Задание 1

Условие:

$$f(z) = \frac{1}{z(z+2)}, z_0 = -1$$

Решение:

Т.к. -2 и 0 симметричны относительно -1, необходимо рассмотреть 2 области

$$z - z_0 = q$$
$$f(z) = \frac{1}{(z+1-1)(z+1+1)} = \frac{1}{(q-1)(q+1)} = \frac{1}{q^2-1} = -\frac{1}{1-q^2}$$
$$t = q^2$$
$$f(z) = -\frac{1}{1-t} = -\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} -t^n = \sum_{n=0}^{\infty} -q^{2n} = \sum_{n=0}^{\infty} -(z+1)^{2n}$$
$$\text{Для } |z+1| > 1:$$
$$f(z) = \frac{1}{z(z+2)} = \frac{1}{(q-1)(q+1)} = \frac{1}{2}\frac{1}{q-1} - \frac{1}{2}\frac{1}{1+q} =$$
$$\frac{1}{q-a} = \frac{\frac{1}{q}}{1-\frac{a}{q}} = \frac{1}{q} \cdot \frac{1}{1-\frac{a}{q}} = \frac{1}{q}\sum_{n=0}^{\infty}\left(\frac{a}{q}\right)^n = \sum_{n=0}^{\infty}\frac{a^n}{q^{n+1}}$$
$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{q^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{q^{n+1}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1-(-1)^n}{q^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{2}\cdot\frac{1-(-1)^n}{(z+1)^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{2(n+1)}}$$

Задание 2

Условие:

$$f(z) = z^3 \cos\frac{1}{z^2}$$

Решение:

$$\text{Особые точки:}$$
$$z = 0 : \lim_{z\to 0} z^3 \cos\frac{1}{z^2} = \nexists$$
$$z^3\cos\frac{1}{z^2} = z^3 \cdot \sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{z^2}\right)^{2n}}{(2n)!} = z^3\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!z^{4n}} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!z^{4n-3}}$$
$$c_{-1} = -1 \cdot \frac{1}{2!} = -\frac{1}{2}$$
$$\operatorname*{Res}_{z=0} f = -\frac{1}{2}$$
$$z = \infty : \text{ т.к. сумма всех вычетов равна 0, то} \operatorname*{Res}_{z=\infty} f = \frac{1}{2}$$
$$\text{Очевидно, что бесконечность - это полюс 3-его порядка}$$

Задание 3

Условие:

$$y = -x$$
$$w = \frac{z+1}{z-1}$$

Решение:

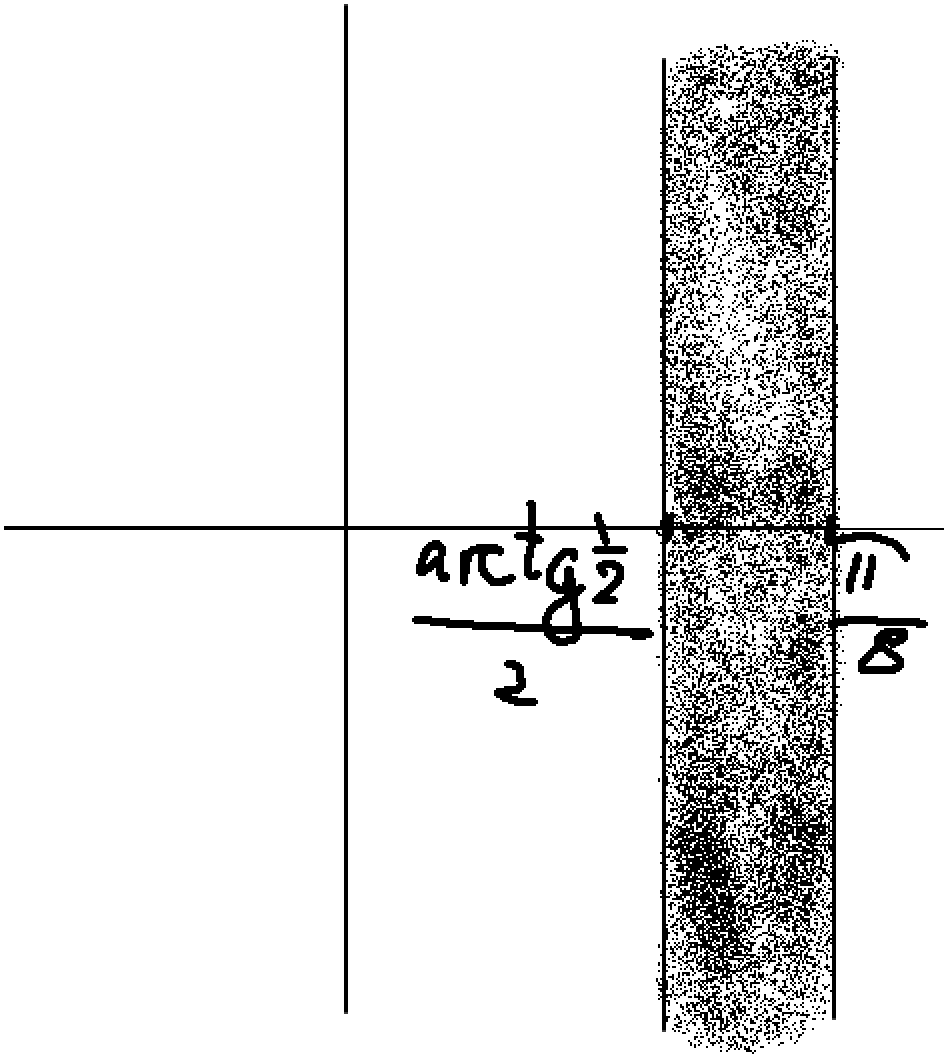
$$w(z-1) = z+1$$
$$(u+vi)(x+yi-1) = x+yi+1$$
$$ux-u-vy+i(vx-v+uy) = x+1+iy$$
$$\begin{cases} ux-u-vy = x+1 \\ vx-v+uy = y \\ y=-x \end{cases} \Rightarrow$$
$$\begin{cases} ux-u+vx = x+1 \\ vx-v-ux = -x \end{cases} \Rightarrow \begin{cases} x(u+v-1) = 1+u \\ x(v-u+1) = v \end{cases} \Rightarrow (1+u)(v-u+1) = v(u+v-1) \Rightarrow$$
$$v \cancel{-u} + 1 \cancel{+uv} - u^2 \cancel{+u} = \cancel{vu} + v^2 - v$$
$$1 = u^2 + v^2 - 2v$$
$$2 = u^2 + (v-1)^2 - \text{окружность}$$

