Optimality Conditions for a Simple Convex Bilevel Programming Problem

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OPTIMALITY CONDITIONS FOR A SIMPLE CONVEX BILEVEL PROGRAMMING PROBLEM

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Dedicated to our friend Boris S. Mordukhovich in occasion of his 60-th birthday

Abstract. The problem to find a best solution within the set of optimal solutions of a convex optimization problem is modeled as a bilevel programming problem. It is shown that regularity conditions like Slater's constraint qualification are never satisfied for this problem. If the lower level problem is replaced with its (necessary and sufficient) optimality conditions it is possible to derive a necessary optimality condition for the resulting problem. An example is used to show that this condition in not sufficient even if the initial problem is a convex one. If the lower level problem is replaced using its optimal value, it is possible to obtain an optimality condition which is both necessary and sufficient in the convex case.

1. Introduction

In this article we are interested in studying optimality conditions for the following bilevel problem (BP)

$$\min f(x) \quad \text{subject to} \quad x \in S,$$

where S is given as

$$S = \operatorname{argmin}\{h(x) : x \in \Theta\}.$$

Here f and h are real-valued convex functions on $\mathbb{R}^n$ and Θ is a convex subset of $\mathbb{R}^n$. Thus it is clear that S is a convex set and, hence, problem (BP) is a convex programming problem. We refer to (BP) as a *simple convex bilevel programming problem.* This problem was first studied by Solodov [12] who developed a nice algorithm for the problem and also gave a convergence criterium for the algorithm. In this paper Solodov also shows that as a special case the problem (BP) contains the standard differentiable convex optimization problem of the form

$$\min f(x) \quad \text{subject to} \quad g(x) \le 0, Ax = b,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function, $g : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable convex function, A is a $l \times n$ matrix and $b \in \mathbb{R}^l$. This problem can be posed as the problem (BP) by simply assuming that the lower-level function h is given as

$$h(x) = ||Ax - b||^2 + ||\max\{0, g(x)\}||^2$$

and the lower-level problem is to minimize the function h over $\mathbb{R}^n$. It is important to note that in the above expression the maximum is taken coordinate-wise.

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In this article we want to analyze optimality conditions for the problem (BP). Given the inherent bilevel structure of the problem it appears that it seems not be straightforward to write down the optimality conditions for (BP). In order to motivate our study it might be a good idea to take a brief tour of the usual bilevel programming problem. In general a bilevel programming problem is given as follows

$$\min_x F(x,y), \quad \text{subject to} \quad x \in X, y \in \Psi(x),$$

where $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, X is a closed subset of $\mathbb{R}^n$ and Ψ is a set-valued map denoting the solution set mapping of the following parametric optimization problem,

$$\min_y f(x,y), \quad \text{subject to} \quad y \in \Theta(x),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\Theta(x)$ is a set depending on the parameter x. The bilevel programming problem has been often referred to as an optimization formulation of the Stackelberg game [13]. The leaders problem is the so-called upper-level problem where the minimization is carried out on the variable x while the followers problem is called the lower-level problem where for each of the input x by the leader the follower optimizes his objective function $f(x,y)$ using the variable y depending on some constraints which depend on the input provided. It must be now apparent why we call the problem at hand a *simple bilevel programming problem*. However, an important point is that, unless for each x the lower-level problem or the follower's problem has an unique solution, the upper-level objective is a set-valued map. So in the general scenario a bilevel programming problem can be viewed as a mathematical programming problem with set-valued maps. In order to avoid the set-valued objective two different approaches, namely the optimistic approach and the pessimistic approach, have been introduced mainly to have the final objective function as a single-valued one. For details on these approaches see for example Dempe [1], Dutta and Dempe [4], Dempe, Dutta and Lohse [2] and Dempe, Dutta and Mordukhovich [3] and the references there in.

