Задание 1

Условие:

$$f(z) = \frac{1}{z(z+2)}, z_0 = -1$$

Решение:

Т.к. -2 и 0 симметричны относительно -1, необходимо рассмотреть 2 области

$$z - z_0 = q$$
$$f(z) = \frac{1}{(z+1-1)(z+1+1)} = \frac{1}{(q-1)(q+1)} = \frac{1}{q^2-1} = -\frac{1}{1-q^2}$$
$$t = q^2$$
$$f(z) = -\frac{1}{1-t} = -\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} -t^n = \sum_{n=0}^{\infty} -q^{2n} = \sum_{n=0}^{\infty} -(z+1)^{2n}$$
$$\text{Для } |z+1| > 1:$$
$$f(z) = \frac{1}{z(z+2)} = \frac{1}{(q-1)(q+1)} = \frac{1}{2}\frac{1}{q-1} - \frac{1}{2}\frac{1}{1+q} =$$
$$\frac{1}{q-a} = \frac{\frac{1}{q}}{1-\frac{a}{q}} = \frac{1}{q} \cdot \frac{1}{1-\frac{a}{q}} = \frac{1}{q}\sum_{n=0}^{\infty}\left(\frac{a}{q}\right)^n = \sum_{n=0}^{\infty}\frac{a^n}{q^{n+1}}$$
$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{q^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{q^{n+1}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1-(-1)^n}{q^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{2}\cdot\frac{1-(-1)^n}{(z+1)^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{2(n+1)}}$$

Задание 2

Условие:

$$f(z) = z^3 \cos\frac{1}{z^2}$$

Решение:

$$\text{Особые точки:}$$
$$z = 0 : \lim_{z \to 0} z^3 \cos\frac{1}{z^2} = \nexists$$
$$z^3 \cos\frac{1}{z^2} = z^3 \cdot \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{z^2}\right)^{2n}}{(2n)!} = z^3 \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)! z^{4n}} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)! z^{4n-3}}$$
$$c_{-1} = -1 \cdot \frac{1}{2!} = -\frac{1}{2}$$
$$\operatorname*{Res}_{z=0} f = -\frac{1}{2}$$
$$z = \infty : \text{ т.к. сумма всех вычетов равна 0, то} \operatorname*{Res}_{z=\infty} f = \frac{1}{2}$$
$$\text{Очевидно, что бесконечность - это полюс 3-его порядка}$$

Задание 3

Условие:

$$y=-x$$
$$w=\frac{z+1}{z-1}$$

Решение:

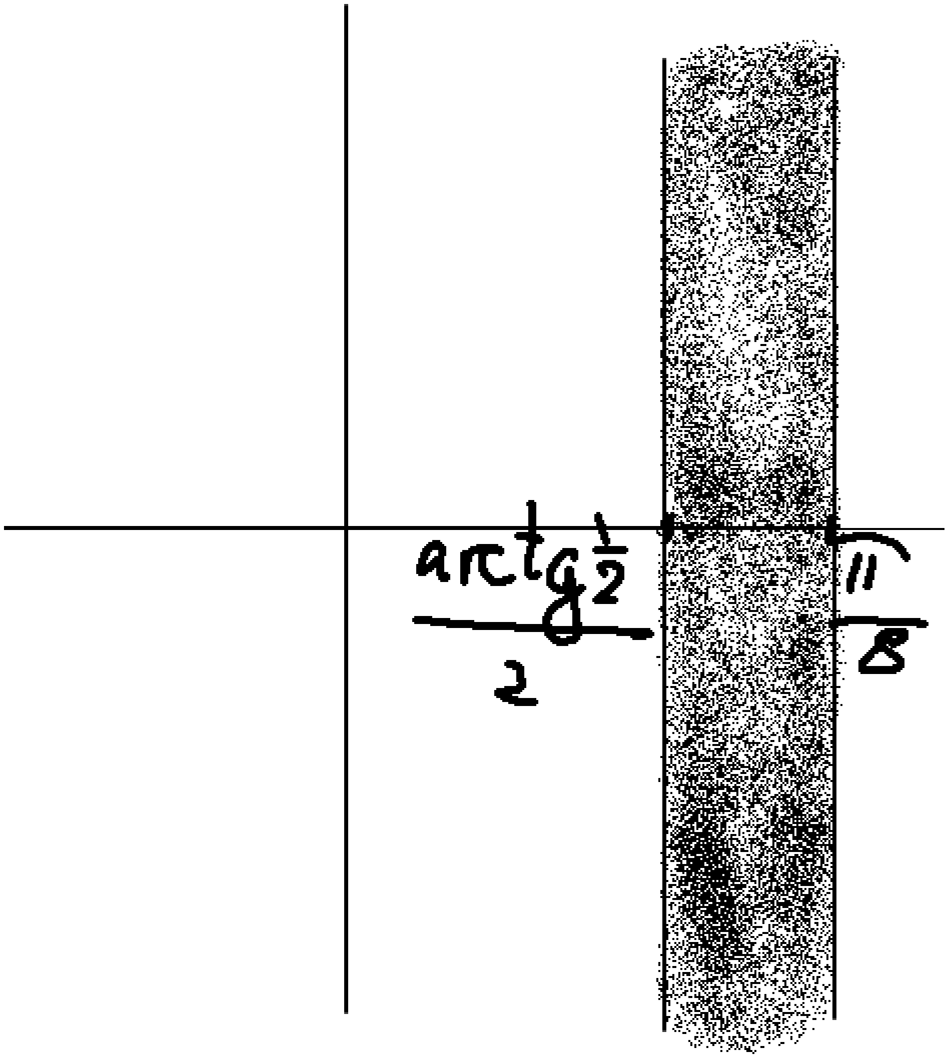
$$w(z-1)=z+1$$
$$(u+vi)(x+yi-1)=x+yi+1$$
$$ux-u-vy+i(vx-v+uy)=x+1+iy$$
$$\begin{cases} ux-u-vy=x+1 \\ vx-v+uy=y \\ y=-x \end{cases} \Rightarrow$$
$$\begin{cases} ux-u+vx=x+1 \\ vx-v-ux=-x \end{cases} \Rightarrow \begin{cases} x(u+v-1)=1+u \\ x(v-u+1)=v \end{cases} \Rightarrow (1+u)(v-u+1)=v(u+v-1) \Rightarrow$$
$$v \cancel{-u}+1 \cancel{+uv}-u^2 \cancel{+u} = \cancel{vu}+v^2-v$$
$$1=u^2+v^2-2v$$
$$2=u^2+(v-1)^2-\text{окружность}$$

