SIGNALS & SYSTEMS (MCT 304)

BY

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SYSTEM MODELLING

Mathematical modeling is the use of mathematical equations to represent the dynamic behaviour of a system or process in order to carry out analysis and design.

Control systems are modeled using 3 major techniques:

1) Differential equation model
2) Transfer function model
3) State space model

System models are generally derived through:

- **WHITE BOX MODELING**: Model is derived through first principles using fundamental rules such as Newton's laws, Kirchoff's law etc.
- **BLACK BOX MODELING**: Model is derived using empirical data typically through system identification

SYSTEM			
ELECTRICAL	CAPACITOR $i = C\frac{dv}{dt}$	INDUCTOR $V = L\frac{di}{dt}$	RESISTOR $V = Ri$
MECHANICAL (TRANSLATIONAL)	MASS $F = ma$	SPRING $F = Kx$	DAMPER (FRICTION) $F = bv$
MECHANICAL (ROTATIONAL)	INERTIA $T = J\alpha$	SPRING (ROTATIONAL) $T = K_R\theta$	DAMPER (ROTATIONAL) $T = b_R\omega$

LAPLACE TRANSFORMS

$$f'(t) \longrightarrow sF(s) - f(0)$$

$$f''(t) \longrightarrow s^2F(s) - sf(0) - f'(0)$$

$$f'''(t) \longrightarrow s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

MECHANICAL SYSTEMS

- They are modeled using D'Alembert's principle.
- D'Alembert's principle states that, "For any body, the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero."

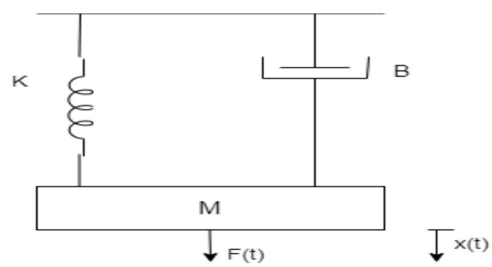
PROCEDURE FOR MODELING MECHANICAL SYSTEMS

1) Draw the Free Body Diagram of forces exerted on each mass in the system.

2) Apply mechanical system laws (Newton's laws of motion, D'Alembert's principle, Hooke's law) to each diagram to derive the differential equation model.

3) Apply Laplace transform with zero initial conditions to convert to transfer function model.

1)

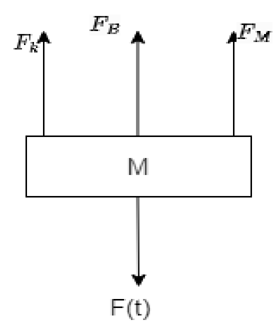


A) Derive the differential equation model of the system

B) Deduce the transfer function model, $\frac{X(s)}{F(s)}$

SOLUTION

A) Free body diagram



External force: F(t)

Resisting forces: F_M, F_B, F_K

$$F_M = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

$$F_B = Bv = B\frac{dx}{dt}$$

$$F_K = Kx$$

The differential equation model becomes

$$\boldsymbol{m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = F}$$

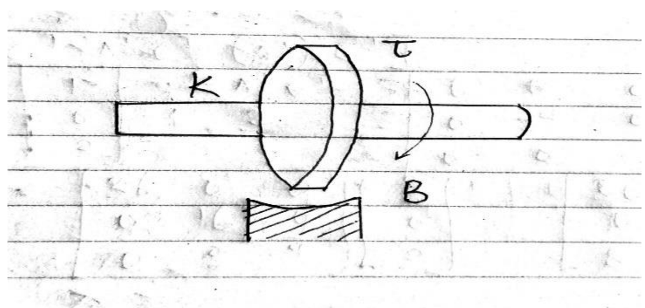
B) Taking the Laplace transform with zero initial conditions

$$ms^2X(s) + BsX(s) + KX(s) = F(s)$$

$$(ms^2 + Bs + K)X(s) = F(s)$$

$$\boldsymbol{\frac{X(s)}{F(s)} = \frac{1}{ms^2+Bs+K}}$$

2)

