$$x^1 \equiv x,\ x^2 \equiv y$$
$$X^1 \equiv X,\ X^2 \equiv Y$$
$$(X,Y):\ \begin{cases} x^1 = \frac{1}{2}X^2 + \frac{1}{2}\ln Y \\ \\ x^2 = \frac{2}{3}X^3 - \frac{1}{2}Y \end{cases},\ M:\ (1,\quad 1),\ \vec{a}:\ \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$$

1. $Q^I_J = \frac{\partial x^I}{\partial X^J}$

$$\begin{cases} \frac{\partial x^1}{\partial X} = X(1,1) = 1 \\ \frac{\partial x^1}{\partial Y} = \frac{1}{2Y}(1,1) = \frac{1}{2} \\ \frac{\partial x^2}{\partial X} = 2X^2(1,1) = 2 \\ \frac{\partial x^2}{\partial X} = -\frac{1}{2} \end{cases} \Rightarrow Q^I_J = \begin{pmatrix} 1 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{pmatrix} \text{ - матрица перехода}$$

$$Q = \begin{vmatrix} 1 & \frac{1}{2} \\ 2 & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} - 2 \cdot \frac{1}{2} = -\frac{3}{2}$$

$$P^I_J = \frac{\partial X^I}{\partial x^J}$$

$$(P^I_J) = (Q^I_J)^{-1}$$

$$(Q^I_J)^{-1} = \frac{1}{Q}\begin{pmatrix} Q^2_2 & -Q^1_2 \\ -Q^2_1 & Q^1_1 \end{pmatrix} = \frac{1}{-\frac{3}{2}}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{pmatrix}$$

$$P^I_J = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \text{ - обратная матрица перехода}$$

$$P = \left(\frac{1}{3} \cdot -\frac{2}{3} - \frac{4}{3} \cdot \frac{1}{3}\right) = -\frac{2}{3} = Q^{-1}$$

2. $\vec{R}_i = Q^j_i \vec{e}_j$

$$\vec{R}_i = {Q_i}^1 \vec{e}_1 + {Q^2}_i \vec{e}_2$$

$$\begin{cases} \vec{R}_1 = {Q_1}^1 \vec{e}_1 + {Q^2}_1 \vec{e}_2 = 1 \cdot \vec{e}_1 + 2 \cdot \vec{e}_2 \\ \vec{R}_2 = {Q_2}^1 \vec{e}_1 + {Q^2}_2 \vec{e}_2 = \frac{1}{2} \cdot \vec{e}_1 - \frac{1}{2} \cdot \vec{e}_2 \end{cases}$$

$$\vec{R}_i = {Q_i}^1 \vec{e}_1 + {Q^2}_i \vec{e}_2$$

$$\begin{cases} \vec{R}_1 = {Q_1}^1 \vec{e}_1 + {Q^2}_1 \vec{e}_2 = 1 \cdot \vec{e}_1 + 2 \cdot \vec{e}_2 \\ \vec{R}_2 = {Q_2}^1 \vec{e}_1 + {Q^2}_2 \vec{e}_2 = \frac{1}{2} \cdot \vec{e}_1 - \frac{1}{2} \cdot \vec{e}_2 \end{cases}$$

$$\vec{R}_i = Q^1_i \vec{e}_1 + Q^2_i \vec{e}_2$$

$$\begin{cases} \vec{R}_1 = Q^1_1 \vec{e}_1 + Q^2_1 \vec{e}_2 = 1 \cdot \vec{e}_1 + 2 \cdot \vec{e}_2 \\ \vec{R}_2 = Q^1_2 \vec{e}_1 + Q^2_2 \vec{e}_2 = \frac{1}{2} \cdot \vec{e}_1 - \frac{1}{2} \cdot \vec{e}_2 \end{cases}$$