Задание 4

Условие:

$$\frac{\operatorname{arctg}\frac{1}{2}}{2} \leq \operatorname{Re} z \leq \frac{\pi}{8}, \quad \omega = \operatorname{tg} z$$

Решение:

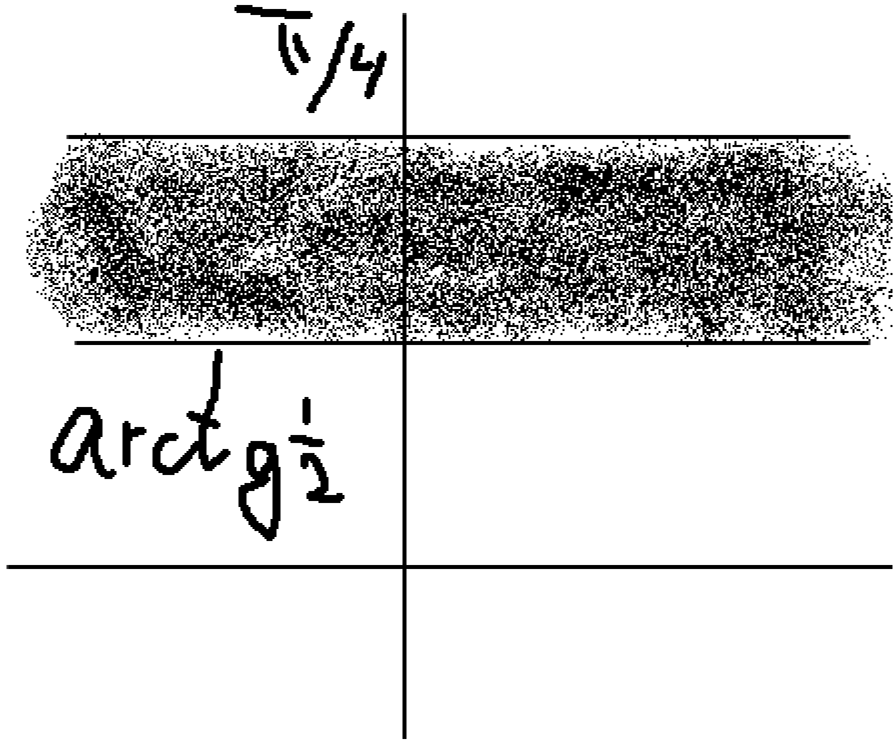


$$w = \operatorname{tg} z = \frac{e^{2iz} - 1}{i(e^{2iz} + 1)} - \text{ композиция линейной, экспоненты и дробно-линейной:}$$

