$$y_i = \beta x_i + \varepsilon_i, \quad \varepsilon_i \underset{iid}{\sim} N(0, \sigma^2), \quad y_i \underset{iid}{\sim} N(\beta x_i, \sigma^2)$$

T1

$$\mathcal{L} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2\sigma^2}(y_i - x_i\beta)^2\right)$$

$$\ln \mathcal{L} = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i \cdot \beta)^2$$

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = -\frac{1 \cdot 2}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i\beta)\cdot x_i = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i x_i - x_i^2\beta) = 0$$

$$\sum y_i x_i = \beta \sum x_i^2 \Rightarrow \hat{\beta}_{ML} = \frac{\sum y_i x_i}{\sum x_i^2}$$

T2

$$\mathbb{E}[\hat{\beta}] = \mathbb{E}\left[\frac{\sum x_i \cdot y_i}{\sum x_i^2}\right] = \frac{1}{\sum x_i^2} \sum x_i \cdot \sum \mathbb{E}[y_i] =$$

$$= \frac{1}{\sum x_i^2} \cdot \sum x_i \cdot \beta \cdot x_i = \frac{1}{\cancel{\sum x_i^2}} \beta \cdot \cancel{\sum x_i^2} = \beta$$

$\Rightarrow \hat{\beta}$ – несмещенная

$$\mathrm{var}[\hat{\beta}] = \mathrm{var}\left[\frac{\sum x_i \cdot y_i}{\sum x_i^2}\right] = \left(\frac{1}{\sum x_i^2}\right)^2 \sum\left[x_i^2 \cdot \mathrm{var}(y_i)\right] =$$

$$= \left(\frac{1}{\sum x_i^2}\right)^2 \sum\left[x_i^2 \cdot \sigma^2\right] = \left(\frac{1}{\sum x_i^2}\right)^2 \sigma^2\left(\sum x_i^2\right) = \frac{1}{\sum x_i^2} \cdot \sigma^2$$

$$\mathbb{P}(|\hat{\beta} - \beta| > \varepsilon) \leq \frac{\mathrm{var}(\hat{\beta})}{\varepsilon^2} = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2} \xrightarrow[n\to\infty]{} 0 \Rightarrow \hat{\beta} \xrightarrow[n\to\infty]{\mathbb{P}} \beta$$

$\Rightarrow \hat{\beta}$ – состоятельная

$$\mathcal{I}(\beta) = \frac{\partial^2}{\partial \beta^2}[\ln \mathcal{L}] = \frac{\partial}{\partial \beta^2}\left[-\frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i x_i - x_i^2\beta)\right] = \frac{1}{\sigma^2} \cdot \sum_{i=1}^{n} x_i^2$$

Нер-во Крамера-Рао-Фреше:

$$\mathrm{var}(\hat{\beta}) \geq \frac{1}{\mathcal{I}(\beta)} = \frac{\sigma^2}{\sum x_i} = \mathrm{var}[\hat{\beta}] \Rightarrow \hat{\beta}$$ – эффективна среди несмещенных

T1 $\quad y_i \sim \text{Poiss}\,(\lambda = \beta \cdot x_i)$

$$\mathbb{P}(Y = y) = \frac{1}{y!} \cdot e^{-\lambda} \cdot \lambda^{y} = \frac{1}{y!} \cdot e^{-\beta x} \cdot (\beta x)^{y}$$

$$\mathcal{L} = \prod_{i=n}^{n} \left[\frac{1}{y!} \cdot e^{-\beta x_i} \cdot (\beta x_i)^{y_i} \right]$$

$$\ln \mathcal{L} = -\sum_{i=1}^{n} \ln(y!) - \beta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i \ln(\beta x_i)$$

$$\frac{\partial}{\partial \beta} \ln \mathcal{L} = \sum x_i - \sum y_i \frac{1}{\beta x_i} \cdot x_i = \sum x_i - \frac{1}{\beta} \sum y_i = 0$$

$$\Rightarrow \hat{\beta}_{ML} = \frac{\sum y_i}{\sum x_i} = \frac{\bar{y}}{x}$$

$$y_i \sim N(\beta_0 + \beta_1 x_i ,\ \sigma^2)$$

T1

$$\mathcal{L} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2\sigma^2}\left(y_i - x_i\beta_1 - \beta_0\right)^2\right)$$

$$\ln \mathcal{L} = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - x_i\cdot\beta_1 - \beta_0\right)^2$$

$$\frac{\partial \ln \mathcal{L}}{\partial \beta_1} = +\frac{1\cdot 2}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - x_i\beta_1 - \beta_0\right)\cdot x_i = +\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i x_i - x_i^2\beta_1 - \beta_0 x_i\right) = 0$$

$$\frac{\partial \ln \mathcal{L}}{\partial \beta_0} = +\frac{1\cdot 2}{2\sigma^2}\sum_{i=1}^{n}\left(y_i - x_i\beta_1 - \beta_0\right) = +\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i - x_i\beta_1 - \beta_0\right) = 0$$

$$\begin{cases} \sum y_i x_i = \beta_1 \sum x_i^2 - \beta_0 \sum x_i & (1) \\ \sum y_i - \beta_1 \sum x_i = n\beta_0 & (2) \end{cases}$$

$$(2) \Rightarrow \beta_0 = \bar{y} - \beta_1 \bar{x} \Rightarrow (1)$$

$$\sum y_i x_i = \beta_1 \sum x_i^2 - (\bar{y} - \beta_1\bar{x})\sum x_i$$

$$\sum y_i \cdot x_i = \beta_1 \sum x_i^2 - \bar{y}\sum x - \beta_1 \bar{x}\sum x_i \quad \Big|\cdot \frac{1}{n}$$

$$\overline{yx} = \beta_1 \overline{x^2} - \bar{y}\cdot\bar{x} - \beta_1(\bar{x})^2$$

$$\Rightarrow \hat{\beta}_1 = \frac{\overline{yx} - \bar{y}\cdot\bar{x}}{\overline{x^2} - (\bar{x})^2}$$

T2

$$\hat{\beta} = \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}$$

$$\mathrm{cov}(X,Y) = \sum (x_i - \bar{x})(y_i - \bar{y}) = \sum \left(x_i y_i - \bar{x} y_i - x_i \bar{y} + \bar{x}\cdot\bar{y}\right) =$$

$$= \sum x_i y_i - n\bar{x}\bar{y} - \cancel{n\bar{x}\bar{y}} + \cancel{\bar{x}\bar{y}\cdot n} = \overline{xy} - \bar{x}\cdot\bar{y}$$

$$\mathrm{var}(X) = \sum (x_i - \bar{x})^2 = \sum \left(x_i^2 - 2x_i\bar{x} + (\bar{x})^2\right) = \overline{x^2} - n\,2(\bar{x})^2 + (\bar{x})^2 n = \overline{x^2} - (\bar{x})^2$$