Ordinary Differential Equations

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Course text: The Qualitative Theory of Ordinary Differential Equations: An Introduction, Bauer and Nohel

Disclaimer: I take full responsibility for any errors, typos, and other flaws in these notes. If you see any or have other suggestions for improvement, feel free to reach out to me at felicialim@berkeley.edu. My notes are a work in progress and I appreciate any ideas on how they can be improved.

Math 123 | Ordinary Differential Equations

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Contents

Overview

What is an **ordinary differential equation** (ODE)? The rudimentary answer is an equation whose unknown is a function $y = y(t)$ which involves ordinary derivatives $\frac{\mathrm{d}}{\mathrm{d}t}y$ (as opposed to partial derivatives). Such equations are called functional equations, where t is the independent variable and y is the dependent variable.

Example

A classic example by Newton is

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y = F$$

where y is displacement, m is mass of the particle, and F is force. This is the second law of mechanics. This can be combined with

$$F = -\frac{GMm}{y^2}$$

where G and M are constants to yield the ODE

$$my'' = -\frac{GMm}{y^2}.$$

Another example is Ptolemaic epicycles in astronomy.

Complex phenomena often omit simple descriptions in terms of rates of changes, which is exactly a derivative. Our goal is to elucidate this relationship.

Terminology

The formal definition of an ODE of order k is a function $\vec{F}(t, \vec{y}, \vec{y}', \dots, \vec{y}^{(k)})$ where $t \in \mathbb{R}$ is the independent variable. $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ where $y_1, \dots, y_n$ are dependent variables (unknown functions) and for a domain $D \subseteq \mathbb{R}^{1+n(k+1)}$, $\vec{F} : D \to \mathbb{R}^n$ where $\vec{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$ and $F_j : D \to \mathbb{R}$. For the domain D, the 1 in the $\mathbb{R}^{1+nk}$ corresponds to the independent variable while the $n(k+1)$ corresponds to the k derivatives of n dependent variables.

By a solution to $\vec{F}(t, \vec{y}, \vec{y}', \dots, \vec{y}^{(k)})$, we mean an interval $I \subseteq \mathbb{R}$ and functions $\vec{\varphi}(t) = \begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_n(t) \end{pmatrix}$ where $\varphi_j : I \to \mathbb{R}$ such that:

1. $\varphi_j', \dots, \varphi_j^{(k)}$ exist on I
2. $(t, \vec{\varphi}(t), \vec{\varphi}'(t), \dots, \vec{\varphi}^{(k)}(t)) \in D \ \forall t \in I$
3. $\vec{F}(t, \vec{\varphi}(t), \vec{\varphi}'(t), \dots, \vec{\varphi}^{(k)}(t)) = 0 \ \forall t \in I$

$\vec{F}(t, \vec{y}, \vec{y}', \dots, \vec{y}^{(k)}) = 0$ is the **implicit form** of an ODE. $\vec{y}^{(k)} = f(t, \vec{y}, \dots, \vec{y}^{(k-1)})$ is the **explicit form** of an ODE.

Example

1. $\frac{1}{y}y' - t = 0$ is in implicit form
2. $y' = ty$ is in explicit form

$\vec{F}(t, \vec{y}, \vec{y}', \dots, \vec{y}^{(k)})$ is **autonomous** if F is independent of t. Similarly, $f(t, \vec{y}, \dots, \vec{y}^{(k-1)})$ is autonomous if f is independent of t.

$\vec{F}(t, \vec{y}, \vec{y}', \dots, \vec{y}^{(k)})$ is a **scalar** if $n = 1$ and is a **system** if $n > 1$.

Example

$y' - y$ is a scalar while $y_1' = y_2$ and $y_2' = y_1$ is a system.

An important special case that we will focus on is systems of first order differential equations.

$$\vec{y}' = \vec{f}(t, \vec{y}) \qquad \Longleftrightarrow \qquad \begin{aligned} y_1' &= f_1(t, y) \\ &\vdots \\ y_n' &= f_n(t, y) \end{aligned}$$

Theorem

Any n^{th} order scalar differential equation can be written as a system of n first-order differential equations.

Example

Take the 3$^{\text{rd}}$ order system $y''' + 2yy'' + (y')^2 + \sin(t) = 0$. Define y_0, y_1, and y_2 such that

$$y_0 = y \qquad \begin{aligned} y_1 &= y_0' \\ &= y' \end{aligned} \qquad \begin{aligned} y_2 &= y_1' \\ &= y_0'' \\ &= y''' \end{aligned}$$

So, we have the system of 3 first-order equations

$$\begin{aligned} y_0' &= y_1 \\ y_1' &= y_2 \\ y_2' &= y''' \\ &= -(2yy'' + (y')^2 + \sin(t)) \\ &= -(2y_0y_2 + y_1^2 + \sin(t)) \end{aligned}$$

In fact, any system of n-many k^{th} order differential equations can be written as nk-many 1$^{\text{st}}$ order differential equations.

Recall the definition of linearity: any $g(\vec{y})$ is linear if it fulfills the superposition principle, which states that

$$g(c_1\vec{y}_1 + c_2\vec{y}_2) = cg(\vec{y}_1) + c_2g(\vec{y}_2)$$

$\forall c_1, c_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^n$.
$\vec{y}' = \vec{f}(t, \vec{y})$ is **linear** if $\vec{f}$ is linear in $\vec{y}$. This implies that $\vec{f} = A(t)\vec{y} + \vec{b}(t)$ where A is an $n \times n$ matrix.

Theorem: Principle of Superposition

If $\vec{\varphi}_{(1)}$ and $\vec{\varphi}_{(2)}$ are solutions to $\vec{y}' = A\vec{y}$, then so is

$$c_1\vec{\varphi}_{(1)} + c_2\vec{\varphi}_{(2)}$$

for $c_1, c_2 \in \mathbb{R}$.

A **homogeneous linear equation** is an equation of the form $\vec{y}' = A\vec{y}$ (where $\vec{b}(t) = 0$). This implies that $(c\vec{y})' = A(c\vec{y}) = c(A\vec{y})$.

A **nonhomogeneous linear equation** is an equation of the form $\vec{y}' = A\vec{y} + \vec{b}(t)$ (where $\vec{b}(t) \neq 0$). This does not imply that $(c\vec{y})' = A(c\vec{y}) + \vec{b}(t)$.

Types of ODE Problems

Take the simplest ODE, $y' = g(t)$. $\varphi(t) = \int g(t)\mathrm{d}t + C$ is a solution, but it is a family of solutions rather than one single solution.

1. **General solution problems**: $\varphi(t) - \int g(t)\mathrm{d}t + C$ is the general solution of $y' = g(t)$. However, in applications, this may not be useful. In problems in mechanics, we are not generally interested in all solutions to a given equation, but rather the solution which gescribes the motion of the particle at hand. In other words, we want uniqueness of a solution.

2. **Initial value problems**: consider $\vec{F}(t, \vec{y}, \vec{y}', \ldots, \vec{y}^{(k)}) = 0$ and consider $\vec{\eta}_0, \vec{\eta}_1, \ldots, \vec{\eta}_{k-1} \in \mathbb{R}^n$. The initial value problem (or the Cauchy problem) for the system asks for a solution $\vec{\varphi}(t)$ which satisfies the following initial conditions at some initial time $t_0 \in I$:

$$\vec{\varphi}(t_0) = \vec{\eta}_0,\ \vec{\varphi}'(t_0) = \vec{\eta}_1, \ldots, \vec{\varphi}^{(k-1)}(t_0) = \vec{\eta}_{k-1}$$

   The IVP arises naturally in evolutionary (dynamic) problems. This will be our main focus in this course.

3. **Boundary value problems**: take $k = 2$ and $n = 1$. We have $F(t, y, y', y'') = 0$ where y and F are scalar, and $t \in I = (a, b)$. Given η_a and $\eta_b \in \mathbb{R}$, find a solution to F such that $\varphi(a) = \eta_a$ and $\varphi(b) = \eta_b$.

Solving First Order ODEs

We will review two main techniques of solving first order ODEs.

Separation of Variables

This is for first order scalar differential equations that are separable:

$$y' = g(t)h(y)$$

Steps for solving:

1. Find all $\ell \in \mathbb{R}$ such that $h(\ell) = 0$. Then, $\varphi(t) = \ell$ are the constant solutions to the ODE.

2. Separate the variables: $\frac{1}{h(y)}y' = g(t)$.

$$\int_{t_0}^{t_1} \frac{1}{h(y)} \frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t = \int_{t_0}^{t_1} g(t)\mathrm{d}t \qquad \text{substitution rule}$$
$$\underbrace{\int_{y(t_0)}^{y(t_1)} \frac{1}{h(y)}\mathrm{d}y}_{\text{expression in } y(t)} = \underbrace{\int_{t_0}^{t_1} g(t)\mathrm{d}t}_{\text{expression in } t}$$

If solving an IVP with $y(t_0) = \eta_0$, then take the integral domain to be $\int_{\eta_0}^{y(t)}$.

Example

Solve the logistic equation. which is used to model population growth:

$$P' = k\left(1 + \frac{P}{M}\right)P, \quad P(0) = P_0$$

k is the growth constant and M is the carrying capacity.

First, find the zeros of $k\left(1 - \frac{P}{M}\right)P = 0$. These are at $P = 0$ and $P = M$, so these are the constant solutions to the logistic equation.

Now, perform separation of variables.

$$\begin{aligned}
\int_0^t \frac{1}{\left(1-\frac{P}{M}\right)P} P' \mathrm{d}t &= \int_0^t k \mathrm{d}t \\
\int_{P_0}^{P(t)} \frac{1}{\left(1-\frac{P}{M}\right)P} \mathrm{d}P &= kt \\
\int_{P_0}^{P(t)} \frac{\frac{1}{M}}{1-\frac{P}{M}} + \frac{1}{P} \mathrm{d}P &= kt \\
-\log\left|1-\frac{P}{M}\right| + \log|P| \Big|_{P_0}^{P(t)} &= kt \\
\log\left|\frac{P}{1-\frac{P}{M}}\right| \Big|_{P_0}^{P(t)} &= kt \\
\log\left|\frac{P(t)}{1-\frac{P(t)}{M}}\right| \cdot \left|\frac{1-\frac{P_0}{M}}{P_0}\right| &= kt \\
\log\left|\frac{P(t)}{1-\frac{P(t)}{M}}\right| &= kt + \log\left|\frac{P_0}{1-\frac{P_0}{M}}\right| \\
\frac{P(t)}{1-P(t)} &= e^{kt} \cdot \frac{P_0}{1-\frac{P_0}{M}}
\end{aligned}$$

From here, solve for $P(t)$.

Integrating Factors

These are to solve linear first-order ODEs of the form

$$y' = a(t)y + b(t).$$

In the homogeneous case ($b = 0$), this can be solved by separation of variables. In the nonhomogeneous case, we want to find the **integrating factor** $I(t)$ such that

$$\begin{aligned}
(I(t)y)' &= I(t)(y' - a(t)y) \\
&= I(t)b(t)
\end{aligned}$$

We can find this using $(Iy)' = Iy' + I'y$, so $-Iay = I'y$. So, we need to solve $-Ia = I'$, which is separable. Take $I(t) = e^{-A(t)}$ where $A(t)$ is any antiderivative of a. $A(t)' = a(t)$, so $A(t) = \int a(t)\mathrm{d}t$. Thus,

$$\begin{aligned}
I' &= e^{-A(t)}(-A'(t)) \\
&= -ae^{-A} \\
&= -aI
\end{aligned}$$

Now, we can use the integrating factor to solve the ODE.

$$I(t)(y' - a(t)y) = I(t)b(t)$$

The LHS is equal to $(I(t)y)'$.

$$\begin{aligned}
I(t)y(t) - I(\alpha)y(\alpha) &= \int_\alpha^t I(s)b(s)\mathrm{d}s \\
y(t) &= \frac{I(\alpha)}{I(t)}y(\alpha) + \int_\alpha^t \frac{I(s)}{I(t)}b(s)\mathrm{d}s
\end{aligned}$$

Example

Solve $y' = y + t$. Here, $a = 1$, so $I(t) = e^{-t}$.

$$\begin{aligned}
y' - y &= t \\
(e^{-t}y)' &= e^{-t}t \\
e^{-t}y(t) &= e^{-\alpha}y(\alpha) + \int_\alpha^t e^{-s}s\mathrm{d}s \\
&= e^{-\alpha}y(\alpha) - e^{-s}s\Big|_\alpha^t + \int_\alpha^t e^{-s}\mathrm{d}s \\
&= e^{-\alpha}y(\alpha) - e^{-s}s\Big|_\alpha^t - e^{-s}\Big|_\alpha^t \\
&= e^{-\alpha}y(\alpha) - e^{-t}t + e^{-\alpha}\alpha - e^{-t} + e^{-\alpha} \\
y(t) &= -t = 1 + e^{t-\alpha}(y(\alpha) + \alpha + 1)
\end{aligned}$$

This is an instance of variation of constants (Duhamel's formula), which we will return to later in the course.

Bernoulli's Equation

Bernoulli's equation is an example of a nonlinear ODE:

$$y' = a(t)y + b(t)y^n$$

Our goal is to perform a change of variables to turn the equation into a first order linear equation. We will assume that $n \neq 1$ since otherwise, the equation is linear. Note that n does not need to be an integer. Assume also that $y \neq 0$ (although $y = 0$ is a solution).

$$\begin{aligned}
y' &= a(t)y + b(t)y^n \\
\frac{1}{y^n}y' &= a(t)y^{1-n} + b(t)
\end{aligned}$$

We have

$$\begin{aligned}
\int \frac{1}{y^n}y'\mathrm{d}t = \int \frac{1}{y^n}\mathrm{d}y &= \frac{1}{1-n}y^{1-n} + C \\
\frac{1}{y^n}y' &= \frac{1}{1-n}(y^{1-n})'
\end{aligned}$$

So, take $z = y^{1-n}$. Then, we can write

$$\frac{1}{1-n}z' = a(t)z + b(t)$$

From here, we can use the integrating factor as usual.

Note that when substituting z and y, we must be careful to avoid zero. If z hits zero, then this marks the end of the interval of validity of y.

Example

Solve $y' = y^2$ where $y(0) = \eta_0 > 0$. We have $z = y^{-1}$ where $z(0) = \frac{1}{y(0)} = \frac{1}{\eta_0}$. So, we want to solve $\frac{1}{1-z}z' = 1$, or $-z' = 1$.

$$z = -t + c = -t + \frac{1}{\eta_0} = \frac{1}{y(t)}$$

So, we have $y(t) = \frac{1}{-t+\frac{1}{\eta_0}}$. This is defined on $\left[0, \frac{1}{\eta_0}\right)$ which is the interval of validity. As $t \to \frac{1}{\eta_0}$, $y \to +\infty$.

Grönwall's Inequality

Theorem: Grönwall's Inequality (differential form)

Let f and g be continuous on $[\alpha, \beta]$. Suppose $g \geq 0$ on $[\alpha, \beta]$ and f is differentiable on (α, β) with $f' \leq g(t)f$ for $t \in (\alpha, \beta)$. Then,

$$f(t) \leq f(\alpha)e^{\int_\alpha^t g(s)\mathrm{d}s}$$

for $t \in [\alpha, \beta]$. In other words, $f(t)$ is bounded from above by the solution to the differential equation $F' = g(t)F$ with $F(\alpha) = f(\alpha)$.

- **Proof**: there is a quick proof using the method of integrating factors.

$$f' - g(t)f \leq 0$$

$I = e^G$ where $G' = -g$. $I \geq 0$ since it is an exponential.

$$\begin{aligned}(If)' &= I(f' - gf) \leq 0 \\ I(t)f(t) &= I(\alpha)f(\alpha) \leq 0 \\ f(t) &\leq \frac{I(\alpha)}{I(t)}f(\alpha)\end{aligned}$$

$\square$

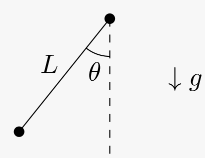
Well-Posedness of ODEs

Many interesting ODEs do not admit solutions expressed in terms of elementary functions (polynomial, exponential, logarithmic, trigonometric).

Example

Observe the single pendulum. The motion can be modeled by the following ODE:

$$\theta'' + \frac{g}{L}\sin(\theta) = 0$$



We want to develop approaches that do not necessarily require explicit expressions of solutions. In doing this, mathematical rigor is crucial to avoid missteps.

A fundamental (basic) issue that has to be sorted out is whether solutions to the problem exist. Suppose we have the following IVP:

$$\begin{cases}\vec{y}' = \vec{f}(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}_0\end{cases}$$

The system is (local) **well-posed** if the following conditions are satisfied:

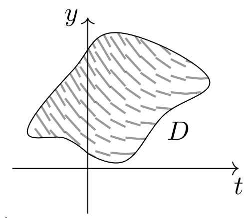
1. **Existence**: $\exists$ an interval I with $t_0 \in I$ and a solution $\vec{\varphi} : I \to \mathbb{R}^n$ which solves the system.
2. **Uniqueness**: if $\vec{\varphi}_1 : I \to \mathbb{R}^n$ and $\vec{\varphi}_2 : I \to \mathbb{R}^n$ are both solutions to the system, then $\vec{\varphi}_1 = \vec{\varphi}_2$.
3. **Continuous dependence on data (continuity)**: the solution $\vec{\varphi}_{\vec{y}_0}$ to the system depends continuously on the initial condition $\vec{y}_0$, i.e. if $\vec{y}_k \to \vec{y}_0$, then $\sup_I |\vec{\varphi}_{\vec{y}_k} - \vec{\varphi}_{\vec{y}_0}| \to 0$.

The third condition is local **stability**, which allows some error in the initial condition. Local refers to an interval around t_0 (as opposed to global).

Existence Theory for Scalar First-Order ODEs

We will prove well-posedness of the IVP for a large class of first-order ODEs. This is called the Picard-Lindelöf theorem, the Cauchy-Lipschitz Theorem, or the Fundamental Theorem of ODEs.

Take the first-order ODE $y' = f(t,y)$ with $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}_t \times \mathbb{R}_y$, which is a visualizable plane. We can draw a slope field to visualize y'. At point (t,y), draw a line segment with slope $f(t,y)$. This is the slope-field representation of f.



The graphical solution at (t_0, y_0) is the graph which matches the slope field at the point.

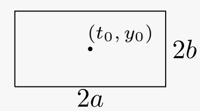
The existence property states that if f is sufficiently regular near (t_0, y_0), then $\exists$ a local solution φ to $y' = f(t,y)$. Sufficient regularity will be Lipschitz continuity in y. $f : D \to \mathbb{R}$ with $d \subseteq \mathbb{R}_t \times \mathbb{R}_y$ is **Lipschitz continuous** in y (satisfies a Lipschitz property) with Lipschitz constant K if $\exists K > 0$ such that

$$|f(t,y_2) - f(t,y_1)| \leq K|y_2 - y_1|$$

$\forall (t,y_1),(t,y_2) \in D$.

Example

Let $f : R \to \mathbb{R}$ with $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$. If f is continuous on R and $\frac{\partial}{\partial y} f$ is continuous on R with bound $\left|\frac{\partial}{\partial y} f(t,y)\right| \leq K$ in R, then f is Lipschitz continuous in y with Lipschitz constant K.



We can prove this using the Fundamental Theorem of Calculus.

$$\begin{aligned} |f(t,y_2) - f(t,y_1)| &= \left|\int_{y_1}^{y_2} \frac{\partial}{\partial y} f(t,y)\mathrm{d}y\right| \\ &\leq \int_{y_1}^{y_2} \left|\frac{\partial}{\partial y} f(t,y)\right| \mathrm{d}y && \text{triangle ineq.} \\ &\leq K(y_2 - y_1) \end{aligned}$$

A non-example is $f(t,y) = y^\alpha$ with $0 < \alpha < 1$ on $I \times [-b,b]$ which is not Lipschitz continuous at 0.

Theorem: Existence Theorem for Scalar First-Order ODEs

Consider $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ and consider the problem

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \tag{1}$$

where $f : R \to \mathbb{R}$. Suppose f is continuous on R and Lipschitz continuous in y on R with Lipschitz constant K. Then, $\exists$ an interval $I = (t_0 - \alpha, t_0 + \alpha)$ and a solution $\varphi : I \to \mathbb{R}$ to the system.

Note that in fact, we will produce a solution φ such that $(t, \varphi(t)) \in R$ $\forall t \in I$. However, we may take $I = (t_0 - \alpha, t_0 + \alpha)$ with $\alpha = \min\left\{a, \frac{b}{m}\right\}$ with $M = \sup_R |f|$. This is because if we have $|f| \leq M$ in R, the slope at each point is between $-M$ and M.

Proving the Existence Theorem

Note that we are trying to find $\varphi(t)$ such that $\varphi'(t) = f(t, \varphi(t))$ with $\varphi(t_0) = y_0$. The difficulty lies in the dependence of f on $\varphi(t)$. Otherwise, $\varphi'(t) = f(t)$ with $\varphi(t_0) = y_0$ can be solved by integration $\varphi(t) = y_0 + \int_{t_0}^t f(s)\mathrm{d}s$. It turns out that integrating the differential equation is stil a useful idea.

Step 1: reduction to an integral form.

Lemma

φ is a solution to (1) on I if and only if $\varphi(t)$ is continuous on I satisfying

$$\varphi(t) = y_0 + \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s.$$

- **Proof**: if $\varphi : I \to \mathbb{R}$ is a solution to (1), then $\varphi(t) - \varphi(t_0) = \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s$, or $\varphi(t) = y_0 + \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s$.

  Conversely, suppose φ is a continuous function on $I \to \mathbb{R}$ satisfying $\varphi(t) = y_0 + \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s$. Note that $f(s, \varphi(s))$ is continuous, so $\varphi(t) = y_0 + \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s$ is continuously differentiable. So, by differentiating the equation, we get our original system. □

Step 2: forming Picard iterates (successive approximations).

Let us try to find reasonable approximation $\varphi_j(t)$ for our solution:

$$\begin{aligned}
\varphi_0(t) &= y_0 \\
\varphi_1(t) &= y_0 + \int_{t_0}^t f(s, \varphi_0(s))\mathrm{d}s \\
&\vdots \\
\varphi_j(t) &= y_0 + \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s
\end{aligned}$$

The idea is to show that $\varphi_j(t) \to \varphi(t)$ as $j \to \infty$, which solves the system. We start by verifying that for each j, $(t, \varphi_j(t)) \in R \ \forall t \in [t_0 - \alpha, t_0 + \alpha]$.

Lemma

Let $a = \min\left\{a, \frac{b}{M}\right\}$. Recall $|f| \leq M$ in R. Let $I = [t_0 - \alpha, t_0 + \alpha]$. Then, $\varphi_j : I \to R$ is well-defined and satisfies $(t, \varphi_j(t)) \in R \ \forall t \in I$.

- **Proof**: use mathematical induction.

  1. Base case: we have $\varphi_0(t) = y_0$ which is continuous and satisfies $(t_0, \varphi_0(t)) = (t_0, y_0) \in R$.
  2. Induction hypothesis: suppose $\varphi_{j-1} : I \to R$ is a continuous function satisfying $(t, \varphi_{j-1}(s)) \in R$.
  3. Inductive step: show that $\varphi_j : I \to R$ is continuous and satisfies $(t, \varphi_j(s)) \in R$. $f(s, \varphi_{j-1}(s))$ is continuous on I, so $\varphi_j(t) = y_0 + \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s$ is continuous on I. Moreover, $(t, y) \in \mathbb{R}$ if and only if $|t - t_0| \leq a$, $|y - y_0| \leq b$.

$$\begin{aligned}
|\varphi_j(t) - y_0| &= \left| \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s \right| \\
&\leq \int_{t_0}^t |f(s, \varphi_{j-1}(s))|\mathrm{d}s && \text{triangle ineq.} \\
&\leq \int_{t_0}^t M\mathrm{d}s \\
&\leq M|t - t_0| \\
&\leq M\alpha \\
&\leq b
\end{aligned}$$

  We also have $|t - t_0| \leq \alpha \leq a$. So, our result is proved. □

Step 3: showing convergence of φ_j.

The idea is to use

$$\begin{aligned}
\varphi_j &= \varphi_0 + \varphi_1 - \varphi_0 + \cdots + \varphi_j - \varphi_{j-1} \\
&= \varphi_0 + \sum_{j'=1}^{j} (\varphi_{j'} - \varphi_{j'-1})
\end{aligned}$$

to prove convergence. We will show that $\varphi_{j'} - \varphi_{j'-1}$ gets small as $j' \to \infty$.

$$
\begin{aligned}
\varphi_{j'}(t) &= y_0 + \int_{t_0}^{t} f(s, \varphi_{j'-1}(s)) \mathrm{d}s \\
\varphi_{j'-1}(t) &= y_0 + \int_{t_0}^{t} f(s, \varphi_{j'-2}(s)) \mathrm{d}s \\
(\varphi_{j'} - \varphi_{j'-1})(t) &= \int_{t_0}^{t} f(s, \varphi_{j'-1}(s)) - f(s, \varphi_{j'-2}(s)) \mathrm{d}s \\
|(\varphi_{j'} - \varphi_{j'-1})(t)| &\leq \int_{t_0}^{t} |f(s, \varphi_{j'-1}(s)) - f(s, \varphi_{j'-2}(s))| \, \mathrm{d}s \\
&\leq \int_{t_0}^{t} K \, |\varphi_{j'-1}(s) - \varphi_{j'-2}(s)| \, \mathrm{d}s
\end{aligned}
$$

This is a recurrence relation. We want to conjecture an upper bound to $|(\varphi_{j'} - \varphi_{j'-1})(t)|$.

$$
\begin{aligned}
(\varphi_1 - \varphi_0)(t) &= \int_{t_0}^{t} f(s, y_0) \mathrm{d}s \\
|(\varphi_1 - \varphi_0)(t)| &\leq M|t - t_0| \\
|(\varphi_2 - \varphi_1)(t)| &\leq \int_{t_0}^{t} K|(\varphi_1 - \varphi_0)(s)| \mathrm{d}s \\
&\leq \int_{t_0}^{t} MK(s - s_0) \mathrm{d}s \\
&\leq \frac{MK(t - t_0)^2}{2} \\
|(\varphi_3 - \varphi_2)(t)| &\leq \int_{t_0}^{t} k|(\varphi_2 - \varphi_1)(s)| \mathrm{d}s \\
&\leq \int_{t_0}^{t} \frac{MK^2(s - s_0)^2}{2} \mathrm{d}s \\
&\leq \frac{MK^2(t - t_0)^3}{2 \cdot 3}
\end{aligned}
$$

So, we can make a conjecture:

$$
|(\varphi_{j'} - \varphi_{j'-1})(s)| \leq \frac{MK^{j'-1}|t - t_0|^{j'}}{(j')!}
$$

This can be easily proved using induction.

Now, note that

$$
\varphi_j(t) = \varphi_0(t) + \sum_{j'=1}^{j} \underbrace{(\varphi_{j'} - \varphi_{j'-1})(t)}_{r_{j'}(t)}
$$

for $t \in I = [t_0 - \alpha, t_0 + \alpha]$ and that

$$
|r_{j'}(t)| \leq \frac{MK^{j'-1}|t - t_0|^{j'}}{(j')!} \leq \frac{MK^{j'-1}\alpha^{j'}}{(j')!}.
$$

If

$$
\sum_{j=0}^{\infty} \frac{MK^{j-1}\alpha^j}{j!}
$$

converges, then by comparison, $\varphi_j(t)$ converges absolutely. It turns out that

$$
\sum_{j=0}^{\infty} \frac{MK^{j-1}\alpha^j}{j!} = \frac{M}{K} e^{K\alpha}.
$$

Step 4: completing the proof.

To finish the proof, we must show 3 things: φ_j is convergent, φ is continuous, and φ satisfies

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\mathrm{d}s.$$

From step 1, this would prove the Existence Theorem.

1. φ_j is convergent.

   We defined $r_{j'}(t) = \varphi_{j'}(t) - \varphi_{j'-1}(t)$. We also have

   $$|r_j| \leq \frac{MK^{j-1}\alpha^j}{j!}$$

   and

   $$\underbrace{\sum_{j=0}^{\infty} \frac{MK^{j-1}\alpha^j}{j!}}_{\text{convergent}} = \frac{M}{K}e^{K\alpha}.$$

   So, by the comparison test, $\sum\limits_{j'} r_{j'}(t)$ converges (absolutely) $\forall t \in I$. So, we define

   $$\varphi(t) = \varphi_0 + \sum_{j'=1}^{\infty} r_{j'}(t).$$

2. φ is continuous.

   $$\begin{aligned}
   |\varphi(t) - \varphi_j(t)| &= \left|\varphi_0 + \sum_{j'=1}^{\infty} r_{j'}(t) - \left(\varphi_0 + \sum_{j'=1}^{j} r_{j'}(t)\right)\right| \\
   &= \left|\sum_{j'=j+1}^{\infty} r_{j'}(t)\right| \\
   &\leq \sum_{j'=j+1}^{\infty} |r_{j'}(t)| \\
   &\leq \sum_{j'=j+1}^{\infty} \frac{MK^{j'-1}\alpha^{j'}}{j'!}
   \end{aligned}$$

   We can show that this $\to 0$ as $j \to \infty$ since $\sum r_j$ converges and this is independent of t. However, we will complete the proof of continuity.

   $$\begin{aligned}
   &\leq \frac{M}{K}\sum_{n=0}^{\infty} \frac{K^{j+1+n}\alpha^{j+1+n}}{(j+1+n)!} \\
   &\leq \frac{M}{K}\frac{K^{j+1}\alpha^{j+1}}{(j+1)!}\sum_{n=1}^{\infty}\frac{K^n\alpha^n}{n!} \\
   &= \frac{MK^j\alpha^{j+1}}{(j+1)!}e^{k\alpha}
   \end{aligned}$$

   From here, we can use the $\varepsilon/3$ argument. Let us prove that φ is continuous on I. We want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|\varphi(t) - \varphi(t')| \leq \varepsilon$ if $|t - t'| < \delta$ for $t, t' \in I$.

   $$\begin{aligned}
   |\varphi(t) - \varphi(t')| &= |\varphi(t) - \varphi_j(t) + \varphi_j(t) - \varphi_j(t') + \varphi_j(t') - \varphi(t')| \\
   &\leq |\varphi(t) - \varphi_j(t)| + |\varphi_j(t) - \varphi_j(t')| + |\varphi_j(t') - \varphi(t')|
   \end{aligned}$$

(i) We know that $\frac{K^{j+1}\alpha^{j+1}}{(j+1)!} \to 0$ as $j \to \infty$ since $\sum_{j=0}^{\infty} \frac{K^{j+1}\alpha^{j+1}}{(j+1)!}$ is convergent. So, we can take $\frac{MK^j\alpha^{j+1}}{(j+1)!} < \frac{\varepsilon}{3}$ for j sufficiently large, meaning

$$|\varphi(t) - \varphi_j(t)| \le \varepsilon/3 \quad \text{and} \quad |\varphi(t') - \varphi_j(t')| \le \varphi/3$$

for $t, t' \in I$.

