PIXEL RECOVERY VIA ℓ_1 MINIMIZATION IN THE WAVELET DOMAIN

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ABSTRACT

This paper uses probability models on expansive wavelet transform coefficients with interpolation constraints to estimate missing blocks in images. We use simple probability models on wavelet coefficients to formulate the estimation process as a linear programming problem and solve it to recover the missing pixels. Our formulation is general and can be augmented with more sophisticated probability models to obtain even better estimates on a variety of image regions. The presented approach has many parallels to recently introduced dictionary based signal representations with which it shares certain optimality properties. We provide simulation examples over edge regions using both critically-sampled and expansive (over-complete) wavelet transforms.

1. INTRODUCTION

In this paper we investigate the problem of estimating missing regions in images with the aid of overcomplete dictionaries and the ℓ_1 norm. In some cases missing regions in an image are due to transmission errors, and many *error concealment* algorithms for filling in the missing region have been developed [16]. The problem is also called *inpainting* where an object is intentionally deleted from an image and the remaining area should be convincingly filled in [1].

One approach to recovering missing image regions is to use an image decomposition that is expected to be sparse. In this case, image recovery can be performed by finding a sparse set of expansion coefficients [9]. Transforms, like the wavelet transform, that provide sparse image representations have been successfully used in several image processing applications, denoising for example [3].

In this paper we describe an approach to image recovery that parallels a common approach to image denoising. Specifically, we show how a probability model for the wavelet coefficients of an image can be used to derive both image denoising algorithms and image recovery algorithms. As the principle example for this paper, we suppose that the wavelet coefficients of an image follow a Laplacian probability distribution:

$$p(w) = \frac{1}{\sqrt{2}\,\sigma}\, e^{-\sqrt{2}\,|w|/\sigma}. \qquad (1)$$

It has been shown previously [10] that if this probability model for the wavelet coefficients is assumed, and if the image has been corrupted by additive independent Gaussian noise with variance σ_n^2, then the maximum aposteriori probability (MAP) estimator for the true wavelet coefficients is given by the soft-thresholding rule:

$$\hat{w}(y) = \mathrm{sign}(y)\, \left(y - \sqrt{2}\,\sigma_n^2/\sigma\right)_+$$

where y is the noisy wavelet coefficient. Below it is shown that the probability model in Equation (1) can also be used to derive a wavelet-based algorithm for pixel recovery — specifically, an image recovery algorithm that entails the minimization of the ℓ_1 norm of the wavelet coefficients subject to interpolation constraints on the known pixels.

Therefore, the estimation of missing pixels in an image via the minimization of the ℓ_1 norm of wavelet coefficients subject to interpolation constraints, is the analogue of the estimation of noisy pixels in an image via soft-thresholding of wavelet coefficients. In addition, both algorithms result in zero-valued wavelet coefficients, or a somewhat sparse representation.

Although the minimization of the number of non-zero coefficients (the ℓ_0 norm) is a non-convex problem, it turns out that in some cases, the solution that minimizes the ℓ_1 norm is also the solution that minimizes the ℓ_0 norm [7, 6, 8]. However, in the general case, one can expect the solutions according to these two criteria to differ. In [9] an algorithm based on sparsity in the transform domain is presented for estimating missing image pixels. The algorithm of [9] is an iterative, nonconvex algorithm that is motivated by nonlinear approximation with a given basis (over-complete

This research was supported by NSF (CAREER award IIS-0093179), and by ONR (grant N00014-03-1-0217).

DCTs, wavelets, etc.). In current work, the results of that algorithm is being compared with the result of minimizing the ℓ_1 norm (which can be minimized exactly via linear programming).

As in the problem of image denoising, superior results can be obtained by using an over-complete, or expansive, transform, in place of a critically-sampled transform. Non-isotropic transforms where the expansion functions (atoms) are oriented can also give superior results.

In the problem of image denoising, the use of different probability models, in place of the Laplacian (1), leads to algorithms that give superior results compared to soft-thresholding. For example, probability models that model dependencies between wavelet coefficients (in addition to the non-Gaussian marginal distributions) lead to improved denoising performance [5, 12, 13, 14]. It is expected that pixel recovery algorithms can likewise be developed based on these more accurate probability models, and that the resulting algorithms will give improved performance. The improved pixel recovery algorithms derived from more accurate probability models will require the minimization not of the ℓ_1 norm of the wavelet coefficients but a modification of that norm.