$$w_1 = 2iz$$
$$w_2 = e^{w_1}$$
$$w = -i \cdot \frac{w_2 - 1}{w_2 + 1}$$

w_1 : i повернёт полосу относительно (0,0) против часовой на $\dfrac{\pi}{2}$

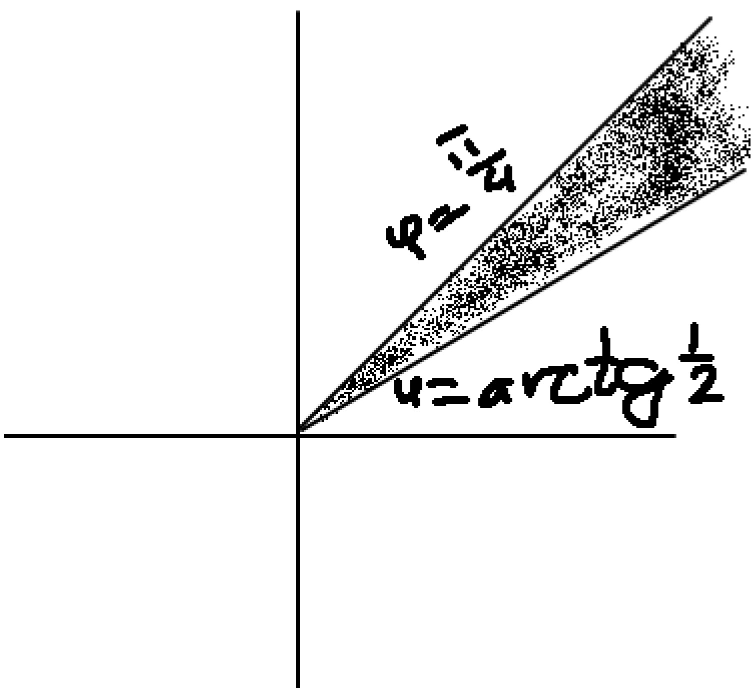
2 "растянет" в 2 раза $\Rightarrow$



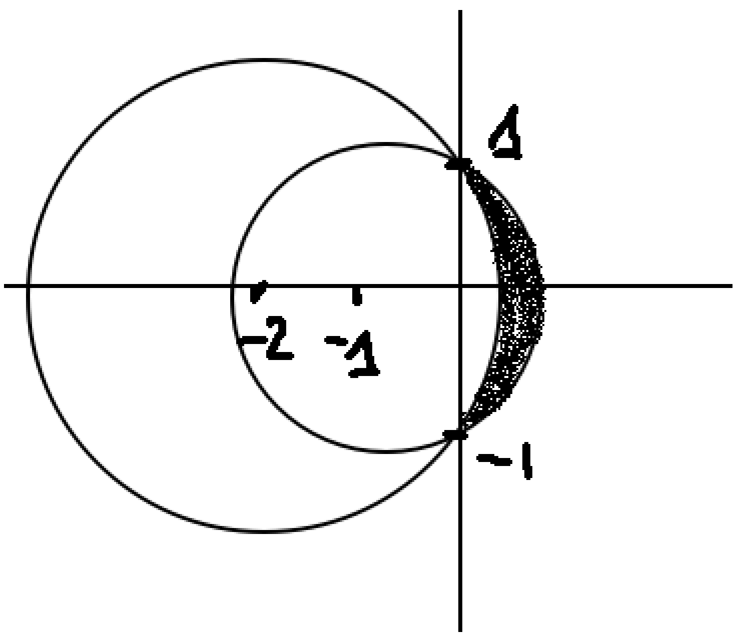
$$w_1:$$
$$\operatorname{arctg}\frac{1}{2} \leq \mathrm{Im} w_1 \leq \frac{\pi}{4}$$
$$w_2:$$
$$w_2 = \exp(w_1) = \exp(\mathrm{Re} w_1 + i \mathrm{Im} w_1) = \exp(\mathrm{Re} w_1) \cdot \exp(i \mathrm{Im} w_1)$$

w_2 задаёт угол с следующим условием:

$$\operatorname{arctg}\frac{1}{2} \leq \varphi \leq \frac{\pi}{4}$$



$$w = -i\frac{w_2-1}{w_2+1}$$
$$w_2 w + w = -i w_2 + i$$
$$w_2(w+i) = i - w$$
$$w_2 = \frac{i-w}{i+w}$$
$$r(\cos\varphi + i\sin\varphi) = \frac{i-x-iy}{i+x+iy}$$
$$r\cos\varphi + ir\sin\varphi = \frac{(-x+i(1-y))(x-i(y+1))}{x^2+(y+1)^2}$$
$$r\cos\varphi + ir\sin\varphi = \frac{-x^2+1-y^2}{x^2+(y+1)^2} + i\frac{x-xy+x+xy}{x^2+(y+1)^2}$$
$$\begin{cases} r\cos\varphi = \frac{1-x^2-y^2}{x^2+(y+1)^2} \\ r\sin\varphi = \frac{2x}{x^2+(y+1)^2} \end{cases} \Rightarrow \begin{cases} \operatorname{arctg}\frac{1}{2} \le \varphi \le \frac{\pi}{4} \\ \frac{2x}{\sin\varphi} \ge 0 \\ \frac{1-x^2-y^2}{\cos\varphi} \ge 0 \\ \operatorname{tg}\varphi = \frac{2x}{1-x^2-y^2} \end{cases} \overset{\substack{\sin\varphi>0 \\ \cos\varphi>0}}{\Rightarrow} \begin{cases} \frac{1}{2} \le \frac{2x}{1-x^2-y^2} \le 1 \\ x \ge 0 \\ 1-x^2-y^2 \ge 0 \end{cases} \Rightarrow$$
$$\begin{cases} x \ge 0 \\ 1-x^2-y^2 \le 4x \\ 2x \le 1-x^2-y^2 \end{cases} \Rightarrow \begin{cases} x \ge 0 \\ (x+2)^2+y^2 \ge 5 = \sqrt{5}^2 \\ (x+1)^2+y^2 \le 2 = \sqrt{2}^2 \end{cases}$$



Ответ: $\begin{cases} \operatorname{abs}(w+2) \ge \sqrt{5} \\ \operatorname{abs}(w+1) \le \sqrt{2} \end{cases}$

Предыдущие попытки:

$$y = \alpha x$$
$$\begin{cases} xu + u - \alpha x v = \alpha x \\ \alpha x u + x v + v + x = 1 \end{cases} \Rightarrow \begin{cases} x(u - \alpha v - \alpha) = -u \\ x(\alpha u + v + 1) = 1 - v \end{cases} \Rightarrow$$
$$(1-v)(u-\alpha v-\alpha) = -u(\alpha u + v + 1)$$
$$u - \cancel{\alpha v} - \alpha \cancel{-vu} + \alpha v^2 \cancel{+\alpha v} = -\alpha u^2 \cancel{-uv} - u$$
$$\alpha v^2 + \alpha u^2 + 2u = \alpha$$
$$v^2 + \left(u + \frac{1}{\alpha}\right)^2 = 1 + \frac{1}{\alpha^2}$$
$$\alpha = 1: \;\; v^2 + (u+1)^2 = 2$$
$$\alpha = \frac{1}{2}: \;\; v^2 + (u+2)^2 = 5$$