(ii) φ_j is continuous at t', which can be proved by induction. So, $\exists \delta > 0$ such that if $|t - t'| < \delta$, then $|\varphi_j(t) - \varphi_j(t')| < \varepsilon/3$.

So, for $|t - t'| < \delta$, we have

$$\begin{aligned} |\varphi(t) - \varphi(t')| &\le \underbrace{|\varphi(t) - \varphi_j(t)|}_{\le \varepsilon/3 \text{ by (i)}} + \underbrace{|\varphi_j(t) - \varphi_j(t')|}_{\le \varepsilon/3 \text{ by (ii)}} + \underbrace{|\varphi_j(t') - \varphi(t')|}_{\le \varepsilon/3 \text{ by (i)}} \\ &\le \varepsilon \end{aligned}$$

So, φ is continuous.

3. φ satisfies $\varphi(t) = y_0 + \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s$.

$$\begin{aligned} \varphi(t) = \lim_{j\to\infty} \varphi_j(t) &= \lim_{j\to\infty} \left[y_0 + \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s \right] \\ &= y_0 + \lim_{j\to\infty} \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s \end{aligned}$$

We want to show that

$$\lim_{j\to\infty} \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s = \int_{t_0}^t \lim_{j\to\infty} f(s, \varphi_{j-1}(s))\mathrm{d}s = \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s.$$

To do this, we can show that as $j \to \infty$, $\left| \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s - \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s \right| \to 0$.

$$\begin{aligned} \left| \int_{t_0}^t f(s, \varphi_{j-1}(s))\mathrm{d}s - \int_{t_0}^t f(s, \varphi(s))\mathrm{d}s \right| &\le \int_{t_0}^t |f(s, \varphi_{j-1}(s)) - f(s, \varphi(s))| \, \mathrm{d}s \\ &\le \int_{t_0}^t K \, |\varphi_{j-1}(s) - \varphi(s)| \, \mathrm{d}s \\ &\le \int_{t_0}^t K \frac{MK^{j-1}\alpha^j}{j!} e^{K\alpha} \mathrm{d}s \\ &\le \alpha \cdot \frac{M(K\alpha)^j}{j!} e^{K\alpha} \end{aligned}$$

This approaches 0 as $j \to \infty$ since $\sum_{j=0}^{\infty} \frac{(K\alpha)^j}{j!} = e^{K\alpha}$, so $\frac{(K\alpha)^j}{j!} \to 0$ as $j \to \infty$.

Follow-up Discussions

1. Our proof to the Existence Theorem in fact gives us more information on how we can approximate φ using φ_j:

$$|\varphi(t) - \varphi_j(t)| \le \frac{M(K\alpha)^{j+1}}{K(j+1)!} e^{K\alpha}$$

2. This theorem is not the most general of its kind. Peano's Theorem has a similar statement: let f be continuous on $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ with $|f| \le M$ in R (without the Lipschitz condition). Then, $\exists$ a solution $\varphi : I \to \mathbb{R}$ to $y' = f(t, y)$ with $y(t_0) = y_0$ where $I = [t_0 - \alpha, t_0 + \alpha]$ and $\alpha = \min\left\{a, \frac{b}{M}\right\}$.

   But, this assumption is usually insufficient for uniqueness, as Picard iterates generally won't converge. We will see this later. To prove this, we need Euler's method of approximation.

Existence Theorem for Systems of First-Order ODEs

Consider the IVP

$$\vec{y}' = \vec{f}(t,\vec{y}), \quad \vec{y}(t_0) = \vec{y}_0$$

with $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $\vec{f} = \begin{pmatrix} f_1(t,\vec{y}) \\ \vdots \\ f_n(t,\vec{y}) \end{pmatrix}$ where $f : D \to \mathbb{R}^n$ and $D \subseteq \mathbb{R} \times \mathbb{R}^n$.

The continuity of $\vec{f}$ is defined using $|\cdot|$ and $d(\cdot,\cdot)$ from Appendix A on page 62.

Let $\vec{f} : R \to \mathbb{R}$ where $R = [t_0 - a, t_0 + a] \times \{y \in \mathbb{R}^n : |y - y_0| \leq b\}$. $\vec{f}$ is Lipschitz continuous in $\vec{y}$ with constant K if

$$\left|\vec{f}(t,\vec{y}_{(1)}) - \vec{f}(t,\vec{y}_{(2)})\right| \leq K \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right|$$

$\forall (t, \vec{y}_{(i)}) \in R$.

Theorem: Existence Theorem for Systems of First-Order ODEs

Consider $R = [t_0 - a, t_0 + a] \times \{\vec{y} \in \mathbb{R}^n : |\vec{y} - \vec{y}_0| \leq b\}$ and consider the problem

$$\begin{cases} \vec{y}' = \vec{f}(t,\vec{y}) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases} \tag{2}$$

where $\vec{f} : R \to \mathbb{R}$. Suppose $\vec{f}$ is continuous on R and Lipschitz continuous in $\vec{y}$ on R. Then, $\exists$ an interval $I = (t_0 - \alpha, t_0 + \alpha)$ and a solution $\vec{\varphi} : I \to \mathbb{R}^n$ to the system. If $|\vec{f}| \leq M$ on R, then $\alpha = \min\left\{a, \frac{b}{M}\right\}$.

The proof of this is the same as in the scalar case.

Uniqueness of Solutions for Systems of First-Order ODEs

Theorem: Uniqueness Theorem for Systems of First-Order ODEs

Consider $\mathbb{R} = [t_0 - a, t_0 + a] \times \{\vec{y} \in \mathbb{R}^n : |\vec{y} - \vec{y}_0| \leq b\}$ and consider the IVP (2) where $\vec{f} : R \to \mathbb{R}$. Suppose $\vec{f}$ is continuous on R and Lipschitz continuous in $\vec{y}$ on R. Then, there is at most one solution to (2), meaning if $\vec{\varphi}_1 : J_1 \to \mathbb{R}^n$ and $\vec{\varphi}_2 : J_2 \to \mathbb{R}^n$ both solve the system where J_1 and J_2 are intervals containing t_0, then $\vec{\varphi}_1 = \vec{\varphi}_2$ on $J_1 \cap J_2$.

- **Proof**: let $\vec{\varphi}_1$ and $\vec{\varphi}_2$ be two solutions to (2). Let $J = J_1 \cap J_2$ with $J = (\alpha, \beta)$ (either end could be closed, but that isn't important) and $t_0 \in J$. We want to show that $\vec{\varphi}_1 - \vec{\varphi}_2 = 0$.

  Recall the integral form of the system:

  $$\begin{aligned} \vec{y}(t) &= \vec{y}_0 + \int_{t_0}^{t} \vec{f}(s, \vec{y}(s))\mathrm{d}s \\ &= \vec{y}_0 + \begin{pmatrix} \int_{t_0}^{t} f_1(s,\vec{y}(s))\mathrm{d}s \\ \vdots \\ \int_{t_0}^{t} f_n(s,\vec{y}(s))\mathrm{d}s \end{pmatrix} \end{aligned}$$

  So, we can express $\vec{\varphi}_1 - \vec{\varphi}_2$ in this form.

  $$\begin{aligned} (\vec{\varphi}_1 - \vec{\varphi}_2)(t) &= \left(\vec{y}_0 + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}_1(s))\mathrm{d}s\right) - \left(\vec{y}_0 + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}_2(s))\mathrm{d}s\right) \\ &= \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}_1(s)) - \vec{f}(s, \vec{\varphi}_2(s))\mathrm{d}s \end{aligned}$$

Take $|\cdot|$ and use the Lipschitz continuity of $\vec{f}$ in $\vec{y}$.

$$\begin{aligned}|(\vec{\varphi}_1 - \vec{\varphi}_2)(t)| &\le \int_{t_0}^{t} \left|\vec{f}(s, \vec{\varphi}_1(s)) - \vec{f}(s, \vec{\varphi}_2(s))\right| \mathrm{d}s \\ &\le \int_{t_0}^{t} K \underbrace{|\vec{\varphi}_1(s) - \vec{\varphi}_2(s)|}_{\text{same quantity as LHS}} \mathrm{d}s\end{aligned}$$

Rewrite $F(t) = |\vec{\varphi}_1(t) - \vec{\varphi}_2(t)|$.

$$F(t) \le \underbrace{\int_{t_0}^{t} KF(s)\mathrm{d}s}_{H(t)}$$

$H(t)$ is differentiable, so $H'(t) = KF(t) \le KH(t)$. Now, recall Grönwall's inequality which can be proven by integrating factors, as shown in class, or by separation of variables, as we did in homework.

Grönwall's inequality: suppose $g(t)$ is a nonnegative continuous function on $[\mu, \nu]$ and suppose $H(t)$ is a differentiable function on (μ, ν) such that $H'(t) \le g(t)H(t)$ on $[\mu, \nu]$. Then,

$$H(t) \le H(\mu)e^{\int_\mu^t g(s)\mathrm{d}s}$$

where the RHS is the solution to $h' = gh$ where $h(\mu) = H(\mu)$.

By applying Grönwall's inequality to $H'(t) \le KH(t)$ with $H(t) = \int_{t_0}^{t} K\,|\vec{\varphi}_1(s) - \vec{\varphi}_2(s)|\,\mathrm{d}s$, we have $H(t) \le H(\mu)e^{\int_\mu^t K\mathrm{d}s}$. So,

$$|\vec{\varphi}_1(t) - \vec{\varphi}_2(t)| \le H(t) \le H(\mu)e^{\int_\mu^t K\mathrm{d}s}\Big|_{\mu = t_0} = 0$$

since $H(t_0) = 0$. So, $\vec{\varphi}_1 = \vec{\varphi}_2$. □

This proof suggests that the following integral form of Grönwall's inequality is of interest.

Theorem: Grönwall's Inequality (integral form)

Let f and g be continuous functions on $[\alpha, \beta]$ and let $g \ge 0$ on $[\alpha, \beta]$. Let $A > 0$. Suppose

$$f(t) \le A + \int_\alpha^t g(s)f(s)\mathrm{d}s$$

$\forall t \in [\alpha, \beta]$. Then,

$$f(t) \le Ae^{\int_\alpha^t g(s)\mathrm{d}s}.$$

- **Proof**: define $H(t) = A + \int_\alpha^t g(s)f(s)\mathrm{d}s$. Note that H is differentiable and that $f(t) \le H(t)$.

$$\begin{aligned}H(t) &= A + \int_\alpha^t g(s)f(s)\mathrm{d}s \\ H'(t) &= g(s)f(s) \\ &\le g(s)\left(A + \int_\alpha^t g(s)f(s)\mathrm{d}s\right)\end{aligned}$$

by the condition and since $g(s) \ge 0$ on $[\alpha, \beta]$

$$= g(s)H(t)$$

So, we can apply the differential form of Grönwall's inequality.

$$\begin{aligned}f(t) \le H(t) &\le H(\alpha)e^{\int_\alpha^t g(s)\mathrm{d}s} \\ f(t) &\le Ae^{\int_\alpha^t g(s)\mathrm{d}s}\end{aligned}$$

So, by using the differential form of Grönwall's inequality, we have $f(t) \le Ae^{\int_\alpha^t g(s)\mathrm{d}s}$. □

Recall that for the Existence theorem, we only need $\vec{f}$ to be continuous to guarantee existence. However, uniqueness may fail if the Lipschitz continuity in $\vec{y}$ is dropped.

Example

$$y' = f(t,y) = \begin{cases} 3y^{2/3} & y > 0 \\ 0 & y \leq 0 \end{cases}$$

$f(t,y)$ is continuous on $(-\infty, \infty)$ but not Lipschitz continuous in y. Note that $\varphi_1 = 0$ is a solution. Let us also look for a positive solution $\varphi_2 > 0$.

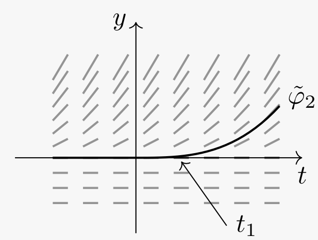
$$y' = 3y^{2/3}$$
$$\int \frac{1}{3} y^{-2/3} \mathrm{d}y = \int 1 \mathrm{d}t$$
$$y(t) = (t+C)^3$$

$y(t) = (t+C)^3$ is a solution for $t + C > 0$, so $\varphi_2(t) = (t-t_1)^3$ for $g > t_0$. Note that the following is also a solution:

$$\tilde{\varphi}_2 = \begin{cases} 0 & t \leq t_1 \\ (t-t_0)^3 & t > t_1 \end{cases}$$

We just need to check that $\tilde{\varphi}_2$ is still continuously differentiable near $t = t_1$ and $\tilde{\varphi}' t_1) = 0 = f(t_1, \tilde{\varphi}(t_1))$. But now, notice that $\forall t_1 > 0$, $\tilde{\varphi}_2$ is a solution to the IVP $y' = f(t,y)$ with $y(0) = 0$.

We can see this graphically. Observe that $\tilde{\varphi}_2$ may "split" from the axis at any t_1.



We can combine the existence theorem and the uniqueness theorem into one theorem known as the Picard-Lindelöf Theorem or the Cauchy-Lipschitz Theorem.

Theorem: Existence and Uniqueness Theorem

Consider $R = [t_0 - a, t_0 + a] \times \{\vec{y} \in \mathbb{R}^n : |\vec{y} - \vec{y}_0| \leq b\}$ and the problem

$$\begin{cases} \vec{y}' = \vec{f}(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases} \tag{2}$$

where $\vec{f} : R \to \mathbb{R}$ and $|\vec{f}| \leq M$ on R. Suppose $\vec{f}$ is continuous on R and Lipschitz continuous in $\vec{y}$ on R with Lipschitz constant K. Then, $\exists$ a unique solution $\vec{\varphi} : I \to \mathbb{R}^n$ to the system whose graph $\{(t, \vec{\varphi}(t)) : t \in I\}$ belongs to R. $I = [t_0 - \alpha, t_0 + \alpha]$ where $\alpha = \min\left\{a, \frac{b}{M}\right\}$.

Note that $\exists M$ such that $|\vec{f}| \leq M$ since $|\vec{f}|$ is continuous on R, which is a closed and bounded subset of $\mathbb{R}^{1+n}$. Why is this true?

Theorem: Extreme Value Theorem

Suppose $h : K \to \mathbb{R}^n$ is continuous on K where $K \subseteq \mathbb{R}^N$ is closed and bounded meaning $\exists L \geq 0$ such that $\forall x, y \in K$, $|x - y| \leq L$. Then, h is bounded in K, meaning $\exists M > 0$ such that $|h(x)| \leq M \ \forall x \in K$.

Recall from calculus that $f(x)$ continuous on a closed interval is bounded. However, $f(x)$ continuous on an open interval may not be bounded. For example, $f(x) = \frac{1}{x}$ is continuous on $(0,1)$ but is not bounded.

Note also that a sufficient condition for the Lipschitz continuity of $\vec{f}$ in $\vec{y}$ on R is that $\frac{\partial}{\partial y_i}\vec{f}$ is continuous on R. The reason this works is because of the EVT, $\partial y_i \vec{f}$ is continuous in $\vec{y}$ on R implies that $\frac{\partial}{\partial y_i}\vec{f}$ is bounded in R. By the fundamental theorem of calculus, this implies that $\vec{f}$ is Lipschitz in $\vec{y}$ in R.

Continuing Solutions

We will start by looking at a motivating example.

Example

Let $y' = y^2$ and $y(0) = 1$. This is in the form of (1) with $f(t, y) = y^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ where $D = \mathbb{R}^2$. Does the theorem capture the maximum interval of validity (lifespan) of the solution φ to the system? To address this, we will do the following:

1. Compute φ explicitly and understand its lifespan
2. Compute $I = [t_0 - \alpha, t_0 + \alpha]$ given by the Existence and Uniqueness Theorem

To find φ explicitly, we will use separation of variables.

$$\begin{aligned} \int_0^{y(t)} \frac{1}{y^2}\mathrm{d}y &= \int_0^t 1\mathrm{d}t \\ -\frac{1}{y(t)} + 1 &= t \\ y(t) &= \frac{1}{1-t} \end{aligned}$$

$\varphi(t) = \frac{1}{1-t}$ is defined on $t \in (-\infty, 1)$.

Now, to find $I = [t_0 - \alpha, t_0 + \alpha]$, we must compute $\alpha = \min\left\{a, \frac{b}{M}\right\}$. More precisely, in order to apply the theorem, we must think of a rectangle $R = [t_0 - \alpha, t_0 + \alpha] \times [y_0 - b, y_0 + b]$ with $t_0 = 0$ and $y_0 = 1$.

$$M = \max_R |f| = \max_R |y^2| = \max_{[1-b,1+b]} y^2 = (1+b)^2$$

So, $\alpha = \min\left\{a, \frac{b}{(1+b)^2}\right\}$. $\frac{b}{(1+b)^2} \le \frac{1}{4}$ which can be seen using the AM-GM inequality.

$$\begin{aligned} \sqrt{b} = 1 \cdot \sqrt{b} &\le \frac{1 + \sqrt{b}^2}{2} \\ &= \frac{1+b}{2} \end{aligned}$$

So, $\frac{\sqrt{b}}{1+b} \le \frac{1}{2}$ meaning $\frac{b}{(1+b)^2} \le \frac{1}{4}$ and we have $\alpha \le \frac{1}{4}$. The theorem gives $\varphi : I \to \mathbb{R}$ with $I = \left[-\frac{1}{4}, \frac{1}{4}\right]$. The interval $I = \left[-\frac{1}{4}, \frac{1}{4}\right]$ is clearly shorter than the lifespan from $\varphi(t) = \frac{1}{1-t}$, which is $(-\infty, 1)$.

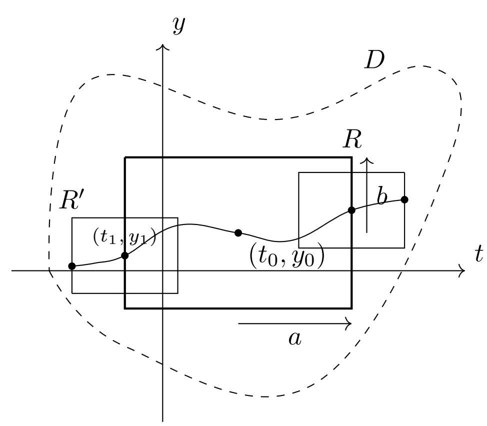
Our goal is to give a theoretical approach to capture the lifespan of the solution to the system.

Suppose we have $\vec{f} : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^2$. We start with original rectangle

$$R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b].$$

To continue the solution past the rectangle, we find the solution on the edges of the rectangle and fit more rectangles. Let the endpoint of φ obtained by applying the theorem to R be (t_1, y_1). By applying the theorem again with initial condition (t_1, y_1), we obtain new rectangle R'.

How much will this solution expand?



Theorem

Let D be an open set in $\mathbb{R}^{1+n}$. Consider $\vec{y}' = \vec{f}(t, \vec{y})$ and $\vec{y}(t_0) = \vec{y}_0$ with $\vec{f} : D \to \mathbb{R}^n$ where $\vec{f}$ and $\frac{\partial}{\partial y_j}\vec{f}$ are continuous in D. Assume also that f is bounded in D, i.e. $\exists M > 0$ such that $|\vec{f}| \leq M$ in D. Then, the solution $\vec{\varphi}$ to the system given by the Existence and Uniqueness Theorem can be extended until its graph reaches the boundary of D.

The last statement means the following:

$$\partial D = \{(t, \vec{y}) : \exists \text{ a sequence } (t_n, \vec{y}_n) \in D \text{ such that } (t_n, \vec{y}_n) \to (t, \vec{y})\} \backslash D$$

where ∂D is the boundary of D and D is an open set. This definition only works for D open, otherwise the set will be empty as ∂D would only contain points in D.

So, we can rewrite the statement of the theorem: if $\vec{\varphi}$ is defined on (t_-, t_+), then $\vec{\varphi}$ can be continued past t_+ as long as $\left(t_+, \lim\limits_{t \to (t_+)^-} \vec{\varphi}(t)\right) \notin \partial D$. Similarly, $\vec{\varphi}$ can be continued past t_- as long as $\left(t_-, \lim\limits_{t \to (t_-)^+} \vec{\varphi}(t)\right) \notin \partial D$.

Example

Let $y' = \cos^2(y) + \varepsilon \cos(t)$ with $y(t_0) = y_0$. This system cannot be solved explicitly if $\varepsilon \neq 0$, but the theorem applies with $D = R^2$. So, we have existence on $(-\infty, \infty)$, which is global existence.

Lemma

Consider the system (2). Assume that $\vec{f}$ is continuous on D and $\exists M > 0$ such that $|\vec{f}| \leq M$ in D. Then, for any solution $\vec{\varphi} : (c, d) \to \mathbb{R}^n$ to the system, $\lim\limits_{t \to d^-} \vec{\varphi}(t)$ and $\lim\limits_{t \to c^+} \vec{\varphi}(t)$ exist.

- **Proof**: recall Cauchy's criterion for existence of a (left) limit: the following are equivalent.

  1. $\lim\limits_{t \to d^-} \varphi(t)$ exists.
  2. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $t_1, t_2 \in (d - \delta, d)$, then $|\varphi(t_1) - \varphi(t_2)| < \varepsilon$.

  An analagous statement holds for $\lim\limits_{t \to c^+} \varphi(t)$. So, given $t_1, t_2 \in (c, d)$ (assume WLOG that $t_1 \leq t_2$), note that by the integral formulation of (2), we have the following:

  $$\vec{\varphi}(t_1) = y_0 + \int_{t_0}^{t_1} \vec{f}(s, \vec{\varphi}(s))\mathrm{d}s$$
  $$\vec{\varphi}(t_2) = y_0 + \int_{t_0}^{t_2} \vec{f}(s, \vec{\varphi}(s))\mathrm{d}s$$

  So, we can take the difference:

  $$\begin{aligned} |\vec{\varphi}(t_1) - \vec{\varphi}(t_2)| &= \left| \int_{t_1}^{t_2} \vec{f}(s, \vec{\varphi}(s))\mathrm{d}s \right| \\ &\leq \int_{t_1}^{t_2} \left| \vec{f}(s, \vec{\varphi}(s)) \right| \mathrm{d}s \\ &\leq M|t_2 - t_1| \end{aligned}$$

  Recall that from the Cauchy criterion, we have $t_1, t_2 \in (d - \delta, d)$ meaning $|t_1 - d|, |t_2 - d| < \delta$.

  $$|t_2 - t_1| = |t_2 - d + d - t_1| \leq |t_2 - d| + |d - t_1| < 2\delta < \frac{\varepsilon}{M}$$

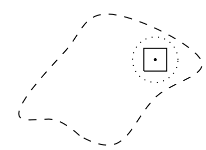
  So, for $\varepsilon > 0$, choosing $\delta = \frac{\varepsilon}{2M}$ satisfies the Cauchy criterion. Thus, $\lim\limits_{t \to d^-} \varphi(t)$ exists. We can use the analogous statement to show that $\lim\limits_{t \to c^+} \varphi(t)$ exists. $\square$

Now, we have the tools necessary to prove the theorem.

Proving the Continuation Theorem

Let $\vec{\varphi}$ be a solution to the system with maximum interval of validity (t_-, t_+). Let us focus on t_+, as t_- is handled similarly. Suppose by contradiction that $t_+ < +\infty$, $\lim\limits_{t \to t^+} \vec{\varphi}(t) := \vec{\varphi}(t_+)$ exists (by the lemma) but $(t_+, \vec{\varphi}(t_+)) \notin \partial D$. But since $(t_+, \vec{\varphi}(t_+))$ is the limit of a sequence in D by the definition of ∂D. $(t_+, \vec{\varphi}(t_+)) \in D$.

But since D is open, $\exists a, b > 0$ such that $R = [t_+ - a, t_+ + a] \times \{|\vec{y} - \vec{y}(t_+)| \leq b\}$ lies in D. Thus, the Existence and Uniqueness Theorem applies and gives a solution $\vec{\psi}$ on $[t_+ - \alpha, t_+ + \alpha]$ which agrees with $\vec{\varphi}$ on $(t_-, t_+) \cap [t_+ - \alpha, t_+ + \alpha]$. So, $\vec{\varphi}$ can be continued past t_+ until $t_+ + \alpha$, which is a contradiction.



Thus, if $t_+ < \infty$, $\lim\limits_{t \to t^+} \vec{\varphi}(t) = \vec{\varphi}(t_+)$ exists and $(t_+, \vec{\varphi}(t_+)) \in \partial D$. The analogous statement holds for t_-, thus proving the theorem. $\square$

Addressing the Boundness Assumption

In the theorem, requiring the function to be bounded is quite restrictive. The following is a quick consequence to the theorem which does not require boundness.

Corollary

If $D = \mathbb{R}^{1+n}$ and $\vec{f}$, $\frac{\partial}{\partial y_j} \vec{f}$ are continuous in D, then the solution $\vec{\varphi}$ to the system can be continued as long as $|\vec{\varphi}|$ is bounded. Equivalently, if (t_-, t_+) is the maximum interval of validity of $\vec{\varphi}$, then either $t_+ = +\infty$ or $t_+ < +\infty$ and $\limsup\limits_{t \to t_+^-} |\vec{\varphi}(t)| = +\infty$. An analogous statement holds for t_-.

- **Proof**: write $D = \mathbb{R}^{1+n} = \bigcup\limits_{A > 0} D_A$ where $D_A = \big\{(t, \vec{y}) \in \mathbb{R}^{1+n} : |\vec{y}| < A\big\}$. Apply the theorem to each D_A. If $\vec{\varphi}(t)$ is bounded by some A' as $t \to t^+$, then by the theorem with D_A and $A > A'$, the solution can be extended past t_+. $\square$

Note that the corollary does not say that if $\vec{f}$, $\frac{\partial}{\partial y_j} \vec{f}$ is continuous on $\mathbb{R}^{1+n}$, then every solution is global. This can be seen for $y' = y^2$.

Continuous Dependence on Initial Conditions

Recall that for a solution to be well-posed, three conditions must be satisfied: existence, uniqueness, and continuous dependence on initial condition.

Theorem

Consider the system

$$\begin{cases} \vec{y}' = \vec{f}(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases} \tag{2}$$

where $\vec{f} : D \to \mathbb{R}^n$ for $D \subseteq \mathbb{R}^{1+n}$ open and $|\vec{f}| \leq M$ in D. Assume $\vec{f}$ is continuous in D and Lipschitz continuous in $\vec{y}$ on D with Lipschitz constant K, meaning $\forall (t, \vec{y}_{(1)}), (t, \vec{y}_{(2)}) \in D$,

$$\left|\vec{f}(t, \vec{y}_{(1)}) - \vec{f}(t, \vec{y}_{(2)})\right| \leq K \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right|.$$

Let $\vec{\varphi}_{(1)}$ and $\vec{\varphi}_{(2)}$ be solutions to (2) with initial conditions

$$\vec{\varphi}_{(1)}(t_0) = \vec{y}_{(1)}, \qquad \vec{\varphi}_{(2)}(t_0) = \vec{y}_{(2)}$$

defined on a common interval $[\alpha, \beta]$ where $-\infty < \alpha < \beta < +\infty$. Then,

$$\left|\vec{\varphi}_{(1)}(t) - \vec{\varphi}_{(2)}(t)\right| \leq \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| e^{K|t - t_0|}.$$

- **Proof**: we will use the integral formulation and Grönwall's inequality. WLOG, assume $t > t_0$.

$$\vec{\varphi}_{(1)}(t) = \vec{y}_{(1)} + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}_{(1)}(s))\mathrm{d}s$$
$$\vec{\varphi}_{(2)}(t) = \vec{y}_{(2)} + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}_{(2)}(s))\mathrm{d}s$$

Now, take the difference an the absolute value.

$$\begin{aligned}
\vec{\varphi}_{(1)}(t) - \vec{\varphi}_{(2)}(t) &= \vec{y}_{(1)} - \vec{y}_{(2)} + \int_{t_0}^{t} \left[\vec{f}(s, \vec{\varphi}_{(1)}(s)) - \vec{f}(s, \vec{\varphi}_{(2)}(s))\right] \mathrm{d}s \\
\left|\vec{\varphi}_{(1)}(t) - \vec{\varphi}_{(2)}(t)\right| &\leq \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| + \int_{t_0}^{t} \left|\vec{f}(s, \vec{\varphi}_{(1)}(s)) - \vec{f}(s, \vec{\varphi}_{(2)}(s))\right| \mathrm{d}s \\
&\leq \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| + \int_{t_0}^{t} K \left|\vec{\varphi}_{(1)}(s) - \vec{\varphi}_{(2)}(s)\right| \mathrm{d}s
\end{aligned}$$

Now, apply Grönwall's inequality.

$$\begin{aligned}
\left|\vec{\varphi}_{(1)}(t) - \vec{\varphi}_{(2)}(t)\right| &\leq \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| e^{\int_{t_0}^{t} K \mathrm{d}t} \\
&= \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| e^{K|t-t_0|}
\end{aligned}$$

□

We can make the following remarks on this theorem.

1. In the textbook, the hypothesis is more general. The initial times may be different: $\vec{\varphi}_{(1)}(t_{(1)0}) = \vec{y}_{(1)}$ and $\vec{\varphi}_{(2)}(t_{(2)0}) = \vec{y}_{(2)}$ where $|t_{(1)0} - t_{(2)0}|$ is small. The conclusion is about $\left|\vec{\varphi}_{(1)}(t) - \vec{\varphi}_{(2)}(t)\right|$ and is written in $\varepsilon - \delta$ form. But, the idea of the proof is the same.

2. In what precise sense is this continuously dependent on the initial condition? Consider the data-to-solution map $S : \{\text{initial condition}\} \to \{\text{solutions}\} \subseteq C(I) = \{\vec{h} : I \to \mathbb{R}^n \mid \vec{h} \text{ is continuous}\}$ where $\vec{y}_i \mapsto \vec{\varphi}_{\vec{y}_0}(t)$ with solution to (2) defined on I. We can define the supremum/uniform norm on $C(I)$ by

$$\|h\| = \sup_{t \in I} |h(t)|$$

and a distance

$$d(h, g) = \|h - g\| = \sup_{t \in I} |h(t) - g(t)|.$$

Note that this is called the uniform norm since for any sequence g_n, $d(g_n, g) \to 0$ as $n \to \infty$ if and only if g_n converges to g uniformly on I. Recall the definition of uniform convergence: $\forall \varepsilon > 0$, $\exists N > 0$ such that if $n \geq N$, then $|g_n(t) - g(t)| < \varepsilon \; \forall t \in I$.