Let us note that the overall bilevel programming problem is in general a non-convex problem even if the problem data are convex. Further a major drawback is that for a bilevel programming problem most standard constraint qualification conditions like the Mangasarian- Fromovitz constraint qualification are never satisfied. Let us note that our problem (BP) which is a convex problem is also not free from such a drawback. If we assume that the set S is non-empty and put $\alpha = \inf_{x\in\Theta} h$ then the problem (BP) is equivalent to the reformulated problem (RP)

$$\min f(x), \quad \text{subject to} \quad h(x) \leq \alpha, x \in \Theta.$$

It is simple to notice that the Slater's constraint qualification does not hold for this problem and, since the Slater's constraint qualification is equivalent to the Mangasarian-Fromowitz constraint qualification for a convex programming problem, thus even for this simple bilevel problem we are faced with the same issues of the usual bilevel programming problem. As we had stated earlier our main aim is to study optimality conditions for the problem (BP). In fact our main aim would be to develop a necessary and sufficient optimality condition. In our first approach we will consider the problem data to be smooth and in fact we will assume that the lower-level function h is twice continuously differentiable while the upper-level objective f is just differentiable and hence continuously differentiable since f is convex. In this setting we will take the approach as considered for a bilevel programming problem with convex lower-level problems in Dutta and Dempe [4]. However the optimality condition that we will get is a necessary one and not sufficient even though the problem (BP) is a convex problem. In fact using this approach we will see that the Lagrange multipliers are related to the coderivative of the normal cone map to the set Θ. This co-derivative appears to be quiet difficult to compute though very recently some advances have been made by Henrion, Mordukhovich and Nam [10]. This approach will be demonstrated in Section 2. The natural question is whether it is possible to develop an optimality condition which is both necessary and sufficient for problem (BP). This will be achieved through an alternative approach of reformulating the bilevel program (BP) as the convex programming problem (RP) which never satisfies Slater's constraint qualification. We use very recent results in convex optimization to develop a simple necessary and sufficient optimality condition for the problem (BP) using this reformulation. Our notations are standard. Further, instead of collecting all the preliminary definitions and results in one section, we present the basic tools in the main sections as and when needed.

2. Optimality Conditions: The Bilevel Approach

In this section we will use techniques from variational analysis to develop a necessary optimality condition for the problem (BP). Assume that the convex function f is differentiable and the convex function h is twice continuously differentiable. We will show through an example that the optimality condition that we develop in this section is necessary but not sufficient. We begin our study by observing that the set S can equivalently be written as

$$S = \{x \in \Theta : 0 \in \nabla h(x) + N_\Theta(x)\}.$$

Let us denote by $F(x) = (x, -\nabla h(x))^T$, where T denotes transpose. Then we can write S as

$$S = \{x \in \Theta : F(x) \in \mathrm{gph} N_\Theta\}, \tag{2.1}$$

where $\mathrm{gph}N_\Theta$ is the graph of the normal cone map N_Θ which is given as

$$N_\Theta(x) = \{v \in \mathbb{R}^n : \langle v, y - x\rangle \leq 0, \text{for all } y \in \Theta\}$$

if $x \in \Theta$ and $N_\Theta(x) = \emptyset$ if $x \notin \Theta$. In what follows we also need to consider the limiting normal cone or the Mordukhovich normal cone to the graph of the normal cone map to the feasible set Θ of the lower-level problem. Hence, we now briefly describe the limiting normal cone. For more details, see Rockafellar and Wets [11].

Given a set $C \subseteq \mathbb{R}^n$ and an element $\bar{x} \in C$ an element $v \in \mathbb{R}^n$ is called a regular normal vector or a Frechét normal vector to C at $\bar{x}$ if

$$v \in \hat{N}_C(\bar{x}) := \{v \in \mathbb{R}^n : \langle v, x - \bar{x}\rangle \leq o(||x - \bar{x}||),$$

where $\dfrac{o(t)}{t} \to 0$ as $t \to 0$. The set of all regular or Frechét normal vectors to C at $\bar{x}$ is denoted by $\hat{N}_C(\bar{x})$ and forms a cone which is known as the regular of Frechét normal cone to C at $\bar{x}$. This cone is a closed and convex but suffers from the drawback that it can reduce to the trivial cone containing only the zero element at some points of the boundary of C. This problem is eliminated by defining the limiting normal cone or the Mordukhovich normal cone.