3. $g_{ij} = \vec{R}_i \cdot \vec{R}_j$

$$\begin{cases} g_{11} = (1\cdot\vec{e}_1 + 2\cdot\vec{e}_2)(1\cdot\vec{e}_1 + 2\cdot\vec{e}_2) = 1^2 + 2^2 = 5 \\ g_{12} = (1\cdot\vec{e}_1 + 2\cdot\vec{e}_2)\left(\frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2\right) = \frac{1}{2} - 1 = -\frac{1}{2} \\ g_{21} = g_{12} = -\frac{1}{2} \\ g_{22} = \left(\frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2\right)\left(\frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{cases}$$

$$g_{IJ} = \begin{pmatrix} 5 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$g = \begin{vmatrix} 5 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$$

$$(g^{IJ}) = (g_{IJ})^{-1} = \frac{4}{9}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 5 \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{20}{9} \end{pmatrix}$$

$$q^{IJ} = g^{IJ}$$

$$q = \frac{16}{81}\cdot\left(\frac{5}{2} - \frac{1}{4}\right) = \frac{16}{81}\cdot\frac{9}{4} = \frac{4}{9}$$

$$qg = 1$$

4. $\overrightarrow{R^i} = g^{ij}\overrightarrow{R_j}$

$$\overrightarrow{R^i} = g^{i1}\overrightarrow{R_1} + g^{i2}\overrightarrow{R_2}$$

$$\begin{cases} \overrightarrow{R^1} = g^{11}\overrightarrow{R_1} + g^{12}\overrightarrow{R_2} = \frac{2}{9}(1\cdot\vec{e}_1 + 2\cdot\vec{e}_2) + \frac{2}{9}\left(\frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2\right) = \frac{1}{3}\vec{e}_1 + \frac{1}{3}\vec{e}_2 \\ \overrightarrow{R^2} = g^{21}\overrightarrow{R_1} + g^{22}\overrightarrow{R_2} = \frac{2}{9}\left(\vec{e}_1 + 2\vec{e}_2\right) + \frac{20}{9}\left(\frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2\right) = \frac{4}{3}\vec{e}_1 - \frac{2}{3}\vec{e}_2 \end{cases}$$

5. $\vec{a} = a^I\vec{e}_I = b^I\vec{R}_I = b_I\vec{R}^I$

$$b^I = P^I_J a^J = P^I_1 a^1 + P^I_2 a^2$$

$$\begin{cases} b^1 = \frac{1}{3}\cdot\frac{3}{2} + \frac{1}{3}\cdot\frac{3}{2} = 1 \\ b^2 = \frac{4}{3}\cdot\frac{3}{2} - \frac{2}{3}\cdot\frac{3}{2} = 1 \end{cases}$$

$$b_I = Q^J_I a_J = Q^J_I \delta_{JK} a^K = Q^1_I\cdot a^1 + Q^2_I\cdot a^2$$

$$\begin{cases} b_1 = 1\cdot\frac{3}{2} + 2\cdot\frac{3}{2} = \frac{9}{2} \\ b_2 = \frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot\frac{3}{2} = 0 \end{cases}$$

Проверка:

$$\overrightarrow{R^I}\cdot\overrightarrow{R_J} = \delta^I_J$$

$$\begin{cases} \overrightarrow{R^1}\cdot\overrightarrow{R_1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix} \circ \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{3} + \frac{2}{3} = 1 = \delta^1_1 \\ \overrightarrow{R^1}\cdot\overrightarrow{R_2} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix} \circ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = 0 = \delta^1_2 \\ \overrightarrow{R^2}\cdot\overrightarrow{R_1} = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} \end{pmatrix} \circ \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{4}{3} - \frac{4}{3} = 0 = \delta^2_1 \\ \overrightarrow{R^2}\cdot\overrightarrow{R_2} = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} \end{pmatrix} \circ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{2}{3} + \frac{1}{3} = 1 = \delta^2_2 \end{cases}$$