The theorem states that $S : \{\text{initial condition}\} \to C(I)$ is continuous where $\{\text{initial condition}\} \subseteq \mathbb{R}$.

3. In the case that the error $\left|\vec{\varphi}_{(1)}(t) - \vec{\varphi}_{(2)}(t)\right|$ does grow exponentially in time, we say that the system is sensitive to the initial condition. This is the "butterfly effect", which is one of the defining properties of chaos.

Additional Parameters

Often times, an ODE might involve some parameter α:

$$\begin{cases} y'(t) = \vec{f}(t, \vec{y}, \alpha) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases} \tag{3}$$

$\vec{f} : D \times J \to \mathbb{R}^n$ where $\alpha \in J$. One may ask about the continuous dependence of $\vec{\varphi}$ on α.

Theorem

Consider

$$\vec{y}(t) = \vec{f}(t, \vec{y}) \qquad\qquad \vec{y}(t) = \vec{g}(t, \vec{y})$$
$$\vec{y}(t_0) = \vec{y}_{(1)} \qquad\qquad \vec{y}(t_0) = \vec{y}_{(2)}$$

Assume that both $\vec{f}$ and $\vec{g}$ satisfy the hypothesis of the theorem, and also that $|\vec{g}(t,\vec{y}) - \vec{f}(t,\vec{z})| \leq K|\vec{y} - \vec{z}|$. Consider solutions $\vec{\varphi}$ and $\vec{\psi}$ to the two systems respectively and define a common interval $[\alpha, \beta]$ with $-\infty < \alpha < t < \beta < +\infty$. Then,

$$\left|\vec{\varphi}(t) - \vec{\psi}(t)\right| \leq \left(\left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| + \varepsilon(\beta - \alpha)\right) e^{K|t-t_0|}$$

where $\left|\vec{f}(t,\vec{y}) - \vec{g}(t,\vec{y})\right| < \varepsilon$ in D.

- **Proof**: like before, write the integral form of $\vec{\varphi}(t)$ and $\vec{\psi}(t)$.

$$\vec{\varphi}(t) = \vec{y}_{(1)} + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}(s))\mathrm{d}s \qquad\qquad \vec{\psi}(t) = \vec{y}_{(2)} + \int_{t_0}^{t} \vec{g}(s, \vec{\psi}(s))\mathrm{d}s$$

Now, take the difference.

$$\begin{aligned}
\vec{\varphi}(t) - \vec{\psi}(t) &= \vec{y}_{(1)} - \vec{y}_{(2)} + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}(s)) - \vec{g}(s, \vec{\psi}(s))\mathrm{d}s \\
&= \vec{y}_{(1)} - \vec{y}_{(2)} + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}(s)) - \vec{f}(s, \vec{\psi}(s)) + \vec{f}(s, \vec{\psi}(s)) - \vec{g}(s, \vec{\psi}(s))\mathrm{d}s \\
&= \vec{y}_{(1)} - \vec{y}_{(2)} + \int_{t_0}^{t} \left[\vec{f}(s, \vec{\varphi}(s)) - \vec{f}(s, \vec{\psi}(s))\right]\mathrm{d}s + \int_{t_0}^{t} \left[\vec{f}(s, \vec{\psi}(s)) - \vec{g}(s, \vec{\psi}(s))\right]\mathrm{d}s \\
\left|\vec{\varphi}(t) - \vec{\psi}(t)\right| &\leq \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| + \int_{t_0}^{t} \left|\vec{f}(s, \vec{\varphi}(s)) - \vec{f}(s, \vec{\psi}(s))\right|\mathrm{d}s + \int_{t_0}^{t} \left|\vec{f}(s, \vec{\psi}(s)) - \vec{g}(s, \vec{\psi}(s))\right|\mathrm{d}s \\
&\leq \left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| + \int_{t_0}^{t} K\left|\vec{\varphi}(s) - \vec{\psi}(s)\right|\mathrm{d}s + \int_{t_0}^{t} \varepsilon\mathrm{d}s \\
&\leq \left|\vec{y}_{(1)} + \vec{y}_{(2)}\right| + \varepsilon(\beta - \alpha) + \int_{t_0}^{t} K\left|\vec{\varphi}(s) - \vec{\psi}(s)\right|\mathrm{d}s \\
&\leq \left(\left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| + \varepsilon(\beta - \alpha)\right) e^{\int_{t_0}^{t} K\mathrm{d}s} && \text{Grönwall's Inequality} \\
&\leq \left(\left|\vec{y}_{(1)} - \vec{y}_{(2)}\right| + \varepsilon(\beta - \alpha)\right) e^{K|t-t_0|} && \square
\end{aligned}$$

Denote the solution $\vec{\varphi} : I \to \mathbb{R}^n$ to (3) as $\vec{\varphi}(t; \vec{y}_0, \alpha)$. In fact, one can prove that if $\vec{f}$ is C^k with $k \geq 1$ in $\vec{y}$ and α, then $\vec{\varphi}(\cdot;, \vec{y}_0, \alpha)$ is C^k in $\vec{y}_0$ and α. C^k allows us to use Taylor expansion for $\alpha \ll 1$:

$$\vec{\varphi}(t; \vec{y}_0, \alpha) = \underbrace{\vec{\varphi}(t, \vec{y}_0, 0)}_{\text{(a)}} + \underbrace{\alpha\frac{\partial}{\partial\alpha}\vec{\varphi}(t, \vec{y}_0, 0)}_{\text{(b)}} + \mathcal{O}(\alpha^2)$$

(a) solves $\vec{y}' = \vec{f}(t, \vec{y}, 0)$ which is easier to solve than $\vec{y}' = \vec{f}(t, \vec{y}, \alpha)$. (b) solves a linear equation which is also easier to solve than $\vec{y}' = \vec{f}(t, \vec{y}, \alpha)$.

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\vec{\varphi}'(t; \vec{y}_0, \alpha) &= \frac{\partial}{\partial\alpha}\vec{f}(t, \vec{\varphi}(t; \vec{y}_0, \alpha)\alpha) \\
\underbrace{\left(\frac{\partial}{\partial\alpha}\vec{\varphi}\right)'} &= \frac{\partial}{\partial y_i}\vec{f}(t; \vec{\varphi}(t; \vec{y}_0, 0), 0)\underbrace{\frac{\partial}{\partial\alpha}\vec{\varphi}(t; \vec{y}_0, 0)} + \left(\frac{\partial}{\partial\alpha}\vec{f}\right)(t; \vec{\varphi}(t; \vec{y}_0, 0), 0)
\end{aligned}$$

By observing the indicated parts, we can see that this is an inhomogeneous linear equation in $\frac{\partial}{\partial\alpha}\vec{\varphi}$.

Linear Systems

Existence, Uniqueness, and Continuation of Solutions

Theorem

Consider the system

$$\begin{cases} \vec{y}' = A(t)\vec{y} + \vec{g}(t) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases} \tag{4}$$

where A is a $n \times n$ matrix, $\vec{g} : I \to \mathbb{R}^n$ for some interval I, and $\vec{y}_0 \in \mathbb{R}^n$. Suppose $A(t)$ and $\vec{g}(t)$ are continuous on $[a, b]$ with $t_0 \in [a, b]$. Then, $\exists$ a unique solution $\vec{\varphi} : [a, b] \to \mathbb{R}^n$ to the IVP (4).

- **Proof**: note that this theorem states existence, uniqueness, and continuation of solutions. So, we can apply our theorems from before. We have $\vec{y}' = \vec{f}(t, \vec{y})$ where $\vec{f}(t, \vec{y}) = A(t)\vec{y} + \vec{g}(t)$. This is a proof in three parts.

  1. Show that $\vec{f}$ is continuous on $[a, b] \times \mathbb{R}^n$. This follows directly from the fact that $A(t)$ and $\vec{g}(t)$ are continuous on $[a, b]$ and $\vec{f}$ is a linear combination of $A(t)$, $\vec{g}(t)$, and $\vec{y}$.
  2. Show that $\vec{f}$ is Lipschitz continuous in $\vec{y}$ on $[a, b] \times \mathbb{R}^n$. Recall that we are using the ℓ^1 norm which states that for a vector $\vec{y} \in \mathbb{R}^n$, $|\vec{y}| = |y_1| + \cdots + |y_n|$. Now, suppose we have matrix A with entry A_{ij} in row i and column j. We define $|A| = \sum_{j=1}^{n} \sum_{k=1}^{n} |A_{jk}|$. In the ℓ^1 norm, we have $|AB| \leq |A||B|$ and $|A\vec{y}| \leq |A||\vec{y}|$. We can prove the second statement:

     $$|A\vec{y}| = \sum_{j=1}^{n} |(A\vec{y})_j| = \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{jk} y_k \right| \leq \sum_{j=1}^{n} \sum_{k=1}^{n} |A_{jk}||y_k| \leq |\vec{y}| \sum_{j=1}^{n} \sum_{k=1}^{n} |A_{jk}| = |\vec{y}||A|$$

     We can apply this same theory for $|AB| \leq |A||B|$, but it is not necessary for this proof. Now, $\forall$ $t \in [a, b]$ and $\vec{y}_{(1)}, \vec{y}_{(2)} \in \mathbb{R}^n$:

     $$\begin{aligned} \left| \vec{f}(t, \vec{y}_{(1)}) - \vec{f}(t, \vec{y}_{(2)}) \right| &= \left| A\vec{y}_{(1)} - A\vec{y}_{(2)} \right| \\ &= \left| A \cdot \left(\vec{y}_{(1)} - \vec{y}_{(2)} \right) \right| \\ &\leq |A| \left| \vec{y}_{(1)} - \vec{y}_{(2)} \right| \end{aligned}$$

     So, we have Lipschitz continuity with Lipschitz constant $|A|$.
  3. Show that on any interval $[\alpha, \beta] \ni t_0$ on which $\vec{\varphi}$ is defined, $\exists B > 0$ such that $|\vec{\varphi}(t)| \leq B$ $\forall t \in [\alpha, \beta]$. WLOG, let $t > t_0$.

     $$\begin{aligned} \vec{\varphi}(t) &= \vec{y}_0 + \int_{t_0}^{t} [A(s)\vec{\varphi}(s) + \vec{g}(s)] \, \mathrm{d}s \\ |\vec{\varphi}(t)| &\leq |\vec{y}_0| + \int_{t_0}^{t} |A(s)||\vec{\varphi}(s)| + |\vec{g}(s)| \mathrm{d}s \\ &= |\vec{y}_0| + \int_{t_0}^{t} |\vec{g}(s)| \mathrm{d}s + \int_{t_0}^{t} |A(s)||\vec{\varphi}(s)| \mathrm{d}s \end{aligned}$$

     Since $|A|$ and $|\vec{g}|$ are continuous on $[a, b] \supseteq [\alpha, \beta]$, by the EVT, $\exists M > 0$, $G > 0$ such that $|A(t)| \leq M$ and $|\vec{g}(t)| \leq G$ $\forall t \in [a, b]$.

     $$\begin{aligned} &\leq |\vec{y}_0| + G(\beta - \alpha) + \int_{t_0}^{t} M |\vec{\varphi}(s)| \mathrm{d}s \\ &\leq (|\vec{y}_0| + G(\beta - \alpha)) \, e^{M|t - t_0|} && \text{Gronwall's inequality} \\ &\leq (|\vec{y}_0| + G(\beta - \alpha)) \, e^{M|\beta - \alpha|} \end{aligned}$$

  So, all three conditions are satisfied and we have proven existence, uniqueness, and continuation of solution. □

We can make the following remarks.

1. The theorem also holds if any of the bounds of $[a, b]$ is open or are $-\infty$ or ∞. The proof is the same by applying it to every closed subinterval of I containing t_0. However, the bound on $\vec{\varphi}$ would in general not hold.
2. We can also prove this theorem where $\mathbb{R}$ is replaced by $\mathbb{C}$. We can think of $\mathbb{C}$ as $\mathbb{R}^2$.

Now that existence, uniqueness, and continuation of solutions has been clarified, we want to study the structure of space of solutions. From the theorem, we have

$$\vec{\varphi} \text{ solution to } \vec{y}' = A(t)\vec{y} + \vec{g}(t) \Longleftrightarrow \vec{y}_0 = \vec{\varphi}(t_0)$$

Linearity tells us that the space of solutions have "vector space" structure.

Linear Homogeneous Systems

Theorem

Consider the system

$$\vec{y}' = A(t)\vec{y} \tag{5}$$

where $A(t)$ is a $n \times n$ matrix in $\mathbb{R}$ which is continuous on $[a, b]$. The set of all solutions to the system is an n-dimensional vector space.

By a vector space, we mean that the set of solutions S to (5) is a linear subspace of $C(a, b)$, the set of continuous functions on (a, b). By dimension, we mean the number of elements in a basis $\{\vec{\varphi}_1, \ldots, \vec{\varphi}_k\}$ where $\{\vec{\varphi}_1, \ldots, \vec{\varphi}_k\}$ is linearly independent (i.e. if $c_1\vec{\varphi}_1 + \cdots + c_k\vec{\varphi}_k = 0$ for $c_1, \ldots, c_k \in \mathbb{R}$, then $c_1 = \cdots = c_k = 0$) and $\{\vec{\varphi}_1, \ldots, \vec{\varphi}_k\}$ spans S (i.e. $\forall \vec{\psi} \in S$, $\vec{\psi} = c_1\vec{\varphi}_1 + \cdots + c_k\vec{\varphi}_k$ for some $c_1, \ldots, c_k \in \mathbb{R}$). A list of linear algebra terms can be found in Appendix B on page 64.

- **Proof**: we must prove two things, that the set of solutions S is a vector space and that it is n-dimensional.
  1. Show that S is a vector space, i.e. if $\vec{\varphi}$ and $\vec{\psi}$ are solutions to the system, then $\forall c, d \in \mathbb{R}$, $c\vec{\varphi} + d\vec{\psi}$ is also a solution.
  $$(c\vec{\varphi} + d\vec{\psi})' = c\vec{\varphi}' + d\vec{\psi}' = cA(t)\vec{\varphi} + dA(t)\vec{\psi} = A(t)(c\vec{\varphi} + d\vec{\psi})$$
  So, $c\vec{\varphi} + d\vec{\psi}$ is also a solution to the system.
  2. Show that S is n-dimensional, i.e. show that $\exists$ a basis for S consisting of n elements. Use the existance and uniqueness theorem. Fix $t_0 \in (a, b)$ and let $\vec{\varphi}_{(k)}$ be the solution to the IVP
  $$\begin{aligned} \vec{y}' &= A(t)\vec{y} \\ \vec{y}(t_0) &= \vec{e}_{(k)} \end{aligned}$$
  where $\vec{e}_{(k)} = (0, \ldots, 1, \ldots, 0)^\top$ where 1 is the k^{th} component. $\{\vec{e}_{(1)}, \ldots, \vec{e}_{(n)}\}$ is the standard basis for $\mathbb{R}^n$. Note that $\vec{\varphi}_{(k)}$ exists on (a, b) by the existence and uniqueness theorem.
     i. Show that $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$ is linearly independent. Suppose $c\vec{\varphi}_{(1)}(t) + \cdots + c_n\vec{\varphi}_{(n)}(t) = 0$. Then, at $t = t_0$,
     $$\begin{aligned} 0 &= c_1\vec{\varphi}_{(1)}(t_0) + \cdots + c_n\vec{\varphi}_{(n)}(t_0) \\ &= c_1\vec{e}_{(1)} + \cdots + c_n\vec{e}_{(n)} \\ &= (c_1, \ldots, c_n)^\top \end{aligned}$$
     So, we must have $c_1 = \cdots = c_n = 0$ meaning $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$ is linearly independent.
     ii. Show that $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$ spans S. Let $\vec{\psi} \in S$ and write $\vec{\psi}(t_0) = (c_1, \ldots, c_n)^\top$. Now consider $\vec{\psi}_* = c_1\vec{\varphi}_{(1)} + \cdots + c_n\vec{\varphi}_{(n)}$. $\vec{\psi}_* \in S$ meaning $\vec{\psi}_*$ solves the system and $\vec{\psi}_*(t_0) = c_1\vec{e}_{(1)} + \cdots + c_n\vec{e}_{(n)} = (c_1, \ldots, c_n)^\top = \vec{\psi}(t_0)$, so by uniqueness, $\vec{\psi} = \vec{\psi}_*$. So, $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$ spans S.

  So, S is an n-dimensional vector space. □

In view of this theorem, we make the following definitions. Any n-tuple of solutions to $\vec{y}' = A(t)\vec{y}$, $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$, is called a **fundamental set of solutions** if it is a basis for the set of solutions S.

Given $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(k)}\}$ solutions to (5), we form an $n \times k$ matrix:

$$\Phi(t) = \left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\varphi}_{(k)}\right)$$

$\Phi(t)$ satisfies $\Phi' = A(t)\Phi$ by matrix multiplication since

$$\left(\vec{\varphi}'_{(1)}\middle|\cdots\middle|\vec{\varphi}'_{(k)}\right) = \begin{pmatrix} A_n & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} \left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\varphi}_{(n)}\right).$$

We have $c_1\vec{\varphi}_{(1)} + \cdots + c_k\vec{\varphi}_{(k)} = \Phi(c_1, \ldots, c_k)^\top$. If $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(k)}\}$ is a fundamental set of solutions, then $\Phi = \left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\varphi}_{(k)}\right)$ is a **fundamental** (principal) **matrix solution**.

We now want to discuss a criterion for Φ to be a fundamental matric solution. Note that a fundamental set is not unique.

Theorem: Abel's Formula

Let $\Phi = \left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\varphi}_{(k)}\right)$ where each $\vec{\varphi}_{(j)}$ solves (5). So, $\Phi' = A\Phi$. Let $t_0, t \in (a, b)$. Then,

$$\det(\Phi(t)) = \det(\Phi(t_0))e^{\int_{t_0}^{t} \operatorname{tr}(A(s))\mathrm{d}s}$$

where $\operatorname{tr}(A(t)) = \sum_{i=1}^{n} A_{ii}(t)$.

Before proving this, we will first state and prove a corollary.

Corollary: Criterion for Fundamental Set

$\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$ is a fundamental set of solutions if and only if $\det(\Phi(t)) \neq 0$ at some $t \in (a, b)$. In fact, in this case, $\det(\Phi(t')) \neq \forall t' \in (a, b)$.

- **Proof**: Remember that for matrix M, $\det(M) \neq 0$ if and only if $\exists M^{-1}$ such that $MM^{-1} = M^{-1}M = I$ where I is the identity matrix. If $\det(\Phi(t_0)) \neq 0$, $\exists\Phi(t_0)^{-1}$ such that $\Phi(t_0)\Phi(t_0)^{-1} = \Phi(t_0)^{-1}\Phi(t_0) = I$. Hence, given any $\vec{\psi} \in S$, for $(c_1, \ldots, c_n)^\top = \Phi(t_0)^{-1}\vec{\psi}(t_0)$, form $\vec{\psi}_* = \Phi(c_1, \ldots, c_n)^\top$. Then,

  $$\vec{\psi}_*(t_0) = \Phi(t_0)(c_1, \ldots, c_n)^\top = \Phi(t_0)\Phi(t_0)^{-1}\vec{\psi}(t_0) = \vec{\psi}(t_0).$$

  So, by uniqueness, $\vec{\psi}_* = \vec{\psi}$ and $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$ spans S. Since $\dim(S) = n$, this already tells us that $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(n)}\}$ is a basis. We can also prove linear dependence directly. The last statement follows directly from Abel's Formula.

  In the other direction, we can see that if $\{\vec{\varphi}_{(1)}, \ldots, \vec{\varphi}_{(k)}\}$ is a basis, then at any $t \in S$, $\Phi(t)$ must be invertable, which is true if and only if $\det(\Phi(t)) \neq 0$. ☐

To prove Abel's formula, we will first go over the characterization of the determinant.

1. **Multilinearity**

   $$\det\left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\underbrace{c\vec{\varphi}_{(j)} + d\vec{\psi}_{(j)}}_{j^{\text{th}}\text{ column}}\middle|\cdots\middle|\vec{\varphi}_{(n)}\right) = c\det\left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\varphi}_{(j)}\middle|\cdots\middle|\vec{\varphi}_{(n)}\right) + d\det\left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\psi}_{(j)}\middle|\cdots\middle|\vec{\varphi}_{(n)}\right)$$

2. **Total anti-symmetry**

   $$\det\left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\varphi}_{(j)}\middle|\vec{\varphi}_{(j+1)}\middle|\cdots\middle|\vec{\varphi}_{(n)}\right) = -\det\left(\vec{\varphi}_{(1)}\middle|\cdots\middle|\vec{\varphi}_{(j+1)}\middle|\vec{\varphi}_{(j)}\middle|\cdots\middle|\vec{\varphi}_{(n)}\right)$$

3. **Normalization**

   $$\det(I) = 1$$

Observe that we can use rows instead of columns which still yields the same determinant function. So, we have the following properties.

1.′ **Multilinearity for rows**

$$\det\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & & \vdots \\ c\Phi_{j1}+d\Psi_{j1} & \cdots & c\Phi_{jn}+d\Psi_{jn} \\ \vdots & & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix} = c\det\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & & \vdots \\ \Phi_{j1} & \cdots & \Phi_{jn} \\ \vdots & & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix} + d\det\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & & \vdots \\ \Psi_{j1} & \cdots & \Psi_{jn} \\ \vdots & & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix}$$

2.′ **Total antisymmetry for rows**

$$\det\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & & \vdots \\ \Phi_{j1} & \cdots & \Phi_{jn} \\ \Phi_{j+1,1} & \cdots & \Phi_{j+1,n} \\ \vdots & & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix} = -\det\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & & \vdots \\ \Phi_{j+1,1} & \cdots & \Phi_{j+1,n} \\ \Phi_{j1} & \cdots & \Phi_{jn} \\ \vdots & & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix}$$

These two characterizations imply the following property.

4.′ **Elementary column operation**

$$\det\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & & \vdots \\ \sum_k c_k\Phi_{k1} & \cdots & \sum_k c_k\Phi_{kn} \\ \vdots & & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix} = c_j\det(\Phi)$$

The row with the sums is the j^{th} row.

Now, we can prove Abel's Formula.

- **Proof**: we want to compute $\frac{\mathrm{d}}{\mathrm{d}t}\det(\Phi(t))$. We can use the property that $\frac{\mathrm{d}}{\mathrm{d}t}f(t) = g(t)$ if and only if $f(t+h) = f(t) = hg(t) + o(h)$ where $o(h)$ is some quantity that approaches 0 faster than h:

$$\lim_{h\to 0}\frac{f(t+h) - (f(t) + hg(t))}{h} = 0$$

So, $f = g + o(h)$ is equivalent to $\lim\limits_{h\to 0}\frac{f-g}{h} = 0$.

If we set $\frac{\mathrm{d}}{\mathrm{d}t}\det(\Phi(t)) = g(t)$, this is equivalent to $\det\Phi(t+h) = \det\Phi(t) + hg(t) + o(h)$, which is easier to prove.

$$\begin{aligned}\det\Phi(t+h) &= \det\left(\Phi(t) + h\Phi'(t) + o(h)\right) \\ &= \det\begin{pmatrix} \Phi_{11}+h\Phi'_{11}+o(h) & \cdots & \Phi_{1n}+h\Phi'_{1n}+o(h) \\ \vdots & \ddots & \vdots \\ \Phi_{n1}+h\Phi'_{n1}+o(h) & \cdots & \Phi_{nn}+h\Phi'_{nn}+o(h) \end{pmatrix} \\ &= \det\Phi(t) + h\det\begin{pmatrix} \Phi'_{11} & \cdots & \Phi'_{1n} \\ \Phi_{21} & \cdots & \Phi_{2n} \\ \vdots & \vdots & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix} + \cdots + h\det\begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \Phi_{21} & \cdots & \Phi_{2n} \\ \vdots & \vdots & \vdots \\ \Phi'_{n1} & \cdots & \Phi'_{nn} \end{pmatrix} + o(h)\end{aligned}$$

Use the equation $\Phi' = A\Phi$.

$$
\begin{aligned}
&= \det \Phi(t) + h \det \begin{pmatrix} \sum_k A_{1k}\Phi_{k1} & \cdots & \sum_k A_{1k}\Phi_{kn} \\ \vdots & \ddots & \vdots \\ \Phi_{n1} & \cdots & \Phi_{nn} \end{pmatrix} \\
&\quad + \cdots + h \det \begin{pmatrix} \Phi_{11} & \cdots & \Phi_{1n} \\ \vdots & \ddots & \vdots \\ \sum_k A_{nk}\Phi_{k1} & \cdots & \sum_k A_{nk}\Phi_{kn} \end{pmatrix} + o(h) \\
&= \det \Phi(t) + h \sum_k A_{kk} \det \Phi(t) + o(h)
\end{aligned}
$$

So, $[\det \Phi(t)]' = \sum\limits_k A_{kk}(t) \det \Phi(t)$. By separation of variables, we have

$$\det \Phi(t) = \det \Phi(t_0) \cdot e^{\int_{t_0}^t \operatorname{tr} A(s) \mathrm{d}s}. \qquad \square$$

Let Φ be a $n \times n$ be a matrix solution to the linear homogeneous system (5). For $\vec{c}$ constant n-vector,

$$\Phi \cdot \vec{c} = c_1 \vec{\varphi}_{(1)} + \cdots + c_n \vec{\varphi}_{(n)}$$

is also a solution to (5) by linearity. Additionally, if C is a constant $n \times n$ matrix, then ΦC is a $n \times n$ matrix solution to (5), although $C\Phi$ may not be a solution.

Theorem

Let Φ be a fundamental matrix. Then for any nonsingular constant $n \times n$ matrix C, ΦC is a fundamental matrix. Moreover, any fundamental matrix Ψ is of the form $\Psi = \Phi C$.

- **Proof**: for the first statement, note

$$\det(\Phi(t_0)C) = \underbrace{\det(\Phi(t_0))}_{\neq 0} \underbrace{\det(C)}_{\neq 0} \neq 0$$

meaning $\Phi(t_0)C$ is a fundamental matrix.

For the second statement, it suffices to show that $\forall t_0 \in (a,b)$, $\forall$ fundamental matrices Ψ on (a,b), $\exists C$ such that $\Psi(t_0)C = I$. If Ψ and Φ are fundamental matrices, then $\exists C$ and D such that

$$
\begin{aligned}
\Psi(t_0)C &= \Phi(t_0)D \\
\Psi(t_0)CD^{-1} &= \Phi(t_0)
\end{aligned}
$$

Let $\Psi(t)CD^{-1} = \left(\vec{\theta}_{(1)}(t) \middle| \cdots \middle| \vec{\theta}_{(n)}(t)\right)$ and $\Phi(t) = \left(\vec{\varphi}_{(1)}(t) \middle| \cdots \middle| \vec{\varphi}_{(n)}(t)\right)$. Each column is a solution and $\vec{\theta}_{(j)}(t_0) = \vec{\varphi}_{(j)}(t_0)$ for every j. By uniqueness, $\vec{\theta}_{(j)}(t) = \vec{\varphi}_{(j)}(t)$.

$$\Psi(t)CD^{-1} = \Phi(t)$$

So, the claim is proven by noting that $C = \Psi(t_0)^{-1}$ works. $\square$

Given any fundamental matrix Φ, observe that the solution $\vec{\varphi}$ to the IVP

$$\begin{cases} \vec{y}' = A(t)\vec{y} \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

is given by

$$\vec{\varphi}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{y}_0.$$

It is a solution by the theorem since $\Phi^{-1}(t_0)$ is a constant matrix and at $t = t_0$, $\vec{\varphi}(t_0) = \vec{y}_0$.

Linear Nonhomogeneous Systems

Theorem

Consider the system

$$\vec{y}' = A(t)\vec{y} + \vec{g}(t) \tag{6}$$

where $A(t)$ is a $n \times n$ matrix and $\vec{g}$ is a n-vector. Let $\vec{\varphi}_p$ be a particular solution to the system. Then, any solution $\vec{\varphi}$ to the system can be written as

$$\vec{\varphi} = \vec{\varphi}_h + \vec{\varphi}_p$$

where $\vec{\varphi}_h$ solves the associated homogeneous equation $\vec{y}' = A(t)\vec{y}$.

- **Proof**: $\vec{\varphi} - \vec{\varphi}_p$ solves the following:

$$\begin{aligned}(\vec{\varphi} - \vec{\varphi}_p)' &= \vec{\varphi}' - \vec{\varphi}_p' \\ &= (A(t)\vec{\varphi} + \vec{g}) - (A(t)\vec{\varphi}_p + \vec{g}) \\ &= A(t)(\vec{\varphi} - \vec{\varphi}_p)\end{aligned}$$

So, $\vec{\varphi}_h = \vec{\varphi} - \vec{\varphi}_p$ solves $\vec{y}' = A(t)\vec{y}$. $\square$

By the theorem, we can see that in order to understand the structure of the space of solutions to (6), we just need to find a particular solution. It turns out that there is a way to find a particular solution using the fundamental matrix for $\vec{y}' = A(t)\vec{y}$. This is called variation of constants.

Variation of Constants

Let Φ be a fundamental matrix for $\vec{y}' = A(t)\vec{y}$. Recall that $\Phi\vec{c}$ for any constant n-vector $\vec{c}$ would also solve $\vec{y}' = A(t)\vec{y}$. To solve $\vec{y}' = A(t)\vec{y} + \vec{g}(t)$, try $\vec{\varphi}_p(t) = \Psi(t)\vec{u}(t)$ where $\vec{u}(t)$ depends on time.

$$\vec{\varphi}_p'(t) = A(t)\vec{\varphi}_p(t) + \vec{g}(t)$$

Observe the LHS where $\vec{\varphi}_p(t) = \Phi(t)\vec{u}(t)$.

$$\begin{aligned}\vec{\varphi}_p'(t) &= (\Psi(t)\vec{u}(t))' \\ &= \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) \\ &= A(t)\Phi(t)\vec{u}(t) + \Phi(t)\vec{u}'(t)\end{aligned}$$

Now, the RHS.

$$A(t)\vec{\varphi}_p(t) + \vec{g}(t) = A(t)\Phi(t)\vec{u}(t) + \vec{g}(t)$$

Now, combine these two to solve for $\vec{u}(t)$.

$$\begin{aligned}A(t)\Phi(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) &= A(t)\Phi(t)\vec{u}(t) + \vec{g}(t) \\ \Phi(t)\vec{u}'(t) &= \vec{g}(t) \\ \vec{u}'(t) &= \Phi(t)^{-1}\vec{g}(t)\end{aligned}$$

From this, we get the following theorem.

