Introduction to Partial Differential Equations

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Course text: Introduction to Partial Differential Equations (Borthwick)

Disclaimer: I take full responsibility for any errors, typos, and other flaws in these notes. If you see any or have other suggestions for improvement, feel free to reach out to me at felicialim@berkeley.edu. My notes are a work in progress and I appreciate any ideas on how they can be improved.

Math 126 | Partial Differential Equations

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Contents

Introduction to Differential Equations

Ordinary Differential Equations

Ordinary differential equations (ODEs) are differential equations with funtions of one variable. ODEs are a subset of partial differential equations (PDEs).

Linear ODEs

Let the notation $\partial_t u$ mean $u_t = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t}$ where $u : t \to \mathbb{C}$ where $t \in \mathbb{R}$. Linear ODEs are in the form

$$\partial_t u = \alpha u$$

where α is a constant. This ODE can then be solved:

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= \alpha u \\
\frac{1}{u}\,\mathrm{d}u &= \alpha\,\mathrm{d}t \\
\int_{u(0)}^{u(t)} \frac{1}{u}\,\mathrm{d}u &= \int_0^t \alpha\,\mathrm{d}t \\
\ln\left(u(t)\right) - \ln(u(0)) &= \alpha t \\
\ln(u(t)) &= \ln(u(0)) + \alpha t \\
u(t) &= u(0)e^{\alpha t}
\end{aligned}$$

Linear ODEs satisfy the **superposition principle**.

- If $\partial_t u = \alpha u$ and $\partial_t v = \alpha v$, then $\partial_t(u+v) = \alpha(u+v)$.
- If u and v solve the system of equations, so does $u+v$.
- Similarly, cu also solves the system of equations for constant c.

Nonlinear ODEs

An example of a nonlinear ODE is $\partial_t u = u^3$, which can be solved.

$$\begin{aligned}
\partial_t u &= u^3 \\
\int_{u(0)}^{u(t)} \frac{1}{u^3}\,\mathrm{d}u &= \int_0^t \mathrm{d}t \\
-\frac{1}{2u(t)^2} + \frac{1}{2u(0)^2} &= t \\
-\frac{1}{2u(t)^2} &= -\frac{1}{2u(0)^2} + t \\
\frac{1}{u(t)^2} &= \frac{1}{u(0)^2} - 2t \\
u(t) &= \left(\frac{1}{u(0)^2} - 2t\right)^{-1/2} \\
&= \left(\frac{1}{u(0)^2}\left(1 - 2tu(0)^2\right)\right)^{-1/2} \\
&= u(0)\left(1 - 2tu(0)^2\right)^{-1/2}
\end{aligned}$$

As t approaches $\frac{1}{2(u(0))^2}$, the solution “blows up” since the denominator approaches 0.

ODEs with complex numbers

Let $i\partial_t u = |u|^2 u$ where u is complex. We must find a **conserved quantity** as t varies, meaning the quantity does not change in time. In this example, $|u(t)|^2$ is the conserved quantity. To show this, show that its derivative is 0. Recall the following properties of complex numbers: if $z \in \mathbb{C}$, $|z|^2 = z \cdot \overline{z}$ where $z = x + iy$ and $\overline{z} = x - iy$. Additionally, $z + \overline{z} = 2x$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t)\cdot\overline{u(t)}\right] \\
&= u'(t)\overline{u(t)} + u(t)\overline{u'(t)} \\
&= 2\,\mathrm{Re}\left(u'(t)\overline{u(t)}\right)
\end{aligned}$$

Using the equation $\partial_t u = \frac{1}{i}|u|^2 u$ (and knowing $\frac{1}{i} = -i$ since $i^2 = -1$):

$$\begin{aligned}
&= 2\,\mathrm{Re}\left(\frac{1}{i}|u(t)|^2 u(t)|u(t)|^2\right) \\
&= 2\,\mathrm{Re}\left(-i|u(t)|^2|u(t)|^2\right) \\
&= 2\,\mathrm{Re}\underbrace{\left(-i|u(t)|^4\right)}_{\text{imaginary}} \\
&= 0
\end{aligned}$$

So, $|u(t)|^2 = |u(0)|^2$ meaning $|u(t)|^2$ is the conserved quantity. Now, we can write the ODE as a linear ODE.

$$i\partial_t u = |u(0)|^2 u(t)$$

Using the solution from the earlier with $\alpha = \frac{|u(0)|^2}{i}$, we can solve for $u(t)$.

$$u(t) = u(0)e^{-i|u(0)|^2 t}$$

Partial Differential Equations

Partial differential equations (PDEs) are equations involving partial derivatives, meaning they involve functions of multiple variables.

Linear PDEs

Professor Zworski's favorite PDE is the Schrödinger Equation, specifically the free Schrödinger Equation on a line which describes a free particle in one dimension in quantum mechanics.

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$$

The notation $\partial_t u$ means $\frac{\partial u}{\partial t}$. Let P be an operator. As a linear PDE, this equation can be written as

$$P(u) = i\partial_t u + \frac{1}{2}\partial_x^2 u$$

with properties

$$\begin{aligned}
P(u+v) &= P(u) + P(v) \\
P(cu) &= cP(u) \qquad c \in \mathbb{C}
\end{aligned}$$

If $P(u) = 0$ and $P(v) = 0$, then $P(au + bv) = 0$ for $a, b \in \mathbb{C}$.

We want to find some solutions to

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = 0.$$

Separation of variables is a technique that can be used to reduce PDEs to ODEs. In this case, we want to write $u(t,x) = v(t)w(x)$, meaning u is a product of single variable equations.

$$\begin{aligned} 0 &\stackrel{?}{=} i\partial_t u + \frac{1}{2}\partial_x^2 u \\ &= i\partial_t \left[v(t)w(x)\right] + \frac{1}{2}\partial_x^2 \left[v(t)w(x)\right] \\ &= iv'(t)w(x) + \frac{1}{2}v(t)w''(x) \\ iv'(t)w(x) &= -\frac{1}{2}v(t)w''(t) \\ \underbrace{2i\frac{v'(t)}{v(t)}}_{\text{func. of } t} &= \underbrace{-\frac{w''(x)}{w(x)}}_{\text{func. of } x} \end{aligned}$$

Since both sides are equations of a single variable, both must be constant. Let this constant be μ.

$$2i\frac{v'(t)}{v(t)} = -\frac{w''(x)}{w(x)} = \mu$$

Let $\mu = \lambda^2 > 0$. Solve for $v(t)$ and $w(x)$ separately.

$$\begin{aligned} 2i\frac{v'(t)}{v(t)} &= \lambda^2 \\ v'(t) &= \frac{\lambda^2}{2i}v(t) \\ v(t) &= e^{-(i\lambda^2/2)t} \end{aligned}$$

$$\begin{aligned} -\frac{w''(x)}{w(x)} &= \lambda^2 \\ w''(x) + \lambda^2 w(x) &= 0 \\ w(x) &= e^{\pm i\lambda x} \end{aligned}$$

Since Euler's Formula states $e^{i\xi} = \cos\xi + i\sin\xi$ and $e^{-i\xi} = \cos\xi - i\sin\xi$.

So, we have our solution for the linear Schr odinger Equation.

$$\begin{aligned} u(t,x) &= v(t)w(x) \\ &= e^{-i\lambda(\lambda t/2 \pm x)} \end{aligned}$$

Nonlinear PDEs

The nonlinear Schrödinger Equation is

$$i\partial_t u + \frac{1}{2}\partial_x^2 u + |u|^2 u = 0,$$

We have already found solutions to $iu_t + |u|^2 u = 0$ and $i\partial_t u + \frac{1}{2}\partial_x^2 u = 0$. Our task is to find some interesting solutions. We will try

$$u(t,x) = \left(e^{i\lambda x + i\mu t}\right)\left(\psi(x - \sigma t)\right).$$

This is an example of an **ansatz**, or a guess of the form of a solution. A crazy solution to the formula is

$$u(x,t) = 2e^{it/2}\operatorname{sech} x \frac{4 + \operatorname{sech}^2\left(e^{4it} - 1\right)}{4 - e\operatorname{sech}^4 x \sin^2 2t}$$

where $\operatorname{sech} x = \frac{1}{\cosh x}$ and $\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$. Going back to the ansatz, we want to find ψ, a function of one variable. To do this, we will use the original equation and find every term in it in terms of ψ.

$$\begin{aligned} i\partial_t u &= e^{i\lambda x + i\mu t}(-\mu\psi - i\sigma\psi') \\ \frac{1}{2}\partial_x^2 u &= \frac{1}{2}\partial_x\left(e^{i\lambda x + i\mu t}\left(i\lambda\psi + \psi'\right)\right) \\ &= \frac{1}{2}\left(-\lambda^2\psi + 2i\lambda\psi' + \psi''\right)e^{i\lambda x + i\mu t} \\ |u|^2 u &= |\psi|^2 e^{i\lambda x + i\mu t}\psi \end{aligned}$$

So, we can rewrite the equation in terms of ψ.

$$\begin{aligned} e^{i\lambda x+i\mu t}(-\mu\psi - i\sigma\psi') + \frac{1}{2}\left(-\lambda^2\psi + 2i\lambda\psi' + \psi''\right)e^{i\lambda x+i\mu t} + |\psi|^2 e^{i\lambda x+i\mu t}\psi &= 0 \\ -\mu\psi - i\sigma\psi' - \frac{1}{2}\lambda^2\psi + i\lambda\psi' + \frac{1}{2}\psi'' + |\psi|^2\psi &= 0 \\ \left(-\mu - \frac{\lambda^2}{2}\right)\psi + (-i\sigma + i\lambda)\psi' + \frac{1}{2}\psi'' + |\psi|^2\psi &= 0 \end{aligned}$$

We want to get rid of the ψ' term, so set $\sigma = \lambda$. Let $\gamma := \mu + \frac{\lambda^2}{2}$. We want a real solution, so $|\psi|^2 = \psi^2$.

$$\begin{aligned} \frac{1}{2}\psi'' - \gamma\psi + \psi^3 &= 0 \\ \frac{1}{2}\psi''\psi' - \gamma\psi\psi' + \psi'\psi^3 &= 0 \end{aligned}$$

Use the law $(u^p)' = pu'u^{p-1}$:

$$\begin{aligned} \frac{1}{4}\left((\psi')^2\right)' - \frac{\gamma}{2}\left(\psi^2\right)' + \frac{1}{4}\left(\gamma^4\right)' &= 0 \\ \frac{1}{4}\left((\psi')^2 + 2\gamma\psi^2 + \psi^4\right)' &= 0 \\ (\gamma')^2 - 2\gamma\psi^2 + \psi^4 &= \text{ constant} = 0 \end{aligned}$$

Let $A^2 := 2\gamma$.

$$\begin{aligned} \psi' &= \left(A^2\psi^2 - \psi^4\right)^{1/2} \\ \frac{1}{\left(A^2\psi^2 - \psi^4\right)^{1/2}}\,\mathrm{d}\psi &= \mathrm{d}x \end{aligned}$$

To evaluate the integral of the LHS, use the substitution $u = -\frac{1}{\psi}$ meaning $\mathrm{d}\psi = \frac{1}{u^2}\,\mathrm{d}u$.

$$\begin{aligned} \int \frac{1}{\left(A^2\psi^2 - \psi^4\right)^{1/2}}\,\mathrm{d}\psi &= -\int \frac{1}{u^2\frac{1}{u}\left(A^2 - \frac{1}{u^2}\right)^{1/2}}\,\mathrm{d}u \\ &= -\int \frac{1}{u\left(A^2 - \frac{1}{u^2}\right)^{1/2}}\,\mathrm{d}u \\ &= -\int \frac{1}{\left(A^2u^2 - 1\right)^{1/2}}\,\mathrm{d}u \end{aligned}$$

Recall the definition of hyperbolic functions:

$$\begin{aligned} \cosh y &= \frac{1}{2}\left(e^y + e^{-y}\right) \\ \sinh y &= \frac{1}{2}\left(e^y - e^{-y}\right) \\ \cosh^2 y - 1 &= \sinh^2 y \\ (\cosh y)' &= \sinh y \end{aligned}$$

Substitute $u(x) = \frac{1}{A}\cosh(Ax)$. This yields

$$\psi(x) = A\operatorname{sech}(Ax)$$

Now, plug this back into the original equation along with the substitutions $\sigma = \lambda$, $\mu = -\frac{\lambda^2}{2} + \gamma$ and $\gamma = \frac{A^2}{2}$:

$$u(t,x) = e^{i\lambda x + i(A^2/2 - \lambda^2/2)t} A\operatorname{sech}(A(x - \lambda t))$$

Some Preliminaries

Real and Complex Numbers

The set of **real numbers** is $\mathbb{R}$. A real number $r \in \mathbb{R}$ is an element of $\mathbb{R}$. A set $A \subset \mathbb{R}$ is a subset of $\mathbb{R}$. $x_0 = \sup A$ is the smallest number $\geq$ all numbers in A, and $y_0 = \inf A$ is the largest number $\leq$ all numbers in A.

The set of **complex numbers** is $\mathbb{C}$. For $z \in \mathbb{C}$, $z = x + iy$ where $x, y \in \mathbb{R}$, $\overline{z} = x - iy$, and $i = \sqrt{-1}$.

$$x = \operatorname{Re} z = \frac{1}{2}(z + \overline{z})$$
$$y = \operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$$

For any operator $*$, $\overline{z * w} = \overline{z} * \overline{w}$. Additionally, the magnitude of a complex number is defined as

$$|z| = (z \cdot \overline{z})^{1/2} = \left(x^2 + y^2\right)^{1/2}.$$

Limits and Series

For $z_j \in \mathbb{C}$, $z = \lim_{j \to \infty} z_j$ (or $z_j \to z$) if $\forall \varepsilon > 0$, $\exists J$ such that for every $j \geq J$, $|z - z_j| < \varepsilon$. An **infinite series** is of the form

$$\sum_{k=0}^{\infty} a_k.$$

The series converges to s if each partial sum s_n converges to s, where

$$s_n := \sum_{k=0}^{n} a_k.$$

An infinite series converges absolutely if $\sum_{k=0}^{\infty} |a_k| < \infty$. Absolute convergence implies convergence, but the converse is not true.

Professor Zworski's favorite absolutely convergent series:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for $x \in \mathbb{C}$.

$$\begin{aligned} e^{iy} &= \sum_{k=0}^{\infty} \frac{i^k y^k}{k!} \\ &= \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k}}{(2k)!}}_{\cos y} + i \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k y^{2k+1}}{(2k+1)!}}_{\sin y} \\ &= \cos y + i \sin y \end{aligned}$$

Taylor series of $\cosh y$ and $\sinh y$:

$$\cosh y = \frac{1}{2}\left(e^y + e^{-y}\right) \qquad \sinh y = \frac{1}{2}\left(e^y - e^{-y}\right)$$
$$\cosh y = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!} \qquad \sinh y = \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!}$$

$\mathbb{R}^n$ and its Subsets

$\mathbb{R}^n$ is the set $\{(x_1, \dots, x_n) : x_j \in \mathbb{R}, 1 \le j \le n\}$. An element $x \in \mathbb{R}^n$ has the form $(x_1, \dots, x_n)$.

Let $x, y \in \mathbb{R}^n$.

$$\begin{aligned} x \cdot y &= \sum_{j=1}^{n} x_j y_j \\ |x| &= \sqrt{x \cdot x} = \left(x_1^2 + \cdots + x_n^2\right)^{1/2} \\ x + y &= (x_1 + y_1, \dots, x_n + y_n) \\ cx &= (cx_1, \dots, cx_n) \text{ for } c \in \mathbb{R} \end{aligned}$$

A **ball** of radius R centered at x_0 is $B(x_0, R) := \{x \mid |x - x_0| < R\}$. For set $A \subset \mathbb{R}^n$, x is a **boundary point** of A if $\exists$ a sequence $x_j \in A$ and $\tilde{x}_j \notin A$ such that both x_j and $\tilde{x}_j$ converge to x. The set of boundary points of A is ∂A.

A is **closed** if and only if the boundary of A is contained in A ($\partial A \subset A$). A is **open** if and only if A contains none of its boundary points ($\partial A \cap A = \emptyset$). The **closure** of A is $\overline{A} = A \cup \partial A$, which is the smallest closed set containing A. A is closed if and only if $A = \overline{A}$. The **interior** of A is $\text{int}(A) = A \backslash \partial A$ (also written as A^0).

Example

1. Let $A = B(x_0, R) = \{x \mid |x - x_0| < R\}$. In this case, $\partial A = \{x \mid |x - x_0| = R\}$. This set is open since the set of points points $\{x \mid |x - x_0| = R\}$ has no elements in $A = B(x_0, R)$.
2. Let $A = \{x \mid 0 < |x - x_0| \le R\}$. In this case, $\partial A = \{x_0\} \cup \{x \mid |x - x_0| = R\}$. This set is neither open nor closed since although $\{x \mid |x - x_0| = R\}$ is contained in A, $\{x_0\}$ is not in A.
3. The closure of the ball $B(x_0, R)$ is $\overline{B(x_0, R)} = \{x \mid |x - x_0| \le R\}$.

A is **connected** if any 2 points in A can be connected by a continuous curve contained in A. A is **convex** if any $x, y \in A$, $tx + (1-t)y \in A$ where $0 \le t \le 1$ (the shortest path from x to y is contained in A). If A is convex, then it must be connected, but the converse is not true.

$\Omega \subset \mathbb{R}^n$ is a **domain** if it is open and connected. $K \subset \mathbb{R}^n$ is compact if K is closed and bounded, meaning $\exists x_0, R$ such that $K \subset B(x_0, R)$.

Differentiability

Let Ω be a domain in $\mathbb{R}^n$ and f be a function $f : \Omega \to \mathbb{R}$. The set of **continuous** functions in Ω is

$$C^0(\Omega) := \{f : \Omega \to \mathbb{R} \mid \text{ if } x_j, x \in \Omega \text{ exist such that } x_j \to x, \text{ then } f(x_j) \to f(x)\}.$$

If the derivative exists, the **partial derivative** is defined as

$$\frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(x_1, \dots, x_{j+h}, x_{j+1}, \dots, x_n) - f(x_1, \dots, x_n)}{h}.$$

The set of continuously differentiable functions in Ω is

$$C^1(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \frac{\partial f}{\partial x_j} \in C^0(\Omega) \right\}.$$

The set of continuously m-times differentiable functions in Ω is

$$C^m(\Omega) := \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \frac{\partial f}{\partial x_j} \in C^{m-1}(\Omega) \right\}.$$

The set of continuously infinitely differentiable functions (smooth functions) in Ω is

$$C^\infty(\Omega) := \bigcap_{m \ge 0} C^m(\Omega).$$

Lemma: Clairaut's Theorem

For a twice differentiable function f,
$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

The same is true for higher derivatives as long as everything is continuous.

Example

1. Let $f(x) = x^\alpha$ and $\Omega = (0, \infty)$. Find the values of α such that $f \in C^\infty(\Omega)$.

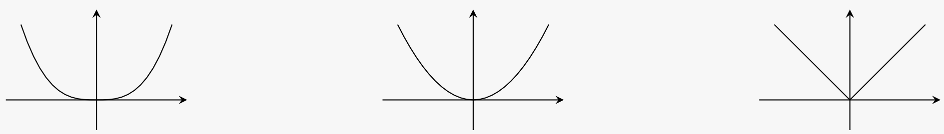
   We know that $f'(x) = \alpha x^{\alpha-1}$, so $f \in C^\infty(\Omega)$ for all values of α.

2. Let $f(x) = |x|^\alpha \in C^\infty(\Omega)$ where $\Omega = \mathbb{R}$. Find the values of α such that $f \in C^\infty(\Omega)$.

   For f to be in $\mathbb{C}^\infty(\Omega)$, α must be of the form $\alpha = 2n$ where $n \in \mathbb{Z}_{\geq 0}$. Otherwise, we will end up with some term with $|x|$ which is not differentiable.

3. Let $f(x) = |x|^{2k+1} \in C^n(\Omega)$. Find the values of n such that the statement is true.

   When $k = 0$, $f(x)$ is not differentiable and when $k = 1$, $f(x)$ is twice differentiable. So, $n = 2k$ since we always end up with $f^{(\alpha)}(x) = \beta|x|$, which is not differentiable.

   So, $f(x) = |x|^{2k+1}$ is $2k$ differentiable.

4. Let $f(x)$ be an even function. Since $f(-x) = f(x)$, $f(|x|) = f(x)$ meaning if $f(x) \in C^m(\Omega)$, then $f(|x|) \in C^m(\Omega)$. In particular, if $f(x) \in C^\infty(\Omega)$, then $f(|x|) \in C^\infty(\Omega)$.

Let $f \in C^m(\Omega)$ where $m \geq 0$. The **support** of f is
$$\text{supp}(f) = \overline{\{x \mid f(x) \neq 0\}}$$
where as before, the line indicates the closure of the set.

Example

1. Let $m = 0$ and f be the function
$$f(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| \geq 1 \end{cases}$$

   Find supp(f).
$$\begin{aligned} \text{supp}(f) &= \overline{\{x \mid f(x) \neq 0\}} \\ &= \overline{\{x \mid |x < 1\}} \\ &= \{x \mid |x| \leq 1\} \end{aligned}$$

2. Let $\Omega = (0, \infty)$ and $f \in C^\infty(\Omega)$ where
$$f(x) = \sin \frac{1}{x}.$$

   Find supp(f).
$$\begin{aligned} \text{supp}(f) &= \left\{ x \;\middle|\; x \geq 0,\ x \neq \frac{1}{\pi k} \text{ where } k \in \mathbb{N} \right\} \\ &= [0, \infty) \end{aligned}$$

The set of compact support functions is $C_c^\infty(\mathbb{R}^n)$ or $C_{\text{comp}}^\infty(\mathbb{R}^n)$, where

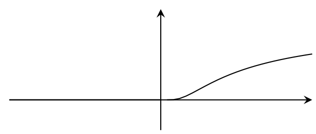
$$C_c^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp}(f) \text{ is compact}\}.$$

Recall that compact means closed and bounded. By definition, all supports are closed, but not all are bounded. A trivial example of one such function is $f(x) = 0$ for all values of x, so $\operatorname{supp}(f) = \emptyset$.

A nontrivial example is

$$h(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

We can view the plot of this function.



We want to show that

$$h^{(k)}(x) = \frac{q_k(x)}{x^{2k}} e^{-1/x}$$

for $x > 0$ where $q_k(x)$ is a polynomial of degree $2k$. We can use proof by induction.

1. Base case: when $k = 0$, we have

$$\begin{aligned} h^{(0)}(x) &= e^{-1/x} \\ &= \frac{q_0(x)}{x^0} e^{-1/x} \end{aligned}$$

   where $q_0(x) = 1$.

2. Induction hypothesis: suppose $h^{(k)}(x) = \frac{q_k(x)}{x^{2k}} e^{-1/x}$ for some $k \geq 0$.

3. Inductive step: show that the formula holds for $k+1$.

$$\begin{aligned} h^{(k+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{q_k(x)}{x^{2k}} e^{-1/x}\right) \\ &= \frac{q_k'(x)}{x^{2k}} e^{-1/x} - \frac{2kq_k(x)}{x^{2k+1}} e^{-1/x} + \frac{q_k(x)}{x^{2k}x^2} e^{-1/x} \\ &= \frac{q_k'(x)x^2 - 2kq_k(x)x + q_k(x)}{x^{2k+2}} e^{-1/x} \\ &= \frac{q_{k+1}(x)}{x^{2(k+1)}} e^{-1/x} \end{aligned}$$

Now, to show that $h \in C^\infty(\mathbb{R})$, we must match the derivatives at $x = 0$. From the definition of h, we know that $h^{(k)}(x) = 0$ for $x < 0$, so for $x > 0$, we need $h^{(k)}(x) \to 0$ as $x \to 0$.

$q_k(x)$ is a polynomial of degree $2k$, so for $|x| < 1$, it is bounded by a constant which we can show using the reverse triangle inequality:

$$|q_k(x)| = |A_{2k}x^{2k} + \cdots + x_0| \leq |A_{2k}| + \cdots + |A_0| \leq C_k$$

We can assume $|x| < 1$ since x is approaching 0. So, we have the following bound for $h^{(k)}(x)$.

$$|h^{(k)}(x)| \leq \frac{|q_k(x)|}{x^{2k}} e^{-1/x} \leq \frac{C_k}{x^{2k}} e^{-1/x}$$

To show that this approaches 0, write it as $x^{-2k}e^{-1/x} = y^{2k}e^{-y}$ where $y = 1/x$. As $x \to 0$, $y \to \infty$.

$$e^y = \sum_0^\infty \frac{y^m}{m!} \geq \frac{y^m}{m!}$$

where $y > 0$. So, $e^{-y} \leq \frac{m!}{y^m}$.

$$x^{-2k}e^{-1/x} = y^{2k}e^{-y} \leq \frac{m!}{y^m}y^{2k} = m!y^{2k-m}$$

for $m > 2k$. This approaches 0 as $y \to \infty$. So, $\frac{C_k}{x^{2k}}e^{-1/x} \to 0$ as $x \to 0$. This proves that $h \in C^\infty(\mathbb{R})$.

Now, we can show that $f(x) = h(1 - |x|^2)$ also satisfies the required conditions. To show that $f \in C_c^\infty(\mathbb{R}^n)$, we must first show that $f \in C^\infty$. The chain rule can be used here.

Lemma: Chain Rule

Let $F : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $F, h \in C^\infty$. $G(x) \in \mathbb{R}^n$ is defined as $G(x) = h(F(x))$, where

$$\frac{\partial}{\partial x_j}G(x) = \frac{\partial F(x)}{\partial x_j}h'(F(x))$$

Combined with the product rule, we can see that $G \in C^\infty$. For higher order derivatives, the same principal holds in the Faà di Bruno formula.

For this case, we have $f(x) = h(g(x))$ where

$$\begin{aligned} g(x) &= 1 - |x|^2 \\ &= 1 - x_1^2 - x_2^2 - \cdots - x_n^2 \in C^\infty(\mathbb{R}) \end{aligned}$$

So, $f(x) \in C^\infty$.

Now, show that $\operatorname{supp}(f)$ is bounded. To do this, show that $f(x) = 0$ when $|x| > R$ for some R. But, $h(y) = 0$ when $y \leq 0$. $y = 1 - |x|^2 < 0$ when $x| > 1$, so $h(1 - |x|^2) = 0$ when $|x| > 1$.

Ordinary Differential Equations

For a function $y : t \mapsto y(t)$ where $y, t \in \mathbb{R}$, the general form of the scalar equation is

$$y^{(n)}(t) = F(t, y, y', \ldots, y^{(n-1)}).$$

Example

1. Let $y'' = y + y^3$. $F = F(y(t)) = y(t) + y(t)^3$.
2. Let $y' = \alpha y$. $F(y) = \alpha y$.
3. Let $y'' = -\omega^2 + \beta y'$. $F(y, y') = -\omega^2 y + \beta y'$.

We can "solve" $y'(t) = g(y(t))$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= g(y) \\ \frac{\mathrm{d}y}{g(y)} &= \mathrm{d}t \\ \int_{y(0)}^{y(t)} \frac{\mathrm{d}y}{g(y)} &= \int_0^t \mathrm{d}t = t \end{aligned}$$

The issue is that the integral on the left may be difficult to solve.

First order ODEs

The general form for a first order ODE is $f \mapsto \underline{w}(t)$ where $t \in \mathbb{R}$ and $\underline{w} \in \mathbb{R}^n$. The notation $\underline{w}$ represents a vector.

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{w}(t) = \begin{pmatrix} w_1'(t) \\ \vdots \\ w_n'(t) \end{pmatrix} = G(t, \underline{w}(t))$$

or $\underline{w}'(t) = G(t, \underline{w}(t))$ where $G : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

The general form of an ODE of order n can be transformed into a first order ODE.

$$\begin{aligned} y^{(n)}(t) &= F(t, y, y', \dots, y^{(n-1)}) \\ \underline{w}(t) &:= (y(t), y'(t), \dots, y^{(n-1)}(t))^\intercal \in \mathbb{R}^n \text{ (transpose)} \\ \underline{w}'(t) &= (y'(t), y''(t), \dots, y^{(n)}(t))^\intercal \\ &= (w_2(t), w_3(t), \dots, F(t, w(t), \dots, w_n(t)))^\intercal \\ &=: G(t, \underline{w}(t)) \end{aligned}$$

Example

Let $y'' + \omega^2(y) = 0$, or $y'' = -\omega^2 y$.

$$\begin{aligned} \underline{w} &= \begin{pmatrix} y \\ y' \end{pmatrix} \\ \underline{w}' &= \begin{pmatrix} w_2 \\ -\omega^2 w_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \underline{w} \end{aligned}$$

First order ODEs can be written as integral equations.

$$\begin{aligned} \int_{t_0}^{t} \frac{\mathrm{d}\underline{w}}{\mathrm{d}\tau} &= \int_{t_0}^{t} G(\tau, \underline{w}(\tau))\,\mathrm{d}\tau \\ \underline{w}(t) - \underline{w}(t_0) &= \int_{t_0}^{t} G(\tau, \underline{w}(\tau))\,\mathrm{d}\tau \\ \underline{w}(t) &= \underline{w}(t_0) + \int_{t_0}^{t} G(\tau, \underline{w}(\tau))\,\mathrm{d}\tau \end{aligned}$$

This is an integral representation of $\underline{w}'(t) = G(t, \underline{w}(t))$.

