$X_1, \dots, X_n \sim \underline{iid}\ N(\mu, \sigma^2)$ • identically • independent •distributed

Метод моментов:

$\mathbb{E}[X] = \mu = \overline{X} \Rightarrow \hat{\mu} = \overline{X}$

$\mathbb{E}[X^2] = \sigma^2 + \mu^2 = \overline{X^2} \Rightarrow \hat{\sigma}^2 = \overline{X^2} - (\overline{X})^2$

1) Несмещенность

$\mu$ $\mathbb{E}[\hat{\mu}] = \mathbb{E}[\overline{X}] = \mathbb{E}\left[\frac{1}{n}(X_1 + \dots + X_n)\right] =$

$= \frac{1}{n} \cdot \left(\underbrace{\mathbb{E}[X_1]}_{\mu} + \dots + \underbrace{\mathbb{E}[X_n]}_{\mu}\right) = \frac{1}{\cancel{n}} \cdot \cancel{n}\mu = \mu$

$\Rightarrow \hat{\mu}$ — несмещенная оценка $\mu$

$\sigma^2$

$\mathbb{E}[\hat{\sigma}^2] = \mathbb{E}[\overset{①}{\overline{X^2}} - \overset{②}{(\overline{X})^2}] = \mathbb{E}[\overline{X^2}] - \mathbb{E}[(\overline{X})^2]$

① $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i^2] = \frac{1}{n}(\sigma^2 + \mu^2)n = \sigma^2 + \mu^2$

② $\mathbb{E}[\overline{X}^2] = \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] = \frac{1}{n^2}\mathbb{E}\left[(X_1 + X_2 + \dots + X_n)^2\right] =$

$= \frac{1}{n^2}\mathbb{E}\left[X_1^2 + X_2^2 + \dots X_n^n + 2\underbrace{X_1 \cdot X_2}_{=0} + 2\underbrace{X_1 X_3}_{=0} + \dots \underbrace{X_{n-1} X_n}_{=0}\right] =$

$= \frac{1}{n^2}\mathbb{E}\left[\sum_{i=1}^{n} X_i^2 + 2\sum_{i<j} X_i X_j\right] =$

$\mathbb{E}[X_i \cdot X_j] = \mathbb{E}[X_i] \cdot \mathbb{E}[X_j] = \mu^2$

т.к $X_i$ и $X_j$ независимы

$= \frac{1}{n^{\cancel{2}}}\left(\cancel{n}(\sigma^2 + \mu^2) + \cancel{2} \cdot \frac{\cancel{n}(n-1)}{\cancel{2}} \cdot \mu^2\right) =$

$C_n^2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$

$$= \frac{1}{n}\left(\sigma^2 + \cancel{\mu^2} + \mu^2 \cdot n - \cancel{\mu^2}\right) = \frac{1}{n}\left(\sigma^2 + \mu n\right) =$$

$$= \frac{\sigma^2}{n} + \mu^2$$

$$\mathbb{E}[\hat{\sigma}^2] = \mathbb{E}[\overline{X^2}] - \mathbb{E}[(\overline{X})^2] =$$

$$= \sigma^2 + \cancel{\mu^2} - \left(\frac{\sigma^2}{n} + \cancel{\mu^2}\right) = \sigma^2 - \frac{\sigma^2}{n} = \sigma^2\left(1 - \frac{1}{n}\right) =$$

$$= \sigma^2 \cdot \frac{n-1}{n} \neq \sigma^2$$

$\Rightarrow$ $\hat{\sigma}^2_{MM}$ – смещенная оценка дисперсии

$$\hat{\sigma}^2 = \overline{X^2} - (\overline{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

$$\left(\frac{1}{\cancel{n}}\sum_{i=1}^{n}(X_i - \overline{X})^2\right) \cdot \frac{\cancel{n}}{n-1} =$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \hat{S}^2$$

несмещенная оценка дисперсии

ddof = delta degrees of freedom = 1

$$\mathbb{E}[\hat{\sigma}^2] = \sigma^2 \cdot \frac{n-1}{n} \xrightarrow[n \to \infty]{} \sigma^2$$

$\hat{\sigma}^2$ – ассимптотическая несмещенная

$\mathbb{E}[\hat{\sigma}^2+5] \underset{n\to\infty}{=} \sigma^2 + 5$

пример смещенности всегда

2) состоятельность

если $\mathbb{P}(|\hat{\theta}-\theta|>\varepsilon) \xrightarrow[n\to\infty]{} 0$, то говорят

$\hat{\theta} \xrightarrow[n\to\infty]{\mathbb{P}} \theta$ ($\hat{\theta}$ сходится по вероятности к $\theta$)

$$\mathbb{P}(|\hat{\theta}-\theta|>\varepsilon) \leq \frac{var(\hat{\theta})}{\varepsilon^2}$$

$\hat{\mu} = \overline{X}$

$$var(\hat{\mu}) = var(\overline{X}) = var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} var\left(\sum_{i=1}^{n} X_i\right) =$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} var(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

$$\mathbb{P}(|\overline{X}-\mu|>\varepsilon) \leq \frac{\sigma^2}{n\cdot\varepsilon^2} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ $\hat{\mu} = \overline{X}$ – состоятельная оценка для $\mu$

3) эффективность

$$J(\theta) = \mathbb{E}\left[\left(\frac{\partial \ln f(X,\theta)}{\partial \theta}\right)^2\right] = -\mathbb{E}\left[\frac{\partial^2 \ln f(X,\theta)}{\partial \theta^2}\right]$$

$$f(x,\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2}\cdot\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

$$\ln f(X,\mu) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}(X-\mu)^2$$

$$\frac{\partial \ln f(X,\mu)}{\partial \mu} = +\frac{1}{\cancel{2}\sigma^2}(X-\mu)^1\cdot\cancel{2} = \frac{X-\mu}{\sigma^2}$$

$$\frac{\partial^2 \ln f(X,\mu)}{\partial \mu^2} = \left(\frac{X-\mu}{\sigma^2}\right)'_\mu = \frac{0-1}{\sigma^2} = -\frac{1}{\sigma^2}$$

$$J(\mu) = -\mathbb{E}\left[\frac{\partial^2 \ln f(X,\mu)}{\partial \mu^2}\right] = -\mathbb{E}\left[-\frac{1}{\sigma^2}\right] = \frac{1}{\sigma^2}$$

нер-во Рао-Крамера-Фреше:

$$var(\hat{\mu}) \geqslant \frac{1}{n\cdot J(\mu)} = \frac{\sigma^2}{n}$$

$\Rightarrow$ $\hat{\mu}$ - эффективная

в классе всех несмещенных

(у $\hat{\mu}$ самая низкая дисперсия)