$$\omega = \operatorname{tg}(x+yi)$$

$$\begin{cases} y \in \mathbb{R} \\ \frac{\operatorname{arctg}\frac{1}{2}}{2} \leq x \leq \frac{\pi}{8} \\ \omega = \operatorname{tg}(x+iy) \\ \omega = u+vi \end{cases} \Rightarrow \begin{cases} y \in \mathbb{R} \\ \frac{\operatorname{arctg}\frac{1}{2}}{2} \leq x \leq \frac{\pi}{8} \\ u+vi = \frac{\sin(x+yi)}{\cos(x+iy)} = \frac{\sin x \operatorname{ch} y + \cos x \operatorname{sh} y \cdot i}{\cos x \operatorname{ch} y - \sin x \operatorname{sh} y \cdot i} = \frac{\operatorname{tg} x + \operatorname{th} y \cdot i}{1 - \operatorname{tg} x \operatorname{th} y \cdot i} \end{cases} \Rightarrow$$

$$(1 - \operatorname{tg} x \operatorname{th} y \cdot i)(u+vi) = \operatorname{tg} x + \operatorname{th} y \cdot i$$

$$u + \operatorname{tg} x \operatorname{th} y\, v + i(v - \operatorname{tg} x \operatorname{th} y\, u) = \operatorname{tg} x + \operatorname{th} y\, i$$

$$\begin{cases} u + \operatorname{tg} x \operatorname{th} y v = \operatorname{tg} x \\ v - \operatorname{tg} x \operatorname{th} y u = \operatorname{th} y \end{cases} \Rightarrow \begin{cases} u + v = \operatorname{tg} x + \operatorname{th} y \\ u - v + \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) = \operatorname{tg} x - \operatorname{th} y \end{cases} \Rightarrow$$

$$\begin{cases} u = \operatorname{tg} x - \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \\ v = \operatorname{th} y + \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \end{cases}$$

$$\begin{cases} y \in R \\ \frac{\operatorname{arctg}\frac{1}{2}}{2} \leq x \leq \frac{\pi}{8} \\ u = \operatorname{tg} x - \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \\ v = \operatorname{th} y + \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \end{cases} \Rightarrow \begin{cases} \theta \in [-1,1] \\ \operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right) \leq \xi \leq \operatorname{tg}\frac{\pi}{8} \\ u = \xi - \xi\theta(\xi+\theta) \\ v = \theta + \xi\theta(\xi+\theta) \end{cases}$$

$$\eta = \operatorname{tg}(2\operatorname{arctg}(\xi))$$

$$\xi = \operatorname{tg}\left(\frac{\operatorname{arctg}\eta}{2}\right)$$

$$\begin{cases} -1 \leq \theta \leq 1 \\ \frac{1}{2} \leq \eta \leq 1 \\ u = \operatorname{tg}\left(\frac{\operatorname{arctg}\eta}{2}\right) - \operatorname{tg}\frac{\operatorname{arctg}\eta}{2}\theta\left(\operatorname{tg}\frac{\operatorname{arctg}\eta}{2} + \theta\right) \\ v = \theta + \operatorname{tg}\frac{\operatorname{arctg}\eta}{2}\theta\left(\operatorname{tg}\frac{\operatorname{arctg}\eta}{2} + \theta\right) \end{cases}$$

$$\operatorname{tg} 2x = \frac{2\operatorname{tg} x}{1 - \operatorname{tg}^2 x}$$

$$(1-t^2)p = 2t$$

$$t^2 + \frac{2}{p}t - 1 = 0$$

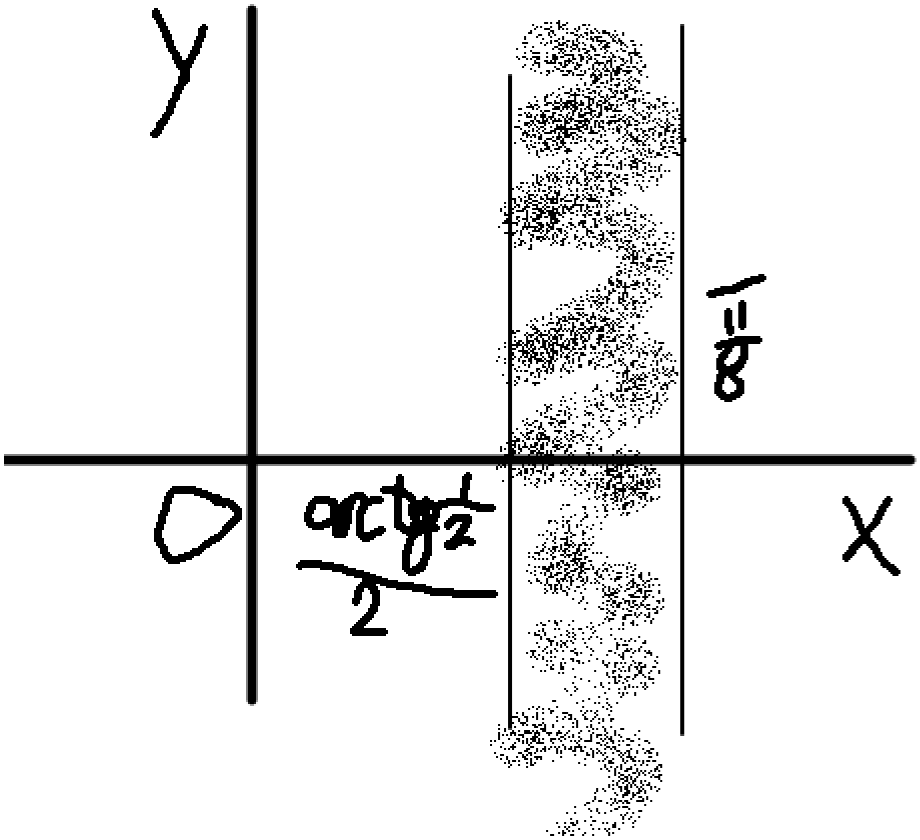
$$\frac{D}{4} = \frac{1}{p^2} + 1 = \frac{1+p^2}{p^2}$$

