$X_1, \dots, X_n \sim \text{iid } U[0;\theta]$, $f_X(x) = \frac{1}{\theta} \cdot I\{x \in [0;\theta]\}$

T1 $\mathcal{L}(X_1,\dots,X_n;\theta) =$

$$= \prod_{i=1}^{n} \left[\frac{1}{\theta} \cdot I\{x_i \in [0;\theta]\}\right] =$$

$$= \frac{1}{\theta^n} \cdot \prod_{i=1}^{n} I\{x_i \in [0;\theta]\} = \begin{cases} \frac{1}{\theta^n}, & \text{если все } x_i \in [0;\theta] \\ 0, & \text{если хотя бы } 1\ x_i \notin x_i \in [0;\theta] \end{cases}$$

$\Rightarrow$ θ должно быть таким, что все $x_i \in [0;\theta]$, следовательно $\hat{\theta}_{ML} = \max(X_1,\dots,X_n)$

T2 $\mathbb{E}[X_i] = \frac{\theta}{2} \equiv \bar{X} \Rightarrow \hat{\theta}_{MM} = 2\bar{X}$

T3 $\hat{\theta}_{ML} = \max(X_1,\dots,X_n)$

$\mathbb{E}[\hat{\theta}_{ML}] = \mathbb{E}[\max(X_1,\dots,X_n)] = ?$

$$\mathbb{P}(\max(X_1,\dots,X_n) \le x) = \left(\frac{x}{\theta}\right)^n = F_{\hat{\theta}_{ML}}(x)$$

$$f_{\hat{\theta}_{ML}}(x) = \left[\left(\frac{x}{\theta}\right)^n\right]'_x = n\left(\frac{x}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}$$

$$\mathbb{E}[\hat{\theta}_{ML}] = \int_0^\theta n\left(\frac{x}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \cdot x\,dx = \frac{n}{\theta^n}\int_0^\theta x^n\,dx =$$

$$= \frac{n}{\theta^n}\left[\frac{x^{n+1}}{n+1}\right]\Bigg|_0^\theta = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1} \cdot \theta \ne \theta -$$

$\hat{\theta}_{ML}$ - смещенная

но $\lim_{n\to\infty}\left[\frac{n}{n+1} \cdot \theta\right] = \theta \Rightarrow \hat{\theta}_{ML}$ - асимптотически несмещенная

$$\mathbb{P}\left(|\hat{\theta}_{ML}-\theta|>\varepsilon\right)\le\frac{var[\hat{\theta}_{ML}]}{\varepsilon^2}$$

$$var[\hat{\theta}_{ML}]=\mathbb{E}[\hat{\theta}_{ML}^2]-\mathbb{E}^2[\hat{\theta}_{ML}]$$

$$\mathbb{E}[\hat{\theta}_{ML}^2]=\int_0^\theta n\left(\frac{x}{\theta}\right)^{n-1}\cdot\frac{1}{\theta}\cdot x^2\,dx=\frac{n}{\theta^n}\int_0^\theta x^{n+1}\,dx=$$

$$=\frac{n}{\theta^n}\left[\frac{x^{n+2}}{n+2}\right]_0^\theta=\frac{n}{\theta^n}\cdot\frac{\theta^{n+2}}{n+2}=\frac{n}{n+2}\cdot\theta^2$$

$$\Rightarrow var[\hat{\theta}_{ML}]=\frac{n}{n+2}\cdot\theta^2-\left[\frac{n}{n+1}\cdot\theta\right]^2=\theta^2\left(\frac{n}{n+2}-\frac{n^2}{(n+1)^2}\right)=$$

$$=\theta^2 n\left(\frac{(n+1)^2-n(n+2)}{(n+2)(n+1)^2}\right)=\theta^2\cdot n\cdot\left(\frac{\cancel{n^2}+\cancel{2n}+1\cancel{-n^2}\cancel{-2n}}{(n+2)(n+1)^2}\right)=$$

$$=\theta^2\cdot\frac{n}{(n+2)(n+1)^2}$$

$$\mathbb{P}\left(|\hat{\theta}_{ML}-\theta|>\varepsilon\right)\le\frac{var[\hat{\theta}_{ML}]}{\varepsilon^2}=\theta^2\cdot\frac{n}{(n+2)(n+1)^2}\cdot\frac{1}{\varepsilon^2}\xrightarrow[n\to\infty]{}0$$

$\Rightarrow \hat{\theta}_{ML}$ – состоятельна

$$\hat{\theta}_{MM}=2\overline{X}$$

$$\mathbb{E}[\hat{\theta}_{MM}]=\mathbb{E}[2\overline{X}]=2\cdot\frac{\theta}{2}=\theta\Rightarrow$$ $\hat{\theta}_{MM}$ – несмещенная

$$var[\hat{\theta}_{MM}]=var[2\overline{X}]=4var[\overline{X}]=4\cdot\frac{var[X_i]}{n}=\frac{4}{n}\cdot\frac{\theta^2}{12}=\frac{\theta^2}{3n}$$

$$\mathbb{P}\left(|\hat{\theta}_{MM}-\theta|>\varepsilon\right)\le\frac{var[\hat{\theta}_{MM}]}{\varepsilon^2}=\theta^2\cdot\frac{1}{3n}\cdot\frac{1}{\varepsilon^2}\xrightarrow[n\to\infty]{}0$$