Given a set $C \subseteq \mathbb{R}^n$ and $\bar{x} \in C$ a vector $v \in \mathbb{R}^n$ is said to be a limiting normal vector or a Mordukhovich normal vector to C at $\bar{x}$ if there exist sequences $v_k \to v$ and $x_k \to \bar{x}$ with $x_k \in C$ and $v_k \in \hat{N}_C(x_k)$. The set of all limiting normal vectors to the set C at $\bar{x}$ forms a cone and is denoted by $N_C^L(\bar{x})$ wich is known as the limiting normal cone or the Mordukhovich normal cone. This cone is not convex in general but it is always closed in our finite dimensional setting. Moreover, the limiting normal cone never reduces to the trivial cone containing only the zero element at the boundary points of C. Of course if $\bar{x}$ is an interior point of C then $N_C^L(\bar{x}) = \{0\}$. We will now state the main result of this section.

Theorem 2.1. *Let us consider the bilevel programming problem (BP) where the upper level objective function f is convex and differentiable, the lower-level objective function is convex and twice continuously differentiable and Θ is a convex set. Let $\bar{x}$ be a solution of the problem (BP). Assume that the following qualification condition holds at $\bar{x}$: $(w, v) \in N^L_{gphN_\Theta}(\bar{x}, -\nabla h(\bar{x}))$ satisfying the condition*

$$0 \in w - \nabla^2 h(\bar{x})^T w + N_\Theta(\bar{x}),$$

implies that $w = 0, v = 0$. Then there exists $(\bar{w}, \bar{v}) \in N^L_{gphN_\Theta}(\bar{x}, -\nabla h(\bar{x}))$ such that

$$0 \in \nabla f(\bar{x}) + \bar{w} - \nabla^2 h(\bar{x})^T \bar{w} + N_\Theta(\bar{x}).$$

Proof. Once we have expressed S as in (2.1) the result follows easily by an application Theorem 6.14 in Rockafellar and Wets [11]. Observe that the qualification condition given in the above theorem is the same as the one given in Theorem 6.14 of Rockafellar and Wets [11]. □

It is important to note that the above optimality condition is just necessary and not sufficient. We demonstrate this fact through the following example.

Example 2.2. Consider the problem (BP) with the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and the lower-level objective $h : \mathbb{R} \to \mathbb{R}$ given as follows: $h(x) = x^3$ when $x \geq 0$ and $h(x) = 0$, $x \leq 0$. The lower-level constraint set is $\Theta = [-1, +1]$. Observe that $S = [-1, 0]$. Thus $x = 0$ is the only solution to the problem. However, the optimality condition given in Theorem 2.1 is satisfied at the point $x = -1$ which we know is not a solution of the problem. This fact can be seen by noting that $(-1, 0) \in graphN_\Theta$ and also observing that $\nabla f(-1) = -2$, $\nabla^2 h(-1) = 0$ and $(4, 0) \in N^L_{graphN_\Theta}(-1, 0)$. Now the optimality condition is satisfied by choosing the element -2 from $N_\Theta(-1) = (-\infty, 0]$.

However observe that (BP) is overall a convex optimization problem. So naturally one would like to develop necessary and sufficient optimality conditions for the problem (BP). As we have seen in this section that formulation of the problem as done above only produces a necessary optimality condition. In the next section we demonstrate how we can develop a necessary and sufficient optimality condition.