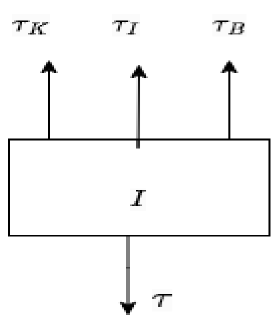


A) Determine the differential equation model of the rotational mechanical system

B) Derive the transfer function model, $\frac{\Theta(s)}{T(s)}$

SOLUTION

A) Free body diagram



External torque: T

Resisting forces: T_I, T_B, T_K

$T_I = J\alpha = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$

$T_B = B\omega = B\frac{d\theta}{dt}$

$T_K = K\theta$

The differential equation model becomes

$\boldsymbol{J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = T}$

B) Taking the Laplace transform with zero initial conditions

$Js^2\theta(s) + Bs\theta(s) + K\theta(s) = T(s)$

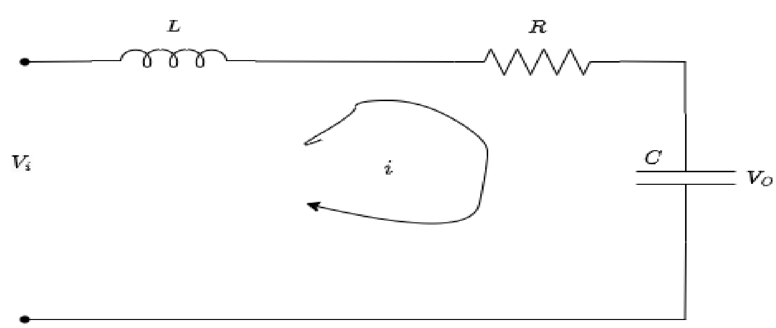
$(Js^2 + Bs + K)\theta(s) = T(s)$

$\frac{\theta(s)}{\boldsymbol{T(s)}} = \frac{\mathbf{1}}{\boldsymbol{Js^2+Bs+K}}$

ELECTRICAL SYSTEMS

- Kirchoff's laws are used for modeling Electrical systems.
- Kirchoff's current law states that, "The sum of currents entering a node is equal to the sum of currents leaving the same node."
  Kirchoff's voltage law states that, "The algebraic sum of the voltages around any loop in an electrical circuit is zero."

3) Consider the following electrical circuit



A) Derive the differential equation model of the electrical system

B) Deduce the transfer function model, $\frac{V_o(s)}{V_i(s)}$

SOLUTION

A) Applying KVL, we obtain the differential model as:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = V_i \qquad (1)$$

$$V_o = \frac{1}{C}\int idt \qquad (2)$$

B) Converting to transfer function, by taking Laplace transform with zero initial conditions

$$LsI(s) + RI(s) + \frac{1}{C}\frac{I(s)}{s} = V_i(s) \qquad (3)$$

$$\frac{1}{C}\frac{I(s)}{s} = V_o(s) \qquad (4)$$

$$I(s)(Ls + R + \frac{1}{Cs}) = V_i(s)$$

$$I(s) = \frac{V_i(s)}{Ls+R+\frac{1}{Cs}} \qquad (5)$$

Substituting (5) into (4)

$$\frac{1}{Cs}.\left(\frac{V_i(s)}{Ls+R+\frac{1}{Cs}}\right) = V_o(s)$$

$$\frac{V_i(s)}{LCs^2+RCs+1} = V_o(s)$$

$$\boldsymbol{\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

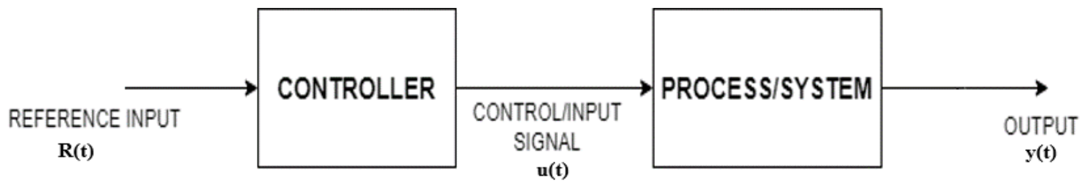
CONTROL STRATEGIES

A control system is a set of mechanical or electronic devices that regulates the output of a process/plant/system to a desired value.

Control systems are broadly classified into 2 main categories

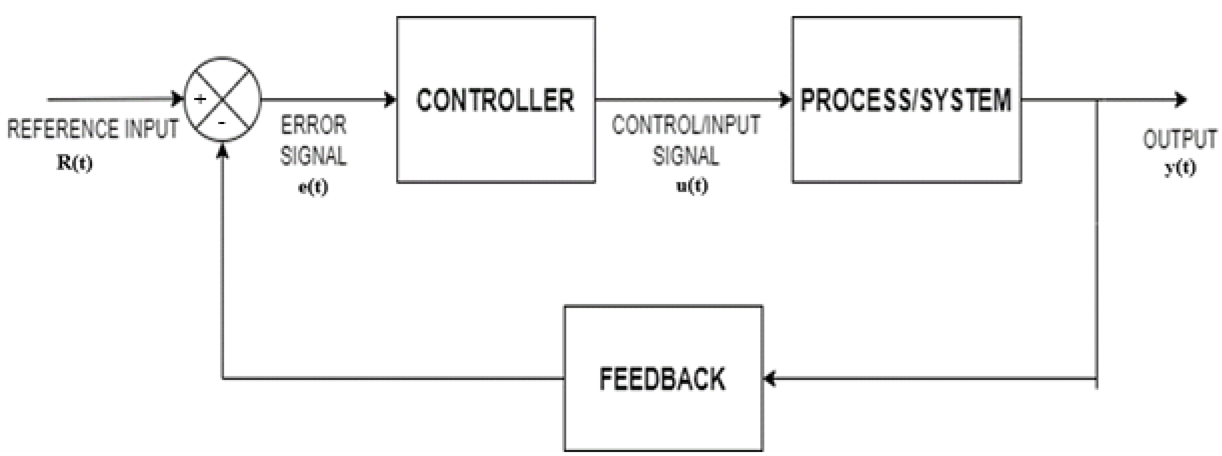
- Open Loop control system
- Closed Loop control system