Theorem: Picard's Theorem

Suppose we have the following conditions:

1. $(t, \underline{w}) \to G(t, \underline{w})$ is in $C^0((t_0 - c, t_0 + c) \times \Omega; \mathbb{R}^n)$ where $\Omega \subset \mathbb{R}^n$ is a domain (connected open set).
2. G is Lipschitz continuous, so
$$|G(t, \underline{w}) - G(t, \underline{v})| \leq M(\underline{v}, \underline{w})$$
for some constant $M > 0$.

Then, $\forall \underline{w}_0 \in \Omega$, $\exists \varepsilon$ such that the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\underline{w}(t) &= G(t, \underline{w}(t)) \\ \underline{w}(t_0) &= \underline{w}_0 \in \mathbb{R}^n \end{aligned}$$

has a unique solution for t such that $|t - t_0| < \varepsilon$.

Note that Picard's Theorem does not solve the differential equation, it merely proves the existance of a solution.

Example

Let $w \in \mathbb{R}$ and $w'1 + w^2$. $n = 1$ in this case. $G(t, w) = 1 + w^2 = G(w)$ since G is not a function of t.

We want to show that this satisfies the condition for Picard's Theorem.

First, show that $w \mapsto 1 + w^2$ is continuous. This is intuitive since it is a polynomial.

Next, show that G is Lipschitz continuous.

$$\begin{aligned} |G(w) - G(v)| &= |w^2 - v^2| \\ &= |w + v||w - v| \\ &\leq (|w| + |v|)\,|w - v| \end{aligned}$$

Let $\Omega = (-R, R)$. If $w, v \in \Omega$, then $|w|, |v| \leq R/$ So, $|w| + |v| \leq 2R$.

$$|G(w) - G(v)| \leq M|v - w|$$

where $M = 2R$. So, Picard's Theorem is satisfied.

This example is quite simple, so we can solve the differential equation numerically:

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t} &= 1 + w^2 \\ \int_0^{w(t)} \frac{1}{1 + w^2}\,\mathrm{d}w &= \int_0^t \mathrm{d}t \\ \arctan w(t) &= t \\ w(t) &= \tan t \end{aligned}$$

By Picard's Theorem, this solution is unique.

Proving Picard's Theorem requires a method known as Picard's iteration.

$$\begin{aligned} \underline{w}_0(t) &:= \underline{w}_0 \\ \underline{w}_1(t) &:= \underline{w}_0 + \int_{t_0}^t G(s, \underline{w}_0(s))\,\mathrm{d}s \\ &\vdots \\ \underline{w}_{k+1}(t) &:= \underline{w}_0 + \int_{t_0}^t G(s, \underline{w}_k(s))\,\mathrm{d}s \\ &\vdots \\ \underline{w}(t) &:= \underline{w}_0 + \int_{t_0}^t G(s, \underline{w}(s))\,\mathrm{d}s \end{aligned}$$

As $k \to \infty$, $\underline{w}_k$ converges to $\underline{w}$.

Example

Let's apply Picard's iteration to the previous example: $w' = 1 + w^2$, $w(0) = 0$.

$$\begin{aligned} w_0 &= 0 \\ w_1(t) &= 0 + \int_0^t \underbrace{G(s, w_0(s))}_{1 + w_0(s)^2}\,\mathrm{d}s \\ &= \int_0^t 1\,\mathrm{d}s \\ &= t \end{aligned}$$

$$
\begin{aligned}
w_2(t) &= 0 + \int_0^t (1+s^2)\,\mathrm{d}s \\
&= t + \frac{t^3}{3} \\
w_3(t) &= 0 + \int_0^t \left(1 + \left(s + \frac{s^3}{3}\right)^2\right)\mathrm{d}s \\
&= \int_0^t \left(1 + s^2 + \frac{2s^4}{3} + \frac{s^6}{9}\right)\mathrm{d}s \\
&= t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{t^7}{63}
\end{aligned}
$$

We can continue this process for higher values of k. This is the Taylor expansion of $\tan t$.

To prove that Picard's Theorem works, we must show that it converges. Let $g : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n$ be a function. We define

$$||g|| := \sup_{|t-t_0|\le\varepsilon} |g(t)|.$$

For 2 continuous functions on $[t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n$ g and h, the distance between g and h is $||g-h||$. Why does Picard's iteration converge? Let sup represent $\sup_{|t-t_0|\le\varepsilon}$.

$$
\begin{aligned}
||\underline{w}_{k+1} - \underline{w}_k|| &= \sup\left|\int_{t_0}^t G(s, \underline{w}_k(s))\,\mathrm{d}s - \int_{t_0}^t G(s, \underline{w}_{k-1}(s))\,\mathrm{d}s\right| \\
&= \sup\left|\int_{t_0}^t G(s, \underline{w}_k(s)) - G(s, \underline{w}_{k-1}(s))\,\mathrm{d}s\right|
\end{aligned}
$$

Recall that by triangle inequality, $\left|\int f(t)\,\mathrm{d}t\right| \le \int |f(t)|\,\mathrm{d}t$.

$$\le \sup \int_{t_0}^t \left|G(s, \underline{w}_k(s)) - G(s, \underline{w}_{k-1}(s))\right|\mathrm{d}s$$

We know that $|G(s,v)_G(s,w)| \le M|v-w|$ for some constant $M > 0$.

$$\le \sup \int_{t_0}^t M\left|\underline{w}_k(s) - \underline{w}_{k-1}(s)\right|\mathrm{d}s$$

There is a property that $\sup \int_{t_0}^t |F(s)|\,\mathrm{d}s \le \varepsilon \sup F(t)|$ for the defined sup.

$$
\begin{aligned}
&\le \varepsilon \sup M\left|\underline{w}_k(t) - \underline{w}_{k-1}(t)\right| \\
&= \varepsilon M ||\underline{w}_k - \underline{w}_{k-1}|| \\
&\le \varepsilon M \left(\varepsilon M ||\underline{w}_{k-1} - \underline{w}_{k-2}||\right) \\
&\vdots \\
&\le (\varepsilon M)^k ||\underline{w}_1 - \underline{w}_0|| \\
&\le A(\varepsilon M)^2
\end{aligned}
$$

where $A = \sup \int_{t_0}^t G(s, \underline{w}_0)\,\mathrm{d}s$ is constant. For $\varepsilon = \frac{1}{2M}$, we have $||\underline{w}_{k+1} - \underline{w}_k|| \le A2^{-k}$.

$$\underline{w}(t) = \underline{w}_0 + \sum_{k=0}^{\infty} \underbrace{\left(\underline{w}_{k+1}(t) - \underline{w}_k(t)\right)}_{\le A2^{-k} \text{ when } |t-t_0|\le\varepsilon}$$

By the comparison test, the sum converges. So, we have a telescoping series:

$$\underline{w}_0 + \underline{w}_1 - \underline{w}_0 + \underline{w}_2 - \underline{w}_1 + \cdots + \underline{w}_{k+1} - \underline{w}_k = \underline{w}_{k+1}$$

So, this converges to $\underline{w}$.

Example

Using an example from earlier, let $y'' + \omega y = 0$. $y(0) = y_0$ and $y'(0) = y_1$.

$$\underline{w}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \qquad G(t, \underline{w}) = G(\underline{w}) = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{A} \underline{w}$$

$$\begin{aligned} \underline{w}'(t) &= \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix} & \qquad |G(v) - G(w)| &= |Av - Aw| \\ &= \begin{pmatrix} w_2(t) \\ -\omega^2 w_1(t) \end{pmatrix} & &= |A(v-w)| \\ &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \underline{w}(t) & &\leq M|v-w| \end{aligned}$$

So, we have the following:

$$\underline{w}_0(t) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \underline{w}_0, \qquad \underline{w}_1(t) = \underline{w}_0 + \int_0^t A\underline{w}_0 \,\mathrm{d}s = \underline{w}_0 + tA\underline{w}_0$$

We want to prove that $\underline{w}_{k+1}(t) = \sum\limits_{j=0}^{k+1} \frac{t^j}{j!} A^j \underline{w}_0$. We can use induction to prove this.

1. Base case: this is shown above.
2. Induction hypothesis: suppose that for some $k \geq 0$, we have $\underline{w}_k(t) = \sum\limits_{j=0}^{k} \frac{t^j}{j!} A^j \underline{w}_0$.
3. Inductive step: show that the statement holds for $k+1$.

$$\begin{aligned} \underline{w}_{k+1}(t) &= \underline{w}_0 + \int_0^t A\underline{w}_k \,\mathrm{d}s \\ &= \underline{w}_0 + \int_0^t A \sum_{j=0}^{k} \frac{s^j}{j!} A^j \underline{w}_0 \,\mathrm{d}s \\ &= \underline{w}_0 + \sum_{j=0}^{k} \int_0^t \frac{s^j}{j!} A^{j+1} \underline{w}_0 \,\mathrm{d}s \\ &= \underline{w}_0 + \sum_{j=0}^{k} \frac{s^{j+1}}{(j+1)!} A^{j+1} \underline{w}_0 \\ &= \sum_{\ell=0}^{k+1} \frac{t^\ell}{\ell!} A^\ell \underline{w}_0 \text{ where } \ell = j+1 \end{aligned}$$

So, the statement holds for $k+1$.

We have the following:

$$\begin{aligned} A^j &= \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}^j \\ &= \begin{cases} \begin{pmatrix} (-\omega^2)^\ell & 0 \\ 0 & (-\omega^2)^\ell \end{pmatrix} & j = 2\ell \\ \begin{pmatrix} 0 & (-\omega^2)^\ell \\ (-\omega^2)^{\ell+1} & 0 \end{pmatrix} & j = 2\ell + 1 \end{cases} \end{aligned}$$

So, $y(t) = \cos(t\omega)y_0 + \frac{1}{\omega}\sin(t\omega)y_1$.

Vector Calculus

Differentiation

Let $f : \Omega \to \mathbb{R}$ where Ω is a domain and $f \in C^1$. We define the **gradient** of f as

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \in \mathbb{R}^n$$

where $\nabla f \in C^0(\Omega; \mathbb{R}^n)$. Note that $\Omega; \mathbb{R}^n$ implies that ∇f is a function $\Omega \to \mathbb{R}^n$.

Let $\underline{v} : \Omega \to \mathbb{R}^n$ where $\underline{v}(x) = (v_w(x), \dots, v_n(x))$. The **divergence** of $\underline{v}(x)$ is

$$\boldsymbol{\nabla} \cdot \underline{v}(x) = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n} \in C^0(\Omega : \mathbb{R})$$

Note that the divergence of $\underline{v}(x)$ sends a vector to a scalar, while the gradient of f sends a scalar to a vector.

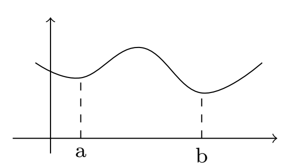
$$\begin{array}{ccc} f \in C^m(\Omega; \mathbb{R}) & \to & \nabla f(x) \in C^{m-1}(\Omega; \mathbb{R}^n) \\ \underline{v} \in C^m(\Omega; \mathbb{R}^n) & \to & \boldsymbol{\nabla} \cdot \underline{v} \in C^m(\Omega; \mathbb{R}) \end{array}$$

Note that we can set $m = \infty$ meaning $m - 1$ is also ∞. This is the case that we care about the most for the purpose of this class.

Integration

Let $f \in C^0([a, b]; \mathbb{R})$. The **Riemann integral** of f is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \frac{1}{N} \sum_{j/N \in [a,b]} f\left(\frac{j}{N}\right)$$



For a more general $f \in C^0(\overline{\Omega}; \mathbb{R})$, the Riemann integral is

$$\int_\Omega f(x)\,\mathrm{d}^n x = \lim_{N\to\infty} \frac{1}{N^n} \sum_{j/N \in \Omega} f\left(\frac{j}{N}\right)$$

where $\mathrm{d}^n x$ is the n dimensional element of integration $\mathrm{d}x_1\,\mathrm{d}x_2 \dots \mathrm{d}x_n$.

Change of Variables

Let Ω be a domain such that $F(\Omega) = \Omega'$. $f \in C^0(\Omega'; \mathbb{R})$ and $f \circ F(y) = f(F(y))$.

$$\int_{F(\Omega)} f(x)\,\mathrm{d}^n x = \int_\Omega f(F(y)) \left|\frac{\partial f}{\partial y}\right| \mathrm{d}y$$

where

$$\left|\frac{\partial F}{\partial y}\right| = \left|\det \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}\right|$$

where $F(x) = (F_1(x), \dots, F_n(x))$. Note that the matrix that we calculate the determinant of is called the Jacobian.

Example

Let $u(t,x)$ represent the density at time t and position x where $t \in \mathbb{R}_{\geq 0}$ and $x \in \mathbb{R}^n$.

$$V(t) = \int_\Omega u(t,x)\,\mathrm{d}^n x$$

where V is the mass in Ω at time t.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_\Omega \partial_t u(t,x)\,\mathrm{d}^n x$$

represents how mass changes over time.

Let $f \in C_c^\infty(\mathbb{R}^n)$, meaning $f \in C^\infty(\mathbb{R}^n)$ and $f = 0$ for $|x| > R$.

$$\begin{aligned}
u(t,x) &= f(tx) \\
V(t) &= \int_{\mathbb{R}^n} f(tx)\,\mathrm{d}^n x \\
V'(t) &= \int_{\mathbb{R}^n} \partial_t[f(tx)]\,\mathrm{d}^n x
\end{aligned}$$

Apply the chain rule: $\partial_t[f(G(t,x))] = \left(\frac{\partial G_1}{\partial t}\frac{\partial f}{\partial x_1} + \cdots + \frac{\partial G_n}{\partial t}\frac{\partial f}{\partial x_n}\right)(G(t,x))$ where $G = (G_1, \ldots, G_n)$ and $G_i(t,x) \in \mathbb{R}$.

$$\begin{aligned}
V'(t) &= \int_{\mathbb{R}^n} x \cdot \nabla f(tx)\,\mathrm{d}^n x \\
F(y) &= \frac{y}{t} \\
g(x) &= f(x,t) \\
V(t) &= \int_{\mathbb{R}^n} g(x)\,\mathrm{d}^n x \\
&= \int g(F(y)) \left|\frac{\partial F}{\partial y}\right| \mathrm{d}^n y \\
g(F(y)) &= f(tF(y)) = f(t(y/t)) = f(y) \\
\left|\frac{\partial F}{\partial y}\right| &= \left|\det \begin{bmatrix} \frac{1}{t} & 0 & \cdots & 0 \\ 0 & \frac{1}{t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{t} \end{bmatrix}\right| = t^{-n} \\
V(t) &= \int_{\mathbb{R}^n} f(y) t^{-n}\,\mathrm{d}^n y \\
V(t) &:= \int_{\mathbb{R}^n} f(tx)\,\mathrm{d}^n x = \frac{1}{t^n}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}^n x \\
\frac{\mathrm{d}}{\mathrm{d}t} V(t) &= -nt^{-n-1}\int_{\mathbb{R}^n} f(x)\,\mathrm{d}^n x \\
&= \int_{\mathbb{R}^n} x \cdot \nabla f(tx)\,\mathrm{d}^n x
\end{aligned}$$

We can set $t = 1$, which yields

$$\int_{\mathbb{R}^n} x \cdot \nabla f(x)\,\mathrm{d}^n x = -n\int_{\mathbb{R}^n} f(x)\,\mathrm{d}^n x\,.$$

Divergence theorem

The divergence theorem relies on **surface integrals**, which relates integrals over different dimensions. If σ is the surface integral function, for $U \in \mathbb{R}^{n-1}$, $\sigma : U \to \mathbb{R}^n$. $\Sigma = \sigma(U) \subset \mathbb{R}^n$ is an $n-1$ dimensional surface and $\underline{\nu}$ is a unit vector. For $n = 3$, we have

$$\int_\Sigma f \mathrm{d}S = \int_U f(\sigma(w)) |\sigma_{w_1} \times \sigma_{w_2}| \,\mathrm{d}u_1 \,\mathrm{d}u_2$$

For the general case, we have

$$\int_\Sigma f \,\mathrm{d}S = \int_U f(\sigma(w)) \left|\det\left(\sigma_{w_1}\middle|\sigma_{w_2}\middle|\cdots\middle|\sigma_{w_{n-1}}\middle|\nu\right)\right| \mathrm{d}^{n-1} w$$

Example

Let $\sigma(w) = (w, g(w))$ where $g : \mathbb{R}^{n-1} \to \mathbb{R}$. We can solve for the unit vector ν:

$$\nu(w) = \frac{\nabla g(w) - 1}{(1 + |\nabla g(w)|^2)^{1/2}}$$

Now, solve for the determinant:

$$\begin{aligned}
\det \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & & \frac{\partial g}{\partial w_1} \\ 0 & 1 & 0 & \cdots & & \frac{\partial g}{\partial w_2} \\ 0 & 0 & 1 & \cdots & & \frac{\partial g}{\partial w_3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \hline \frac{\partial g}{\partial w_1} & \frac{\partial g}{\partial w_2} & \cdots & \frac{\partial g}{\partial w_n} & & -1 \end{array}\right] &= \det \left[\begin{array}{c|c} \mathcal{I}_{n-1} & \nabla g^\intercal \\ \hline \nabla g & -1 \end{array}\right] \\
&= \det \left[\begin{array}{c|c} \mathcal{I}_{n-1} & (v_1 \cdots v_{n-1})^\intercal \\ \hline v_1 \cdots v_{n-1} & -1 \end{array}\right] \text{ where } v_j = g w_j \\
&= \det \begin{bmatrix} 1 & 0 & \cdots & v_1 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & v_{n-1} \\ 0 & 0 & \cdots & -1 - |v|^2 \end{bmatrix} \\
&= 1 + |\nabla g|^2
\end{aligned}$$

$$\int_\Sigma f \,\mathrm{d}S = \int_U f(w, g(w)) \left(1 + |\nabla g(w)|^2\right)^{1/2} \mathrm{d}^{n-1} w$$

A special case of the surface integral is where

$$\Sigma = \left\{(u, g(u))\middle| u \in U, g \in C^1(\overline{U}, \mathbb{R})\right\}.$$

In this case, we have

$$\int_\Sigma f \mathrm{d}s = \int \left(1 + |\nabla g(u)|^2\right)^{1/2} f(u, g(u)) \,\mathrm{d}^{n-1} u\,.$$

Theorem: Divergence Theorem

Let $\Omega \subset \mathbb{R}^n$. $\partial\Omega = \bigcup_{k=1}^{K} \sigma_k(U_k)$ is "piecewise C^1", where $U_k(\mathbb{R}^{k-1})$ and $\sigma_k \in C^1(\mathbb{R}^{n-1}; \mathbb{R}^n)$. Let $\underline{F} \in C^1(\overline{\Omega}; \mathbb{R}^n)$.

$$\int_\Omega \operatorname{div} \underline{F} \,\mathrm{d}^n x = \int_{\partial\Omega} \underline{\nu} \cdot \underline{F} \,\mathrm{d}s$$

where $\underline{\nu}$ is an outward pointing unit vector.

Unit Balls and Spheres

We can define the n dimensional **unit ball** as

$$B_n(0,1) = B_n := \{x \in \mathbb{R}^n \,||x| < 1\}$$

and the n dimensional **unit sphere** as

$$\mathbb{S}^n(0,1) = \mathbb{S}^n := \partial B_{n+1}(0,1) = \left\{x \in \mathbb{R}^{n+1} \middle| |x| = 1\right\}.$$

So, we have $V_n = \int_{B_n} \mathrm{d}^n x$ and $S_n = \int_{\mathbb{S}^n} \mathrm{d}s$. We want to relate V_n and S_n. Calculate the first few values of each:

$$\begin{aligned} V_1 &= 2 & S_0 &= 2 \\ V_2 &= \pi & S_1 &= 2\pi \\ V_3 &= \frac{4\pi}{3} & S_2 &= 4\pi \end{aligned}$$

Proposition
$V_{n+1} = \frac{S_n}{n+1}$.

- **Proof**: we want to relate $\int_{B_{n+1}} \mathrm{d}^{n+1} x$ and $\int_{\mathbb{S}^n} \mathrm{d}S$.

$$\begin{aligned} F(x) &= (x_1, \ldots, x_{n+1}) \\ \operatorname{div} F(x) &= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_{n+1}}{\partial x_{n+1}} \text{ where } F_i = x_i \\ &= 1 + \cdots + 1 \\ &= n+1 \end{aligned}$$

A unit vector of a point x on the unit sphere is equal to the point intself, so $\underline{\nu}(x) = x$.

$$\begin{aligned} F(x) \cdot x &= x^2 + \cdots + x_{n+1}^2 \\ &= 1 \qquad\qquad |x| = (x_1^2 + \cdots + x_{n+1}^2)^{1/2} = 1 \\ \int_{B_{n+1}} \underbrace{\operatorname{div} F(x)}_{n+1} \mathrm{d}^{n+1} x &= \int_{\mathbb{S}^n} \underbrace{F \cdot \underline{\nu}}_{1} \mathrm{d}S \\ (n+1) V_{n+1} &= S_n \end{aligned}$$

So, $V_{n+1} = \frac{S_n}{n+1}$. □

Suppose we have a sphere of radius r. We will observe the top hemisphere.

$$\mathbb{S}_r^k = \left\{x \in \mathbb{R}^{k+1} \middle| |x| = r\right\}$$

Let $g(y) = (1-|y|^2)^{1/2}$, $y = (x_1, \ldots, x_n)$. Since $x_1^2 + \cdots + x_{n+1}^2 = 1$, $x_{n+1} = (1 - x_1^2 - \cdots - x_n^2)^{1/2} = (1-|y|^2)^{1/2}$.

$$\int_{\mathbb{S}_r^k} 1 \,\mathrm{d}S = r^k S_k$$

Since our equation is only for the top hemisphere, we must multiply by 2.

$$\begin{aligned} \int_{\mathbb{S}^n} \mathrm{d}S &= 2 \int_{B_n} \left(1 + |\nabla g(y)|^2\right) \mathrm{d}^n y \\ &= 2 \int_{B_n} \left(1 - |y|^2\right)^{-1/2} \mathrm{d}^n y \\ &= 2 S_{n-1} I_n \\ I_n &= \int_0^1 (1-r^2)^{-1/2} r^{n-1} \,\mathrm{d}r \end{aligned}$$

Let $r = |y|$.

$$\begin{aligned}
\int_{B_n} f(|y|)\,\mathrm{d}^n y &= \int_0^1 f(r)\left(\int_{S^{n-1}} 1\,\mathrm{d}S\right)\mathrm{d}r \\
&= \int_0^1 f(r) r^{n-1} S_{n-1}\,\mathrm{d}r \\
f(r) &= (1-r^2)^{-1/2}
\end{aligned}$$

Use the substitution $r = \sin\theta$. $\mathrm{d}r = \cos\theta\,\mathrm{d}\theta$ and $1 - r^2 = \cos\theta$.

$$I_n = \int_0^{\pi/2} \sin^{n-1}\theta \mathrm{d}\theta$$

When $n = 1$ and $n = 2$, we can easily solve for $I_1 = \pi/2$ and $I_2 = 1$. For $n > 2$, we use integration by parts.

$$\begin{aligned}
I_n &= \int_0^{\pi/2} \sin^{n-2}\theta(-\cos\theta)'\mathrm{d}\theta \\
&= (n-2)\int_0^{\pi/2} \sin^{n-3}\theta\cos^2\theta\mathrm{d}\theta \\
&= (n-2)\int_0^{\pi/2} \sin^{n-3}\theta(1-\sin^2\theta)\mathrm{d}\theta \\
&= (n-2)\underbrace{\int_0^{\pi/2} \sin^{n-3}\theta\mathrm{d}\theta}_{I_{n-2}} - (n-2)\underbrace{\int_0^{\pi/2} \sin^{n-1}\theta\,\mathrm{d}\theta}_{I_n} \\
&= (n-2)I_{n-2} - (n-2)I_n \\
I_n &= \frac{n-2}{n-1} I_{n-2}
\end{aligned}$$

We have solved for I_1 and I_2, so we can solve for some higher values of I_n:

$$\begin{aligned}
I_3 &= \frac{3-2}{3-1} I_1 = \frac{\pi}{4} \\
I_4 &= \frac{4-2}{4-1} I_2 = \frac{2}{3} \\
I_5 &= \frac{5-2}{5-1} I_3 = \frac{3\pi}{16} \\
I_6 &= \frac{6-2}{6-1} I_4 = \frac{8}{15}
\end{aligned}$$

Now, solve for values of S_n using $S_n = 2S_{n-1}I_n$ and V_n using $V_{n+1} = \frac{S_n}{n+1}$.

$$\begin{aligned}
S_0 &= 4 & V_1 &= 2 \\
S_1 &= 2\pi & V_2 &= \pi \\
S_2 &= 4\pi & V_3 &= \frac{4\pi}{3} \\
S_3 &= 2\pi^2 & V_4 &= \frac{\pi^2}{2} \\
S_4 &= \frac{8}{3}\pi^2 & V_5 &= \frac{8}{15}\pi^2 \\
S_5 &= \pi^3 & V_6 &= \frac{\pi^3}{6}
\end{aligned}$$

So, we have the general formula

$$V_{n+2} = \frac{2\pi}{n+2} V_n.$$

Observe that odd and even dimensional spheres behave differently.

First Order Semilinear PDEs

General First Order PDEs in $\mathbb{R}$

A general first order PDE is

$$\begin{aligned} \partial_t u(t,x) + v(t,x)\partial_x u(t,x) + w(t,x,u(t,x)) &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

for $t, x \in \mathbb{R}$ where $v(t,x)$ is given, $w(t,x,u(t,x))$ is given, and $u(t,x)$ is unknown. $u(0,x) = f(x)$ is the initial condition and is given. Semilinear refers to the fact that the PDE is linear, at least in w.

The simplest case of this PDE is if $v(t,x)$ is constant and $w(t,x,u(t,x)) = 0$. This gives us the system of equations

$$\begin{aligned} \partial_t u + v\partial_x u &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

Proposition

The system of equations

$$\begin{aligned} \partial_t u + v\partial_x u &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

has unique solution

$$u(t,x) = f(x - tv).$$

- **Proof**: first, verify that this value of $u(t,x)$ is a solution of the PDE. Compute the values of $\partial_t u$ and $\partial_x u$.

  Note that this requires use of the chain rule. For any function $f(x)$, if we want to evaluate $\partial_t f(g(t,x))$, we have $\partial_t f(g(t,x)) = f'(g(t,x)) \cdot \partial_t g(t,x)$.

$$\begin{aligned} \partial_t u &= f'(x - tv) \cdot \partial_t \left[x - tv\right] \\ &= -vf'(x - tv) \\ \partial_x u &= f'(x - tv) \cdot \partial_x \left[x - tv\right] \\ &= f'(x - tv) \\ \partial_t u + v\partial_x u &= 0 \end{aligned}$$

  So, $u(t,x) = f(x - tv)$ is a solution to the PDE.

  Now, verify that this is the only solution. We can use change of variables.

$$\begin{aligned} s &= t & t &= s \\ y &= x - vt & x &= y + vs \end{aligned}$$

  By the chain rule, we have

$$\begin{aligned} \partial_s &= \frac{\partial t}{\partial s}\frac{\partial}{\partial t} + \frac{\partial x}{\partial s}\frac{\partial}{\partial x} \\ &= \partial_t + v\partial x \end{aligned}$$

  So, we have

$$\begin{aligned} \partial_s u(s, y + vs) &= \partial_t u + v\partial_x u \\ &= 0 \end{aligned}$$

  For $w(s,y) = u(s, y + vs)$, $\partial_s w = 0$ and $w(0,y) = f(y)$. Since $\partial_s w = 0$, $w(s,y) = f(y)$ meaning $u(t,x) = w(t, x - tv)$. So, the only solution is $u(t,x) = f(x - tv)$. $\square$

Now, for a less simple case. Suppose $w = 0$ but v is no longer constant. We are solving the system of equations:

$$\begin{aligned} \partial_t u + v(t)\partial_x u &= 0 \\ u(0,x) &= f(x) \end{aligned}$$

Proposition

The system of equations $\partial_t u + v(t)\partial_x u = 0$ with initial condition $u(0,x) = f(x)$ has unique solution

$$u(t,x) = f(x - X(t))$$

where $X(t) = \int_0^t v(s)\mathrm{d}s$.