Задание 4

Условие:

$$\frac{\operatorname{arctg}\frac{1}{2}}{2} \leq \operatorname{Re} z \leq \frac{\pi}{8}, \quad \omega=\operatorname{tg} z$$

Решение:

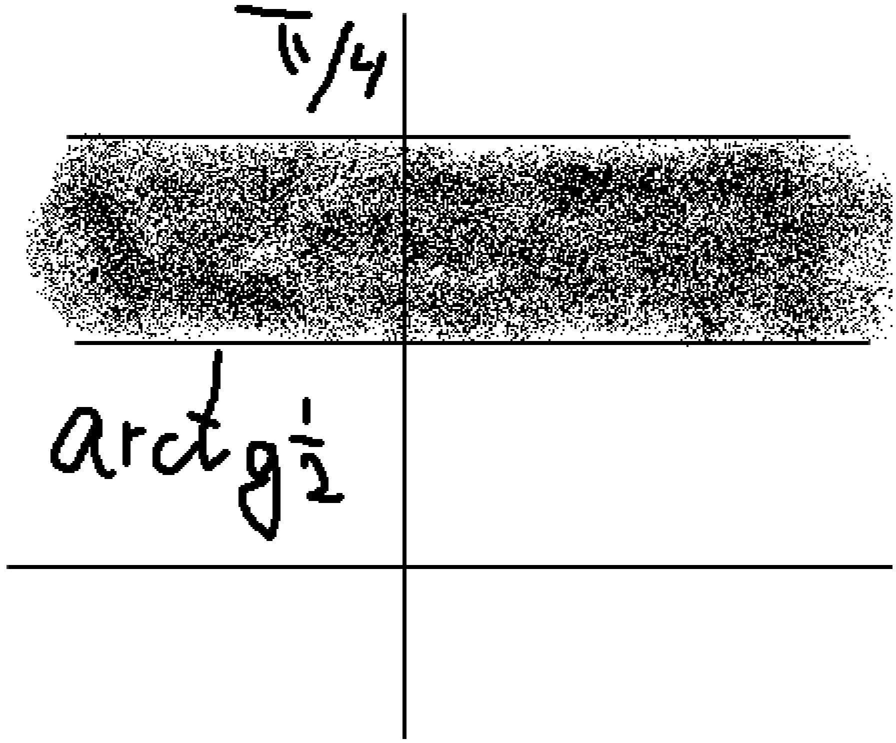


$$w = \operatorname{tg} z = \frac{e^{2iz} - 1}{i(e^{2iz} + 1)} -$$ композиция линейной, экспоненты и дробно-линейной:

$$w_1 = 2iz$$
$$w_2 = e^{w_1}$$
$$w = -i \cdot \frac{w_2 - 1}{w_2 + 1}$$

w_1 : i повернёт полосу относительно $(0{,}0)$ против часовой на $\frac{\pi}{2}$

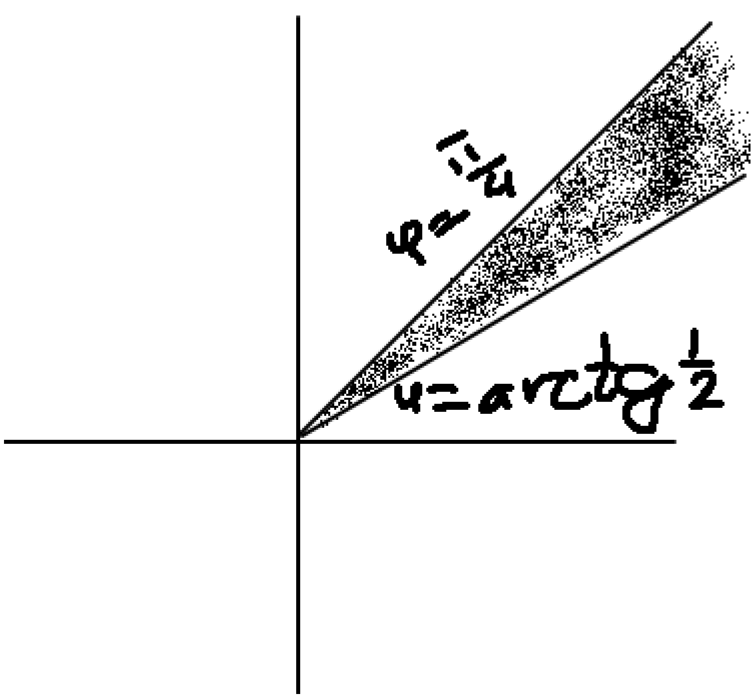
2 "растянет" в 2 раза $\Rightarrow$



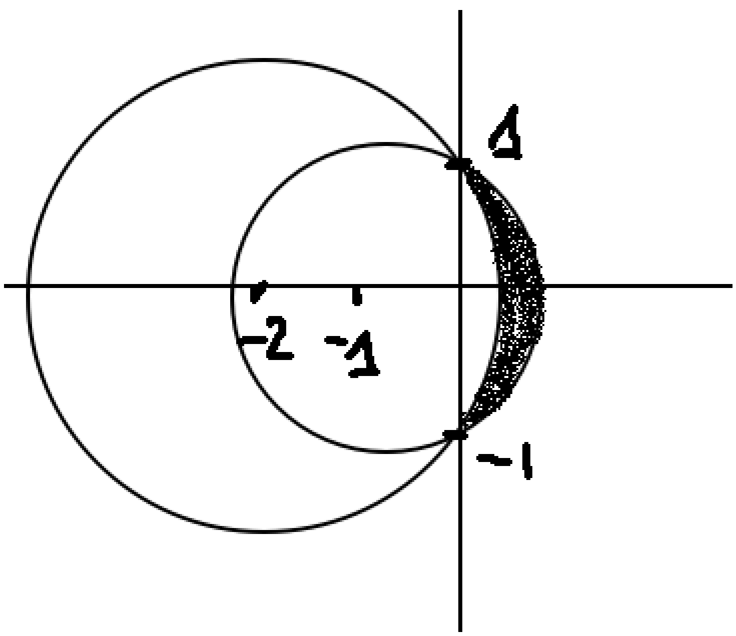
$$w_1:$$
$$\operatorname{arctg}\frac{1}{2} \le \mathrm{Im} w_1 \le \frac{\pi}{4}$$
$$w_2:$$
$$w_2 = \exp(w_1) = \exp(\mathrm{Re} w_1 + i\mathrm{Im} w_1) = \exp(\mathrm{Re} w_1)\cdot\exp(i\mathrm{Im} w_1)$$

w_2 задаёт угол с следующим условием:

$$\operatorname{arctg}\frac{1}{2} \le \varphi \le \frac{\pi}{4}$$



$$w=-i\frac{w_2-1}{w_2+1}$$
$$w_2w+w=-iw_2+i$$
$$w_2(w+i)=i-w$$
$$w_2=\frac{i-w}{i+w}$$
$$r(\cos\varphi+i\sin\varphi)=\frac{i-x-iy}{i+x+iy}$$
$$r\cos\varphi+ir\sin\varphi=\frac{(-x+i(1-y))(x-i(y+1))}{x^2+(y+1)^2}$$
$$r\cos\varphi+ir\sin\varphi=\frac{-x^2+1-y^2}{x^2+(y+1)^2}+i\frac{x-xy+x+xy}{x^2+(y+1)^2}$$
$$\begin{cases}r\cos\varphi=\frac{1-x^2-y^2}{x^2+(y+1)^2}\\ r\sin\varphi=\frac{2x}{x^2+(y+1)^2}\end{cases}\Rightarrow\begin{cases}\operatorname{arctg}\frac{1}{2}\le\varphi\le\frac{\pi}{4}\\ \frac{2x}{\sin\varphi}\ge 0\\ \frac{1-x^2-y^2}{\cos\varphi}\ge 0\\ \operatorname{tg}\varphi=\frac{2x}{1-x^2-y^2}\end{cases}\stackrel{\substack{\sin\varphi>0\\ \cos\varphi>0}}{\Rightarrow}\begin{cases}\frac{1}{2}\le\frac{2x}{1-x^2-y^2}\le 1\\ x\ge 0\\ 1-x^2-y^2\ge 0\end{cases}\Rightarrow$$
$$\begin{cases}x\ge 0\\ 1-x^2-y^2\le 4x\\ 2x\le 1-x^2-y^2\end{cases}\Rightarrow\begin{cases}x\ge 0\\ (x+2)^2+y^2\ge 5=\sqrt{5}^2\\ (x+1)^2+y^2\le 2=\sqrt{2}^2\end{cases}$$



Ответ: $\begin{cases}\operatorname{abs}(w+2)\ge\sqrt{5}\\ \operatorname{abs}(w+1)\le\sqrt{2}\end{cases}$

Предыдущие попытки:

$$y=\alpha x$$
$$\begin{cases}xu+u-\alpha xv=\alpha x\\ \alpha xu+xv+v+x=1\end{cases}\Rightarrow\begin{cases}x(u-\alpha v-\alpha)=-u\\ x(\alpha u+v+1)=1-v\end{cases}\Rightarrow$$
$$(1-v)(u-\alpha v-\alpha)=-u(\alpha u+v+1)$$
$$u-\cancel{\alpha v}-\alpha\cancel{-vu}+\alpha v^2\cancel{+\alpha v}=-\alpha u^2\cancel{-uv}-u$$
$$\alpha v^2+\alpha u^2+2u=\alpha$$
$$v^2+\left(u+\frac{1}{\alpha}\right)^2=1+\frac{1}{\alpha^2}$$
$$\alpha=1:\ \ v^2+(u+1)^2=2$$
$$\alpha=\frac{1}{2}:\ \ v^2+(u+2)^2=5$$