Theorem: Variation of Constants

Let $\Phi(t)$ be a fundamental matrix for $\vec{y}' = A(t)\vec{y}$. Let $t_0 \in (a, b)$. Then,

$$\vec{\varphi}_p(t) = \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}\vec{g}(s)\mathrm{d}s$$

solves $\vec{y}' = A(t)\vec{y} + \vec{g}(t)$.

Note that t_0 is arbitrary and $\vec{\varphi}_p(t)$ satisfies $\vec{\varphi}_p(t_0) = 0$.

Example

Consider the scalar case where $n = 1$. We will see that the method of integrating factors is exactly variation of constants.

$$y' = a(t)y + g(t)$$

1. Homogeneous case: $y' = a(t)y$ yields $\phi(t) = ce^{\int_{t_0}^{t} a(s)\mathrm{d}s}$ by separation of variables, so the fundamental solution is $\Phi(t) = e^{\int_{t_0}^{t} a(s)\mathrm{d}s}$.

2. Nonhomogeneous case: $y' = a(t)y + g(t)$ yields

$$\begin{aligned}\phi_p(t) &= \Phi(t)\int_{t_0}^{t}\Phi(s)^{-1}g(s)\mathrm{d}s \\ &= e^{\int_{t_0}^{t} a(s)\mathrm{d}s}\int_{t_0}^{t} e^{-\int_{t_0}^{s} a(s')\mathrm{d}s'}g(s)\mathrm{d}s\end{aligned}$$

   by variation of constants where $e^{-\int_{t_0}^{s} a(s')\mathrm{d}s}$ is the integrating factor.

Observe the following:

$$\begin{aligned}\vec{\varphi}_p(t) &= \Phi(t)\int_{t_0}^{t}\Phi(s)^{-1}\vec{g}(s)\mathrm{d}s \\ &= \int_{t_0}^{t}\Phi(t)\Phi(s)^{-1}\vec{g}(s)\mathrm{d}s\end{aligned}$$

$\Phi(t)\Phi(s)^{-1}$ is the solution to $\vec{y}' = A(t)\vec{y}$ with $\vec{y}(s) = \vec{g}(s)$. So, we have Duhamel's Principle, which states that nonhomogeneous solutions can be written as continuous sums of homogeneous solutions with initial data given by the nonhomogeneous term.

Linear Systems with Constant Coefficients

Consider the system

$$\vec{y}' = A\vec{y} + \vec{g}(t) \tag{7}$$

where A is a $n \times n$ real-valued matrix. As we have seen, the key to describing the structure of the space of solutions to a linear system is to find a fundamental matrix Φ. For the case with constant coefficients, we want to write Φ in terms of A.

Example

Consider the case where $n = 1$. Consider the system

$$y' = ay$$

where a is some constant. By separation of variables, we have solutions in the form $y = Ce^{at}$ where e^{at} is the fundamental solution. So, $\phi = e^{at}$ is a fundamental solution to the system.

Now, we return to the general case but we will try to make sense of "$\Phi = e^{At}$".

Matrix Exponentials

An idea to make sense of this is to recall the Maclaurin series for e^u.

$$e^u = \sum_{n=0}^{\infty}\frac{1}{n!}u^n = 1 + \frac{1}{1}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \cdots$$

Observe that even for a matrix M, higher powers of M make sense and we also know how to take limits of sequences of matrices.

Theorem: Matrix Exponential

Given any $n \times n$ (possibly complex-valued) matrix M, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} M^n = I + \frac{1}{1!}M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

is convergent (i.e. $S_N = \sum_{n=0}^{N} \frac{1}{n!} M^n$ is convergent as $N \to \infty$). We call its limit the **matrix exponential** of M, which we write as

$$e^M = I + \frac{1}{1!}M + \frac{1}{2!}M^2 + \cdots .$$

- **Proof**: the proof is similar to the proof that absolute convergence implies convergence. Consider

$$U_N = \sum_{n=0}^{N} \frac{1}{n!} |M|^n$$

which converges to $e^{|M|}$. Going back to S_N, we want to show Cauchy's criterion for convergence: $\forall \varepsilon > 0$, $\exists N_0$ such that $\forall N_0 \geq M \geq N$, $|S_N - S_M| < \varepsilon$.

Fix $\varepsilon > 0$.

$$|S_N - S_M| = \left| \sum_{n=M+1}^{N} \frac{1}{n!} M^n \right| \leq \sum_{n=M+1}^{N} \frac{1}{n!} |M^n| \leq \sum_{n=M+1}^{N} \frac{1}{n!} |M|^n = |U_N - U_M|$$

U_N is convergent, so $|U_N - U_M| \to 0$ as $N, M \to \infty$. We can choose N_0 so that $|U_N - U_M| < \varepsilon$ for $M, N \geq N_0$ and we are done. $\square$

Lemma: Properties of e^M

Let A, M, N be $n \times n$ (complex-valued) matrices.

1. $\frac{\mathrm{d}}{\mathrm{d}t} \left[e^{At} \right] = Ae^{At}$
2. If M and N commute (meaning $MN = NM$), then $e^{M+N} = e^M e^N = e^N e^M$
3. Let P be a nonsingular $n \times n$ matrix. Then $e^{P^{-1}MP} = P^{-1} e^M P$. Matrices N and M are similar if $\exists P$ with $\det P \neq 0$ such that $N = P^{-1}MP$.

- **Proof**: we can prove each property separately.

  1. Use the definition of e^M.

$$\begin{aligned} e^{At} &= I + \frac{1}{1}At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots \\ &= I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{At}\right] &= 0 + A + \frac{2}{2!}A^2t + \frac{3}{3!}A^3t^2 + \cdots \\ &= A\left(I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \cdots\right) \\ &= Ae^{At} \end{aligned}$$

  2. We know $MN = NM$.

$$eM + N = I + \frac{1}{1}(M+N) + \frac{1}{2!}(M+N)^2 + \frac{1}{3!}(M+N)^3 + \cdots$$

If M and N are numbers, then by the Binomial Theorem:

$$(M+N)^n = M^n + \binom{n}{1} M^{n-1}N + \cdots + \binom{n}{n} N^n$$

Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. So, $e^{M+N} = e^M \cdot e^N$.

The key observation in the matrix case is that the Binomial Theorem holds if $MN = NM$.

3. Again, use the definition of the matrix exponential.

$$e^{P^{-1}MP} = I + \frac{1}{1}P^{-1}MP + \frac{1}{2!}(P^{-1}MP)^2 + \frac{1}{3!}(P^{-1}MP)^3 + \cdots$$

Note that $(P^{-1}MP)^n = P^{-1}MPP^{-1}MP \cdots P^{-1}MP = P^{-1}M^nP$ since the P and P^{-1} in the middle cancel out, leaving n M's.

$$\begin{aligned} &= P^{-1}P + \frac{1}{1!}P^{-1}MP + \frac{1}{2!}P^{-1}M^2P + \cdots \\ &= P^{-1}e^MP \end{aligned}$$

□

By this lemma, we have:

Theorem

$\Phi = e^{At}$ is a fundamental matrix for $\vec{y}' = A\vec{y}$ with $\Phi(0) = I$.

Example

Let $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then, $A^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$ and $A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$.

$$\begin{aligned} e^{At} &= I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{1}\lambda_1 t & 0 \\ 0 & \frac{1}{1}\lambda_2 t \end{pmatrix} + \begin{pmatrix} \frac{1}{2!}\lambda_1^2t^2 & 0 \\ 0 & \frac{1}{2!}\lambda_2^2t^2 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \end{aligned}$$

Example

Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. To find e^{At}, we can set $A = D + N$ where $D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Observe that $DN = ND = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$.

$$\begin{aligned} e^{At} &= e^{Dt}e^{Nt} \\ &= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} e^{Nt} \end{aligned}$$

Observe that $N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$, so any higher power of N is also the zero matrix.

$$\begin{aligned} e^{Nt} &= I + \frac{1}{1}Nt + \frac{1}{2!}N^2t^2 + \cdots \\ &= I + Nt \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ e^{At} &= e^{3t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \end{aligned}$$

We will see that in general, the computation of e^{At} can be reduced to these cases if we use some ideas from algebra.

Change of Basis

A basic idea is choosing the right basis. Let X be a vector space for X. $\forall \vec{v} \in X$, $\exists! c_1, \dots, c_n$ such that $\vec{v} = c_1\vec{e}_1 + \cdots + c_n\vec{e}_n$. Existence is from $\vec{e}_1, \dots, \vec{e}_n$ spanning X and uniqueness is from the linear independence of $\vec{e}_1, \dots, \vec{e}_n$.

If $(\vec{f}_1, \dots, \vec{f}_n)$ is another basis for X, then $\forall \vec{v} \in X$, $\exists! d_1, \dots, d_n$ such that $\vec{v} = d_1\vec{f}_1 + \cdots + d_n\vec{f}_n$. We want to know how $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ and $\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ are related to each other. The answer is a change of basis matrix P from $(\vec{e}_i)$ to $(\vec{f}_i)$.

For simplicity, let $X = \mathbb{R}^n$ and let $\vec{f}_1, \dots, \vec{f}_n$ be the standard basis of $\mathbb{R}^n$:

$$\vec{f}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{f}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

For $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $\vec{v} = v_1\vec{f}_1 + \cdots + v_n\vec{f}_n$. Now, consider $P = \left(\vec{e}_1 \middle| \cdots \middle| \vec{e}_n\right)$ where $\vec{e}_i$ is expressed in terms of $\vec{f}_1, \dots, \vec{f}_n$.

$$\begin{aligned} P\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} &= \left(\vec{e}_1 \middle| \cdots \middle| \vec{e}_n\right)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ &= c_1\vec{e}_1 + c_2\vec{e}_2 + \cdots + c_n\vec{e}_n \end{aligned}$$

This is now expressed in terms of $\vec{f}_1, \dots, \vec{f}_n$.

Example

Let $X = \mathbb{R}^2$. $\vec{f}_1 = \binom{1}{0}$ and $\vec{f}_2 = \binom{0}{1}$. Let $\vec{e}_1 = \binom{1}{1/2} = 1 \cdot \vec{f}_1 + \frac{1}{2} \cdot \vec{f}_2$ and $\vec{e}_2 = \binom{0}{1} = 0 \cdot \vec{f}_1 + 1 \cdot \vec{f}_2$. $P = \left(\vec{e}_1 \middle| \vec{e}_2\right) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}$.

If $\vec{v} \in \mathbb{R}^2$ is $\vec{v} = c_1\vec{e}_1 + c_2\vec{e}_2$, then:

$$\vec{v} = P\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ 1/2c_1 + c_2 \end{pmatrix} = c_1\vec{f}_1 + \left(\tfrac{1}{2}c_1 + c_2\right)\vec{f}_2$$

To summarize, if $\vec{v} = c_1\vec{e}_1 + \cdots + c_n\vec{e}_n = d_1\vec{f}_1 + \cdots + d_n\vec{f}_n$, then $\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = P\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ where $P = \left(\vec{e}_1 \middle| \cdots \middle| \vec{e}_n\right)$ where each $\vec{e}_i$ is expressed in terms of $\vec{f}_1, \dots, \vec{f}_n$. This can also be written as $P = (P_{ij})$ where $\vec{e}_j = \sum\limits_{i=1}^{n} P_{ij}\vec{f}_i$.

Note that the change-of-basis matrix from $(\vec{f}_i)$ to $(\vec{e}_i)$ is P^{-1} where P is as before. This can be seen by taking it as some matrix Q. We observe that $QP = PQ = I$, so $Q = P^{-1}$.

Now, given any matrix $M = (M_{ij})$ written in terms of $(\vec{f}_1, \dots, \vec{f}_n)$, for $\vec{f} = d_1\vec{f}_1 + \cdots + \vec{f}_n$,

$$M\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} M_{1j}d_j \\ \vdots \\ \sum_{j=1}^{n} M_{nj}d_j \end{pmatrix}$$

represents $\left(\sum\limits_{j=1}^{n} M_{1j}d_j\right)\vec{f}_1 + \cdots \left(\sum\limits_{j=1}^{n} M_{nj}d_j\right)\vec{f}_n$. In the basis $\vec{e}_1, \dots, \vec{e}_n$, $P^{-1}MP$ is the same transformation written in terms of $(\vec{e}_1, \dots, \vec{e}_n)$.

Procedure for Computing e^{At}

Coming back to $\vec{y}' = A\vec{y}$, we want to compute e^{At}, or equivalently, $e^{At}\vec{v}_j$ where $\vec{v}_j$ forms a basis for $\mathbb{R}^n$. The idea is to start with $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v}$, $A^2\vec{v}$, $A^3\vec{v}, \dots$ are easy to compute.

If $A\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq 0$, we say that λ is an **eigenvalue** of A and $\vec{v}$ is a corresponding **eigenvector**. Indeed, if $A\vec{v} = \lambda\vec{v}$, then

$$A^2\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v} \quad A^3\vec{v} = \lambda^3\vec{v} \cdots\cdots \quad A^n\vec{v} = \lambda^n\vec{v}$$

Hence,

$$\begin{aligned}
e^{At}\vec{v} &= I\vec{v} + tA\vec{v} + \frac{1}{2!}t^2A^2\vec{v} + \cdots \\
&= \vec{v} + t\lambda\vec{v} + \frac{1}{2!}t^2\lambda^2\vec{v} + \cdots \\
&= \left(1 + t\lambda + \frac{1}{2!}t^2\lambda^2 + \cdots\right)\vec{v} \\
&= e^{\lambda t}\vec{v}
\end{aligned}$$

How do we compute eigenvalues and eigenvectors?

$$\begin{aligned}
\exists \vec{v} \neq 0 \text{ such that } A\vec{v} = \lambda\vec{v} &\iff \ker(A - \lambda I) \neq \{0\} \\
&\iff \det(\lambda I - A) \neq 0
\end{aligned}$$

We define the **characteristic polynomial** as $P(\lambda) = \det(\lambda I - A)$, which is a polynomial of degree n. λ is an eigenvalue of A if and only if λ is a root of $P(\lambda)$.

Recall from the fundamental theorem of algebra that $P(\lambda)$ has n many roots, $\lambda_1, \dots, \lambda_k$, so

$$P(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}$$

where $n_1 + \cdots + n_k = n$. Next, we ask: is there a basis consisting of eigenvectors? The answer is no. For example, $\left(\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right)$ has only one eigenvector. But, if we consider generalized eigenvectors, the answer is yes. We say $\vec{v}$ is a **generalized eigenvector** associated with λ if $(A - \lambda_k I)^{n_k}\vec{v} = 0$ where $n_k \geq 0$ is the **algebraic multiplicity** of λ_k, or the multiplicity of the root λ_k in $P(\lambda)$.

Theorem

$\mathbb{R}^n$ is spanned by generalized eigenvectors.

Let A be a $n \times n$ real-valued matrix. To solve $\vec{y}' = A\vec{y}$, we want to compute e^{At}. To do this, we can follow the general procedure.

1. **Characteristic polynomial**: consider the characteristic polynomial $P(\lambda) = \det(\lambda I - A)$ which is a polynomial of degree n in λ with real coefficients. Solve $P(\lambda) = 0$ and find all roots. Note that the fundamental theorem of algebra guarantees that $\exists n$ many roots (with possible repetitions). The roots of $P(\lambda)$ are exactly the eigenvalues of A.

2. **Generalized eigenvectors**: find solutions $\vec{v}$ to $(A - \lambda_k I)^{n_k}\vec{v} = 0$, where λ_k is an eigenvalue of A and n_k is the algebraic multiplicity of λ_k, to find all generalized eigenvectors associated with λ_k. It is a theorem in linear algebra that this would find all generalized eigenvectors (the Cayley-Hamilton Theorem, which states that $P(A) = 0$).

3. **Basis**: find n_j linearly independent vectors $\vec{v}_{j,1}, \dots, \vec{v}_{j,n_j}$ that solve $(A - \lambda_j I)^{n_j} = 0$. Then, $\{\vec{v}_{1,1}, \dots, \vec{v}_{1,n_1}, \dots, \vec{v}_{k,1}, \dots, \vec{v}_{k,n_k}\}$ is a basis for our vector space. For $\vec{v}_{j,\ell}$ solving $(A - \lambda_j I)^{n_j}\vec{v}_{j,\ell} = 0$, we have $At = \lambda_j It + (A - \lambda_j I)t$ and $\lambda_j I(A - \lambda_j I) = (A - \lambda_j I)\lambda_j I$. So, $e^{At} = e^{\lambda_j It}e^{(A-\lambda_j I)t}$ and on $\vec{v}_{j,\ell}$, $(A - \lambda_j I)^{n_j}\vec{v} = 0$. So, using the definition of e^{At}, we have

$$e^{At}\vec{v}_{j,\ell} = e^{\lambda_i t}\left(I + \frac{1}{1!}(A - \lambda_j I)t + \cdots + \frac{1}{(n_j - 1)!}(A - \lambda_j I)^{n_j - 1}t^{n_i - 1}\right)\vec{v}_{j,\ell}.$$

We can make the following remarks.

1. Even if A is real-valued, it is possible that λ_j's are not real-valued (e.g. $P(\lambda) = \lambda^2 + 1$). So, $\vec{v}$'s may also not be real-valued. We will discuss how to come back to real-valued objects.

2. It may turn out that a generalized eigenvector associated with λ_j satisfies $(A - \lambda_j I)^q \vec{v} = 0$ with $q < n_j$. For example, $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ yields $P(\lambda) = (\lambda - 1)^2$, so $n_1 = 2$. But, $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ already solve $(I - \lambda I)\vec{v} = 0$.

Example

Consider the case where all eigenvalues are real and $\exists n$ linearly independent eigenvectors. So, A is real diagonalizable.

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find e^{At} by following the steps outlined previously.

1. Find $P(\lambda)$.
$$P(\lambda) = \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - A\right) = \det\begin{pmatrix} \lambda & -1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$
So, we have eigenvalues -1 and 1.

2. $(A - I)\vec{v} = 0$ is associated with $\lambda = 1$, yielding $\left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)\vec{v} = 0$ and $\vec{v} = \alpha\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ for some α.
$(A + I)\vec{v} = 0$ is associated with $\lambda = -1$, yielding $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)\vec{v} = 0$ and $\vec{v} = \beta\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$ for some β.

3. $\left\{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\right\}$ is a basis consisting of eigenvectors.
$$e^{At}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad e^{At}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
So, we can form the change of basis matrix $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, meaning
$$e^{At}P = \left(e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$$
is a fundamental matrix.

Alternatively, we can observe:
$$e^{At}P = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} P \iff e^{At}P = P\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$
$$\iff P^{-1}e^{At}P = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Note that $AP = A\left(\vec{v}_1 \middle| \vec{v}_2\right) = \left(\vec{v}_1 \middle| -\vec{v}_2\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. So, $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We can set this equal to D. So, $P^{-1}e^{At}P = e^{P^{-1}APt} = e^{Dt} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} D$.

In general, if $\{\vec{v}_{1,1}, \ldots, \vec{v}_{1,n_1}, \ldots, \vec{v}_{k,1}, \ldots, \vec{v}_{k,n_k}\}$ consist of eigenvectors meaning $(A - \lambda_j I)\vec{v}_{j,\ell} = 0$, then if we form $P = \left(\vec{v}_{1,1}\middle|\cdots\middle|\vec{v}_{1,n_1}\middle|\cdots\middle|\vec{v}_{k,1}\middle|\cdots\middle|\vec{v}_{k,n_k}\right)$, then
$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & & & & \lambda_k & \\ & & & & & \ddots \\ & & & & & & \lambda_k \end{pmatrix}, \qquad P^{-1}e^{At}P = e^{(P^{-1}AP)t} = \begin{pmatrix} e^{\lambda_1 t} & & & & & \\ & \ddots & & & & \\ & & e^{\lambda_1 t} & & & \\ & & & \ddots & & \\ & & & & e^{\lambda_k t} & \\ & & & & & \ddots \\ & & & & & & e^{\lambda_k t} \end{pmatrix}$$

Example

Now, consider the case where there are complex eigenvalues and all n $\vec{v}_{j,\ell}$'s are eigenvectors.

Let $A = \begin{pmatrix} 3 & 4 \\ -4 & -3 \end{pmatrix}$.

$$P(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 3 & 4 \\ -4 & \lambda + 3 \end{pmatrix} = \lambda^2 - 9 + 16 = \lambda^2 + 17$$

So, $\lambda_1 = i\sqrt{7}$ and $\lambda_2 = -i\sqrt{7}$. Now, we want to find eigenvectors. Start with $\vec{v}_1$ such that $(A - \lambda_1 I)\vec{v}_1 = 0$.

$$\begin{pmatrix} 3 - i\sqrt{7} & 4 \\ -4 & -3 - i\sqrt{7} \end{pmatrix} \vec{v}_1 = 0$$

We want $\vec{v}_1 = \begin{pmatrix} 1 \\ x \end{pmatrix}$ such that $-4 + x(-3 - i\sqrt{7}) = 0$. Solving this, we get $\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-3+i\sqrt{7}}{4} \end{pmatrix}$. In fact, $\vec{v}_2 = \overline{\vec{v}_1} = \begin{pmatrix} 1 \\ \frac{-3-i\sqrt{7}}{4} \end{pmatrix}$ since $0 = \overline{(A - \lambda_1 I)\vec{v}_1} = (A - \overline{\lambda_1} I)\overline{\vec{v}_1} = (A - \lambda_2 I)\overline{\vec{v}_1}$. So, $P = \begin{pmatrix} 1 & 1 \\ \frac{-3+i\sqrt{7}}{4} & \frac{-3-i\sqrt{7}}{4} \end{pmatrix}$.

$$P^{-1}AP = \begin{pmatrix} i\sqrt{7} & 0 \\ 0 & -i\sqrt{7} \end{pmatrix} \qquad \Longrightarrow \qquad P^{-1}e^{At}P = \begin{pmatrix} e^{i\sqrt{7}t} & 0 \\ 0 & e^{-i\sqrt{7}t} \end{pmatrix}$$

In many cases, we want everything to be real-valued. Now, we will look for a new basis $\vec{v}_2 = \overline{\vec{v}_1}$. Define the following:

$$\begin{aligned} \vec{u}_1 &= \mathrm{Re}(\vec{v}_1) = \frac{1}{2}(\vec{v}_1 - \overline{\vec{v}_1}) = \frac{1}{2}(\vec{v}_1 + \vec{v}_2) \\ \vec{u}_2 &= \mathrm{Im}(\vec{v}_2) = \frac{1}{2i}(\vec{v}_1 - \overline{\vec{v}_1}) = \frac{1}{2i}(\vec{v}_1 - \vec{v}_2) \end{aligned}$$

$\vec{u}_1$ and $\vec{u}_2$ span $\{\vec{v}_1, \vec{v}_2\}$ and are linearly independent.

$$\begin{aligned} A\vec{u}_1 &= \frac{1}{2}(A\vec{v}_1 + A\vec{v}_2) \\ &= \frac{1}{2}i\sqrt{7}\vec{v}_1 - \frac{1}{2}i\sqrt{7}\vec{v}_2 \\ &= -\sqrt{7}\frac{1}{2i}(\vec{v}_1 - \vec{v}_2) \\ &= -\sqrt{7}\vec{u}_2 \end{aligned}$$

$$\begin{aligned} A\vec{u}_2 &= \frac{1}{2i}(A\vec{v}_1 - A\vec{v}_2) \\ &= \frac{1}{2i}(i\sqrt{7}\vec{v}_2 + i\sqrt{7}\vec{v}_2) \\ &= \sqrt{7}\frac{1}{2}(\vec{v}_1 + \vec{v}_2) \\ &= \sqrt{7}\vec{u}_1 \end{aligned}$$

Set $Q = (\vec{u}_1 | \vec{u}_2)$. $Q^{-1}AQ = \begin{pmatrix} 0 & \sqrt{7} \\ -\sqrt{7} & 0 \end{pmatrix}$ and $Q^{-1}e^{At}Q = \begin{pmatrix} \cos(\sqrt{7}t) & \sin(\sqrt{7}t) \\ -\sin(\sqrt{7}t) & \cos(\sqrt{7}t) \end{pmatrix}$.

One method is to use $e^{At}\vec{u}_1 = \frac{1}{2}\left(e^{At}\vec{v}_1 + e^{At}\vec{v}_2\right)$ and proceed as previously done. Otherwise, we can observe:

$$M = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} -b^2 & 0 \\ 0 & -b^2 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 0 & -b^3 \\ -b^3 & 0 \end{pmatrix}, \quad M^4 = \begin{pmatrix} b^4 & 0 \\ 0 & b^4 \end{pmatrix}$$

So, we can find e^{Mt}:

$$\begin{aligned} e^{Mt} &= I + Mt + \frac{1}{2!}M^2t^2 + \frac{1}{3!}M^3t^3 + \cdots \\ &= \left(1 + \frac{b^{2k}t^{2k}}{(2k)!}\right) I + \begin{pmatrix} 0 & b + \frac{1}{3!}b^3t^3 + \frac{1}{5!}b^5t^5 + \cdots \\ -\left(b + \frac{1}{3!}b^3t^3 + \frac{1}{5!}b^5t^5 + \cdots\right) & 0 \end{pmatrix} \\ &= \cos(bt)I + \begin{pmatrix} 0 & \sin(bt) \\ -\sin(bt) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix} \end{aligned}$$

We can generalize this process for 2×2 matrices. Let A be a 2×2 real-valued matrix where $\lambda=\mu+i\nu$ is an eigenvalue of A with $\nu\neq 0$. $\overline{\lambda}=\mu-i\nu$ is also an eigenvalue of A since A is a real-valued matrix, so $P(\lambda)$ has real coefficients. We can set $P(\lambda)=\lambda^2+a_1\lambda+a_0=0$ where a_0 and a_1 are real.

$$\begin{aligned}P(\overline{\lambda}) &= \overline{\lambda}+a_1\overline{\lambda}+a_0\\ &= \overline{\lambda^2+a_1\lambda+a_0}\\ &= \overline{P(\lambda)}=0\end{aligned}$$

So, $\overline{\lambda}$ is also a eigenvalue of A. Moreover, if $\vec{v}=\vec{u}+i\vec{w}$ where $\vec{u},\vec{w}$ are real-valued is an eigenvector associated with λ, then $\overline{\vec{v}}=\vec{u}-i\vec{w}$ is an eigenvector associated with $\overline{\lambda}$ since, again, A is real-valued.

$$\begin{aligned}(A-\lambda I)\vec{v}=0 &\iff \overline{(A-\lambda I)\vec{v}}=0\\ &\iff (A-\overline{\lambda}I)\overline{\vec{v}}=0\end{aligned}$$

So, we can set $P=\left(\vec{v}\middle|\overline{\vec{v}}\right)$ where $P^{-1}AP=\begin{pmatrix}\lambda & 0\\ 0 & \overline{\lambda}\end{pmatrix}=\begin{pmatrix}\mu+i\nu & 0\\ 0 & \mu-i\nu\end{pmatrix}$.

In order to get real-valued objects, consider the basis instead:

$$\{\operatorname{Re}\vec{v},\operatorname{Im}\vec{v}\}=\{\vec{u},\vec{w}\}$$

It can easily be checked that linear independence of $\vec{v}$ and $\overline{\vec{v}}$ implies linear independence of $\operatorname{Re}\vec{v}$ and $\operatorname{Im}\vec{v}$. Let $Q=\left(\operatorname{Re}\vec{v}\middle|\operatorname{Im}\vec{v}\right)=\left(\vec{u}\middle|\vec{w}\right)$. So, $Q^{-1}AQ=\begin{pmatrix}\mu & \nu\\ -\nu & \mu\end{pmatrix}$ since:

$$\begin{aligned}A\operatorname{Re}\vec{v} &= \frac{1}{2}A(\vec{v}+\overline{\vec{v}})\\ &= \frac{1}{2}\left((\mu+i\nu)\vec{v}+(\mu-i\nu)\overline{\vec{v}}\right)\\ &= \frac{1}{2}\mu(\vec{v}+\overline{\vec{v}})+\frac{i}{2}\nu(\vec{v}-\overline{\vec{v}})\\ &= \frac{1}{2}\mu(\vec{v}+\overline{\vec{v}})-\frac{1}{2i}\nu(\vec{v}-\overline{\vec{v}})\\ &= \mu\operatorname{Re}\vec{v}-\nu\operatorname{Im}\vec{v}\end{aligned}$$

Similarly, $A\operatorname{Im}\vec{v}=\nu\operatorname{Re}\vec{v}+\mu\operatorname{Im}\vec{v}$. Thus, we have:

$$\begin{aligned}A\left(\operatorname{Re}\vec{v}\middle|\operatorname{Im}\vec{v}\right) &= \left(A\operatorname{Re}\vec{v}\middle|A\operatorname{Im}\vec{v}\right)\\ &= \left(\mu\operatorname{Re}\vec{v}-\nu\operatorname{Im}\vec{v}\middle|\nu\operatorname{Re}\vec{v}+\mu\operatorname{Im}\vec{v}\right)\\ &= \left(\operatorname{Re}\vec{v}\middle|\operatorname{Im}\vec{v}\right)\begin{pmatrix}\mu & \nu\\ -\nu & \mu\end{pmatrix}\end{aligned}$$

We know that $P^{-1}e^{At}P=\begin{pmatrix}e^{\mu t+i\nu t} & 0\\ 0 & e^{\mu t-i\nu t}\end{pmatrix}$. We can also find $Q^{-1}e^{At}Q$, using the fact that $\lambda=\mu+i\nu$. So, $e^{\lambda t}=e^{\mu t}e^{i\nu t}=e^{\mu t}(\cos(\nu t)+i\sin(\nu t))$. Similarly, $e^{\overline{\lambda}t}=e^{\mu t}(\cos(\nu t)-i\sin(\nu t))$.

$$\begin{aligned}e^{At}\operatorname{Re}\vec{v} &= \frac{1}{2}\left(e^{At}\vec{v}+e^{At}\overline{\vec{v}}\right)\\ &= \frac{1}{2}\left(e^{\lambda t}\vec{v}+e^{\overline{\lambda}t}\overline{\vec{v}}\right)\\ &= e^{\mu t}\frac{1}{2}\left[(\cos(\nu t)+i\sin(\nu t))\,\vec{v}+(\cos(\nu t)-i\sin(\nu t))\,\overline{\vec{v}}\right]\\ &= e^{\mu t}\left(\cos(\nu t)\operatorname{Re}\vec{v}-\sin(\nu t)\operatorname{Im}\vec{v}\right)\\ e^{At}\operatorname{Im}\vec{v} &= e^{\mu t}\left(\sin(\nu t)\operatorname{Re}\vec{v}+\cos(\nu t)\operatorname{Im}\vec{v}\right)\end{aligned}$$

So, we have $Q^{-1}e^{At}Q=e^{\mu t}\begin{pmatrix}\cos(\nu t) & \sin(\nu t)\\ -\sin(\nu t) & \cos(\nu t)\end{pmatrix}$.