$$g_{IJ} = Q_I^K Q_J^L \delta_{KL} = Q_I^1 \cdot Q_J^1 + Q_I^2 \cdot Q_J^2$$

$$\begin{cases} g_{11} = 1 \cdot 1 + 2 \cdot 2 = 5 \\ g_{12} = 1 \cdot \frac{1}{2} + 2 \cdot -\frac{1}{2} = -\frac{1}{2} \\ g_{21} = g_{21} \\ g_{22} = \frac{1}{2} \cdot \frac{1}{2} + -\frac{1}{2} \cdot -\frac{1}{2} = \frac{1}{2} \end{cases}$$

$$g^{IJ} = P_K^I P_L^J \delta^{KL} = P_1^I \cdot P_1^J + P_2^I \cdot P_2^J$$

$$\begin{cases} g^{11} = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9} \\ g^{12} = \frac{1}{3} \cdot \frac{4}{3} + \frac{1}{3} \cdot -\frac{2}{3} = \frac{2}{9} \\ g^{21} = g^{12} \\ g^{22} = \frac{4}{3} \cdot \frac{4}{3} + -\frac{2}{3} \cdot -\frac{2}{3} = \frac{20}{9} \end{cases}$$

$$g^{IJ} = \overrightarrow{R^I} \cdot \overrightarrow{R^J}$$

$$\begin{cases} g^{11} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix} \circ \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \frac{2}{9} \\ g^{12} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix} \circ \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix} = \frac{2}{9} \\ g^{21} = g^{12} \\ g^{22} = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} \end{pmatrix} \circ \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix} = \frac{20}{9} \end{cases}$$

$$\vec{R}_J = g_{IJ} \vec{R}^I = g_{1J} \vec{R}^1 + g_{2J} \vec{R}^2$$

$$\begin{cases} \vec{R}_1 = 5 \cdot \left(\frac{1}{3}\vec{e}_1 + \frac{1}{3}\vec{e}_2\right) + -\frac{1}{2} \cdot \left(\frac{4}{3}\vec{e}_1 - \frac{2}{3}\vec{e}_2\right) = \left(\frac{5}{3} - \frac{2}{3}\right)\vec{e}_1 + \left(\frac{5}{3} + \frac{1}{3}\right)\vec{e}_2 = \vec{e}_1 + 2\vec{e}_2 \\ \vec{R}_2 = -\frac{1}{2} \cdot \left(\frac{1}{3}\vec{e}_1 + \frac{1}{3}\vec{e}_2\right) + \frac{1}{2} \cdot \left(\frac{4}{3}\vec{e}_1 - \frac{2}{3}\vec{e}_2\right) = \left(-\frac{1}{6} + \frac{2}{3}\right)\vec{e}_1 + \left(-\frac{1}{6} - \frac{1}{3}\right)\vec{e}_2 = \frac{1}{2}\vec{e}_1 - \frac{1}{2}\vec{e}_2 \end{cases}$$

$$b_I = g_{IJ} b^J = g_{I1} b^1 + g_{I2} b^2$$

$$\begin{cases} b_1 = 5 - \frac{1}{2} = \frac{9}{2} \\ b_2 = -\frac{1}{2} + \frac{1}{2} = 0 \end{cases}$$

$$b^I = g^{IJ} b_J = g^{I1} b_1 + g^{I2} b_2$$

$$\begin{cases} b^1 = \frac{2}{9} \cdot \frac{9}{2} + 0 = 1 \\ b^2 = \frac{2}{9} \cdot \frac{9}{2} = 1 \end{cases}$$

$$b_I = \vec{a} \cdot \vec{R}_I$$

$$b^I = \vec{a} \cdot \vec{R}^I$$

$$\begin{cases} b_1 = \begin{pmatrix} 1 & 2 \end{pmatrix} \circ \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = \frac{9}{2} \\ b_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \circ \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = 0 \\ b^1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix} \circ \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = 1 \\ b^2 = \begin{pmatrix} \frac{4}{3} & -\frac{2}{3} \end{pmatrix} \circ \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = 1 \end{cases}$$

$$X^i = \begin{pmatrix} r \\ \varphi \\ z \end{pmatrix}$$

$$\tilde{x}^i = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

$$e^i = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$X^i = e^j$$