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A Homogenization Method for Nonlinear Inhomogeneous Elastic Materials

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Abstract

In this paper, we propose a homogenization method for efficient simulation of nonlinear inhomogeneous elastic materials. The state of art homogenization approach in computer graphics only addresses linear materials, while our approach allows for a faithful approximation of fine, heterogeneous nonlinear materials with very coarse discretizations. Based on the fact that modal analysis provides the basis of linear deformation space and modal derivatives extend the space to nonlinear regime, we exploit modal derivatives as the input characteristic deformations for homogenization. We also present a simple elastic material model that is nonlinear and anisotropic to represent the homogenized materials. Nonlinearity of material deformation can be represented properly with this model. The material properties for the coarsened model are solved via a constrained optimization that minimizes the weighted sum of strain energy deviations for all input deformation modes. Arbitrary number of basis can be used as input for homogenization and greater weights are put on the more important low-frequency modes. Several examples are given to illustrate that the coarse simulation resulting from our method well captures the nonlinear dynamics of the original dynamical system, and saves orders of magnitude in computational time.

Categories and Subject Descriptors (according to ACM CCS): I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism—Animation

1. Introduction

Ever since Terzopoulos et al.'s seminal paper [TPBF87] on elastically deformable models, physically based deformable simulation has popularized itself in computer graphics over recent years with tremendous development. Despite the progress that has been made, so far most of the work has focused on objects made of a single, homogeneous material. This is because simulation methods with this simplification are more applicable to interactive applications. However, many real-world objects are composed of heterogeneous materials. Simulation of such complex objects using the currently available techniques usually requires a high resolution spatial discretization to resolve the fine-scale heterogeneity. This requirement yields overwhelming computation costs, thereby making interactive simulation impractical.

Our goal is to efficiently simulate inhomogeneous deformable objects with very coarse discretizations, yet effectively capture the physical behavior. Homogenization theory [Jik94, Far02, Glo12] is a perfect match for this purpose, which studies exactly how to extract information from fine scales to perform efficient computation. However, the application of homogenization theory in computer graphics is surprisingly rare. To the best of our knowledge, Kharevych et al. [KMOD09] introduced the only homogenization method so far for graphics applications. From the exact matching of potential energies for a set of characteristic displacements, their method achieves a coarse approximation of deformable objects that are composed of inhomogeneous linear elastic materials. The coarsened material properties allow for real-time simulation that captures the proper dynamical behavior of the original materials. Their method is limited to linear elasticity and therefore it's not applicable to the ubiquitous nonlinear materials. In this paper, we address this problem and propose a novel homogenization method for nonlinear inhomogeneous elastic materials.

We employ the same strategy as Kharevych et al.'s method, obtaining the coarsened material properties by matching the potential energies for a set of characteristic displacements. The homogenization of nonlinear materials imposes several key challenges compared to linear homogenization. First, the space of nonlinear deformations is significantly larger than the space of infinitesimal linear deformations. Therefore a set of displacements have to be found that are characteristic of the typical nonlinear deformations. Second, a proper anisotropic nonlinear material model needs to be defined for the coarse representation so that the homogenized material properties exhibit sufficient anisotropy and nonlinearity. Finally, exact matching of potential energies for the chosen displacements requires that the number of equations equals the number of unknowns in the material properties to be solved. This requirement is overly restrictive for nonlinear homogenization because nonlinear materials possess a larger family of characteristic deformations and it is good practice to account for as many of them as possible during homogenization.

To address each of these problems, respective contributions are presented in this paper. We exploit the idea of using deformation modes from modal analysis as the typical deformations for homogenization. Although modal basis have been explored extensively by previous methods for reduced simulation [NMK*06, SB12] , using their inherent properties for homogenization is new. Nonlinear deformations are beyond the spans of linear modal basis. We construct the deformation basis employing the modal derivatives technique proposed by Barbič and James [BJ05], and thereby nonlinear deformations are covered by the basis.

The second contribution of ours is a simple anisotropic nonlinear material model. Based on the isotropic St. Venant-Kirchhoff model [BW08] as commonly used in graphics [OH99,DDCB01,CGC*02b,BJ05], we define its anisotropic extension. This anisotropic material model is used in our coarse representation and expresses sufficient nonlinearity for graphics simulations.

