EEE 352 AND MCT 312 LECTURE NOTE

REVISION

FREQUENCY DOMAIN REPRESENTAION OF RLC CIRCUITS

Voltage–current relationships of network elements can also be represented in the frequency domain.

1. Resistor For the resistor, the v–i relationship in time domain is

$$v(t) = R\,i(t)$$

The corresponding frequency–domain relation are given as

$$V(s) = RI(s)$$

The transformed network is shown in Fig R1.

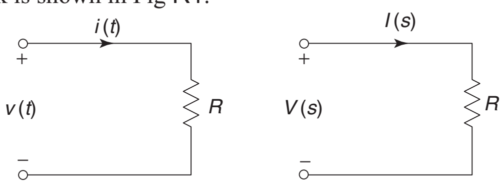

Fig. R1 *Resistor*

2. Inductor For the inductor, the v–i relationships in time domain are

$$v(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

The corresponding frequency-domain relation are given as

$$V(s) = Ls\,I(s) - L\,i(0)$$

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

The transformed network is shown in Fig R2.

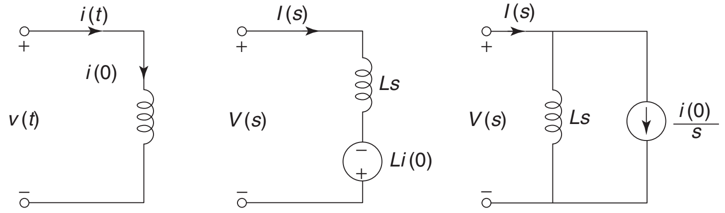

Fig. R2 *Inductor*

3. Capacitor For capacitor, the v–i relationships in time domain are

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

$$i(t) = C\frac{dv}{dt}$$

The corresponding frequency–domain relations are given as

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

$$I(s) = CsV(s) - Cv(0)$$

The transformed network is shown in Fig R3.

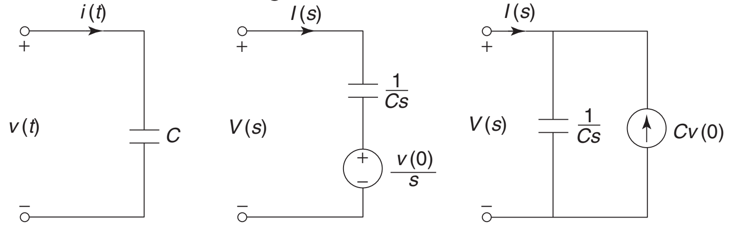

Fig. R3 *Capacitor*

Network Functions

INTRODUCTION

A network function gives the relation between currents or voltages at different parts of the network. It is broadly classified as *driving point* and *transfer function*. It is associated with terminals and ports.

Any network may be represented schematically by a rectangular box. Terminals are needed to connect any network to any other network or for taking some measurements. Two such associated terminals are called *terminal pair* or *port*. If there is only one pair of terminals in the network, it is called a one-port network. If there are two pairs of terminals, it is called a two-port network. The port to which energy source is connected is called the *input port*. The port to which load is connected is known as the *output port*. One such network having only one pair of terminals (1 – 1′) is shown in Fig. 1 (a) and is called *one-port network*. Figure 1 (b) shows a two-port network with two pairs of terminals. The terminals 1 – 1′ together constitute a port. Similarly, the terminals 2 – 2′ constitute another port.

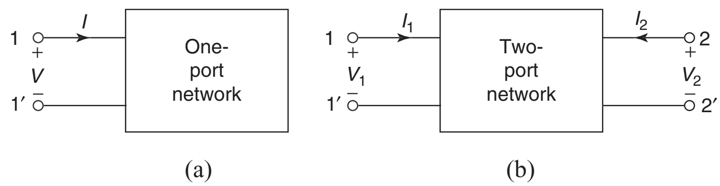

Fig. 1 *(a) One-port network (b) Two-port network*

A voltage and current are assigned to each of the two ports. V_1 and I_1 are assigned to the input port, whereas V_2 and I_2 are assigned to the output port. It is also assumed that currents I_1 and I_2 are entering into the network at the upper terminals 1 and 2 respectively.

DRIVING-POINT FUNCTIONS

If excitation and response are measured at the same ports, the network function is known as the driving-point function. For a one-port network, only one voltage and current are specified and hence only one network function (and its reciprocal) can be defined.

1. ***Driving-point Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at the same port. It is denoted by $Z(s)$.

$$Z(s) = \frac{V(s)}{I(s)}$$

2. ***Driving-point Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at the same port. It is denoted by $Y(s)$.

$$Y(s) = \frac{I(s)}{V(s)}$$

For a two-port network, the driving-point impedance function and driving-point admittance function at port 1 are

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}$$

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

Similarly, at port 2,

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

$$Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$$

TRANSFER FUNCTIONS

The transfer function is used to describe networks which have at least two ports. It relates a voltage or current at one port to the voltage or current at another port. These functions are also defined as the ratio of a response transform to an excitation transform. Thus, there are four possible forms of transfer functions.

1. ***Voltage Transfer Function*** It is defined as the ratio of the voltage transform at one port to the voltage transform at another port. It is denoted by $G(s)$.

$$G_{12}(s) = \frac{V_2(s)}{V_1(s)}$$

$$G_{21}(s) = \frac{V_1(s)}{V_2(s)}$$

2. ***Current Transfer Function*** It is defined as the ratio of the current transform at one port to the current transform at another port. It is denoted by $\alpha(s)$.

$$\alpha_{12}(s) = \frac{I_2(s)}{I_1(s)}$$

$$\alpha_{21}(s) = \frac{I_1(s)}{I_2(s)}$$

3. ***Transfer Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at another port. It is denoted by $Z(s)$.

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)}$$

$$Z_{21}(s) = \frac{V_1(s)}{I_2(s)}$$

4. ***Transfer Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at another port. It is denoted by Y (s).

$$Y_{12}(s) = \frac{I_2(s)}{V_1(s)}$$

$$Y_{21}(s) = \frac{I_1(s)}{V_2(s)}$$

Example 1 *Determine the driving-point impedance function of a one-port network shown in Fig. 2.*

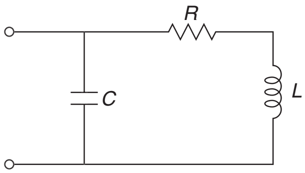

Fig. 2

Solution The transformed network is shown in Fig. 3.

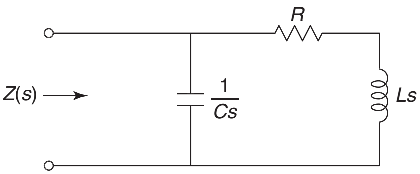

Fig. 3

$$Z(s) = \frac{\frac{1}{Cs}(R+Ls)}{\frac{1}{Cs}+(R+Ls)} = \frac{R+Ls}{LCs^2+RCs+1} = \frac{1}{C}\frac{s+\frac{R}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Example 2 *Determine the driving-point impedance of the network shown in Fig. 4.*

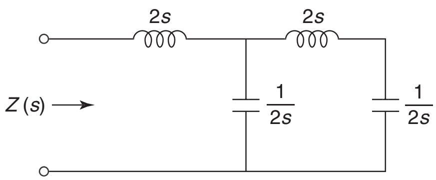

Fig. 4

Solution

$$Z(s) = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{1}{2s}+2s+\frac{1}{2s}} = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{2+4s^2}{2s}} = 2s + \frac{2s+\frac{1}{2s}}{2+4s^2} = \frac{4s+8s^3+2s+\frac{1}{2s}}{2+4s^2} = \frac{16s^4+12s^2+1}{8s^3+4s}$$

Example 3 *Determine the driving-point impedance of the network shown in Fig. 5.*

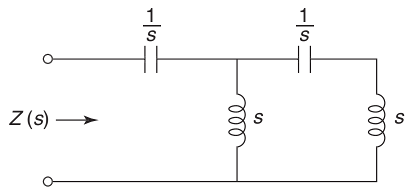

Fig. 5

Solution

$$Z(s) = \frac{1}{s} + \frac{s\left(\frac{1}{s}+s\right)}{s+\frac{1}{s}+s} = \frac{1}{s} + \frac{(1+s^2)s}{2s^2+1} = \frac{1}{s} + \frac{s+s^3}{2s^2+1} = \frac{s^4+3s^2+1}{2s^3+s}$$

Example 4 *Find the driving-point admittance function of the network shown in Fig. 6.*

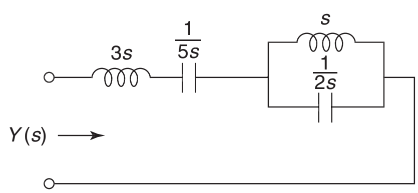

Fig. 6

Solution

$$Z(s) = 3s + \frac{1}{5s} + \frac{s\left(\frac{1}{2s}\right)}{s+\frac{1}{2s}} = 3s + \frac{1}{5s} + \frac{s}{2s^2+1} = \frac{30s^4+15s^2+2s^2+1+5s^2}{5s\,(2s^2+1)} = \frac{30s^4+22s^2+1}{5s\,(2s^2+1)}$$

$$Y(s) = \frac{1}{Z(s)} = \frac{5s\,(2s^2+1)}{30s^4+22s^2+1}$$

Example 5 *Find the transfer impedance function $Z_{12}(s)$ for the network shown in Fig. 7.*

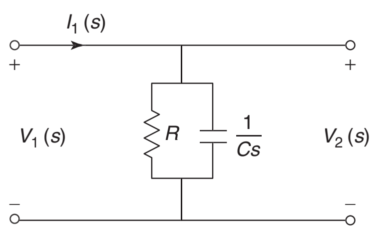

Fig. 7

Solution

$$V_2(s) = I_1(s)\frac{R\left(\frac{1}{Cs}\right)}{R+\frac{1}{Cs}}$$

$$\frac{V_2(s)}{I_1(s)} = \frac{R}{RCs+1}$$

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)} = \frac{1}{C\left(s+\frac{1}{RC}\right)}$$

Example 6 *Find voltage transfer function of the two-port network shown in Fig. 8.*

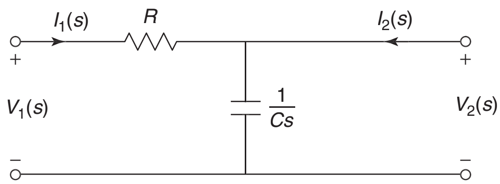

Fig.8

Solution By voltage division rule,

$$V_2(s) = V_1(s)\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}} = V_1(s)\frac{1}{RCs+1} = V_1(s)\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

Voltage transfer function

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

ANALYSIS OF LADDER NETWORKS

The network functions of a ladder network can be obtained by a simple method. This method depends upon the relationships that exist between the branch currents and node voltages of the ladder network. Consider the network shown in Fig. 9 where all the impedances are connected in series branches and all the admittances are connected in parallel branches.

