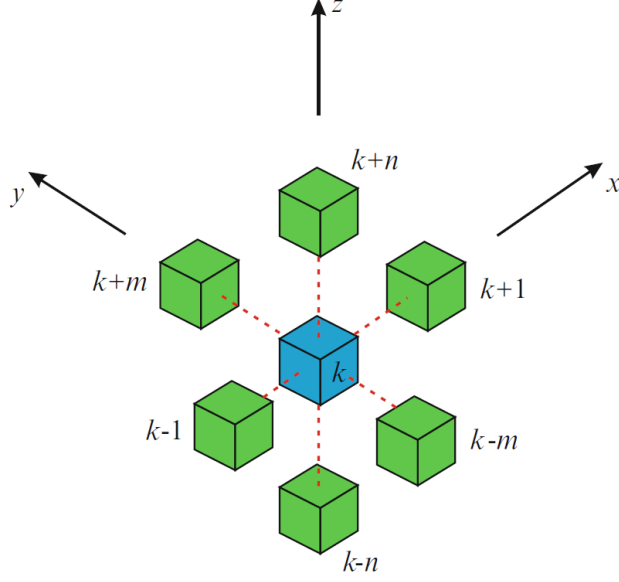


Figure 1: Material point k interacts with its immediate neighborhood. Image from [MO14].

where $\mathbf{t}_{(k)(j)}$ denotes the internal force density that material point j exerted on point k , and $\mathbf{t}_{(j)(k)}$ is the other way around.

Let's recall the form of governing equations in continuum mechanics:

$$\rho_{(k)} \ddot{\mathbf{u}}_{(k)} = \nabla \cdot \sigma_{(k)} + \mathbf{b}_{(k)}, \quad (4)$$

where $\sigma_{(k)}$ is the cauchy stress at point k . We could further write it in a component-wise form and approximate the spatial derivatives with central difference. Take the x -component as an example:

$$\begin{aligned} \rho_{(k)} \ddot{u}_{x(k)} = & \frac{1}{2} \frac{\sigma_{xx(k)} - \sigma_{xx(k-l)}}{\Delta x} + \frac{1}{2} \frac{\sigma_{xx(k+l)} - \sigma_{xx(k)}}{\Delta x} \\ & + \frac{1}{2} \frac{\sigma_{xy(k)} - \sigma_{xy(k-m)}}{\Delta y} + \frac{1}{2} \frac{\sigma_{xy(k+m)} - \sigma_{xy(k)}}{\Delta y} \\ & + \frac{1}{2} \frac{\sigma_{zx(k)} - \sigma_{zx(k-n)}}{\Delta z} + \frac{1}{2} \frac{\sigma_{zx(k+n)} - \sigma_{zx(k)}}{\Delta z} \\ & + \mathbf{b}_{x(k)}. \end{aligned} \quad (5)$$

Each term in the above equation involves only material point k and one immediate neighbor. We could recast Equation 3 in a similar form:

$$\begin{aligned} \rho_{(k)} \ddot{\mathbf{u}}_{(k)} = & (\mathbf{t}_{(k)(k-l)} - \mathbf{t}_{(k-l)(k)})V_{(k-l)} + (\mathbf{t}_{(k)(k+l)} - \mathbf{t}_{(k+l)(k)})V_{(k+l)} \\ & + (\mathbf{t}_{(k)(k-m)} - \mathbf{t}_{(k-m)(k)})V_{(k-m)} + (\mathbf{t}_{(k)(k+m)} - \mathbf{t}_{(k+m)(k)})V_{(k+m)} \\ & + (\mathbf{t}_{(k)(k-n)} - \mathbf{t}_{(k-n)(k)})V_{(k-n)} + (\mathbf{t}_{(k)(k+n)} - \mathbf{t}_{(k+n)(k)})V_{(k+n)} \\ & + \mathbf{b}_{(k)}. \end{aligned} \quad (6)$$

By equating Equation 5 (and its counterparts) with Equation 6, we get the relationships between the cauchy stress and the peridynamics internal force density:

$$\sigma_{\alpha\beta(k)} = 2t_{\beta(k)(q_\alpha)}\Delta\alpha V_{(q_\alpha)} \quad \text{with} \quad q_x = (k+l), q_y = (k+m), q_z = (k+n) \quad (7)$$

$$\sigma_{\alpha\beta(k)} = -2t_{\beta(k)(q_\alpha)}\Delta\alpha V_{(q_\alpha)} \quad \text{with} \quad q_x = (k-l), q_y = (k-m), q_z = (k-n), \quad (8)$$

where $\alpha, \beta = x, y, z$. Especially, for normal stress, we would have:

$$\sigma_{\alpha\alpha} = 2\mathbf{t}_{(k)(q_\alpha)} \cdot (\mathbf{x}_{(q_\alpha)} - \mathbf{x}_k)V_{(q_\alpha)}. \quad (9)$$

The subsequent equation would also be useful for the derivation in following sections:

$$\begin{aligned} \sum_{\beta=x,y,z} \sigma_{\alpha\beta(k)}^2 &= \sum_{\beta=x,y,z} 4t_{\beta(k)(q_\alpha)}^2 (\Delta\alpha)^2 V_{(q_\alpha)}^2 \\ &= 4(\mathbf{t}_{(k)(q_\alpha)}|\mathbf{x}_{(q_\alpha)} - \mathbf{x}_k|V_{(q_\alpha)}) \cdot (\mathbf{t}_{(k)(q_\alpha)}|\mathbf{x}_{(q_\alpha)} - \mathbf{x}_k|V_{(q_\alpha)}). \end{aligned} \quad (10)$$

2.3 Strain energy density and internal force density

In continuum mechanics the hyperelastic constitutive models are generally represented as strain energy densities. Therefore it is necessary to define the relationship between strain energy density and peridynamics internal force density. According to the book by Madenci and Oterkus[MO14], the relationship is defined as below:

$$\mathbf{t}_{(k)(j)} = \frac{1}{V_{(j)}} \frac{\partial W_{(k)}}{\partial (|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}|)} \frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}|}, \quad (11)$$

in which $W_{(k)}$ is the strain energy density at material point k . The direction of the force density vector is aligned with the relative position vector in the deformed configuration.

3 Derivation for general hyperelastic materials

4 Derivation for linear elastic materials

References

- [Bon08] BONET J.: *Nonlinear continuum mechanics for finite element analysis*. Cambridge University Press, Cambridge, UK New York, 2008.
- [MO14] MADENCI E., OTERKUS E.: *Peridynamic theory and its applications*. Springer, 2014.