- **Proof**: first, show correctness of the solution $u(t,x) = f(x - X(t))$. Note that $X'(t) = v(t)$.

$$\begin{aligned} \partial_t u &= \partial_t\left[x - X(t)\right] \cdot f'(x - X(t)) \\ &= -X'(t)f'(x - X(t)) \\ &= -v(t)f'(x - X(t)) \end{aligned} \qquad \begin{aligned} \partial_x u &= \partial_x\left[x - X(t)\right] \cdot f'(x - X(t)) \\ &= f'(x - X(t)) \end{aligned}$$

So, $\partial_t u + v(t)\partial_x u = 0$ meaning $u(x,t) = f(x - X(t))$ is a solution to the PDE. Once again, to determine if the solution is unique, we can use the same change of variables $s = t$ and $y = x - vt$. Note that for $t = s$, $X'(s) = v(s)$ and $X(0) = 0$.

$$\begin{aligned} \partial_s &= \frac{\partial t}{\partial s}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} \\ &= \partial_t + X'(s)\partial_x \\ &= \partial_t + v(t)\partial_x \\ \partial_s u(s, y + X(s)) &= \partial_t u + v\partial_x u = 0 \\ u(0, y + X(0)) &= f(y) \\ w(s,y) &= u(s, y + X(s)) \end{aligned}$$

$\partial_s w = 0$, so $w(s,y) = f(y)$ meaning the solution $u(t,x) = w(t, x - X(t)) = f(x - X(t))$ is unique. $\square$

Now for the general case:

$$\partial_t u + v(t,x)\partial_x u + w(t,x,u(t,x)) = 0$$

$u = u(t,x)$ is the unknown. We want to convert this into a (nonlinear) ODE

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(t, x(t))$$

where the solution $t \mapsto x(t)$ is a **characteristic**. For this, we will use the **Lagrangian derivative**:

$$\frac{\mathrm{D}u}{\mathrm{D}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{u(t,x(t))}_{h(t)} = h'(t)$$

Theorem

On each characteristic $t \mapsto x(t)$ which solves $\frac{\mathrm{d}x}{\mathrm{d}t} = v(t,x)$, the 1^{st} order PDE

$$\frac{\partial u}{\partial t} + v(t,x)\frac{\partial u}{\partial x} + w(t,x,u(t,x)) = 0$$

reduces to an ODE

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \tilde{w} = 0$$

where $\tilde{w} = w(t, x(t), u(t, x(t)))$. Note that $\tilde{w} = \tilde{w}(t)$, so $\tilde{w}$ is a function of only t.

- **Proof**: apply the chain rule where $x = x(t)$.

$$\begin{aligned}\frac{\mathrm{D}u}{\mathrm{D}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[u(t, x(t))\right] \\ &= \partial_t u + x'(t)\partial_x u \\ &= \partial_t u + v(t,x)\partial_x u\end{aligned}$$

Substitute this back into the original equation.

$$\begin{aligned}\partial_t u + v(t,x)\partial_x u + w(t,x,u(t,x)) &= 0 \\ \frac{\mathrm{D}u}{\mathrm{D}t} + w(t,x,u(t,x)) &= 0\end{aligned}$$

So, for $\tilde{w} = w(t, x, u(t, x(t)))$ and $x = x(t)$, we have $\frac{\mathrm{D}u}{\mathrm{D}t} + \tilde{w} = 0$. $\square$

Example

Let $v = a + bx$ and $w = bu$ where $a, b > 0$ and $x \geq 0$. We want to solve the system of equations

$$\begin{aligned}\partial_t u + (a+bx)\partial_x u + bu &= 0 \\ u(t, 0) &= g(t)\end{aligned}$$

First, solve for the characteristic.

$$\begin{aligned}x'(t) &= v(t,x) \\ &= ax(t) + b \\ \int \frac{1}{a+bx(t)}\mathrm{d}x &= \int \mathrm{d}t \\ \frac{1}{b}\ln(a+bx(t)) &= t + C \\ x(t) &= \frac{1}{b}\left(e^{b(t+C)} - a\right)\end{aligned}$$

Use initial condition $x(t_0) = 0$. This yields $C = -t_0 - \frac{\ln(-a)}{b}$, so the equation can be rewritten as

$$x(t) = \frac{a}{b}\left(e^{b(t-t_0)} - 1\right).$$

We know that $u(t, x(t))$ is of the form

$$\begin{aligned}u(t,x(t)) &= Ae^{-bt}. \\ u(t_0, x(t_0)) &= g(t_0)e^{-b(t-t_0)} && (x(t_0) = 0)\end{aligned}$$

Using our equation for $x(t)$ from above:

$$u\left(t, \frac{a}{b}\left(e^{b(t-t_0)} - 1\right)\right) = g(t_0)e^{-b(t-t_0)}$$

Solve for t_0 in terms of t and x from the equation of $x(t)$ using $x = x(t)$.

$$\frac{a}{b}\left(e^{b(t-t_0)} - 1\right) \Longrightarrow t_0 = t + \frac{1}{b}\ln\left(\frac{a}{a+bx}\right)$$

So, our final solution for $u(t, x)$ is

$$u(t,x) = \frac{a}{a+bx}g\left(t + \frac{1}{b}\ln\left(\frac{a}{a+bx}\right)\right).$$

Outline of Steps

There are 3 main steps to solving a first order PDE of the form

$$\frac{\partial u}{\partial t} + v(t,x)\frac{\partial u}{\partial x} + w(t,x,u(t,x)) = 0$$
$$u(0,x) = f(x)$$

1. Solve $x'(t) = v(t, x(t))$ for characteristic $x(t)$.
2. Solve $h'(t) + w(t, x(t), h(t)) = 0$ for $h(t)$.
3. Match $u(t, x(t)) = h(t)$ and $h(0) = f(x(0))$ where $x(t) = x$. Solve for any constants of integration and plug it back in to $u(t,x) = u(t,x(t)) = h(t)$.

Examples

Solve the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \gamma u$$
$$u(x,0) = g(x)$$

where c and γ are constant. Note that this is a linear PDE since if we have $\frac{\partial u_j}{\partial t} + c\frac{\partial u_j}{\partial x} = \gamma u_j$ for $j = 1, 2$, $u = u_1 + u_2$ is also a solution.
Here, we have $v(t,x) = c$ and $w(t,x,u(t,x)) = -\gamma u$. Follow the steps outlined above.

1. Solve $x'(t) = v(t, x(t))$.
$$x'(t) = v(x,t) = c$$
$$x(t) = ct + A$$
2. Solve $h'(t) + w(t, x(t), h(t)) = 0$.
$$h'(t) + w(t, x(t), h(t)) = h'(t) - \gamma h(t) = 0$$
$$h(t) = Be^{\gamma t}$$
3. Match $u(t, x(t)) = h(t)$ and $h(0) = f(0)$ where $x(t) = x$.
$$u(t, x(t)) = h(t)$$
$$u(t, \underbrace{ct + A}_{x}) = Be^{\gamma t}$$
$$x = ct + A$$
$$A = x - ct$$
Use the initial condition.
$$h(0) = g(x(0))$$
$$B = g(A) = g(x - ct)$$

So, we have our final answer:

$$u(t,x) = g(x - ct)e^{\gamma t}$$

Check that our answer satisfies the original equation and initial condition:

$$\partial_t u = -cg'(x - ct)e^{\gamma t} + \gamma g(x - ct)e^{\gamma t}$$
$$c\partial_x u = cg'(x - ct)e^{\gamma t}$$
$$\partial_t u + c\partial_x u = \gamma g(x - ct)e^{\gamma t} = \gamma u \quad \checkmark$$
$$u(0,x) = g(x - c(0))e^{\gamma \cdot 0} = g(x) \quad \checkmark$$

So, our solution $u(t,x) = g(x - ct)e^{\gamma t}$ is a solution to the PDE.

Example

Solve the PDE

$$\frac{\partial u}{\partial t} + 2t\frac{\partial u}{\partial x} + u^2 = 0$$
$$u(0,x) = g(x)$$

Note that this is a nonlinear PDE since $w(t,x,u) = u^2$. Here, $v(t,x) = 2t$. Follow the same steps:

1. Solve $x'(t) = v(t,x(t))$.
$$x'(t) = 2t \Longrightarrow x(t) = t^2 + A$$
2. Solve $h'(t) + w(t,x(t),h(t)) = 0$.
$$\begin{aligned} h'(t) + h(t)^2 &= 0 \\ \frac{1}{h^2}\mathrm{d}h &= -\mathrm{d}t \\ -\frac{1}{h} = -t - B &\Longrightarrow h(t) = \frac{1}{t+B} \end{aligned}$$
3. Match $u(t,x(t)) = h(t)$ and $h(0) = f(0)$ where $x(t) = x$.
$$\begin{aligned} u(t, t^2 + A) &= \frac{1}{t+B} \\ x &= t^2 + A \\ A &= x - t^2 \end{aligned}$$
Use the initial condition.
$$\begin{aligned} h(0) &= g(x(0)) \\ \frac{1}{B} = g(A) &\Longrightarrow B = \frac{1}{g(x-t^2)} \end{aligned}$$

So, we have our final answer:

$$\begin{aligned} u(t,x) &= \frac{1}{t + \frac{1}{g(x-t^2)}} \\ &= \frac{g(x-t^2)}{1+tg(x-t^2)} \end{aligned}$$

Check that our answer satisfies the original equation and initial condition. Here, let $g = g(x-t^2)$ and $g' = g'(x-t^2)$.

$$\begin{aligned} \partial_t u &= \frac{-2tg' \cdot (1+tg) - g \cdot (g - 2t^2 g')}{(1+tg)^2} \\ &= \frac{-2tg' - g^2}{(1+tg)^2} \\ \partial_x u &= \frac{g' \cdot (1+tg) - g \cdot (tg')}{(1+tg)^2} \\ &= \frac{g'}{(1+tg)^2} \\ \partial_t u + 2t\partial_x u + u^2 &= \frac{-tg' - g^2}{(1+tg)^2} + \frac{2tg'}{(1-tg)^2} + \frac{g'}{(1+tg)^2} = 0 && \checkmark \\ u(0,x) &= \frac{g(x)}{1+0} = g(x) && \checkmark \end{aligned}$$

So, our solution $u(t,x) = \frac{g(x-t^2)}{1+tg(x-t^2)}$ is a solution to the PDE.

Higher Dimension PDEs

A general PDE of dimension n is

$$\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w = 0$$
$$u(0, \underline{x}) = f(\underline{x})$$

where u is unknown, ∇u is in terms of x, and $\underline{v}, w$ are given:

$$\begin{aligned}
u &: [0,T] \times \Omega \to \mathbb{R}, & \Omega &\in \mathbb{R}^n \\
\underline{v} &: [0,T] \times \Omega \to \mathbb{R}^n, & \underline{v} &= \underline{v}(t, \underline{x}) \\
w &: \underbrace{[0,T]}_{t} \times \underbrace{\Omega}_{\underline{x}} \times \underbrace{\mathbb{R}}_{u} \to \mathbb{R}, & w &= w(t, \underline{x}, u)
\end{aligned}$$

The steps to solving higher dimensional PDEs is exactly the same as solving a first order PDE. We solve the characteristic equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{x}(t) = \underline{v}(t, \underline{x}(t))$$

where $t \mapsto \underline{x}(t)$ is a **characteristic**. The Lagrangean derivative is again defined as

$$\frac{\mathrm{D}u}{\mathrm{D}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\underbrace{u(t, \underline{x}(t))}_{h(t)} = h'(t)$$

Theorem

If $u \in C^1([0,T] \times \Omega; \mathbb{R})$ solves the PDE

$$\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w = 0,$$

then $h(t) := u(t, \underline{x}(t))$ solves $h'(t) + w(t, \underline{x}(t), h(t)) = 0$.

- **Proof**: the proof of this is the same as that for first order PDEs.

$$\begin{aligned}
h'(t) + w(t, \underline{x}(t), h(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}[u(t, \underline{x}(t))] + w(t, \underline{x}(t), u(t, \underline{x}(t))) \\
&= \frac{\partial u}{\partial t} + \frac{\mathrm{d}\underline{x}}{\mathrm{d}t} \cdot \nabla u + w(t, \underline{x}(t), u(t, \underline{x}(t))) \\
&= \frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w(t, \underline{x}(t), u(t, \underline{x}(t))) \\
&= 0
\end{aligned}$$

since by assumption, this u solves the PDE. □

Outline of Steps

Like with first dimension PDEs, there are 3 main steps to solving a higher dimension first order PDE of the form

$$\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u + w(t, \underline{x}, u) = 0$$
$$u(0, \underline{x}) = f(\underline{x})$$

1. Solve $\underline{x}'(t) = \underline{v}(t, \underline{x}(t))$ for characteristic $\underline{x}(t)$.
2. Solve $h'(t) + w(t, \underline{x}(t), h(t)) = 0$ for $h(t)$.
3. Match $u(t, \underline{x}(t)) = h(t)$ and $h(t) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$. Solve for any constants of integration and plug it back in to $u(t, \underline{x}) = w(t, \underline{x}(t)) = h(t)$.

Example

Let $\Omega = \mathbb{R} \times [-1, 1]$. Solve the PDE

$$\frac{\partial u}{\partial t} + \underline{v} \cdot \nabla u = 0$$
$$u(0, \underline{x}) = f(\underline{x})$$

where $\underline{x} = (x_1, x_2)$ and $\underline{v}(t, \underline{x}) = (1 - x_2^2, 0)$. Note that this is a linear PDE. Follow the same steps as the first order PDEs:

1. Solve $\underline{x}'(t) = \underline{v}(t, \underline{x}(t))$. Let $\underline{x}(t) = (x_1(t), x_2(t))$.

$$\begin{aligned} \underline{x}'(t) &= (x_1'(t), x_2'(t)) \\ &= \underline{v} = (1 - x_2^2, 0) \\ x_1'(t) &= 1 - x_2(t)^2 \\ x_2'(t) &= 0 \\ x_2(t) &= A_2 \end{aligned}$$

Use this to solve for $x_1(t)$.

$$\begin{aligned} x_1'(t) &= 1 - A_2^2 \\ x_1(t) &= A_1 + (1 - A_2^2)t \end{aligned}$$

So, we have

$$\underline{x}(t) = (x_1(t), x_2(t)) = (A_1 + (1 - A_2^2)t, A_2)$$

2. Solve $h'(t) + w(t, \underline{x}(t), h(t)) = 0$.

$$\begin{aligned} h'(t) + 0 &= 0 \\ h(t) &= B \end{aligned}$$

3. Match $u(t, \underline{x}(t)) = h(t)$ and $h(0) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$.

$$\begin{aligned} u(t, \underline{x}(t)) &= B \\ \underline{x} &= (x_1, x_2) = (A_1 + (1 - A_2^2)t, A_2) \\ x_1 &= A_1 + (1 - A_2^2)t \\ &= A_1 + (1 - x_2^2)t \\ x_2 &= A_2 \\ A_1 &= x_1 - (1 - x_2^2)t \end{aligned}$$

Use the initial condition.

$$\begin{aligned} h(0) &= B = f(\underline{x}(0)) \\ \underline{x}(t) &= (A_1 + (1 - A_2^2)t, A_2) \\ \underline{x}(0) &= (A_1, A_2) \\ B &= f(A_1, A_2) \\ &= f(x_1 - (1 - x_2^2)t, x_2) \end{aligned}$$

So, we have our final answer:

$$u(t, \underline{x}) = f(x_1 - (1 - x_2^2)t, x_2)$$

Check that our answer satisfies the original equation and initial condition.

$$\begin{aligned}
\partial_t u &= -(1-x_2^2)\cdot f_{x_1}(x_1-(1-x_2^2)t, x_2)\\
\underline{v}\cdot\nabla u &= (1-x_2^2, 0)\cdot\nabla u\\
&= (1-x_2^2)\cdot\partial_{x_1} u\\
&= (1-x_2^2) f_{x_1}(x_1(1-x_2^2)t, x_2)\partial_t u + \underline{v}\cdot\nabla u = 0 && \checkmark\\
u(0,x) &= f(x_1-(1-x_2^2)\cdot 0, x_2)\\
&= f(x_1, x_2) = f(\underline{x}) && \checkmark
\end{aligned}$$

So, our solution $u(t,\underline{x}) = f(x_1-(1-x_2^2)t, x_2)$ is a solution to the PDE.

Example

Solve the PDE

$$\begin{aligned}
x_1\frac{\partial u}{\partial x_1} - 2x_2\frac{\partial u}{\partial x_2} - u^2 = 0\\
u(y,y) = y^3
\end{aligned}$$

where $\underline{x} = (x_1, x_2)$, y is constant where if $\underline{x}$ falls on the line $x_1 = x_2$ then $u(\underline{x}) = y^3$, and $u = u(x)$ is independent of t. This can be rewritten as

$$(x_1, -2x_2)\cdot\nabla u - u^2 = 0$$

where $\underline{v}(t,\underline{x}) = (x_1, -2x_2)$ and $w(t,\underline{x},u) = -u^2$. Follw the steps outlined previously.

1. Solve $\underline{x}'(t) = (x_1(t), x_2(t)) = \underline{v}(t, \underline{x}(t))$.

$$\begin{aligned}
\underline{x}'(t) &= (x_1'(t), x_2'(t))\\
&= \underline{v}(t, \underline{x}(t))\\
&= (x_1(t), -2x_2(t))\\
x_1'(t) &= x_1(t)\\
x_1(t) &= A_1 e^t\\
x_2'(t) &= -2x_2(t)\\
x_2(t) &= A_2 e^{-2t}
\end{aligned}$$

2. Solve $h'(t) + w(t, \underline{x}(t), h(t)) = 0$ for $h(t)$.

$$\begin{aligned}
h'(t) - h(t)^2 &= 0\\
\frac{1}{h^2}\mathrm{d}h &= \mathrm{d}t\\
-\frac{1}{h} &= t - B\\
h(t) &= \frac{1}{B-t}
\end{aligned}$$

3. Match $u(t, \underline{x}(t)) = h(t)$ and $h(0) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$.

$$\begin{aligned}
u(\underline{x}(t)) &= \frac{1}{B-t}\\
\underline{x}(t) &= (x_1(t), x_2(t))\\
&= (A_1 e^t, A_2 e^{-2t})
\end{aligned}$$

Use the initial condition $u(y,y) = y^3$. Set $A_1 = A_2 = y$. Note that $f(0,0) = y^3$.

$$\begin{aligned}
\underline{x}(t) &= (ye^t, ye^{-2t}) \\
h(0) &= f(0,0) = y^3 = \frac{1}{B} \\
B &= \frac{1}{y^3} \\
u(x(t)) &= \frac{1}{\frac{1}{y^3} - t} = \frac{y^3}{1 - y^3 t} \\
x_1 &= x_1(t) = ye^t \\
x_2 &= x_2(t) = ye^{-2t}
\end{aligned}$$

We do not want y in our final answer, so solve for y in terms of $\underline{x}$ and t.

$$\begin{aligned}
y &= x_1 e^{-t} = x_2 e^{2t} \\
x_2 &= x_1 e^{-3t} \\
e^{3t} &= \frac{x_1}{x_2}
\end{aligned}$$

$u = u(\underline{x})$ is not a function of t, so solve for t in terms of $\underline{x}$.

$$\begin{aligned}
t &= \frac{1}{3} \ln\left(\frac{x_1}{x_2}\right) \\
y^3 &= e^{-3t} x_1^3 = x_1^2 x_2 \\
u(x) &= \frac{y^3}{1 - y^3 t} = \frac{x_1^2 x_2}{1 - \frac{1}{3}\ln\left(\frac{x_1}{x_2}\right) x_1^2 x_2}
\end{aligned}$$

So, our final answer is $u(t,x) = u(t) = \frac{x_1^2 x_2}{1 - \frac{1}{3}\ln\left(\frac{x_1}{x_2}\right) x{-}1^2 x_2}$. We will not check this answer since it will take longer than solving the problem.

Classifying PDEs

Suppose we have the following PDE:

$$Lu = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + Fu = 0$$

where $F = f(x,t)$ and $v = v(x,t)$. This is a **1st order linear homogeneous PDE**.

- **1st order**: the highest order derivative is of the 1st order.
- **linear**: solutions can be added together to form another solution:

$$\begin{gathered}
L(u_1, u_2) = Lu_1 + Lu_2 \\
\frac{\partial(u_1 + u_2)}{\partial t} + v\frac{\partial(u_1 + u_2)}{\partial x} + F(u_1 + u_2) = \left(\frac{\partial u_1}{\partial t} + v\frac{\partial u_1}{\partial x} + Fu_1\right) + \left(\frac{\partial u_2}{\partial t} + v\frac{\partial u_2}{\partial x} + Fu_2\right)
\end{gathered}$$

- **homogeneous**: if u_1 and u_2 are solutions of Lu, then $Lu_1 = Lu_2 = 0$ meaning $L(u_1 + u_2) = 0$.
- **PDE**: the equation uses partial differential equations.

If $Lu = f \neq 0$, then it is a 1st order linear **nonhomogeneous** PDE.

If u_1 and u_2 solve $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + w(t,x,u) = 0$ but $w(t,x,u_1+u_2) \neq w(t,x,u_1) + w(t,x,u_2)$, then it is a 1st order **semilinear** PDE (linear in w, but not outside w).

First Order Quasilinear PDEs

Quasilinear 1^{st} order PDEs are of the form

$$\begin{aligned} \partial_t u + \underline{a}(u) \cdot \nabla u &= 0 \\ u(0,x) &= f(z) \end{aligned}$$

where $u = u(t,x)$ is unknown, ∇u is in terms of x, and $\underline{a} \in C^1(\mathbb{R};\mathbb{R}^n)$ is a given function of u.

Theorem

Suppose $u \in C^1([0,t] \times \Omega; \mathbb{R})$ solves the PDE

$$\begin{aligned} \partial_t u + \underline{a}(u) \cdot \nabla u &= 0 \\ u(0,x) &= f(z) \end{aligned}$$

Then, $\forall x_0 \in \Omega$, u is constant along the characteristics of the system given by

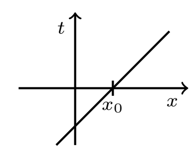
$$\underline{x}(t) = \underline{x}_0 + \underline{a}(u(0,\underline{x}_0))t.$$

This is equivalent to the statement $u(t, \underline{x}_0 + g\underline{a}(f(\underline{x}_0))) = f(\underline{x}_0)$.

So, if we have a solution to the system, the solution is constant on the line

$$\begin{aligned} \underline{x}(t) &= \underline{x}_0 + \underline{a}(u(0,\underline{x}_0))t \\ &= \underline{x}_0 + \underline{a}_0(f(\underline{x}_0))t \end{aligned}$$

The theorem states that $u(t, \underline{x}_0 + t\underline{a}(f(\underline{x}_0))) = f(\underline{x}_0)$. The line can be seen in the plot.



The slope of the line is $\frac{1}{\underline{a}f(\underline{x}_0)}$, so we have the equation of the line:

$$\underline{x} = \underline{x}_0 + t\underline{a}(f(\underline{x}_0))$$

Example

Let $\underline{a} = (a_1, \ldots, a_n)$ be a constant vector. Solve the PDE

$$\begin{aligned} \partial_t u + \underline{a} \cdot \nabla u &= 0 \\ u(0,\underline{x}) &= f(\underline{x}) \end{aligned}$$

This is a simple case that follows the steps outline previously for semilinear PDEs.

1. Solve $\underline{x}'(t) = \underline{a}(u(t,\underline{x}(t)))$.
$$\underline{x}'(t) = \underline{a} \Longrightarrow \underline{x}(t) = \underline{a}t + A$$
2. Solve $h'(t) = w(t, \underline{x}(t), h(t))$ for $h(t)$. In this form of quasilinear PDEs, $w(t,x,u) = 0$.
$$h'(t) = 0 \Longrightarrow h(t) + B$$
3. Match $u(t,\underline{x}(t)) = h(t)$ and $h(0) = f(\underline{x}(0))$ where $\underline{x}(t) = \underline{x}$.
$$\begin{aligned} u(t,\underline{x}(t)) &= B \\ \underline{x} &= \underline{a}t + A \\ h(0) = B &= f(\underline{x}(0)) \\ \underline{x}(t) &= \underline{a}t + A \\ \underline{x}(0) = A &= \underline{x} - \underline{a}t \\ B &= f(\underline{x} - \underline{a}t) \end{aligned}$$

So, we have our final answer:

$$u(t,\underline{x}) = f(\underline{x} - \underline{a}t)$$

- **Proof**: prove the theorem from earlier. First, consider $\underline{x}'(t) = \underline{a}(u(t\underline{x}(t)))$ where $\underline{x}(0) = \underline{x}_0$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u(t,\underline{x}(t)) &= \partial_t u + \underline{a}(u)\cdot\nabla u \\ &= \partial_t u + \underline{a}(u)\cdot\nabla u \\ &= 0\end{aligned}$$

by assumption that u is a solution to the PDE. This implies

$$u(t,\underline{x}(t)) = u(0,\underline{x}(0)) = u(0,\underline{x}_0)$$

is constant. Back to the ODE $\underline{x}'(t) = \underline{a}(u(t),\underline{x}(t))$ where $\underline{x}(0) = \underline{x}_0$:

$$\begin{aligned}\underline{x}'(t) &= \underline{a}(u(0,\underline{x}_0)) \\ &= a(f(\underline{x}_0)) \\ x(t) &= \underline{x}_0 + \underline{a}(f(\underline{x}_0))t \\ \Rightarrow \quad u(t,\underline{x}_0 + a(f(\underline{x}_0))t) &= f(x_0)\end{aligned}$$

So, we have $u(t,\underline{x}_0 + \underline{a}(f(\underline{x}_0))t) = f(\underline{x}_0)$. $\square$

Notable Examples

Burger's Equation

Burger's equation is a simple quasilinear PDE related to models of gas dynamics. For $n = 1$ and $\underline{a}(u) = u$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Let us first consider a case of a simple initial condition.

$$u(0,x) = ax + b$$

where a and b are constants, For an arbitrary $x(t)$, let $u(t,x(t)) = f(\underline{x}_0)$.

$$x(t) = \underline{x}_0 + t(a\underline{x}_0 + b)$$

Solve for x_0.

$$\begin{aligned}\underline{x} &= \underline{x}_0 + t(a\underline{x}_0 + b) \\ \underline{x}_0 &= \frac{\underline{x} - tb}{1 + ta} \\ u(t,\underline{x}) &= f\left(\frac{\underline{x} - tb}{1 + ta}\right) \\ &= a\frac{\underline{x} - tb}{1 + ta} + b \\ &= \frac{a\underline{x} - tab + b + tab}{1 + ta} \\ &= \frac{a\underline{x} + b}{1 + ta}\end{aligned}$$

If $a < 0$, then the solution exists when $t < -\frac{1}{a}$. If $a > 0$, the solution exists for $t > 0$.

Now for arbitrary initial condition, we have

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0 \\ u(0,\underline{x}) &= f(\underline{x}) \\ u(t,\underline{x}_0 + tf(\underline{x}_0)) &= f(\underline{x}_0)\end{aligned}$$

We want to solve $\underline{x}_0 + tf(\underline{x}_0) - \underline{x} = 0$ with $t, \underline{x}$ given. To do this, we must use implicit differentiation.

Lemma: Implicit Function Theorem

$$F(x,y,t) = 0$$
$$F(x_0,y_0,t_0) = 0$$

When can we find y as a function of x and t such that $F(x, y(t,x), t) = 0$ and $y(t_0, x_0) = y_0$?

If $\frac{\partial}{\partial y} F(x_0, y_0, t_0) \neq 0$, then

$$\frac{\partial y}{\partial t} = -\frac{F_t(x_0,y_0,t_0)}{F_y(x_0,y_0,t_0)}$$
$$\frac{\partial y}{\partial x} = -\frac{F_x(x_0,y_0,t_0)}{F_y(x_0,y_0,t_0)}$$

Here, let $y = x_0$.So, we have $F(x,y,t) = y + ty - x = 0$ where $x_0 = x$, $y_0 = x$, and $t_0 = 0$.

$$\begin{aligned}\frac{\partial}{\partial y} F(x_0,y_0,t_0) &= \frac{\partial}{\partial y} F(x,x,0) \\ &= 1 \neq 0\end{aligned}$$

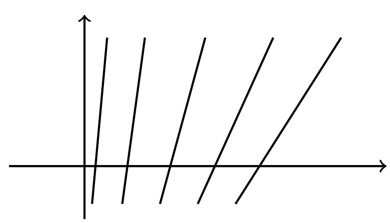
For our example, we have slope $= \frac{1}{f(x(0))}$. We want to find

$$u(t, \underbrace{y + tf(y)}_{x}) = f(y).$$

To solve this, we must find y such that $y + tf(y) = x$, or $y + tf(y) - x = 0$. We have $y = y(t,x)$ and $u(t,x) = f(y(t,x))$.

Suppose $\tilde{t} = t$ and $\tilde{x} = \tilde{y} + tf(\tilde{y})$. We have $F(t,x,y) = y + tf(y) - x$. We need $\partial_y F \neq 0$, or $1 + tf'(\tilde{y}) \neq 0$.

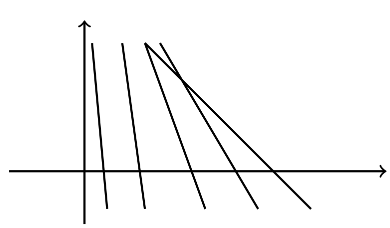
If $f' \geq 0$, f is non decreasing meaning the slope $\frac{1}{f(x)}$ is non increasing.