$\Rightarrow \hat{\theta}_{MM}$ – состоятельна

T5

$$\hat{\theta}_{ML} = \max(X_1, \dots, X_n)$$

$$\text{cdf}(\hat{\theta}_{ML}) = \left(\frac{x}{\theta}\right)^n = \alpha = \mathbb{P}(\hat{\theta}_{ML} < x)$$

$$\frac{x}{\hat{\theta}_{ML}} = \alpha^{\frac{1}{n}}$$

$$x_\alpha = \hat{\theta}_{ML} \cdot \alpha^{\frac{1}{n}}$$

$$\mathbb{P}(\hat{\theta}_{ML} \leq q_u) = 1 - \frac{\alpha}{2} \Rightarrow q_u = \hat{\theta}_{ML} \cdot \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{n}}$$

$$\mathbb{P}(\hat{\theta}_{ML} \leq q_L) = \frac{\alpha}{2} \Rightarrow q_L = \hat{\theta}_{ML} \cdot \left(\frac{\alpha}{2}\right)^{\frac{1}{n}}$$

$$\Rightarrow \mathbb{P}\left(\theta \in \left[\hat{\theta}_{ML} \cdot \left(\frac{\alpha}{2}\right)^{\frac{1}{n}} ; \hat{\theta}_{ML} \cdot \left(1 - \frac{\alpha}{2}\right)^{\frac{1}{n}}\right]\right) = 1 - \alpha$$

T6 способ 1

$$\bar{X} \underset{n\to\infty}{\sim} N\left(E[X_i], \frac{var[X_i]}{n}\right) = N\left(\frac{\theta}{2}, \frac{\theta^2}{12n}\right)$$

$$\hat{\theta}_{MM} = 2\bar{X} \underset{n\to\infty}{\sim} N\left(2\cdot\frac{\theta}{2}, 4\cdot\frac{\theta^2}{12n}\right) = N\left(\theta, \frac{\theta^2}{3n}\right)$$

$$\mathbb{P}\left(\left|\frac{2\bar{X} - \theta}{\theta/\sqrt{3n}}\right| \leq z_{1-\frac{\alpha}{2}}\right) = 1-\frac{\alpha}{2}$$

$$\left|\frac{2\bar{X}}{\theta} - 1\right| \leq z/\sqrt{3n}$$

$$-z/\sqrt{3n} \leq \frac{2\bar{X}}{\theta} - 1 \leq z/\sqrt{3n}$$

$$1 - z/\sqrt{3n} \leq \frac{2\bar{X}}{\theta} \leq z\sqrt{3n} + 1$$

$$\frac{1-z/\sqrt{3n}}{2\bar{X}} \leq \frac{1}{\theta} \leq \frac{z/\sqrt{3n}+1}{2\bar{X}}$$

$$\Rightarrow \left\{\begin{array}{l} \theta \geq \dfrac{2\bar{X}}{z/\sqrt{3n}+1} \\ \theta \leq \dfrac{2\bar{X}}{-z/\sqrt{3n}+1} \end{array}\right\} \Rightarrow$$

$$\Rightarrow P\left(\theta \in \left[\frac{2\bar{X}}{1+z/\sqrt{3n}}; \frac{2\bar{X}}{1-z/\sqrt{3n}}\right]\right) = 1-\alpha$$

$$P\left(\theta \in \left[\frac{\hat{\theta}_{MM}}{1+z/\sqrt{3n}}; \frac{\hat{\theta}_{MM}}{1-z/\sqrt{3n}}\right]\right) = 1-\alpha$$

T6 способ 2 (не совсем корректный)

оценим $var(\theta_{мм})$ по выборке

$$\hat{\theta}_{мм} = 2\bar{X} \underset{n\to\infty}{\sim} N\left(\theta, \frac{\hat{\sigma}^2}{n}\right), \quad \hat{\sigma}^2 \text{ - оценка дисперсии}$$

$$\mathbb{P}\left(\left| \frac{2\bar{X} - \theta}{\hat{\sigma}/\sqrt{n}} \right| \leq z_{1-\frac{\alpha}{2}} \right) = 1-\alpha$$

$$|2\bar{X} - \theta| \leq z \cdot \frac{\hat{\sigma}}{\sqrt{n}}$$

$$-z \cdot \frac{\hat{\sigma}}{\sqrt{n}} \leq 2\bar{X} - \theta \leq z \cdot \frac{\hat{\sigma}}{\sqrt{n}}$$

$$-2\bar{X} - z \cdot \frac{\hat{\sigma}}{\sqrt{n}} \leq -\theta \leq z \cdot \frac{\hat{\sigma}}{\sqrt{n}} - 2\bar{X}$$

$$2\bar{X} - z \cdot \frac{\hat{\sigma}}{\sqrt{n}} \leq \theta \leq 2\bar{X} + z \cdot \frac{\hat{\sigma}}{\sqrt{n}}$$

$$\Rightarrow \theta \in 2\bar{X} \pm z \cdot \frac{\hat{\sigma}}{\sqrt{n}},$$

$$\mathbb{P}\left(\theta \in \left[2\bar{X} - z\frac{\hat{\sigma}}{n}; 2\bar{X} + z \cdot \frac{\hat{\sigma}}{n} \right] \right) = 1-\alpha$$