3. Optimality Conditions: An Alternative Approach

In this section we proceed to develop necessary and sufficient optimality conditions for the bilevel problem (BP) by reformulating it as the single-level convex optimization problem (RP). We have mentioned that the reformulated problem never satisfies Slater's constraint qualification and, hence, we need modern tools of convex optimization to address this issue. We shall divide this section into three subsections. In the first one we will describe the tools from convex optimization needed for our study. In the second subsection we show how this can be used to develop necessary and sufficient optimality conditions for the problem (BP) while in the last one we consider the case where the feasible set Θ of the lower-level problem is described through cone-constraints and an abstract constraint.

3.1. **Recent Tools from Convex Optimization.** We deal with a class of *cone-convex programs* given as:

$$\min \vartheta(x) \quad \text{subject to} \quad g(x) \in -D, \text{ and } x \in C, \tag{3.1}$$

where $\vartheta\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ is a *proper, convex, lower semicontinuous* (l.s.c.) function with values in the *extended* real line $\overline{\mathbb{R}}$, $g\colon \mathbb{R}^n \to \mathbb{R}^m$ is a continuous $D-$convex mapping with D is a closed convex cone in $\mathbb{R}^m$ and $C \subset \mathbb{R}^n$ is a closed and convex subset.

For a set $C \subset \mathbb{R}^n$, the indicator function δ_C is defined as $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin D$. Let us recall that if C is nonempty, closed and convex, then δ_C is a proper l.s.c. convex function. The normal cone of C at x is given by

$$N_C\left(x\right) = \left\{u \in \mathbb{R}^n \mid \langle u, y - x\rangle \leq 0 \text{ for all } y \in C\right\},$$

when $x \in C$, and $N_C\left(x\right) = \emptyset$, otherwise.

Let $A = \{x \in C \mid g(x) \in -D\}$. Further, let D^+ be the positive dual cone of D, i.e.,

$$D^+ := \{s^* \in \mathbb{R}^m \mid \langle s^*, s\rangle \geq 0, \ \forall s \in D\}.$$

Assume that $\operatorname{dom} f \cap A \neq \emptyset$.

Considering further an extended-real-valued function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ with the *domain* $\operatorname{dom}\varphi := \{x \in \mathbb{R}^n |\ \varphi(x) < \infty\}$, we always assume that it is *proper*, i.e., $\varphi(x) \not\equiv \infty$ on $\mathbb{R}^n$. The *conjugate function* $\varphi^*\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ to φ is defined by

$$\begin{aligned} \varphi^*(x^*) &:= \sup\big\{\langle x^*, x\rangle - \varphi(x)\big|\ x \in \mathbb{R}^n\big\} \\ &= \sup\big\{\langle x^*, x\rangle - \varphi(x)\big|\ x \in \operatorname{dom}\varphi\big\}. \end{aligned} \tag{3.2}$$

Definition 3.1. (Farkas-Minkowski (FM) constraint qualification). We say that problem (3.1) satisfies the Farkas-Minkowski constraint qualification, (FM) in brief, if the cone

$$K := \operatorname{cone}\Big\{ \bigcup_{\lambda \in D^+} \operatorname{epi}(\lambda g)^* \Big\} + \operatorname{epi}\delta_C^* \tag{3.3}$$

is closed in the space $\mathbb{R}^n \times \mathbb{R}$.

Definition 3.2. ((CC) constraint qualification). We say that problem (3.1) satisfies the (CC) constraint qualification, if the

$$\operatorname{epi}(\vartheta)^* + K \tag{3.4}$$

is closed in the space $\mathbb{R}^n \times \mathbb{R}$, where K is given in (3.3).

Remark 3.3. The constraint qualification conditions (CC) and (FM) are often known as *closedness conditions* and *closed cone constraint qualification* which are used extensively in [6], [7], [8], [9] to establish optimality conditions, duality results for convex/DC programs subject to cone-constraints or to infinitely many convex constraints. These are also used to establish necessary conditions for bilevel programs [6]. It was proved in the papers mentioned above that these conditions are much weaker than the classical constraint qualification of Slater-type. As we will see below, when reformulated the simple bilevel program as an optimization problem, the new problems never satisfies the Slater's condition while the (CC) and (FM) may hold.