Example

Suppose $f(x) = x^7$. Then, $y + ty^7 - x = 0$. We want to solve for y. Although we cannot find an equation for y explicitly, by the Implicit Function Theorem, we know that $y(t,x)$ exists. So, we have our solution in the form $u(t,x) = y(t,x)^7$.

Now, suppose $f' < 0$. We have the reverse of the plot we had earlier, so at some point, the lines must intersect.



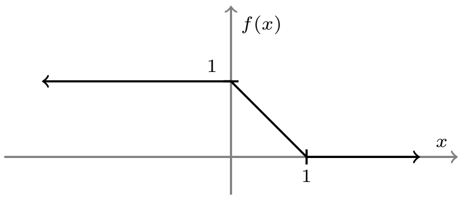
We stop at the first intersection since at this point, we cannot determine $x(0)$ uniquely. However, by the Implicit Function Theorem, $F = y + f(y) - x = 0$ meaning $F_y = 1 + tf'(y) \neq 0$ if t is small. So, some solution $u(t,x)$ must exist.

Telegrapher's Equation

The Telegrapher's Equation is another famous quasilinear PDE. It takes the form

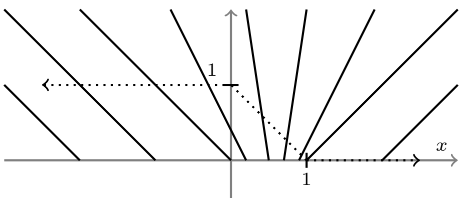
$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0$$
$$u(0,x) = f(x)$$

with $a(u) = 1-2u$. We solve this for



$$f(x) = \begin{cases} 1 & x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

f is differentiable everywhere except for $x=0$ and $x=1$. To solve this, we follow the same procedure. We have $a(u) = 1-2u$ and $x(t) = y + ta(f(y))$. Since $f(y)$ varies depending on the value of $y = x(0)$, we have the following $x(t)$:



$$x(t) = \begin{cases} x(0) - t & x(t) \le -t \\ x(0) + t(2x(0)-1) & -t \le x(t) \le 1+t \\ x(0)+t & x(t) \ge 1+t \end{cases}$$

We have $u(t,x(t)) = f(x(0))$. We want to find $x(0)$ such that $x(t) = x$, which is given.

$$\begin{aligned} &\text{For } x \le -t: & x(0) &= x+t \\ &\text{For } -t \le x \le t+1: & x(0) + t(2x(0)-1) &= x \\ & & x(0) &= \frac{x+t}{1+2t} \\ &\text{For } x \ge 1+t: & x(0) &= x-t \end{aligned}$$

So, we can calculate each $f(x(0))$. For $x \le -t$, $f(x) = 1$ and for $x \ge 1+t$, $f(x) = 0$. For $-t \le x \le t+1$:

$$\begin{aligned} f(x) = 1-x \implies f(x(0)) &= 1 - x(0) \\ &= 1 - \frac{x+t}{1+2t} = \frac{-x+t+1}{1+2t} \end{aligned}$$

So, we have our final answer for $u(t,x)$:

$$u(t,x) = \begin{cases} 1 & x \le -t \\ \frac{-x+t+1}{1+2t} & -t \le x \le t+1 \\ 0 & x \ge 1+t \end{cases}$$

Note that Telegrapher's Equation is simply Burger's Equation "in disguise". Let $v(t,x) := 1-2u(t,x)$. If $u(t,x)$ solves the Telegrapher's Equation, $v(t,x)$ solves Burger's Equation. We have $\partial_t + (1-2u)\partial_x u = 0$, and we can calculate $\partial_t u$ and $\partial_x u$ in terms of $\partial_t v$ and $\partial_x v$. $\partial_t v = -2\partial_t u$ and $\partial_x v = -2\partial_x u$, so $\partial_t u = -\frac{1}{2}\partial_t v$ and $\partial_x u = -\frac{1}{2}\partial_x v$. So:

$$\begin{aligned} -\frac{1}{2}\partial_t v + \underbrace{(1-2u)}_{v}\left(-\frac{1}{2}\partial_x v\right) &= 0 \\ \partial_t v + v\partial_x v &= 0 \end{aligned}$$

So, $v(t,x)$ solves Burger's Equation. Observe that our answer is always of the form $\underline{y} + t\underline{a}(f(\underline{y})) - \underline{x} = 0$, so it will always have some solution for some t.

Eikonal Equation

Let $u = u(x)$ and $x \in \mathbb{R}^2$. Here, u is a function of strictly x, so we can treat $\frac{\partial u}{\partial t} = 0$. This equation has many solutions, such as

$$\begin{aligned} u(x) &= \cos(\theta)x_1 + \sin(\theta)x_2 \\ &= \underline{\omega} \cdot \underline{x} \end{aligned}$$

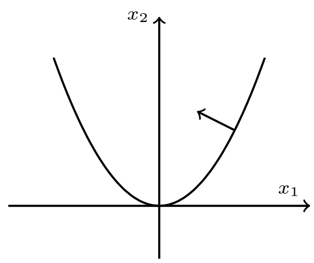
where

$$\underline{\omega} = (\cos(\theta), \sin(\theta)).$$

For this PDE, we have $\underline{a}(u) = \nabla u(x)$ so our equation is $\nabla u(x) \cdot \nabla u(x) = 0$.

$$\begin{aligned} \underline{x}'(t) &= \nabla u(\underline{x}(t)) \\ \underline{x}''(t) &= \nabla \cdot (|\nabla u|^2) \\ &= 0 \\ \underline{x}'(t) &= \underline{v}_0 \\ \underline{x}(t) &= \underline{x}(0) + \underline{v}_0 t \end{aligned}$$

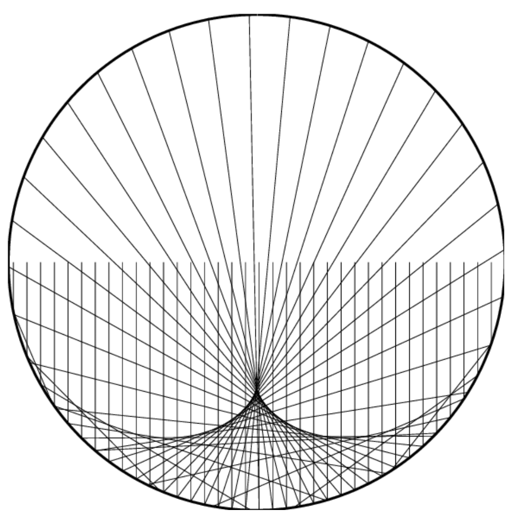
for some $\underline{v}_0$. Let the initial condition be a parabola $u(x_1, x_1^2) = 1$.



The gradient must have length 1, so it is the normal unit vector of the parabola. So:

$$\underline{v}_0 = \frac{(-2x_1, 1)}{(1 + 4x_1^2)^{1/2}}$$

We have $u(\underline{x}(t)) = u(\underline{x}(0)) = 1$ where $\underline{x}(0) = (x_1, x_1^2)$.



The unit vectors focus to a caustic, which can be seen in the figure above. This figure was obtained from a paper on wave propagation.

The Wave Equation

The wave equation takes the form

$$\left(\partial_t^2 - c^2\Delta\right) u = 0$$

where the Laplace operator is

$$\begin{aligned}\Delta &= \partial_{x_1}^2 + \cdots + \partial_{x_n}^2 \\ &= \nabla \cdot \nabla\end{aligned}$$

c is a constant known as the "speed of propagation" and $c > 0$.

Example

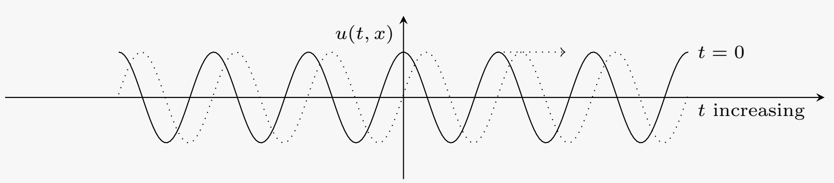
We will look at an important solution to the wave equation, known as the standing wave solution. The function

$$u(t,x) = e^{ic\lambda t + i\lambda x\cdot\omega}$$

solves the wave equation where $\omega \in \mathbb{R}^n$ and $|\omega| = 1$. We write $x \cdot \omega = x_1\omega_1 + \cdots + x_n\omega_n$. Let the exponent of e, $ic\lambda t + i\lambda x \cdot \omega$, be $\mu(t,x)$.

$$\begin{aligned}\partial_t^2 u(t,x) &= \partial_t\left[ic\lambda e^{\mu(t,x)}\right] \\ &= -c^2\lambda^2 e^{\mu(t,x)} \\ \partial_{x_j}^2 u(t,x) &= \partial_{x_j}\left[i\lambda\omega_j e^{\mu(t,x)}\right] \\ &= -\lambda^2\omega_j^2 e^{\mu(t,x)} \\ \partial_t^2 u - c^2\Delta u &= \partial_t^2 u - c^2\sum_{j=1}^{n}\partial_{x_j}^2 u \\ &= -c^2\lambda^2 e^{\mu(t,x)} - c^2(-\lambda^2\underbrace{(\omega_1^2 + \cdots + \omega_n^2)}_{=1 \text{ since } |\omega|=1})e^{\mu(t,x)} \\ &= -c^2\lambda^2 e^{\mu(t,x)} + c^2\lambda^2 e^{\mu(t,x)} \\ &= 0\end{aligned}$$

In dimension 1, $\omega = \pm 1$. Let $\omega = -1$. Then, we have $u(t,x) = e^{i\lambda(ct-x)}$. We have $\mathrm{Re}(u) = \cos(\lambda(ct-x)) = \cos(\lambda(x-ct))$. We can plot this solution and observe the effect as t increases.



From the Eikonal equation, $\varphi(x) = \omega \cdot x$ is a solution to $|\nabla\varphi(x)|^2 = 1$. Try $u(t,x) = e^{i\lambda t + i\lambda\varphi(x)}$ as a solution (take $c = 1$ for simplicity).

$$\begin{aligned}\partial_t^2 u &= -\lambda^2 e^{it + i\lambda\varphi(x)} \\ \partial_{x_j}^2 u &= \partial_{x_j}\left[i\lambda\partial_{x_j}\varphi(x)e^{i\lambda t + i\lambda\varphi(x)}\right] \\ &= \left(-\lambda^2\left(\partial_{x_j}\varphi(x)\right)^2 + i\lambda\partial_{x_j}^2\varphi(x)\right)e^{-\lambda t + i\lambda\varphi(x)} \\ \left(\partial_t^2 - \nabla\right)\left(e^{i\lambda t + i\lambda\varphi(x)}\right) &= \left[-\lambda^2 + \lambda^2|\nabla\varphi|^2 + i\lambda\Delta\varphi\right]e^{i\lambda t + i\lambda\varphi(x)}\end{aligned}$$

For high frequency oscilations, λ is large meaning $\lambda^2 \gg \lambda$. At $|\nabla\varphi|^2 = 1$, we get $i\lambda\Delta\varphi e^{i\lambda t + i\lambda\varphi(x)}$.

First Dimension Wave Equation

Theorem: d'Alembert's Formula

Suppose $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Then, $\exists$ a unique function $u \in C^2(\mathbb{R} \times \mathbb{R})$ such that the following are true:

$$\begin{aligned} \left(\frac{1}{c^2}\partial_t^2 - \partial_x^2\right) u = \left(\partial_t^2 - c^2\partial_x^2\right) u = 0 \\ u(0,x) &= g(x) \\ \partial_t u(0,x) &= h(x) \\ u(t,x) &= \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\mathrm{d}s \end{aligned}$$

This states existsance of a unique solution with the formula.

- **Proof**: note that the wave equation can be written as

$$(\partial_t^2 - c^2\partial_x^2) = (\partial_t + c\partial_x)(\partial_t - c\partial_x).$$

Let

$$w(t,x) = (\partial_t - c\partial_x)u(t,x).$$

Then, if u exists, we have

$$(\partial_t + c\partial_x)w = (\partial_t^2 - c^2\partial_x^2)u(t,x) = 0$$

So, we are solving the first order system

$$(\partial_t + c\partial_x)w = 0.$$

To do this, we can use the method of characteristics.

$$\begin{aligned} x'(t) &= c \\ x(t) &= ct + x_0 \\ x_0 &= x - ct \\ w(t,x) &= w_0(x-ct) \text{ for some function } w_0 \\ (\partial_t - c\partial_x)u(t,x) &= w(t,x) \\ &= w_0(x-ct) \end{aligned}$$

So, we want to solve the PDE

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w_0(x-ct)$$

for some function $w_0(x-ct)$.

Note: for a general PDE

$$\partial_t u + c\partial_x u = f(t,x)$$

where $u(0,x) = g(x)$, we can solve for $u(t,x)$.

1. Solve $x'(t) = v(t, x(t))$ for $x(t)$.

$$\begin{aligned} x'(t) &= c \\ x(t) &= ct + A \end{aligned}$$

2. Solve $h'(t) + w(t, x(t), h(t)) = 0$ for $h(t)$.

$$\begin{aligned} h'(t) - f(t, x(t)) = 0 \\ h'(t) &= f(t, x(t)) \\ h(t) &= B + \int_0^t f(s, x(s))\mathrm{d}s \end{aligned}$$

3. Match $u(t, x(t)) = h(t)$ and $h(0) = g(x(0))$.

$$\begin{aligned}
u(t, x(t)) &= B + \int_0^t f(s, x(s))\mathrm{d}s \\
x(t) = x &= ct + A \\
A &= x - ct \\
x(0) &= A = x - ct \\
x(s) &= A + cs \\
&= x - ct + cs \\
&= x + c(s - t) \\
h(0) &= B + \int_0^0 f(s, x(s))\mathrm{d}s \\
&= B \\
&= g(x(0)) \\
&= g(x - ct)
\end{aligned}$$

So, we have $u(t, x) = g(x - ct) + \int_0^t f(s, x + c(s - t))\mathrm{d}s$.

In this case, we have $-c$ instead of $+c$, so we reverse the signs:

$$u(t, x) = g(x + ct) + \int_0^t f(s, s + c(t - s))\mathrm{d}s$$

Apply this function where $f(t, x) = w_0(x - ct)$.

$$u(t, x) = g(x + ct) + \int_0^t w_0(\underbrace{x - 2cs + ct}_{\tau})\mathrm{d}s$$

Write the integral in terms of τ.

$$\begin{aligned}
\tau &= x - 2cs + ct \\
c &= \frac{1}{2c}(x - \tau + ct) \\
\mathrm{d}s &= -\frac{1}{2c}\mathrm{d}\tau
\end{aligned}$$

To convert the bounds, note that at $s = 0$, $\tau = x + ct$ and at $s = \tau$, $\tau = x - ct$.

$$\begin{aligned}
u(t, x) &= g(x + ct) - \frac{1}{2c}\int_{x+ct}^{x-ct} w_0(\tau)\mathrm{d}\tau \\
&= g(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\mathrm{d}\tau
\end{aligned}$$

Find w_0 using the initial conditions.

$$\begin{aligned}
u(0, x) &= g(x) + \frac{1}{2c}\int_x^x w_0(\tau)\mathrm{d}\tau \\
&= g(x) \\
\partial_t u(0, x) &= cg'(x) + \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\mathrm{d}\tau\right]
\end{aligned}$$

The chain rule can be used to calculate the derivative of the integral.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(\tau)\mathrm{d}\tau = b'(t)F(b(t)) - a'(t)F(a(t))$$

In our case, $a(t) = x - ct$, $b(t) = x + ct$, and $F(\tau) = w_0(\tau)$.

$$\begin{aligned}
\partial_t u &= cg'(x) + \frac{1}{2c}(cw_0(x)) - \frac{1}{2c}(-cw_0(x)) \\
&= cg'(x) + w_0(x) \\
&= h(x) \\
w_0(x) &= h(x) - cg'(x)
\end{aligned}$$

Plug this back into the equation for $u(t, x)$.

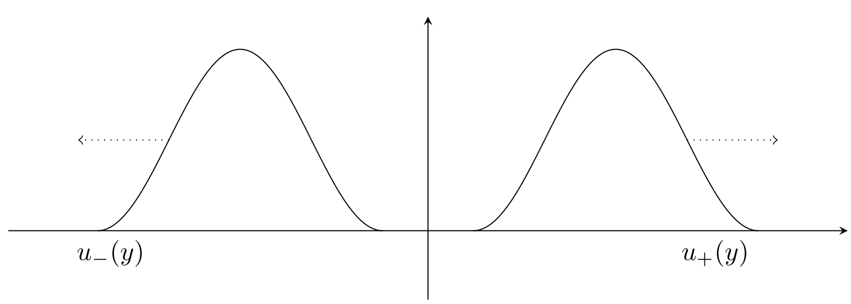
$$\begin{aligned}
u(t,x) &= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\mathrm{d}\tau \\
&= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \left[h(\tau) - cg'(\tau)\right]\mathrm{d}\tau \\
&= g(x+ct) - \frac{1}{2}\int_{x-ct}^{x+ct} g'(\tau)\mathrm{d}\tau + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau \\
&= g(x+ct) - \frac{1}{2}[g(\tau)]_{x-ct}^{x+ct} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau \\
&= g(x+ct) - \frac{1}{2}g(x+ct) + \frac{1}{2}g(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau \\
&= \frac{1}{2}(g(x+ct) + g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\mathrm{d}\tau
\end{aligned}$$

So, we have proven the theorem. The solution is unique since if u exists, then all proved conditions are satisfied. A known solution to $(\partial_t - c\partial_x)u(t,x) = w_0(t,x)$ is unique. $\square$

Note that we can write $u(t, x)$ as a sum of two parts:

$$\begin{aligned}
u(t,x) &= u_-(x+ct) + u_+(x-ct) \\
u_-(y) &= \frac{1}{2}g(y) + \frac{1}{2}\int_0^y h(\tau)\mathrm{d}\tau \\
u_+(y) &= \frac{1}{2}g(y) - \frac{1}{2}\int_0^y h(\tau)\mathrm{d}\tau
\end{aligned}$$

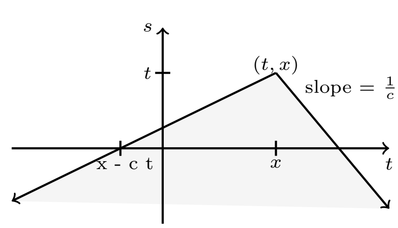
This solution can be plotted:



The two curves move apart from each other.

Huygens' Principle

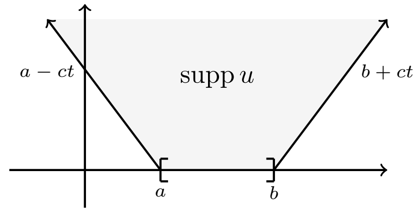
$x(0) = x + ct$ is constant on the line $x = x(0) - ct$. $u(t, x)$ depends only on the data in the "backwards cone" shown in the diagram.



Huygens' Principle states that if $\operatorname{supp} g, \operatorname{supp} h \subset [a, b]$, then

$$\operatorname{supp} u(t, \bullet) \subseteq [a - ct, b + ct]$$

for fixed $t \geq 0$ and varying $\bullet$.

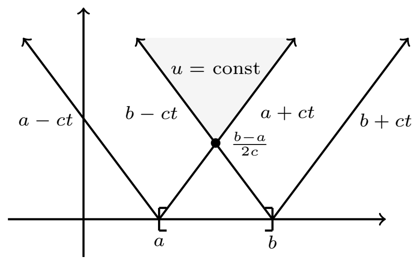


- **Proof**: if $a < a + ct$, then $x + ct < a$. This means $x - ct < a - 2ct < a$, so $g(x \pm ct) = 0$ and $h(s) = 0$ for $s \in [x - ct, x + ct]$. A similar argument can be made for $x > b + ct$.

Strong Huygens' Principle states that for $\operatorname{supp} g, \operatorname{supp} h \subset [a, b]$,

$$u(t, x) = \frac{1}{2c} \int_a^b h(\tau) \mathrm{d}\tau = \text{constant}$$

for $b - ct < x < a + ct$.



Boundary Problems

First order wave equation boundary problems have form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\text{Initial conditions} \begin{cases} u(0, x) = g(x) \\ \partial_t u(0, x) = h(x) \end{cases}$$

$$\text{Boundary conditions} \begin{cases} u(t, 0) = 0 \\ u(t, L) = 0 \end{cases}$$

This type of boundary condition is called a Dirichlet boundary condition. We only care about $u(t, x)$ between $x = 0$ and $x = L$.

Using linearity of the PDE, we can solve for u_1 and u_2 separately where $u = u_1 + u_2$ solves the PDE.

$$\begin{aligned} \frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} &= 0 \\ u_1(0, x) &= g(x) \\ \partial_t u_1(0, x) &= 0 \\ u_1(t, 0) = u_1(t, L) &= 0 \end{aligned} \qquad \begin{aligned} \frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} &= 0 \\ u_2(0, x) &= 0 \\ \partial_t u_2(0, x) &= h(x) \\ u_2(t, 0) = u_2(t, L) &= 0 \end{aligned}$$

We split it up so that solving for u_1 and u_2 separately will be easier. Solve for u_1. We use the equation from the previous theorem where $h(s) = 0$, so the integral is 0.

$$u_1(t, x) = \frac{1}{2}(\tilde{g}(x + ct) + \tilde{g}(x - ct))$$

We postulate that $u(0,x) = \tilde{g}(x) = g(x)$ for $0 \le x \le L$.

$$\partial_t u_1(0,x) = \frac{1}{2}\left(c\tilde{g}'(x+ct) - c\tilde{g}'(x-ct)\right)\big|_{t=0} = 0$$

We need $\tilde{g} \in C^2(\mathbb{R})$ such that $\partial_t^2 u_1 - c^2 \partial_x^2 u_1 = 0$.

$$\begin{aligned} u_1(t,0) &= \frac{1}{2}(\tilde{g}(tc) + \tilde{g}(-tc)) = 0 \\ u_1(t,L) &= \frac{1}{2}(\tilde{g}(L+tc) + \tilde{g}(L-tc)) = 0 \end{aligned}$$

Since $t \ge 0$ is arbitrary, we can set $tc = y$ for arbitrary y.

$$\begin{aligned} \tilde{g}(y) + \tilde{g}(-y) &= 0 \\ \tilde{g}(-y) &= -\tilde{g}(y) \end{aligned}$$

So, $\tilde{g}$ is an odd function.

$$\tilde{g}(L+y) + \tilde{g}(L-y) = 0$$

y is arbitrary, so we can set $w = L + y$ for arbitrary w. $y = w - L$ and $L - y = 2L - w$.

$$\begin{aligned} \tilde{g}(w) &= -\tilde{g}(2L-w) \\ \tilde{g}(-w) &= \tilde{g}(2L-w) \\ \tilde{g}(y) &= \tilde{g}(2L+y) \text{ where } y = -w \end{aligned}$$

So, $\tilde{g}$ is both odd and periodic on $2L$. If we have $\tilde{g}$ defined on 0 to L, we can extend $\tilde{g}$ to $[-L,0]$ since $\tilde{g}$ is odd and then extend it in both directions to $[-3L,-L]$ and $[L,3L]$ and so on since $\tilde{g}$ is periodic on $2L$, meaning it repeats itself every $2L$.

$\exists u_1 \in C^2$ if the odd $2L$-periodic extension of g, called $\tilde{g}$, is C^2.

$$u_1(t,x) = \frac{1}{2}\left(\tilde{g}(x-ct) + \tilde{g}(x+ct)\right)$$

By the same argument as u_1, $\exists u_2 \in C^2$ if the odd $2L$-periodic extension of h, called $\tilde{h}$, is C^1. Here, the g function is 0, so we only have the integral.

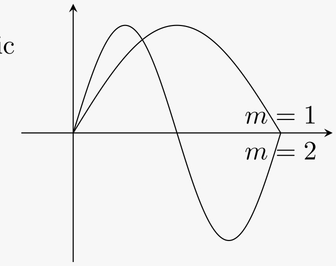
$$u_2(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(s)\mathrm{d}s$$

So, we can solve the initial PDE with $u \in C^2$ if g and h have C^2 and C^1 odd $2L$-periodic extensions respectively. This u has the form

$$u(t,x) = \frac{1}{2}\left(\tilde{g}(x-ct) + \tilde{g}(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(s)\mathrm{d}s$$

Example

Let $g(x) = \sin\left(\frac{\pi}{L}mx\right)$ and $h(x) = 0$. g is already odd and $2L$-periodic since sin is and odd function and



$$g(x+2L) = \sin\left(\frac{\pi}{L}mx + 2\pi mx\right) = \sin\left(\frac{\pi}{L}mx\right)$$

So, $\tilde{g}(x) = g(x)$ and we have our formula for u:

$$u(t,x) = \frac{1}{2}\left[\sin\left(\frac{\pi}{L}m(x-ct)\right) + \sin\left(\frac{\pi}{L}m(x+ct)\right)\right]$$

Nonhomogeneous Problems

Nonhomegenous problems are where the wave equation is no longer equal to 0. The general nonhomogeneous wave equation is given by

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= f(t,x) \\ u(0,x) &= g(x) \\ \partial_t u(0,x) &= h(x) \end{aligned}$$

where $f \in C^1(\mathbb{R}^2)$, $g \in C^2(\mathbb{R})$, and $h \in C^1(\mathbb{R})$.

Like with solving boundary problems, we can use linearity of u to split this PDE into two parts.

$$\begin{aligned} \left(\partial_t^2 - c^2\partial_x^2\right) u_1 &= 0 \\ u_1(0,x) &= g(x) \\ \partial_t u_1(0,x) &= h(x) \end{aligned} \qquad \begin{aligned} \left(\partial_t^2 - c^2\partial_x^2\right) u_2 &= 0 \\ u_2(0,x) &= 0 \\ \partial_t u_2(0,x) &= 0 \end{aligned}$$

Once again, $u = u_1 + u_2$. So, we can solve

$$\begin{aligned} \left(\partial_t^2 - c^2\partial_x^2\right) u &= f(t,x) \\ u(0,x) &= 0 \\ \partial_t u(0,x) &= 0 \end{aligned}$$

Theorem: Duhamel's Formula

For $f \in C^1(\mathbb{R}^2)$, $g \in C^2(\mathbb{R})$, and $h \in C^1(\mathbb{R})$, $\exists$ a unique $u \in C^2$ which solves the nonhomegenous system

$$\begin{aligned} \left(\partial_t^2 - c^2\partial_x^2\right) u &= f(t,x) \\ u(0,x) &= g(x) \\ \partial_t u(0,x) &= h(x) \end{aligned}$$

which is given by

$$u(t,x) = \frac{1}{2}\left(g(x-ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(s)\mathrm{d}s + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$$

- **Proof**: As stated previously, we have already proven this to be true for $f = 0$, so by the superposition principle, it suffices to consider $g = h = 0$. So, we must show that

  $$u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$$

  is the unique solution to the system where $g = h = 0$.

  First prove uniqueness. Suppose $\tilde{u}$ also solves the system.

  $$\begin{aligned} \left(\partial_t^2 - c^2\partial_x^2\right) \tilde{u} &= f(t,x) \\ \tilde{u}(0,x) &= 0 \\ \partial_t \tilde{u}(0,x) &= 0 \end{aligned}$$

  We want to show that $u = \tilde{u}$. Let $U = u - \tilde{u}$.

  $$\begin{aligned} \left(\partial_t^2 - c^2\partial_x^2\right) U &= f(t,x) - f(t,x) = 0 \\ U(0,x) &= 0 \\ \partial_t U(0,x) &= 0 \end{aligned}$$

  We already know that the unique solution for this system. Since $g(x) = h(x) = 0$, we have $U = 0$ meaning $u = \tilde{u}$.

Now, we can prove correctness by verification. We want to show that

$$u(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$$

solves the system where $g = h = 0$. To simplify the formula, let

$$F(t,s,x) := \int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y$$

So, we want to show

$$u(t,x) = \frac{1}{2c}\int_0^t F(t,s,x)\mathrm{d}s$$

solves the system. First, verify the bounds.

$$\begin{aligned}
\partial_t u(t,x) &= \partial_t\left(\frac{1}{2c}\int_0^t F(t,s,x)\mathrm{d}s\right) \\
&= \frac{1}{2c}F(t,t,x) + \frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s \\
&= \frac{1}{2c}\int_x^x f(s,y)\mathrm{d}s + \frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s \\
&= \frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s
\end{aligned}$$

This is 0 when $t = 0$, so $\partial_t u(0,x) = 0$ as needed. We can also observe that $u(0,x) = 0$ as needed.