OPEN LOOP CONTROL SYSTEM



- This is a control system without feedback.
- The control action is independent of the output.
- The output is not compared with the reference input.
- An example is a washing machine, that operates based on time and not the output i.e. the cleanliness of the clothes.

CLOSED LOOP CONTROL SYSTEM



- This is a control system with feedback. It is also called FEEDBACK CONTROL SYSTEM.
- The control action is dependent on the desired output.

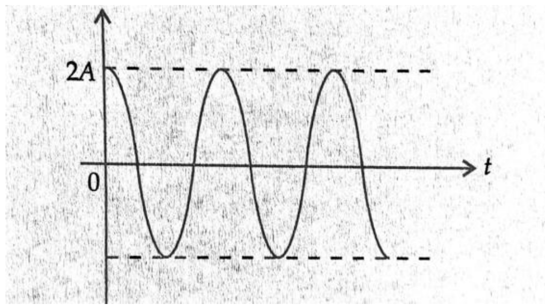
- The output is compared with the reference input and an error signal and control signal are produced.

COMPARISON BETWEEN OPEN-LOOP AND CLOSED-LOOP CONTROL SYSTEM

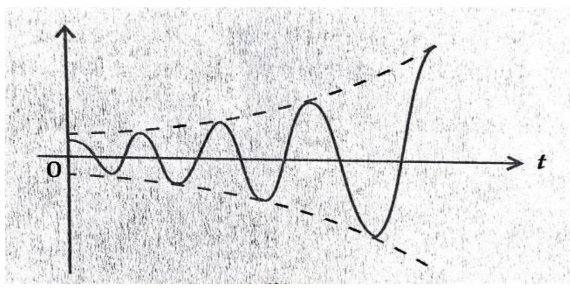
S/N	OPEN LOOP SYSTEM	CLOSED LOOP SYSTEM
1	There is no feedback	There is feedback
2	It is less expensive	It is more expensive
3	It is less accurate	It is more accurate
4	It is easier to construct	It is more difficult to construct
5	They are not reliable	They are reliable

STABILITY

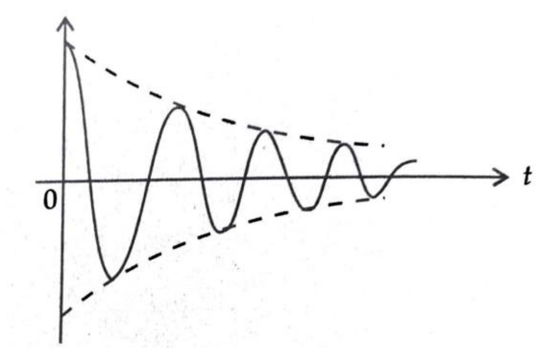
A system is said to be stable, if the output is bounded with respect to a bounded input.



By bounded, it means that it has an upper limit and lower limit. Hence, it is STABLE

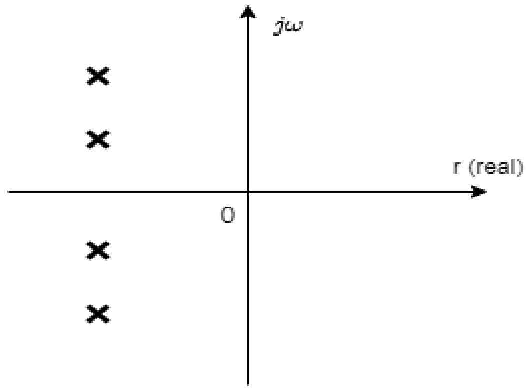


No upper limit, output is unbounded. Hence, it is UNSTABLE



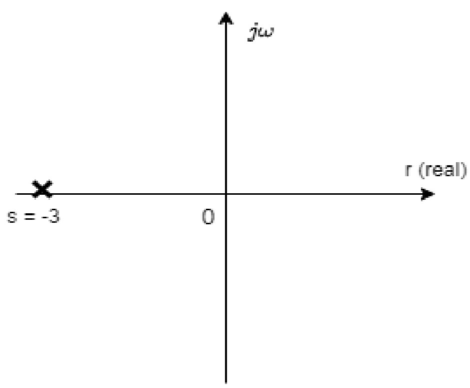
System is STABLE because output is bounded

- The stability of a system depends on the POLES. If all the poles are located in the left half of the S-plane, then the system is STABLE. In other words, poles should have NEGATIVE REAL PART.