Furthermore, unlike the previous method that solves for the coarsened material properties with exact matching of the potential energies, we formulate the homogenization process as an optimization. The weighted sum of strain energy deviations for the input deformation modes is minimized. We also take the frequencies of the deformation modes into account and put greater weights on the more important low-frequency modes. With this optimization strategy, arbitrary number of basis can be used as input and therefore the restriction on the number of displacements is overcomed.

2. Related Work

A complete review of the vast methods on physically based deformable simulation is beyond the scope of this paper, and we refer the readers to the excellent survey by Nealen et al. [NMK*06]. In this section, we focus on previous research closest to ours on fast simulation techniques and simulation of inhomogeneous materials.

Fast Deformable Simulation. Fast simulation techniques are strongly favored in computer graphics because real-time/interactive frame rates are crucial to applications such as video games and virtual surgery. Many strategies have been devised for the purpose of fast simulation. Running simulations at larger time steps [BW98, GS14] is one strategy that seeks to reduce the overall computation cost by using less simulation steps. Another kind of methods [MDM*02, MG04, GW08] exploit the simplicity of linear material models while circumvent the artifacts with large deformations by factoring out rotation from the deformation before applying the material model and rotating it back after. These methods are called stiffness warping methods or linear corotated methods.

Multi-resolution approaches [DDBC99, DDCB00, DDCB01, GKS02] are intuitive solutions that adaptively refine the simulation for more intensive deformations and coarsen it otherwise. Since objects typically interact through their surfaces, another way of reducing the number of degrees of freedom in simulation is to express the physical equations at the surface points only [JP99,JP02,JP03]. Such methods however are generally limited to small deformations. Domain embedding [FVDPT97,CGC*02a,THMG04] is one widely used technique in graphics that achieves fast computation and detailed rendering by embedding high resolution visual meshes into coarse deformable simulation.

Model reduction methods limit the possible deformations in a low-dimensional subspace to achieve efficiency. The reduced space is constructed using a set of deformation basis, typically obtained through eigen-analysis of the stiffness matrix [PW89] or of empirical simulation data [KLM01]. Among the two strategies, methods using eigen-analysis of the stiffness matrix, or so-called modal analysis, are more prevailing. Linear modal analysis provides deformation basis only for small deformations away from the configuration at which the stiffness matrix is evaluated, thereby not suitable for simulation of large deformations. Several solutions were proposed to resolve this problem. Choi and Ko [CK05] exploited the idea of stiffness warping for reduced simulation and developed a procedure to express and update the rotation component of deformation in subspace. Barbič and James [BJ05] generalized linear modal analysis and proposed to enhance the linear deformation modes with their directional derivatives. These so-called modal derivatives contain substantial nonlinear content that are sufficient to describe large deformations in practice. The idea of model reduction is extensively used by subsequent methods for reducing computation complexity [AKJ08, BP08, NACC08, BdSP09, KJ09, BZ11, HSvTP12, HZ13, vTSSH13].

Instead of using modal basis for reduced simulation, in this paper we exploit their inherent properties as characteristic deformations and use them for homogenization. To our knowledge, modal analysis has not been used for this purpose before.

Inhomogeneous Material Simulation. Simulation of deformable objects that are composed of inhomogeneous materials is less explored in computer graphics due to the complexity of material structure. Nesme et al. [NPF06] proposed to approximate non-uniform stiffness on cubical grids with spatial average of the elasticity tensor. Yet, such a simple average does not accurately coarsen the original materials. In another work of theirs, Nesme et al. [NKJF09] employed domain embedding technique and computed shape functions for coarse elements which consider the varying materials in the elements based on high-resolution mechanical analysis in precomputation step. They also took void space inside the elements into account and preserved the branch structure with element duplication. Unfortunately, their method only applies to linear elasticity. Bickel et al. [BBO*09] presented a data-driven approach to simulate nonlinear heterogeneous materials. With a set of stress-strain relationships captured from example deformations, they modeled the material by nonlinear interpolation of the relationships at runtime. As the captured data are restricted to simple deformations with a single contact probe, their method can not accurately capture the material behavior in complex scenarios. Faure and colleagues [FGBP11] introduced material-aware shape functions that efficiently resolve non-uniform stiffness for sparse meshless skinning. These shape functions failed to ideally resolve complex three-dimensional deformations since they only used one scalar stiffness value for computing the shape functions. Kharevych et al. [KMOD09] novelly adopted homogenization theory in computer graphics and properly approximated heterogeneous materials on very coarse discretizations with orders of magnitude speedup. Their method is based on linear elasticity and therefore restricted to infinitesimal deformations of linear materials. In this paper, we address these limitations with a nonlinear homogenization method.