Remark 3.4. It is worth noting that if f is continuous at one point in A then [5]

$$\operatorname{epi}(f+\delta_A)^* = \operatorname{cl}\{\operatorname{epi} f^* + \operatorname{epi}\delta_A^*\} = \operatorname{epi} f^* + \operatorname{epi}\delta_A^* = \operatorname{epi} f^* + \operatorname{cl}K,$$

where cl$A$ denotes the closure of the set $\mathbb{R}^n$. So, if (FM) holds (i.e. K is closed) then (CC) holds.

The following optimality condition for (3.1) was established in [8].

Theorem 3.5. [8] **(necessary and sufficient optimality conditions for cone-convex programs).** *Let the qualification condition (CC) hold for the convex program* (3.1). *Then $\bar{x} \in A \cap \operatorname{dom}\vartheta$ is a* (*global*) *solution to* (3.1) *if and only if there is $\lambda \in D^+$ such that*

$$0 \in \partial\vartheta(\bar{x}) + \partial(\lambda g)(\bar{x}) + N_C(\bar{x}) \tag{3.5}$$

$$\lambda g(\bar{x}) = 0. \tag{3.6}$$

3.2. **Applications to the simple Bilevel Problem.** Let us consider the simple bilevel programming problem (BP) given in Section 1. We have already stated in Section 1 that the problem (BP) is equivalent to the following convex optimization problem (RP)

$$\min f(x) \quad \text{subject to} \quad h(x) - \alpha \leq 0,\ x \in \Theta.$$

We would just like to recall that $\alpha = \inf_{x\in\Theta} h$ and further we also assume that α is a finite real number.

It is worth noting that Slater's constraint qualification (and other interior-types of constraint qualification conditions) never hold for the problem (RP).

We now give an optimality condition for (RP) (which is an optimality condition for problem (BP) as well) which is a consequence of Theorem 3.5.

Theorem 3.6. *For the problem (RP), assume that*

$$\operatorname{cone}\,[(0,\alpha) + \operatorname{epi} h^*] + \operatorname{epi}\delta_\Theta^*$$

is closed. Then $\bar{x} \in \Theta$ is a (*global*) *solution to (RP) if and only if there is $\lambda \in \mathbb{R}_+$ such that*

$$0 \in \partial f(\bar{x}) + \lambda\partial h(\bar{x}) + N_\Theta(\bar{x}) \tag{3.7}$$

$$\lambda[h(\bar{x}) - \alpha] = 0. \tag{3.8}$$

Proof. We observe firstly that problem (RP) is of the type (3.1) with $D = D^+ = \mathbb{R}_+$ and $C = \Theta$. Secondly, for each $u^* \in \mathbb{R}^n$,

$$(h(.) - \alpha)^*(u^*) = \alpha + h^*(u^*).$$

It then follows that

$$\operatorname{epi}(h(.) - \alpha)^* = (0,\alpha) + \operatorname{epi} h^*.$$

Since

$$\operatorname{cone}\,[(0,\alpha) + \operatorname{epi} h^*] + \operatorname{epi}\delta_\Theta^*$$

is closed, the problem (RP) satisfies (FM) and hence, it satisfies (CC) since f is continuous (see Remark 2).

It now follows from Theorem 3.5 that there is $\lambda \in \mathbb{R}_+$ such that

$$0 \in \partial f(\bar{x}) + \lambda\partial[h(.) - \alpha](\bar{x}) + N_\Theta(\bar{x}) \tag{3.9}$$

$$\lambda[h(\bar{x}) - \alpha] = 0.$$

Since $\partial[h(.) - \alpha](\bar{x}) = \partial h(\bar{x})$, the conclusion follows. ☐

Example 3.7. Let us consider the bilevel problem of the model (3.1) where $f(x) = x^2+1$, $\Theta = [-1,1]$, and $h(x) = \max\{0, x\}$.