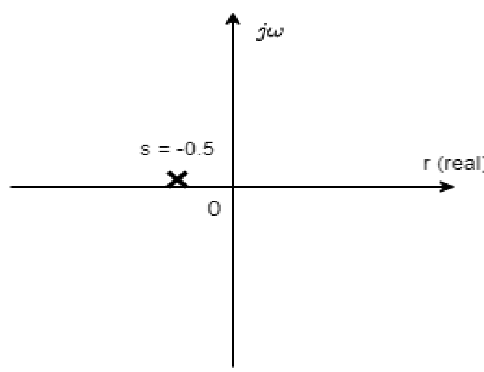


- As poles approach zero, the stability DECREASES

SYSTEM A

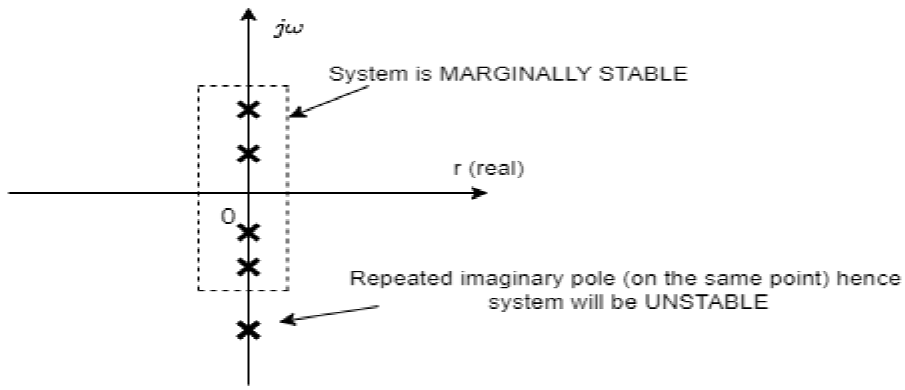


SYSTEM B

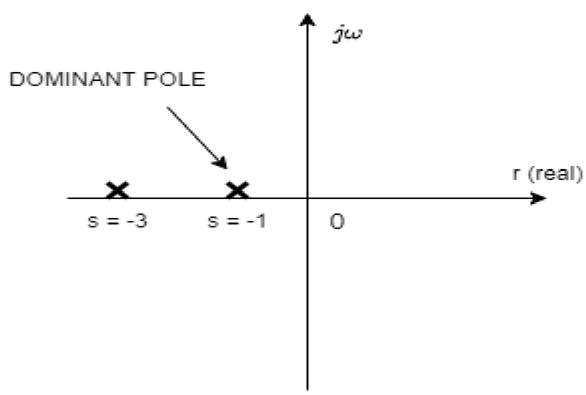


In this case, the pole of system A is farther away from zero with respect to system B, hence, system A is more stable than B.

- When poles are located on the IMAGINARY AXIS, then the system is MARGINALLY STABLE.
  Additionally, if those poles are repeated, then the system becomes UNSTABLE.



- The poles which are closest to the origin are called DOMINANT POLES.



TECHNIQUES USED TO CALCULATE STABILITY

- Routh-Hurwitz criteria
- Root locus
- Bode plot
- Nyquist plot
- Nicholas chart
- Polar plot

ROUTH'S STABILITY CRITERION

- It states that a system is stable if and only if all the elements in the first column of the Routh array have the same algebraic sign.
- It enables us to determine the number of closed-loop poles that lie in the right-half s-plane. It tells us if there are unstable roots in a polynomial equation without solving for them.

PROCEDURE FOR ROUTH'S STABILITY CRITERION

Given the closed loop transfer function form

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

1) Write the characteristic equation in the following form

$$\boldsymbol{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0}$$

2) All the coefficients must be positive and there should be no missing term.

3) Create the Routh matrix according to the following pattern.