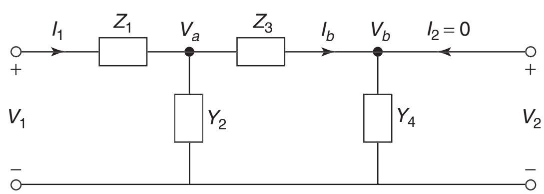

Fig. 9 *Ladder network*

Analysis is done by writing the set of equations. In writing these equations, we begin at the port 2 of the ladder and work towards the port 1.

$$V_b = V_2$$
$$I_b = Y_4\, V_2$$
$$V_a = Z_3\, I_b + V_2 = (Z_3 Y_4 + 1)\, V_2$$
$$I_1 = Y_2\, V_a + I_b = [Y_2\,(Z_3\, Y_4 + 1) + Y_4]\, V_2$$
$$V_1 = Z_1\, I_1 + V_a = [Z_1\,\{Y_2\,(Z_3\, Y_4 + 1) + Y_4\} + (Z_3\, Y_4 + 1)]\, V_2$$

Thus, each succeeding equation takes into account one new impedance or admittance. Except the first two equations, each subsequent equation is obtained by multiplying the equation just preceding it by imittance (either impedance or admittance) that is next down the line and then adding to this product the equation twice preceding it. After writing these equations, we can obtain any network function.

Example 7 *For the network shown in Fig. 10, determine transfer function* $\dfrac{V_2}{V_1}$.

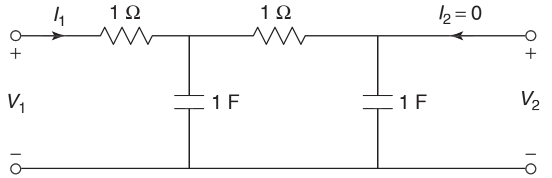

Fig. 10

Solution The transformed network is shown in Fig. 11.

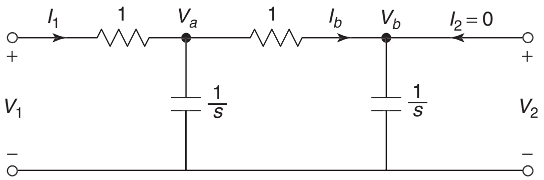

Fig. 11

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_a = 1I_b + V_2$$

$$= sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s+1)V_2 + sV_2 = (s^2 + 2s)V_2$$

$$V_1 = 1I_1 + V_a = (s^2 + 2s)V_2 + (s+1)V_2 = (s^2 + 3s + 1)V_2$$

Hence,
$$\frac{V_2}{V_1} = \frac{1}{s^2 + 3s + 1}$$

Example 8 *For the network shown in Fig. 12, determine the voltage transfer function* $\dfrac{V_2}{V_1}$.

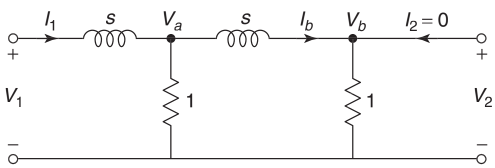

Fig. 12

Solution

$$V_b = V_2$$

$$I_b = \frac{V_2}{1} = V_2$$

$$V_a = sI_b + V_2 = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{1} + I_b = (s+1)V_2 + V_2 = (s+2)V_2$$

$$V_1 = sI_1 + V_a = s(s+2)V_2 + (s+1)V_2 = (s^2 + 3s + 1)V_2$$

Hence,
$$\frac{V_2}{V_1} = \frac{1}{s^2 + 3s + 1}$$

Example 9 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$ *and* $\frac{V_2}{I_1}$ *for the network shown in Fig. 13.*

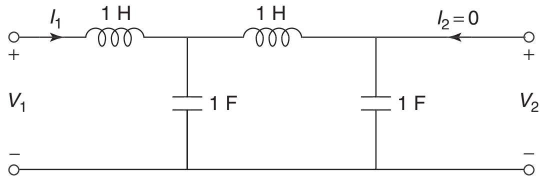

Fig. 13

Solution The transformed network is shown in Fig. 14.

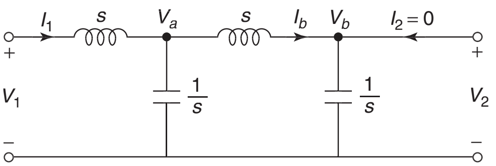

Fig. 14

$$V_b = V_2$$
$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$
$$V_a = sI_b + V_2 = s(sV_2) + V_2 = (s^2 + 1)V_2$$
$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s^2 + 1)V_2 + sV_2 = (s^3 + 2s)V_2$$
$$V_1 = sI_1 + V_a = s(s^3 + 2s)V_2 + (s^2 + 1)V_2 = (s^4 + 2s^2 + s^2 + 1)V_2 = (s^4 + 3s^2 + 1)V_2$$

Hence,
$$\frac{V_1}{I_1} = \frac{s^4 + 3s^2 + 1}{s^3 + 2s}$$
$$\frac{V_2}{V_1} = \frac{1}{s^4 + 3s^2 + 1}$$
$$\frac{V_2}{I_1} = \frac{1}{s^3 + 2s}$$

Example 10 *Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$, *and* $\frac{V_2}{I_1}$ *for the network in Fig. 15.*

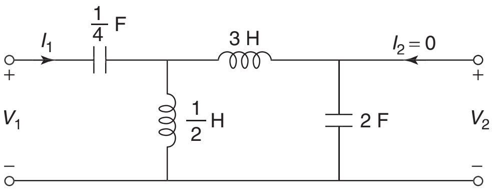

Fig. 15

Solution The transformed network is shown in Fig. 16.

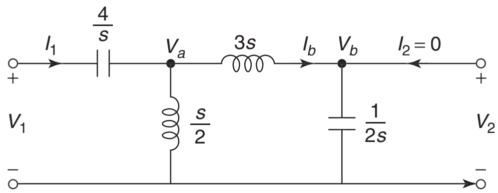

Fig. 16

$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{2s}} = 2sV_2$$

$$V_a = 3s\,I_b + V_2 = 3s(2sV_2) + V_2 = (6s^2+1)V_2$$

$$I_1 = \frac{V_a}{\frac{s}{2}} + I_b = \frac{2}{s}(6s^2+1)V_2 + 2sV_2 = \left(\frac{14s^2+2}{s}\right)V_2$$

$$V_1 = \frac{4}{s}I_1 + V_a = \frac{4}{s}\left(\frac{14s^2+2}{s}\right)V_2 + (6s^2+1)V_2 = \left(\frac{6s^4+57s^2+8}{s^2}\right)V_2$$

Hence,
$$\frac{V_1}{I_1} = \frac{6s^4+57s^2+8}{14s^3+2s}$$

$$\frac{V_2}{V_1} = \frac{s^2}{6s^4+57s^2+8}$$

$$\frac{V_2}{I_1} = \frac{s}{14s^2+2}$$

Example 11 *For the ladder network of Fig. 17, find the driving point-impedance at the 1 – 1′ terminal with 2 – 2′ open.*

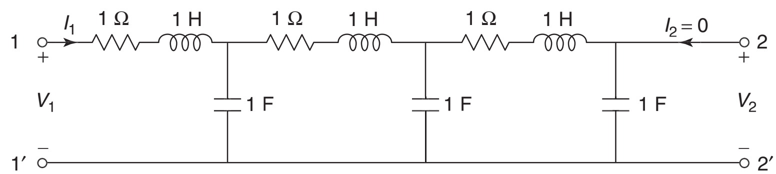

Fig. 17

Solution The transformed network is shown in Fig. 18.

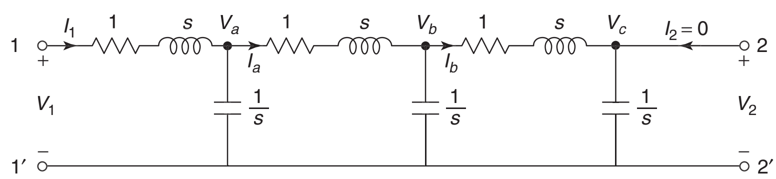

Fig. 18

$$V_c = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = s\,V_2$$

$$V_b = (s+1)\,I_b + V_2 = (s+1)\,sV_2 + V_2 = (s^2 + s + 1)V_2$$

$$I_a = \frac{V_b}{\frac{1}{s}} + I_b = sV_b + I_b = s(s^2 + s + 1)V_2 + sV_2 = (s^3 + s^2 + 2s)V_2$$

$$V_a = (s+1)\,I_a + V_b = (s+1)(s^3 + s^2 + 2s)V_2 + (s^2 + s + 1)V_2 = (s^4 + 2s^3 + 4s^2 + 3s + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(s^4 + 2s^3 + 4s^2 + 3s + 1)V_2 + (s^3 + s^2 + 2s)V_2$$

$$= (s^5 + 2s^4 + 5s^3 + 4s^2 + 3s)V_2$$

$$V_1 = (s+1)\,I_1 + V_a = (s+1)(s^5 + 2s^4 + 5s^3 + 4s^2 + 3s)V_2 + (s^4 + 2s^3 + 4s^2 + 3s + 1)V_2$$

$$= (s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1)V_2$$

Hence, $$Z_{11} = \frac{V_1}{I_1} = \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

Example 12 *Determine the voltage transfer function* $\dfrac{V_2}{V_1}$ *for the network shown in Fig. 19.*

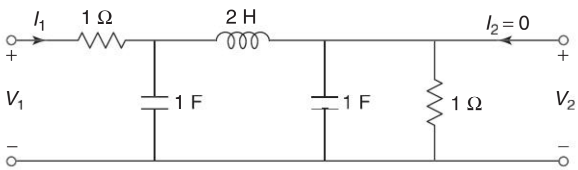

Fig. 19

Solution The transformed network is shown in Fig. 20.