$$t = -\frac{1 \pm \sqrt{1+p^2}}{p} \Rightarrow \operatorname{tg}\frac{x}{2} = \frac{\sqrt{1+\operatorname{tg}^2 x} - 1}{\operatorname{tg} x}$$

$$\frac{\sqrt{1+\frac{1}{4}} - 1}{\frac{1}{2}} = \sqrt{5} - 2$$

$$\frac{\sqrt{1+1} - 1}{1} = \sqrt{2} - 1$$

$$\begin{cases} -1 \leq \theta \leq 1 \\ \sqrt{5} - 2 \leq \xi \leq \sqrt{2} - 1 \\ u = \xi - \xi\theta(\xi+\theta) \\ v = \theta + \xi\theta(\xi+\theta) \end{cases}$$



Из графических соображений понятно, что образ искомого множества лежит между кривыми, которые являются образами его границы.

$$\operatorname{tg}(\alpha+\beta i)=\frac{\sin(\alpha)\cos(\beta i)+\sin(\beta i)\cos(\alpha)}{\cos(\alpha)\cos(\beta i)-\sin(\alpha)\sin(\beta i)}=\frac{\sin\alpha\operatorname{ch}\beta+\cos\alpha\operatorname{sh}\beta i}{\cos\alpha\operatorname{ch}\beta-\sin\alpha\operatorname{sh}\beta i}=\frac{\operatorname{tg}\alpha+\operatorname{th}\beta i}{1-\operatorname{tg}\alpha\operatorname{th}\beta i}$$

$$\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}+yi\right)=\frac{\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right)+\operatorname{th}yi}{1-\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right)\operatorname{th}yi}=u+vi$$

$$T=\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right),\ \theta=\operatorname{th}y$$

$$\frac{T+\theta i}{1-T\theta i}=u+vi\Rightarrow T+\theta i=u+T\theta v+i(v-T\theta u)$$

$$\begin{cases}u+\theta Tv=T\\ v-\theta Tu=\theta\end{cases}\Rightarrow\begin{cases}u+v=\theta+T\\ u-v=-2\theta T(\theta+T)+T-\theta\end{cases}\Rightarrow\begin{cases}u=T-\theta T(\theta+T)\\ v=\theta+\theta T(\theta+T)\end{cases}$$

$$\operatorname{tg}2x=\frac{2\operatorname{tg}x}{1-\operatorname{tg}^2x}$$

$$(1-t^2)p=2t$$

$$t^2+\frac{2}{p}t-1=0$$

$$\frac{D}{4}=\frac{1}{p^2}+1=\frac{1+p^2}{p^2}$$

$$t=-\frac{1\pm\sqrt{1+p^2}}{p}\Rightarrow\operatorname{tg}\frac{x}{2}=\frac{\sqrt{1+\operatorname{tg}^2x}-1}{\operatorname{tg}x}$$

$$\frac{\sqrt{1+\frac{1}{4}}-1}{\frac{1}{2}}=\sqrt{5}-2$$

$$\frac{\sqrt{1+1}-1}{1}=\sqrt{2}-1$$

$$\begin{cases} u + \theta T v = T \\ v - \theta T u = \theta \end{cases} \Rightarrow \begin{cases} \theta = \frac{T-u}{Tv} \\ \theta = \frac{v}{1+Tu} \end{cases} \Rightarrow (T-u)(1+Tu) = Tv^2$$
$$T - u + T^2 u - Tu^2 = Tv^2$$
$$Tv^2 + Tu^2 + (1-T^2)u = T$$
$$v^2 + u^2 + 2 \cdot \frac{1-T^2}{2T} u + \frac{(1-T^2)^2}{4T^2} = T + \frac{(1-T^2)^2}{4T^2}$$
$$v^2 + \left(u + \frac{1-T^2}{2T}\right)^2 = T + \frac{(1-T^2)^2}{4T^2}$$
$$T = \sqrt{5} - 2 : T^2 = 9 - 4\sqrt{5}$$
$$v^2 + \left(u + \frac{-8+4\sqrt{5}}{2\sqrt{5}-4}\right)^2 = \sqrt{5} - 2 + 2^2$$
$$v^2 + (u+2)^2 = \sqrt{5} + 2$$
$$T = \sqrt{2} - 1 : T^2 = 3 - 2\sqrt{2}$$
$$v^2 + \left(u + \frac{-2+2\sqrt{2}}{2\sqrt{2}-2}\right)^2 = \sqrt{2} - 1 + 1^2$$
$$v^2 + (u+1)^2 = \sqrt{2}$$

2 окружности с центрами в $(-2, 0)$ и $(-1, 0)$ и радиусами $\sqrt{\sqrt{5}+2}$ и $\sqrt{\sqrt{2}}$.