$$\omega = \operatorname{tg}(x+yi)$$

$$\begin{cases} y\in\mathbb{R} \\ \frac{\operatorname{arctg}\frac{1}{2}}{2} \le x \le \frac{\pi}{8} \\ \omega = \operatorname{tg}(x+iy) \\ \omega = u+vi \end{cases} \Rightarrow \begin{cases} y\in\mathbb{R} \\ \frac{\operatorname{arctg}\frac{1}{2}}{2} \le x \le \frac{\pi}{8} \\ u+vi = \frac{\sin(x+yi)}{\cos(x+iy)} = \frac{\sin x \operatorname{ch} y + \cos x \operatorname{sh} y\cdot i}{\cos x \operatorname{ch} y - \sin x \operatorname{sh} y \cdot i} = \frac{\operatorname{tg} x + \operatorname{th} y\cdot i}{1-\operatorname{tg} x \operatorname{th} y \cdot i} \end{cases} \Rightarrow$$

$$(1-\operatorname{tg} x \operatorname{th} y\cdot i)(u+vi) = \operatorname{tg} x + \operatorname{th} y \cdot i$$

$$u + \operatorname{tg} x \operatorname{th} y\, v + i(v - \operatorname{tg} x \operatorname{th} y\, u) = \operatorname{tg} x + \operatorname{th} y\, i$$

$$\begin{cases} u + \operatorname{tg} x \operatorname{th} y v = \operatorname{tg} x \\ v - \operatorname{tg} x \operatorname{th} y u = \operatorname{th} y \end{cases} \Rightarrow \begin{cases} u + v = \operatorname{tg} x + \operatorname{th} y \\ u - v + \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) = \operatorname{tg} x - \operatorname{th} y \end{cases} \Rightarrow$$

$$\begin{cases} u = \operatorname{tg} x - \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \\ v = \operatorname{th} y + \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \end{cases}$$

$$\begin{cases} y \in R \\ \frac{\operatorname{arctg}\frac{1}{2}}{2} \le x \le \frac{\pi}{8} \\ u = \operatorname{tg} x - \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \\ v = \operatorname{th} y + \operatorname{tg} x \operatorname{th} y(\operatorname{tg} x + \operatorname{th} y) \end{cases} \Rightarrow \begin{cases} \theta \in [-1, 1] \\ \operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right) \le \xi \le \operatorname{tg}\frac{\pi}{8} \\ u = \xi - \xi\theta(\xi+\theta) \\ v = \theta + \xi\theta(\xi+\theta) \end{cases}$$

$$\eta = \operatorname{tg}(2\operatorname{arctg}(\xi))$$

$$\xi = \operatorname{tg}\left(\frac{\operatorname{arctg}\eta}{2}\right)$$

$$\begin{cases} -1 \le \theta \le 1 \\ \frac{1}{2} \le \eta \le 1 \\ u = \operatorname{tg}\left(\frac{\operatorname{arctg}\eta}{2}\right) - \operatorname{tg}\frac{\operatorname{arctg}\eta}{2}\theta\left(\operatorname{tg}\frac{\operatorname{arctg}\eta}{2} + \theta\right) \\ v = \theta + \operatorname{tg}\frac{\operatorname{arctg}\eta}{2}\theta\left(\operatorname{tg}\frac{\operatorname{arctg}\eta}{2} + \theta\right) \end{cases}$$

$$\operatorname{tg} 2x = \frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}$$

$$(1-t^2)p = 2t$$

$$t^2 + \frac{2}{p}t - 1 = 0$$

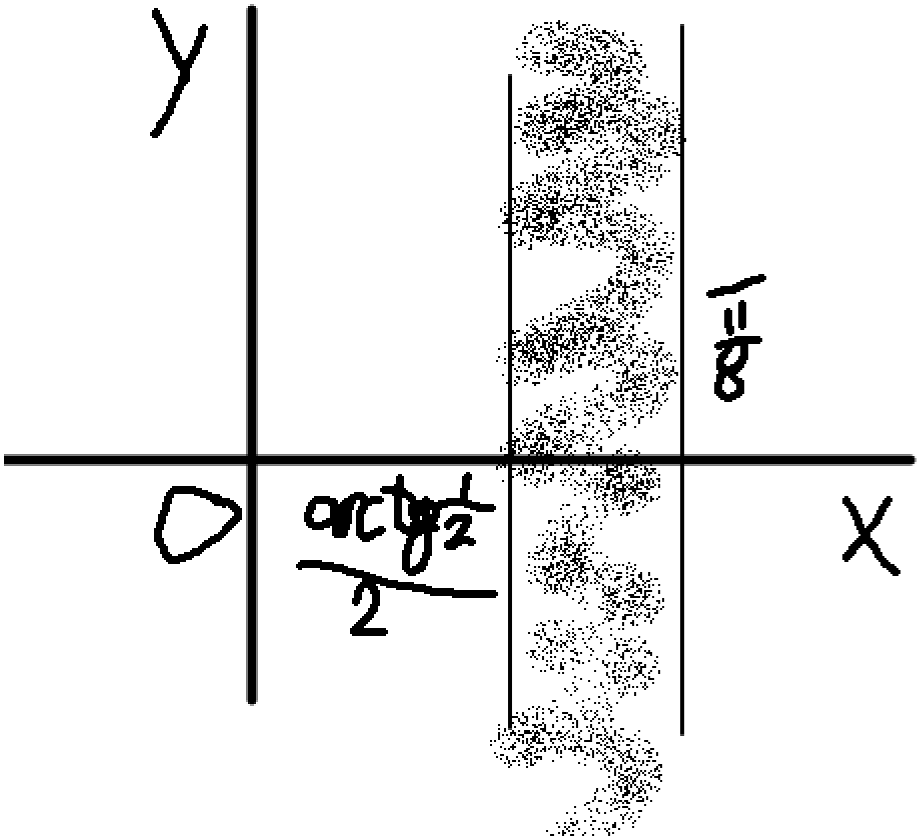
$$\frac{D}{4} = \frac{1}{p^2} + 1 = \frac{1+p^2}{p^2}$$