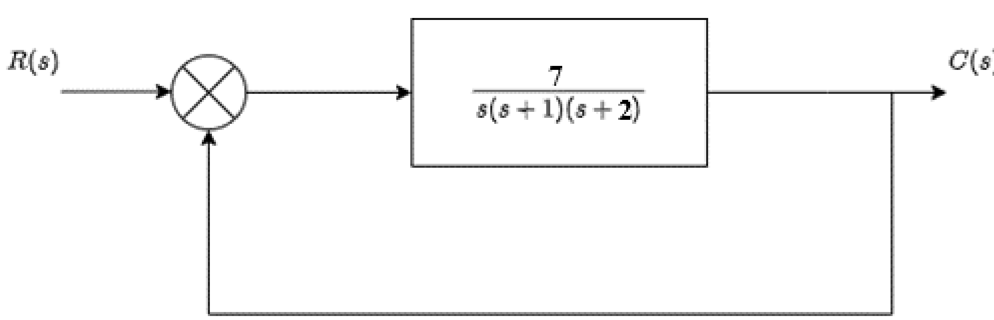
$$\begin{array}{lllll} s^n & a_0 & a_2 & a_4 & \ldots \\ s^{n-1} & a_1 & a_3 & a_5 & \ldots \\ s^{n-2} & b_1 & b_2 & b_3 & \ldots \\ s^{n-3} & c_1 & c_2 & c_3 & \ldots \end{array}$$

Where $\boldsymbol{b_1} = \frac{1}{a_1}[a_1 a_2 - a_0 a_3]$; $\boldsymbol{b_2} = \frac{1}{a_1}[a_1 a_4 - a_0 a_5]$

$\boldsymbol{c_1} = \frac{1}{b_1}[b_1 a_3 - b_2 a_1]$; $\boldsymbol{c_2} = \frac{1}{b_1}[b_1 a_5 - a_1 b_3]$

4) From the 1st column, if there are any sign changes, then the system is unstable.

4) Check whether the given system is stable or not using Routh's criterion



SOLUTION

The Characteristic equation => **1 + G(s)H(s) = 0**

$1 + \frac{7}{s(s+1)(s+2)} = 0$

$s^3 + 3s^2 + 2s + 7 = 0$

Formulate the Routh Matrix

s^3	1	2
s^2	3	7
s^1	$-\frac{1}{3}$	0
s^0	7	

$b_1 = \frac{1}{3}\big((3 * 2) - (1 * 7)\big) = \frac{1}{3}(6 - 7) = -\frac{1}{3}$

$b_2 = \frac{1}{3}\big((3 * 0) - (0 * 1)\big) = 0$

$c_1 = -\frac{1}{\frac{1}{3}}((-\frac{1}{3} * 7) - (0 * 3)) = 7$

Considering the first column, there are 2 sign changes ($3 \rightarrow -\frac{1}{3} \rightarrow 7$) . Thus, there are 2 roots in the right half of the s-plane.

Hence, the system is UNSTABLE

N/B: For there to be stability, there should be no sign changes in the first column of the Routh Matrix.

5) Verify the stability of the system whose characteristic equation is given by

$2s^4 + 4s^3 + 12s^2 + 8s + 2 = 0$

SOLUTION

s^4 2 12 2

s^3 4 8

s^2 8 2

s^1 7 0

s^0 2

$b_1 = \frac{1}{4}\big((4*12)-(8*2)\big) = \frac{1}{4}(48-16) = 8$

$b_2 = \frac{1}{4}\big((4*2)-(0*2)\big) = \frac{1}{4}(8-0) = 2$

$c_1 = \frac{1}{8}((8*8)-(2*4)) = \frac{1}{8}(64-8) = 7$

$c_2 = \frac{1}{8}((7*0)-(0*8)) = \frac{1}{8}(0-0) = 0$

$d_1 = \frac{1}{7}((7*2)-(0*8)) = \frac{1}{7}(14-0) = 2$

There are no sign changes, hence, the system is stable

5) A unity feedback control system has the following open loop transfer function

$$G(s) = \frac{k(s+13)}{s(s+3)(s+7)}$$

Using Routh criterion, calculate the range of values of k for the system to be stable

SOLUTION

The C.E => 1 + G(s)H(s) = 0

$1 + \frac{k(s+13)}{s(s+3)(s+7)} = 0$

$\frac{s(s+3)(s+7)+ k(s+13)}{s(s+3)(s+7)} = 0$

$s(s+3)(s+7) + k(s+13) = 0$

$s^3 + 10s^2 + 21s + ks + 13k = 0$

$s^3 + 10s^2 + (21 + k)s + 13k = 0$

Formulate the Routh Matrix

s^3 1 21 + k

s^2 10 13k

s^1 $\frac{210-3k}{10}$ 0

s^0 13k

$b_1 = \frac{1}{10}(210 + 10k - 13k) = \frac{210-3k}{10}$

$c_1 = \frac{1}{\frac{210-3k}{10}}\left(13k(\frac{210-3k}{10})\right) = \frac{10}{210-3k}\left(13k(\frac{210-3k}{10})\right) = 13k$