Задание 5

Условие:

$$\omega = \operatorname{Artg} z$$
$$z_0 = \frac{4}{3} i$$

Решение:

$$\operatorname{tg}\omega = \frac{\sin\omega}{\cos\omega} = \frac{e^{i\omega} - e^{-i\omega}}{i(e^{i\omega} + e^{-i\omega})}$$

$$e^{i\omega} = q, \operatorname{tg} w = t$$

$$t = \frac{q - \frac{1}{q}}{i\left(q + \frac{1}{q}\right)}$$

$$it\left(q + \frac{1}{q}\right) = q - \frac{1}{q}$$

$$q(it-1) + \frac{1}{q}(it+1) = 0$$

$$q^2 = -\frac{it+1}{it-1}$$

$$q = \left(-\frac{it+1}{it-1}\right)^{\frac{1}{2}} = e^{i\omega}$$

$$\omega = \frac{1}{i}\left(\operatorname{Ln}\left(\left(-\frac{it+1}{it-1}\right)^{\frac{1}{2}}\right)\right)$$

$$t = \frac{4}{3}i \Rightarrow$$

$$\omega = \frac{1}{i}\left(\operatorname{Ln}\left(\left(-\frac{1-\frac{4}{3}}{-1-\frac{4}{3}}\right)^{\frac{1}{2}}\right)\right) =$$

$$= -i\operatorname{Ln}\left[\left(-\frac{1}{7}\right)^{\frac{1}{2}}\right] = -\frac{i}{2}\operatorname{Ln}\left(-\frac{1}{7}\right) =$$

$$l = \operatorname{Ln}\left(-\frac{1}{7}\right) = \ln\left(\frac{1}{7}\right) + \pi i + 2\pi k i, k \in \mathbb{Z}$$

$$= -\frac{i}{2}(-\ln 7 + \pi i + 2\pi k i) = \frac{\ln 7}{2}i + \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$$

Прошлые попытки:

$$\omega = \operatorname{Artg} z_0 \Leftrightarrow \operatorname{tg}\omega = z_0$$

$$\begin{cases} \omega = u + vi \\ z_0 = \frac{4}{3}i \end{cases} \Rightarrow \operatorname{tg}(u+vi) = \frac{4}{3}i$$

$$\frac{\sin(u)\cos(vi) + \sin(vi)\cos(u)}{\cos(u)\cos(vi) - \sin(u)\sin(vi)} = \frac{\operatorname{tg} u + \operatorname{th} vi}{1 - \operatorname{tg} u \operatorname{th} vi} = \frac{4}{3}i$$

$$\operatorname{tg} u + \operatorname{th} vi = \frac{4}{3}\operatorname{tg} u \operatorname{th} v + \frac{4}{3}i$$

$$\begin{cases} \operatorname{tg} u = \frac{4}{3}\operatorname{tg} u \operatorname{th} v \\ \operatorname{th} v = \frac{4}{3} \end{cases} \Rightarrow \begin{cases} \operatorname{tg} u\left(\frac{16}{9} - 1\right) = 0 \\ \operatorname{th} v = \frac{4}{3} \end{cases} \Rightarrow \begin{cases} \operatorname{tg} u = 0 \\ \operatorname{th} v = \frac{4}{3} \end{cases} \Leftrightarrow \begin{cases} u = \pi k, k \in \mathbb{Z} \\ v = \operatorname{arth}\left(\frac{4}{3}\right) \end{cases}$$

Задание 6

Условие:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (z - 2 + 2i)^n}{3^n \left(1 + \frac{1}{n}\right)^n}$$

Решение:

$$q = z - 2 + 2i$$
$$\cdot \sum_{n=0}^{\infty} \frac{|q|^n}{3^n \left(1 + \frac{1}{n}\right)^n}$$
$$a_n = \frac{|q|^n}{3^n \left(1 + \frac{1}{n}\right)^n}$$
$$\frac{a_{n+1}}{a_n} = \frac{|q|^{n+1}}{3^{n+1} \left(1 + \frac{1}{n+1}\right)^{n+1}} \cdot \frac{3^n \left(1 + \frac{1}{n}\right)^n}{|q|^n} = \frac{|q|}{3} \cdot \frac{1}{1 + \frac{1}{n+1}} \cdot \left(\frac{1 + \frac{1}{n}}{1 + \frac{1}{n+1}}\right)^n = \frac{|q|}{3} \cdot \frac{1}{1 + \frac{1}{n+1}} \cdot \left(\frac{(n+1)^2}{n(n+2)}\right)^n =$$
$$= \frac{|q|}{3} \cdot \frac{1}{1 + \frac{1}{n+1}} \cdot \left(1 + \frac{1}{n^2 + 2n}\right)^{(n^2+2n) \cdot \frac{1}{n+2}}$$
$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{|q|}{3} \cdot \frac{1}{1 + \frac{1}{n+1}} \cdot e^{\frac{1}{n+2}} = \frac{|q|}{3} \Rightarrow |q| < 3 \text{ - область абсолютной сходимости}$$
$$|q| = 3:$$
$$\sum_{n=0}^{\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$
$$b_n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$
$$\lim_{n \to \infty} b_n = \frac{1}{e} \neq 0 \Rightarrow \text{не сходится даже условно}$$
$$3^2 = 9$$
$$2^2 + 2^2 = 8 < 9$$
$$3^2 + 0 = 9$$
$$3^2 + 1^2 > 9$$