Now, find $\partial_t^2 u(t,x)$.

$$\begin{aligned}
\partial_t^2 u(t,x) &= \partial_t\left(\frac{1}{2c}\int_0^t \partial_t F(t,s,x)\mathrm{d}s\right) \\
&= \frac{1}{2c}\partial_t F(t,t,x) + \frac{1}{2c}\int_0^t \partial_t^2 F(t,s,x)\mathrm{d}s
\end{aligned}$$

We must find $\partial_t F(t,s,x)$ and $\partial_t^2 F(t,s,x)$.

$$\begin{aligned}
\partial_t F(t,s,x) &= \partial_t\left(\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\right) \\
&= c\left(f(s,x+c(t-s)) + f(s,x-c(t-s))\right) \\
\partial_t^2 F(t,s,x) &= c^2\left(\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right)
\end{aligned}$$

Plug this back into the formula for $\partial_t^2 u(t,x)$.

$$\begin{aligned}
\partial_t^2 u(t,x) &= \frac{1}{2c}\left[c\left(f(s,x+c(t-s)) + f(s,x-c(t-s))\right)\right]|_{s=t} \\
&\quad + \frac{1}{2c}(c^2)\left(\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s\right) \\
&= \frac{1}{2}\left(f(s,x+c(0)) + f(s,x-c(0))\right) \\
&\quad + \frac{c}{2}\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s \\
&= f(t,x) + \frac{c}{2}\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s
\end{aligned}$$

Now, find $c^2\partial_x^2 u(t,x)$.

$$c^2\partial_x^2 u(t,x) = c^2\frac{1}{2c}\int_0^t \partial_x^2 F(t,s,x)\mathrm{d}s$$

We must find $\partial_x^2 F(t,s,x)$.

$$\begin{aligned}\partial_x F(t,s,x) &= \partial_x\left(\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\right)\\ &= f(s,x+c(t-s)) - f(s,x-c(t-s))\\ \partial_x^2 F(t,s,x) &= \partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\end{aligned}$$

Plug this back into the equation for $c^2\partial_x^2 u(t,x)$.

$$c^2\partial_x^2 u(t,x) = \frac{c}{2}\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s$$

So, we can plug these values back into the original equation.

$$\begin{aligned}\left(\partial_t^2 - c^2\partial_x^2\right)u(t,x) &= f(t,x) + \frac{c}{2}\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s\\ &\quad - \frac{c}{2}\int_0^t \left[\partial_x f(s,x+c(t-s)) - \partial_x f(s,x-c(t-s))\right]\mathrm{d}s\\ &= f(t,x)\end{aligned}$$

Thus, $u(t,x) = \frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(s,y)\mathrm{d}y\mathrm{d}s$ solves the system $\left(\partial_t^2 - c^2\partial_x^2\right)u = f(t,x)$ where $u(0,x) = \partial_t u(0,x) = 0$. This proves correctness of Duhamel's formula. □

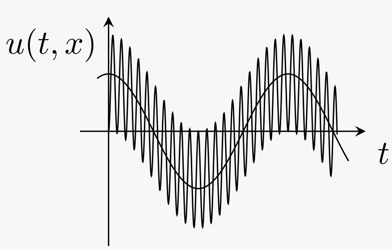
Example

Let $g = h = 0$ and $f(x,t) = (\sin(\omega_0 x))(\cos(\omega t))$. Suppose $c = 1$.

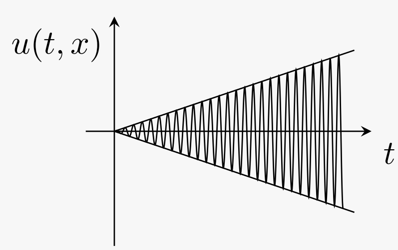
$$\begin{aligned}u(t,x) &= \frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)} \sin(\omega_0 y)\mathrm{d}y\cos(\omega s)\mathrm{d}s\\ &= \frac{1}{2}\int_0^t \left[-\frac{\cos(\omega_0 y)}{\omega_0}\right]_{x-(t-s)}^{x+(t-s)}\cos(\omega s)\mathrm{d}s\\ &= \frac{1}{2}\int_0^t \frac{(\cos(\omega_0(x-(t-s))) - \cos(\omega_0(x+(t-s))))\cos(\omega s)}{w_0}\mathrm{d}s\\ &= \begin{cases}\frac{\sin(\omega_0 x)}{\omega_0^2-\omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right] & \omega \neq \pm\omega_0\\ \frac{t}{2\omega_0}\sin(\omega_0 t)\sin(\omega_0 x) & \omega = \pm\omega_0\end{cases}\end{aligned}$$

Let's plot these cases for fixed x.

For $\omega \ll \omega_0$:



For $\omega = \omega_0$:



Third Dimension Wave Equation

The homogeneous third dimension wave equation for $c = 1$ is

$$\left(\partial_t^2 - \Delta\right) u = 0$$

where $u \in C^2$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$. To solve this, we want to reduce this to an equation in one dimension.

Lemma: Darboux's Lemma

Suppose $u(t,x)$ solves $\left(\partial_t^2 - \Delta\right) u = 0$. Define

$$\tilde{u}(t,x,\rho) := \frac{1}{4\pi\rho} \int_{\partial B(x,\rho)} u(t,w) \mathrm{d}S(w)$$

where $\partial B(x,\rho)$ is the boundary of the ball of radius ρ centered at x and $S(w)$ is the surface element of integration. Then,

$$\left(\partial_t^2 - \partial_\rho^2\right) \tilde{u}(t,x,\rho) = 0$$

meaning $\tilde{u}$ solves the 1-D wave equation.

Note that

$$\frac{1}{\rho}\tilde{u}(t,x,\rho) = \frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} u(t,w) \mathrm{d}S(w).$$

$\frac{1}{4\pi\rho^2}$ is the area of the sphere of radius ρ. We can use change of coordinates to take the integral over the boundary of the ball of radius ρ centered at 0 instead of x.

$$\begin{aligned}
\tilde{u}(t,x,\rho) &= \frac{1}{4\pi\rho} \int_{\partial B(0,\rho)} u(t,x+w) \mathrm{d}S(w) \\
&= \frac{1}{4\pi\rho} \int_{\partial B(0,\rho)} (u(t,x) + \underbrace{\varepsilon(t,x\rho)}_{\to 0 \text{ as } \rho \to 0}) \mathrm{d}S \\
&= \frac{\rho}{4\pi\rho} \int_{\partial B(0,\rho)} u(t,x) \mathrm{d}S
\end{aligned}$$

We have $\frac{1}{4\pi\rho^2} \int_{\partial B(0,\rho)} \mathrm{d}S = 1$. So, as $\rho \to 0$,

$$\frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} u(t,w) \mathrm{d}S(w) \to u(t,x)$$

meaning $\tilde{u}(t,x,0) = 0$.

- **Proof** of Darboux's Lemma: we want to show that $\partial_\rho^2 \tilde{u} = \partial_t^2 \tilde{u}$. First, calculate $\partial_\rho \tilde{u}(t,x,\rho)$. Use the change of variables $w = x + \rho y$ where $y \in \partial B(0,1)$. So, $y = \frac{w-x}{y}$.

$$\begin{aligned}
\partial_\rho \left[\frac{1}{\rho}\tilde{u}(t,x,\rho)\right] &= \partial_\rho \left[\frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} u(t,w) \mathrm{d}S(w)\right] \\
&= \partial_\rho \left[\int_{\partial B(0,1)} u(t,x+\rho y) \mathrm{d}S(y)\right] \\
&= \int_{\partial B(0,1)} y \cdot (\nabla u(t,x+\rho y)) \, \mathrm{d}S(y) \\
&= \frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} \left(\frac{w-x}{\rho}\right) \cdot (\nabla u(t,w)) \, \mathrm{d}S(w) \\
&= \frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} \nu(w) \cdot \nabla u(t,w) \mathrm{d}S(w)
\end{aligned}$$

where $\nu(w) = \frac{w-x}{\rho}$ is the outward pointing vector that is normal to $\partial B(x,\rho)$.

To solve for this, use the divergence theorem. For a surface Ω with outward pointing normal vector $\nu(w)$, let $\vec{F} = (a, b, c)$. We have

$$\int_{\partial\Omega} \nu(w) \cdot \vec{F}(w) \mathrm{d}S(w) = \int_{\Omega} \nabla \cdot \vec{F}(w)\, \mathrm{d}^3 w$$

We know

$$\nabla \cdot \vec{F}(w) = \frac{\partial a}{\partial x_1} + \frac{\partial b}{\partial x_2} + \frac{\partial c}{\partial x_3}$$

In our equation, we have $\vec{F}(w) = \nabla u(t, w)$. So,

$$\begin{aligned} \nabla \cdot \vec{F}(w) &= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \\ &= \Delta u \end{aligned}$$

So, we have

$$\begin{aligned} \partial_\rho \left[\frac{1}{\rho}\tilde{u}(t, x, \rho)\right] &= \frac{1}{4\pi\rho^2} \int_{B(x,\rho)} \Delta u(t, w)\, \mathrm{d}^3 w \\ \partial_\rho \tilde{u}(t, x, \rho) &= \partial_\rho \left[\rho \frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} u(t, w) \mathrm{d}S(w)\right] \\ &= \frac{1}{4\pi\rho^2} \int_{\partial B(x,\rho)} u(t, w) \mathrm{d}S(w) + \rho \cdot \frac{1}{4\pi\rho^2} \int_{B(x,\rho)} \Delta u\, \mathrm{d}^3 w \\ \partial_\rho^2 \tilde{u}(t, x, \rho) &= \frac{1}{4\pi\rho^2} \int_{B(x,\rho)} \Delta u(t, w)\, \mathrm{d}^3 w - \frac{1}{4\pi\rho^2} \int_{B(x,\rho)} \Delta u(t, w)\, \mathrm{d}^3 w \\ &\quad + \frac{1}{4\pi\rho} \partial_\rho \int_{B(x,\rho)} \Delta u(t, w)\, \mathrm{d}^3 w \\ &= \frac{1}{4\pi\rho} \partial_\rho \int_{B(x,\rho)} \Delta u(t, w)\, \mathrm{d}^3 w \end{aligned}$$

For any function $f(w)$, suppose we use the change of variables $w = x + \rho y$ where $y \in B(0, 1)$.

$$\begin{aligned} \partial_\rho \int_{B(x,\rho)} f(w)\, \mathrm{d}^3 w &= \partial_\rho \left[\rho^3 \int_{B(0,1)} f(x + \rho y)\, \mathrm{d}^3 y\right] \\ &= 3\rho^2 \int_{B(0,1)} f(x + \rho y)\, \mathrm{d}^3 y + \rho^3 \int_{B(0,1)} y \cdot \nabla f(x + \rho y)\, \mathrm{d}^3 y \end{aligned}$$

We have $y \cdot \nabla f(x + \rho y) = \nabla_y (y f(x + \rho y)) - \frac{3}{\rho} f(x + \rho y)$ since $\nabla_y \cdot (y f(x + \rho y)) = 3f + \rho y \nabla f(x + \rho y)$.

$$\begin{aligned} &= 3\rho^2 \int_{B(0,1)} f(x + \rho y)\, \mathrm{d}^3 y - \rho^3 \left(\frac{3}{\rho}\right) \int_{B(0,1)} f(x + \rho y)\, \mathrm{d}^3 y \\ &\quad + \rho^2 \int_{B(0,1)} \nabla \cdot (y f(x + \rho y))\, \mathrm{d}^3 y \\ &= \rho^2 \int_{B(0,1)} \nabla \cdot (y f(x + \rho y))\, \mathrm{d}^3 y \end{aligned}$$

Let $\vec{F}(y) = y f(x + \rho y)$. By the divergence theorem, we have

$$\begin{aligned} &= \rho^2 \int_{\partial B(0,1)} \nu(y) \cdot y f(x + \rho y) \mathrm{d}S(y) \\ &= \int_{\partial B(x,\rho)} f(w) \mathrm{d}S(w) \end{aligned}$$

where $\nu(y) = y$.

So, we have

$$\partial_\rho^2 \tilde{u}(t,x,\rho) = \frac{1}{4\pi\rho}\int_{\partial B(x,\rho)} \Delta u(t,w)\mathrm{d}S(w)$$

We know that $u(t,x)$ solves $\left(\partial_t^2 \Delta\right) u = 0$, or $\Delta u(t,x) = \partial_t^2 u(t,x)$.

$$\begin{aligned}
&= \frac{1}{4\pi\rho}\int_{\partial B(x,\rho)} \partial_t^2 u(t,w)\mathrm{d}S(w) \\
&= \partial_t^2 \left[\frac{1}{4\pi\rho}\int_{\partial B(x,\rho)} u(t,w)\mathrm{d}S(w)\right] \\
&= \partial_t^2 \tilde{u}(t,x,\rho)
\end{aligned}$$

So, $\left(\partial_t^2 - \partial_\rho^2\right)\tilde{u}(t,w,\rho) = 0$. □

We can use Darboux's Lemma to solve for the wave equation in 3-D.

Theorem: Kirchhoff's Formula

The system

$$\begin{aligned}
\left(\partial_t^2 - \Delta_x\right) u &= 0 \\
u(0,x) &= g(x) \in C^2(\mathbb{R}^3) \\
\partial_t u(0,x) &= h(x) \in C^1(\mathbb{R}^3)
\end{aligned}$$

is solved by $u(t,x)$ where B is the ball in 3 dimensions:

$$u(t,x) = \partial_t \left[\frac{1}{4\pi t}\int_{\partial B(x,t)} g(w)\mathrm{d}S(w)\right] + \frac{1}{4\pi t}\int_{\partial B(x,t)} h(w)\mathrm{d}S(w)$$

- **Proof**: use Darboux's Lemma to rewrite the system of equations as $\left(\partial_t^2 - \partial_\rho^2\right)\tilde{u}(t,x,\rho) = 0$ where

$$\begin{aligned}
\tilde{u}(0,x,\rho) &= \tilde{g}(x,\rho) = \frac{1}{4\pi\rho}\int_{\partial B(x,\rho)} g(w)\mathrm{d}S(w) \\
\partial_t \tilde{u}(0,x,\rho) &= \tilde{h}(x,\rho) = \frac{1}{4\pi\rho}\int_{\partial B(x,\rho)} h(w)\mathrm{d}S(w) \\
\tilde{u}(t,x,0) &= 0
\end{aligned}$$

Take the odd extension of $\tilde{g}$ and $\tilde{h}$ in ρ.

$$\tilde{u}(\rho) = \begin{cases} \tilde{g}(\rho) & \rho \geq 0 \\ -\tilde{g}(-\rho) & \rho \leq 0 \end{cases} \qquad \tilde{h}(\rho) = \begin{cases} \tilde{h}(\rho) & \rho \geq 0 \\ -\tilde{h}(-\rho) & \rho \leq 0 \end{cases}$$

This defines $\tilde{g}$ and $\tilde{h}$ on $\mathbb{R}$. Apply the 1-D formula.

$$\tilde{u}(t,x,\rho) = \frac{1}{2}\left(\tilde{g}(x,\rho+t) + \tilde{g}(x,\rho-t)\right) + \frac{1}{2}\int_{\rho-t}^{\rho+t} \tilde{h}(x,s)\mathrm{d}s$$

We take ρ small $(0 \leq \rho < t)$. Since $\tilde{g}$ is an odd extension, we have

$$\begin{aligned}
\tilde{u}(t,x,\rho) &= \frac{1}{2}\left(\tilde{g}(x,\rho+t) - \tilde{g}(x,t-\rho)\right) + \frac{1}{2}\int_{\rho-t}^{\rho+t} \tilde{h}(x,s)\mathrm{d}s \\
u(t,x) &= \lim_{\rho\to 0}\frac{1}{2\rho}\left[\left(\tilde{g}(x,t+\rho) - \tilde{g}(x,t-\rho)\right) + \left(\int_0^{t+\rho} \tilde{h}(x,s)\mathrm{d}s - \int_0^{t-\rho} \tilde{h}(x,s)\mathrm{d}s\right)\right] \\
&= \partial_\rho \tilde{g}(x,\rho) + \tilde{h}(x,\rho)
\end{aligned}$$

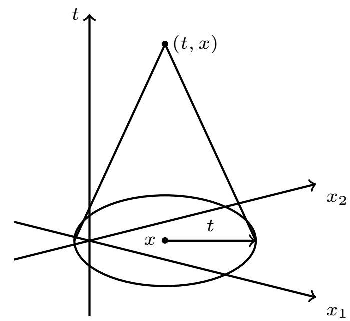
Substituting the definitions of $\tilde{g}(x,\rho)$ and $\tilde{h}(x,\rho)$ yields Kirchoff's Formula. □

The Light Cone

We can illustrate this, although we can only plot in two dimensions. Although there is an additional x_3 axis, we cannot plot it. So, the circle on the x plane is actually the boundary $\partial B(t, x)$, which is a sphere.

We consider the case where $g = 0$. So, we have

$$u(t, x) = \frac{1}{4\pi t} \int_{\partial B(x,t)} h(w) \mathrm{d}S(w).$$



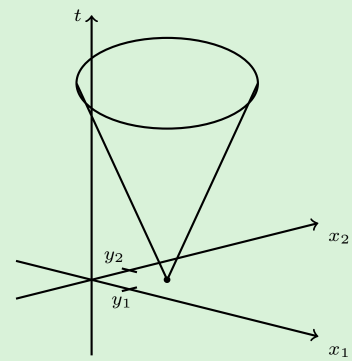
Theorem

Let

$$\Gamma_+(y) = \{(x, t) \mid t \geq 0,\ |x - y| = t\}.$$

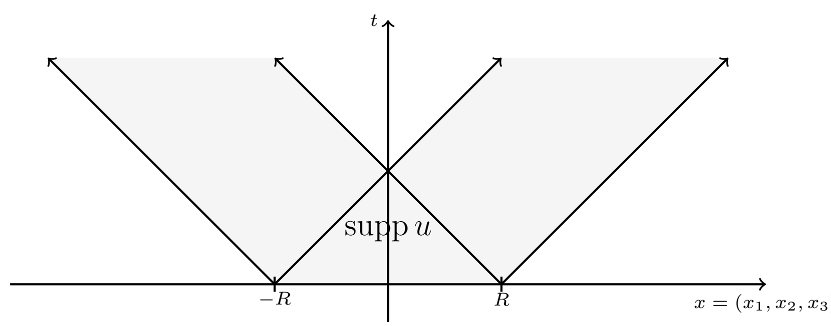
Suppose $g(x) = 0$ and $h(x) = 0$ for $|x| \geq R$ and $|x - y| = t$. Then,

$$\operatorname{supp}_{t\geq 0}(u) \subset \bigcup_{|y|\leq R} \Gamma_+(y) = \{(x, t) \mid -R + t < |x| < R + t\}$$

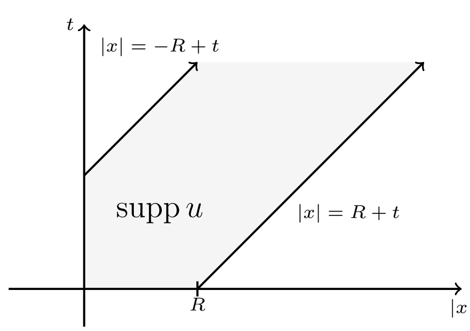


The illustration of the forward light cone is once again in 2 dimensions. Again, there is a third axis x_3, but we can only illustrate x_1 and x_2. The vertical axis represents time and the horizontal plane represents space. It is important to remember that we are illustrating a 3-D space since the result for 2 dimensions is different.

The region of $\operatorname{supp}_{t\geq 0}(u)$ can be illustrated.



Observe the strictly positive x values of this.



Second Dimension Wave Equation

The homogeneous second dimension wave equation for $c = 1$ is

$$\left(\partial_t^2 \Delta\right) u = 0$$

where $u \in C^2$ and $\Delta = \partial_{x_1}^2 \partial_{x_2}^2$. The solution to the second dimension wave equation is given by Poisson's Formula.

Theorem: Poisson's Formula

The system

$$\begin{aligned} \left(\partial_t^2 - \Delta_x\right) u &= 0 \\ u(0,x) &= g(x) \in C^2(\mathbb{R}^2) \\ \partial_t u(0,x) &= h(x) \in C^1(\mathbb{R}^2) \end{aligned}$$

is solved by $u(t,x)$ where B is the ball in 2 dimensions:

$$u(t,x) = \partial_t \left[\frac{t}{2\pi} \int_{B(0,1)} \frac{g(x+tw)}{\left(1-|w|^2\right)^{1/2}} \, \mathrm{d}^2 w \right] + \frac{t}{2\pi} \int_{B(0,1)} \frac{h(x+tw)}{\left(1-|w|^2\right)^{1/2}} \, \mathrm{d}^2 w$$

- **Proof**: To solve this, we will use the method of descent using Kirchhoff's formula in 3-D to solve Poisson's formula in 2-D. Let

$$\begin{aligned} g &= g(x_1, x_2) \in C^2(\mathbb{R}^3) \\ h &= h(x_1, x_2) \in C^1(\mathbb{R}^3) \end{aligned}$$

meaning g and h are functions in $\mathbb{R}^3$, but they do not depend on the third variable. Solve this system using Kirchhoff's Formula and set $x_3 = 0$:

$$u(t, x_1, x_2) = \partial_t \left[\frac{1}{4\pi t} \int_{\partial B_3((x_1,x_2,0),t)} g(y_1,y_2) \mathrm{d}S(y) \right] + \frac{1}{4\pi t} \int_{\partial B_3((x_1,x_2,0),t)} h(y_1,y_2) \mathrm{d}S(y)$$

We want to write this in 2 dimensions. Let $y_3 \geq 0$. Then, we can solve for where y_3 is any value on the LHS and $y_3 \geq 0$ on the RHS:

$$\frac{1}{4\pi t} \int_{\partial B_3((x_1,x_2,0),t)} h(y_1,y_2) \mathrm{d}S(y) = \frac{1}{2\pi t} \int_{\partial B_3((x_1,x_2,0),t)} h(y_1,y_2) \mathrm{d}S(y)$$

We have $y_3 = \sqrt{t^2 - (y_1^2 + y_2^2)}$ where $(y_1, y_2) \in B_2((x_1,x_2),t)$. We can use change of variables $g = \sqrt{t^2 - |y-x|^2}$. So, we have

$$\begin{aligned} \nabla g &= \frac{-(y-x)}{\sqrt{t^2 - |y-x|^2}} \\ 1 + |\nabla g|^2 &= 1 + \frac{|y-x|^2}{t^2 - |y-x|^2} \\ &= \frac{t^2}{t^2 - |y-x|^2} \end{aligned}$$

Plugging this back into the original equation for $y_3 \geq 0$, we have

$$\frac{1}{2\pi t} \int_{\partial B_3((x_1,x_2,0),t)} h(y_1,y_2) \mathrm{d}S(y) = \frac{1}{2\pi t} \int_{B_2((x_1,x_2),t)} h(y) \frac{t}{\left(t^2 - |y-x|^2\right)^{1/2}} \, \mathrm{d}^2 y$$

We can use the same process with the same result for the integral of $g(y_1, y_2)$. Plugging these back into the formula for u, we have

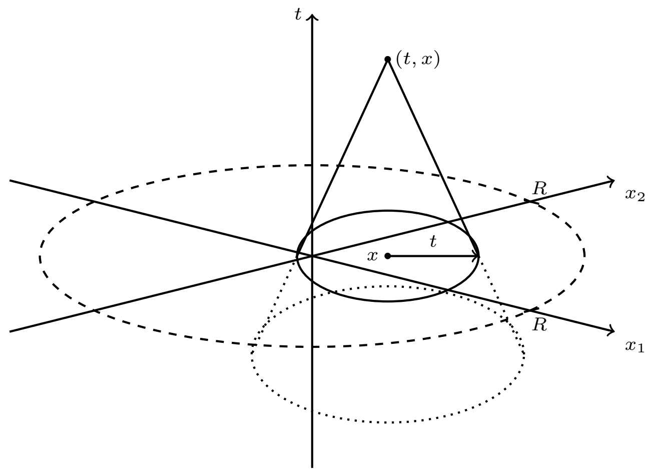
$$u(t,x_1,x_2) = \partial_t \left[\frac{1}{2\pi}\int_{B_2(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}^2y\right] + \frac{1}{2\pi}\int_{B_2(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}^2y$$

Using the change of variables $y = x + tw$ where $\mathrm{d}^2y = t^2\,\mathrm{d}^2w$, we have

$$u(t,x) = \partial_t \left[\frac{t}{2\pi}\int_{B_2(0,1)} \frac{g(x+tw)}{(1-|w|^2)^{1/2}}\,\mathrm{d}^2w\right] + \frac{t}{2\pi}\int_{B_2(0,1)} \frac{h(x+tw)}{(1-|w|^2)^{1/2}}\,\mathrm{d}^2w$$

where $\mathrm{d}S = \frac{t\mathrm{d}^2y}{(t^2-|y-x|^2)^{1/2}}$. □

We can visualize this solution. Suppose $g(x) = 0$ and $h(x) = 0$ for $|x| \le R$ for some $R \in \mathbb{R}_{>0}$. Fix $x_0 \in \mathbb{R}^2$.



Observe $u(t,x_0)$ as $t \to \infty$.

$$u(t,x_0) = \frac{1}{2\pi t}\int_{B_2(x_0,t)} h(y)\left(1-\frac{|x-y|^2}{t^2}\right)^{-1/2}\mathrm{d}^2y$$

We take t large, so $t > |x_0| + R$. $B_2(0,R) \subset B_2(x_0,t)$ since $|y| \le R$, so $|x_0 - y| \le |x_0| + |y| \le |x_0| + R < t$.

$$= \frac{1}{2\pi t}\int_{\mathbb{R}^2} h(y)\left(1-\frac{|x-y|^2}{t^2}\right)^{1/2}\mathrm{d}^2y$$

We have $q = \frac{|x-y|^2}{t^2} \to 0$ as $t \to \infty$ since x is fixed and w is bounded. We want to approximate $(1-q)^{-1/2}$.

$$(1+q)^a = 1 + aq + \frac{a(a-1)}{2}q^2 + \frac{a(a-1)(a-2)}{3!}q^3 + \cdots$$
$$(1-q)^{-1/2} = 1 + \frac{1}{2}q + \frac{3}{8}q^2 - \cdots$$

So, we have

$$u(t,x_0) = \frac{1}{2\pi t}\int_{\mathbb{R}^2} h(y)\left[1+\frac{1}{2}\frac{|x-y|^2}{t^2}+\frac{3}{8}\frac{|x-y|^4}{t^4}+\cdots\right]\mathrm{d}y$$
$$= \frac{1}{2\pi t}\int_{\mathbb{R}^2} h(y)\mathrm{d}y + E(t,x)$$

where $E(t,x)$ is the error term bounded by $\frac{1}{t^3}$. The wave decays at approximately $\frac{1}{t}$, which is far slower than the decay rate in 3-D.

Summary of Wave Equations

Recall that the general form of the wave equation for $c = 1$ is

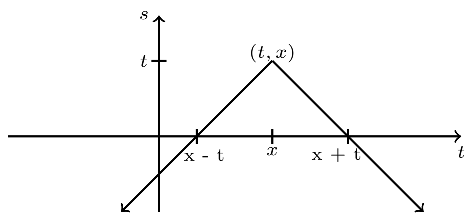
$$\left(\partial_t^2 - \Delta\right) u = 0$$

with initial conditions $u(0, x) = g(x) \in C^2(\mathbb{R}^n)$ and $\partial_t u(0, x) = h(x) \in C^1(\mathbb{R}^n)$ where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$.

- For $n = 1$, the wave equation is solved by

$$u(t, x) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\mathrm{d}s$$

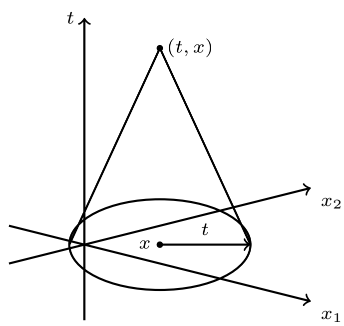
The value of $u(t, x)$ is determined only by the values between $x - t$ and $x + t$.



- For $n = 2$, the wave equation is solved by

$$\begin{aligned} u(t, x) &= \partial_t\left[\frac{1}{2\pi}\int_{B(x,t)} \frac{g(w)}{\left(t^2 - |x-w|^2\right)^{1/2}}\,\mathrm{d}^2 w\right] + \frac{1}{2\pi}\int_{B(x,t)} \frac{h(w)}{\left(t^2 - |x-w|^2\right)^{1/2}}\,\mathrm{d}^2 w \\ &= \partial_t\left[\frac{t}{2\pi}\int_{B(0,1)} \frac{g(x+ty)}{\left(1 - |y|^2\right)^{1/2}}\,\mathrm{d}^2 y\right] + \frac{t}{2\pi}\int_{B(0,1)} \frac{h(x+ty)}{\left(1 - |y|^2\right)^{1/2}}\,\mathrm{d}^2 y \end{aligned}$$

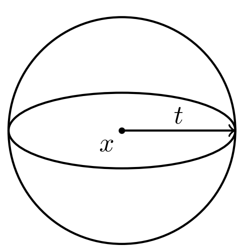
The value of $u(t, x)$ is determined only by the values on the disk of radius t centered at x.