Proposition: General Case

Let A be a $n \times n$ real-valued matrix.

1. Suppose $\lambda = \mu + i\nu$ is an eigenvalue. Then, $\overline{\lambda}$ is also an eigenvalue.
2. Suppose $\vec{v}$ is an eigenvector associated with λ. Then, $\overline{\vec{v}}$ is an eigenvector associated with $\overline{\lambda}$.
3. Suppose the eigenvectors of A form a basis (A is diagonalizable). Order the eigenvalues and eigenvectors as follows:
   - $\lambda_1, \dots, \lambda_n$ are real eigenvalues
   - $\vec{v}_{1,1}, \dots, \vec{v}_{1,n_1}, \dots, \vec{v}_{m,1}, \dots, \vec{v}_{m,n_m}$ are eigenvectors associated with real eigenvalues
   - $\lambda_{m+1}, \lambda_{m+2} = \overline{\lambda_{m+1}}, \lambda_{m+3}, \lambda_{m+4} = \overline{\lambda_{m+3}}, \dots$ are complex eigenvalues
   - $\vec{v}_{m+1,1}, \overline{\vec{v}_{m+1,1}}, \dots, \vec{v}_{m+1,n_{m+1}}, \overline{\vec{v}_{m+1,n_{m+1}}}$ are eigenvectors associated with λ_{m+1} and $\overline{\lambda_{m+1}}$
     $\vec{v}_{m+3,1}, \overline{\vec{v}_{m+3,1}}, \dots, \vec{v}_{m+3,n_{m+3}}, \overline{\vec{v}_{m+3,n_{m+3}}}$ are eigenvectors associated with λ_{m+3} and $\overline{\lambda_{m+3}}$.
     $\vdots$

   So, $P = \big(\vec{v}\text{'s in above order}\big)$. We have the following (where $\lambda_1, \dots, \lambda_m$ contains n_i number of repetitions of each λ_i):

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_m & & & & & \\ & & & \lambda_{m+1} & & & & \\ & & & & \overline{\lambda_{m+1}} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_{k-1} & \\ & & & & & & & \overline{\lambda_{k-1}} \end{pmatrix}$$

$$P^{-1}e^{At}P = \begin{pmatrix} e^{\lambda_1 t} & & & & & & & \\ & \ddots & & & & & & \\ & & e^{\lambda_m t} & & & & & \\ & & & e^{\lambda_{m+1} t} & & & & \\ & & & & e^{\overline{\lambda_{m+1}} t} & & & \\ & & & & & \ddots & & \\ & & & & & & e^{\lambda_{k-1} t} & \\ & & & & & & & e^{\overline{\lambda_{k-1}} t} \end{pmatrix}$$

   Take $Q = \big(\vec{v}_{1,1}\big|\cdots\big|\vec{v}_{m,n_m}\big|\operatorname{Re}\vec{v}_{m+1,1}\big|\operatorname{Im}\vec{v}_{m+1,1}\big|\cdots\big)$.

$$Q^{-1}AQ = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_m & & & & & \\ & & & \mu_{m+1} & \nu_{m+1} & & & \\ & & & -\nu_{m+1} & \mu_{m+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \mu_{n_k-1} & \nu_{n_k-1} \\ & & & & & & -\nu_{n_k-1} & \mu_{n_k-1} \end{pmatrix}$$

$$Q^{-1}e^{At}Q = \begin{pmatrix} e^{\lambda_1 t} & & & & & \\ & \ddots & & & & \\ & & e^{\lambda_m t} & & & \\ & & & e^{\mu_{m+1} t}\cos(\nu_{m+1} t) & e^{\mu_{m+1} t}\sin(\nu_{m+1} t) & \\ & & & -e^{\mu_{m+1} t}\sin(\nu_{m+1} t) & e^{\mu_{m+1} t}\cos(\nu_{m+1} t) & \\ & & & & & \ddots \end{pmatrix}$$

The last example we will look at is the case in which there are generalized eigenvectors. First, we must define some terms. Suppose $\vec{v}$ is a generalized eigenvector associated with λ. If $(A-\lambda I)^p\vec{v} = 0$ but $(A-\lambda I)^{p-1}\vec{v} \neq 0$, then $\vec{v}$ is a generalized eigenvector of **index** p. Note that if $\vec{v}$ has index 1, then $\vec{v}$ is an eigenvector. The **chain** of generalized eigenvectors associated with $\vec{v}$ is $\vec{v}, (A-\lambda I)\vec{v}, \dots, (A-\lambda I)^{p-1}\vec{v}$. Observe that $\vec{v}$ has index p, $(A-\lambda I)\vec{v}$ has order $p-1$, and so on until $(A-\lambda I)^{p-1}\vec{v}$ which has index 1 and is an eigenvector.

Lemma

The chain of any generalized eigenvectors is linearly independent.

- **Proof**: let $c_0\vec{v}+c_1(A-\lambda I)\vec{v}+\cdots+c_{p-1}(A-\lambda I)^{p-1}\vec{v}=0$. We want to show that $c_0,c_1,\dots,c_{p-1}=0$. We know that $(A-\lambda I)^p\vec{v}=0$, as well as any higher powers.

$$\begin{aligned}
\text{Multiply by } (A-\lambda I)^{p-1} &\longrightarrow c_0(A-\lambda I)^{p-1}\vec{v}=0 \longrightarrow c_0=0\\
\text{Multiply by } (A-\lambda I)^{p-2} &\longrightarrow c_1(A-\lambda I)^{p-1}\vec{v}=0 \longrightarrow c_1=0\\
&\vdots\\
\text{Multiply by } (A-\lambda I)^{0}=I &\longrightarrow c_{p-1}(A-\lambda I)^{p-1}\vec{v}=0 \longrightarrow c_{p-1}=0
\end{aligned}$$

So, $c_0=c_1=\cdots=c_{p-1}=0$. $\square$

We claim that if we take a basis to consist of chains, $P^{-1}AP$ would look simple. We can define the following:

$$\begin{aligned}
\vec{v}_1 &= (A-\lambda I)^{p-1}\vec{v}\\
&\vdots\\
\vec{v}_{p-1} &= (A-\lambda I)\vec{v}\\
\vec{v}_p &= \vec{v}
\end{aligned}$$

Set $P=\big(\vec{v}_1\big|\cdots\big|\vec{v}_p\big)$.We want $P^{-1}AP$.

$$\begin{aligned}
A\vec{v}_1 &= (\lambda I+(A-\lambda I))\vec{v}_1 = \lambda\vec{v}_1\\
A\vec{v}_2 &= (\lambda I+(A-\lambda I))\vec{v}_2 = \vec{v}_1+\lambda\vec{v}_2\\
A\vec{v}_3 &= (\lambda I+(A-\lambda I))\vec{v}_3 = \vec{v}_2+\lambda\vec{v}_3\\
&\vdots\\
A\vec{v}_p &= (\lambda I+(A-\lambda I))\vec{v}_p = \vec{v}_{p-1}+\lambda\vec{v}_p
\end{aligned}$$

So, we have the following:

$$P^{-1}AP=\begin{pmatrix}\lambda & 1 & & 0\\ & \lambda & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda\end{pmatrix}, \qquad AP=P\begin{pmatrix}\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & \lambda\end{pmatrix}$$

We have $P^{-1}e^{At}P=e^{P^{-1}APt}=e^{\lambda tI+tN}$ where N is the matrix with 1 along the diagonal above the main diagonal. We found e^{Nt} in the homework, which can be found in Appendix C on page 65.

$$\begin{aligned}
P^{-1}e^{At}P &= e^{\lambda tI}e^{Nt}\\
&= e^{\lambda t}\begin{pmatrix}1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(n-1)!}t^{n-1}\\ & & & \ddots & \vdots\\ & & \ddots & & \frac{1}{2}t^2\\ & & & & t\\ 0 & & & & 1\end{pmatrix}
\end{aligned}$$

We can ask, can we always choose our basis to consist of chains? If so, how many? It turns out that the answer is yes, and the number is the geometric multiplicity of λ.

Let A be a (complex) $n\times n$ matrix and let λ be an eigenvalue of A. So, $P(s)=(s-\lambda)^n(\cdots\cdots)$. The **algebraic multiplicity** of λ is the order of the root λ in the characteristic polynomial (n_j for λ_j). The **geometric multiplicity** is the dimension of the set of eigenvectors associated with λ:

$$\dim\{\text{eigenvectors associated with }\lambda\}=\dim\ker(A-\lambda I)$$

Theorem

Let $X_j = \ker(A - \lambda_j I)^{n_j}$ where n_j is the algebraic multiplicity of λ_j. Then, $\dim X_j = n_j$. Moreover, $\exists$ a basis of X_j consisting of chains where the number of chains is the geometric multiplicity m_j, e.g.:

$$\vec{v}_{j,1}, \dots, \vec{v}_{j,n_j} \text{ where } (A - \lambda I)\vec{v}_{j,\ell} = \begin{cases} \vec{v}_{j,\ell-1} \\ 0 \end{cases}$$

Note that at the end of every chain, $\exists$ an eigenvector (which is why the number of chains is m_j). How do we find this basis in practice? Start from a linear independent set of eigenvectors $\vec{v}_1, \dots, \vec{v}_{m_j}$. Take $\vec{v}_{j,1} = \vec{v}_1$ and find $(A - \lambda I)\vec{v}_{j,2} = \vec{v}_{j,1}$ and continue until impossible. Then, take $\vec{v}_{j,\ell} = \vec{v}_2$ and repeat.

Example

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$.

$$P(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 4 \end{pmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

So, $\lambda = 3$ is the only eigenvalue and has multiplicity 2.

$$\begin{aligned} (A - 3I)\vec{v} &= 0 \\ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \vec{v} &= 0 \\ \vec{v} &= \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \qquad \begin{aligned} (A - 3I)^2\vec{v} &= 0 \\ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \vec{v} &= 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{v} &= 0 \end{aligned}$$

We want a basis such that $P^{-1}AP$ is simple. We can take $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We want $(A - 3I)\vec{v}_2 = \vec{v}_1$, or $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So, we can choose $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. So, $P = (\vec{v}_1 | \vec{v}_2) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $P^{-1}AP = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ which we have seen before. So, $P^{-1}e^{At}P = e^{P^{-1}APt} = e^{3t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ meaning $e^{At}P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is a fundamental matrix.

We can make the following concluding remarks.

1. Our discussion so far leads naturally to Jordan Canonical Form.

Theorem: Jordan Canonical Form

Let A be a $n \times n$ (complex) matrix and $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues with algebraic multiplicities $n_1, \dots, n_k$ where $n_1 + \cdots + n_k = n$. Choose basis $\{\vec{v}_{1,1}, \dots, \vec{v}_{k,n_k}\}$ of $\mathbb{C}^n$ such that:

1. $\vec{v}_{j,\ell}$ is a generalized eigenvector of λ_j
2. $(A - \lambda_j I)\vec{v}_{j,\ell} = \begin{cases} \vec{v}_{j,\ell-1} \\ 0 \end{cases}$

The theorem states that this always exists. For $P = \left(\vec{v}_{1,1} \middle| \cdots \middle| \vec{v}_{k,n_k}\right)$, $P^{-1}AP = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_m \end{pmatrix}$ where J_m is of the form $J_m = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}$, which is called a Jordan block.

2. When A has generalized eigenvectors and $\exists$ complex eigenvalues, then by a procedure similar to the case with distinct complex eigenvalues (taking Re and Im of complete basis elements), we can get a real-valued basis. This is called the real canonical form.

Asymptotic Behavior of Solutions

Recall that we have ODEs of the form $\vec{y}' = A\vec{y}$ where A is a $n \times n$ real-valued constant matrix with initial conditions $\vec{y}(0) = \vec{y}_0$. We want to explore the behavior of solution $\vec{\varphi}$ as $t \to \infty$.

Theorem

Consider the system $\vec{y}' = A\vec{y}$ with initial conditions $\vec{y}(0) = \vec{y}_0$ where A is a real-valued $n \times n$ matrix. Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of A. Suppose $\rho \in \mathbb{R}$ where $\rho > \operatorname{Re} \lambda_j$ for $j = 1, \dots, k$. Then, $\exists K > 0$ such that for every $t \geq 0$,

$$|e^{At}| \leq K e^{\rho t}.$$

This is important since e^{At} is a fundamental matrix for $\vec{y}' = A\vec{y}$ and $\vec{\varphi}(t) = e^{At}\vec{y}_0$.

- **Proof**: this is a proof in 2 steps.
    1. Show that it suffices to show that $\exists K' > 0$ and a basis $\vec{v}_1, \dots, \vec{v}_n$ such that
    $$|e^{At}\vec{v}_j| \leq K' e^{\rho t}.$$
    Indeed, if the inequality holds, then for $P = \left(\vec{v}_1 \middle| \cdots \middle| \vec{v}_n\right)$, $e^{At}P = \left(e^{At}\vec{v}_1 \middle| \cdots \middle| e^{At}\vec{v}_n\right)$. Thus, from the inequality:
    $$\begin{aligned} |e^{At}P| &\leq K'e^{\rho t} + \cdots + K'e^{\rho t} = nK'e^{\rho t} \\ |e^{At}| &= |e^{At}PP^{-1}| \\ &\leq |e^{At}P||P^{-1}| \\ &\leq n|P^{-1}|K'e^{\rho t} \end{aligned}$$
    By setting $K = n|P^{-1}|K'$, we have our desired result.
    2. Now, show that $\exists K' > 0$ and a basis $\vec{v}_1, \dots, \vec{v}_n$ such that $|e^{At}\vec{v}_j| \leq K'e^{\rho t}$ for $j = 1, \dots, k$. We will choose the basis to consist of generalized eigenvectors of A $\vec{v}_{1,1}, \dots, \vec{v}_{1,n_1}, \dots, \vec{v}_{k,1}, \dots, \vec{v}_{k,n_k}$ where n_j is the algebraic multiplicity of λ_j and $(A - \lambda_j I)^{n_j}\vec{v}_{j,\ell} = 0$ with $\vec{v}_{j,\ell} \neq 0$.
    $$\begin{aligned} e^{At}\vec{v}_{j,\ell} &= \left|e^{\lambda_j t}\right| e^{(A-\lambda_j I)t}\vec{v}_{j,\ell} \\ &= \left|e^{\lambda_j t}\right| \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}(A - \lambda_j I)^n\right)\vec{v}_{j,\ell} \\ &= \left|e^{\lambda_j t}\right| \sum_{m=0}^{n_j-1} \frac{t^m}{m!}(A - \lambda_j I)^m \vec{v}_{j,\ell} \\ |e^{At}\vec{v}_{j,\ell}| &\leq \left|e^{\lambda_j t}\right| \left|\sum_{m=0}^{n_j-1} \frac{t^m}{m!}(A - \lambda_j I)^m \vec{v}_{j,\ell}\right| \\ &\leq \left|e^{\lambda_j t}\right| \sum_{m=0}^{n_j-1} \frac{t^m}{m!}\left|(A - \lambda_j I)^m \vec{v}_{j,\ell}\right| \end{aligned}$$
    Observe that the summation term is a polynomial in t of degree $n_j - 1$. Let this be $Q(t)$. We can find $K' > 0$ such that $Q(t) \leq K'e^{(\rho - \max_j \operatorname{Re} \lambda_j)t}$ since by assumption, $\rho - \max_j \operatorname{Re} \lambda_j > 0$. We also know that $|e^{\lambda_j t} = e^{\operatorname{Re} \lambda_j t}$.
    $$\begin{aligned} &\leq \left|e^{\lambda_j t}\right| K' e^{(\rho - \max_j \operatorname{Re} \lambda_j)t} \\ &\leq e^{\operatorname{Re} \lambda_j t} K' e^{(\rho - \max_j \operatorname{Re} \lambda_j)t} \\ &\leq K'e^{\rho t} \end{aligned}$$
    So, we have $\left|e^{At}\vec{v}_{j,\ell}\right| \leq K'e^{\rho t}$ which proves the theorem. $\square$

> **Corollary**
>
> If $\operatorname{Re}\lambda_1, \ldots, \operatorname{Re}\lambda_k < 0$, then all solutions to $\vec{y}' = A\vec{y}$ approach 0 as $t \to \infty$.

- **Proof**: we want to find ρ such that $\max_j \operatorname{Re}\lambda_j < \rho < 0$. By the theorem, $\exists K > 0$ such that $\left|e^{At}\right| \leq Ke^{\rho t}$. Any solution $\vec{\varphi}$ to the system is of the form $\vec{\varphi}(t) = e^{At}\vec{y}_0$.

$$\begin{aligned} |\vec{\varphi}(t)| &\leq \left|e^{At}\right| |\vec{y}_0| \\ &\leq |\vec{y}_0| K e^{\rho t} \end{aligned}$$

  $\rho < 0$, so this goes to 0 as $t \to \infty$. $\square$

Autonomous Linear Systems

A first-order autonomous system of ODEs is written as

$$\vec{y}' = \vec{g}(\vec{y}) \tag{8}$$

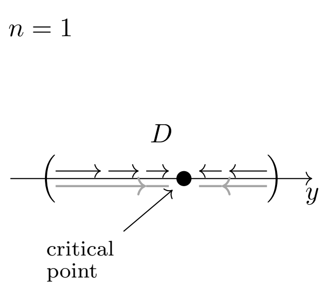
where $\vec{g} : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R}^n$ (as opposed to $\mathbb{R}^{n+1}$ for non-autonomous systems, since D is a function of only $\vec{y}$).

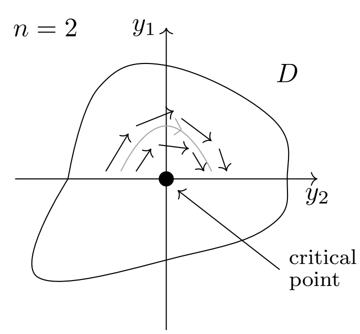
> **Example**
>
> The following are examples of autonomous ODEs.
>
> 1. $y' = y(1-y)$, which is the logistic equation
> 2. $\vec{y}' = A\vec{y}$ where A is a constant $n \times n$ matrix
> 3. $y'' + \sin(y) = 0$, which is the equation for a nonlinear simple pendulum

Autonomous ODEs are invariant under time translations, meaning if $\vec{\varphi}$ is a solution to (8), then $\forall t_0 \in \mathbb{R}$, $\vec{\varphi}(t - t_0)$ is also a solution to (8).

Phase Portraits

To visualize solutions to autonomous ODEs, we draw **phase portraits**. These are vector fields (as opposed to slope fields). At every $\vec{y} \in D$, draw an arrow (vector) $\vec{g}(\vec{y})$. For $n = 1$, this is called a phase space and for $n = 2$, this is called a phase plane. The vectors form a phase trajectory which has arrows to indicate the direction that solution $\vec{\varphi}(t)$ goes as t increases. These are drawn in grey in the diagrams.





$\vec{y}_0 \in D$ is a **critical point** of $\vec{y}' = \vec{g}(\vec{y})$ if $\vec{g}(\vec{y}_0) = 0$. These are exactly when $\vec{\varphi}(t) = \vec{y}_0$ is a solution. This is called a time-independent solution or equilibrium of the system.

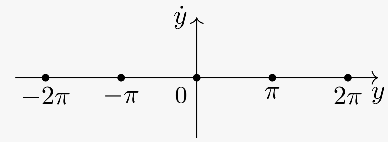
A linear homogeneous first order autonomous system is equivalent to a homogeneous constant coefficient system. $\vec{y} = A(t)\vec{y}$ is the general linear homogeneous system, and time independence of $A(t)$ means that A is constant. The motivation for studying the linear case in detail is that the nonlinear case can be approximated by the linear case.

Example

Consider the system $y'' + \sin(y) = 0$. This can be written as

$$\begin{pmatrix} y' \\ \dot{y}' \end{pmatrix} = \begin{pmatrix} \dot{y} \\ -\sin(y) \end{pmatrix}.$$

So, we have $\vec{y}' = \vec{g}(\vec{y})$ with $\vec{y} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$ and $\vec{g}(\vec{y}) = \begin{pmatrix} \dot{y} \\ -\sin(y) \end{pmatrix}$. The critical points are at $\dot{y} = 0$ and $-\sin(y) = 0$, which are at $\vec{y} = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}$ with $n \in \mathbb{Z}$.



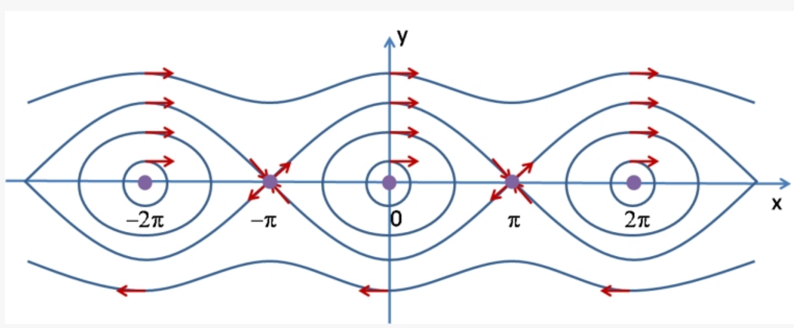
Let's observe what happens near $\vec{y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\vec{g}(\vec{y}) = \vec{g}(\vec{0}) + D\vec{g}(\vec{0})\vec{y} + \mathcal{O}(|\vec{y}|^2)$$

$\vec{g}(\vec{0}) = 0$ since $\vec{0}$ is a critical point. $D\vec{g}(\vec{0})\vec{y}$ is the first term of $\begin{pmatrix} \dot{y} \\ -y + \frac{1}{3}y^3 + \cdots \end{pmatrix}$.

$$\begin{aligned} &= 0 + \begin{pmatrix} \dot{y} \\ -y \end{pmatrix} + \mathcal{O}(|\vec{y}|^2) \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y} \\ -y \end{pmatrix} + \mathcal{O}(|\vec{y}|^2) \end{aligned}$$

We hope that by understanding the phase portrait of $\vec{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{y}$, we can understand the phase portrait of $\begin{pmatrix} y \\ \dot{y} \end{pmatrix}' = \begin{pmatrix} \dot{y} \\ -\sin(y) \end{pmatrix}$, which is as follows (source).



Our goal is to essentially draw all possible phase portraits for 2×2 linear homogeneous autonomous systems:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Given a change of basis matrix P, if we change $\vec{y} = P\vec{z}$ (meaning if $P = \left(\vec{v}_1 \middle| \vec{v}_2\right)$, then setting $\vec{y} = z_1\vec{v}_1 + z_2\vec{v}_2$ is equivalent to $\vec{y} = P\vec{z}$), then $\vec{z}' = P^{-1}A\vec{y} = P^{-1}AP\vec{z}$.

If we choose $\vec{v}_1$ and $\vec{v}_2$ to consist of generalized eigenvectors of A (and we do additional manipulations if λ is complex to make $\vec{v}_1$ and $\vec{v}_2$ real), then there are 6 possible forms of $P^{-1}AP$ that each have corresponding phase diagrams. We assume A is nonsingular, meaning all its eigenvalues are nonzero.

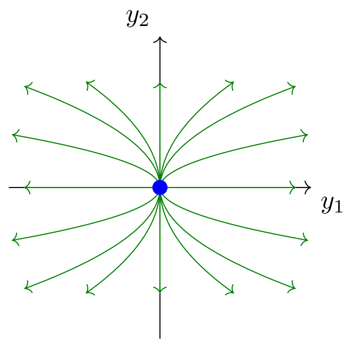
In cases 1-3, the eigenvalues λ and μ are real and $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors. In case 4, λ is the only eigenvalue of A and is real, $\vec{v}_1$ is an eigenvector, and $\vec{v}_2$ is a generalized eigenvector with $(A - \lambda I)\vec{v}_2 = \vec{v}_1$. In cases 5-6, $\lambda_1 = \sigma + i\nu$ and $\lambda_2 = \sigma - i\nu$ and $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to λ_1 and λ_2 respectively where $\vec{v}_2 = \overline{\vec{v}_1}$ and $P = \left(\operatorname{Re}\vec{v}_1 \middle| \operatorname{Im}\vec{v}_1\right)$.

1. $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $0 < \mu < \lambda$ or $\mu < \lambda < 0$. We have solution $\vec{\varphi} = e^{At}\vec{\eta} = \begin{pmatrix} e^{\lambda t}\eta_1 \\ e^{\mu t}\eta_2 \end{pmatrix}$ to $\vec{y}' = A\vec{y}$. We can write y_2 in terms of y_1 to sketch the phase portrait.

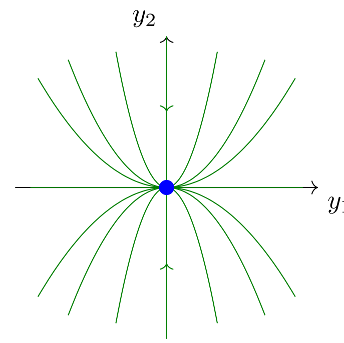
$$y_2 = e^{\mu t}\eta_2 = (e^{\lambda t})^{\mu/\lambda}\eta_2 = \underbrace{(e^{\lambda t}\eta_1)}_{y_1}^{\mu/\lambda}\frac{\eta_2}{\eta_1^{\mu/\lambda}}$$

By setting the non y_1-term as α, we get $y_2 = \alpha y_1^{\beta}$ where $\beta = \frac{\mu}{\lambda}$. Here, the equilibrium point is a **proper node**.

$0 < \mu < \lambda$



$\mu < \lambda < 0$

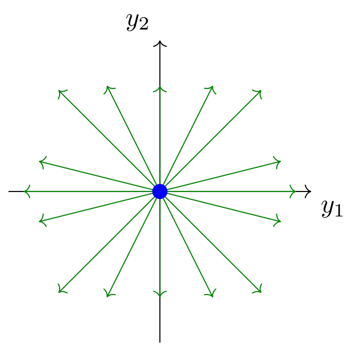


2. $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ where $\lambda > 0$ or $\lambda < 0$. We have solution $\vec{\varphi} = e^{At}\eta = \begin{pmatrix} e^{\lambda t}\eta_1 \\ e^{\lambda t}\eta_2 \end{pmatrix}$ to $\vec{y}' = A\vec{y}$. We can write y_2 in terms of y_1.

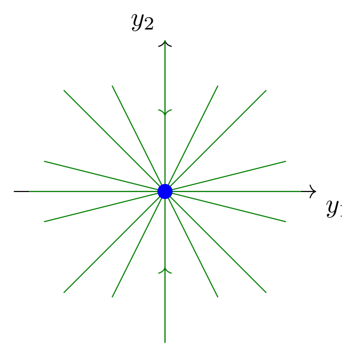
$$y_2 = e^{\lambda t}\eta_2 = e^{\lambda t}\eta_1 \cdot \frac{\eta_2}{\eta_1} = \frac{\eta_2}{\eta_1}y_1$$

Here, the equilibrium point is also a **proper node**.

$0 < \lambda$

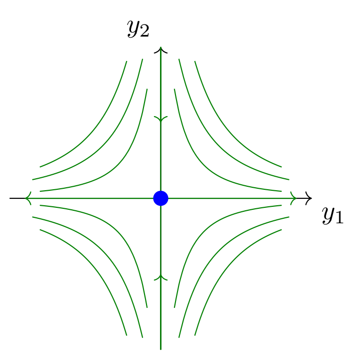


$\lambda < 0$



3. $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\mu < 0 < \lambda$. We have the same solution $\vec{\varphi}$ to $\vec{y}' = A\vec{y}$ as case 1 as well as the same y_2 in terms of y_1. However, since λ and μ have opposite signs, we can set $-\beta = \frac{\mu}{\lambda}$ where $\beta > 0$. So, $y_2 = \alpha y_1^{-\beta}$. Here, the equilibrium point is called a **saddle point**.

$\mu < 0 < \lambda$

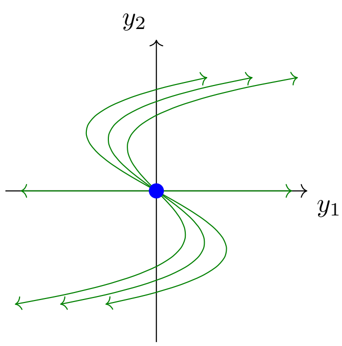


4. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ where $\lambda > 0$ or $\lambda < 0$. We have solution

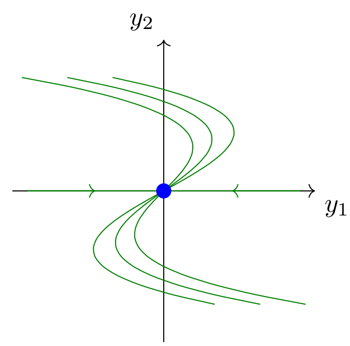
$$\vec{\varphi} = e^{At}\vec{\eta} = e^{\lambda t}\left(I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}\right)\vec{\eta} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} = \begin{pmatrix} e^{\lambda t}\eta_1 + te^{\lambda t}\eta_2 \\ e^{\lambda t}\eta_2 \end{pmatrix}$$

to $\vec{y}' = A\vec{y}$. Here, the equilibrium point is called an **improper node**.

$0 < \lambda$



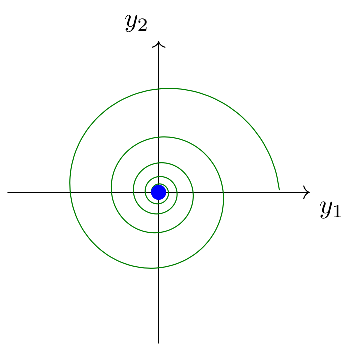
$\lambda < 0$



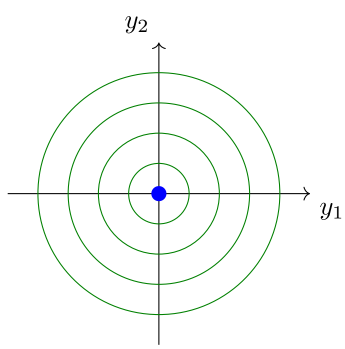
5. $\begin{pmatrix} \sigma & \nu \\ -\nu & \sigma \end{pmatrix}$ where $\nu, \sigma \neq 0$. We have solution

$$\vec{\varphi} = e^{At}\vec{\eta} = e^{\left(\begin{smallmatrix} \sigma & 0 \\ 0 & \sigma \end{smallmatrix}\right)t} e^{\left(\begin{smallmatrix} 0 & \nu \\ -\nu & 0 \end{smallmatrix}\right)t}\vec{\eta} = e^{t\sigma}\begin{pmatrix} \cos(\nu t) & \sin(\nu t) \\ -\sin(\nu t) & \cos(\nu t) \end{pmatrix}$$

to $\vec{y}' = A\vec{y}$. We will draw the case where $\sigma < 0$ and $\nu > 0$, meaning the vectors asymptotes to 0 and are clockwise respectively. Here, the equilibrium point is called a **spiral point**.