3. Rationale and Overview

In this section, we present a brief overview of our homogenization algorithm. For consistency and clarity, we use **ROMAN** characters to refer to quantities on the fine discretizations, and $\mathbb{BLACKBOARD}$ characters to refer to their coarse counterparts.

We employ FEM discretization for simulations, therefore our method starts from a fine mesh $\mathbf{D}$ with varying material properties and obtains approximation of the material properties on a coarse mesh $\mathbb{D}$ whose element count is much smaller. Following Kharevych et al.'s approach [KMOD09], we compute the coarsened material parameters by matching the potential energies between two scales for a set of characteristic deformations. Instead of enforcing exact matching, we tolerate some deviations of the energies. We defer explanation of the benefits to the section that follows, and merely describe the methodology here. For any element $\mathbb{T}_p$ on the coarse mesh, several elements $\mathbf{T}_q$ on the fine mesh occupy the same space. Hence, the potential energy of the coarse element $\mathbb{T}_p$ should approximately match the sum of potential energies over the fine elements $\mathbf{T}_q$ for all deformations.

$$W(\mathbb{T}_p) \approx W(\mathbf{T}_q)$$

Computing the coarse potential energy $W(\mathbb{T}_p)$ requires a proper constitutive model defined on the coarse mesh. As inhomogeneous materials generally lead to anisotropic behavior, the material model must be anisotropic. In order to represent homogenized nonlinear materials, the material model needs to be nonlinear as well. We define one simple material model in this paper to achieve these aims.

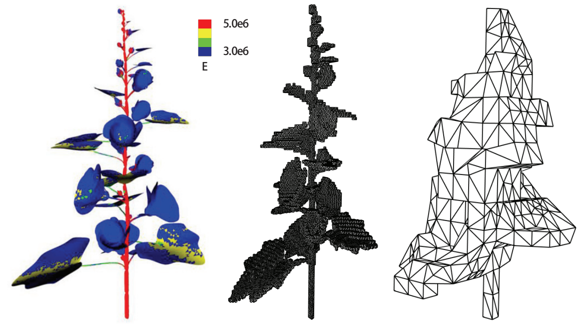

Figure 1: Example of our homogenization procedure setup. Stiffness distribution of the flower model is visualized using color map (left), our homogenization algorithm obtains material properties on a very coarse mesh (right) from material properties on a highly detailed mesh (middle).

Kharevych et al. used the material deformations subjected to linear tractions on the boundary as the input displacements. In contrast, we employ nonlinear modal basis to enforce the energy matching. As typical nonlinear deformations are taken into account by our homogenization process, we wish to obtain graceful approximations for general nonlinear deformations. Given a displacement field $\mathbf{u}$ on the fine mesh, we can easily downsample it via interpolation and obtain the displacement field $\mathbb{U}$ on the coarse mesh. For boundary nodes of the coarse mesh that lie outside of the fine mesh domain, we find one closest fine element closest to it and use extrapolation. The potential energies for meshes of different resolutions are evaluated using $\mathbf{u}$ and $\mathbb{U}$ respectively.

4. Nonlinear Homogenization

The strength of our method is achieved from the combination of three contributions: the use of nonlinear deformation modes, an anisotropic nonlinear material model, and an optimization-based homogenization strategy. We now present detailed descriptions of our work.

4.1. Nonlinear Deformation Modes

Kharevych et al.'s approach and ours share the same basic idea of enforcing potential energy matching between two scales for a set of characteristic deformations, hoping that the coarsened material matches the original heterogeneous one for all possible deformations. It is important to find a good set of deformations that are typical enough to represent all deformations, i.e. they form the basis of deformation space.