$$t = -\frac{1 \pm \sqrt{1+p^2}}{p} \Rightarrow \operatorname{tg}\frac{x}{2} = \frac{\sqrt{1+\operatorname{tg}^2 x} - 1}{\operatorname{tg} x}$$

$$\frac{\sqrt{1+\frac{1}{4}} - 1}{\frac{1}{2}} = \sqrt{5} - 2$$

$$\frac{\sqrt{1+1} - 1}{1} = \sqrt{2} - 1$$

$$\begin{cases} -1 \le \theta \le 1 \\ \sqrt{5} - 2 \le \xi \le \sqrt{2} - 1 \\ u = \xi - \xi\theta(\xi+\theta) \\ v = \theta + \xi\theta(\xi+\theta) \end{cases}$$



Из графических соображений понятно, что образ искомого множества лежит между кривыми, которые являются образами его границы.

$$\operatorname{tg}(\alpha+\beta i)=\frac{\sin(\alpha)\cos(\beta i)+\sin(\beta i)\cos(\alpha)}{\cos(\alpha)\cos(\beta i)-\sin(\alpha)\sin(\beta i)}=\frac{\sin\alpha\operatorname{ch}\beta+\cos\alpha\operatorname{sh}\beta i}{\cos\alpha\operatorname{ch}\beta-\sin\alpha\operatorname{sh}\beta i}=\frac{\operatorname{tg}\alpha+\operatorname{th}\beta i}{1-\operatorname{tg}\alpha\operatorname{th}\beta i}$$

$$\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}+yi\right)=\frac{\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right)+\operatorname{th}yi}{1-\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right)\operatorname{th}yi}=u+vi$$

$$T=\operatorname{tg}\left(\frac{\operatorname{arctg}\frac{1}{2}}{2}\right),\ \theta=\operatorname{th}y$$

$$\frac{T+\theta i}{1-T\theta i}=u+vi\Rightarrow T+\theta i=u+T\theta v+i(v-T\theta u)$$

$$\begin{cases}u+\theta Tv=T\\ v-\theta Tu=\theta\end{cases}\Rightarrow\begin{cases}u+v=\theta+T\\ u-v=-2\theta T(\theta+T)+T-\theta\end{cases}\Rightarrow\begin{cases}u=T-\theta T(\theta+T)\\ v=\theta+\theta T(\theta+T)\end{cases}$$

$$\operatorname{tg}2x=\frac{2\operatorname{tg}x}{1-\operatorname{tg}^2x}$$

$$(1-t^2)p=2t$$

$$t^2+\frac{2}{p}t-1=0$$

$$\frac{D}{4}=\frac{1}{p^2}+1=\frac{1+p^2}{p^2}$$

$$t=-\frac{1\pm\sqrt{1+p^2}}{p}\Rightarrow\operatorname{tg}\frac{x}{2}=\frac{\sqrt{1+\operatorname{tg}^2x}-1}{\operatorname{tg}x}$$

$$\frac{\sqrt{1+\frac{1}{4}}-1}{\frac{1}{2}}=\sqrt{5}-2$$

$$\frac{\sqrt{1+1}-1}{1}=\sqrt{2}-1$$

$$\begin{cases} u + \theta T v = T \\ v - \theta T u = \theta \end{cases} \Rightarrow \begin{cases} \theta = \frac{T-u}{Tv} \\ \theta = \frac{v}{1+Tu} \end{cases} \Rightarrow (T-u)(1+Tu) = Tv^2$$

$$T - u + T^2 u - T u^2 = T v^2$$

$$T v^2 + T u^2 + (1 - T^2) u = T$$

$$v^2 + u^2 + 2 \cdot \frac{1 - T^2}{2T} u + \frac{(1 - T^2)^2}{4T^2} = T + \frac{(1 - T^2)^2}{4T^2}$$

$$v^2 + \left(u + \frac{1 - T^2}{2T}\right)^2 = T + \frac{(1 - T^2)^2}{4T^2}$$

$$T = \sqrt{5} - 2 : T^2 = 9 - 4\sqrt{5}$$

$$v^2 + \left(u + \frac{-8 + 4\sqrt{5}}{2\sqrt{5} - 4}\right)^2 = \sqrt{5} - 2 + 2^2$$

$$v^2 + (u + 2)^2 = \sqrt{5} + 2$$

$$T = \sqrt{2} - 1 : T^2 = 3 - 2\sqrt{2}$$

$$v^2 + \left(u + \frac{-2 + 2\sqrt{2}}{2\sqrt{2} - 2}\right)^2 = \sqrt{2} - 1 + 1^2$$

$$v^2 + (u + 1)^2 = \sqrt{2}$$

2 окружности с центрами в $(-2, 0)$ и $(-1, 0)$ и радиусами $\sqrt{\sqrt{5} + 2}$ и $\sqrt{\sqrt{2}}$.