6. $\begin{pmatrix} 0 & \nu \\ -\nu & 0 \end{pmatrix}$ where $\nu \neq 0$. We have solution $\vec{\varphi} = e^{At}\eta = \begin{pmatrix} \cos(\nu t) & \sin(\nu t) \\ -\sin(\nu t) & \cos(\nu t) \end{pmatrix}$ to $\vec{y}' = A\vec{y}$. We will draw the case where $\nu > 0$, meaning the vectors point clockwise. Here, the equilibrium point is called a **center**.



Stability & Instability of Equilibria

Consider $\vec{y}' = \vec{g}(\vec{y})$ with critical point $\vec{y}_0$. $\vec{y}_0$ is **stable** if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall \vec{\eta}$ with $|\vec{\eta} - \vec{y}_0| < \delta$, the solution $\vec{\varphi}$ with $\vec{\varphi}(0) = \vec{\eta}$ satisfies $|\vec{\varphi}(t) - \vec{y}_0| < \varepsilon \; \forall t \geq 0$. If $\vec{y}_0$ is not stable, we say it is **unstable**.

$\vec{y}_0$ is **asymptotically stable** if $\exists \delta > 0$ such that $\forall \vec{\eta}$ with $|\vec{\eta} - \vec{y}_0| < \delta$, then the solution $\vec{\varphi}$ with $\vec{\varphi}(0) = \vec{\eta}$ satisfies $\lim_{t \to \infty} \vec{\varphi}(t) = \vec{y}_0$. Observe that this definition implies that all asymptotically stable critical points are also stable. If we observe the equilibria from the 6 cases, a center is stable but not asymptotically stable.

Now, look at $\vec{y}' = A\vec{y}$ where A is a $n \times n$ real-valued constant matrix. $\vec{y}_0 = 0$ is the only critical point.

Theorem

Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of A.

1. If $\operatorname{Re} \lambda_j \leq 0 \; \forall j$ and λ_j with $\operatorname{Re} \lambda_j = 0$ are simple (meaning its algebraic multiplicity and geometric multiplicity are equal), then 0 is stable.
2. If (and only if) $\operatorname{Re} \lambda_j < 0 \; \forall j$, then 0 is asymptotically stable.
3. If $\exists \lambda_j$ with $\operatorname{Re} \lambda_j > 0$, then 0 is unstable.

This follows from the theorem on asymptotic behavior of solutions, which in turn is based on the form of e^{At} in the basis consisting of chains of eigenvectors.

$$e^{(\mu + i\nu)t} = \underbrace{e^{\mu t}}_{\text{Re}} \underbrace{(\cos(\nu t) + i \sin(\nu t))}_{\text{bounded}}$$

We will not discuss the case when $\operatorname{Re} \lambda_j = 0$ and λ_j is not simple, as this has to do with the real canonical form.

We can now classify every equilibrium solution of 2×2 homogeneous autonomous systems.

1. Asymptotically stable when $\mu < \lambda < 0$, unstable when $0 < \mu < \lambda$.
2. Asymptotically stable when $\lambda < 0$, unstable when $0 < \lambda$.
3. Always unstable.
4. Asymptotically stable when $\lambda < 0$, unstable when $0 < \lambda$.
5. Asymptotically stable when $\sigma < 0$, unstable when $\sigma > 0$ (both for either $\nu > 0$ or $\nu < 0$).
6. Always stable but not asymptotically stable.

Nonautonomous Linear Equations

Stability & Instability of Equilibria

Consider the system

$$\vec{y}' = \vec{f}(t, \vec{y})$$

where $\vec{f} : D \to \mathbb{R}^n$ and $D \in \mathbb{R}^{n+1}$. Let $\vec{f}$ be continuous on D and Lipschitz continuous in $\vec{y}$ on D. Suppose $\vec{\varphi}$ is a solution to the equation on $t \in [0, \infty)$. We define the following.

1. $\vec{\varphi}$ is **stable** if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall \vec{\eta}$ with $|\vec{\varphi}(0) - \vec{\eta}| < \delta$, the solution $\vec{\psi}(t)$ with $\vec{\psi}(0) = \vec{\eta}$ exists $\forall t \in [0, \infty)$ and $\left[\vec{\psi}(t) - \vec{\varphi}(t)\right| < \varepsilon \; \forall t \in [0, \infty)$.
2. $\vec{\varphi}$ is **asymptotically stable** if $\vec{\varphi}$ is stable and $\exists \delta_0 > 0$ such that $\forall \vec{\eta}$ with $|\vec{\varphi}(0) - \vec{\eta}| < \delta_0$, the solution $\vec{\psi}(t)$ with $\vec{\psi}(0) = \vec{\eta}$ exists $\forall t \in [0, \infty)$ and $\lim_{t \to \infty} \left|\vec{\psi}(t) - \vec{\varphi}(t)\right| = 0$.
3. $\vec{\varphi}$ is **unstable** if it is not stable.

In the textbook, these definitions are formulated with potentially different initial times t_0. By the continuous dependence theory that we covered, these definitions are equivalent.

Perturbation Methods

The "base case" is at $\vec{y}' = A\vec{y}$, where $n \times n$ constant matrix. Recall that if we know the real part of the eigenvalues of A, we can determine if 0 is stable, asymptotically stable, or unstable. We want to extend this result to "perturbations" of $\vec{y}' = A\vec{y}$.

Example

$$y' = \left(-1 + \frac{1}{1+t}\right) y$$

Observe that as $t \to \infty$, $\frac{1}{1+t} \to 0$. So, our problem resembles $y' = -y$ more and more as $t \to \infty$. Since 0 is stable for $y' = -y$, we expect the same for $y' = \left(-1 + \frac{1}{1+t}\right) y$. Indeed, in this particular page, we can verify that 0 is asymptotically stable by solving the equation by using separation of variables which yields $y = Ce^{-t}(1+t)$. As $t \to \infty$, the exponential term becomes smaller faster than t increases meaning y goes to 0 and 0 is asymptotically stable.

Theorem

Consider the system

$$\vec{y}' = (A + B(t))\vec{y} \tag{9}$$

where A is a $n \times n$ real-valued constant matrix and $B(t)$ is a continuous $n \times n$ matrix-valued function on $t \geq 0$. Assume that all eigenvalues λ_j of A satisfy $\operatorname{Re} \lambda_j < 0$ meaning 0 is an asymptotically stable critical point of $\vec{y}' = A\vec{y}$. Additionally, assume $\lim_{t\to\infty} B(t) = 0$. Then, 0 is an asymptotically stable critical point for (9).

- **Proof**: we want to show that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|\vec{\eta}| < \delta$, then the solution $\vec{\psi}(t)$ with $\vec{\psi}(0) = \vec{\eta}$ satisfies stability, meaning $\left|\vec{\psi}(t)\right| < \varepsilon \ \forall t \geq 0$, and asymptotic stability, meaning $\left|\vec{\psi}(t)\right| \to 0$ as $t \to \infty$. Note that the existence of $\vec{\psi}(t)$ for $t \geq 0$ is already guaranteed. We must show two steps.

  1. Show that if $B(t)$ is sufficiently small, then $\left|\vec{\psi}(t)\right| \to 0$ as $t \to \infty$. Suppose $|B(t)| < \delta$ for $t \geq T$. We claim that $\exists k > 0$, $\sigma > 0$ such that for all solutions $\vec{\psi}$ to (9), $\left|\vec{\psi}(t)\right| \leq K \left|\vec{\psi}(T)\right| e^{-\sigma(t-T)}$. Choose ρ so that $\operatorname{Re} \lambda_j < -\rho < 0$. Then, $\exists k_0 > 0$ such that $\left|e^{At}\right| \leq k_0 e^{-\rho t}$.

     $$\vec{\psi}' = A\vec{\psi} + B(t)\vec{\psi}$$

     Let $B(t)\vec{\psi} = \vec{g}(t)$. By variation of constants starting from $t = T$:

     $$\begin{aligned}
     \vec{\psi}(t) &= e^{(t-T)A}\vec{\psi}(T) + \int_T^t e^{A(t-s)} B(s)\vec{\psi}(s)\mathrm{d}s \\
     \left|\vec{\psi}(t)\right| &\leq k_0 e^{-\rho(t-T)} \left|\vec{\psi}(T)\right| + \int_T^t k_0 e^{-\rho(t-T)} |B(s)| \left|\vec{\psi}(s)\right| \mathrm{d}s \\
     &\leq k_0 e^{-\rho(t-T)} \left|\vec{\psi}(t)\right| + e^{-\rho t} \int_T^t k_0 \delta e^{\rho s} \left|\vec{\psi}(s)\right| \mathrm{d}s \\
     e^{\rho t} \left|\vec{\psi}(t)\right| &\leq k_0 e^{\rho T} \left|\vec{\psi}(T)\right| + \int_T^t k_0 \delta e^{\rho s} \left|\vec{\psi}(s)\right| \mathrm{d}s
     \end{aligned}$$

     Apply Gronwall's inequality.

     $$\begin{aligned}
     &\leq k_0 e^{\rho T} \left|\vec{\psi}(T)\right| e^{\int_T^t k_0 \delta \mathrm{d}s} \\
     &= k_0 e^{\rho T} \left|\vec{\psi}(T)\right| e^{k_0 \delta (t-T)} \\
     \left|\vec{\psi}(t)\right| &\leq k_0 e^{-\rho(t-T) + k_0 \delta (t-T)} \left|\vec{\psi}(T)\right|
     \end{aligned}$$

     If we choose δ such that $k_0 \delta < \rho$ and choose $\sigma = \rho - k_0 \delta > 0$, then our claim is proven.

2. Observe that $\exists T > 0$ such that $|B(t)| < \delta$ on $t \geq T$ (which is the hypothesis in step 1) since $\lim\limits_{t\to\infty} |B(t)| = 0$. We need to control the solution $\vec{\psi}(t)$ for $t \in [0,T]$. We will use the integral formulation and Gronwall's inequality.

$$\begin{aligned}\vec{\psi}' &= (A + B(t))\vec{\psi} \\ \vec{\psi}(t) &= \vec{\psi}(0) + \int_0^t (A + B(s))\vec{\psi}(s)\mathrm{d}s\end{aligned}$$

By continuity of B on $[0,T]$ and the intermediate value theorem, $\exists M'$ such that $|B(s)| \leq M$ for $s \in [0,T]$. So, we have $M = M' + |A|$ with $|A| + |B(s)| \leq M$ for $s \in [0,T]$.

$$\begin{aligned}\left|\vec{\psi}(t)\right| &\leq \left|\vec{\psi}(0)\right| + \int_0^t (|A| + |B(s)|)\left|\vec{\psi}(s)\right|\mathrm{d}s \\ &\leq \left|\vec{\psi}(0)\right| + \int_0^t M\left|\vec{\psi}(s)\right|\mathrm{d}s\end{aligned}$$

Apply Gronwall's inequality

$$\begin{aligned}\left|\vec{\psi}(t)\right| &\leq \left|\vec{\psi}(0)\right| e^{Mt} \\ &\leq \left|\vec{\psi}(0)\right| e^{MT}\end{aligned}$$

for $t \in [0,T]$.

From step 1, for $t \geq T$, $\left|\vec{\psi}(t)\right| \leq K\left|\vec{\psi}(T)\right| e^{-\sigma(t-T)}$ is bounded by $Ke^{MT}\left|\vec{\psi}(0)\right| e^{-\sigma(t-T)}$. So, we have:

$$\left|\vec{\psi}(t)\right| \leq \begin{cases}\left|\vec{\psi}(0)\right| e^{MT} & t \in [0,T] \\ Ke^{MT}\left|\vec{\psi}(0)\right| e^{-\sigma(t-T)} & t \geq T\end{cases}$$

Given $\varepsilon > 0$, if we choose δ so that $\delta < \min\left\{\frac{\varepsilon}{e^{MT}}, \frac{\varepsilon}{Ke^{MT}}\right\}$, then $\left|\vec{\psi}(0)\right| < \delta$ implies $\left|\vec{\psi}(t)\right| < \varepsilon\ \forall t \geq 0$ (stability) and $\left|\vec{\psi}(t)\right| \to 0$ as $t \to \infty$ (asmyptotic stability). Hence, the theorem is proven. □

Almost Linear Systems

We want to study the stability of critical points for almost linear systems. The motivation of this is to study nonlinear systems $\vec{y}' = \vec{g}(\vec{y})$ near a critical point $\vec{y}_0$ where $\vec{g}(\vec{y}_0) = 0$. To do this, introduce $\vec{z}(t)$ such that $\vec{\psi}(t) = \vec{y}_0 + \vec{z}(t)$ where $\vec{\psi}(t)$ solves $\vec{y}' = \vec{g}(\vec{y})$. So, $\vec{\psi}' = \vec{z}'(t)$ meaning $\vec{z}' = \vec{g}(\vec{y}_0 + \vec{z})$.

If $\vec{z}$ is small, it is reasonable to use Taylor expansion to write

$$\vec{g}(\vec{y}_0 + \vec{z}) = \vec{g}(\vec{y}_0) + D\vec{g}(\vec{y}_0)\vec{z} + \vec{G}(\vec{z}).$$

$\vec{g}(\vec{y}_0) = 0$ since $\vec{y}_0$ is a critical point, and we can observe that $\left|\vec{G}(\vec{z})\right| \leq c|\vec{z}|^2$ for $\vec{z} \leq k$ for some k. $D\vec{g}(\vec{y}_0)$ is the gradient of $\vec{g} : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R}^n$ of $\vec{y}_0$, which yields a $n \times n$ matrix.

$$\begin{aligned}(D\vec{g}(\vec{y}_0))_{jk} &= \partial_{y_k} g_j(\vec{y}_0) \\ (D\vec{g}(\vec{y}_0)\vec{z})_j &= \sum_{k=1}^n z_k \cdot \partial_{y_k} g_j(\vec{y}_0)\end{aligned}$$

$D\vec{g}(\vec{y}_0)$ is a constant real-valued matrix. By writing $A = D\vec{g}(\vec{y}_0)$, we see that $\vec{z}' = A\vec{z} + \vec{G}(\vec{z})$ where $\lim\limits_{|\vec{z}|\to 0} \frac{|\vec{G}(\vec{z})|}{|\vec{z}|} = 0$.

Motivated by this computation, we introduce almost linear systems.

Theorem

Consider the system

$$\vec{y}' = A\vec{y} + \vec{f}(t, \vec{y}) \tag{10}$$

with the following assumptions:

1. All eigenvalues of A have negative real parts, meaning 0 is an asymptotically stable critical point of $\vec{y}' = A\vec{y}$.
2. $\vec{f}$ is continuous and Lipschitz continuous in $\vec{y}$ on D where $D = [0, \infty) \times \{|\vec{y}| < k\}$ for $k > 0$.
3. $\lim_{\vec{y} \to 0} \sup_{t \geq 0} \frac{|\vec{f}(t,\vec{y})|}{|\vec{y}|} = 0$, which is the "almost linear" assumption. Observe that this implies $\vec{f}(t, 0) = 0$ meaning 0 is a solution to (10).

Then, $\exists \delta_0 > 0$ such that $\forall |\vec{\eta}_0| < \delta_0$, the unique solution $\vec{\psi}(t)$ to (10) with $\vec{\psi}(0) = \vec{\eta}_0$ exists $\forall t \geq 0$ and $\left|\vec{\psi}(t)\right| < \varepsilon \ \forall t \geq 0$. Moreover, $\lim_{t \to \infty} \vec{\psi}(t) = 0$, that is, 0 is asymptotically stable.

- **Proof**: the core idea of this proof is similar to the preceding theorem, variation of constants and Gronwall's inequality. However, we now need to be careful since $\vec{\psi}$ may not exist $\forall t \geq 0$.

  1. A-priori estimate: let us assume that $\exists$ a solution $\vec{\psi}(t)$ on $t \in [0, T]$. Note that by the hypothesis, $\forall \delta_1 > 0$, $\exists \alpha > 0$ such that if $|\vec{y}| < \alpha$, then $\frac{|\vec{f}(t,\vec{y})|}{|\vec{y}|} < \delta_1 \ \forall t \geq 0$. Equivalently, $\left|\vec{f}(t, \vec{y})\right| \leq \delta_1 |\vec{y}|$. We can assume that $|\vec{\varphi}(t)| < \alpha$ for $t \in [0, T]$.

  $$\vec{\psi}' = A\vec{\psi} + \vec{f}(t, \vec{y})$$

  By variation of constants,

  $$\vec{\psi}(t) = e^{At}\vec{\psi}(0) + \int_0^t e^{(t-s)A} \vec{f}(s, \vec{\psi}(s)) \mathrm{d}s.$$

  If we also assume $\left|\vec{\psi}(t)\right| < \alpha \ \forall t \geq 0$,

  $$\begin{aligned} \left|\vec{\psi}(t)\right| &\leq \left|e^{At}\right| \left|\vec{\psi}(0)\right| + \int_0^t \left|e^{(t-s)A}\right| \left|\vec{f}(s, \vec{\psi}(s))\right| \mathrm{d}s \\ &\leq \left|e^{At}\right| \left|\vec{\psi}(0)\right| + \int_0^t \delta_1 \left|e^{(t-s)A}\right| \left|\vec{\psi}(s)\right| \mathrm{d}s \end{aligned}$$

  This is similar to step 1 of the previous theorem. Let $\rho > 0$, $k > 0$ such that $\max_j \operatorname{Re} \lambda_j < -\rho < 0$ and $\left|e^{At}\right| \leq k e^{-\rho t}$.

  $$\begin{aligned} &\leq k e^{-\rho t} \left|\vec{\psi}(0)\right| + \int_0^t k e^{-\rho(t-s)} \delta_1 \left|\vec{\psi}(s)\right| \mathrm{d}s \\ e^{\rho t} \left|\vec{\psi}(t)\right| &\leq k \left|\vec{\psi}(0)\right| + \int_0^t k \delta_1 e^{\rho s} \left|\vec{\psi}(s)\right| \mathrm{d}s \end{aligned}$$

  Apply Gronwall's inequality.

  $$\begin{aligned} &\leq k \left|\vec{\psi}(0)\right| e^{k\delta_1 t} \\ \left|\vec{\psi}(t)\right| &\leq k e^{-(\rho - k\delta_1)t} \left|\vec{\psi}(0)\right| \end{aligned}$$

  Observe that this statement is stronger than what we assumed.

2. Bootstrap argument: our goal is to show that $\forall \varepsilon > 0$, $\exists \delta_0 > 0$ such that if $\left|\vec{\psi}(0)\right| < \delta_0$, then $\vec{\psi}(t)$ exists $\forall t \geq 0$ and and $\left|\vec{\psi}(t)\right| < \delta e^{-\sigma t}$ for some $\sigma > 0$ $\forall t \geq 0$. In view of our result from step 1, we can choose δ_1 so that $\rho - k\delta_1 > 0$. Then, we can choose $\sigma = \rho - k\delta_1 > 0$, $\delta_0 < \frac{\varepsilon}{k}$, and $\delta_0 < \frac{\alpha}{k}$. If $\vec{\psi}$ satisfies the a-priori assumptions made, then the result from step 1 holds.

$$\begin{aligned}\left|\vec{\psi}(t)\right| &\leq ke^{-(\rho - k\delta_1)t}\left|\vec{\psi}(0)\right| \\ &\leq k\delta_0 e^{-\sigma t} \\ &< \varepsilon e^{-\sigma t}\end{aligned}$$

The bootstrap assumption (BA) is that $\vec{\psi}(t)$ exists for $t \in [0, T]$ and satisfies $\left|\vec{\psi}(t)\right| \leq \varepsilon e^{-\sigma t}$ and $\left|\vec{\psi}(t)\right| \leq \alpha$. The bootstrap conclusion (BC) is that $\vec{\psi}(t)$ satisfies $\left|\vec{\psi}(t)\right| \leq \frac{\varepsilon}{2}e^{-\sigma t}$ and $\left|\vec{\psi}(t)\right| < \frac{\alpha}{2}$. If we choose $\delta_0 < \frac{\varepsilon}{2k}$ and $\delta < \frac{\alpha}{2k}$, then by step 1, the bootstrap assumption implies the bootstrap conclusion.

We can perform a continuous induction on time. Consider $\mathscr{B} = \{T \in [0, \infty) :\ \text{BC holds on } [0, T]\}$. If δ_0 is small enough, then $\mathscr{B} \neq \emptyset$, meaning $\exists T > 0$ such that $\vec{\psi}$ exists on $[0, T]$ (local existence) and $\left|\vec{\psi}(t)\right| < \frac{\varepsilon}{2}e^{-\sigma t}$ for $t \in [0, T]$ (continuity of $\vec{\psi}$ in t). Moreover, define $T_* := \sup \mathscr{B} \neq \infty$. By the definition of $\mathscr{B}$, $\vec{\psi}$ exists on any $[0, T]$ for $T < T_*$ and $\left|\vec{\psi}(t)\right| < \frac{\varepsilon}{2}e^{-\sigma t}$ on $[0, T]$. So, $\vec{\psi}$ exists on $[0, T_*)$ and $\left|\vec{\psi}(t)\right| \leq \frac{\varepsilon}{2}e^{-\sigma t}$ on $[0, T_*)$. By the theory of continuity of solutions, we can continue the solution to T_* and a bit past T_* since T_* is finite. So, $\exists T' > T_*$ such that $\left|\vec{\psi}(t)\right| < \varepsilon e^{-\sigma t}$ on $[0, T']$. Since the bootstrap assumption implies the bootstrap conclusion, $\left|\vec{\psi}(t)\right| < \frac{\varepsilon}{2}e^{-\sigma t}$ on $[0, T']$ which is a contradiction since this would mean $T^* \neq \sup \mathscr{B}$. $\square$

Conditional Stability

Consider the system

$$\vec{y}' = A\vec{y} + \vec{g}(\vec{y}).$$

What happens if there is an eigenvalue A of A where $\operatorname{Re}\lambda > 0$? We will focus on a special case, where A is a 2×2 matrix and A has 2 eigenvalues, $\lambda > 0$ and $-\mu < 0$. In the linear case when $A = \left(\begin{smallmatrix} -\mu & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, we have a saddle point (case 3). What happens in the nonlinear case?

Theorem

Consider the system

$$\vec{y}' = A\vec{y} + \vec{g}(\vec{y})$$

where $A = \left(\begin{smallmatrix} -\mu & 0 \\ 0 & \lambda \end{smallmatrix}\right)$ with $\mu, \lambda > 0$. Let $\vec{g}$ and $\partial_{y_1}\vec{g}$, $\partial_{y_2}\vec{g}$ be continuous on $\{\vec{y} \in \mathbb{R}^2 : |\vec{y}| < k\}$. Additionally, let $\partial_{y_1}\vec{g}(0) = \partial_{y_2}\vec{g}(0) = 0$ and $\vec{g}(0) = 0$ (which is the almost linear assumption). Then, $\exists$ a curve C in the phase plane that passes through 0 such that:

1. If $\vec{\varphi}$ is any solution with $\vec{\varphi}(0)$ on C and $|\vec{\varphi}(0)|$ small enough, then $\vec{\varphi}(t) \to 0$.
2. No solution $\vec{\varphi}(t)$ with $|\vec{\varphi}(0)|$ small enough but not on C can remain small $\forall t \geq 0$.

We can make the following remarks.

- If we apply the theorem with the direction of time reversed ($s = -t$), then $\exists C'$ such that the theorem holds in the reverse direction.
- This can easily be generalized to A with eigenvalues λ and $-\mu$ via a change in basis.
- This is called the **conditional stability** of 0. The theorem generalizes to the case when A is a $n \times n$ with k negative eigenvalues and $n - k$ positive eigenvalues. In that case, C would be a k-dimensional surface passing through 0. This has to do with the stable manifold theorem

Now, we can prove the theorem.

- **Proof**: we will first set up the proof.

0. Set up: we have $A = \begin{pmatrix} -\mu & 0 \\ 0 & \lambda \end{pmatrix}$, so $e^{At} = \begin{pmatrix} e^{-\mu t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$. Let $U_1(t) = \begin{pmatrix} e^{-\mu t} & 0 \\ 0 & 0 \end{pmatrix}$ and $U_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$. So, $e^{At} = U_1(t) + U_2(t)$ with $U_1(t) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} e^{-\mu t}\eta_1 \\ 0 \end{pmatrix}$ and $U_2(t) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\lambda t}\eta_2 \end{pmatrix}$. We have $\vec{g}(0) = 0$ and $\lim\limits_{\vec{y} \to 0} \partial_{y_1} \vec{g}(\vec{y}) = 0$. So, by the fundamental theorem of calculus, we have:

$$\begin{aligned}
\vec{g}(\vec{y}) - \vec{g}(\vec{y}^*) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \vec{g}((1-s)\vec{y}^* + s\vec{y})\mathrm{d}s \\
&= \int_0^1 \sum_j \left(y - y_j^*\right) \partial_{y_j} g_j((1-s)\vec{y}^* + s\vec{y})\mathrm{d}s \, |\vec{g}(\vec{y}) - \vec{g}(\vec{y}^*)| \\
&\le c \left|\vec{y} - \vec{y}_j^*\right| \sup_j \left|\partial_{y_j} \vec{g}(\vec{y}')\right|
\end{aligned}$$

For $\vec{y}'$ between $\vec{y}$ and $\vec{y}^*$, the term in the sup goes to 0 as $\vec{y}' \to \infty$. So, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|\vec{y}|, |\vec{y}^*| < \delta$, then $|\vec{g}(\vec{y}) - \vec{g}(\vec{y}^*)| < \varepsilon |\vec{y} - \vec{y}^*|$. ε will be specified later and $\delta = \delta(\varepsilon)$.

1. Show that if $\vec{\varphi}$ is a solution which exists on $t \ge 0$ and satisfies $|\vec{\varphi}(t)| < \delta \ \forall t \ge 0$, then:

$$\vec{\varphi}(t) = U_1(t) \begin{pmatrix} \vec{\varphi}_1(0) \\ 0 \end{pmatrix} + \int_0^t U_1(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s - \int_t^\infty U_2(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s$$

To derive this, we use variation of constants.

$$\begin{aligned}
\vec{\varphi}(t) &= e^{At}\vec{\varphi}(0) + \int_0^t e^{(t-s)A}\vec{g}(\vec{\varphi}(s))\mathrm{d}s \\
&= U_1(t)\vec{\varphi}(0) + \int_0^t U_1(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s + U_2(t)\vec{\varphi}(0) + \int_0^t U_2(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s
\end{aligned}$$

We can think about the condition that $|\vec{\varphi}| < \delta$ entails. We worry about $U_2(t)\vec{\varphi}(0) = \begin{pmatrix} 0 \\ e^{\lambda t}\varphi_2(0) \end{pmatrix}$ and $\int_0^t U_2(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s = \begin{pmatrix} 0 \\ e^{\lambda t} \int_0^t e^{-s\lambda} g_2(\vec{\varphi}(s))\mathrm{d}s \end{pmatrix}$ since both grow without bound. We hope they may cancel out each other.

$$\begin{aligned}
\vec{\varphi}(t) &= U_1(t)\vec{\varphi}(0) + \int_0^t U_1(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s \\
&\quad + \begin{pmatrix} 0 \\ e^{\lambda t}\left(\vec{\varphi}_2(0) + \int_0^\infty e^{-\lambda s} g_2(\vec{\varphi}(s))\mathrm{d}s - \int_t^\infty e^{-\lambda s} g_2(\vec{\varphi}(s))\mathrm{d}s\right) \end{pmatrix} \\
&= U_1(t)\vec{\varphi}(0) + \int_0^t U_1(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s - \int_t^\infty \begin{pmatrix} 0 \\ e^{\lambda(t-s) g_2(\vec{\varphi}(s))} \end{pmatrix} \mathrm{d}s \qquad (\star) \\
&\quad + \begin{pmatrix} 0 \\ e^{\lambda t}\left(\varphi_2(0) + \int_0^\infty e^{-\lambda s} g_2(\vec{\varphi}(s))\mathrm{d}s\right) \end{pmatrix}
\end{aligned}$$

Observe that the LHS, $\vec{\varphi}(t)$, and the first expression $(\star)$ are both bounded $\forall t \ge 0$ (which is slightly nontrivial to prove). The remainder of the RHS will not be bounded $\forall t \ge 0$ unless

$$\varphi_2(0) + \int_0^\infty e^{-\lambda s} g_2(\vec{\varphi}(s))\mathrm{d}s = 0.$$

So, to keep the LHS bounded $\forall t \ge 0$, we have the second expression as 0 meaning we have our result as desired.

2. We now want to solve the integral equation in step 1. using Picard iteration. We want to show that if ε is sufficiently small so that $\delta > 0$ is correspondingly small, then $\exists \sigma > 0$, $\delta_1 > 0$, and $c_1 > 0$ such that $\forall \eta \in \mathbb{R}$ with $|\eta| < \delta_1$, $\exists$ a solution $\vec{\varphi}$ to the integral equation from step 1 on $t \geq 0$ with $\varphi_1(0) = \eta$ and
$$|\vec{\varphi}(t)| < c_1|\eta|e^{-\sigma t}$$
and $|\vec{\varphi}(t)| < \delta\ \forall t \geq 0$. Observe two differences from the Picard-Lindelöf theorem: only $\varphi_1(0)$ is specified and there is both $\int_0^t$ and $\int_t^\infty$. Even so, the basic idea is the same.