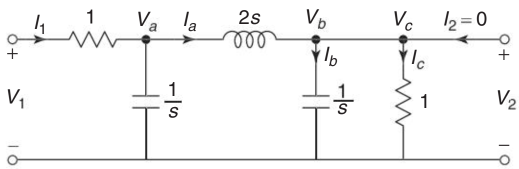

Fig. 20

$$V_c = V_b = V_2$$

$$I_a = I_b + I_c = \frac{V_2}{\frac{1}{s}} + \frac{V_2}{1} = s\,V_2 + V_2 = (s+1)V_2$$

$$V_a = 2s\,I_a + V_2 = 2s(s+1)V_2 + V_2$$

$$= (2s^2 + 2s + 1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = s(2s^2 + 2s + 1)V_2 + (s+1)V_2 = (2s^3 + 2s^2 + 2s + 1)V_2$$

$$V_1 = 1I_1 + V_a = (2s^3 + 2s^2 + 2s + 1)V_2 + (2s^2 + 2s + 1)V_2 = (2s^3 + 4s^2 + 4s + 2)V_2$$

Hence, $$\frac{V_2}{V_1} = \frac{1}{2s^3 + 4s^2 + 4s + 2}$$

Example 13 *For the network shown in Fig. 21, determine the transfer function* $\frac{I_2}{V_1}$.

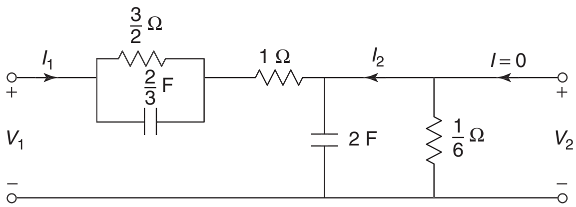

Fig. 21

Solution The transformed network is shown in Fig. 22.

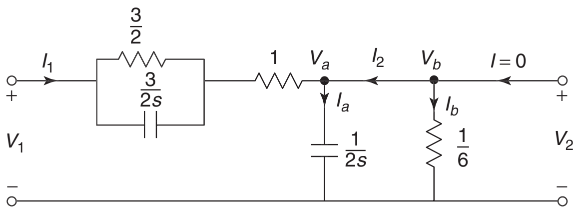

Fig. 22

$$V_b = V_a = V_2$$

$$I_1 = I_a + I_b = \frac{V_2}{\frac{1}{2s}} + \frac{V_2}{\frac{1}{6}} = 2sV_2 + 6V_2 = (2s+6)V_2$$

$$V_1 = \left(\frac{\frac{3}{2}\times\frac{3}{2s}}{\frac{3}{2}+\frac{3}{2s}} + 1\right)I_1 + V_2 = \left(\frac{9}{6s+6}+1\right)I_1 + V_2 = \frac{(6s+15)}{6s+6}(2s+6)V_2 + V_2 = \left[\frac{(2s+5)(s+3)}{(s+1)}+1\right]V_2$$

$$= \left[\frac{(2s+5)(s+3)+(s+1)}{s+1}\right]V_2 = \left(\frac{2s^2+6s+5s+15+s+1}{s+1}\right)V_2 = \left(\frac{2s^2+12s+16}{s+1}\right)V_2$$

$$= \frac{2(s^2+6s+8)}{s+1}V_2 = \frac{2(s+4)(s+2)}{s+1}V_2$$

Also, $I_2 = -I_b = -6\,V_2$

$$\frac{I_2}{V_1} = -\frac{3(s+1)}{(s+4)(s+2)}$$

Example 14 *For the network shown in Fig. 23, compute* $\alpha_{12}(s) = \frac{I_2}{I_1}$ *and* $Z_{12}(s) = \frac{V_2}{I_1}$.

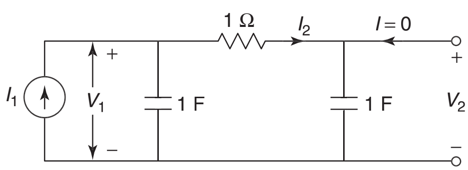

Fig. 23

Solution The transformed network is shown in Fig. 24.

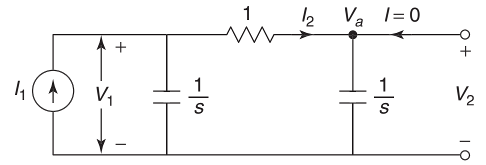

Fig. 24

$$V_a = V_2$$

$$I_2 = \frac{V_2}{\frac{1}{s}} = sV_2$$

$$V_1 = 1 I_2 + V_a = sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_1}{\frac{1}{s}} + I_2 = sV_1 + I_2 = s(s+1)V_2 + sV_2 = (s^2 + 2s)V_2$$

Hence, $\alpha_{12}(s) = \frac{I_2}{I_1} = \frac{1}{s+2}$

and $Z_{12}(s) = \frac{V_2}{I_1} = \frac{1}{s^2 + 2s}$

Example 15 *Determine the voltage ratio* $\frac{V_2}{V_1}$, *current ratio* $\frac{I_2}{I_1}$, *transfer impedance* $\frac{V_2}{I_1}$ *and driving- point impedance* $\frac{V_1}{I_1}$ *for the network shown in Fig. 25.*

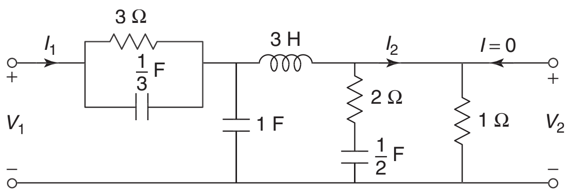

Fig. 25

Solution The transformed network is shown in Fig. 26.

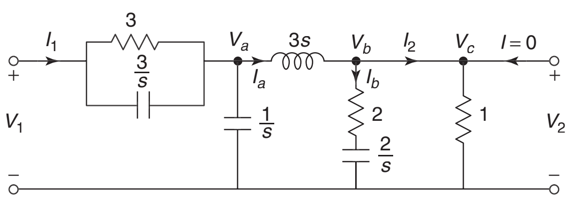

Fig. 26

$$V_c = V_b = V_2$$

$$I_2 = \frac{V_2}{1} = V_2$$

$$I_a = I_b + I_2 = \frac{V_2}{2 + \frac{2}{s}} + \frac{V_2}{1} = \frac{s}{2s+2}V_2 + V_2 = \left(\frac{3s+2}{2s+2}\right)V_2$$

$$V_a = 3s\,I_a + V_2 = \frac{3s(3s+2)}{2s+2}V_2 + V_2 = \left(\frac{9s^2 + 8s + 2}{2s+2}\right)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_a = sV_a + I_a = \frac{s(9s^2+8s+2)}{2s+2}V_2 + \left(\frac{3s+2}{2s+2}\right)V_2 = \left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2$$

$$V_1 = \left(\frac{3\times\frac{3}{s}}{3+\frac{3}{s}}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)I_1 + V_a = \left(\frac{3}{s+1}\right)\left(\frac{9s^3+8s^2+5s+2}{2s+2}\right)V_2 + \left(\frac{9s^2+8s+2}{2s+2}\right)V_2$$

$$= \left[\frac{27s^3+24s^2+15s+6+9s^3+8s^2+2s+9s^2+8s+2}{(s+1)(2s+2)}\right]V_2 = \frac{(36s^3+41s^2+25s+8)}{(s+1)(2s+2)}V_2$$

$$= \left(\frac{36s^3+41s^2+25s+8}{2s^2+4s+2}\right)V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{2s^2+4s+2}{36s^3+41s^2+25s+8}$$

$$\frac{I_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_2}{I_1} = \frac{2s+2}{9s^3+8s^2+5s+2}$$

$$\frac{V_1}{I_1} = \frac{36s^3+41s^2+25s+8}{(s+1)(9s^3+8s^2+5s+2)} = \frac{36s^3+41s^2+25s+8}{9s^4+17s^3+13s^2+7s+2}$$

Example 16

For the resistive two-port network of Fig. 27, find $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

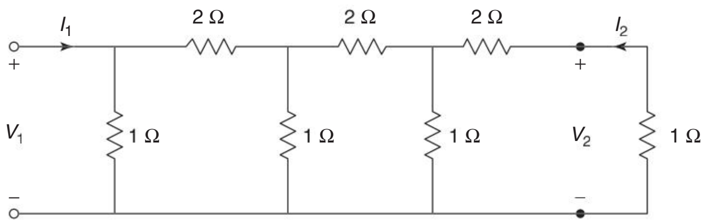

Fig. 27

Solution The network is redrawn as shown in Fig. 28.

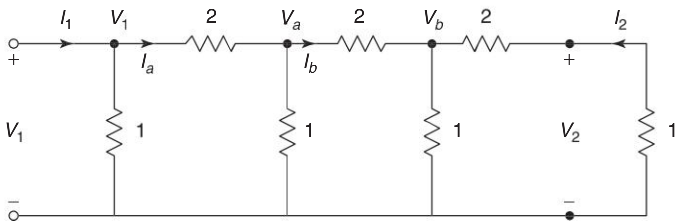

Fig. 28

$$I_2 = -\frac{V_2}{1} = -V_2$$

$$V_b = -3 I_2 = 3 V_2$$

$$I_b = \frac{V_b}{1} + \frac{V_b}{3} = \frac{4}{3} V_b = 4 V_2$$

$$V_a = 2 I_b + V_b = 8 V_2 + 3 V_2 = 11 V_2$$

$$I_a = \frac{V_a}{1} + I_b = 11 V_2 + 4 V_2 = 15 V_2$$

$$V_1 = 2 I_a + V_a = 30 V_2 + 11 V_2 = 41 V_2$$

$$I_1 = \frac{V_1}{1} + I_a = 41 V_2 + 15 V_2 = 56 V_2$$

Hence,

$$\frac{V_2}{V_1} = \frac{1}{41}$$

$$\frac{V_2}{I_1} = \frac{1}{56}\ \Omega$$

$$\frac{I_2}{V_1} = -\frac{1}{41}\ \mho$$

$$\frac{I_2}{I_1} = -\frac{1}{56}$$

Example 17 *Find the network function $\frac{V_2}{V_1}$ for the network shown in Fig. 29.*

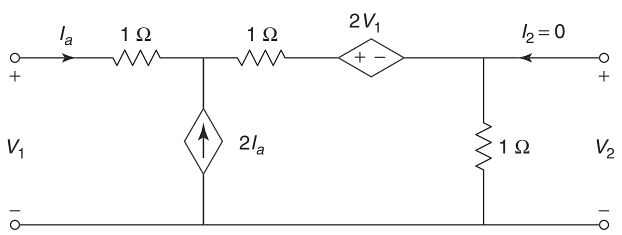

Fig. 29

Solution The network is redrawn as shown in Fig. 30.