Задание 7

Условие:

$$\oint_{|z|=8} \frac{\sin z}{z^4 (z + 4i)} dz$$

Решение:

$$f(z) = \frac{\sin z}{z^4(z+4i)}$$

Особые точки:

$$z = 0 - \text{полюс } n = 3$$

$$z = -4i \text{ - полюс } n = 1$$

$$\oint_L f(z)dz = -2\pi i \cdot \operatorname*{Res}_{z=\infty} f$$

$$\operatorname*{Res}_{z=\infty} f:$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\frac{1}{z+4i} = \sum_{n=0}^{\infty} \frac{(-4i)^n}{z^{n+1}}$$

$$f = \frac{1}{z^4}\left[\sum_{n=0}^{\infty} \frac{(-4i)^n}{z^{n+1}}\right]\left[\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\right]$$

$$\operatorname*{Res}_{z=\infty} f = -z \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \cdot \frac{1}{z^4} \cdot \frac{(-4i)^m}{z^{m+1}} =$$

$$2n + 1 - 4 - (m+1) = -1 \Rightarrow 2n - 3 - m = 0 \Rightarrow m = 2n - 3 \Rightarrow \min(n) = 2$$

$$= -z \sum_{n=2}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \cdot \frac{1}{z^4} \cdot \frac{(-4i)^{2n-3}}{z^{2n-2}} = -\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+5)!} \cdot (-4i)^{2n+1} =$$

$$= -\sum_{n=0}^{\infty} -4i \cdot \cancel{(-1)^n}\ \cancel{(-1)^{2n}}\ \cancel{i^{2n}}\ \cancel{(-1)^n} \cdot \frac{4^{2n}}{(2n+5)!} = i \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n+5)!} =$$

$$\operatorname{sh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \Rightarrow \operatorname{sh} 4 = \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n+1)!} = 4 + \frac{4^3}{6} + \sum_{n=2}^{\infty} \frac{4^{2n+1}}{(2n+1)!} \Rightarrow$$

$$\operatorname{sh} 4 - 4 - \frac{32}{3} = \sum_{n=0}^{\infty} \frac{4^{2n+5}}{(2n+5)!} = 4^4 \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n+5)!}$$

$$= \frac{i}{4^4}\left(\operatorname{sh} 4 - 4 - \frac{32}{3}\right) = i\left(\frac{\operatorname{sh} 4}{256} - \frac{11}{64 \cdot 3}\right)$$

$$\oint_L f(z)dz = -2\pi i i\left(\frac{\operatorname{sh} 4}{256} - \frac{11}{64 \cdot 3}\right) = 2\pi\left(\frac{\operatorname{sh} 4}{256} - \frac{11}{64 \cdot 3}\right)$$

Прошлые попытки:

$$\operatorname*{Res}_{z=0} f = \frac{1}{6}\lim_{z\to 0}\left(\frac{\sin z}{z+4i}\right)'''$$

$$a' = \cos z$$

$$a'b = z\cos z + 4i\cos z$$

$$b' = 1$$

$$ab' = \sin z$$

$$a'b - ab' = z\cos z + 4i\cos z - \sin z$$

$$\left(\frac{z\cos z + 4i\cos z - \sin z}{z^2+8iz-16}\right)''$$

$$a' = \cos z - z\sin z - 4i\sin z - \cos z = -z\sin z - 4i\sin z$$

$$a'b = -z^3\sin z - 12iz^2\sin z + 48z\sin z + 16\sin z$$

$$b' = 2z+8i$$

$$ab' = 2z^2\cos z + 16iz\cos z - 32\cos z - 2z\sin z - 8i\sin z$$

$$a'b - ab' = -z^3\sin z - 12iz^2\sin z - 2z^2\cos z + 50z\sin z - 16iz\cos z + (16+8i)\sin z + 32\cos z$$

$$\left(\frac{-z^3\sin z - 12iz^2\sin z - 2z^2\cos z + 50z\sin z - 16iz\cos z + (16+8i)\sin z + 32\cos z}{z^4+16iz^3-96z^2-256iz+256}\right)$$

$$a' = \cancel{-3z^2\sin z} - \cancel{z^3\cos z} - \cancel{24iz\sin z} - \cancel{12iz^2\cos z} - \cancel{4z\cos z} + \cancel{2z^2\cos z} + \cancel{50\sin z} + \cancel{50z\cos z}$$

$$-\cancel{16i\cos z} + \cancel{16iz\sin z} + \cancel{(16+8i)\cos z} - \cancel{32\sin z} =$$

$$(0)z^3\sin z + (-1)z^3\cos z + (-3)z^2\sin z + (-12i+2)z^2\cos z +$$

$$(-24i+16i)z\sin z + (-4+50)z\cos z + (50-32)\sin z + (-16i+16+8i)\cos z =$$

$$\cancel{-z^3\cos z} - \cancel{3z^2\sin z} + \cancel{(-12i+2)z^2\cos z} + (-24i+16i)z\sin z + 46z\cos z + 18\sin z + (16-8i)\cos z$$

$$a'b =$$

$$(-1)z^7\cos z +$$

$$(-16i-12i+2)z^6\cos z +$$

$$(+96-192i^2+32i+46)z^5\cos z +$$

$$(+256i+1152i-192+736i_16-8i)z^4\cos z +$$

$$(-256+3072i^2-512i-4416+256i-128i^2)z^3\cos z +$$

$$(-3072i+512-11776i-1536+768i)z^2\cos z +$$

$$(11776-4096i+2048i^2)z\cos z +$$

$$(4096-2048i)\cos z +$$

$$(0)z^7\sin z +$$

$$(-3)z^6\sin z +$$

$$(-48i-24i+16)z^5\sin z +$$

$$(288-384i^2+256i+18)z^4\sin z +$$

$$(768i+2304i-1536+288i)z^3\sin z +$$

$$(-768+6144i^2-4096i-1728)z^2\sin z +$$

$$(-6144i+4096-4608i)z\sin z +$$

$$(4608)\sin z =$$

$$= -z^7\cos z +$$

$$(2-28i)z^6\cos z +$$

$$(334+32i)z^5\cos z +$$

$$(-176+2136i)z^4\cos z +$$

$$(-7616-256i)z^3\cos z +$$

$$(-1024-14080i)z^2\cos z +$$

$$(9728-4096)z\cos z +$$

$$(4096-2048)\cos z +$$

$$0+$$

$$-3z^6\sin z +$$

$$(16-72i)z^5\sin z +$$

$$(690+256i)z^4\sin z +$$

$$(-1536+3360i)z^3\sin z +$$

$$(-8640-4096i)z^2\sin z +$$

$$(4096-10752)z\sin z +$$

$$4608\sin z$$

....