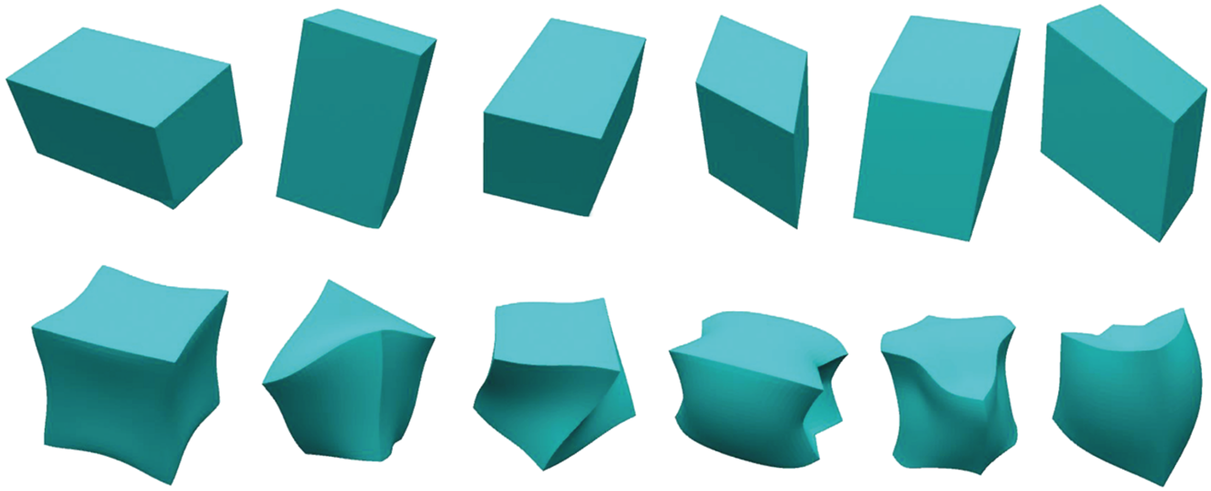

Figure 2: Comparison of the displacements used by our method and the ones by Kharevych et al.'s approach [KMOD09]. The harmonic displacements by Kharevych et al. (top) exhibit only linear stretch and shear, while our nonlinear deformation modes (bottom) contain substantial nonlinearity.

Deformation basis generation is a hard open problem, especially for deformations under general forcing. Kharevych et al. used a set of so-called harmonic displacements as the typical deformations for homogenization. These displacements are computed by solving a set of static boundary value problems with linear surface tractions prescribed as Neumann boundary conditions. As depicted in Figure 2 (top), harmonic displacements merely consist of linear stretch and shear, therefore encode no information of nonlinear deformations.

Inspired by recent methods that use modal analysis to construct a reduced deformation space, we exploit the fact that modal basis represent typical deformations from the frequency perspective. Standard linear modal basis can only express small deformations, and we employ the solution provided by Barbič and James [BJ05] which constructs a set of nonlinear deformation modes by including directional derivatives of linear modals. Please refer to their paper for a detailed description of the procedures to compute modal derivatives and to generate the low dimensional deformation basis with mass-PCA. Our contribution lies in the novel application of modal analysis for homogenization.

4.2. Anisotropic Nonlinear Material Model

As we mentioned earlier, the material model on the coarse mesh needs to be anisotropic and nonlinear in order to express the complex behaviors due to heterogeneity and nonlinearity. However, the wide variety of material models in literature such as the popular Neo-Hookean and Mooney-Rivlin models [Ogd97] are mostly isotropic. Here we define an anisotropic and nonlinear material model based on the simple yet widely used St. Venant-Kirchhoff model.

St. Venant-Kirchhoff material, or StVK in short, is perhaps the simplest kind of material model. It extends the isotropic linear elasticity model $\sigma = \lambda(tr\varepsilon)\mathbf{I} + 2\mu\varepsilon$ with geometric nonlinearity, replacing the Cauchy strain ε with the Green strain $\mathbf{E}$. Its material behavior can be expressed by the relationship between $\mathbf{E}$ and the second Piola-Kirchhoff stress $\mathbf{S}$ as: $\mathbf{S} = \lambda(tr\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$. The stress-strain relationship is linear, but the nonlinear strain tensor injects sufficient nonlinearity between displacements and stress for applications in computer graphics.