Задание 5

Условие:

$$\omega = \operatorname{Artg} z$$

$$z_0 = \frac{4}{3} i$$

Решение:

$$\operatorname{tg}\omega = \frac{\sin\omega}{\cos\omega} = \frac{e^{i\omega} - e^{-i\omega}}{i(e^{i\omega} + e^{-i\omega})}$$
$$e^{i\omega} = q, \operatorname{tg} w = t$$
$$t = \frac{q - \frac{1}{q}}{i\left(q + \frac{1}{q}\right)}$$
$$it\left(q + \frac{1}{q}\right) = q - \frac{1}{q}$$
$$q(it - 1) + \frac{1}{q}(it + 1) = 0$$
$$q^2 = -\frac{it + 1}{it - 1}$$
$$q = \left(-\frac{it+1}{it-1}\right)^{\frac{1}{2}} = e^{i\omega}$$
$$\omega = \frac{1}{i}\left(\operatorname{Ln}\left(\left(-\frac{it+1}{it-1}\right)^{\frac{1}{2}}\right)\right)$$
$$t = \frac{4}{3}i \Rightarrow$$
$$\omega = \frac{1}{i}\left(\operatorname{Ln}\left(\left(-\frac{1 - \frac{4}{3}}{-1 - \frac{4}{3}}\right)^{\frac{1}{2}}\right)\right) =$$
$$= -i\operatorname{Ln}\left[\left(-\frac{1}{7}\right)^{\frac{1}{2}}\right] = -\frac{i}{2}\operatorname{Ln}\left(-\frac{1}{7}\right) =$$
$$l = \operatorname{Ln}\left(-\frac{1}{7}\right) = \ln\left(\frac{1}{7}\right) + \pi i + 2\pi k i, k \in \mathbb{Z}$$
$$= -\frac{i}{2}(-\ln 7 + \pi i + 2\pi k i) = \frac{\ln 7}{2}i + \frac{\pi}{2} + \pi k, k \in \mathbb{Z}$$

Прошлые попытки:

$$\omega = \operatorname{Artg} z_0 \Leftrightarrow \operatorname{tg}\omega = z_0$$
$$\begin{cases} \omega = u + vi \\ z_0 = \frac{4}{3}i \end{cases} \Rightarrow \operatorname{tg}(u + vi) = \frac{4}{3}i$$
$$\frac{\sin(u)\cos(vi) + \sin(vi)\cos(u)}{\cos(u)\cos(vi) - \sin(u)\sin(vi)} = \frac{\operatorname{tg} u + \operatorname{th} vi}{1 - \operatorname{tg} u \operatorname{th} vi} = \frac{4}{3}i$$
$$\operatorname{tg} u + \operatorname{th} vi = \frac{4}{3}\operatorname{tg} u \operatorname{th} v + \frac{4}{3}i$$
$$\begin{cases} \operatorname{tg} u = \frac{4}{3}\operatorname{tg} u \operatorname{th} v \\ \operatorname{th} v = \frac{4}{3} \end{cases} \Rightarrow \begin{cases} \operatorname{tg} u\left(\frac{16}{9} - 1\right) = 0 \\ \operatorname{th} v = \frac{4}{3} \end{cases} \Rightarrow \begin{cases} \operatorname{tg} u = 0 \\ \operatorname{th} v = \frac{4}{3} \end{cases} \Leftrightarrow \begin{cases} u = \pi k, k \in \mathbb{Z} \\ v = \operatorname{arth}\left(\frac{4}{3}\right) \end{cases}$$

Задание 6

Условие:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (z - 2 + 2i)^n}{3^n \left(1 + \frac{1}{n}\right)^n}$$

Решение:

$$q = z - 2 + 2i$$
$$\cdot \sum_{n=0}^{\infty} \frac{|q|^n}{3^n\left(1+\frac{1}{n}\right)^n}$$
$$a_n = \frac{|q|^n}{3^n\left(1+\frac{1}{n}\right)^n}$$
$$\frac{a_{n+1}}{a_n} = \frac{|q|^{n+1}}{3^{n+1}\left(1+\frac{1}{n+1}\right)^{n+1}} \cdot \frac{3^n\left(1+\frac{1}{n}\right)^n}{|q|^n} = \frac{|q|}{3} \cdot \frac{1}{1+\frac{1}{n+1}} \cdot \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^n = \frac{|q|}{3} \cdot \frac{1}{1+\frac{1}{n+1}} \cdot \left(\frac{(n+1)^2}{n(n+2)}\right)^n =$$
$$= \frac{|q|}{3} \cdot \frac{1}{1+\frac{1}{n+1}} \cdot \left(1 + \frac{1}{n^2+2n}\right)^{(n^2+2n)\cdot\frac{1}{n+2}}$$
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{|q|}{3} \cdot \frac{1}{1+\frac{1}{n+1}} \cdot e^{\frac{1}{n+2}} = \frac{|q|}{3} \Rightarrow |q| < 3 \text{ - область абсолютной сходимости}$$
$$|q| = 3:$$
$$\sum_{n=0}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$
$$b_n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$
$$\lim_{n\to\infty} b_n = \frac{1}{e} \neq 0 \Rightarrow \text{не сходится даже условно}$$
$$3^2 = 9$$
$$2^2 + 2^2 = 8 < 9$$
$$3^2 + 0 = 9$$
$$3^2 + 1^2 > 9$$