Задание 8

Условие:

$$\int_{C:\begin{cases} t\in[0,\pi] \\ x=\pi-t \\ y=t\end{cases}} \cos\overline{z}dz$$

Решение:

$$\int_{C:\begin{cases}t\in[0,\pi]\\x=\pi-t\\y=t\end{cases}}\cos\overline{z}dz=\int_0^{\pi}\cos(\pi-t-ti)(-1+i)dt=$$

$$z=\pi-t+it$$
$$\overline{z}=\pi-t-ti$$
$$dz=(-1+i)dt$$
$$\cos(\pi-\alpha)=-\cos\alpha$$

$$=-\int_0^{\pi}\cos((1+i)t)\frac{-1+i}{i+1}d((1+i)t)=-\frac{-1+i}{i+1}(\sin((1+i)\pi)-0)=\frac{(1-i)(i-1)}{2}(-i\,\mathrm{sh}\,\pi)=$$

$$\sin(\pi+\alpha)=-\sin\alpha$$
$$\sin(i\alpha)=i\,\mathrm{sh}\,\alpha$$
$$=\mathrm{sh}\,\pi$$

Задание 9

Условие:

$$\int_{C:\text{ первая часть эллипса } |z-3-6i|+|z-3+2i|=10}(\overline{z}-3+2i)dz$$

Решение:

$$\text{Фокусы эллипса: } z_1=3+6i, z_2=3-2i$$
$$c=\frac{|z_1-z_2|}{2}=4$$
$$a=7-6+4=5, b=\sqrt{a^2-c^2}=3$$
$$\begin{cases}x=3+3\cos t\\y=2+5\sin t\end{cases}$$
$$\text{Первая часть} \Rightarrow t\in\left[0,\frac{\pi}{2}\right]$$
$$z=3+3\cos t+i(2+5\sin t)$$
$$\overline{z}-3+2i=3\cos t-i(5\sin t)$$
$$dz=(-3\sin t+5i\cos t)dt$$
$$\int_C f(z,\overline{z})dz=\int_0^{\frac{\pi}{2}}(3\cos t-5\sin ti)(-3\sin t+5\cos ti)dt=$$
$$=\int_0^{\frac{\pi}{2}}[-9\cos t\sin t+25\sin t\cos t+i(15\sin^2 t+15\cos^2 t)]dt=$$
$$=\int_0^{\frac{\pi}{2}}4\sin(2t)d(2t)+\frac{15\pi}{2}i=8+\frac{15\pi}{2}i$$

Задание 10

Условие:

$$\int_{C:\text{ -верхняя половина окружности } |z-3i|=5}(z\overline{z}+i)dz$$

Решение:

$$C: \begin{cases} x = 5\cos t \\ y = 3 + 5\sin t \\ t \in [0, \pi] \end{cases} \Rightarrow \begin{matrix} z = 5\cos t + (3 + 5\sin t)i \\ dz = (-5\sin t + 5\cos t i)dt \\ z\overline{z} = 25 + 9 + 30\sin t \end{matrix}$$

$$\int_C (z\overline{z} + i)dz = \int_0^\pi (34 + 30\sin t + i)(-5\sin t + 5\cos t i)dt =$$

$$= \int_0^\pi [-170\sin t - 5\cos t - 150\sin^2 t + i(170\cos t - 5\sin t + 150\sin t\cos t)]dt =$$

$$= -340 + 0 - 75\pi + i(0 - 10 + 0) = -75\pi - 340 - 10i$$

$$\int_0^\pi \sin^2 t dt = \frac{\pi}{2}$$

$$\int_0^\pi \sin t \cos t dt = 0$$

Ответ: $-75\pi - 340 - 10i$

Задание 11

Условие:

$$\int_{\frac{\pi}{3} \to \ln_5} z \operatorname{sh} z dz$$

Решение:

$$(z \operatorname{ch} z)' = \operatorname{ch} z + z \operatorname{sh} z$$

$$(z \operatorname{ch} z - \operatorname{sh} z)' = \operatorname{ch} z + z \operatorname{sh} z - \operatorname{ch} z = z \operatorname{sh} z$$

$$\int_C z \operatorname{sh} z dz = z \operatorname{ch} z - \operatorname{sh} z|_{\frac{i\pi}{3}}^{\ln 5} = \ln 5 \cdot \frac{e^{\ln 5} + e^{-\ln 5}}{2} - \frac{e^{\ln 5} - e^{-\ln 5}}{2} - \frac{i\pi}{3} \operatorname{ch}\left(\frac{i\pi}{3}\right) + \operatorname{sh}\left(\frac{i\pi}{3}\right) =$$

$$= \ln 5 \cdot \frac{1}{2}\left(5 + \frac{1}{5}\right) - \frac{1}{2}\left(5 - \frac{1}{5}\right) - \frac{i\pi}{3} \cdot \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} =$$

$$= \frac{\ln 5}{2}\left(\frac{26}{5}\right) - \frac{12}{5} + i\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$$