Characterizing knockoff distributions

Theorem 1 (Bates, Candès, Janson and **Wang**, 2019)

For a random vector $(X, \tilde{X}) \in \mathbb{R}^{2p}$, pairwise exchangeability holds if and only if both of the following conditions hold:

- (Conditional exchangeability)

$$(X_j, \tilde{X}_j) \overset{d}{=} (\tilde{X}_j, X_j) \mid X_{-j}, \tilde{X}_{1:(j-1)}, 1 \leq j \leq p$$

- (Knockoff symmetry)

$$\mathbb{P}((X_j, \tilde{X}_j) \in A \mid X_{-j}, \tilde{X}_{1:(j-1)})$$

is $\sigma(X_{(j+1):p}, \{X_1, \tilde{X}_1\}, \ldots, \{X_{j-1}, \tilde{X}_{j-1}\})$-measurable for any Borel set A, where $\{\ \cdot\ ,\ \cdot\ \}$ is the unordered pair. In other words, the conditional distribution does not change when swapping X_k and $\tilde{X}_k$, $1 \leq k \leq j-1$, $1 \leq j \leq p$.

Intuition