For stability, there should be no sign change in the first column

i) $\frac{210-3k}{10} > 0$

210 – 3k > 0

-3k > -210

K < 70

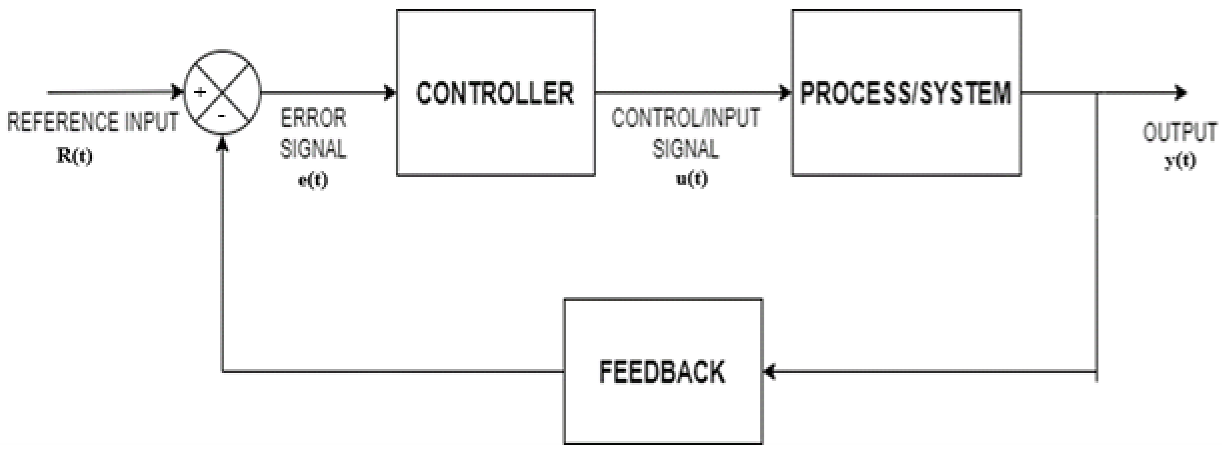
ii) 13k > 0

k > 0

So, for the system to be stable, the range of k is

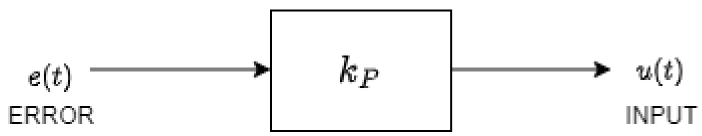
0 < k < 70

PID CONTROL



The goal is to MINIMIZE ERROR by using a control system/controller.

PROPORTIONAL CONTROLLER



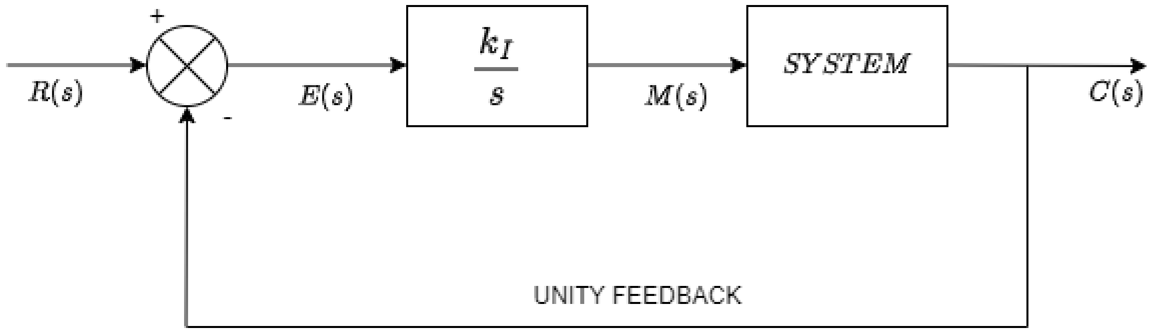
- There is a continuous linear/proportional relationship between the output of the controller (manipulated output) and the error signal.
- For higher values of K_p, there is faster response of the system but this results in increase in MAXIMUM PEAK OVERSHOOT.
- It amplifies the error signal.

$$m(t) = K_p e(t)$$

$$M(s) = K_p E(s)$$

$$\boldsymbol{\frac{M(s)}{E(s)} = K_p}$$

INTEGRAL CONTROLLER



- It produces an output which is the integration of error signal.
- It decreases steady state error.