It is easy to see that $\operatorname{epi}\delta^*_\Theta = \operatorname{epi}|.|$, $S = [-1,0]$, and $\alpha = 0$. The optimization problem reformulated from this bilevel problem is

$$\min f(x) := x^2+1 \ \text{ subject to } h(x) = \max\{0,x\} \leq 0,\ x \in [-1,1]. \tag{3.10}$$

Note that for each $u \in \mathbb{R}$,

$$h^*(u) = \begin{cases} +\infty & \text{if } \ u<0 \ \text{ or } \ u>1 \\ 0 & \text{if } \ u \in [0,1]. \end{cases}$$

We have

$$\operatorname{epi} h^* = \{(u,r) \mid u \in [0,1], r \geq 0\} = [0,1] \times \mathbb{R}_+,$$

and

$$\operatorname{cone}\{(0,\alpha) + \operatorname{epi} h^*\} + \operatorname{epi}\delta^*_\Theta = \mathbb{R}^2_+ \cup \{(u,r) \mid u \leq 0,\ r \geq -u\}$$

is a closed subset of $\mathbb{R}^2$. This shows that for the problem (3.10), (FM) holds. Since f is continuous, (CC) holds as well (note that the Slater's condition fails to hold for (3.10)). It is easy to see that $\bar{x} = 0$ is a solution of the bilevel problem. Since $N_\Theta(0) = \{0\}$, $\partial f(0) = \{0\}$, and $\partial h(0) = [0,1]$, (3.7) - (3.9) are satisfied with $\lambda = 0$.

3.3. **Lower-level Problem with Explicit Constraints.** We now consider the bilevel problem of the type (3.1),

$$\inf_{x \in S} f(x), \tag{3.11}$$

where S is the solution set of the lower level problem:

$$\min h(x) \quad \text{subject to} \quad g_1(x) \in -D_1, \quad x \in C. \tag{3.12}$$

Here the data are as in Subsection 3.1. Concretely, $h \colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ is a *proper, convex, lower semicontinuous* (l.s.c.) function, and $g_1 \colon \mathbb{R}^n \to \mathbb{R}^m$ is a D_1-convex mapping with D_1 is a closed convex cone in $\mathbb{R}^m$ and $C \subset \mathbb{R}^n$ is a closed and convex set. Further assume that the mapping g_1 is continuous on $\mathbb{R}^n$.

Suppose that

$$\alpha = \inf_{g_1(x) \in -D_1, x \in C} h(x) < +\infty.$$

For the sake of simplicity, assume that $\alpha = 0$. This can be achieved by setting $h(x) := h(x) - \alpha$. Then the bilevel program (3.11) is equivalent to the following optimization problem:

$$\min f(x) \quad \text{subject to} \quad h(x) \leq 0,\ g_1(x) \in -D_1, \quad x \in C. \tag{3.13}$$

Now let

$$D := \mathbb{R}_+ \times D_1,\ g : \mathbb{R}^n \to \mathbb{R}^{m+1},\ g(x) = (h(x), g_1(x)).$$

Then, problem (3.13) can be reformulated as

$$\min f(x) \quad \text{subject to} \quad g(x) \in -D, \quad x \in C, \tag{3.14}$$

which is of the type (3.1).

The next theorem gives necessary and sufficient conditions for optimality of the bilevel programming problem (3.11).

Theorem 3.8. *For the problem* (3.11), *assume that*

$$\operatorname{cone}\operatorname{epi} h^* + \bigcup_{\lambda \in D_1^+} \operatorname{epi}(\lambda g_1)^* + \operatorname{epi}\delta_C^*$$

is closed. Then $\bar{x} \in g^{-1}(-D) \cap C$ *is a (global) solution to* (3.11) *if and only if there is* $r \in \mathbb{R}_+$ *and* $\lambda \in D_1^+$ *such that*

$$0 \in \partial f(\bar{x}) + r\partial h(\bar{x}) + \partial(\lambda g_1)(\bar{x}) + N_C(\bar{x}) \tag{3.15}$$