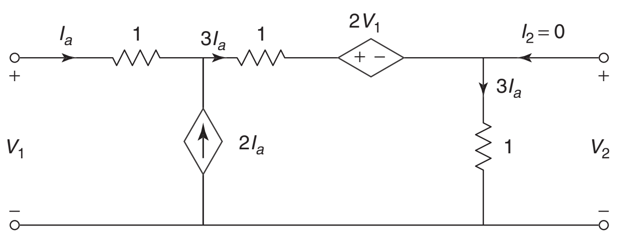

Fig. 30

From Fig. 30,

$$V_2 = 1\,(3\,I_a) = 3\,I_a$$

Applying KVL to the outermost loop,

$$V_1 - 1\,(I_a) - 1\,(3I_a) - 2\,V_1 - 1\,(3I_a) = 0$$

$$V_1 = -7I_a$$

Hence,

$$\frac{V_2}{V_1} = -\frac{3}{7}$$

Example 18 *Find the network function* $\frac{I_2}{I_1}$ *for the network shown in Fig. 31.*

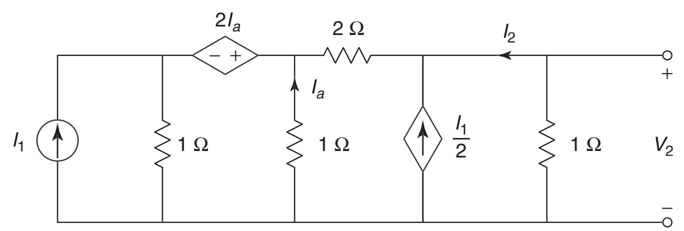

Fig. 31

Solution The network is redrawn as shown in Fig. 32.

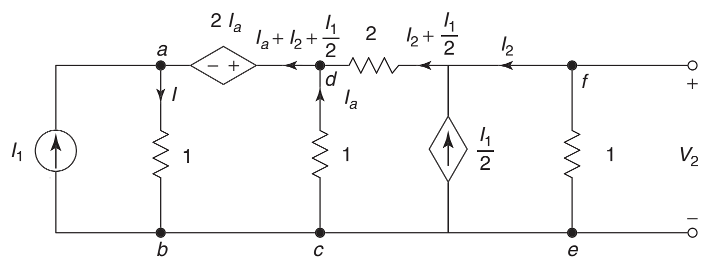

Fig. 32

From Fig. 32,

$$I = I_1 + I_a + I_2 + \frac{I_1}{2}$$

$$= \frac{3}{2}I_1 + I_a + I_2 \qquad \text{...(i)}$$

Applying KVL to the loop *abcda*,

$$-1I - 1I_a - 2I_a = 0$$

$$-I - 3I_a = 0$$

$$I + 3I_a = 0 \qquad \text{...(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$\frac{3}{2}I_1 + I_a + I_2 + 3I_a = 0$$

$$\frac{3}{2}I_1 + I_2 + 4I_a = 0 \qquad \text{...(iii)}$$

Applying KVL to the loop *dcefd,*

$$1I_a - 1I_2 - 2\left(I_2 + \frac{I_1}{2}\right) = 0$$

$$I_a - 3I_2 - I_1 = 0$$

$$I_a = 3I_2 + I_1 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (iii),

$$\frac{3}{2}I_1 + I_2 + 4(3I_2 + I_1) = 0$$

$$\frac{3}{2}I_1 + I_2 + 12I_2 + 4I_1 = 0$$

$$\frac{11}{2}I_1 + 13I_2 = 0$$

$$13I_2 = -\frac{11}{2}I_1$$

Hence,

$$\frac{I_2}{I_1} = -\frac{11}{26}$$

ANALYSIS OF NON-LADDER NETWORKS

The above method is applicable for ladder networks. There are other network configurations to which the technique described is not applicable. Figure 33 shows one such network.

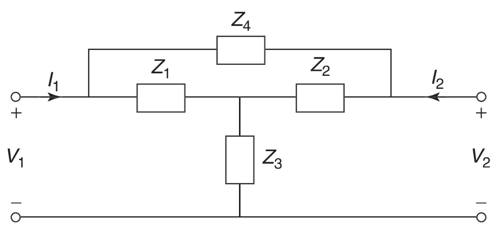

Fig. 33 *Non-ladder network*

For such a type of network, it is necessary to express the network functions as a quotient of determinants, formulated on KVL and KCL basis.

Example 19 *For the resistive bridged T network shown in Fig. 34, find* $\frac{V_2}{V_1}, \frac{V_2}{I_1}, \frac{I_2}{V_1}$ *and* $\frac{I_2}{I_1}$.

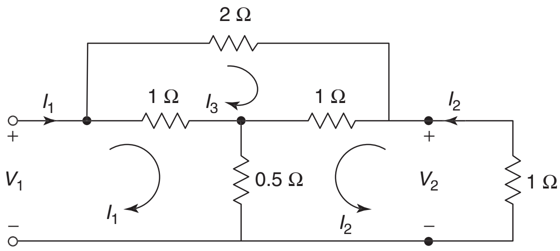

Fig. 34

Solution Applying KVL to Mesh 1,

$$V_1 = 1.5I_1 + 0.5I_2 - I_3 \quad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = 0.5I_1 + 2.5I_2 + I_3 \quad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -I_1 + I_2 + 4I_3 \quad \text{...(iii)}$$

Writing these equations in matrix form,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \quad \text{...(iv)}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} V_1 & 0.5 & -1 \\ 0 & 2.5 & 1 \\ 0 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{V_1(10-1)}{9} = V_1 \quad \text{...(iv)}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 1.5 & V_1 & -1 \\ 0.5 & 0 & 1 \\ -1 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1.5 & 0.5 & -1 \\ 0.5 & 2.5 & 1 \\ -1 & 1 & 4 \end{vmatrix}} = \frac{-V_1(2+1)}{9} = -\frac{1}{3}V_1 \quad \text{...(v)}$$

From Fig. 8.34, $V_2 = -1(I_2) = -I_2$

From Eq. (v), $V_1 = -3I_2$

From Eqs. (iv) and (v), $I_2 = -\frac{1}{3}V_1 = -\frac{1}{3}I_1$

$$I_1 = -3I_2$$

Hence, $\frac{I_2}{V_1} = -\frac{1}{3}\,\mho$

$$\frac{I_2}{I_1} = \frac{-\frac{1}{3}V_1}{V_1} = -\frac{1}{3}$$

$$\frac{V_2}{V_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}$$

$$\frac{V_2}{I_1} = \frac{-I_2}{-3I_2} = \frac{1}{3}\,\Omega$$

Example 20 *For the network of Fig. 35, find Z_{11}, Z_{12} and G_{12}.*

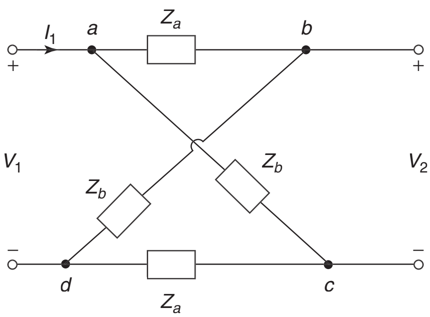

Fig. 35

Solution The network can be redrawn as shown in Fig. 36. Since the network consists of two identical impedances connected in parallel, the current in I_1 divides equally in each branch.

$$V_1 = (Z_a + Z_b)\frac{I_1}{2}$$

$$Z_{11} = \frac{V_1}{I_1} = \frac{Z_a + Z_b}{2}$$

$$V_2 = Z_b\frac{I_1}{2} - Z_a\left(\frac{I_1}{2}\right) = (Z_b - Z_a)\frac{I_1}{2}$$

$$Z_{12} = \frac{V_2}{I_1} = \frac{Z_b - Z_a}{2}$$

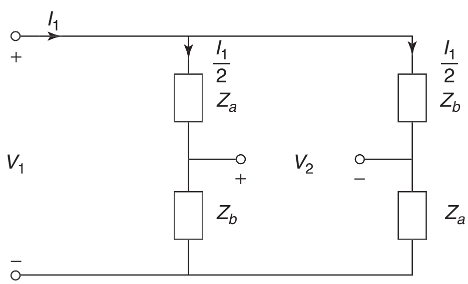

Fig. 36

By voltage-division rule,

$$V_2 = \frac{Z_b}{Z_a + Z_b}V_1 - \frac{Z_a}{Z_a + Z_b}V_1 = \frac{Z_b - Z_a}{Z_a + Z_b}V_1$$

$$G_{12} = \frac{V_2}{V_1} = \frac{Z_b - Z_a}{Z_a + Z_b}$$

Example 21 *For the network shown in Fig. 37, determine $Z_{11}(s)$, $G_{12}(s)$ and $Z_{12}(s)$.*

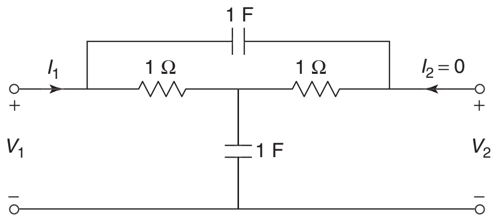

Fig. 37

Solution The transformed network is shown in Fig. 38.