$$m(t) \alpha \int e(t)\, dt$$

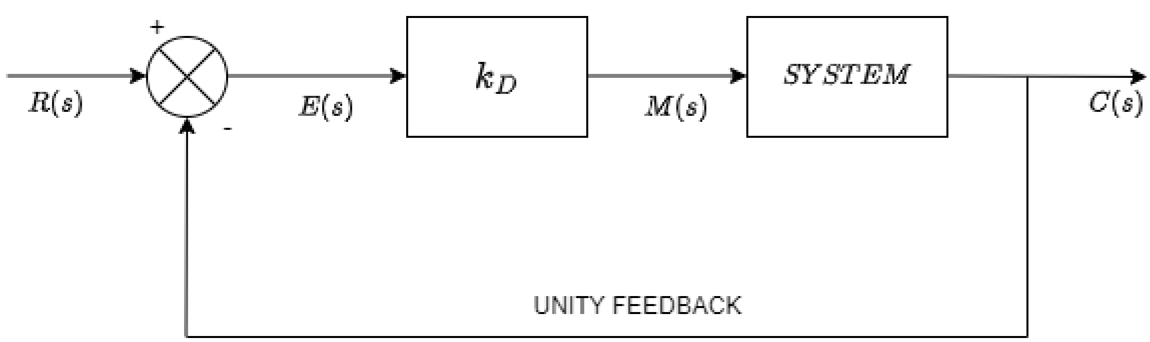
$$m(t) = K_I \int e(t)\, dt$$

Taking laplace transform

$$M(s) = \frac{K_I E(s)}{s}$$

$$\boldsymbol{\frac{M(s)}{E(s)} = \frac{K_I}{s}}$$

DERIVATIVE CONTROLLER



- It produces an output which is the derivative of the error signal.
- It reduces the system’s overshoot and improves stability.

$$m(t)\alpha\ \frac{de(t)}{d(t)}$$

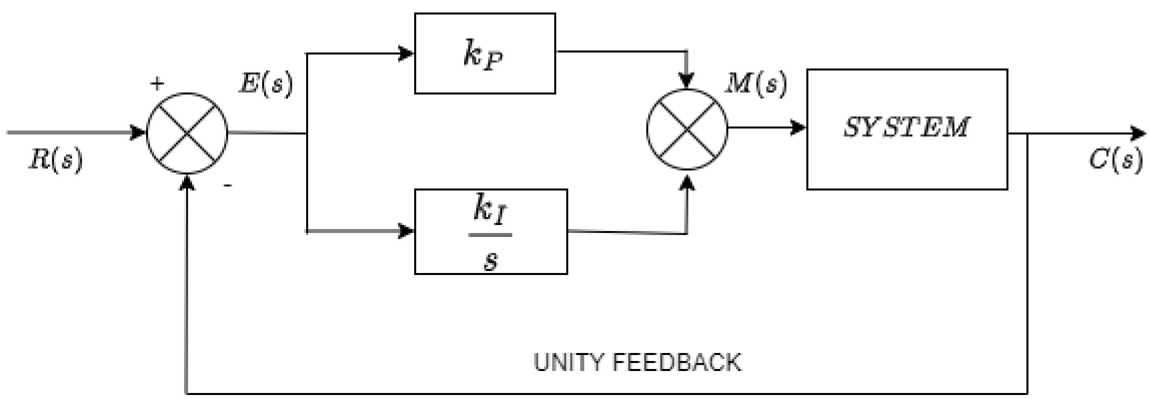
$$m(t) = \ K_D \frac{de(t)}{d(t)}$$

Taking Laplace transform

$$M(s) = K_D sE(s)$$

$$\boldsymbol{\frac{M(s)}{E(s)} = K_D \mathrm{s}}$$

PROPORTIONAL INTEGRAL CONTROLLER (PI CONTROLLER)



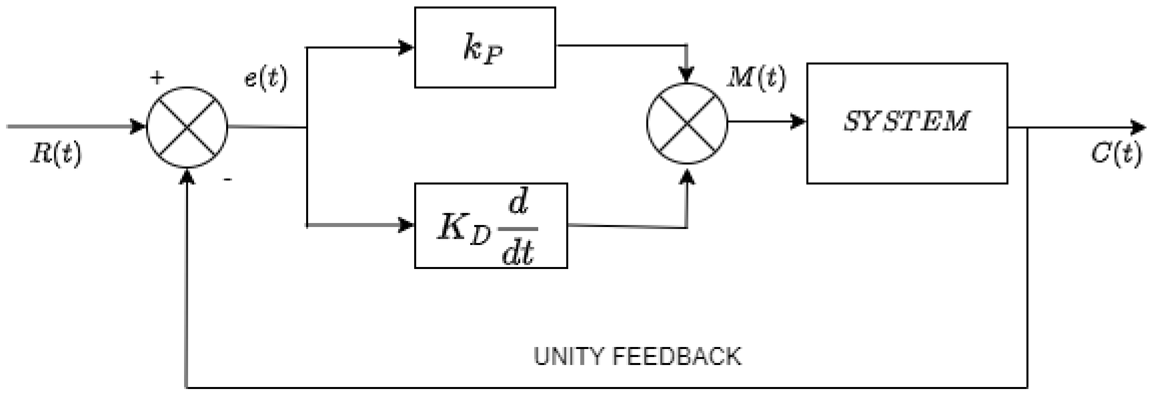
$$m(t) = \ K_p e(t) + K_I \int e(t)\, dt$$