$$rh(\bar{x}) = 0 \text{ and } \lambda g_1(\bar{x}) = 0. \tag{3.16}$$

Proof. Observe that $D^+ = \mathbb{R}_+ \times D_1^+$ and for any $\widetilde{\lambda} = (r, \lambda) \in D^+$,

$$(\widetilde{\lambda} g)(x) = rh(x) + (\lambda g_1)(x).$$

Moreover,

$$\begin{aligned} \operatorname{epi}(\widetilde{\lambda} g)^* &= \operatorname{cl}\{\operatorname{epi}(rh)^* + \operatorname{epi}(\lambda g_1)^*\} \\ &= \operatorname{epi}(rh)^* + \operatorname{epi}(\lambda g_1)^* \\ &= r \cdot \operatorname{epi} h^* + \operatorname{epi}(\lambda g_1)^*. \end{aligned}$$

Therefore,

$$\bigcup_{\widetilde{\lambda} \in D^+} \operatorname{epi}(\widetilde{\lambda} g)^* + \operatorname{epi}\delta_C^* = \operatorname{cone}\operatorname{epi} h^* + \bigcup_{\lambda \in D_1^+} \operatorname{epi}(\lambda g_1)^* + \operatorname{epi}\delta_C^*.$$

By assumption, this set is closed. Hence, (FM) holds for the problem (3.14). Since f is continuous, (CC) holds for (3.14) as well (see Remark 2).

Since the problem (3.11) is equivalent to (3.14) (also, (3.13)), by Theorem 3.5, $\bar{x} \in A$ is an optimal solution of (3.11) if and only if there exists $\widetilde{\lambda} = (r, \lambda) \in D^+$ such that

$$0 \in \partial f(\bar{x}) + \partial(\widetilde{\lambda} g)(\bar{x}) + N_C(\bar{x}), \tag{3.17}$$

$$(\widetilde{\lambda} g)(\bar{x}) = 0. \tag{3.18}$$

It is obvious that

$$\partial(\widetilde{\lambda} g)(\bar{x}) = r\partial h(\bar{x}) + \partial(\lambda g_1)(\bar{x}).$$

On the other hand,

$$(\widetilde{\lambda} g)(\bar{x}) = rh(\bar{x}) + \lambda g_1(\bar{x}) = 0.$$

Since $r \geq 0$, $h(\bar{x}) \leq 0$, $\lambda g_1(\bar{x}) \leq 0$, we get $rh(\bar{x}) = 0$ and $\lambda g_1(\bar{x}) = 0$. It then follows from (3.17)-(3.18) that

$$\begin{aligned} &0 \in \partial f(\bar{x}) + r\partial h(\bar{x}) + \partial(\lambda g_1)(\bar{x}) + N_C(\bar{x}) \\ &rh(\bar{x}) = 0 \text{ and } \lambda g_1(\bar{x}) = 0, \end{aligned}$$

which is desired. □

4. Conclusion

The problem (BP) of finding a "best" optimal solution of a convex optimization problem

$$\min\{h(x) : x \in \Theta\}, \tag{4.1}$$

where Θ is a convex set and h a convex function defined on Θ with respect to a convex function f is modeled as a bilevel programming problem. Using the necessary and sufficient optimality conditions

$$0 \in \nabla h(x) + N_\Theta(x)$$

for the lower level problem, optimality conditions for problem (BP) can be derived. An example shows that these optimality conditions are necessary but not sufficient in general, even if problem (BP) is a convex optimization problem.

To formulate necessary and sufficient optimality conditions for problem (BP) this needs to be transformed using the optimal function value of problem (4.1). Then, using tools from cone-convex optimization optimality conditions of Karush-Kuhn-Tucker type can be developed provided some weak constraint qualification is satisfied.

The results presented again show that the reformulation of the bilevel programming problem using the optimal value function for the lower level problem is more promising than using the (necessary and sufficient) optimality conditions of the lower level problem itself, see also [3, 4].

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