Задание 7

Условие:

$$\oint_{|z|=8} \frac{\sin z}{z^4(z+4i)} dz$$

Решение:

$$f(z)=\frac{\sin z}{z^4(z+4i)}$$

Особые точки:

$z=0$ – полюс $n=3$

$z=-4i$ - полюс $n=1$

$$\oint_L f(z)dz=-2\pi i\cdot\operatorname*{Res}_{z=\infty}f$$

$$\operatorname*{Res}_{z=\infty}f:$$

$$\sin z=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

$$\frac{1}{z+4i}=\sum_{n=0}^{\infty}\frac{(-4i)^n}{z^{n+1}}$$

$$f=\frac{1}{z^4}\left[\sum_{n=0}^{\infty}\frac{(-4i)^n}{z^{n+1}}\right]\left[\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\right]$$

$$\operatorname*{Res}_{z=\infty}f=-z\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\cdot\frac{1}{z^4}\cdot\frac{(-4i)^m}{z^{m+1}}=$$

$$2n+1-4-(m+1)=-1\Rightarrow 2n-3-m=0\Rightarrow m=2n-3\Rightarrow\min(n)=2$$

$$=-z\sum_{n=2}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\cdot\frac{1}{z^4}\cdot\frac{(-4i)^{2n-3}}{z^{2n-2}}=-\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+5)!}\cdot(-4i)^{2n+1}=$$

$$=-\sum_{n=0}^{\infty}-4i\cdot\cancel{(-1)^n}\ \cancel{(-1)^{2n}}\ \cancelto{(-1)^n}{i^{2n}}\cdot\frac{4^{2n}}{(2n+5)!}=i\sum_{n=0}^{\infty}\frac{4^{2n+1}}{(2n+5)!}=$$

$$\operatorname{sh}x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\Rightarrow\operatorname{sh}4=\sum_{n=0}^{\infty}\frac{4^{2n+1}}{(2n+1)!}=4+\frac{4^3}{6}+\sum_{n=2}^{\infty}\frac{4^{2n+1}}{(2n+1)!}\Rightarrow$$

$$\operatorname{sh}4-4-\frac{32}{3}=\sum_{n=0}^{\infty}\frac{4^{2n+5}}{(2n+5)!}=4^4\sum_{n=0}^{\infty}\frac{4^{2n+1}}{(2n+5)!}$$

$$=\frac{i}{4^4}\left(\operatorname{sh}4-4-\frac{32}{3}\right)=i\left(\frac{\operatorname{sh}4}{256}-\frac{11}{64\cdot 3}\right)$$

$$\oint_L f(z)dz=-2\pi ii\left(\frac{\operatorname{sh}4}{256}-\frac{11}{64\cdot 3}\right)=2\pi\left(\frac{\operatorname{sh}4}{256}-\frac{11}{64\cdot 3}\right)$$

Прошлые попытки:

$$\operatorname*{Res}_{z=0} f = \frac{1}{6} \lim_{z\to 0} \left(\frac{\sin z}{z+4i} \right)'''$$

$$a' = \cos z$$

$$a'b = z\cos z + 4i\cos z$$

$$b' = 1$$

$$ab' = \sin z$$

$$a'b - ab' = z\cos z + 4i\cos z - \sin z$$

$$\left(\frac{z\cos z + 4i\cos z - \sin z}{z^2 + 8iz - 16} \right)''$$

$$a' = \cos z - z\sin z - 4i\sin z - \cos z = -z\sin z - 4i\sin z$$

$$a'b = -z^3\sin z - 12iz^2\sin z + 48z\sin z + 16\sin z$$

$$b' = 2z + 8i$$

$$ab' = 2z^2\cos z + 16iz\cos z - 32\cos z - 2z\sin z - 8i\sin z$$

$$a'b - ab' = -z^3\sin z - 12iz^2\sin z - 2z^2\cos z + 50z\sin z - 16iz\cos z + (16+8i)\sin z + 32\cos z$$

$$\left(\frac{-z^3\sin z - 12iz^2\sin z - 2z^2\cos z + 50z\sin z - 16iz\cos z + (16+8i)\sin z + 32\cos z}{z^4 + 16iz^3 - 96z^2 - 256iz + 256} \right)$$

$$a' = \cancel{-3z^2\sin z} - \cancel{z^3\cos z} - \cancel{24iz\sin z} - \cancel{12iz^2\cos z} - \cancel{4z\cos z} + \cancel{2z^2\cos z} + \cancel{50\sin z} + \cancel{50z\cos z}$$

$$- \cancel{16i\cos z} + \cancel{16iz\sin z} + \cancel{(16+8i)\cos z} - \cancel{32\sin z} =$$

$$(0)z^3\sin z + (-1)z^3\cos z + (-3)z^2\sin z + (-12i+2)z^2\cos z +$$

$$(-24i+16i)z\sin z + (-4+50)z\cos z + (50-32)\sin z + (-16i+16+8i)\cos z =$$

$$\cancel{-z^3\cos z} - \cancel{3z^2\sin z} + \cancel{(-12i+2)z^2\cos z} + (-24i+16i)z\sin z + 46z\cos z + 18\sin z + (16-8i)\cos z$$