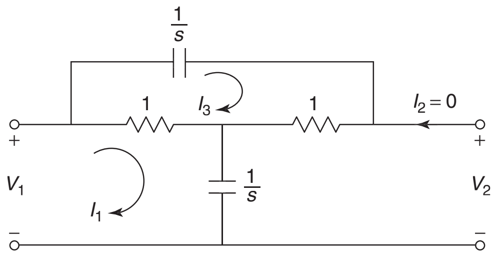

Fig. 38

Applying KVL to Mesh 1,

$$V_1 = \left(1+\frac{1}{s}\right) I_1 - I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$V_2 = \frac{1}{s} I_1 + I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$-I_1 + \left(2+\frac{1}{s}\right) I_3 = 0$$

$$I_3 = \left(\frac{s}{2s+1}\right) I_1 \qquad \text{...(iii)}$$

Substituting Eq. (iii) in Eqs. (i) and (ii),

$$V_1 = \left(1+\frac{1}{s}\right) I_1 - \left(\frac{s}{2s+1}\right) I_1 = \left(\frac{s+1}{s} - \frac{s}{2s+1}\right) I_1 = \left[\frac{s^2+3s+1}{s(2s+1)}\right] I_1$$

$$V_2 = \frac{1}{s} I_1 + \frac{s}{2s+1} I_1 = \left[\frac{s^2+2s+1}{s(2s+1)}\right] I_1$$

Hence,

$$Z_{11}(s) = \frac{V_1}{I_1} = \frac{s^2+3s+1}{s(2s+1)}$$

$$Z_{12}(s) = \frac{V_2}{I_1} = \frac{s^2+2s+1}{s(2s+1)}$$

$$G_{12}(s) = \frac{V_2}{V_1} = \frac{s^2+2s+1}{s^2+3s+1}$$

Example 22 *For the network shown in Fig. 39, find the driving-point admittance Y_{11} and transfer admittance Y_{12}.*

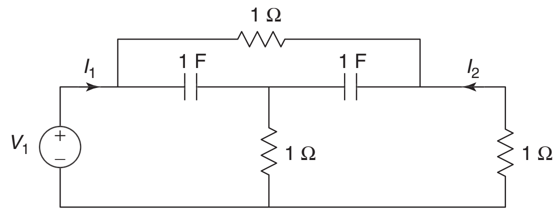

Fig. 39

Solution The transformed network is shown in Fig. 40.

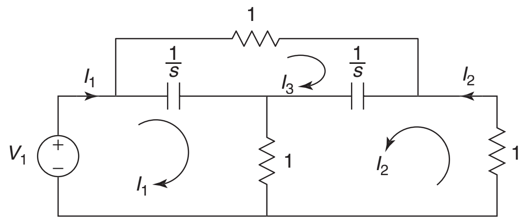

Fig. 40

Applying KVL to Mesh 1,

$$V_1 = \left(\frac{1}{s}+1\right)I_1 + I_2 - \frac{1}{s}I_3 \qquad \text{...(i)}$$

Applying KVL to Mesh 2,

$$0 = I_1 + \left(2+\frac{1}{s}\right)I_2 + \frac{1}{s}I_3 \qquad \text{...(ii)}$$

Applying KVL to Mesh 3,

$$0 = -\frac{1}{s}I_1 + \frac{1}{s}I_2 + \left(\frac{2}{s}+1\right)I_3 \qquad \text{...(iii)}$$

Writing these equations in matrix from,

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta}$$

$$\Delta = \begin{vmatrix} \frac{1}{s}+1 & 1 & -\frac{1}{s} \\ 1 & 2+\frac{1}{s} & \frac{1}{s} \\ -\frac{1}{s} & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = \left(\frac{1}{s}+1\right)\left[\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right]-1\left[(1)\left(\frac{2}{s}+1\right)+\frac{1}{s^2}\right]-\frac{1}{s}\left[(1)\left(\frac{1}{s}\right)+\left(\frac{1}{s}\right)\left(2+\frac{1}{s}\right)\right]$$

$$= \frac{s^2+5s+2}{s^2}$$

$$\Delta_1 = \begin{vmatrix} V_1 & 1 & -\frac{1}{s} \\ 0 & 2+\frac{1}{s} & \frac{1}{s} \\ 0 & \frac{1}{s} & \frac{2}{s}+1 \end{vmatrix} = V_1\left[\left(2+\frac{1}{s}\right)\left(\frac{2}{s}+1\right)-\frac{1}{s^2}\right] = V_1\left(\frac{2s^2+5s+1}{s^2}\right)$$

$$I_1 = V_1\left(\frac{2s^2+5s+1}{s^2+5s+2}\right)$$

Hence, $$Y_{11} = \frac{I_1}{V_1} = \frac{2s^2+5s+1}{s^2+5s+2}$$

$$I_2 = \frac{\Delta_2}{\Delta}$$

$$\Delta_2 = \begin{vmatrix} \frac{1}{s}+1 & V_1 & -\frac{1}{s} \\ 1 & 0 & \frac{1}{s} \\ -\frac{1}{s} & 0 & \frac{2}{s}+1 \end{vmatrix} = -V_1\left(\frac{2}{s}+1+\frac{1}{s^2}\right) = -V_1\left(\frac{s^2+2s+1}{s^2}\right)$$

$$I_2 = -V_1\left(\frac{s^2+2s+1}{s^2+5s+2}\right)$$

Hence, $$Y_{12} = \frac{I_2}{V_1} = -\frac{s^2+2s+1}{s^2+5s+2}$$

Synthesis of RLC Circuits

INTRODUCTION

In the study of electrical networks, broadly there are two topics: 'Network Analysis' and 'Network Synthesis'. Any network consists of excitation, response and network function. In network analysis, network and excitation are given, whereas the response has to be determined. In network synthesis, excitation and response are given, and the network has to be determined. Thus, in network synthesis we are concerned with the realisation of a network for a given excitation-response characteristic. Also, there is one major difference between analysis and synthesis. In analysis, there is a unique solution to the problem. But in synthesis, the solution is not unique and many networks can be realised.

The first step in synthesis procedure is to determine whether the network function can be realised as a physical passive network. There are two main considerations; causality and stability. By *causality* we mean that a voltage cannot appear at any port before a current is applied or vice-versa. In other words, the response of the network must be zero for $t < 0$. For the network to be stable, the network function cannot have poles in the right half of the s-plane. Similarly, a network function cannot have multiple poles on the $j\omega$ axis.

HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when s is real.
(b) The roots of $P(s)$ have real parts which are zero or negative.

Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

   are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.
2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

Example 23 *State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.*

(a) $s^4 + 4s^3 + 3s + 2$

(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

Solution (a) In the given polynomial, the term s^2 is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

Example 24 *Test whether the polynomial* $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 3s)\,s^4 + 5s^2 + 4\,(s$$
$$\underline{s^4 + 3s^2}$$
$$2s^2 + 4\Big)\,s^3 + 3s\left(\frac{1}{2}s\right.$$
$$\underline{s^3 + 2s}$$
$$s)\,2s^2 + 4\,(2s$$
$$\underline{2s^2}$$
$$4\Big)\,s\left(\frac{1}{4}s\right.$$
$$\underline{s}$$
$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 25 *Test whether the polynomial* $P(s) = s^3 + 4s^2 + 5s + 2$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)\ n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$\begin{array}{l}
4s^2+2\,\big)\, s^3+5s\,\big(\, \frac{1}{4}s \\
\qquad\quad \underline{s^3+\frac{2}{4}s} \\
\qquad\qquad \frac{9}{2}s\,\big)\, 4s^2+2\,\big(\, \frac{8}{9}s \\
\qquad\qquad\qquad \underline{4s^2} \\
\qquad\qquad\qquad\quad 2\,\big)\, \frac{9}{2}s\,\big(\, \frac{9}{4}s \\
\qquad\qquad\qquad\qquad \underline{\frac{9}{2}s} \\
\qquad\qquad\qquad\qquad\ 0
\end{array}$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Example 26 *Test whether the polynomial* $P(s) = s^4 + s^3 + 3s^2 + 2s + 12$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 3s^2 + 12$

Odd part of $P(s) = n(s) = s^3 + 2s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$\begin{array}{l}
s^3+2s\,\big)\, s^4+3s^2+12\,\big(\, s \\
\qquad\quad \underline{s^4+2s^2} \\
\qquad\qquad s^2+12\,\big)\, s^3+\ 2s\,\big(\, s \\
\qquad\qquad\qquad \underline{s^3+12s} \\
\qquad\qquad\qquad -10s\,\big)\, s^2+12\,\big(\, -\frac{1}{10}s \\
\qquad\qquad\qquad\qquad \underline{s^2} \\
\qquad\qquad\qquad\qquad 12\,\big)\, -10s\,\big(\, -\frac{10}{12}s \\
\qquad\qquad\qquad\qquad\quad \underline{-10s} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 27 *Prove that polynomial* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is not Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 2s^2 + 2$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
s^3+3s)\,s^4+2s^2+2\;(s \\
\qquad\quad \underline{s^4+3s^2} \\
\qquad\qquad -s^2+2)\,s^3+3s\,(-s \\
\qquad\qquad\qquad \underline{s^3-2s} \\
\qquad\qquad\qquad\quad 5s\Big)-s^2+2\Big(-\frac{1}{5}s \\
\qquad\qquad\qquad\qquad \underline{-s^2} \\
\qquad\qquad\qquad\qquad\quad 2\Big)\,5s\Big(\frac{5}{2}s \\
\qquad\qquad\qquad\qquad\qquad \underline{5s} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Since two quotient terms are negative, $P(s)$ is not Hurwitz.

Example 28 *Prove that polynomial* $P(s) = 2s^4 + 5s^3 + 6s^2 + 3s + 1$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^4 + 6s^2 + 1$

Odd part of $P(s) = n(s) = 5s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
5s^3+3s\Big)\,2s^4+6s^2+1\;\Big(\frac{2}{5}s \\
\qquad\quad \underline{2s^4+\frac{6}{5}s^2} \\
\qquad\qquad \frac{24}{5}s^2+1\Big)\,5s^3+3s\Big(\frac{25}{24}s \\
\qquad\qquad\qquad\quad \underline{5s^3+\frac{25}{24}s}
\end{array}
$$

$$\frac{47}{24}s\,\Big)\,\frac{24}{5}s^2+1\,\Big(\,\frac{576}{235}s$$

$$\frac{24}{5}s^2$$

$$1\,\Big)\,\frac{47}{24}s\,\Big(\,\frac{24}{47}s$$

$$\frac{47}{24}s$$

$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 29 *Test whether the polynomial* $P(s)=s^4+7s^3+6s^2+21s+8$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+6s^2+8$

Odd part of $P(s)=n(s)=7s^3+21s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$7s^3+21s\,\Big)\,s^4+6s^2+8\,\Big(\,\frac{1}{7}s$$

$$s^4+3s^2$$

$$3s^2+8\,\Big)\,7s^3+21s\,\Big(\,\frac{7}{3}s$$

$$7s^3+\frac{56}{3}s$$

$$\frac{7}{3}s\,\Big)\,3s^2+8\,\Big(\,\frac{9}{7}s$$

$$3s^2$$

$$8\,\Big)\,\frac{7}{3}s\,\Big(\,\frac{7}{24}s$$

$$\frac{7}{3}s$$

$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 30 *Check whether* $P(s)=s^4+5s^3+5s^2+4s+10$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+5s^2+10$

Odd part of $P(s)=n(s)=5s^3+4s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$5s^3 + 4s \overline{\big) \; s^4 + 5s^2 + 10 \big(\frac{1}{5}s}$$

$$\frac{s^4 + \frac{4}{5}s^2}{\frac{21}{5}s^2 + 10 \big) \; 5s^3 + 4s \big(\frac{25}{21}s}$$

$$\frac{5s^3 + \frac{250}{21}s}{-\frac{166}{21}s \big) \; \frac{21}{5}s^2 + 10 \big(-\frac{441}{830}s}$$

$$\frac{\frac{21}{5}s^2}{10 \big) -\frac{166}{21}s \big(-\frac{166}{210}s}$$

$$\frac{-\frac{166}{21}s}{0}$$

Since the last two quotient terms are negative, the polynomial $P(s)$ is not Hurwitz.