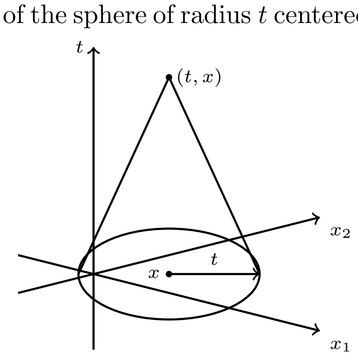


- For $n = 3$, the wave equation is solved by

$$u(t, x) = \partial_t\left[\frac{1}{4\pi t}\int_{\partial B(x,t)} g(y)\mathrm{d}S(y)\right] + \frac{1}{4\pi t}\int_{\partial B(x,t)} h(y)\mathrm{d}S(y)$$

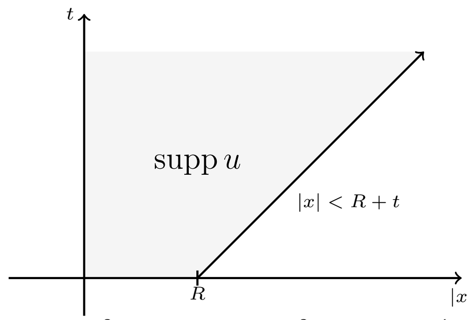
The value of $u(t, x)$ is determined only by the values on the surface of the sphere of radius t centered at x.



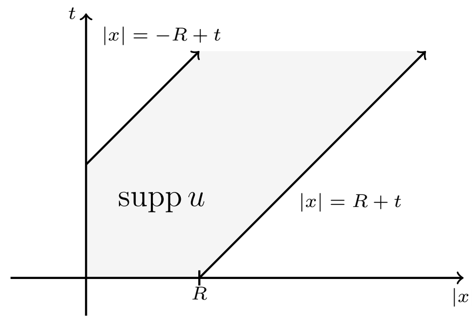


If $g(x) = 0$ and $h(x) = 0$ for $|x| > R$, then for $|x| > R + t$:

For $n = 1, 2$:



For $n = 3$:



To have $u \in C^2$, we need $g \in C^2$ and $h \in C^1$.

Example

Let $n = 2$ and $g(x) = 0$, where

$$h(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

So, we have

$$u(t, 0) = \begin{cases} \frac{1}{2\pi} \int_{B(0,t)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w & t \leq 1 \\ \frac{1}{2\pi} \int_{B(0,1)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w & t > 1 \end{cases}$$

Let $w = t\,(r\cos(\theta), r\sin(\theta))$. Then, we have $|w|^2 = t^2 r^2$ meaning $r = \frac{|w|}{t}$ and $\mathrm{d}^2 w = t^2 r \mathrm{d}r \mathrm{d}\theta$. Solve for each condition of $u(t, 0)$. For $t \leq 1$, we have $|w| \leq t \leq 1$ meaning $r \leq 1$.

$$\begin{aligned} \frac{1}{2\pi} \int_{B(0,t)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{1}{(t^2 - t^2 r^2)^{1/2}} t^2 r \mathrm{d}r \mathrm{d}\theta \\ &= t \int_0^1 \frac{r}{(1 - r^2)^{1/2}} \mathrm{d}r \end{aligned}$$

Use u substitution: $s = r^2$, $\mathrm{d}s = 2r$.

$$\begin{aligned} &= \frac{t}{2} \int_0^t \frac{1}{(1 - s)^{1/2}} \mathrm{d}s \\ &= t \end{aligned}$$

Now, for $t > 1$, we have $|w| \geq t > 1$ meaning $0 \leq r < \frac{1}{t}$.

$$\frac{1}{2\pi} \int_{B(0,1)} \frac{1}{(t^2 - |w|^2)^{1/2}} \, \mathrm{d}^2 w = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{1/t} \frac{t}{(1 - r^2)^{1/2}} r \mathrm{d}r \mathrm{d}\theta$$

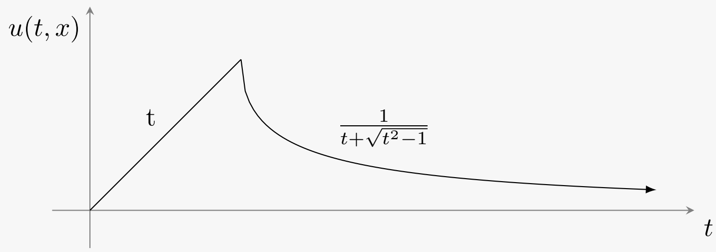
Use the same u substitution parameters.

$$\begin{aligned} &= \frac{t}{2} \int_0^{1/t^2} \frac{1}{(1 - s)^{1/2}} \mathrm{d}s \\ &= t \left(1 - \left(1 - \tfrac{1}{t^2}\right)^{1/2}\right) \\ &= t - \sqrt{t^2 - 1} \\ &= \frac{1}{t + \sqrt{t^2 - 1}} \\ &\approx \frac{1}{2t} \text{ as } t \to \infty \end{aligned}$$

So, we have our formula for $u(t,0)$:

$$u(t,0) = \begin{cases} t & t \le 1 \\ \dfrac{1}{t + \sqrt{t^2 - 1}} & t > 0 \end{cases}$$

We can plot this solution:



We can repeat this process for $n = 3$. We still have $g(x) = 0$ and

$$h = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Similarly, we can find $u(t,0)$.

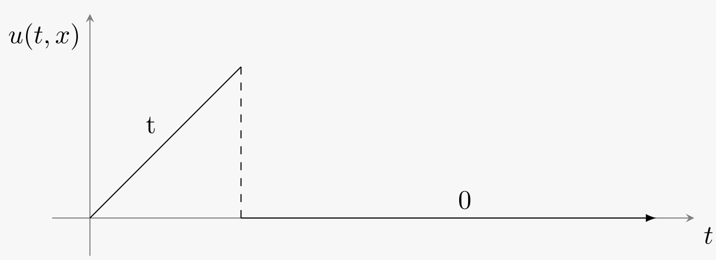
$$u(t,0) = \begin{cases} \dfrac{1}{4\pi t} \displaystyle\int_{\partial B(x,t)} 1 \mathrm{d}S(y) & t \le 1 \\ 0 & t > 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{4\pi t} \cdot 4\pi t^2 & t \le 1 \\ 0 & t > 1 \end{cases}$$

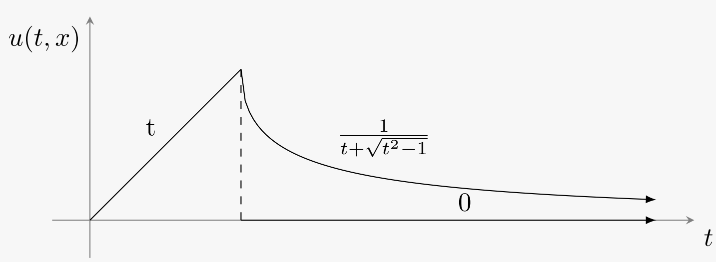
where $4\pi t^2$ is the surface area of the sphere of radius t

$$= \begin{cases} t & t \le 1 \\ 0 & t > 1 \end{cases}$$

We can plot this solution as well.



Observe the difference between the two plots. In 2 dimensions, after passing a certain time, the wave slowly decays while in 3 dimensions, the wave immediately disappears.



Energy

Given the wave equation

$$\left(\partial_t^2 - \Delta\right) u = 0$$

with $u(t,0) = g \in C^2$ and $\partial_t u(t,0) = h \in C^1$, u exists with $u \in C^2$. For $g(x) = h(x)$ for $|x| \geq R$ for some R, we have $u(t,x) = 0$ for $|x| \geq R + t$. Then,

$$E(u)[t] = \frac{1}{2} \int_{\mathbb{R}^n} \left[(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2 \right] \mathrm{d}^n x$$

is the **energy** of a solution $u(t,x)$ at time t. This makes sense since $u = 0$ for $|x| > R + t$ and $u \in C^2$.

Theorem

Under the assumptions made above,

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left[(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2 \right] \mathrm{d}^n x$$

is independent of t.

- **Proof**: it suffices to show that $\partial_t E(u) = 0$, meaning $E(u)$ is not a function of t.

$$\begin{aligned} \partial_t E(u) &= \partial_t \left[\frac{1}{2} \int_{\mathbb{R}^n} \left[(\partial_t u(t,x))^2 + |\nabla u(t,x)|^2 \right] \mathrm{d}^n x \right] \\ &= \partial_t \left[\frac{1}{2} \int_{\mathbb{R}^2} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial u}{\partial x_n}\right)^2 \right] \mathrm{d}^n x \right] \end{aligned}$$

Evaluate the components separately.

$$\begin{aligned} \partial_t \left(\left(\frac{\partial u}{\partial t}\right)^2 \right) &= 2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} \\ \partial_t \left(\left(\frac{\partial u}{\partial x_j}\right)^2 \right) &= 2 \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial t} \end{aligned}$$

So, we have

$$\partial_t E(u) = \int_{\mathbb{R}^n} \left[\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x_1} \frac{\partial^2 u}{\partial x_1 \partial t} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial^2 u}{\partial x_n \partial t} \right] \mathrm{d}^n x$$

To evaluate $\int_{\mathbb{R}^n} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} \frac{\partial u}{\partial t} \mathrm{d}x_1 \cdots \underline{\mathrm{d}x_j} \cdots \mathrm{d}x_n$, use integration by parts.

$$\int_{\mathbb{R}} \frac{\partial w}{\partial y} v \mathrm{d}y = - \int_{\mathbb{R}} w \frac{\partial v}{\partial y} \mathrm{d}y$$

since $\int \partial_y (u,v) \mathrm{d}y = \int \left[\partial_y u \cdot v + u \cdot \partial_y v\right] \mathrm{d}y$ and $u, v = 0$ for $|y| > R$. So, we have

$$\int_{\mathbb{R}^n} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} \frac{\partial u}{\partial t} \mathrm{d}x_1 \cdots \underline{\mathrm{d}x_j} \cdots \mathrm{d}x_n = - \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial x_j^2} \frac{\partial u}{\partial t} \, \mathrm{d}^n t$$

meaning

$$\begin{aligned} \partial_t E(u) &= \int_{\mathbb{R}^n} \frac{\partial u}{\partial t} \left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2} \right) \mathrm{d}^n x \\ &= 0 \end{aligned}$$

So, $\partial_t E(u) = 0$ meaning $E(u)$ is not a function of t. □

Recall that for $\Omega \subset \mathbb{R}^n$ bounded and $\partial\Omega$ piecewise C^1, $u \in C^2(\mathbb{R}_+ \times \overline{\Omega})$ solves $\left(\partial_t^2 - \Delta\right) u = 0$. Suppose $u(t,x) = 0$ when $g \geq 0$ and $x \in \partial\Omega$.

Theorem

Define

$$E(u) = \frac{1}{2} \int_\Omega |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 \, \mathrm{d}^n x \,.$$

Then, $E(u)$ is independent of t.

The proof of this is the same as that of the previous theorem except that instead of using integration by parts, we use the divergence theorem.

Theorem

Suppose $u \in C^2(\mathbb{R}_+ \times \overline{\Omega})$ solves $\left(\partial_t^2 - \Delta\right) u = f(t,x)$ with $u(0,x) = g(x) \in C^2$, $\partial_t u(0,x) = h(x) \in C^1$, and $u(t,x) = 0$ for $t \geq 0$ and $x \in \partial\Omega$. Then, u is unique.

- **Proof**: suppose that $\tilde{u}$ is another solution that satisfies the conditions. Consider $U(t,x) = u(t,x) - \tilde{u}(t,x)$. Then, we have

  $$\left(\partial_t^2 - \Delta\right) U = 0$$

  with $U(0,x) = 0$ and $\partial_t U(0,x) = 0$. From the previous theorem,

  $$\begin{aligned} \int |\partial_t U(t,x)|^2 + |\nabla U(t,x)|^2 \, \mathrm{d}^n x &= \int \underbrace{|\partial_t U(0,x)|^2}_{=0} + \underbrace{|\nabla U(0,x)|^2}_{=0} \mathrm{d}^n x \\ &= 0 \end{aligned}$$

  So, $U(t,x) = 0$ meaning $u(t,x) = \tilde{u}(t,x)$, or u is unique. $\square$

Lemma: Green's Identity

$$\int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}^n x = -\int v \Delta u \mathrm{d}x + \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S$$

where ν is the outward pointing normal vector.

- **Proof**: let $F = v \cdot \nabla u$. Then, $\operatorname{div}(F) = \nabla u \cdot \nabla v + v\Delta u$. We can apply the divergence theorem in reverse from this.

  $$\begin{aligned} \nabla \cdot (v \cdot \nabla u) &= \nabla v \cdot vu + v \cdot \nabla \cdot \nabla u \\ &= \nabla v \cdot \nabla u + v \Delta u \end{aligned}$$

  So,

  $$\begin{aligned} \int_\Omega (v\Delta u + \nabla v \cdot \nabla u) \, \mathrm{d}^n x &= \int_\Omega \nabla \cdot (v \cdot \nabla u) \, \mathrm{d}^n x \\ &= \int_{\partial\Omega} \partial_\nu \left(v \cdot \nabla u\right) \mathrm{d}S && \text{Divergence Theorem} \\ &= \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S \\ \int_\Omega v \Delta u \mathrm{d}x + \int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}^n x &= \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S \\ \int_\Omega \nabla v \cdot \nabla u \mathrm{d}^n x &= -\int v \Delta u \mathrm{d}x + \int_{\partial\Omega} v \cdot \partial_\nu u \mathrm{d}S \end{aligned}$$ $\square$

Separation of Variables

Separation of variables is a technique used to reduce PDEs to ODEs or simpler PDEs.

Proposition

Suppose $u \in C^2([a,b] \times \Omega)$ solves the wave equation $\left(\partial_t^2 - \Delta\right) u = 0$ and

$$u(t,x) = v(t)\varphi(x)$$

Then, in any set where $u \neq 0$, v and φ satisfy

$$\begin{aligned} \partial_t^2 v(t) &= \kappa v(t) \\ \Delta\varphi(x) &= \kappa\varphi(x) \end{aligned}$$

for some constant κ.

- **Proof**: by plugging $u(t,x) = v(t)\varphi(x)$ into the wave equation, we have

$$\begin{aligned} \left(\partial_t^2 - \Delta\right)(v(t)\varphi(x)) &= 0 \\ \partial_t^2 v(t)\varphi(x) - v(t)\Delta\varphi(x) &= 0 \\ \partial_t^2 v(t)\varphi(x) &= v(t)\Delta\varphi(x) \end{aligned}$$

  $u \neq 0$, so $v(t), \varphi(x) \neq 0$.

$$\frac{\partial_t^2 v(t)}{v(t)} = \frac{\Delta\varphi(x)}{\varphi(x)}$$

  The LHS is independent of x and the RHS is independent of t. So,

$$\frac{\partial_t^2 v(t)}{v(t)} = \frac{\Delta\varphi(x)}{\varphi(x)} = \text{constant}$$

  Let this constant be κ.

$$\begin{aligned} \partial_t^2 v(t) &= \kappa v(t) \\ \Delta\varphi(x) &= \kappa\varphi(x) \end{aligned} \qquad \square$$

This is useful since $\partial_t^2 v = \kappa v$ is easy to solve:

$$v(t) = Ae^{\sqrt{\kappa}t} + Be^{-\sqrt{\kappa}t}$$

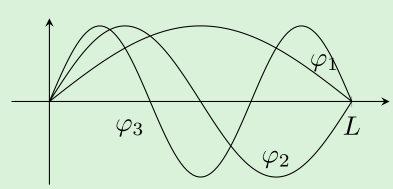
$\Delta\varphi = \kappa\varphi$ may be more difficult to solve depending on the degree of x.

1-D Case

Theorem

The following two statements are equivalent (meaning the first is true if and only if the second is also true).

1. $\varphi \in C^2([0,L])$, $\varphi \neq 0$, $\varphi(0) = \varphi(L) = 0$, and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x) = \lambda\varphi(x)$
2. $\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $\varphi = \varphi_n = A\sin\left(\sqrt{\lambda_n}x\right)$ for $n \in \mathbb{N}$



So, we have $v(t) = Ae^{i\sqrt{\lambda_n}t} + Be^{-i\sqrt{\lambda_n}t}$. This yields a single frequency solution

$$u_n(t,x) = \left(Ae^{i\sqrt{\lambda_n}t} + Be^{-i\sqrt{\lambda_n}t}\right)\sin\left(\sqrt{\lambda_n}x\right)$$

- **Proof**: in the reverse direction, we have

$$\begin{aligned} -\partial_x^2\varphi_n &= -\partial_x^2\left[A\sin\left(\sqrt{\lambda_n}x\right)\right] & \varphi(0) &= 0 \\ &= -\partial_x\left(A\sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n}x\right)\right) & \varphi(L) &= A\lambda_n\sin\left(\sqrt{\lambda_n}L\right) \\ &= A\lambda_n\sin\left(\sqrt{\lambda_n}x\right) & &= A\lambda_n\sin\left(\frac{n\pi}{L}\cdot L\right) \\ &= \lambda\varphi_n & &= 0 \end{aligned}$$

So, if 2. is true, then 1. is true.

Now, in the forward direction, we first want to show that $\lambda > 0$.

$$\begin{aligned} -\frac{\mathrm{d}^2}{\mathrm{d}x^2}|\varphi|^2 &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi\cdot\overline{\varphi} \\ &= \lambda\varphi\overline{\varphi} \\ -\int_0^L\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi\right)\overline{\varphi}\mathrm{d}x &= \int_0^L\lambda\varphi\overline{\varphi}\mathrm{d}x \end{aligned}$$

Evaluate both sides separately.

$$\int_0^L\lambda\varphi\overline{\varphi}\mathrm{d}x = \lambda\int_0^L|\varphi(x)|^2\mathrm{d}x$$

For the LHS, use integration by parts. For any u and v, we have

$$-\int_0^L\frac{\mathrm{d}}{\mathrm{d}x}u\cdot v = u(0)v(0) - u(L)v(L) + \int_0^L u\cdot\frac{\mathrm{d}}{\mathrm{d}x}v$$

In our case, $u = \frac{\mathrm{d}}{\mathrm{d}x}\varphi$ and $v = \overline{\varphi}$. We know $v(0) = v(L) = 0$, so

$$-\int_0^L\left(\frac{\mathrm{d}}{\mathrm{d}x}\varphi\right)\overline{\varphi} = \int_0^L\left|\frac{\mathrm{d}}{\mathrm{d}}\varphi(x)\right|^2\mathrm{d}x$$

So, we have

$$\underbrace{\int_0^L\left|\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\right|^2\mathrm{d}x}_{\geq 0} = \lambda\underbrace{\int_0^L|\varphi(x)|^2\mathrm{d}x}_{\geq 0}$$

So, we know $\lambda \geq 0$. If $\lambda = 0$, then $\frac{\mathrm{d}\varphi}{\mathrm{d}x} = 0$ and $\varphi(0) = 0$ meaning $\varphi = 0$ which is not allowed. So, $\lambda > 0$.

Now, we have $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi = \lambda\varphi$ where $\lambda > 0$. So, we have

$$\begin{aligned} \varphi(x) &= A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) \\ \varphi(0) &= 0 = B \\ \varphi(L) &= 0 = A\sin\left(\sqrt{\lambda}L\right) \\ \sin\left(\sqrt{\lambda}L\right) &= 0 \end{aligned}$$

$\sin(y) = 0$ when $y = n\pi$ for $n \in \mathbb{Z}$.

$$\begin{aligned} \sqrt{\lambda}L &= n\pi \\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \qquad \text{for } n \in \mathbb{N} \end{aligned}$$

So, if 1. is true, then 2. is true. □

Theorem

The following two statements are equivalent (meaning the first is true if and only if the second is also true).

1. $\varphi \in C^2([0,L])$, $\varphi \neq 0$, $\varphi(0) = \varphi'(L) = 0$, and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi(x) = \lambda\varphi(x)$
2. $\lambda = \lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$ and $\varphi = \varphi_n = A\sin\left(\sqrt{\lambda_n}x\right)$ for $n \in \mathbb{N}$

- **Proof**: this has the same proof as the previous theorem except

$$\varphi'(L) = 0 = A\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)$$

$\cos(y) = 0$ when $y = \left(n + \frac{1}{2}\right)\pi$ for $n \in \mathbb{Z}$.

$$\lambda = \left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}\right)^2 \qquad \square$$

λ_n are eigenvalues of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $[0, L]$ with

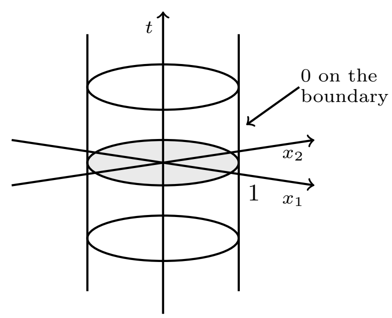
$$\begin{cases} \varphi(0) = \varphi(L) = 0 & \text{Dirichlet boundary conditions} \\ \varphi(0) = \varphi'(L) = 0 & \text{Mixed boundary conditions} \end{cases}$$

2-D Case

In the 2-D case, we have $\left(\partial_t^2 - \Delta\right)u(t,x) = 0$ for $t \in \mathbb{R}$ and $B_{\mathbb{R}^2}(0,1)$. $u(t,x) = 0$ if $|x| = 1$.

Once again, we have $u(t,x) = v(t)\varphi(x)$.

$$\begin{aligned} -\Delta\varphi(x) &= \lambda\varphi(x) && \text{for } |x| < 1 \\ \varphi(x) &= 0 && \text{for } |x| = 1 \\ -\partial_t^2 v(t) &= \lambda v(t) \\ v(t) &= Ae^{i\sqrt{\lambda}t} + Be^{-i\sqrt{\lambda}t} \end{aligned}$$



Lemma

If $\varphi \in C^2(\overline{\Omega})$ and $-\Delta\varphi = \lambda\varphi$, if $\varphi|_{\partial\Omega} = 0$ where $\varphi \neq 0$, then $\lambda > 0$.

- **Proof**: we know $-\Delta\varphi \cdot \overline{\varphi} = \lambda\varphi\overline{\varphi}$, so $\int -\Delta\varphi \cdot \overline{\varphi} = \int \lambda\varphi\overline{\varphi}$. By applying the divergence theorem, we have

$$\operatorname{div}(\overline{\varphi} \cdot \nabla\varphi) = \nabla\overline{\varphi}\nabla\varphi + \overline{\varphi}\Delta\varphi$$

$$\int_\Omega |\nabla\varphi(x)|^2 \,\mathrm{d}x + \int_\Omega \overline{\varphi}\Delta\varphi = \sum \left|\frac{\partial u}{\partial x_j}\right|^2 + \int_{\partial\Omega} \underbrace{\overline{\varphi} \cdot \nu}_{=0} \cdot \nabla\varphi$$

$$= \sum \left|\frac{\partial u}{\partial x_j}\right|^2$$

So, we have

$$0 \leq \int_\Omega |\nabla\varphi(x)|^2 = \lambda \int_\Omega |\varphi(x)|^2$$

Since $\varphi \neq 0$, the inequality is strict and we have $\lambda > 0$. $\square$

Example

Let $\Omega = [0, L_1] \times [0, L_2]$. $-\Delta\varphi = \lambda\varphi$, and we have Dirichlet boundary condition $\varphi|_{\partial\Omega} = 0$. We postulate that $\varphi(x) = \varphi_1(x_1)\varphi_2(x_2)$.

$$-\Delta\varphi = -\left(\partial_{x_1}^2\varphi_1\right)\varphi_2 - \left(\partial_{x_2}^2\varphi_2\right)\varphi_1 = \lambda\varphi_1\varphi_2$$
$$-\frac{\partial_{x_1}^2\varphi_1(x_1)}{\varphi_1(x_1)} = \frac{\partial_{x_2}^2\varphi_2(x_2)}{\varphi_2(x_2)} + \lambda = \kappa$$

So, we can solve for φ_1 and φ_2.

$$-\partial_{x_1}^2\varphi_1(x_1) = \kappa\varphi_1(x_1) \qquad -\partial_{x_2}^2\varphi_2(x_2) = (\lambda-\kappa)\varphi_2(x_2)$$
$$\varphi_1(0) = \varphi_1(L_1) = 0 \qquad \varphi_2(0) = \varphi_2(L_2) = 0$$

These are 1-D equations that we have already solved.

$$\kappa = \left(\tfrac{n\pi}{L_1}\right)^2 \qquad \lambda - \kappa\left(\tfrac{m\pi}{L_2}\right)^2$$
$$\varphi_1(x_1) = \sin\left(\tfrac{n\pi}{L_1}x_1\right) \qquad \varphi_2(x_2) = \sin\left(\tfrac{m\pi}{L_2}x_2\right)$$

So, we have

$$\lambda_{m,n} = \left(\tfrac{n\pi}{L_1}\right)^2 + \left(\tfrac{m\pi}{L_2}\right)^2$$

for $m, n \in \mathbb{N}$.

$-\Delta\varphi = \lambda\varphi$ is the **eignnvalue problem** where $\lambda_{m,n}$ are the **eigenvalues** and φ is the eigenfunction or **eigenvector**.

Separation of Variables in Polar Coordinates

In polar coordinates, we have $x_1 = r\cos(\theta)$ and $x_2 = r\sin(\theta)$. So, we are looking for solutions of the form

$$\varphi(r\cos(\theta), r\sin(\theta)) = f(r)g(\theta)$$

where $-\Delta\varphi = \lambda\varphi$ and $\varphi|_{\partial\Delta=0}$. Consider $\Omega = D(0,1)$ which is the disk of radius 1 centered at (0, 0). We have the following conditions for $f(r)$ and $g(\theta)$.

$$f(1) = 0$$
$$g(\theta) = g(\theta + 2\pi)$$

Recall the Laplacian for polar coordinates:

$$\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$$

So, we have

$$-\Delta\left(f(r)g(\theta)\right) = \lambda f(r)g(\theta)$$
$$\left[\left(-\partial_r^2 - \frac{1}{r}\partial_r\right)f(r)\right]g(\theta) - f(r)\frac{1}{r^2}\partial_\theta^2 g(\theta) = \lambda f(r)g(\theta)$$
$$\left(-\partial_r^2 - \frac{1}{r}\partial_r - \lambda\right)f(r)g(\theta) - f(r)\frac{1}{r^2}\partial_\theta^2 g(\theta) = 0$$
$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \lambda\right)f(r)g(\theta) = -f(r)\frac{1}{r^2}\partial_\theta^2 g(\theta)$$
$$\frac{\left(r^2\partial_r^2 + r\partial_r + \lambda r^2\right)f(r)}{f(r)} = -\frac{\partial_\theta^2 g(\theta)}{g(\theta)} = \kappa$$

Solve for $g(\theta)$.

$$-\partial_\theta^2 g(\theta) = \kappa g(\theta)$$
$$g(\theta + 2\pi) = g(\theta)$$

since $g(\theta)$ is 2π-periodic. So, the general solution is

$$\begin{aligned} g(\theta) &= Ae^{i\sqrt{\kappa}\theta} + Be^{-i\sqrt{\kappa}\theta} \\ &= g(\theta + 2\pi) \\ &= Ae^{i\sqrt{\kappa}\theta + 2\pi i\sqrt{\kappa}} + Be^{-i\sqrt{\kappa}\theta - 2\pi i\sqrt{\kappa}} \\ e^{i\theta + 2\pi i\kappa} &= e^{i\kappa\theta} \\ e^{2\pi i\kappa} &= 1 \end{aligned}$$

So, $\sqrt{\kappa} \in \mathbb{Z}$ meaning $k = n^2$ for $n \in \mathbb{Z}$.

Now, solve for $f(r)$.

$$\begin{aligned} \left(r^2\partial_r^2 + r\partial_r + \lambda r^2 - n^2\right) f(r) &= 0 \\ f(1) &= 0 \end{aligned}$$

from the boundary conditions. Use change of variables $\lambda r^2 = z^2$, so $z = \sqrt{\lambda} r$.

$$\begin{aligned} \partial_r^2 &= \left(\frac{\partial z}{\partial r}\right)^2 \partial_z^2 = \lambda \partial_z^2 \\ r\partial_r^2 &= \left(\frac{z}{\lambda}\right)^2 \lambda\partial_z^2 = z^2\partial_z^2 \end{aligned}$$

Let $F(z) = f\left(\frac{z}{\sqrt{\lambda}}\right)$.

$$f(r) = F(\sqrt{\lambda} r)$$

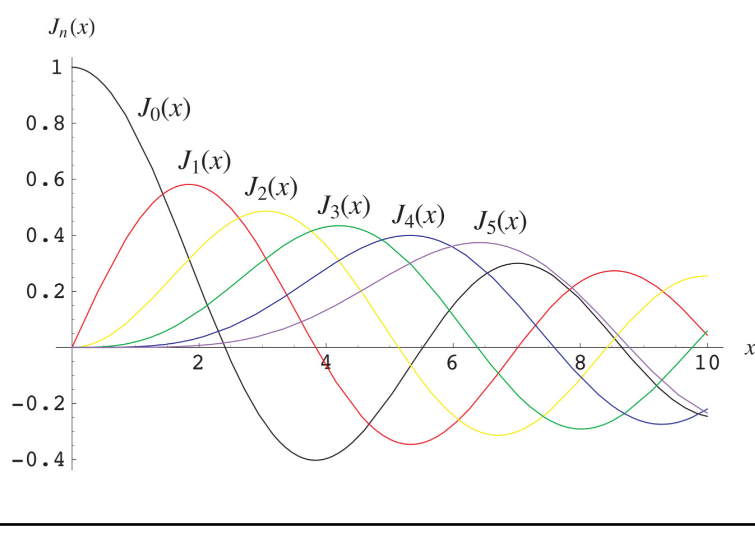
So, we have the system of equations

$$\begin{aligned} \left(z^2\partial_z^2 + z\partial_z + z^2 - n^2\right) F &= 0 \\ F(\sqrt{\lambda}) &= 0 \end{aligned}$$

This is Bessel's Equation. It has 2 independent solutions $J_n(z)$ and $Y_n(z)$ where

$$\begin{aligned} J_n(z)\big|_{z\approx 0} &\approx z^n \\ Y_n(z)\big|_{z\approx 0} &\approx z^{-n} \end{aligned}$$

as $z \to 0$. The plot of J_n can be viewed:



We want λ_n such that $J_n\left(\sqrt{\lambda_{n,k}}\right) = 0$.

$$\varphi^{\pm}_{n,k}(r\cos(\theta), r\sin(\theta)) = e^{\pm in\theta} J_n\left(\sqrt{\lambda_{n,k}}r\right)$$

for $\lambda_{n,k} = j^2_{n,k}$ where $j_{n,1} < j_{n,2} < \cdots$ are the zeros of J_n.