2. WAVELET-BASED PIXEL RECOVERY

For convenience, the notation below is for 1-dimensional signals. The image is denoted as $\mathbf{x}$,

$$\mathbf{x} = (x(1), \ldots, x(N))^t.$$

$x(n)$ are known for $n \in \mathcal{K}$. $\mathcal{K}$ is the set of indices of known pixels. The wavelet (or other) transform is denoted by F, and the its inverse is denoted by G:

$$G \cdot F = I.$$

The matrices F and G will be rectangular when the transform is expansive (tall and wide respectively). The wavelet coefficients of the image are given by $\mathbf{w} = F\,\mathbf{x}$,

$$\mathbf{w} = (w(1), \cdots, w(N))^t.$$

If it is assumed that the wavelet coefficients are independent Laplacian, then

$$p_{\mathbf{w}}(\mathbf{w}) = \prod_{i=1}^{N} p_w(w(i)) \tag{2}$$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2}\,\sigma} e^{-\sqrt{2}\,|w(i)|/\sigma} \tag{3}$$

$$= \left(\frac{1}{\sqrt{2}\,\sigma}\right)^N \exp\left\{-\frac{\sqrt{2}}{\sigma}\sum_{i=1}^{N}|w(i)|\right\}. \tag{4}$$

The independence assumption is not totally accurate — a more accurate probability model will likely give superior results.

We seek the solution $\hat{\mathbf{w}}$ that maximizes the likelihood of the wavelet coefficients

$$\hat{\mathbf{w}} = \arg\max_{\mathbf{w}}\; p_{\mathbf{w}}(\mathbf{w})$$

subject to the constraint that the pixel values of the image obtained from $\hat{\mathbf{w}}$ agree with the known pixels:

$$\hat{x}(n) = x(n) \quad \text{for all} \quad n \in \mathcal{K}$$

where $\hat{\mathbf{x}} = G\,\hat{\mathbf{w}}$. Equivalently,

$$\hat{\mathbf{w}} = \arg\max_{\mathbf{w}}\; \log\{p_{\mathbf{w}}(\mathbf{w})\}$$

such that

$$\hat{x}(n) = x(n) \quad \text{for all} \quad n \in \mathcal{K}.$$

Equivalently,

$$\hat{\mathbf{w}} = \arg\min_{\mathbf{w}}\; \sum_{i=1}^{N}|w(i)| = \arg\min_{\mathbf{w}}\; ||\mathbf{w}||_1$$

such that

$$\hat{x}(n) = x(n) \quad \text{for all} \quad n \in \mathcal{K}.$$

Therefore, under the independent Laplacian model, the likelihood of the wavelet coefficients is maximized by minimizing the ℓ_1 norm of the wavelet coefficients. In the following examples, this constrained minimization problem is solved using linear programming.

Note that the variance of the wavelet coefficient, σ^2, appears in Equation (4). In many cases, it is useful to use a different variance σ^2 for different scales or subbands. In some probability models the variance of the wavelet is also spatially varying, as in [12]. In this case, where the variance $\sigma(i)^2$ is different for different coefficients, Equation (4) becomes

$$p_{\mathbf{w}}(\mathbf{w}) = K\,\exp\left\{-\sqrt{2}\sum_{i=1}^{N}\frac{|w(i)|}{\sigma(i)}\right\} \tag{5}$$

and the maximization of the likelihood requires the minimization of the *weighted* ℓ_1 norm:

$$\hat{\mathbf{w}} = \arg\min_{\mathbf{w}}\; \sum_{i=1}^{N}\frac{|w(i)|}{\sigma(i)}$$

such that

$$\hat{x}(n) = x(n) \quad \text{for all} \quad n \in \mathcal{K}$$

which can also be solved using linear programming. For natural images, generally the variance of the wavelet coefficients is smaller for the finer scales (higher frequency coefficients). Consequently, the fine scale wavelet coefficients should be weighted more heavily in the ℓ_1 norm than the coarse scale coefficients.

3. SOLVING THE CONSTRAINED ℓ_1 MINIMIZATION PROBLEM

The problem of minimizing the weighted ℓ_1 norm

$$\epsilon = \sum_{i=1}^{N} c(i)\,|w(i)|$$

subject to the linear constraints

$$(G\,\mathbf{w})_n = x(n), \quad n \in \mathcal{K}$$

where $c(i)$ are costs, or weights, can be formulated as a linear program [2]. However, the image processing problem considered here gives rise to a dense, potentially large, linear program. Efficient distributed algorithms for dense linear programs that can run on multi-processor platforms have been developed in [17]. [1]

4. EXAMPLES

The following example illustrates the result of pixel recovery by constrained minimization of the wavelet domain ℓ_1 norm. The test image, shown in Figure 1, is a 20 by 20 pixel image patch (taken from the standard test image Lena) that is missing a 6 by 6 block of pixels. [2] [3]

As the first example, we use two stages of a critically-sampled discrete wavelet transform (DWT). Consequently there are 400 wavelet coefficients, 25 of which comprise the lowest frequency subband. It is important that these 25 low frequency coefficients not be included in the ℓ_1 norm minimization. Indeed, the probability model in Equation (4) does not apply to the lowest frequency coefficients of a DWT. There are 364 interpolation constraints — one for each of the known pixels. The image obtained by using linear programming to minimize the (unweighted) ℓ_1 norm of the wavelet coefficients is illustrated in Figure 2. It is interesting to note that of the 400 wavelet coefficients, 36 of them are exactly zero. This is due to the general properties of ℓ_1 solutions obtained using Simplex-based linear program solvers — there will not be more non-zero coefficients than there are known pixels. Although the image in Figure 2 agrees with the known pixels in Figure 1, the result can be improved by using an over-complete (or expansive) wavelet transform. This is to be expected as the application of the soft-threshold rule for image denoising gives improved performance when it is used in conjunction with an expansive transform [4].