Our anisotropic material model exploits the simplicity of StVK model on geometric nonlinearity, and adds anisotropy by relating the stress and strain with a rank-4 elasticity tensor $\mathbf{C}$ instead of two scalars:

$$\mathbf{S} = \mathbf{C} : \mathbf{E}, \tag{1}$$

where : is the double contraction of tensors. This material model can also be regarded as the extension of the linear model used by Kharevych et al., but with geometric nonlinearity. The density of potential energy is computed as $\Psi = \mathbf{S} : \mathbf{E}$. As a result, our homogenization algorithm also solves for the elasticity tensor $\mathbb{C}_{\mathbb{T}_p}$ on elements $\mathbb{T}_p$ of coarse mesh. The inclusion of geometric nonlinearity in homogenization process turns out to work well, which we will demonstrate with comparison of simulations between two scales under large deformations.

4.3. Optimization-based Homogenization

Enforcing exact equivalence of strain energies between each coarse element and its corresponding fine elements is unnecessary because the error introduced by discretization is inevitable. The best we can hope is to minimize the deviation between the two scales. Besides, exact equivalence requires that the number of equations equals the number of unknowns in the material properties to be solved. In the case of solving for a symmetric elasticity tensor **C**, the number of input displacements must be six. Six displacements might be sufficient to describe typical linear deformations, but it's not for large deformations because of the significantly larger deformation space.

Consequently, we formulate homogenization as an optimization problem with respect to the elasticity tensor **C**. We further explore the importance of different deformation modes and put greater weight on the low-frequency modes. The objective to be minimized for each coarse element $\mathbb{T}_p$ is the weighted sum of potential energy deviations over all input deformations. The weight for each deformation mode is determined using the same strategy as Barbič and James [BJ05] to select basis during mass-PCA. Considering the physical property of elasticity tensor, **C** is represented as an invertible symmetric 6×6 matrix with non-negative entries. As a result, the optimization with constraints for coarse element $\mathbb{T}_p$ is:

$$\begin{aligned} &\min_{\mathbb{C}_{\mathbb{T}_p}} \sum_{i=1}^{r} \frac{1}{2} \omega_i \Big(W(\mathbb{U}_i) - \sum_{\substack{\mathbf{T}_q \in \mathbf{D} \\ \mathbf{T}_q \cap \mathbb{T}_p \neq 0}} \frac{|\mathbf{T}_q \cap \mathbb{T}_p|}{|\mathbb{T}_p|} W(\mathbf{u}_i)\Big)^2 \\ &s.t. \\ &\qquad \mathbb{C}_{\mathbb{T}_p}(k,l) \geq 0, \quad 1 \leq k,l \leq 6 \\ &\qquad \mathbb{C}_{\mathbb{T}_p} = \mathbb{C}_{\mathbb{T}_p}^T \\ &\qquad \det(\mathbb{C}_{\mathbb{T}_p}) \neq 0 \end{aligned} \tag{2}$$

where r is the number of input deformation modes, $\mathbf{u}_i$ and $\mathbb{U}_i$ are respectively the i-th deformation mode on the fine mesh and the coarse mesh. $|\cdot|$ operator computes the volume of underlying region. The i-th deformation mode is either a linear mode or a modal derivative, and the weight ω_i is determined as follows:

$$\omega_i = \begin{cases} \dfrac{\lambda_1}{\lambda_j}, & if\ \mathbf{u}_i = \Psi_j \\ \dfrac{\lambda_1^2}{\lambda_j \lambda_k}, & if\ \mathbf{u}_i = \Phi_{jk} \end{cases} \tag{3}$$

here λ are eigenvalues corresponding to the linear deformation modes, Ψ and Φ stand for linear modes and modal derivatives.

We use the solver provided by NLopt [Joh] to solve this optimization, and different coarse elements can be solved in parallel to save computation time.