This turns out to be all of the solutions. So, we can solve for $u(t,x)$.

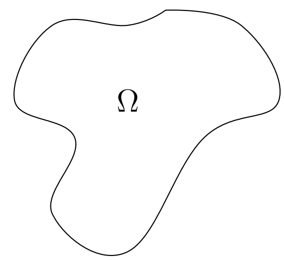
$$u(t,x) = u(r\cos(\theta), r\sin(\theta)) = \left(A\cos\left(\sqrt{\lambda_{n,k}}t\right) + B\sin\left(\sqrt{\lambda_{n,k}}\right)\right) J_k\left(\sqrt{\lambda_{n,k}}r\right)$$

Domains in 2-D

Suppose we have a domain Ω which is associated with the sequence of eigenvalues (frequencies)

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

where the first two inequalities are strict and the rest are not.



These are λ's for which $\exists\varphi \neq 0$ such that $-\Delta\varphi = \lambda\varphi$ and $\varphi|_{\partial\Omega} = 0$. We have found these λ's for two cases:

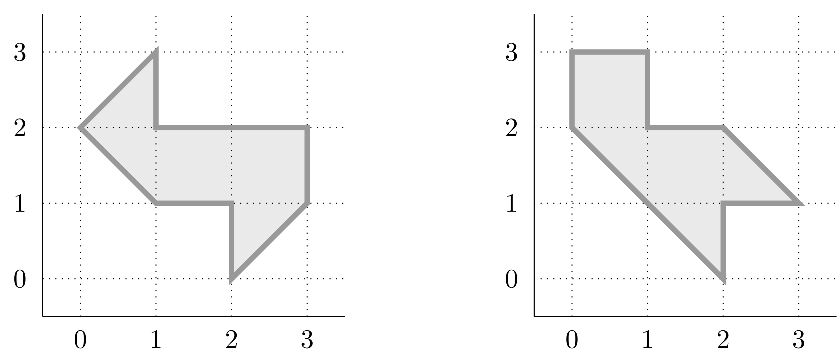
1. For $\Omega = [0, L_1] \times [0, L_2]$,
$$\lambda_{n,m} = \left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2$$
2. For $\Omega = D(0,1)$,
$$\lambda_{n,k} = j^2_{n,k} \text{ where } J_n(j_{n,k}) = 0$$

The λ's are nodes of oscillation over Ω. Each domain has a sequence of λ.

Question: if we have 2 different domains Ω, do we have 2 different sequences?

This question was popularized in the article "Can One Hear the Shape of a Drum?" written by Mark Kac in 1967. The idea is that if we know the λ_n sequences (all frequencies of possible oscillations) of a given domain, can we tell what the domain is? In other words, can we recover the shape of a domain from knowing the frequency of oscillations?

It turns out that multiple different domains can have the same sequence. It is still unknown if there are different smooth domains that have the same sequence. It is also unknown if there are different convex domains with the same sequence.



These two polygons have the same sequence. The reason for this is complex, they are both projected by the same shape in higher dimensions. Note that they are both not smooth and are both not convex.

3-D Case

In 3 dimensions, we have 2 main motivations.

1. Solve the wave equation of a ball with Dirichlet boundary conditions given by

$$\left(\partial_t^2 - \Delta\right) u = 0$$

for $t \geq 0$ and $x \in B_3(0,1)$ where $u(t,x) = 0$ for $x \in \partial B_3(0,1)$. The pure mode mode solutions are

$$u(t,x) = \left(Ae^{it\sqrt{\lambda}} + Be^{-it\sqrt{\lambda}}\right)\varphi(x)$$

where $-\Delta\varphi = \lambda\varphi$ for $\varphi|_{\partial B(0,1)} = 0$.

2. Find the energies of the hydrogen atom given by the the Schrödinger equation

$$-\Delta - \frac{1}{|x|}$$

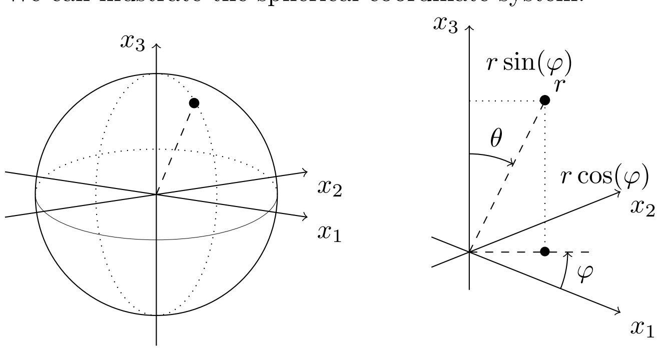
where $1 + \psi = \lambda\psi$ and $\psi \to 0$ as $|x| \to \infty$.

Separation of Variables in Spherical Coordinates

Observe the change of variables from Cartesian to spherical coordinates.

$$(x_1, x_2, x_3) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

We can illustrate the spherical coordinate system.



$$0 \leq \varphi \leq \pi$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq r$$

The Laplacian in spherical coordinates is given by

$$\Delta = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right) + \frac{1}{r^2\sin^2(\varphi)}\partial_\theta^2$$

So, for function

$$w(r,\theta,\varphi) = u(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

we can write the Laplacian in spherical coordinates in terms of the Laplacian in Cartesian coordinates Δ_x:

$$\Delta w(r,\theta,\varphi) = (\Delta_x u)(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$$

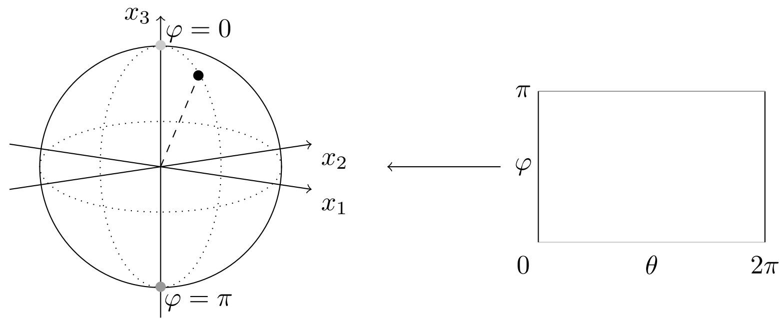
We can rewrite

$$\Delta = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2}\Delta_{\mathbb{S}^2}$$

where the spherical Laplacian is

$$\Delta_{\mathbb{S}^2}\frac{1}{\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right) + \frac{1}{\sin^2(\varphi)}\partial_\theta^2$$

Observe the relationship between 2 and 3 dimensions.



We know the following:

$$\begin{array}{lrl} S^2 & -\partial_\theta^2 g = \lambda g, & g(\theta + 2\pi) = g(\theta) \\ & \lambda = n^2, & g(\theta) = e^{in\theta},\ n \in \mathbb{Z} \\ S^2 & -\Delta_{S^2}\varphi = \lambda\varphi, & \varphi \text{ is a ``nice'' function on } S^2 \\ S^3 & ? & \end{array}$$

To find S^3, we want to solve

$$-\left[\frac{1}{\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right) + \frac{1}{\sin^2(\varphi)}\partial_\theta^2\right]\psi = \lambda\psi$$

We write $\psi(\theta, \varphi) = h(\varphi)g(\theta)$ and $\lambda = \nu(\nu+1)$. We have $-\partial_\theta^2 g = \kappa g$ where g is periodic, so $\kappa = m^2$ where $g = e^{\pm im\theta}$. So, we have

$$\begin{aligned} -\left[\left(\frac{1}{\sin(\varphi)}\partial_\varphi\left(\sin(\varphi)\partial_\varphi\right)\right)h(\varphi)\cdot g(\varphi) + \frac{h(\varphi)}{\sin^2(\varphi)}\partial_\theta^2 g(\theta)\right] &= \nu(\nu+1)g(\theta)h(\varphi) \\ \frac{\sin(\varphi)\partial_\varphi(\sin(\varphi)\partial_\varphi h(\varphi)) + \nu(\nu+1)\sin^2(\varphi)h(\varphi)}{h(\varphi)} &= -\frac{\partial_\theta^2 g(\theta)}{g(\theta)} = \kappa \end{aligned}$$

Note that the left fraction is independent of θ and the right is independent of φ. We know $\kappa = m^2$ and $g(\theta) = e^{im\theta}$ for $m \in \mathbb{Z}$.

$$\sin(\varphi)\partial_\varphi(\sin(\varphi)\partial_\varphi h(\varphi)) - \left(m^2 - \nu(\nu+1)\sin^2(\varphi)\right)h(\varphi) = 0$$

Use the change of variables $z = \cos(\varphi) \in [-1, 1]$. This means

$$\begin{aligned} \partial_\varphi &= \frac{\partial z}{\partial\varphi}\partial_z \\ &= -\sin(\varphi)\partial_z \\ \sin(\varphi)\partial_\varphi &= -\sin^2(\varphi)\partial_z \\ &= -(1-z^2)\partial_z \end{aligned}$$

Plugging this back into the equation with $h(\varphi) = f(\cos(\varphi))$, we have

$$\begin{aligned} \left[(1-z^2)\partial_z(1-z^2)\partial_z - m^2 + \nu(\nu+1)(1-z^2)\right]f &= 0 \\ \left[(1-z^2)^2\partial_z^2 - (1-z^2)2z\partial_z - m^2 + \nu(\nu+1)(1-z^2)\right]f &= 0 \\ \left[(1-z^2)\partial_z^2 - 2z\partial_z + \nu(\nu+1) - \frac{m^2}{1-z^2}\right]f &= 0 \end{aligned}$$

This is the **Associated Legendre Equation**. This has nice solutions that are smooth on the curve where $\nu = \ell \in \mathbb{Z}_{\geq 0}$ and $m \in \{-\ell, \ldots, 0, \ldots, \ell\} \in \mathbb{Z}$. Finding nice solutions to this allows us to find all possible λ's for the sphere and their associated eigenfunctions.

The solutions to the Associated Legendre Equation are

$$P_\ell^m(z) := \frac{(-1)^m}{2^\ell \ell!}(1-z^2)^{m/2}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}z^{\ell+m}}\left(z^2-1\right)^\ell$$

Note that when m is even, P_ℓ^m is a polynomial. So, we have solutions to $-\Delta_{\mathbb{S}^2}\psi = \nu(\nu+1)\psi$ for $v = \ell \in \mathbb{Z}_{\geq 0}$ and $m \in \{-\ell, \ldots, \ell\} \in \mathbb{Z}$.

$$\begin{aligned}\psi(\theta,\varphi) =& Y_\ell^m(\varphi,\theta)\\ :=& P_\ell^m(\cos(\varphi))e^{im\theta}\end{aligned}$$

These are spherical harmonics. The multiplicity $\ell(\ell+1)$ as en eigenvalue of $-\Delta_{\mathbb{S}^2}$ is the number of independent φ's it corresponds to, which is $2\ell+1$.

Hydrogen Atom

Emission of light happens at very specific sequences. This can be explained by the Schrödinger equation. We have $H = -\Delta - \frac{1}{|x|}$.

$$\begin{aligned}\left[-\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{\Delta_{\mathbb{S}^2}}{r^2} - \frac{1}{r}\right] h(r)Y_\ell^m(\varphi,\theta) &= \lambda h(r)Y_\ell^m(\varphi,\theta)\\ \left[-\frac{1}{r^2}\partial_r(r^2\partial_r h(r)) + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r}\right] h(r)Y_\ell^m(\varphi,\theta) &= \lambda h(r)Y_\ell^m(\varphi,\theta)\\ \left[-\frac{1}{r^2}\partial_r(r^2\partial_r h(r)) + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r} - \lambda\right] h(r) &= 0\end{aligned}$$

First, we want to understand the behavior as $r \to 0$. We postulate that $h(r) \approx r^\alpha$ for some $\alpha > 0$.

$$\begin{aligned}&\left[-\partial_r^2 - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r} - \lambda\right] r^\alpha\\ =&\underbrace{-\alpha(\alpha-1)r^{\alpha-2} - 2\alpha r^{\alpha-2} + \frac{\ell(\ell+1)}{r^{\alpha-2}}}_{=0} + ar^{\alpha-1} + br^\alpha\end{aligned}$$

for some constants a and b.

$$\begin{aligned}\alpha(\alpha-1) + 2\alpha - \ell(\ell+1) &= 0\\ \alpha(\alpha+1) - \ell(\ell+1) &= 0\end{aligned}$$

We have $\alpha = \ell$ or $\alpha = -\ell-1$. $-\ell-1 < 0$ which is not allowed, so we have $\alpha = \ell$. So, $h(r) = r^\ell$ as $r \to 0$.

Now, we will observe behavior as $r \to +\infty$. We have $-\partial_r^2 h - \lambda h \approx 0$. Let $\lambda = -\sigma^2$. So, $h(r) = e^{-\sigma r} \to 0$ only when $\sigma > 0$.

$$h(r) = e^{-\sigma r} = r^\ell e^{-\sigma r}q(r)$$

where $e^{-\sigma r}q(r) \to 0$ as $r \to \infty$. We want to find q.

$$\begin{aligned}-\partial_r^2\left[r^\ell e^{-\sigma r}q(r)\right] &= -\partial_r\left[\ell r^{\ell-1}e^{-\sigma r}q(r) - \sigma r^\ell e^{-\delta r} + r^\ell e^{-\sigma r}q'(r)\right]\\ &= -\ell(\ell-1)r^{\ell-2}e^{-\sigma r}q(r) + \sigma\ell r^{\ell-1}e^{-\sigma r}q(r) - \ell r^{\ell-1}e^{-\sigma r}q'(r)\\ &\quad + \sigma\ell r^{\ell-1}e^{-\sigma r}q(r) - \sigma^2 r^\ell e^{-\sigma r}q(r) + \sigma r^\ell e^{-\sigma r}q'(r)\\ &\quad - \ell r^{\ell-1}e^{-\sigma r}q'(r) + \sigma r^\ell e^{-\sigma r}q'(r) - r^\ell e^{-\sigma r}q''(r)\end{aligned}$$

Plug this back into the equation.

$$rq''(r) + 2(\ell+1-r\sigma)q'(r) + (1-2\sigma(\ell+1))q(r) = 0$$

Let $q(r) = \sum_{k=0}^{\infty} a_k r^k$ with $a_0 = 1$.

$$\sum_{k=0}^{\infty} \left[k(k-1)a_k r^{k-1} + 2k(\ell+1-r\sigma)a_k r^{k-1} + (1-2\sigma(\ell+1))a_k r^k\right] = 0$$
$$\sum_{k=0}^{\infty} \left[k(k-1)a_k r^{k-1} + 2k(\ell+1)a_k r^{k-1} + (1-\sigma k-2\sigma(\ell+1))a_k r^k\right] = 0$$

Let $k-1=j$, meaning $k=j+1$.

$$\sum_{j=0}^{\infty} \left[\left((j+1)j + 2(j+1)(\ell+1)\right) a_{j+1} r^j\right] + \sum_{k=0}^{\infty} \left[\left(-\sigma k - 2\sigma(\ell+1)\right) a_k r^k\right] = 0$$
$$\sum_{j=0}^{\infty} \left[(j+1)(j+2(\ell+1))a_{j+1} + \left(1-\sigma j - 2\delta(\ell+1)\right) a_j\right] r^j = 0$$

So, we have
$$a_{j+1} = \frac{2\delta(j+\ell+1)-1}{(j+1)(j+2(\ell+1))} a_j$$

We have $H\psi = \lambda\psi$ for $\int_{\mathbb{R}^3} |\psi|^3 \, \mathrm{d}^3x = 1$. All possible λs are given by $-\frac{1}{4n^2}$ where $n \in \mathbb{N}$.

The ground state is given by the smallest eigenvalue which is $-\frac{1}{4}$. At the ground state,

$$\psi = r^\ell e^{-\frac{1}{2n}r} q(r)$$

The smallest possible ℓ and n is 0 and $-\frac{1}{4}$ respectively.

$$\begin{aligned}\psi &= e^{-\frac{1}{4}r} q(r) \\ &= e^{-r/2}\end{aligned}$$

where $q(r) = 1$ is the simplest possible $q(r)$.

$$\left(-\Delta - \frac{1}{|x|}\right) e^{-\frac{|x|}{2}} = 0$$
$$\begin{aligned}\left(-\partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{2}\right) e^{-r/2} &= -\frac{1}{4}e^{-r/2} + \frac{1}{r}e^{-r/2} - \frac{1}{r}e^{-r/2} \\ &= -\frac{1}{4}re^{-r/2}\end{aligned}$$

So, the ground state of a hydrogen atom is $-\frac{1}{4}e^{-r/2}$.

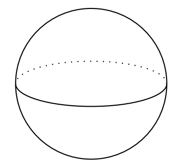
The Heat Equation

The heat equation takes the form

$$(\partial_t - \alpha\Delta)\, u = 0$$

where $\alpha > 0$ is the thermal diffusion constant and $u(t,x)$ is the temperature at time t and position x.

Suppose we model a ball of radius 1. We can heat this ball then submerge it in freezing water. Then, we have



$$(\partial_t - \alpha\Delta u)\, u = 0$$

for $t \geq 0$ and $x \in B(0,1) \subset \mathbb{R}^3$. The boundary conditions are given by $u(t,x) = 0$ for $x \in \partial B(0,1)$.

In spherical coordinates, we have

$$(x_1, x_2, x_3) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)$$

where $\theta \in [0, 2\pi]$ and $\varphi \in [0, \pi]$. So, we have

$$u(t, r, \varphi, \theta) = v(t)\, \underbrace{h(r) Y_\ell^m(\varphi, \theta)}_{\varphi(x)}$$

Recall

$$-\Delta_{\mathbb{S}^2} Y_\ell^m(\varphi, \theta) = \ell(\ell+1) Y_\ell^m(\varphi, \theta)$$

for $\ell = 0, 1, \ldots$ and $m = -\ell, \ldots, 0, \ldots, \ell$.

$$Y_\ell^m(\varphi, \theta) = P_\ell^m(\cos(\theta)) e^{m\theta}$$

Plug this into the equation.

$$\begin{gathered}
(\partial_t - \alpha\Delta)\,(v(t)\varphi(x)) = \partial_t v(t) \cdot \varphi(x) - v(t)\alpha\Delta\varphi(x) = 0 \\
\frac{\partial_t v(t)}{\alpha v(t)} = \frac{\Delta\varphi(x)}{\varphi(x)} = -\lambda \\
\partial_t v + (\alpha\lambda) v = 0 \Longrightarrow v(t) = e^{-\lambda\alpha t}
\end{gathered}$$

So, we want to find φ and λ such that $-\Delta\varphi = \lambda\varphi$ in $B(0,1)$ and $\varphi|_{\partial B(0,1)} = 0$. As shown before, $\lambda > 0$.

$$\begin{aligned}
\varphi &= h(r) Y_\ell^m \\
-\Delta &= -\partial_r^2 - \frac{2}{r}\partial_r - \frac{\Delta_{\mathbb{S}}}{r^2} \\
-\Delta\varphi &= \left[\left(-\partial_r^2 - \frac{2}{r}\partial_r\right) h(r) + \frac{\ell(\ell+1)}{r^2} h(r)\right] Y_\ell^m \\
&= \lambda h(r) Y)\ell^m
\end{aligned}$$

So, we have

$$\begin{aligned}
\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2} + \lambda\right) h &= 0 \\
\left(r^2\partial_r^2 + 2r\partial_r + \lambda r^2 - \ell(\ell+1)\right) h(r) &= 0
\end{aligned}$$

This is almost Bessel's Equation. Let $H(r) := \sqrt{r} h(r)$, which satisfies

$$\left(r^2\partial_r^2 + r\partial_r + r^2\lambda - (\ell + \tfrac{1}{2})^2\right) H(r) = 0$$

This is Bessel's Equation. Use the substitution $z = \sqrt{\lambda} r$ and $f(\sqrt{\lambda} r) = H(r)$.

$$\left(z^2 \partial_z^2 + z\partial_z + z^2 - (\ell + \tfrac{1}{2})^2\right) f = 0$$

This has solutions $J_{\ell+\frac{1}{2}}(z)|_{z\approx 0} \approx z^{\ell+\frac{1}{2}}$ and $Y_{\ell+\frac{1}{2}}(z)|_{z\approx 0} \approx z^{-\ell-\frac{1}{2}}$. So, we have $J_{\ell+\frac{1}{2}}(\sqrt{\lambda} r) = 0$ at $r = 1$.

$$\begin{aligned} h(r) &= r^{-1/2} J_{\ell+\frac{1}{2}}(\sqrt{\lambda} r) \\ \sqrt{\lambda} &= j_{\ell+\frac{1}{2},k} \\ \lambda &= j^2_{\ell+\frac{1}{2},k} \end{aligned}$$

where $J_{\ell+\frac{1}{2}}\left(j_{\ell+\frac{1}{2},k}\right) = 0$. So, our final solution is

$$u(t,x) = e^{-\alpha j_{\ell+\frac{1}{2},k} t} r^{-1/2} J_{\ell+\frac{1}{2}}\left(j_{\ell+\frac{1}{2},k} r\right) Y_\ell^m(\varphi, \theta)$$

The eigenvalue has multiplicity of $2\ell + 1$.

1-D Case

The Scale-Invariant Solution

Consider the initial value problem

$$(\partial_t - \Delta)\, u(t,x) = 0, \quad u(0,x) = \varphi(x)$$

for $t > 0$ and $x \in \mathbb{R}^n$. When $n = 1$, we have

$$\left(\partial_t - \partial_x^2\right) u(t,x) = 0$$

The units of $\partial_t = \partial_x^2 = \frac{1}{\mathrm{m}^2}$ (meters). So, the variable $y = \frac{x}{\sqrt{t}}$ is dimensionless. We will look for solutions $u(t,x) = q\left(\frac{x}{\sqrt{t}}\right)$.

$$\begin{aligned} \partial_t u &= \partial_t q\left(\tfrac{x}{\sqrt{t}}\right) \\ &= -\frac{1}{2}\frac{x}{t}\frac{1}{\sqrt{t}} q'\left(\tfrac{x}{\sqrt{t}}\right) \end{aligned}$$

$$\begin{aligned} \partial_x^2 u &= \partial_x\left(\partial_x\left(q\left(\tfrac{x}{\sqrt{t}}\right)\right)\right) \\ &= \partial_x\left(\tfrac{1}{\sqrt{t}} q'\left(\tfrac{x}{\sqrt{t}}\right)\right) \\ &= \frac{1}{t} q''\left(\tfrac{x}{\sqrt{t}}\right) \end{aligned}$$

$\left(\partial_t - \partial_x^2\right) u = 0$, so we have

$$\begin{aligned} -\frac{1}{2}\frac{x}{\sqrt{t}}\frac{1}{t} q'\left(\tfrac{x}{\sqrt{t}}\right) - \frac{1}{t} q''\left(\tfrac{x}{\sqrt{t}}\right) &= 0 \\ -\frac{1}{2}\frac{x}{\sqrt{t}} q'\left(\tfrac{x}{\sqrt{t}}\right) - q''\left(\tfrac{x}{\sqrt{t}}\right) &= 0 \\ q''(y) &= -\frac{1}{2} y q'(y) \end{aligned}$$

where $y = \frac{x}{\sqrt{t}}$. Let $v(y) = q'(y)$.

$$\begin{aligned} v'(y) &= -\frac{1}{2} y v(y) \\ v(y) &= v(0) e^{-y^2/4} \\ q(y) &= \underbrace{q(0)}_{C_2} + \underbrace{v(0)}_{C_1} \int_0^y e^{-w^2/4} \mathrm{d}w \end{aligned}$$

As $t \to 0^+$, we have

$$u(t,x) = \begin{cases} C_2 + C_1 \int_0^{\infty} e^{-w^2/4}\mathrm{d}w & x > 0 \\ C_2 & x = 0 \\ C_2 + C_1 \int_0^{-\infty} e^{-w^2/4}\mathrm{d}w & x < 0 \end{cases}$$

Evaluate these integrals.

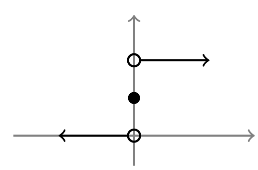
$$\begin{aligned} \int_0^{\infty} e^{-w^2}\mathrm{d}w &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-w^2}\mathrm{d}w \\ &= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{-w^2}\mathrm{d}w \cdot \int_{-\infty}^{\infty} e^{-w^2}\mathrm{d}w\right)^{1/2} \\ &= \frac{1}{2}\left(\int_{\mathbb{R}^2} e^{-|x|^2}\,\mathrm{d}^2x\right)^{1/2} \\ &= \frac{1}{2}\left(\int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty} re^{-r^2}\mathrm{d}r\right)^{1/2} \\ &= \frac{1}{2}\left(\frac{2\pi}{2}\int_0^{\infty}(2r)e^{-r^2}\mathrm{d}r\right)^{1/2} \\ &= \frac{1}{2}\left(\pi\int_0^{\infty} e^{-s}\mathrm{d}s\right)^{1/2} \\ &= \frac{1}{2}(\pi)^{1/2} \\ &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

So, $\int_0^{\infty} e^{-w^2/4}\mathrm{d}w = \frac{1}{2}\left(4\pi\right)^{1/2} = \sqrt{\pi}$.

$$u(t,x) = \begin{cases} C_2 + C_1\sqrt{\pi} & x > 0 \\ C_2 & x = 0 \\ C_2 - C_1\sqrt{\pi} & x < 0 \end{cases}$$

We want to convert this to the Heaviside Function.

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$



For this to be true, we have $C_2 = \frac{1}{2}$ and $C_1 = \frac{1}{2\sqrt{\pi}}$.

This yields

$$U(t,x) := \frac{1}{2} + \frac{1}{\sqrt{4\pi}}\int_0^{x/\sqrt{t}} e^{-y^2/4}\mathrm{d}y$$

where $\left(\partial_t - \partial_x^2\right)U = 0$ and $U(t,x) \to H(x)$ as $t \to 0^+$.