Example 31 *Test whether the polynomial* $s^5 + 3s^3 + 2s$ *is Hurwitz.*

Solution Since the given polynomial contains odd functions only, it is not possible to perform a continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 9s^2 + 2$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$5s^4 + 9s^2 + 2 \big) \; s^5 + 3s^3 + 2s \big(\frac{1}{5}s$$

$$\frac{s^5 + \frac{9}{5}s^3 + \frac{2}{5}s}{\frac{6}{5}s^3 + \frac{8}{5}s \big) \; 5s^4 + 9s^2 + 2 \big(\frac{25}{6}s}$$

$$5s^4 + \frac{20}{3}s^2$$

$$\frac{7}{3}s^2 + 2\,\Big)\,\frac{6}{5}s^3 + \frac{8}{5}s\,\Big(\,\frac{18}{35}s$$

$$\frac{6}{5}s^3 + \frac{36}{35}s$$

$$\frac{20}{35}s\,\Big)\,\frac{7}{3}s^2 + 2\,\Big(\,\frac{49}{12}s$$

$$\frac{7}{3}s^2$$

$$2\,\Big)\,\frac{20}{35}s\,\Big(\,\frac{10}{35}s$$

$$\frac{20}{35}s$$

$$0$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Example 32 *Test whether the polynomial P(s) is Hurwitz.*

$$P(s) = s^5 + s^3 + s$$

Solution Since the given polynomial contains odd functions only, it is not possible to perform continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 3s^2 + 1$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$5s^4 + 3s^2 + 1\,\Big)\,s^5 + s^3 + s\,\Big(\,\frac{1}{5}s$$

$$s^5 + \frac{3}{5}s^3 + \frac{1}{5}s$$

$$\frac{2}{5}s^3 + \frac{4}{5}s\,\Big)\,5s^4 + 3s^2 + 1\,\Big(\,\frac{25}{2}s$$

$$5s^4 + 10s^2$$

$$-7s^2 + 1\,\Big)\,\frac{2}{5}s^3 + \frac{4}{5}s\,\Big(\,-\frac{2}{35}s$$

$$\frac{2}{5}s^3 - \frac{2}{35}s$$

$$\frac{26}{35}s\,\Big)\, -7s^2+1\,\Big(-\frac{245}{26}s$$

$$\underline{-7s^2}$$

$$1\,\Big)\,\frac{26}{35}s\,\Big(\frac{26}{35}s$$

$$\underline{\frac{26}{35}s}$$

$$0$$

Since the third and fourth quotient terms are negative, $P(s)$ is not Hurwitz.

Example 33 *Test the polynomial P(s) of Hurwitz property.*

$$P(s)=s^6+3s^5+8s^4+15s^3+17s^2+12s+4$$

Solution Even part of $P(s)=m(s)=s^6+8s^4+17s^2+4$

Odd part of $P(s)=n(s)=3s^5+15s^3+12s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$3s^5+15s^3+12s\,\Big)\,s^6+8s^4+17s^2+4\,\Big(\frac{1}{3}s$$

$$\underline{s^6+5s^4+4s^2}$$

$$3s^4+13s^2+4\,\Big)\,3s^5+15s^3+12s\,\Big(s$$

$$\underline{3s^5+13s^3+4s}$$

$$2s^3+8s\,\Big)\,3s^4+13s^2+4\,\Big(\frac{3}{2}s$$

$$\underline{3s^4+12s^2}$$

$$s^2+4\,\Big)\,2s^3+8s\,\Big(2s$$

$$\underline{2s^3+8s}$$

$$0$$

The division has terminated abruptly (i.e., the number of partial quotients (that is four) is not equal to the order of polynomial (that is six) with common factor (s^2+4).

$$P(s)=s^6+3s^5+8s^4+15s^3+17s^2+12s+4=(s^2+4)(s^4+3s^3+4s^2+3s+1)$$

If both the factors are Hurwitz, $P(s)$ will be Hurwitz.

Let $$P_1(s) = s^2 + 4$$

Since it contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 2s$$

$$\frac{P_1(s)}{P_1'(s)} = \frac{s^2+4}{2s} = \frac{s^2}{2s} + \frac{4}{2s} = \frac{s}{2} + \frac{1}{\dfrac{s}{2}}$$

Since all the quotient terms are positive, $P_1(s)$ is Hurwitz.

Now, let $$P_2(s) = s^4 + 3s^3 + 4s^2 + 3s + 1$$

$$m_2(s) = s^4 + 4s^2 + 1$$

$$n_2(s) = 3s^3 + 3s$$

By continued fraction expansion,

$$3s^3 + 3s \,\Big)\, s^4 + 4s^2 + 1 \,\Big(\, \frac{1}{3}s$$

$$\underline{s^4 + \ s^2}$$

$$3s^2 + 1)\ 3s^3 + 3s\ (s$$

$$\underline{3s^3 + s}$$

$$2s \,\Big)\, 3s^2 + 1 \,\Big(\, \frac{3}{2}s$$

$$\underline{3s^2}$$

$$1)\ 2s\ (2s$$

$$\underline{2s}$$

$$0$$

Since all the quotient terms are positive, $P_2(s)$ is Hurwitz.

Hence, $P(s) = (s^2 + 4)(s^4 + 3s^3 + 4s^2 + 3s + 1)$ is Hurwitz.

Example 34 *Test whether the polynomial* $P(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 2s^6 + s^4 + 8s^2 + 4$

Odd part of $P(s) = n(s) = s^7 + 2s^5 + 4s^3 + 8s$

$$Q(s) = \frac{n(s)}{m(s)}$$

By continued fraction expansion,

$$2s^6+s^4+8s^2+4\overline{\big)\,s^7+2s^5+4s^3+8s\,}\left(\frac{1}{2}s\right.$$

$$\underline{s^7+\frac{1}{2}s^5+4s^3+2s}$$

$$\frac{3}{2}s^5 \qquad +6s\,\big)\,2s^6+s^4+8s^2+4\left(\frac{4}{3}s\right.$$

$$\underline{2s^6 \qquad +8s^2}$$

$$s^4 \qquad +4\,\big)\,\frac{3}{2}s^5+6s\left(\frac{3}{2}s\right.$$

$$\underline{\frac{3}{2}s^5+6s}$$

$$0$$

Since the division has terminated abruptly it indicates a common factor s^4+4. The polynomial can be written as

$$P(s)=(s^4+4)(s^3+2s^2+2s+1)$$

If both the factor are Hurwitz, $P(s)$ will be Hurwitz.
In the polynomial (s^4+4), the terms s^3, s^2 and s are missing. Hence, it is not Hurwitz.
Therefore, $P(s)$ is not Hurwitz.

Example 35 *Test whether the polynomial* $2s^6+s^5+13s^4+6s^3+56s^2+25s+25$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=2s^6+13s^4+56s^2+25$

Odd part of $P(s)=n(s)=s^5+6s^3+25s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^5+6s^3+25s)\,2s^6+13s^4+56s^2+25(2s$$

$$\underline{2s^6+12s^4+50s^2}$$

$$s^4+6s^2 \quad +25)\,s^5+6s^3+25s(s$$

$$\underline{s^5+6s^3+25s}$$

$$0$$

The division has terminated abruptly.

$$P(s)=2s^6+s^5+13s^4+6s^3+56s^2+25s+25=(s^4+6s^2+25)\,(2s^2+s+1)$$

Let $$P_1(s)=s^4+6s^2+25$$

Since $P_1(s)$ contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 4s^3 + 12s$$

By continued fraction expansion,

$$
\begin{array}{l}
4s^3 + 12s \,\big)\, s^4 + 6s^2 + 25 \,\Big(\, \frac{1}{4}s \\
\qquad\qquad\quad \underline{s^4 + 3s^2 \qquad\quad} \\
\qquad\qquad\qquad 3s^2 + 25 \,\big)\, 4s^3 + \quad 12s \,\Big(\, \frac{4}{3}s \\
\qquad\qquad\qquad\qquad\qquad \underline{4s^3 + \frac{100}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\quad -\frac{64}{3}s \,\Big)\, 3s^2 + 25 \,\Big(\, -\frac{9}{64}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{3s^2 \qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 25 \,\Big)\, -\frac{64}{3}s \,\Big(\, -\frac{64}{75}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{-\frac{64}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Since two of the quotient terms are negative, $P_1(s)$ is not Hurwitz.
We need not test the other factor $(2s^2 + s + 1)$ for being Hurwitz.
Hence, $P(s)$ is not Hurwitz.

There is another method to test a Hurwitz polynomial. In this method, we construct the Routh–Hurwitz array for the required polynomial.

Let $\qquad P(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

The Routh–Hurwitz array is given by,

$$
\begin{array}{c|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_n & b_{n-1} & b_{n-2} & \ldots \\
s^{n-3} & c_n & c_{n-1} & & \ldots \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
s^1 & . & & & \\
s^0 & . & & &
\end{array}
$$

The coefficients of s^n and s^{n-1} rows are directly written from the given equation.

where
$$b_n = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-2} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$c_n = \frac{b_n a_{n-3} - a_{n-1}b_{n-1}}{b_n}$$

$$c_{n-1} = \frac{b_n a_{n-5} - a_{n-1}b_{n-2}}{b_n}$$

Hence, for polynomial $P(s)$ to be Hurwitz, there should not be any sign change in the first column of the array.

Example 36 *Test whether* $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 8 \\ s^3 & 7 & 21 & \\ s^2 & 3 & 8 & \\ s^1 & \frac{7}{3} & 0 & \\ s^0 & 8 & & \end{array}$$

Since all the elements in the first column are positive, the polynomial $P(s)$ is Hurwitz.