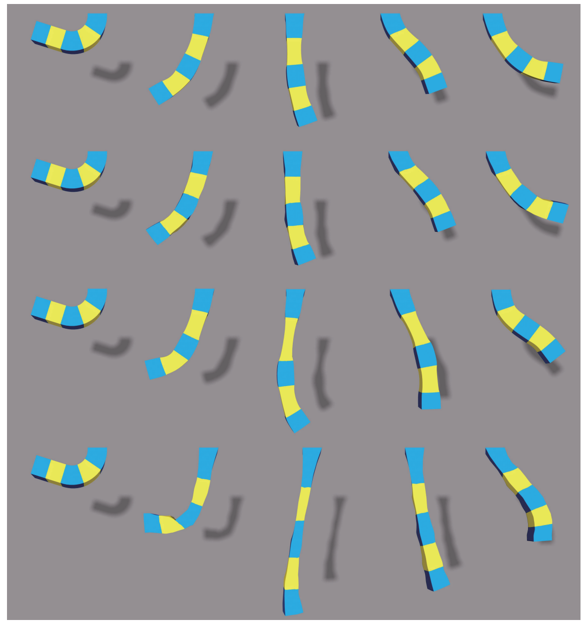

Figure 3: Inhomogeneous layered bar fixed on top swings under gravity. The regions in dodger blue are stiffer than the ones in yellow. Coarse simulation using our homogenized material properties (row 2) matches the high-resolution nonlinear simulation (row 1) very well, while coarse simulation with the results obtained from Kharevych et al.'s method (row 4) deviates noticeably from fine-scale corotated simulation (row 3).

5. Implementation Details

FEM Simulation. In order to validate our algorithm, we use the Vega FEM library [BSS12] to run simulations at different resolutions. As the high resolution mesh is sufficiently dense, we assume the material model for each fine mesh element is isotropic. The global behavior is still anisotropic because parameters of the elements are different. We employ the StVK model for each fine element in high-resolution simulations of nonlinear materials, while our anisotropic material model is used in corresponding coarse simulations. For comparison with Kharevych et al.'s method, we run simulations of linear materials using corotated methods with isotropic materials at high resolutions and anisotropic materials at coarse resolutions. Backward Euler integration method with uniform time step of $\triangle t = 0.001s$ is used for all simulations.

Rendering. We embed surface meshes with high-quality details in the simulation meshes for rendering purpose. The surface meshes are deformed during simulation via interpolation of simulation mesh positions. The deformed surface