[1] The linear programming codes used here are available on the web at `http://cis.poly.edu/rvslyke/retrolp.htm`

[2] The wavelet transform software used here is available on the web at `http://taco.poly.edu/WaveletSoftware/`

[3] The DCT based recovery software is available on the web at `http://eeweb.poly.edu/~onur/source.html`.

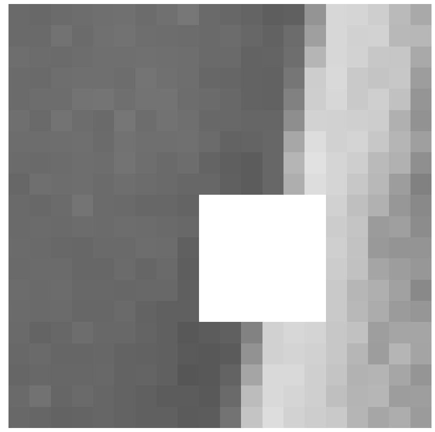

Fig. 1. Image with missing pixels. A 20 by 20 pixel image block is shown with a 6 by 6 pixel missing block.

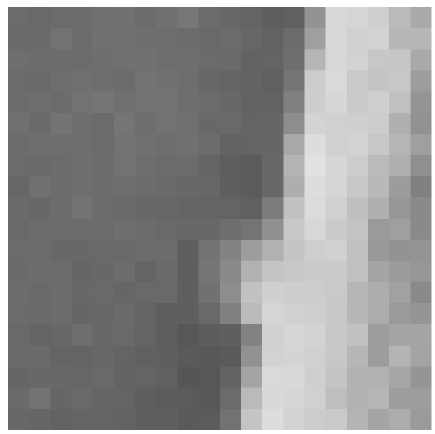

Fig. 2. Pixel recovery using ℓ_1 minimization of wavelet coefficients. A non-expansive DWT was used.

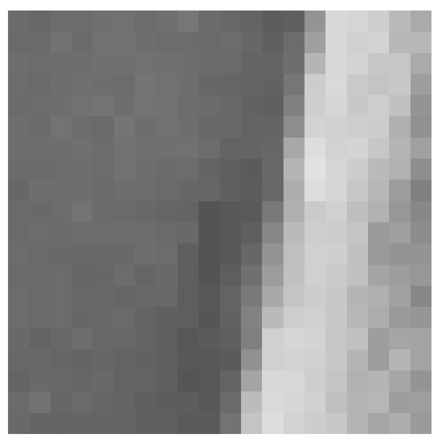

Fig. 3. Pixel recovery using ℓ_1 minimization of wavelet coefficients. A 2x-expansive DWT was used.

The second example uses the two-times expansive real dual-tree DWT. (This is the real part of the complex dual-tree DWT [11]. The complex dual-tree DWT is four-times expansive.) In this case, there are 800 wavelet coefficients, 50 of which comprise the lowest frequency subband which is excluded from the ℓ_1 minimization. Again, there are 364 interpolation constraints — one for each known pixel. The minimal ℓ_1 norm solution, obtained by linear programming, is illustrated in Figure 3, which shows the visual improvement. Of the 800 wavelet coefficients, exactly 364 of them are non-zero as expected (the solution is somewhat sparse in this over-complete wavelet expansion).

5. CONCLUSION

In this paper we have extended the use of the wavelet-domain Laplacian probability model to the problem of recovering missing pixels in images. When this model is used for image denoising, it gives rise to the soft-thresholding nonlinearity. When it is used for recovering missing pixels in images, it gives rise to the minimization of the ℓ_1 norm of the wavelet coefficients subject to interpolation constraints on the known pixels. In this paper we have shown examples to illustrate results of this method where the solution was found using linear programming. It is expected that superior wavelet-based pixel recovery algorithms can be derived by using a more accurate probability model (as is the case for the problem of wavelet-based image denoising). Pixel recovery algorithms derived from a different probability model will not require the minimization of an ℓ_1 norm — a different quantity will need to be minimized.

While the example in this paper illustrates the estimation of a block of pixels, the same approach can be used for image interpolation (image zooming), and potentially for super-resolution imaging from video. Using this approach for interpolation with an oriented wavelet or wavelet-like transform constitutes a type of interpolation algorithm that is implicitly edge-directed.

The main contribution of this paper is the introduction of a new approach that shows how to derive a wavelet-domain pixel recovery algorithm from a probability model. The algorithm given here (the constrained minimization of the ℓ_1 norm) is analogous to soft-thresholding for denoising and is likewise not state-of-the-art for pixel recovery. As in the case of image denoising, more advanced algorithms that utilize local image statistics and inter-coefficient statistical dependencies should give improved performance. It will also be interesting to use recently developed image transforms like the curvelet transform, in conjunction with the wavelet transform, as in [15] for denoising.

6. REFERENCES

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