$$\begin{array}{cc}
a'b = & \\
(-1)z^7\cos z + & (0)z^7\sin z + \\
(-16i - 12i + 2)z^6\cos z + & (-3)z^6\sin z + \\
(+96 - 192i^2 + 32i + 46)z^5\cos z + & (-48i - 24i + 16)z^5\sin z + \\
(+256i + 1152i - 192 + 736i_1 6 - 8i)z^4\cos z + & (288 - 384i^2 + 256i + 18)z^4\sin z + \\
(-256 + 3072i^2 - 512i - 4416 + 256i - 128i^2)z^3\cos z + & (768i + 2304i - 1536 + 288i)z^3\sin z + \\
(-3072i + 512 - 11776i - 1536 + 768i)z^2\cos z + & (-768 + 6144i^2 - 4096i - 1728)z^2\sin z + \\
(11776 - 4096i + 2048i^2)z\cos z + & (-6144i + 4096 - 4608i)z\sin z + \\
(4096 - 2048i)\cos z + & (4608)\sin z = \\
= -z^7\cos z + & 0 + \\
(2 - 28i)z^6\cos z + & -3z^6\sin z + \\
(334 + 32i)z^5\cos z + & (16 - 72i)z^5\sin z + \\
(-176 + 2136i)z^4\cos z + & (690 + 256i)z^4\sin z + \\
(-7616 - 256i)z^3\cos z + & (-1536 + 3360i)z^3\sin z + \\
(-1024 - 14080i)z^2\cos z + & (-8640 - 4096i)z^2\sin z + \\
(9728 - 4096)z\cos z + & (4096 - 10752)z\sin z + \\
(4096 - 2048)\cos z + & 4608\sin z
\end{array}$$

....

Задание 8

Условие:

$$\int_{C:\begin{cases}t\in[0,\pi]\\x=\pi-t\\y=t\end{cases}}\cos\overline{z}dz=\int_0^{\pi}\cos(\pi-t-ti)idt=$$

$$z=\pi-t+it$$

$$\overline{z}=\pi-t-ti$$

$$\cos(\pi-\alpha)=-\cos\alpha$$

$$=-\int_0^{\pi}\cos((1+i)t)\frac{i}{i+1}d((1+i)t)=-\frac{i}{i+1}(\sin((1+i)\pi)-0)=\frac{i(i-1)}{2}(-i\operatorname{sh}\pi)=$$

$$\sin(\pi+\alpha)=-\sin\alpha$$

$$\sin(i\alpha)=i\operatorname{sh}\alpha$$

$$=\frac{i-1}{2}\operatorname{sh}\pi=-\frac{\operatorname{sh}\pi}{2}+i\cdot\frac{\operatorname{sh}\pi}{2}$$

Решение:

Задание 9

Условие:

$$\int_{C:\text{ первая часть эллипса }|z-3-6i|+|z-3+2i|=10}(\overline{z}-3+2i)dz$$

Решение:

$$\text{Фокусы эллипса: }z_1=3+6i,z_2=3-2i$$

$$c=\frac{|z_1-z_2|}{2}=4$$

$$a=7-6+4=5,b=\sqrt{a^2-c^2}=3$$

$$\begin{cases}x=3+3\cos t\\y=2+5\sin t\end{cases}$$

$$\text{Первая часть}\Rightarrow t\in\left[0,\frac{\pi}{2}\right]$$

$$z=3+3\cos t+i(2+5\sin t)$$

$$\overline{z}-3+2i=3\cos t-i(5\sin t)$$

$$dz=(-3\sin t+5i\cos t)dt$$

$$\int_C f(z,\overline{z})dz=\int_0^{\frac{\pi}{2}}(3\cos t-5\sin ti)(-3\sin t+5\cos ti)dt=$$

$$=\int_0^{\frac{\pi}{2}}[-9\cos t\sin t+25\sin t\cos t+i(15\sin^2 t+15\cos^2 t)]dt=$$

$$=\int_0^{\frac{\pi}{2}}4\sin(2t)d(2t)+\frac{15\pi}{2}=8$$

Задание 10

Условие:

$$\int_{C:\text{ -верхняя половина окружности }|z-3i|=5}(z\overline{z}+i)dz$$

Решение:

Задание 11

Условие:

$$\int_{\frac{\pi}{3}\to\ln 5} z\operatorname{sh} z dz$$

Решение:

$$(z\operatorname{ch} z)' = \operatorname{ch} z + z\operatorname{sh} z$$
$$(z\operatorname{ch} z - \operatorname{sh} z)' = \operatorname{ch} z + z\operatorname{sh} z - \operatorname{ch} z = z\operatorname{sh} z$$
$$\int_C z\operatorname{sh} z dz = z\operatorname{ch} z - \operatorname{sh} z|_{\frac{i\pi}{3}}^{\ln 5} = \ln 5\cdot\frac{e^{\ln 5}+e^{-\ln 5}}{2} - \frac{e^{\ln 5}-e^{-\ln 5}}{2} - \frac{i\pi}{3}\operatorname{ch}\left(\frac{i\pi}{3}\right) + \operatorname{sh}\left(\frac{i\pi}{3}\right) =$$
$$= \ln 5\cdot\frac{1}{2}\left(5+\frac{1}{5}\right) - \frac{1}{2}\left(5-\frac{1}{5}\right) - \frac{i\pi}{3}\cdot\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2} =$$
$$= \frac{\ln 5}{2}\left(\frac{26}{5}\right) - \frac{12}{5} + i\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$$