To prove this, we will first form the initial iterate $\varphi_1(0) \to \eta$ and $\vec{g}(\vec{\varphi}(s)) \to \vec{g}(0) = 0$.
$$\vec{\varphi}^1(t) = U_1 \begin{pmatrix} \eta \\ 0 \end{pmatrix} = \begin{pmatrix} \eta e^{-\mu t} \\ 0 \end{pmatrix}$$
We also form $\varphi_1(0) \to \eta$ and $\vec{g}(\vec{\varphi}(s)) \to \vec{g}(\vec{\varphi}^{n-1}(s))$.
$$\vec{\varphi}^n(t) = U_1(t) \begin{pmatrix} \eta \\ 0 \end{pmatrix} + \int_0^t U_1(t-s)\vec{g}(\vec{\varphi}^{n-1}(s))\mathrm{d}s - \int_t^\infty U_2(t-s)\vec{g}(\vec{\varphi}^{n-1}(s))\mathrm{d}s$$
We will show that $\vec{\varphi}^n = \vec{\varphi}^1 + \sum_{n'=2}^{n} \left(\vec{\varphi}^{n'} - \vec{\varphi}^{n'-1}\right)$ goes to $\vec{\varphi}$, our desired solution. For $\vec{\varphi}^1$, we have
$$\left|\vec{\varphi}^1(t)\right| \leq |\eta|e^{-\mu t} \leq \delta_1 e^{-\sigma t}$$
as long as $0 < \sigma \leq \mu$. For $\delta_1 < \delta$, $\left|\vec{\varphi}^1(t)\right| < \delta\ \forall t \geq 0$. For $\vec{\varphi}^n - \vec{\varphi}^{n-1}$, we have:
$$\begin{aligned}
\vec{\varphi}^n(t) - \vec{\varphi}^{n-1}(t) &= \int_0^t U_1(t-s)\left(\vec{g}(\vec{\varphi}^{n-1}(s)) - \vec{g}(\vec{\varphi}^{n-2}(s))\right)\mathrm{d}s \\
&\quad - \int_t^\infty U_2(t-s)\left(\vec{g}(\vec{\varphi}^{n-1}(s)) - \vec{g}(\vec{\varphi}^{n-1}(s))\right)\mathrm{d}s \\
\left|\vec{\varphi}^n(t) - \vec{\varphi}^{n-1}(t)\right| &\leq \int_0^t e^{-\mu(t-s)}\varepsilon\left|\vec{\varphi}^{n-1}(s) - \vec{\varphi}^{n-2}(s)\right|\mathrm{d}s \\
&\quad + \int_t^\infty e^{\lambda(t-s)}\varepsilon\left|\vec{\varphi}^{n-1}(s) - \vec{\varphi}^{n-2}(s)\right|\mathrm{d}s
\end{aligned}$$
For the purpose of induction, assume
$$\left|\vec{\varphi}^{n-1}(s) - \vec{\varphi}^{n-1}(s)\right| \leq \frac{|\eta|}{2^{n-1}}e^{-\sigma t}$$
where σ is to be chosen. So, we have:
$$\begin{aligned}
\left|\vec{\varphi}^n(t) - \vec{\varphi}^{n-1}(t)\right| &\leq \int_0^t e^{-\mu(t-s)}\frac{\varepsilon|\eta|}{2^{n-1}}e^{-\sigma s}\mathrm{d}s + \int_t^\infty e^{\lambda(t-s)}\frac{\varepsilon|\eta|}{2^{n-1}}e^{-\sigma s}\mathrm{d}s \\
&\leq \frac{\varepsilon|\eta|}{2^{n-1}}e^{-\mu t}\int_0^t e^{\mu s - \sigma s}\mathrm{d}s + \frac{\varepsilon|\eta|}{2^{n-1}}e^{\lambda t}\int_t^\infty e^{-\lambda s - \sigma s}\mathrm{d}s \\
&= \frac{\varepsilon|\eta|}{2^{n-1}}e^{-\mu t}\left(\frac{1}{\mu - \sigma}\left(e^{(\mu-\sigma)t} - 1\right)\right) + \frac{\varepsilon|\eta|}{2^{n+1}}e^{\lambda t}\left(\frac{1}{\lambda + \sigma}e^{(-\lambda-\sigma)t}\right)
\end{aligned}$$
This holds if $-\lambda - \sigma < 0$ or $-\lambda < \sigma$. If $\mu - \sigma > 0$, then $\frac{1}{\mu-\sigma}\left(e^{(\mu-\sigma)t} - 1\right) \leq \frac{1}{\mu-\sigma}e^{(\mu-\sigma)t}$.
$$\begin{aligned}
&\leq \frac{\varepsilon|\eta|}{2^{n-1}}e^{-\mu t}\frac{1}{\mu - \sigma}e^{(\mu-\sigma)t} + \frac{\varepsilon|\eta|}{2^{n-1}}e^{\lambda t}\frac{1}{\lambda + \sigma}e^{(-\lambda-\sigma)t} \\
&= \frac{\varepsilon}{2^{n-1}}\left(\frac{1}{\mu - \sigma} + \frac{1}{\lambda + \sigma}\right)|\eta|e^{-\sigma t}
\end{aligned}$$
We have conditions $-\lambda < \sigma < \mu$. If we also take $\varepsilon\left(\frac{1}{\mu-\sigma} + \frac{1}{\lambda+\sigma}\right) < \frac{1}{2}$, then we have:
$$\left|\vec{\varphi}^n(t) - \vec{\varphi}^{n-1}(t)\right| < \frac{|\eta|}{2^n}e^{-\sigma t}$$

By induction, this holds $\forall n$.

$$\begin{aligned}\left|\vec{\varphi}^1\right| + \sum_{n=2}^{\infty}\left|\vec{\varphi}^n - \vec{\varphi}^{n-1}\right| &\leq |\eta|e^{-\sigma t} + \sum_{n=2}^{\infty}\frac{1}{2^n}|\eta|e^{-\sigma t} \\ &\leq \left(1+\frac{1}{2}\right)|\eta|e^{-\sigma t}\end{aligned}$$

So, $\vec{\varphi}^1(t) + \sum_{n=2}^{\infty}\left(\vec{\varphi}^n(t) - \vec{\varphi}^{n-1}(t)\right)$ is summable to $\vec{\varphi}(t)$ and $|\vec{\varphi}(t)| \leq \frac{3}{2}|\eta|e^{-\sigma t}$. Since $|\eta| < \delta_1$, if $\frac{3}{2}\delta_1 < \delta$ and c_1 is chosen to be $\frac{3}{2}$, then we have our desired result.

3. Show uniqueness of the formula from step 1. We want to show that if $\varepsilon > 0$ is sufficiently small (so $\delta = \delta(\varepsilon)$ is accordingly small), then the following holds: if $\vec{\varphi}(t)$ and $\vec{\varphi}_*(t)$ are solutions to the formula defined on $t \geq 0$, $|\vec{\varphi}(t)| < \delta$ and $|\vec{\varphi}_*(t)| < \delta$ $\forall t \geq 0$, and $\varphi_1(0) = (\varphi_*)_1(0)$, then $\vec{\varphi}(t) = \vec{\varphi}_*(t)$ $\forall t \geq 0$.

$$\begin{aligned}\vec{\varphi}(t) &= U_1(t)\begin{pmatrix}\varphi_1(0)\\0\end{pmatrix} + \int_0^t U_1(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s - \int_t^{\infty} U_2(t-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s \\ \vec{\varphi}_*(t) &= U_1(t)\begin{pmatrix}(\varphi_*)_1(0)\\0\end{pmatrix} + \int_0^t U_1(t-s)\vec{g}(\vec{\varphi}_*(s))\mathrm{d}s - \int_t^{\infty} U_2(t-s)\vec{g}(\vec{\varphi}_*(s))\mathrm{d}s \\ \left(\vec{\varphi}-\vec{\varphi}_*\right)(t) &= \int_0^t U_1(t-s)\left(\vec{g}(\vec{\varphi}(s)) - \vec{g}(\vec{\varphi}_*(s))\right)\mathrm{d}s - \int_t^{\infty} U_2(t-s)\left(\vec{g}(\vec{\varphi}(s)) - \vec{g}(\vec{\varphi}_*(s))\right)\mathrm{d}s \\ \left|\left(\vec{\varphi}-\vec{\varphi}_*\right)(t)\right| &\leq \int_0^t e^{-\mu(t-s)}\varepsilon\left|\vec{\varphi}(s) - \vec{\varphi}_*(s)\right|\mathrm{d}s + \int_t^{\infty} e^{\lambda(t-s)}\varepsilon\left|\vec{\varphi}(s) - \vec{\varphi}_*(s)\right|\mathrm{d}s\end{aligned}$$

Let $M = \sup_{t\geq 0}|\vec{\varphi}(t) - \vec{\varphi}_*(t)|$.

$$\begin{aligned}&\leq \int_0^t e^{-\mu(t-s)}\varepsilon M\mathrm{d}s + \int_t^{\infty} e^{\lambda(t-s)}\varepsilon M\mathrm{d}s \\ &\leq \frac{1}{\mu}\varepsilon M + \frac{1}{\lambda}\varepsilon M \\ &= \varepsilon\left(\frac{1}{\mu} + \frac{1}{\lambda}\right)M\end{aligned}$$

This holds $\forall t \geq 0$.

$$M \leq \varepsilon\left(\frac{1}{\mu} + \frac{1}{\lambda}\right)M$$

See notes for the rest.

4. To conclude the proof, we must define the curve C. By steps 2 and 3, given η with $|\eta| < \delta_1$, $\exists!$ solution $\vec{\varphi}$ to the formula from step 1 with $\varphi_1(0) = \eta$ and $|\vec{\varphi}(t)| < \delta$ $\forall t \geq 0$. Moreover, $|\vec{\varphi}(t)| \leq c_1|\eta|e^{-\sigma t}$ for some $c_1, \sigma > 0$. Note that the full initial condition for $\vec{\varphi}(t)$ can be found using the formula from step 1.

$$\vec{\varphi}(0) = \begin{pmatrix}\eta\\ -\int_0^{\infty} U(-s)\vec{g}(\vec{\varphi}(s))\mathrm{d}s\end{pmatrix}$$

Let the bottom term, $\varphi_2(0)$, be $\psi(\eta)$. We define the curve $C = \{(\eta, \psi(\eta)) : |\eta| < \delta_1\}$. 0 is the unique solution that stays close to 0 and $\eta = 0$. So, $\psi(0) = 0$ and C must pass through 0.

We may now conclude the proof. If $\vec{\varphi}(0)$ satisfies $|\vec{\varphi}(0)| < \delta_1$ and if $\vec{\varphi}(0)$ is on C, then $\vec{\varphi}(t)$ satisfies $|\vec{\varphi}(t)| \leq c_1\left|\varphi_1(0)\right|e^{-\sigma t}$ by the previous steps and the uniqueness to the IVP to $\vec{y}' = A\vec{y} + \vec{g}(\vec{y})$. So, we have $\vec{\varphi}(t) \to 0$ as $t \to \infty$. If $\vec{\varphi}(0)$ is not on C but $|\vec{\varphi}(t)| < \delta$ $\forall t \geq 0$, then we reach a contradiction by step 1. So, we have proven our desired result. $\square$

Lypunov's Method

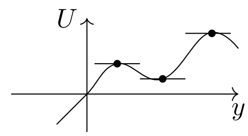
Lypunov's method is another way to study the stability of solutions. As a motivating example, we will look at Newton's equation in one dimension.

$$my'' = f(y)$$

Let $m = 1$ and $f(y) = -\partial_y U(y) = -U'(y)$. This yields $y'' = -U'(y)$. Let $\dot{y} = y'$.

$$\begin{pmatrix} y' \\ \dot{y}' \end{pmatrix} = \begin{pmatrix} \dot{y} \\ -U'(y) \end{pmatrix} \tag{11}$$

Note that $\begin{pmatrix} y_0 \\ \dot{y}_0 \end{pmatrix}$ is a critical point if and only if $\begin{pmatrix} \dot{y}_0 \\ -U'(y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, meaning y_0 is a critical point of $U(y)$ and $\dot{y}_0 = 0$. Which of these critical points are stable?



Theorem

If y_0 satisfies $U'(y_0) = 0$ and $U''(y_0) > 0$ (meaning by the second derivative test, y_0 is a minimum point), then $\left(\begin{smallmatrix} y_0 \\ 0 \end{smallmatrix}\right)$ is a stable solution of (11).

Our approach for this has been to study linearizations. Here, we present a different approach which relies on conservation of energy.

$$H(y, \dot{y}) = \frac{1}{2}\dot{y}^2 + U(y)$$

$H(t, \dot{y})$ is the total energy and the Hamiltonian, $\frac{1}{2}\dot{y}^2$ is the kinetic energy, and $U(y)$ is the potential energy.

Lemma

If $\vec{\varphi} = \begin{pmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{pmatrix}$ is a solution to $\begin{pmatrix} y' \\ \dot{y}' \end{pmatrix} = \begin{pmatrix} \dot{y} \\ -U'(y) \end{pmatrix}$, then $\frac{\mathrm{d}}{\mathrm{d}t} H(\vec{\varphi}(t)) = 0$ ($H(\vec{\varphi}(t))$ is independent of t).

- **Proof**: $\vec{\varphi}$ solves the system, so we have $\varphi'(t) = \dot{\varphi}(t)$ and $\dot{\varphi}'(t) = -U'(\vec{\varphi}(t))$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(\vec{\varphi}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\dot{\varphi}(t)^2 + U(\vec{\varphi}(t))\right] \\ &= \dot{\varphi}(t)\dot{\varphi}'(t) + U'(\vec{\varphi}(t))\vec{\varphi}'(t) \\ &= \dot{\varphi}(t)\left(-U'(\vec{\varphi}(t))\right) + U'(\vec{\varphi}(t))\dot{\varphi}(t) \\ &= 0 \end{aligned}$$

□

Now, we can prove the theorem.

- **Proof**: note that if $\vec{\varphi}(t)$ is a solution to (11), then $H9\vec{\varphi}(t)) = H(\vec{\varphi}(0))$. Let $H_0 = H(y_0, 0)$ where y_0 is the critical point. Define

$$\tilde{H}(y, \dot{y}) = H(y, \dot{y}) - H_0.$$

$\tilde{H}(\vec{\varphi}(t)) = \tilde{H}(\vec{\varphi}(0))$ and $\tilde{H}(y_0, 0) = 0$. Our key claim is that $\exists \delta_0, c, c' > 0$ such that

$$c|(y - y_0, \dot{y})|^2 \le \tilde{H}(y, \dot{y}) \le c'|(y - y_0, \dot{y})|^2$$

for $|(y, \dot{y})| < \delta_0$. If the claim is true, then we can show stability of the specified solution. We want to show that $\forall \varepsilon > 0$, if $|(\varphi(0) - y_0, \dot{\varphi}(0)| < \delta$, then $\tilde{H}(\vec{\varphi}(0)| < c'\delta$ and $|(\varphi(t) - y_0, \dot{\varphi}(t)| < \varepsilon$.

$$\begin{aligned} |(\varphi(t) - y_0, \dot{\varphi}(t)| &\le \frac{1}{c}\tilde{H}(\vec{\varphi}(t)) \\ &= \frac{1}{c}\tilde{H}(\vec{\varphi}(0)) \\ &< \frac{c'}{c}\delta \end{aligned}$$

By setting $\frac{c'}{c}\delta = \varepsilon$, we have our result. Note that $\vec{\varphi}(t)$ would exist $\forall t$ since we have shown that it stays bounded $\forall t$.

To prove the claim, we can use Taylor expansion. WLOG, assume $y_0 = 0$.

$$\begin{aligned}\tilde{H}(y,\dot{y}) &= \frac{1}{2}\dot{y}^2 + U(y) - U(0) \\ &= \frac{1}{2}\dot{y}^2 + U'(0)y + \frac{1}{2}U''(0)y^2 + \mathcal{O}(y^3)\end{aligned}$$

with $U'(0) = 0$, $U''(0) > 0$, and

$$|\mathcal{O}(y^3)| \leq \underbrace{\frac{1}{3!}\sup_{z\in[-1,1]}|U'''(z)|}_{c_0}|y|^3$$

by Taylor's theorem. Choose δ_0 so that $c_0\delta_0 < \frac{1}{4}U''(0)$.

$$|\mathcal{O}(y^3)| \leq c_0|y|^3 \leq c_0\delta_0 y^2 < \frac{1}{4}U''(0)y^2$$

$$-\frac{1}{4}U''(0)y^2 \leq \mathcal{O}(y^3) \leq \frac{1}{4}U''(0)y^2$$

$$c|(y,\dot{y})| \leq \frac{1}{2}\dot{y}^2 + \frac{1}{4}U''(0)y^2 \leq \tilde{H}(y,\dot{y}) \leq \frac{1}{2}\dot{y}^2 + \frac{3}{4}U''(0)y^2 \leq c'|(y,\dot{y})|^2$$

So, we have proven our claim and we are done. □

Lyapunov's method consists of proving the stability of an equilibria by constructing a functional V (Lyapunov's fractal) that is analogous to H in the previous problem. It is simple and powerful, but the challenge is to find the fractal V.

Orbits and Functionals

Orbits

Consider

$$\vec{y}' = \vec{f}(\vec{y})$$

where $\vec{f} : D \to \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ is an open subset. Let $\vec{f}$ be C^1, meaning $\vec{f}$ and $\partial_{y_j}\vec{f}$ are continuous in D. We define the following:

- $C \subseteq D$ is an **orbit** if it is the image of a solution: $C = \{\vec{\varphi}(t) \in D : t \in (T_-, T_+)\}$ where $\vec{\varphi}(t)$ solves $\vec{y}' = \vec{f}(\vec{y})$ and (T_-, T_+) is the maximum interval of validity for $\vec{\varphi}$.
- A **positive semi-orbit** C_+ corresponding to $\vec{\varphi}$ is $C_+ = \{\vec{\varphi}(t) \in D : t \in [0, T_+)\}$.
- A **negative semi-orbit** C_- corresponding to $\vec{\varphi}$ is $C_- = \{\vec{\varphi}(t) \in D : t \in (T_0, 0]\}$.

Orbits have the following properties.

- If $\vec{\eta} \in D$ is not a critical point (meaning it is an **ordinary point**), then there is at most one orbit passing through $\vec{\eta}$. This is due to uniqueness and the fact that if $\vec{\varphi}$ is a solution, then so is any $\vec{\varphi}(t + t_0)$. If $\vec{\varphi}(t_1) = \vec{\eta}$ and $\vec{\psi}(t_2) = \vec{\eta}$, we can define $\vec{\psi}_*(t) = \vec{\varphi}(t + t_1 - t_2)$ which is also a solution. $\vec{\psi}_*(t_2) = \vec{\varphi}(t_1) = \vec{\eta} = \vec{\psi}(t_2)$, so by uniqueness, $\vec{\psi}_*(t) = \vec{\psi}(t)\ \forall t$. So, we have $\vec{\psi}(t) = \vec{\varphi}(t + t_1 - t_2)$.
- If an orbit C passes through an ordinary point $\vec{\eta}$, then it cannot pass through a critical point. More precisely, if
$$\vec{\varphi}(t_1) = \vec{\eta}$$
where $\vec{\eta}$ is an ordinary point and
$$\lim_{t\to b^-}\vec{\varphi}(t) = \vec{\alpha}$$
where $\vec{\alpha}$ is a critical point, then $b = T_+$.
- If an orbit C passes through an ordinary point $\vec{\eta}$, then C cannot intersect itself unless $\vec{\varphi}$ is periodic, in which case C is a **closed orbit**. We can see this by observing $\vec{\varphi}(t_2) = \vec{\varphi}(t_1)$ with $t_2 > t_1$ and $\vec{\varphi}$ is nonconstant. Define $\vec{\varphi}_*(t) = \vec{\varphi}(t + t_2 - t_1)$. $\vec{\varphi}_*(t)$ is a solution and $\vec{\varphi}_*(t_1) = \vec{\varphi}(t_2) = \vec{\varphi}(t_1)$, so by uniqueness, $\vec{\varphi}_*(t) = \vec{\varphi}(t)\ \forall t$. So, $\vec{\varphi}(t) = \vec{\varphi}(t + T)$ with $T = t_2 - t_1$ and $\vec{\varphi}(t)$ is periodic.

Lyapunov Functionals

Again, consider

$$\vec{y}' = \vec{f}(\vec{y})$$

and assume that 0 is a critical point. Let $\Omega \subseteq D$ be an open subset with $\vec{0} \in \Omega$. Consider a **functional** $V : \Omega \to \mathbb{R}$ which is C^1 on Ω. V is a **positive definite** functional on Ω if it satisfies:

1. $V(\vec{y}) \geq 0 \ \forall \vec{y} \in \Omega$
2. $V(\vec{y}) = 0$ if and only if $\vec{y} = \vec{0}$.

Similarly, V is a **negative definite** functional on Ω if the same properties hold except $V(\vec{y}) \leq 0$.

Example

$V = y_1^2 + y_2^2$ is a positive definite functional on $\mathbb{R}^2$. More generally, $V(\vec{y}) = \vec{y}^\top A\vec{y}$ where A is a positive definite matrix (A is symmetric and all eigenvalues of A are positive) is a positive definite functional.

We define the derivative of V with respect to $\vec{y}' = \vec{f}(\vec{y})$ as

$$V^*(\vec{y}) := \vec{f}(\vec{y}) \cdot \boldsymbol{\nabla} V(\vec{y}) = \sum_{j=1}^{n} f_j(\vec{y}) \partial_{y_j} V(\vec{y}).$$

Note that for a solution $\vec{\varphi}$ to $\vec{y}' = \vec{f}(\vec{y})$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(\vec{\varphi}(t)) &= \sum_{j=1}^{n} \partial_{y_j} V(\vec{\varphi}(t)) \cdot \vec{\varphi}'(t) \\
&= \sum_{j=1}^{n} \partial_{y_j} V(\vec{\varphi}(t)) f_j(\vec{\varphi}(t)) \\
&= V^*(\vec{\varphi}(t))
\end{aligned}$$

Lyapunov's Theorems

We are still working with the differential equation

$$\vec{y}' = \vec{f}(\vec{y})$$

where $\vec{0}$ is a critical point. Suppose we have functional V. Lyapunov's Theorems make use of V to determine the stability of critical point $\vec{0}$.

Theorem 1: Stability

Assume that $\exists$ a C^1 functional $V : \Omega \to \mathbb{R}$ where $\Omega \subseteq D$ is open and $\vec{0} \in \Omega$ which satisfies:

- V is positive definite on Ω.
- $V^* \leq 0$ on Ω.

Then, $\vec{0}$ is stable.

Theorem 2: Asymptotic Stability

Assume that $\exists$ a C^1 functional $V : \Omega \to \mathbb{R}$ where $\Omega \subseteq D$ is open and $\vec{0} \in \Omega$ which satisfies:

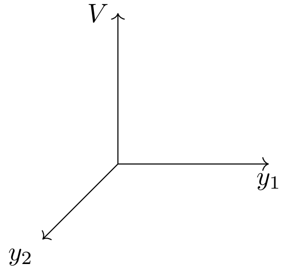
- V is positive definite on Ω.
- V^* is negative definite on Ω, meaning $V^*(\vec{y}) < 0$ for $\vec{y} \in \Omega$, $\vec{y} \neq 0$ and $V^*(\vec{0}) = 0$.

Then, $\vec{0}$ is asymptotically stable.

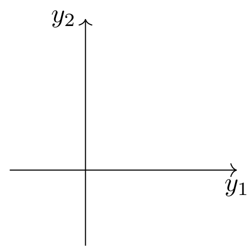
Observe that these theorems do not tell us how to construct V, which poses the greatest difficulty. Additionally, observe that these are sufficient conditions, but not necessary.

Visualizing Lyapunov's Theorems

Lyapunov's Theorems can be expressed graphically.



The contour lines are as follows.



Proof of Lyapunov's Theorems

- **Proof**: starting with the first theorem, we want to show that for $\vec{\varphi}(t)$ solving $\vec{y}' = \vec{f}(\vec{y})$, $\forall\varepsilon > 0$, $\exists\delta > 0$ such that if $|\vec{\varphi}(0)| < \delta$, then $\vec{\varphi}(t)$ exists $\forall t$ and $|\vec{\varphi}(t)| < \varepsilon$. By the fact that V is positive definite, $\exists r > 0$ such that on $B_r(\vec{y}) = \{\vec{y} \in \mathbb{R}^n : |\vec{y}| < r\}$, $B_r(\vec{0}) \subseteq \Omega$. Let $\varepsilon > 0$. Consider $\mu = \min\{V(\vec{y}) : \varepsilon \leq |\vec{y}| \leq r\}$. Note that if $|\vec{y}| \leq r$ and $V(\vec{y}) < \mu$, then $|\vec{y}| < \varepsilon$. So, $\exists\delta > 0$ such that if $|\vec{y}| < \delta$, then $V(\vec{y}) < \mu$. Observe that we need $\delta < \varepsilon$ to ensure that $V(\vec{y}) < \mu$.

  If $|\vec{\varphi}(0)| < \delta$, then $V(\vec{\varphi}(0)) < \mu$. $\forall 0 \leq t < T_+$ where T_+ is the lifespan of $vphi$, $V(\vec{\varphi}(t)) \leq V(\vec{\varphi}(0))$. So, for $V^* \leq 0$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(\vec{\varphi}(t)) &= \vec{\varphi}'(t) \cdot \boldsymbol{\nabla} V(\vec{\varphi}(t)) \\ &= V^*(\vec{\varphi}(t)) \\ &\leq 0\end{aligned}$$

  Thus, $|\vec{\varphi}(t)| < \varepsilon$ $\forall t$ with $0 \leq t < T_+$. By the continuation of solutions, $T_+ = +\infty$. So, we have $|\vec{\varphi}(t)| < \varepsilon$ $\forall t \geq 0$ which proves that $\vec{0}$ is stable. □

- **Proof**: for the second theorem, we know from the previous proof that $\vec{\varphi}(t)$ exists $\forall t$ if $|\vec{\varphi}(0)|$ is small enough. $\frac{\mathrm{d}}{\mathrm{d}t}V(\vec{\varphi}(t)) \leq 0$, so $V(\vec{\varphi}(t))$ is non-increasing. $V \geq 0$, so $V_\infty = \lim\limits_{t\to\infty} V(\vec{\varphi}(t))$ exists.

  By contradiction, assume that $V_\infty \neq 0$. Since V is positive definite, this means $V_\infty > 0$. Define $K = \{\vec{y} : |\vec{y}| < r,\ V_\infty \leq V(\vec{y}) \leq V(\vec{\varphi}(0))\}$. Note that K is closed and bounded, meaning it is compact. So, by the Extreme Value Theorem for V^*, $\exists\vec{y}_{\min} \in K$ with $\min\limits_K V^* = V^*(\vec{y}_{\min}) < 0$ since V^* is negative definite, meaning $V^* \leq -C < 0$ on $\vec{y} \neq 0$. Let $V^*(\vec{y}_{\min}) = -C$.

$$\frac{\mathrm{d}}{\mathrm{d}t}V(\vec{\varphi}(t)) = V^*(\vec{\varphi}(t)) \leq -C \longrightarrow V(\vec{\varphi}(t)) \leq V(\vec{\varphi}(0)) - Ct$$

  This means $V(\vec{\varphi}(t)) < 0$ for $t > \frac{V(\vec{\varphi}(0))}{C}$, which is a contradiction. Thus, $V_\infty = \lim\limits_{t\to\infty} V(\vec{\varphi}(t)) = 0$ meaning $\vec{0}$ is assymptotically stable. □

Examples of Lyapunov's Theorems

Example

Recall our motivating example, Newton's equation with $n = 1$.

$$y'' = -G'(y)$$

G is the potential energy, which is a C^1 function on $\mathbb{R}$. We can write this as an equivalent first order system:

$$\begin{pmatrix} y' \\ \dot{y}' \end{pmatrix} = \begin{pmatrix} \dot{y} \\ -G'(y) \end{pmatrix}$$

The total energy is written as the Hamiltonian, $H(y, \dot{y}) = \frac{1}{2}\dot{y}^2 + G(y)$. Recall that $H^*(y, \dot{y}) = 0$:

$$\begin{aligned} H^*(y, \dot{y}) &= \begin{pmatrix} \dot{y} \\ -G'(y) \end{pmatrix} \cdot \nabla H \\ &= \begin{pmatrix} \dot{y} \\ -G'(y) \end{pmatrix} \begin{pmatrix} G'(y) \\ \dot{y} \end{pmatrix} = 0 \end{aligned}$$

So, H is conserved on each trajectory. If H is also positive definite near a critical point, then we can apply the first theorem to conclude stability. For instance, suppose that:

1. $G'(0) = 0$ (G has a critical point at 0), meaning $\left(\begin{smallmatrix} y \\ \dot{y} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is a critical point of $y'' = -G'(y)$.
2. $G'(y)$ satisfies $yG'(y) > 0 \ \forall y \neq 0$. Equivalently, $G'(y) < 0$ for $y < 0$ and $G'(y) > 0$ for $y > 0$.

Then, H is positive definite. Indeed, $H = \frac{1}{2}\dot{y}^2 + G(y) \geq 0$ since $G \geq 0$ and $\frac{1}{2}\dot{y}^2 \geq 0$. By the seame reason, $H(y, \dot{y}) = 0$ if and only if $\frac{1}{2}\dot{y}^2 = 0$ and $G(y) = 0$, which happens if and only if $(y, \dot{y}) = 0$.

Example

We can generalize the previous example to a Hamiltonian system with $n = 2d$ and $p_j = y_j'$ for $j = 1, \ldots, d$.

$$\begin{aligned} y_j' &= p_j \\ p_j' &= -\partial_j U(y) \end{aligned}$$

$U : \mathbb{R}^d \to R$ is the potential energy. The total energy is

$$H(y, p) = \frac{1}{2}|p|^2 + U(y).$$

Again, we claim that $H^* = 0$.

$$\begin{aligned} H^* &= \vec{f} \cdot \nabla H \\ &= \vec{f} \cdot \begin{pmatrix} \partial_{\vec{y}} U \\ \vec{p} \end{pmatrix} \\ &= \begin{pmatrix} \vec{p} \\ -\partial_{\vec{y}} U \end{pmatrix} \begin{pmatrix} \partial_{\vec{y}} U \\ \vec{p} \end{pmatrix} \\ &= \left(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_{\vec{y}} U \\ \vec{p} \end{pmatrix} \right)^\top \begin{pmatrix} \partial_{\vec{y}} U \\ \vec{p} \end{pmatrix} = 0 \end{aligned}$$

where each I is a $d \times d$ identity matrix. We can write the ODE as $\begin{pmatrix} \vec{y} \\ \vec{p} \end{pmatrix}' = J \nabla H(\vec{y}, \vec{p})$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. For any system of this form, $H^* = 0$. A natural idea is to use H as a Lyapunov function. Moreover, note that if U attains a strict local minimum at $\vec{0}$, then H is positive definite on a neighborhood of $\begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}$. By Theorem 1, if $\vec{0}$ is a strict minimum of U, then it is stable with respect to $\begin{pmatrix} \vec{y} \\ \vec{p} \end{pmatrix} = J \nabla H(\vec{y}, \vec{p})$.

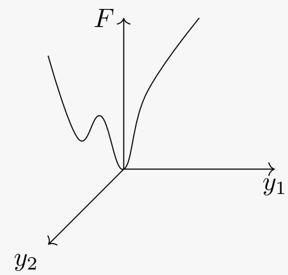
Example

Let us observe gradient flows

$$\vec{y}' = -\boldsymbol{\nabla} F(\vec{y})$$

where $F : \mathbb{R}^n \to \mathbb{R}$. The negative gradient is in the direction of steepest descent.

Note that $\vec{y}_0$ is a critical point for the ODE if and only if $\boldsymbol{\nabla} F(\vec{y}_0) = 0$, meaning $\vec{y}_0$ is a critical point as defined in calculus.