Taking Laplace transform

$M(s) = K_p E(s) + \frac{K_I E(s)}{s} = (K_p + \frac{K_I}{s})E(s)$

$$\boldsymbol{\frac{M(s)}{E(s)} = \ K_p + \frac{K_I}{s}}$$

PROPORTIONAL DERIVATIVE (PD) CONTROLLER



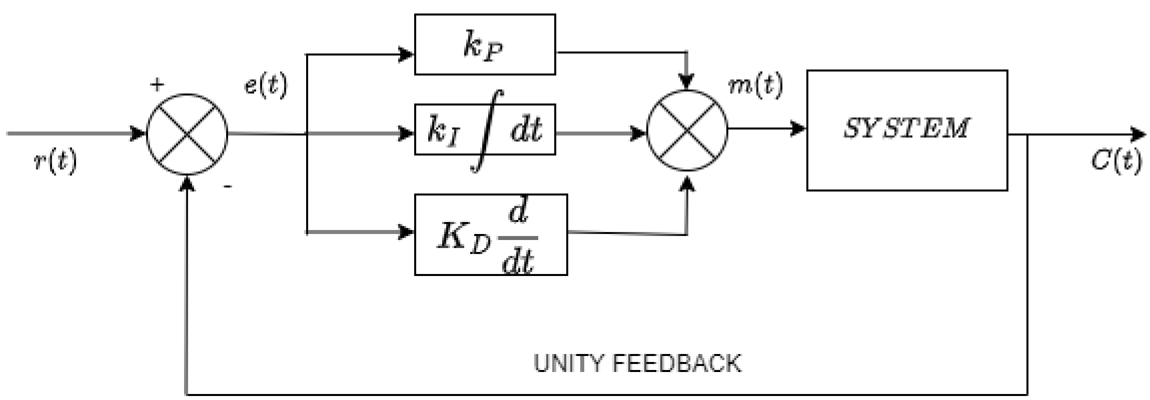
$$m(t) = K_p e(t) + K_D \frac{de(t)}{d(t)}$$

Taking Laplace transform

$$M(s) = K_p E(s) + K_D sE(s) = \left(K_p + K_D s\right)E(s)$$

$$\boldsymbol{\frac{M(s)}{E(s)} = K_p + K_D s}$$

PROPORTIONAL INTEGRAL DERIVATIVE (PD) CONTROLLER



$$m(t) = K_p e(t) + K_I \int e(t)\, dt + K_D \frac{de(t)}{d(t)}$$

Taking Laplace transform

$$M(s) = K_p E(s) + \frac{K_I E(s)}{s} + K_D sE(s) = \left(K_p + K_D s\right)E(s)$$

$$M(s) = (K_p + \frac{K_I}{s} + K_D s)E(s)$$

$$\boldsymbol{\frac{M(s)}{E(s)} = K_p + \frac{K_I}{s} + K_D s}$$

COMPARISON OF P, PI, PD, & PID CONTROLLERS

	P	PI	PD	PID
CONTROLLER EQN IN TIME DOMAIN	K_p	$K_p + K_I \int dt$	$K_p + K_D \frac{d}{d(t)}$	$K_p + K_I \int dt + K_D \frac{d}{d(t)}$
CONTROLLER EQN IN FREQ DOMAIN	K_p	$K_p + \frac{K_I}{s}$	$K_p + K_D s$	$K_p + \frac{K_I}{s} + K_D s$
PROBLEMS	- OFFSET - OVERSHOOT	- SLOW RESPONSE - STABILITY	- OFFSET - STEADY STATE ERROR	
ADVANTAGES	FAST RESPONSE	NO OFFSET OR STEADY STATE ERROR	- REDUCED OVERSHOOT - REDUCED SETTLING TIME	$\boldsymbol{K_p}$- FASTER REPONSE/LOWER RISE TIME $\boldsymbol{K_I}$- ELIMINATES STEADY STATE ERROR/OFFSET $\boldsymbol{K_D}$- REDUCES OVERSHOOT & SETTLING TIME