Example 37 *Determine whether* $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 2 & 2 \\ s^3 & 1 & 3 & \\ s^2 & -1 & 2 & \\ s^1 & 5 & 0 & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 38 *Test whether* $P(s) = s^5 + 2s^4 + 4s^3 + 6s^2 + 2s + 5$ *is Hurwitz.*

Solution The Routh array is given by,

s^5	1	4	2
s^4	2	6	5
s^3	1	–0.5	
s^2	7	5	
s^1	–1.21		
s^0	5		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 39 *Test whether the polynomial $P(s) = s^5 + s^3 + s$ is Hurwitz.*

Solution The given polynomial contains odd functions only.

$$P'(s) = 5s^4 + 3s^2 + 1$$

The Routh array is given by,

s^5	1	1	1
s^4	5	3	1
s^3	0.4	0.8	
s^2	–7	1	
s^1	0.86		
s^0	1		

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 40 *Test whether the polynomial $P(s) = s^8 + 5s^6 + 2s^4 + 3s^2 + 1$ is Hurwitz.*

Solution The given polynomial contains even functions only.

$$P'(s) = 8s^7 + 30s^5 + 8s^3 + 6s$$

The Routh array is given by,

s^8	1	5	2	3	1
s^7	8	30	8	6	0
s^6	1.25	1	2.25	1	
s^5	23.6	–6.4	–0.4	0	
s^4	1.33	2.27	1		
s^3	–46.6	–18.14	0		
s^2	1.75	1			
s^1	8.49				
s^0	1				

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Example 41 *Test whether $P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^5 & 1 & 45 & 44 \\ s^4 & 12 & 60 & 48 \\ s^3 & 40 & 40 & \\ s^2 & 48 & 48 & \\ s^1 & 0 & 0 & \\ s^0 & & & \end{array}$$

Notes: When all the elements in any one row is zero, the following steps are followed:

(i) Write an auxiliary equation with the help of the coefficients of the row just above the row of zeros.
(ii) Differentiate the auxiliary equation and replace its coefficient in the row of zeros.
(iii) Proceed for the Routh test.

Auxiliary equation,

$$A(s) = 48s^2 + 48$$
$$A'(s) = 96s$$

$$\begin{array}{c|ccc} s^5 & 1 & 45 & 44 \\ s^4 & 12 & 60 & 48 \\ s^3 & 40 & 40 & \\ s^2 & 48 & 48 & \\ s^1 & 96 & 0 & \\ s^0 & 48 & & \end{array}$$

Since there is no sign change in the first column of the array, the polynomial $P(s)$ is Hurwitz.

Example 42 *Check whether $P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|cccc} s^6 & 2 & 13 & 56 & 25 \\ s^5 & 1 & 6 & 25 & \\ s^4 & 1 & 6 & 25 & \\ s^3 & 0 & 0 & 0 & \\ s^2 & & & & \\ s^1 & & & & \\ s^0 & & & & \end{array}$$

$$A(s) = s^4 + 6s^2 + 25$$
$$A'(s) = 4s^3 + 12s$$

Now, the Routh array will be given by,

$$\begin{array}{c|cccc} s^6 & 2 & 13 & 56 & 25 \\ s^5 & 1 & 6 & 25 & \\ s^4 & 1 & 6 & 25 & \\ s^3 & 4 & 12 & & \\ s^2 & 3 & 25 & & \\ s^1 & -21.3 & & & \\ s^0 & 25 & & & \end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Example 43 *Determine the range of values of 'a' so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & a & 3 \\ s^3 & 1 & 2 & \\ s^2 & a-2 & 3 & \\ s^1 & \dfrac{2a-7}{a-2} & & \\ s^0 & 3 & & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive, i.e.,

$$a - 2 > 0$$
$$a > 2$$

and

$$\frac{2a-7}{a-2} > 0$$
$$a > \frac{7}{2}$$

Hence, $P(s)$ will be Hurwitz when $a > \frac{7}{2}$.

Example 44 *Determine the range of values of k so that the polynomial $P(s) = s^3 + 3s^2 + 2s + k$ is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \dfrac{6-k}{3} & 0 \\ s^0 & k & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive,

i.e.,
$$\frac{6-k}{3} > 0$$
$$6-k > 0$$

i.e., $k < 6$ and $k > 0$
Hence, $P(s)$ will be Hurwitz for $0 < k < 6$.

POSITIVE REAL FUNCTIONS

A function $F(s)$ is positive real if the following conditions are satisfied:

(a) $F(s)$ is real for real s.
(b) The real part of $F(s)$ is greater than or equal to zero when the real part of s is greater than or equal to zero, i.e.,
Re $F(s) \geq 0$ for Re$(s) \geq 0$

Properties of Positive Real Functions

1. If $F(s)$ is positive real then $\dfrac{1}{F(s)}$ is also positive real.
2. The sum of two positive real functions is positive real.
3. The poles and zeros of a positive real function cannot have positive real parts, i.e., they cannot be in the right half of the s plane.
4. Only simple poles with real positive residues can exist on the $j\omega$-axis.
5. The poles and zeros of a positive real function are real or occur in conjugate pairs.
6. The highest powers of the numerator and denominator polynomials may differ at most by unity. This condition prevents the possibility of multiple poles and zeros at $s = \infty$.
7. The lowest powers of the denominator and numerator polynomials may differ by at most unity. Hence, a positive real function has neither multiple poles nor zeros at the origin.

Necessary and Sufficient Conditions for Positive Real Functions

The necessary and sufficient conditions for a function with real coefficients $F(s)$ to be positive real are the following:

1. $F(s)$ must have no poles and zeros in the right half of the s-plane.
2. The poles of $F(s)$ on the $j\omega$-axis must be simple and the residues evaluated at these poles must be real and positive.
3. Re $F(j\omega) \geq 0$ for all ω.

Testing of the Above Conditions Condition (1) requires that we test the numerator and denominator of $F(s)$ for roots in the right half of the s-plane, i.e., we must determine whether the numerator and denominator of $F(s)$ are Hurwitz. This is done through a continued fraction expansion of the odd to even or even to odd parts of the numerator and denominator.

Condition (2) is tested by making a partial-fraction expansion of $F(s)$ and checking whether the residues of the poles on the $j\omega$-axis are positive and real. Thus, if $F(s)$ has a pair of poles at $s = \pm j\omega_0$, a partial-fraction expansion gives terms of the form

$$\frac{K_1}{s - j\omega_0} + \frac{K_1^*}{s + j\omega_0}$$

Since residues of complex conjugate poles are themselves conjugate, $K_1 = K_1^*$ and should be positive and real.

Condition (3) requires that Re $F(j\omega)$ must be positive and real for all ω.

Now, to compute Re $F(j\omega)$ from $F(s)$, the numerator and denominator polynomials are separated into even and odd parts.

$$F(s) = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)} = \frac{m_1 + n_1}{m_2 + n_2}$$

Multiplying $N(s)$ and $D(s)$ by $m_2 - n_2$,

$$F(s) = \frac{m_1 + n_1}{m_2 + n_2}\,\frac{m_2 - n_2}{m_2 - n_2} = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2} + \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

But the product of two even functions or odd functions is itself an even function, while the product of an even and odd function is odd.

$$\text{Ev } F(s) = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$\text{Od } F(s) = \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

Now, substituting $s = j\omega$ in the even polynomial gives the real part of $F(s)$ and substituting $s = j\omega$ into the odd polynomial gives imaginary part of $F(s)$.

$$\text{Ev } F(s)\big|_{s=j\omega} = \text{Re } F(j\omega)$$

$$\text{Od } F(s)\big|_{s=j\omega} = j\,\text{Im } F(j\omega)$$

We have to test Re $F(j\omega) \geq 0$ for all ω.

The denominator of Re $F(j\omega)$ is always a positive quantity because

$$m_2^2 - n_2^2\big|_{s=j\omega} \geq 0$$

Hence, the condition that Ev $F(j\omega)$ should be positive requires

$$m_1 m_2 - n_1 n_2\big|_{s=j\omega} = A(\omega^2)$$

should be positive and real for all $\omega \geq 0$.

Example 45 *Test whether* $F(s) = \dfrac{s+3}{s+1}$ *is a positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s+3}{s+1}$

The function $F(s)$ has pole at $s = -1$ and zero at $s = -3$ as shown in Fig. 40.

Thus, pole and zero are in the left half of the s-plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

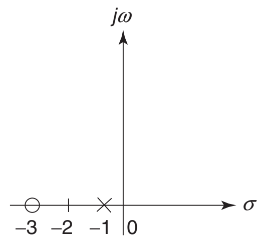

Fig. 40

(c) Even part of $N(s) = m_1 = 3$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 |_{s=j\omega} = (3)(1) - (s)(s)|_{s=j\omega} = 3 - s^2 |_{s=j\omega} = 3 + \omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.
Since all the three conditions are satisfied, the function is positive real.

Example 46 *Test whether* $F(s) = \dfrac{s^2+6s+5}{s^2+9s+14}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+6s+5}{s^2+9s+14} = \dfrac{(s+5)(s+1)}{(s+7)(s+2)}$

The function $F(s)$ has poles at $s = -7$ and $s = -2$ and zeros at $s = -5$ an $s = -1$ as shown in Fig. 41.
Thus, all the poles and zeros are in the left half of the s plane.

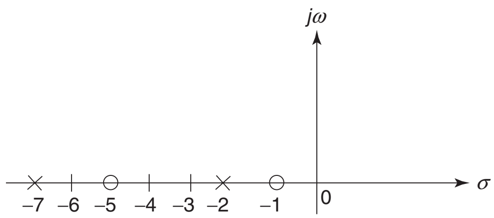

Fig. 41

(b) Since there is no pole on the $j\omega$ axis, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 5$

Odd part of $N(s) = n_1 = 6s$

Even part of $D(s) = m_2 = s^2 + 14$

Odd part of $D(s) = n_2 = 9s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 |_{s=j\omega} = (s^2+5)\,(s^2+14) - (6s)(9s)|_{s=j\omega} = s^4 - 35s^2 + 70\ |_{s=j\omega} = \omega^4 + 35\omega^2 + 70$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.
Since all the three conditions are satisfied, the function is positive real.