We claim that $F^*(\vec{y}) = -\left|\boldsymbol{\nabla} F(\vec{y})\right|^2$. We can prove this directly using the definition.

$$\begin{aligned} F^*(\vec{y}) &= -\boldsymbol{\nabla} F(\vec{y}) \cdot \boldsymbol{\nabla} F \\ &= -\left|\boldsymbol{\nabla} F(\vec{y})\right|^2 \end{aligned}$$

As a corollary, if $\vec{y}_0$ is an isolated minimum of F, then $\vec{y}_0$ is asymptotically stable. To prove this, WLOG, let $\vec{y}_0 = \vec{0}$. Take

$$V(\vec{y}) = F(\vec{y}) - F(\vec{y}_0)$$

and apply Theorem 2.

For a specific example, take $F = (y_1 + y_2)^4 + (y_1 - y_2)^4$.

$$\begin{aligned} y_1' &= -\partial_{y_1} F = -4(y_1 + y_2)^3 - 4(y_1 - y_2)^3 \\ y_2' &= -\partial_{y_2} F = -4(y_1 + y_2)^3 + 4(y_1 - y_2)^3 \end{aligned}$$

Since $\vec{0}$ is a unique minimum of F (which can be checked), it is an asymptotically stable critical point of this ODE.

Example

$$\begin{aligned} y_1' &= -y_1 - y_2 \\ y_2' &= y_1 - y_2^3 \end{aligned}$$

Here, there is no a-priori information for choosing the natural V. In this case, an idea is to try

$$V = \vec{y}^\top A \vec{y}$$

where A is a positive definite matrix. Choose A so that $V^* \leq 0$.

In this case, $V = y_1^2 + y_2^2$ yields $V^* = -2y_1^2 - 2y_2^4$ which is negative definite. So, $\vec{0}$ is asymptotically stable.

Note that in this case, the method of using fundamental matrices also works, since $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ has two eigenvalues with negative real parts.

Instability using Lyapunov's Theorems

Theorem

If $V(\vec{0}) = \vec{0}$, V^* is positive or negative definite definite, and $\exists$ a sequence $\vec{y}_n \to \vec{0}$ such that $V(\vec{y}_n)$ has the same sign as V, then $\vec{0}$ is unstable.

- **Proof**: see Appendix D on page 66.

Example

Let $y' = y^3$. $V = y^2$ is positive definite and $V^* = 2y^4$ is also positive definite, so 0 is unstable.

Invariant Sets & Stability

As a motivating example, let us observe Liénard's equation. We will look at the 1D case. We start with conservative Newton's equation, then add a damping term y'.

$$y'' + y' + g(y) = 0$$

We can compare this with Newton's equation $y'' + G'(y) = 0$. If $G(y) = \frac{1}{2}y^2$, we have $y'' + y = 0$ which is the motion of a particle on a spring.

The energy, or Hamiltonian, of Liénard's equation is

$$H(y, \dot{y}) = \frac{1}{2}\dot{y}^2 + G(y)$$

where $\dot{y} = y'$ and $G(y) = \int_0^y g(y')\mathrm{d}y'$. Note that the energy is not conserved, which can be seen by showing that $H^* \neq 0$. The system can be written as

$$\begin{pmatrix} y \\ \dot{y} \end{pmatrix}' = \begin{pmatrix} \dot{y} \\ -\dot{y} - g(y) \end{pmatrix} =: \vec{f}(y, \dot{y}).$$

So, by definition, we have the following:

$$\begin{aligned} H^* &= \vec{f} \cdot \boldsymbol{\nabla} H \\ &= \begin{pmatrix} \dot{y} \\ -\dot{y} - g(y) \end{pmatrix} \cdot \begin{pmatrix} g(y) \\ \dot{y} \end{pmatrix} \\ &= -\dot{y}^2 \leq 0 \end{aligned}$$

Since $H^* \leq 0$, the energy of the system can only decrease. This follows what we expect.

Assume that $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point, or equivalently, $g(0) = 0$. Assume also that H is positive definite, or equivalently, $yg(y) \neq 0$ for $y \neq 0$. This happens if and only if G has a strict minimum at 0. Since $H^* = -\dot{y}^2 \leq 0$, by Theorem 1, $\vec{0}$ is stable. Is it asymptotically stable? Theorem 2 cannot be applied since H^* is not negative definite (it is 0 at $\dot{y} = 0$, or y is constant). It turns out that $\vec{0}$ is asymptotically stable. We will develop tools for studying the asymptotic behavior solutions to prove it.

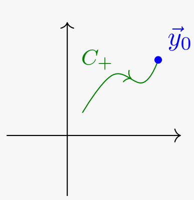
Consider the system

$$\vec{y}' = \vec{f}(\vec{y})$$

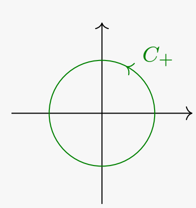
where $\vec{f} : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R}^n$ and $\vec{f}$ C^1 in D. For $\Gamma \subseteq D$, Γ is (positive) **invariant** if $\forall \vec{y} \in \Gamma$, $\vec{\varphi}(t) \in \Gamma$ $\forall t \geq 0$ where $\vec{\varphi}(0) = \vec{y}$. Let $C_+ = \{\vec{\varphi}(t) : t \geq 0\}$ where $\vec{\varphi}$ solves $\vec{y}' = \vec{f}(\vec{y})$. Assume C_+ is bounded. The (positive) **limit set** (or the ω-limit set) of C_+ is

$$L(C_+) = \left\{ \vec{y} : \exists t_n \to \infty \text{ such that } \lim_{n\to\infty} \vec{\varphi}(t_n) = \vec{y} \right\}.$$

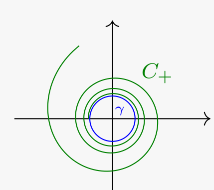
Example



If $\vec{f}(\vec{y})$ is a line approaching a single critical point $\vec{y}_0$, then $L(C_+) = \{\vec{y}_0\}$.



Here, since $\vec{\varphi}(t)$ always stays on C_+, $L(C_+) = C_+$.



Here, $L(C_+) = \gamma$ where γ is the circle that C_+ converges to.

Lemma

If C_+ is bounded, then $L(C_+)$ is nonempty and closed.

- **Proof**: see Appendix D on page 66.

Lemma

$L(C_+)$ is always positive invariant. Moreover, $\forall \vec{y} \in C_+$, $\vec{\varphi}(t)$ with $\vec{\varphi}(0) = \vec{y}$ approaches $L(C_+)$ as $t \to \infty$ in the sense that $d(\vec{\varphi}(t), L(C_+)) \to 0$ as $t \to \infty$.

Recall that we define the distance between point $\vec{y}$ and set A as

$$d(\vec{y}, A) = \inf_{\vec{x} \in A} d(\vec{y}, \vec{x}).$$

- **Proof**: see Appendix D on page 66.

It turns out that $L(C_+)$ is also connected, but this is more difficult to prove.

Theorem

Consider the system

$$\vec{y}' = \vec{f}(\vec{y})$$

where $\vec{f} : D \to \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$, and $\vec{f}$ is C^1 in D. Suppose $\vec{f}(\vec{0}) = \vec{0}$ and $V : \Omega \to \mathbb{R}$ is C^1 in Ω and $\vec{0} \in \Omega$. Suppose the following are true:

1. V is positive definite on Ω
2. $V^* \leq 0$ on Ω

Consider C_λ, the component of $\{\vec{y} : V(\vec{y}) \leq \lambda\}$ containing $\vec{0}$. Assume C_λ is bounded. Let

$$E = \{\vec{y} \in C_\lambda : V^*(\vec{y}) = 0\}$$

and

$$M = \bigcup \{\Gamma : \Gamma \text{ is invariant, } \Gamma \subseteq E\}.$$

Then, $\forall \vec{y} \in C_\lambda$, $\vec{\varphi}(t)$ with $\vec{\varphi}(0) = \vec{y}$ approaches M as $t \to \infty$. This means $d(\vec{\varphi}(t), M) \to 0$ as $t \to \infty$.

To prove this theorem, we must first prove a key lemma.

Lemma

Let $V \in C^1(\Omega)$ and $V^* \leq 0$ on Ω. Let $C_+ = \{\vec{\varphi}(t) : t \geq 0\}$ be a positive orbit and $C_+, L(C_+) \subseteq \Omega$. Then, $V^*(\vec{y}) = 0 \ \forall \vec{y} \in L(C_+)$.

- **Proof**: $V(\vec{\varphi}(t))$ is non-increasing and C_+ is bounded, so $V(\vec{\varphi}(t))$ is bounded from below. Thus, $\exists A = \lim_{t \to \infty} V(\vec{\varphi}(t))$. In particular, $\forall t_n \to \infty$, $V(\vec{\varphi}(t_n)) \to A$. So, $V(\vec{y}) = A \ \forall y \in L(C_+)$.

$$V^*(\vec{y}) = \frac{\mathrm{d}}{\mathrm{d}t} V(\vec{\varphi}(t))\big|_{t=0} = 0$$

for $\vec{\varphi}(0) = \vec{y}$ since $\vec{\varphi}(t) \in L(C_+)$ by the previous lemma, so $V(\vec{\varphi}(t)) = A$. □

Now, we can prove the theorem.

- **Proof**: for any $\vec{y} \in C_\lambda$, $\vec{\varphi}(t) \in C_\lambda$ where $\vec{\varphi}(0) = \vec{y} \ \forall t \geq 0$. C_λ is bounded, so $C_+ = \{\vec{\varphi}(t) : t \geq 0\}$ is bounded. $L(C_+)$ exists, so $L(C_+) \subseteq C_\lambda$. By the key lemma, $\forall \vec{y}' \in L(C_+)$, $V^*(\vec{y}') = 0$. So, $L(C_+) \subseteq E$. But, we also know that $L(C_+)$ is invariant, meaning $L(C_+) \subseteq M$. □

Going back to our motivating example Liénard's equation, $E = \{\dot{y} = 0\}$ and $M = \{\vec{0}\}$. So, by the theorem, $\vec{\varphi}(t)$ approaches $\vec{0}$ meaning $\vec{0}$ is asymptotically stable.

Summary

Overview

We are studying ODEs of the form

$$\vec{F}(t, \vec{y}, \dots \vec{y}^{(k)}) = 0, \qquad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \ \vec{y}^{(i)} = \begin{pmatrix} y_1^{(i)} \\ \vdots \\ y_n^{(i)} \end{pmatrix}, \ \vec{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}$$

where $\vec{F} : D \to \mathbb{R}^n$ and $D \subseteq \mathbb{R}^{1+(1+k)n}$. k is the order of the ODE, t is the independent variable, and y is the dependent variable. If $n = 1$, we have a scalar and if $n > 1$, we have a system.

Solving First Order Scalar ODEs

Suppose we have an ODE in the form $y' = f(t, y)$. We have two methods of explicitly solving the ODE.

1. **Method of separation of variables**: $y' = g(t)h(y)$

   Step 1: find all ℓ such that $h(\ell) = 0$. $\vec{\varphi} = \ell$ are constant solutions.
   Step 2: where $h \neq 0$, write
   $$\int_{y_0}^{y(t)} \frac{1}{h(y)} \mathrm{d}y = \int_{t_0}^{t} g(t) \mathrm{d}t$$
   and evaluate to find $y(t)$.

2. **Method of integrating factors**: $y' = a(t)y + b(t)$ (linear nonhomogeneous 1^{st} order scalar ODEs)

   Introduce $A = \int a(t)\mathrm{d}t$. e^{-A} is the integrating factor.
   $$(e^{-A}y)' = e^{-A}y' + e^{-A}(-a(t)y)$$
   So, $y' - a(t)y = b(t)$ is equivalent to $(e^{-A}y)' = e^{-A}b$, which can then be integrated.

Grönwall's Inequality

There are 2 forms of Grönwall's Inequality.

1. **Differential form**: let $g \geq 0$ be a continuous function on $[\alpha, \beta]$ and let F be a function that is continuous on $[\alpha, \beta]$ and differentiable on (α, β). If $F'(t) \leq g(t)F(t)$ $\forall f \in (\alpha, \beta)$ and $F(a) \leq A$, then
   $$F(t) \leq F(\alpha)e^{\int_\alpha^t g(s)\mathrm{d}s} \leq Ae^{\int_\alpha^t g(s)\mathrm{d}s}$$
   on $t \in [\alpha, \beta]$.

2. **Integral form**: let $g \geq 0$ be a continuous function on $[\alpha, \beta]$ and let f be a continuous function on $[\alpha, \beta]$. If
   $$f(t) \leq A + \int_{t_0}^{t} g(s)f(s)\mathrm{d}s$$
   $\forall t \in [\alpha, \beta]$, then
   $$f(t) \leq Ae^{\int_{t_0}^t g(s)\mathrm{d}s}.$$

Note that our proof for the differential form, we did not need $g \geq 0$. But, in our proof of the integral form, we reduced the hypothesis to differential form:

$$F(t) = A + \int_{t_0}^{t} g(s)f(s)\mathrm{d}s$$
$$F'(t) = g(t)f(t) \leq g(t)F(t)$$

This uses the fact that $g \geq 0$. Then, we used the differential form.

Well-Posedness Theory for First Order IVPs

Picard-Lindelöf/Cauchy-Lipschitz Theorem

Consider the system

$$\begin{cases} \vec{y}'(t) = \vec{f}(t, \vec{y}) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases} \tag{2}$$

where $\vec{f} : D \to \mathbb{R}^n$ with $D \subseteq \mathbb{R}^{1+n}$, D open. Suppose f is continuous on D and $\frac{\partial}{\partial y_1}\vec{f}$ is continuous on D with $j = 1, \dots, n$ (which implies Lipschitz continuity). Then, the following is true.

1. **Existence**

   $\exists$ an interval I where $t_0 \in I$ and a solution $\vec{\varphi} : I \to \mathbb{R}^n$ to (2), i.e. $(t, \vec{\varphi}(t)) \in D$ $\forall t \in I$, $\vec{\varphi}'(t)$ is defined, and (2) is satisfied for $\vec{\varphi} = \vec{f}$.

2. **Uniqueness**

   $\exists$ at most one solution to (2), i.e. if $\vec{\varphi}_{(1)}$ and $\vec{\varphi}_{(2)}$ solve (2) and are defined on a common interval J, then $\vec{\varphi}_{(1)} = \vec{\varphi}_{(2)}$.

3. **Continuity of solution**

   assume in addition that $\vec{f}$ is bounded in D. Then the unique solution $\vec{\varphi}$ to (2) can be continued as long as its graph does not reach the boundary of D ∂D, i.e. if $[t_0, t_+)$ is the maximum interval of validity of $\vec{\varphi}$ to the right, then

   $$\begin{cases} t_+ = +\infty, \text{ or} \\ t_+ < +\infty \text{ and } (t_+, \lim\limits_{t \to t_+^-} \vec{\varphi}(t)) \in \partial D \end{cases}$$

   Alternatively, if $D = \mathbb{R}^{1+n}$ (but $\vec{f}$ is not necessarily bounded), then $\vec{\varphi}$ can be continued as long as $|\vec{\varphi}|$ stays finite.

4. **Continuous dependance on initial condition**

   let $\vec{\varphi}_{(1)}$ and $\vec{\varphi}_{(2)}$ be solutions to (2) with initial conditions

   $$\vec{\varphi}_{(1)}(t_0) = \vec{y}_{(1)0}, \qquad \vec{\varphi}_{(2)}(t_0) = \vec{y}_{(2)0}$$

   defined on a common interval J. Then,

   $$\left|\vec{\varphi}_{(1)}(t) - \vec{\varphi}_{(2)}(t)\right| \le \left|\vec{y}_{(1)0} - \vec{\varphi}_{(2)0}\right| e^{K|t-t_0|}.$$

   Here, K is a Lipschitz constant for $\vec{f}$ on R that contains the graphs of both $\vec{y}_{(1)}$ and $\vec{y}_{(2)}$ on J.

The proofs of these relied on the following key ideas.

1. Writing the equation in integral form:

   $$\vec{y} = \vec{y}_0 + \int_{t_0}^{t} \vec{f}(s, \vec{y}(s)) \mathrm{d}s$$

2. Picard iteration:

   $$\begin{aligned} \vec{\varphi}_0 &= \vec{y}_0 \\ \vec{\varphi}_k &= \vec{y}_0 + \int_{t_0}^{t} \vec{f}(s, \vec{\varphi}_{n-1}(s)) \mathrm{d}s \end{aligned}$$

3. Grönwall's Inequality (integral version).

Linear Systems

A first-order linear system is of the form

$$\vec{y}' = A(t)\vec{y} + \vec{g}(t).$$

Existence and Uniqueness of Solutions

Consider the system

$$\begin{cases} \vec{y}' = A(t)\vec{y} + \vec{g}(t) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

where $A(t) : I \to \mathbb{R}^{n\times n}$ and $\vec{g}(t) : I \to \mathbb{R}^n$ continuous. There exists a unique solution $\vec{\varphi}$ to the system on I.

If $\vec{\varphi}_p$ is a particular solution to $\vec{y}' = A(t)\vec{y} + \vec{g}(t)$ (meaning $\vec{\varphi}_p' = A(t)\vec{\varphi}_p + \vec{g}(t)$ on I), then any solution $\vec{\varphi} : I \to \mathbb{R}^n$ has the form

$$\vec{\varphi} = \vec{\varphi}_h + \vec{\varphi}_p$$

where $\vec{\varphi}_h$ is the homogeneous solution to $\vec{y}' = A(t)\vec{y}$ (meaning $\vec{\varphi}_h' = A(t)\vec{\varphi}_h$).

1. **Homogeneous case**: she space of solutions $\vec{\varphi} : I \to \mathbb{R}^n$ to the homogeneous equation $\vec{y}' = A\vec{y}$ forms a vector space, meaning if $\vec{\varphi}_1$ and $\vec{\varphi}_2$ are both solutions, then so is $c_1\vec{\varphi}_1 + c_2\vec{\varphi}_2$ $\forall c_1, c_2$ scalars. The dimension of this space is n. A basis $\{\vec{\varphi}_1, \dots, \vec{\varphi}_n\}$ of the vector space is called a fundamental system. The matrix $\Phi = \left(\vec{\varphi}_1 \middle| \cdots \middle| \vec{\varphi}_n\right)$ whose columns consist of basis elements is called a fundamental matrix. Φ is a fundamental matrix if and only if $\Phi' = A\Phi$ and $\det \Phi(t_0) \neq 0$ for some $t_0 \in I$.

   Abel's theorem states that if $\Psi' = A\Psi$, then

   $$\det \Psi(t) = \det \Psi(t_0) e^{\int_{t_0}^{t} \operatorname{tr} A(t')\mathrm{d}t'}.$$

   Given constant vector $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$, $\Phi(t)\vec{c}$ is a solution to the homogeneous equation. Additionally, given any nonsingular constant $n \times n$ matrix C, $\Phi(t)C$ is a fundamental matrix. As a consequence,

   $$\vec{\varphi}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{y}_0$$

   solves the homogeneous equation with $\vec{\varphi}(t_0) = \vec{y}_0$. So, knowing the fundamental matrix allows us to find all solutions.

2. **Inhomogeneous case**: variation of constants can be used to find the general solution. To find particular solution $\vec{\varphi}_p$ to $\vec{y}' = A(t)\vec{y} + \vec{g}(t)$, we look for $\vec{\varphi}_p = \Phi(t)\vec{u}(t)$ where $\Phi(t)$ is a fundamental matrix. If $\vec{u}$ is constant, then $\vec{\varphi}_p$ solves the homogeneous case, but we "vary" this constant. The solution to the system is

   $$\vec{\varphi}_p = \Phi(t) \int_{t_0}^{t} \Phi(s)^{-1}\vec{g}(s)\mathrm{d}s$$

   with $\vec{\varphi}_p(t_0) = 0$.

Linear Systems with Constant Coefficients

Consder the system

$$\vec{y}' = A\vec{y}$$

where A is a $n \times n$ real-valued constant matrix. The matrix exponential formula can be used to find the fundamental matrix in terms of A.

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

Given A, to compute e^{At}, we do the following:

1. Find $P(\lambda) = \det(\lambda I - A)$, then find all of its roots which are eigenvalues of A.

$$P(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_k)^{n_k}$$

   The λ_j's are eigenvalues where n_j is the algebraic multiplicity of λ_j with $n_1 + \cdots + n_k = n$.

2. Find generalized eigenvectors associated with λ_j. More specifically, $\ker(A - \lambda_j I)^{n_j}$ is n_j dimensional and if $\vec{v}_{j,\ell}$ is a basis of $\ker(A - \lambda_j I)^{n_j}$, then $\{\vec{v}_{1,1}, \ldots, \vec{v}_{k,n_k}\}$ forms a basis. To find a chain of generalized eigenvectors, follow the steps:

   1. Find all eigenvectors $\vec{v}$ which solve $(A - \lambda_j I)\vec{v} = 0$.
   2. For the first eigenvector $\vec{v}_{j,1}$, find $\vec{v}_{j,2}$ such that $(A - \lambda_j I)\vec{v}_{j,2} = \vec{v}_{j,1}$ and continue until impossible.
   3. Repeat step 2 for each eigenvector.

   Putting these chains together yields a basis:

$$\{\vec{v}_{1,1}, \ldots, \vec{v}_{1,n_1}, \ldots, \vec{v}_{k,1}, \ldots, \vec{v}_{k,n_k}\}$$

   Each $\vec{v}_{j,\ell}$ satisfies $(A - \lambda_j I)\vec{v}_{j,\ell} = \begin{cases} 0 \\ \vec{v}_{j,\ell-1} \end{cases}$.

3. With respect to this basis, e^{At} is easy to compute.

$$e^{At}\vec{v}_{j,\ell} = e^{\lambda_j t}\left(I + (A - \lambda_j I)t + \cdots + \frac{1}{(n_j - 1)!}(A - \lambda_j I)^{n_j - 1} t^{n_j - 1}\right)\vec{v}_{j,\ell}$$

   In fact, if $P = \left(\vec{v}_{1,1}\middle| \cdots \middle|\vec{v}_{k,n_k}\right)$:

$$P^{-1}AP = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_M \end{pmatrix} \qquad J_m = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

$$P^{-1}e^{At}P = \begin{pmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_M T} \end{pmatrix} \qquad e^{J_m T} = e^{\lambda_j T}\begin{pmatrix} 1 & t & \cdots & \frac{1}{(L-1)!}t^{L-1} \\ & \ddots & \ddots & \\ & & \ddots & t \\ 0 & & & 1 \end{pmatrix}$$

   L is the length of the chain of generalized eigenvectors for λ_j and J_m is a $L \times L$ matrix. $e^{At}P$ is a fundamental matrix, but $P^{-1}e^{At}P$ is not a fundamental matrix.

Appendix A: Review of Real Analysis

Let $f : I \to \mathbb{R}$ be a function. f is **continuous** if $\lim_{x\to a} f(x) = f(a)\ \forall a \in I$. This is equivalent to $\forall$ sequences x_n such that $\lim_{n\to\infty} x_n = 1$, $\lim_{n\to\infty} f(x_n) = f(a)\ \forall a \in I$. $\lim_{n\to\infty} x_n = a$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, then $|x_n - a| < \varepsilon$. $|x_n - a|$ is the distance between x_n and a.

Let $X = \mathbb{R}^n$ be a vector space. For $x \in X$, $|x|$ is called a **norm** (length function) if the following properties are true:

1. **Homogeneity**: $\forall c \in \mathbb{R}$, $|cx| = |c| \cdot |x|$.
2. **Positivity**: $|x| \geq 0\ \forall x \in X$ and $|x| = 0$ if and only if $x = 0$.
3. **Triangle inequality**: $|x + y| \leq |x| + |y|\ \forall x, y \in X$.

Let X be a set. $d(x, y)$ for $x, y \in X$ is called a **metric** (distence function) if the following properties are true:

1. **Symmetry**: $d(x, y) = d(y, x)\ \forall x, y \in X$.
2. **Positivity**: $d(x, y) \geq 0\ \forall x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
3. **Triangle inequality**: $d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$.

Proposition

If we have a norm $|\cdot|$ on a set X, $d(x, y) = |x - y|$ is a metric.

A set X equipped with $d(\cdot, \cdot)$ is called a **metric space**. For $X = \mathbb{R}^n$, given $\vec{x} \in \mathbb{R}^n$, define

$$|\vec{x}| = |x_1| + \cdots + |x_n|.$$

This is the ℓ^1 norm of $\vec{x}$. On $S \subseteq \mathbb{R}^n$ with $\vec{x}, \vec{y} \in S$, define

$$d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

This is the ℓ^1 metric of $\vec{x}$. This is what we will use in this course to define convergence and continuity on subsets of $\mathbb{R}^n$.

Note that there is a more geometric norm and metric on $\mathbb{R}^n$, namely

$$\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and

$$\rho(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

This is the Euclidean norm or ℓ^2 norm of $\vec{x}$ and the Euclidean distance or ℓ^2 distance of $\vec{x}$ respectively. The Euclidean distance marks the distance between two points for $n = 1, 2, 3$. One can use these to define continuity and convergence. This will result in equivalent notations in our case. We can show that

$$\|\vec{x}\| \leq |\vec{x}| \leq \sqrt{n}\|\vec{x}\|.$$

But algebraically, $|\cdot|$ is often easier. For instance, the triangle inequality is easy to prove for $|\cdot|$:

$$\begin{aligned} |\vec{x} + \vec{y}| &= |x_1 + y_1| + \cdots + |x_n + y_n| \\ &\leq |x_1| + |y_1 + \cdots + |x_n| + |y_n| \\ &= |x_1| + \cdots + |x_n| + |y_1| + \cdots + |y_n| \\ &= |\vec{x}| + |\vec{y}| \end{aligned}$$

However, for $\|\cdot\|$, we would need to apply Cauchy Schwartz.

Let $(X, d(\cdot,\cdot))$ be a metric space, $x_n \in X$ is said to **converge** to $a \in X$, meaning $\lim_{n\to\infty} x_n = a$, if $\lim_{n\to\infty} d(x_n, a) = 0$.

A subset $S \subseteq X$ is closed if $\forall$ sequences $x_n \in S$ such that $x_n \to a$, $a \in S$. Otherwise, the subset is said to be open.

$B_a(r)$ is the **open ball** of radius r centered at a:

$$B_a(r) = \{x \in X \mid d(x, a) < r\}$$

A set $S \subseteq X$ is open if $\forall a \in S$, $\exists r > 0$ such that $B_a(r) \subseteq S$.

Proposition

$S \subseteq X$ is open if and only if $S^c = X \backslash S$ is closed.

Let $f : X \to Y$ be a function where (X, d_X) and (Y, d_Y) are metric spaces. f is continuous at a if $\forall x_n \in X$ with $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} f(x_n) = f(a)$. If f is continuous at every point $a \in X$, then f is continuous on X.

Proposition: $\varepsilon - \delta$ formulation

$f : X \to Y$ is continuous at a if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $d(x, a) < \delta$, then $d(f(x), f(a)) < \varepsilon$.

Appendix B: Linear Algebra Review

Recall the following definitions.

- Linear subspace: $W \subseteq \mathbb{R}^n$ is a linear subspace if for any $\vec{u}, \vec{w} \in W$ and $c, d \in \mathbb{R}$, the linear combination $c\vec{u} + d\vec{w}$ is also in W.
- Linear dependence: $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is linearly dependent if $\exists c_1, \ldots, c_m \in \mathbb{R}$ of which at least some c_i is nonzero such that $c_1\vec{u}_1 + \cdots + c_m\vec{u}_m = 0$.
- Linear independence: $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is linearly independent if it is not linearly dependent.
- Span: the span of $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is $w = \{\vec{u} = c_1\vec{u}_1 + \cdots + c_m\vec{u}_m \mid \forall c_1, \ldots, c_m \in \mathbb{R}\}$.
- Basis: $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is a basis of linear subspace $W \subseteq \mathbb{R}^n$ if:
  i. $W = \text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$.
  ii. $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is linearly independent.

Appendix C: Relevant Homework Problems

Find a fundamental matrix of the system $\vec{y}' = A\vec{y}$ if

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & 0 & 1 \\ 0 & & \cdots & 0 \end{pmatrix}$$

where A is a $n \times n$ matrix.

Solution

We know that $\Phi = e^{At}$ is a fundamental matrix. So, we can follow the steps outlined in class to find e^{At}.

1. Find the characteristic polynomial of A.

$$\begin{aligned} \det\left(\begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & 0 & 1 \\ 0 & & \cdots & 0 \end{pmatrix}\right) &= \det\begin{pmatrix} \lambda & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & \lambda & -1 \\ 0 & & \cdots & \lambda \end{pmatrix} \\ &= \lambda^n \end{aligned}$$

So, the characteristic polynomial is $\lambda^n = 0$.

2. For $i = 1, \dots, n$, we want $\vec{v}_i$ such that $(A - \lambda I)^i \vec{v}_i = A^i \vec{v}_i = 0$. A^n is the zero matrix. A^2 shifts the diagonal of 1's up by one, A^3 shifts it up by another one, and so on. So, we have:

$$\vec{v}_1 = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{v}_2 = \alpha_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{v}_{n-1} = \alpha_{n-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \vec{v}_n = \alpha_n \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

For each $\vec{v}_i$, we can find $e^{At}\vec{v}_i$.

$$\begin{aligned} e^{At}\vec{v}_i &= e^{\lambda t}\left(I + (A - \lambda I)t + \cdots + \frac{1}{(i-1)!}(A - \lambda I)^{i-1}t^{i-1}\right)\vec{v}_i \\ &= \left(I + At + \cdots + \frac{1}{(i-1)!}A^{i-1}t^{i-1}\right)\vec{v}_i \\ &= \begin{pmatrix} \sum_{n=0}^{i-1} \frac{t^n}{n!} & \sum_{n=0}^{i-2} \frac{t^n}{n!} & \cdots & 1 & 0 & \vdots & 0 \end{pmatrix}^\top \end{aligned}$$

This is by the format of A as mentioned previously. For $i = n$, the last term is 1. Now, we can find the general form for the fundamental matrix, $\Phi = \left(e^{At}\vec{v}_1 \middle| \cdots \middle| e^{At}\vec{v}_n\right)$. Let $\Phi = (\Phi_{ij})$. For any $i, j = 1, \dots, n$,

$$\Phi_{ij} = \begin{cases} 0 & i > j \\ \sum_{n=0}^{j-i} \frac{t^n}{n!} & i \le j \end{cases}$$

In other words, Φ is an upper triangular matrix with nonzero terms in row i and column j $\sum\limits_{n=0}^{j-i} \frac{t^n}{n!}$. This is true for $i = 1, \dots, n$ and $j = 1, \dots, n$ with $i \le j$ to stay in the upper triangle.

Appendix D: Extra Proofs

I'll add these in some other time