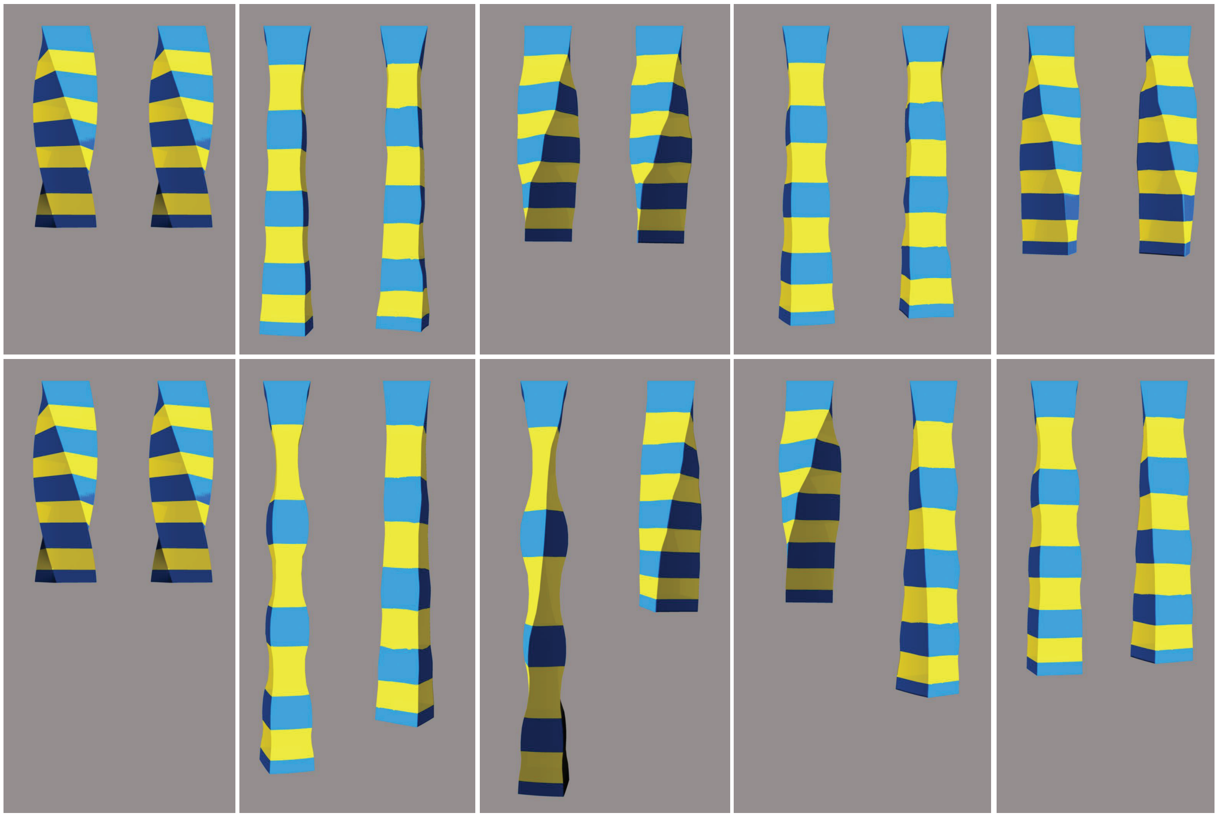

Figure 4: Side-by-side comparison of high-resolution simulation (left) and coarse simulation (right). The top row is the result of our method, while the bottom row is from Kharevych et al.'s. Better consistency between different resolutions is achieved with our method due to more accurate homogenization.

meshes are rendered offline to generate the figures presented in this paper.

6. Results

We demonstrate the power of our method with several carefully designed examples. The high-resolution material models are nonlinear and the objects typically undergo large deformations. By comparing with the results of Kharevych et al.'s method, we show that the homogenized material properties obtained from our method approximate the original nonlinear material behavior much better than Kharevych et al.'s linear approach.

In Figure 3 we fix the top end of an inhomogeneous elastic bar with layered materials and lift its bottom end horizontally. The bar swings with large stretch and bending after the lifted end is released. Side-by-side comparison of high-resolution and low-resolution simulations indicates that the homogenized material of ours is more faithful to the original heterogeneous material.

Figure 4 is another example of inhomogeneous deformable objects undergoing nonlinear deformations. The elastic bar made of 9 layers deforms with coupled twisting and stretch. The homogenized model of Kharevych et al.'s method fails to resolve such nonlinearity and the coarse simulation exhibits great difference with the fine simulation. In contrast, coarse simulation using our results stays consistent with the high-resolution simulation.

Table 1 lists the timing information for all the examples presented. The computations are performed on a commodity PC with a quad-core Intel Core i5, 2.8GHz CPU. The homogenization is conducted in parallel using four CPU cores, while the simulations run sequentially on a single core. As can be seen from the table, with homogenization as a precomputation process, we achieve orders of magnitude speedup in simulation. All the coarse simulations presented in this paper run at real-time frame rates.

7. Conclusions

We have presented a homogenization method that can obtain effective material properties of nonlinear inhomogeneous materials on very coarse discretizations. Coarse simulations with our homogenized materials capture the original mate-

Example	#E$_f$	#E$_c$	$t_h(min)$	$t_f(ms)$	$t_c(ms)$
bar swing	13056	702	19.0	221.9	9.2
bar twist	13056	702	21.0	215.1	10.5

Table 1: Summary of all results presented in the paper. The columns indicate the number of elements on fine mesh and coarse mesh, the time for homogenization (in minutes), the simulation time per time step (in milliseconds) for high-resolution simulation and coarse simulation respectively.

rial behavior with accuracy, and save magnitudes of computation time. Our method is the first in computer graphics to address the homogenization of nonlinear heterogeneous materials.

It would be interesting to explore fast simulation of inhomogeneous materials undergoing topology changes with homogenization techniques. Material properties of corresponding elements need to be recomputed at run-time in case of topology changes. Our optimization-based homogenization strategy is efficient , whereas much work remains to be done in order to re-homogenize local elements on-the-fly without jeopardizing the simulation.

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