Example 47 *Test whether* $F(s) = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s(s+3)(s+5)}{(s+1)(s+4)} = \dfrac{s^3+8s^2+15s}{s^2+5s+4}$

The function $F(s)$ has poles at $s = -1$ and $s = -4$ and zeros at $s = 0$, $s = -3$ and $s = -5$ as shown in Fig. 42.

Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis, hence the residue test is not carried out.

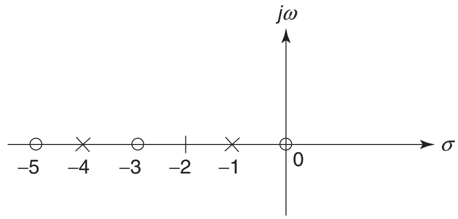

Fig. 42

(c) Even part of $N(s) = m_1 = 8s^2$

Odd part of $N(s) = n_1 = s^3 + 15s$

Even part of $D(s) = m_2 = s^2 + 4$

Odd part of $D(s) = n_2 = 5s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 |_{s=j\omega} = (8s^2)(s^2+4) - (s^3+15s)(5s)|_{s=j\omega} = 3s^4 - 43s^2 |_{s=j\omega} = 3\omega^4 + 43\omega^2$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 48 *Test whether* $F(s) = \dfrac{s^2+1}{s^3+4s}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+1}{s^3+4s} = \dfrac{(s+j1)(s-j1)}{s(s+j2)(s-j2)}$

The function $F(s)$ has poles at $s = 0$, $s = -j2$ and $s = j2$ and zeros at $s = -j1$ and $s = j1$ as shown in Fig. 43.

Thus, all the poles and zeros are on the $j\omega$ axis.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

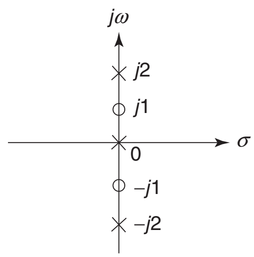

Fig. 43

$$F(s) = \frac{s^2+1}{s^3+4s} = \frac{s^2+1}{s(s^2+4)}$$

By partial-fraction expansion,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+j2} + \frac{K_2^*}{s-j2}$$

The constants K_1, K_2 and K_2* are called residues.

$$K_1 = s\,F(s)|_{s=0} = \left.\frac{s^2+1}{s^2+4}\right|_{s=0} = \frac{1}{4}$$

$$K_2 = (s+j2)F(s)|_{s=-j2} = \left.\frac{s^2+1}{s(s-j2)}\right|_{s=-j2} = \frac{-4+1}{(-j2)(-j2-j2)} = \frac{3}{8}$$

$$K_2^* = K_2 = \frac{3}{8}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = s^2 + 1$

Odd part of $N(s) = n_1 = 0$

Even part of $D(s) = m_2 = 0$

Odd part of $D(s) = n_2 = s^3 + 4s$

$$A\left(\omega^2\right) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+1)(0) - (0)(s^3+4s)\,|_{s=j\omega} = 0$$

$A(\omega^2)$ is zero for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 49 *Test whether* $F(s) = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1} = \dfrac{2s^3 + 2s^2 + 3s + 2}{(s + j1)(s - j1)}$

Since numerator polynomial cannot be easily factorized, we will prove whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 2s^2 + 2$

Odd part of $N(s) = n(s) = 2s^3 + 3s$

By continued fraction expansion,

$$\begin{array}{l}
2s^2+2\,\big)\,2s^3+3s\,\big(\,s \\
\qquad\quad\;\; \underline{2s^3+2s} \\
\qquad\qquad\quad s\,\big)\,2s^2+2\,\big(\,2s \\
\qquad\qquad\qquad\;\; \underline{2s^2} \\
\qquad\qquad\qquad\quad 2\,\big)\,s\,\big(\,\tfrac{1}{2}s \\
\qquad\qquad\qquad\qquad\; \underline{s} \\
\qquad\qquad\qquad\qquad\; 0
\end{array}$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that zeros are in the left half of the s plane.

The function $F(s)$ has poles at $s = -j1$ and $s = j1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$$

As the degree of the numerator is greater than that of the denominator, division is first carried out before partial-fraction expansion.

$$
\begin{array}{r|l}
s^2+1 & 2s^3+2s^2+3s+2 \;\big(\; 2s+2 \\
 & \underline{2s^3 \qquad +2s} \\
 & \qquad 2s^2 + \;\; s+2 \\
 & \qquad \underline{2s^2 \qquad +2} \\
 & \qquad\qquad s
\end{array}
$$

$$F(s) = 2s + 2 + \frac{s}{s^2+1}$$

By partial-fraction expansion,

$$F(s) = 2s + 2 + \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1}$$

$$K_1 = (s+j1)F(s)\,|_{s=-j1} = \frac{-j1}{-j1-j1} = \frac{1}{2}$$

$$K_1^* = K_1 = \frac{1}{2}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = 2s^2 + 2$

Odd part of $N(s) = n_1 = 2s^3 + 3s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 0$

$$A(\omega^2) = m_1 m_2 - n_1 n_2\,|_{s=j\omega} = (2s^2+2)(s^2+1) - (2s^3+3s)(0)\,|_{s=j\omega} = 2s^4 + 4s^2 + 2\;\;|_{s=j\omega} = 2(\omega^4 - 2\omega^2 + 1)$$
$$= 2(\omega^2 - 1)^2$$

$A(\omega^2) \geq 0$ for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Example 50 *Test whether* $F(s) = \dfrac{s^3 + 6s^2 + 7s + 3}{s^2 + 2s + 1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3 + 6s^2 + 7s + 3}{s^2 + 2s + 1} = \dfrac{s^3 + 6s^2 + 7s + 3}{(s+1)(s+1)}$

Since a numerator polynomial cannot be easily factorized, we will test whether $N(s)$ is Hurwitz.

Even part of $N(s) = m(s) = 6s^2 + 3$

Odd part of $N(s) = n(s) = s^3 + 7s$

By continued fraction expansion,

$$6s^2+3\overline{\left)s^3+7s\right.}\left(\frac{1}{6}s\right.$$
$$\underline{s^3+0.5s}$$
$$6.5s\overline{\left)6s^2+3\right.}\left(0.92s\right.$$
$$\underline{6s^2}$$
$$3\overline{\left)6.5s\right.}\left(2.17s\right.$$
$$\underline{6.5s}$$
$$0$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that the zeros are in the left half of the s plane.
The function $F(s)$ has a double pole at $s=-1$.
Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s)=m_1=6s^2+3$

Odd part of $N(s)=n_1=s^3+7s$

Even part of $D(s)=m_2=s^2+1$

Odd part of $D(s)=n_2=2s$

$$A(\omega^2)=m_1m_2-n_1n_2\,|_{s=j\omega}=(6s^2+3)(s^2+1)-(s^3+7s)(2s)\,|_{s=j\omega}=\;4s^4-5s^2+3\,|_{s=j\omega}=4\omega^4+5\omega^2+3$$

$A(\omega^2)$ is positive for all $\omega\geq 0$.
Since all the three conditions are satisfied, the function is positive real.

Example 51 *Test whether* $F(s)=\dfrac{s^2+s+6}{s^2+s+1}$ *is a positive real function.*

Solution

(a) $$F(s)=\frac{N(s)}{D(s)}=\frac{s^2+s+6}{s^2+s+1}=\frac{\left(s+\frac{1}{2}+j\frac{\sqrt{23}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{23}}{2}\right)}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

The function $F(s)$ has zeros at $s=-\frac{1}{2}\pm j\frac{\sqrt{23}}{2}$ and poles at $s=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s)=m_1=s^2+6$

Odd part of $N(s)=n_1=s$

Even part of $D(s)=m_2=s^2+1$

Odd part of $D(s)=n_2=s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+6)(s^2+1) - (s)(s) \,|_{s=j\omega} = s^4 + 6s^2 + 6 \,|_{s=j\omega} = \omega^4 - 6\omega^2 + 6$$

For $\omega = 2$, $A(\omega^2) = 16 - 24 + 6 = -2$
This condition is not satisfied.
Hence, the function $F(s)$ is not positve real.

Example 52 *Test whether* $F(s) = \dfrac{s^2+4}{s^3+3s^2+3s+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+4}{s^3+3s^2+3s+1} = \dfrac{(s+j2)(s-j2)}{(s+1)^3}$

The function $F(s)$ has two zeros at $s = \pm j2$ and three poles at $s = -1$.
Thus, all the poles and zeros are in the left half of the s plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 4$
Odd part of $N(s) = n_1 = 0$
Even part of $D(s) = m_2 = 3s^2 + 1$
Odd part of $D(s) = n_2 = s^3 + 3s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+4)(3s^2+1) - (0)(s^3+3s) \,|_{s=j\omega} = 3s^4 + 13s^2 + 4 \,|_{s=j\omega} = 3\omega^4 - 13\omega^2 + 4$$

For $\omega = 1$, $A(\omega)^2 = 3 - 13 + 4 = -6$
This condition is not satisfied.
Hence, the function $F(s)$ is not positive real.

Example 53 *Test whether* $F(s) = \dfrac{s^3+5s}{s^4+2s^2+1}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^3+5s}{s^4+2s^2+1} = \dfrac{s(s^2+5)}{(s^2+1)^2} = \dfrac{s(s+j\sqrt{5})(s-j\sqrt{5})}{(s\pm j1)(s\pm j1)}$

The function $F(s)$ has zeros at $s = 0$, $s = \pm j\sqrt{5}$ and two poles at $s = j1$ and two poles at $s = -j1$.

Thus, poles on the $j\omega$ axis are not simple.
Hence, the function $F(s)$ not positive real.

Example 54 *Test whether* $F(s) = \dfrac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$ *is positive real function.*

Solution

$$F(s) = \frac{N(s)}{D(s)} = \frac{s^4+3s^3+s^2+s+2}{s^3+s^2+s+1}$$

Here, it is easier to prove that $N(s)$ and $D(s)$ are Hurwitz.

By Routh array,

$$\begin{array}{c|ccc} s^4 & 1 & 1 & 2 \\ s^3 & 3 & 1 & \\ s^2 & \frac{2}{3} & 2 & \\ s^1 & -8 & & \\ s^0 & 2 & & \end{array}$$

Since there is a sign change in the first column of the array, $N(s)$ is not Hurwitz. Thus, all the zeros are not in the left half of the s plane. The remaining two tests need not be carried out.

Hence, the function $F(s)$ is not positive real.