Why do we use the Heaviside Function? For $\varphi \in C_c^1(\mathbb{R})$,

$$\begin{aligned} \int_{\mathbb{R}} \varphi'(z)\underbrace{H(x-z)}_{\substack{0 \text{ at } x-z<0 \\ 1 \text{ at } x-z>0}}\,\mathrm{d}z &= \int_{-\infty}^{x}\varphi'(z)\mathrm{d}z \\ &= \varphi(x) - \varphi(-\infty) \\ &= \varphi(x) \end{aligned}$$

Consider

$$u(t,x) := \int_{\mathbb{R}} \varphi'(z)U(t,x-z)\mathrm{d}z$$

So, we have

$$\int_{\mathbb{R}} \varphi'(x)H(x-z)\mathrm{d}z = \varphi(x) \qquad \text{as } t \to 0^+$$

$$\left(\partial_t - \partial_x^2\right) u(t,x) = \int_{\mathbb{R}} \varphi'(z) \underbrace{\left(\partial_t - \partial_x^2\right) U(t,x-z)}_{=0} \mathrm{d}z$$

$$= 0 \qquad \text{for } t > 0$$

The Integral Solution Formula

A better solution is

$$\int_{\mathbb{R}} \varphi'(z) \left(\frac{1}{2} + \frac{1}{\sqrt{4\pi}} \int_0^{(x-z)/\sqrt{t}} e^{-y^2/4} \mathrm{d}y \right) \mathrm{d}z.$$

So,

$$\frac{1}{2}\int_{\mathbb{R}} \varphi'(z)\mathrm{d}z = 0$$

$$\int_{\mathbb{R}} \varphi'(z) \left(\frac{1}{\sqrt{4\pi}} \int_0^{(x-z)/\sqrt{t}} e^{-y^2/4} \mathrm{d}y \right) \mathrm{d}z = -\int_{\mathbb{R}} \varphi(z)\partial_z \left[\frac{1}{\sqrt{4\pi}} \int_0^{(x-z)/\sqrt{t}} e^{-y^2/4} \right] \mathrm{d}z$$

$$= \int_{\mathbb{R}} \varphi(z) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z$$

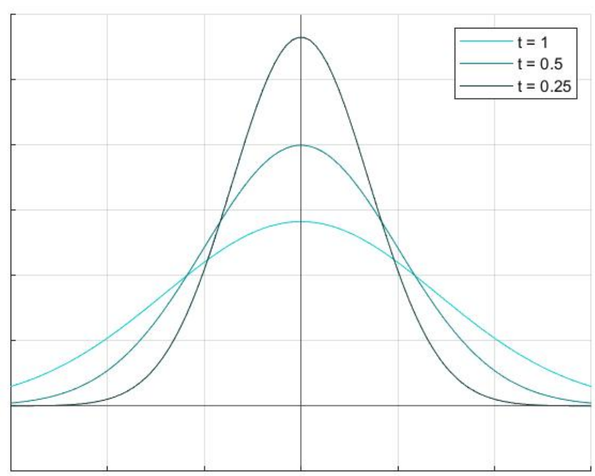
We have $\left(\partial_t - \partial_x^2\right) u(t,x) = 0$ where $u(t,x) \to \varphi(x) \in C_c^1(\mathbb{R})$ as $t \to 0^+$. For now, $\varphi \in C^0(\mathbb{R}^n)$ (bounded and continuous) is enough.

$$u(t,x) = \int_{\mathbb{R}} \varphi(z)H_t(x-z)\mathrm{d}z$$

where

$$H_t(y) := \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}$$
$$:= H_t * \varphi(x)$$

is the heat kernel and $*$ is the convolution.



$$\int_{\mathbb{R}} H_t(y)\mathrm{d}y = 1$$

By change of variables, we have

$$\int_{-\infty}^{\infty} e^{-y^2/4}\mathrm{d}y = 2\sqrt{\pi}$$

Compare this to the Dirac δ_0 "function":

$$\delta_0(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\int \delta_0(x)\mathrm{d}x = 1$$

Our final solution for the 1-D case $\left(\partial_t - \partial_x^2\right) u(t,x) = 0$ for $t > 0$, $u(t,x) \to \varphi(x) \in C_c^1(\mathbb{R})$ as $t \to 0^+$ is

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \varphi(z) e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z \\ &= H_t * \varphi(x) \end{aligned}$$

where the heat kernel is

$$H_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

and the conjugation operation is

$$\begin{aligned} \psi * \varphi(x) &= \int \psi(z)\varphi(x-z)\mathrm{d}z \\ &= \int \psi(x-z')\varphi(z')\mathrm{d}z' \end{aligned}$$

Example

Let $\varphi(x) = e^{-ax^2}$.

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-az^2} e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z$$

Use completing the square to write $az^2 + \frac{(x-z)^2}{4t}$ as a power of z.

$$\begin{aligned} az^2 + \frac{1}{4t}(x-z)^2 &= \left(a + \frac{1}{4t}\right) z^2 - \frac{2}{4t}xz + \frac{1}{4t}x^2 \\ &= \left(a + \frac{1}{4t}\right)\left(z^2 - 2\frac{x}{4ta+1}z + \frac{x^2}{(4ta+1)^2}\right) + \frac{1}{4t}x^2 - \frac{x^2}{4ta+1} \cdot \frac{1}{4t} \\ &= \frac{1+4at}{4t}\left(z - \tilde{x}\right)^2 + \frac{a}{4ta+1}x^2 \end{aligned}$$

where $\tilde{x} = \frac{x}{4at+1}$. Let $z - \tilde{x} = s$.

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-ax^2} e^{-\frac{(x-z)^2}{4t}} \mathrm{d}z \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\left(ax^2 + \frac{(x-z)^2}{4t}\right)} \mathrm{d}z \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{a}{4ta+1}x^2} \int_{\mathbb{R}} e^{-\left(\frac{1+4at}{4t}\right)s^2} \mathrm{d}s \end{aligned}$$

use $r = \left(\frac{1+4at}{4t}\right)^{1/2} s$, where $\mathrm{d}s = \left(\frac{4t}{1+4at}\right)^{1/2} \mathrm{d}w$.

$$\begin{aligned} &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{a}{4ta+1}x^2} \left(\frac{\sqrt{4t}}{(1+4at)^{1/2}}\right) \int_{\mathbb{R}} e^{-r^2} \mathrm{d}w \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{a}{4at+1}x^2} \left(\frac{\sqrt{4t}}{(1+4at)^{1/2}}\right) \cdot \sqrt{\pi} \\ &= \frac{e^{-\frac{a}{4at+1}x^2}}{(1+4at)^{1/2}} \end{aligned}$$

The plot of $u(t,x)$ with varying t is similar to the plot of the heat kernel.

Nonhomegenous problem

The nonhomegenous heat equation in one dimension is given by

$$\left(\partial_t - \partial_x^2\right) u(t,x) = f(t,x)$$

where $u(t,x) \in C_c^2([0,\infty)\times\mathbb{R})$ and $u(0,x)=0$. To find a solution to the nonhomegeous problem, take $\eta_s(t,x)$ which solves $\left(\partial_t - \partial_x^2\right)\eta_s(t,x) = 0$ and $\eta_s(s,x) = f(s,x)$.

$$\eta_s(t,x) = \int H_{t-s}(y) f(s,x-y)\mathrm{d}y$$

So, we postulate

$$\begin{aligned}
u(t,x) &= \int_0^t \nu(t,x)\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} H_{t-s}(y) f(s,x-y)\mathrm{d}y\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} H_s(y) f(t-s,x-y)\mathrm{d}y\mathrm{d}s
\end{aligned}$$

Check this answer.

$$\left(\partial_t - \Delta\right) u(t,x) = \int_{\mathbb{R}} H_t(y) f(0,x-y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} H_s(y)\left(\partial_t - \Delta_s\right) f(t-s,x-y)\mathrm{d}y\mathrm{d}s$$

We want to use integration by parts. Let $\varepsilon > 0$. $|f(t-s,x-y)|$ is bounded by some M since it is in C^2 and it compact.

$$\begin{aligned}
\left|\int_0^\varepsilon \int_{\mathbb{R}} H_s(y)\left(\partial_t - \Delta_x\right) f(t-s,x-y)\mathrm{d}y\mathrm{d}s\right| &\le M \int_0^\varepsilon \underbrace{\int_{\mathbb{R}} H_s(y)\mathrm{d}y}_{=1}\,\mathrm{d}s \\
&= M\varepsilon \to 0 \text{ as } \varepsilon \to 0
\end{aligned}$$

Note that $\partial_t f(t-s,x-y) - \partial_s f(t-s,x-y)$ and $\Delta_x f(t-s,x-y) = \Delta_y f(t-s,x-y)$. So, we have

$$\left(\partial_t - \Delta\right) u(t,x) = \int_{\mathbb{R}} H_t(y) f(0,x-y)\mathrm{d}y + \lim_{\varepsilon\to 0}\int_\varepsilon^t \int_{\mathbb{R}} H_s(y)\left(-\partial_s - \Delta_y\right) f(t-s,x-y)\mathrm{d}y\mathrm{d}s$$

Use integration by parts to evaluate the second integral.

$$\begin{aligned}
&\int_\varepsilon^t H_s(y)\left(-\partial_s f(t-s,x-y)\right)\mathrm{d}s \\
&= - H_s(y) f(t-s,x-y)\Big|_\varepsilon^t + \int_\varepsilon^t \partial_s H_s(y) f(t-s,x-y)\mathrm{d}s \\
&= H_\varepsilon(y) f(t-\varepsilon,x-y) - H_t(y) f(0,x-y) + \int_\varepsilon^t H_s(y) f(t-s,x-y)\mathrm{d}s \\
&\int_{\mathbb{R}}\int_\varepsilon^t H_s(y)\left(-\partial_s f(t-s,x-y)\right)\mathrm{d}s \\
&= \int_{\mathbb{R}} H_\varepsilon(y) f(t-\varepsilon,x-y)\mathrm{d}y - \int_{\mathbb{R}} H_t(y) f(0,x-y)\mathrm{d}y + \int_{\mathbb{R}}\int_\varepsilon^t \partial_s H_s(y) f(t-s,x-y)\mathrm{d}s\mathrm{d}y \\
&= \lim_{\varepsilon\to 0}\left[\int_{\mathbb{R}} H_\varepsilon(y) f(t-\varepsilon,x-y)\mathrm{d}y + \int_\varepsilon^t \int_{\mathbb{R}} \underbrace{\left(\partial_s H_s(y) - \Delta_y H_s(y)\right)}_{=0} f(t-s,x-y)\mathrm{d}y\mathrm{d}s\right]
\end{aligned}$$

since H_s solves the heat equation for $s > 0$

$$= \lim_{\varepsilon\to 0}\int_{\mathbb{R}} H_\varepsilon(y) f(t-\varepsilon,x-y)\mathrm{d}y$$

We want to show that this equals $f(t,x)$. Use the change of variables $y=\varepsilon^{1/2}w$, meaning $\mathrm{d}y=\varepsilon^{1/2}\mathrm{d}w$.

$$\begin{aligned} H_\varepsilon(y) &= \frac{1}{\sqrt{4\pi\varepsilon}}e^{-y^2/4\varepsilon}\mathrm{d}y \\ &= \frac{1}{\sqrt{4\pi}}e^{-w^2/4}\mathrm{d}w \end{aligned}$$

We want to show that as $\varepsilon\to 0$, $\int_{\mathbb{R}} H_\varepsilon(y)f(t-\varepsilon,x-y)\mathrm{d}y - f(t,x)\to 0$.

$$\begin{aligned} \int_{\mathbb{R}} H_\varepsilon(y)f(t-\varepsilon,x-y)\mathrm{d}y - f(t,x) &= \int_{\mathbb{R}} H_1(w)f(t-\varepsilon,x-\varepsilon^{1/2}w)\mathrm{d}w - f(t,x) \\ &= \int_{\mathbb{R}} H_1(w)\left[f(t-\varepsilon,x-\varepsilon^{1/2}w)-f(t,x)\right]\mathrm{d}w \end{aligned}$$

since $\int H_1(w)=1$.
By the Mean Value Theorem and the fact that $f\in C_c^2$,

$$\left|f(t-\varepsilon,x-\varepsilon^{1/2}w)-f(t,x)\right| \le C(\varepsilon+\varepsilon^{1/2}|w|)$$

where $C=\max_{s,y}\left(|\partial_s f(s,y)|+|\partial_y f(s,y)|\right)$.

$$\begin{aligned} \left|\int_{\mathbb{R}} H_\varepsilon(y)f(t-\varepsilon,x-y)\mathrm{d}y - f(t,x)\right| &\le \int_{\mathbb{R}} H_1(w)C(\varepsilon+\varepsilon^{1/2}|w|)\mathrm{d}w \\ &= C\varepsilon + \varepsilon^{1/2}\frac{1}{\sqrt{4\pi}}\underbrace{\int e^{-w^2/4}|w|\mathrm{d}w}_{\text{bounded}} \\ &\to 0 \text{ as } \varepsilon\to 0 \end{aligned}$$

So, the solution to the nonhomegenous heat equation is

$$u(t,x)=\int_0^t\int_{\mathbb{R}^2}\frac{1}{\sqrt{4\pi s}}e^{-y^2/4s}f(t-s,x-y)\mathrm{d}y\mathrm{d}s$$

Higher Dimension Cases

We can perform dimensional analysis on our solution for the 1-D case. Suppose our unit of distance is meters m.

$$\int_0^t\int_{\mathbb{R}}\underbrace{\frac{1}{\sqrt{4\pi s}}}_{\mathrm{m}^{-1}}\ \underbrace{e^{-y^2/4s}}_{\text{dimensionless}}\ \underbrace{f(t-s,x-y)}_{\mathrm{m}^{-2}}\ \underbrace{\mathrm{d}y}_{\mathrm{m}^1}\ \underbrace{\mathrm{d}s}_{\mathrm{m}^2}$$

We want $\mathrm{m}^{-1}\to\mathrm{m}^{-n}$ since $\mathrm{m}^1\to\mathrm{m}^n$, so $\frac{1}{\sqrt{s}}\to\frac{1}{s^{n/2}}$. Our guess is $H_s(y)\in[0,\infty)\times\mathbb{R}^n$ is

$$H_s(y)=\frac{1}{(4\pi s)^{n/2}}e^{-|y|^2/4s}$$

Check that $(\partial_s-\Delta_y)\,H_s(y)=0$ for $s>0$.

Theorem
Suppose $g\in C^0(\mathbb{R}^n)$ and g is bounded. Then,

$$u(t,x)=\int_{\mathbb{R}^n}H_t(x-y)g(y)\,\mathrm{d}^n y$$

solves $(\partial_t-\Delta)\,u(t,x)=0$ for $t>0$ and $u(t,x)\to g(x)$ as $t\to 0^+$.

- **Proof**: for $u(t,x) \to g(x)$, use change of variables.

$$
\begin{aligned}
|u(t,x) - g(x)| &= \left| \int_{\mathbb{R}^n} H_1(w) \left(g(x - \varepsilon^{1/2} w) - g(x) \right) \mathrm{d}w \right| \\
&\leq \int_{|w| \geq R} H_1(w) \left| g(x - \varepsilon^{1/2} w) - g(x) \right| \mathrm{d}w + \int_{|w| \leq R} H_1(w) \left| g(x) - \varepsilon^{1/2} w - g(x) \right| \mathrm{d}w
\end{aligned}
$$

Choose ε_1 such that for some R,

$$\int_{|w| \geq R} H_1(w) \mathrm{d}w < \varepsilon_1$$

Choose ε smaller than ε_1 such that for $|w| \leq R$,

$$\left| g(x - \varepsilon^{1/2} w) - g(x) \right| < \varepsilon_1$$

g is bounded, so $|g(y)| \leq M$ for some M. So, we have

$$|u(t,x) - g(x)| \leq 2M\varepsilon_1 + \varepsilon_1 \to 0 \text{ as } \varepsilon_1 \to 0$$

So, $u(t,x) \to g(x)$. $\square$

Example

Let $g(x) = \sin(kx)$ in $n = 1$.

$$
\begin{aligned}
u(t,x) &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-z^2/4t} \sin(k(x-z)) \mathrm{d}z \\
&= \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \int_{\mathbb{R}} e^{-z^2/4t} e^{ik(x-z)} \mathrm{d}z
\end{aligned}
$$

Use completing the square to put the exponent of e into some form of z^2.

$$
\begin{aligned}
\frac{1}{4t} z^2 - ik(x-z) &= -ikx + \frac{1}{4t} \left(z^2 + 2(i2tk)z + (2it)^2 \right) + \frac{1}{4t} \cdot 4t^2 k^2 \\
&= -ikx + \frac{1}{4t}(z + 2it)^2 + tk^2
\end{aligned}
$$

$$
\begin{aligned}
u(t,x) &= \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \left[e^{ikx} e^{-k^2 t} \int_{\mathbb{R}} e^{-\frac{(z+2it)^2}{4t}} \mathrm{d}z \right] \\
&= \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \left[e^{ikx} e^{-k^2 t} \int_{\mathbb{R}} e^{-z^2 4t} \mathrm{d}z \right] \\
&= \frac{1}{\sqrt{4\pi t}} \operatorname{Im} \left[e^{ikx} e^{-k^2 t} \cdot \sqrt{4\pi t} \right] \\
&= \operatorname{Im} \left(e^{ikt} e^{-k^2 t} \right) \\
&= e^{-k^2 t} \sin(kx)
\end{aligned}
$$

Now, suppose we want to solve the 1-D heat equation for $x \in [0, \pi]$ and $u(t,0) = u(t,\pi) = 0$ with $u(0,x) = g(x)$. We want to answer the question of what $g(x)$ we can solve for $u(t,x)$ with.

We know that if

$$g(x) = \sum_{k=1}^{n} a_n \sin(kx),$$

we have

$$u(t,x) = \sum_{k=1}^{n} e^{-tk^2} a_k \sin(kx).$$

So, we can ask what functions can be written as this $g(x)$. The answer to this requires function spaces.

Function Spaces

Let V be a **complex vector space**. V must satisfy two properties:

1. For $v, w \in V$, $v + w \in V$
2. For $c \in \mathbb{C}$ and $v \in V$, $cv \in V$

Example

$$V = \left\{g \in C^0(\mathbb{R}^n) \mid \exists M \text{ such that } |g(x)| \leq M\right\}$$

is the set of bounded continuous functions in $\mathbb{R}^n$. V is a complex vector space since for any $g, h \in V$, $g + h \in V$ and $cg \in V$ for $c \in \mathbb{C}$.

The **norm** on V is a function $v \mapsto \|v\| \in [0, \infty)$ such that

1. $\|v\| \geq 0$ and if $\|v\| = 0$, then $v = 0$
2. $\|v + w\| \leq \|v\| + \|w\|$
3. $\|cv\| = |c|\|v\|$

Example

$$\|g\| = \sup_{x \in \mathbb{R}^n} |g(x)|$$

for $g \in V$ as previously defined satisfies all properties.

The **inner product** (or scalar product) is a function $u, v \mapsto \langle u, v\rangle \in \mathbb{C}$ such that

1. $\langle v, v\rangle \geq 0$ and if $\langle v, v\rangle = 0$, then $v = 0$
2. $\langle v, w\rangle = \overline{\langle w, v\rangle}$
3. $\langle c_1 v_1 + c_2 v_2, w\rangle = c_1 \langle v_1, w\rangle + c_2 \langle v_2, w\rangle$ for $c_j \in \mathbb{C}$, $v_j \in V$. Note that by the second property, $\langle v, c_1 w_1 + c_2 w_2\rangle = \bar{c}_1 \langle v, w_1\rangle + \bar{c}_2 \langle v, w_2\rangle$

Theorem: Cauchy-Schwarz Inequality

For a complex vector space V with $v, w \in V$,

$$|\langle v, w\rangle|^2 = \langle v, v\rangle \langle w, w\rangle$$

- **Proof**: let $t \in \mathbb{R}$.

$$\begin{aligned}
0 &\leq \langle v + t \langle v, w\rangle w, v + t \langle v, w\rangle w\rangle \\
&= \langle v, v\rangle + t \langle v, w\rangle \langle w, v\rangle + t\overline{\langle v, w\rangle} \langle v, w\rangle + t^2 \langle v, w\rangle \overline{\langle v, w\rangle} \langle w, w\rangle \\
&= \underbrace{\langle v, v\rangle}_{c} + \underbrace{2| \langle v, w\rangle |^2}_{b} t + \underbrace{| \langle v, w\rangle |^2 \langle w, w\rangle}_{a} t^2 \\
&= at^2 + bt + c
\end{aligned}$$

This is ≥ 0 for every $t \in \mathbb{R}$. We know $a \geq 0$ and if $a = 0$, v or $w = 0$ and the equality holds. So, we have $a > 0$. We have cases where $b^2 - 4ac < 0$, $= 0$, and > 0. So, we require $b^2 - 4ac \leq 0$.

$$\begin{aligned}
4| \langle v, w\rangle |^4 - 4 \langle v, v\rangle \langle w, w\rangle | \langle v, w\rangle |^2 \leq 0 \\
4| \langle v, w\rangle |^4 \leq 4 \langle v, v\rangle \langle v, w\rangle | \langle v, w\rangle |^2 \\
| \langle v, w\rangle |^4 \leq \langle v, v\rangle \langle v, w\rangle | \langle v, w\rangle |^2
\end{aligned}$$

If $\langle v, w\rangle = 0$, we have nothing to prove. Otherwise,

$$| \langle v, w\rangle |^2 \leq \langle v, v\rangle \langle w, w\rangle$$

If $| \langle v, w\rangle |^2 = \langle v, v\rangle \langle w, w\rangle$, then $\exists c \in \mathbb{C}$ such that $w = cv$ or $v = cw$, meaning $b^2 - 4ac = 0$. $\square$

Example

Let $V = C^0([0,1])$. We define $\langle g, h\rangle := \int_0^1 g(x)\overline{h(x)}\mathrm{d}x$. By the Cauchy-Schwarz Inequality,

$$\left|\int_0^1 g(x)\overline{h(x)}\mathrm{d}x\right|^2 \leq \int_0^1 |g(x)|^2\mathrm{d}x \cdot \int_0^1 |h(x)|^2\mathrm{d}x$$

Corollary

The norm associated to the inner product $\langle \bullet, \bullet\rangle$,

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

- We want to show that all properties of the norm are satisfied. The first property is automatic and the third follows from the second, so we will just prove the second.

$$\begin{aligned}
\|v+w\|^2 &\stackrel{?}{=} \langle v+w, v+w\rangle \\
&= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\
&= \|v\|^2 + \langle v, w\rangle + \overline{\langle v, w\rangle} + \|w\|^2 \\
&= \|v\|^2 + 2\operatorname{Re}\langle v, w\rangle + \|w\|^2 \\
&\leq \|v\|^2 + 2|\langle v, w\rangle| + \|w\|^2 \\
&\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 && \text{Cauchy-Schwarz} \\
&= (\|v\| + \|w\|)^2 \\
&= \|v+w\|^2
\end{aligned}$$

$\square$

We want to know if every norm comes from an inner product. To test this, use the parallelogram identity.

Lemma: Parallelogram Identity

Let $u, v \in V$ with norm $\|\bullet\|$. Then,

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

- **Proof**: for $u, v \in V$, from the previous corollary, we have

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= (\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle) + (\langle u, u\rangle - \langle v, u\rangle - \langle u, v\rangle + \langle v, v\rangle) \\
&= 2\langle u, u\rangle + 2\langle v, v\rangle \\
&= 2\|u\|^2 + 2\|v\|^2
\end{aligned}$$

$\square$

Example

Let $V = C^0([0,1])$ where $\|g\| = \max_{x\in[0,1]} |g(x)|$ for $g \in V$. Is there an inner product behind it?

Let $g(x) = x$ and $h(x) = 1 - x$. $h + g = 1$ and $h - g = 1 - 2x$. So,

$$\begin{aligned}
\|g\| &= 1 \\
\|h\| &= 1 \\
\|g+h\| &= 1 \\
\|h-g\| &= 1
\end{aligned}$$

But,

$$\|h+g\|^2 + \|h-g\|^2 = 1 + 1 \neq 2 + 2 = 2\|h\|^2 + 2\|g\|^2$$

So, there is no inner product associated to the norm of V.

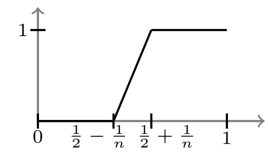
L^p Spaces

For sequence $v_n \in V$, we say v_n **converges** to v ($v_n \to v$) in the norm of V if $\|v - v_n\| \to 0$ as $n \to \infty$.
Take $V = C^0([0,1])$ where the norm $\| \bullet \|_2$ is defined as

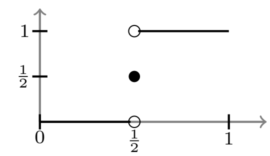
$$\|g\|_2 = \left(\int_0^1 |g(x)|^2 \mathrm{d}x\right)^{1/2}$$

This comes from the inner product $\langle g, h\rangle = \int_0^1 g(x)\overline{h(x)}\mathrm{d}x$. Take $g_n \in C^0([0,1])$ where

$$g_n(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ \frac{n}{2}\left(x - \frac{1}{2} + \frac{1}{n}\right) & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \le x \le 1 \end{cases}$$



$$g(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{2} & x = \frac{1}{2} \\ 1 & x = 1 \end{cases}$$



We have $|g(x) - g_n(x)| \le |g(x)| + |g_n(x)| \le 2$. So,

$$\|g - g_n\| = \left(\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} |g(x) - g_n(x)|\mathrm{d}x\right)^{1/2} \le \left(\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} 2^2\right)^{1/2} = \frac{2}{\sqrt{2n}} = \frac{\sqrt{2}}{\sqrt{n}} \to 0 \text{ as } n \to \infty$$

So, we have

$$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{C} \;\middle|\; \int_0^1 |f(x)|^2 \mathrm{d}x < \infty \right\}$$

where $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x$ is well defined.

$$\|f\| = \left(\int_0^1 |f|^2 \mathrm{d}x\right)^{1/2}$$

We have $C^0([0,1]) \subsetneq L^2([0,1])$. This holds for $C^k([0,1])$ for any k.
An important property is that if $f \in L^2([0,1])$, then $\exists f_n \in C^k([0,1])$ such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$.
For any open connected $\Omega \subseteq \mathbb{R}^n$, we have

$$L^2(\Omega) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)|^2 \,\mathrm{d}^n x < \infty \right\} \qquad L^1(\Omega) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)| \,\mathrm{d}^n x < \infty \right\}$$

$$\langle f, g\rangle := \int_\Omega f(x)\overline{g(x)} \,\mathrm{d}^n x$$

$$\|f\|_{L^2(\Omega)} := \left(\int_\Omega |f(x)|^2 \,\mathrm{d}^n x\right)^{1/2} \qquad \|f\|_{L^1(\Omega)} := \int_\Omega |f(x)| \,\mathrm{d}^n x$$

Proposition
Suppose $\Omega \subset B(0,R)$ with $R < \infty$. Then, $L^2(\Omega) \subset L^1(\Omega)$.

- **Proof**: suppose $f \in L^2(\Omega)$. Note that $1 \in L^2(\Omega)$ since $\int_\Omega 1^2 \le \int_{B(0,R)} 1 < \infty$.

$$\begin{aligned} \int_\Omega |f(x)|\mathrm{d}x &= \int_\Omega |f(x)| \cdot 1\mathrm{d}x \\ &\le \left(\int_\Omega |f(x)|^2\right)^{1/2} \left(\int_\Omega 1\right) \le c\|f\|_2 \end{aligned}$$

by the Cauchy-Schwarz inequality. So, $\|f\|_{L^1(\Omega)} \le c\|f\|_{L^2(\Omega)}$ for some constant c. □

Measure of Sets

Recall for open set $\Omega \subseteq \mathbb{R}^n$, we have

$$L^2(\Omega) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)|^2 \,\mathrm{d}^n x \right\}$$
$$L^1(\Omega) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)| \,\mathrm{d}^n x \right\}$$

For $\Omega \subseteq \mathbb{R}^n$, the **measure** of Ω is

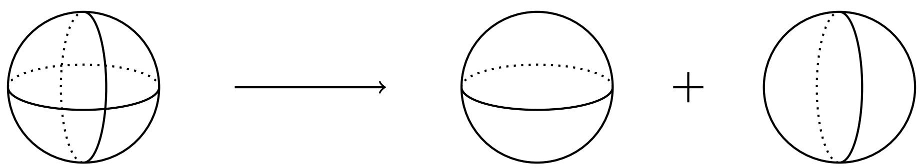
$$m(\Omega) := \int_\Omega 1 \mathrm{d}x.$$

For example, we have $m(\mathbb{R}^n) = \infty$, $m(B(x_0, r)) = c_n r^n$, $m(\{x_0\}) = 0$, and $m(\{(t,t) \mid t \in \mathbb{R}\}) = 0$. Note that not all sets are measurable.

This can be seen in the Banach-Tarski paradox. Take a ball $B(0,1) \subseteq \mathbb{R}^3$. We can take

$$B(0,1) = \bigcup_{k=1}^{5} B_k$$

where $B_k \cap B_j = \emptyset$ for $k \neq j$. We can take $B_1 \cup B_2 = B(0,1)$ and $B_3 \cup B_4 \cup B_5 = B(0,1)$. The pieces can then be rearranged to make two balls.

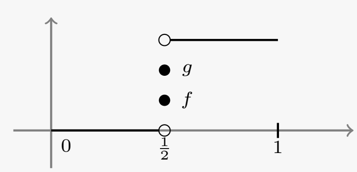


This works because the balls B_k are not measurable.

Example

Let $f \in L^1([0,1])$.

$$\int_0^1 |f - g| \mathrm{d}x = 0$$



We say $f \equiv g$ if $m(\{x \mid f(x) \neq g(x)\}) = 0$. For $f : \Omega \to \mathbb{C}$, we have

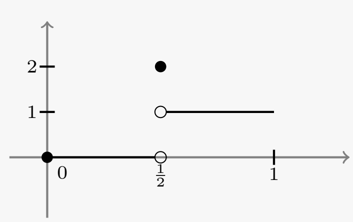
$$\|f\|_\infty = \inf \{a \in \mathbb{R} \mid m(\{x \mid f(x) > a\}) = 0\}$$

So, we have

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{C} \mid \|f\|_\infty < \infty\}$$

Example

Now, for $\Omega = (0,1)$, we have $\|f\|_\infty = 1$.



Banach and Hilbert Spaces

Recall that for a vector space V, we have normed spaces and inner product spaces. We From the Cauchy-Schwarz Inequality, all inner product spaces are normed spaces, but the converse is not true.

For normed vector space V, $v_n \to v$ if and only if $\|v - v_n\|$ converges to 0. We say V is **complete** if and only if "all the limits lie in V".

- A complete normed space is a **Banach space**
- A complete inner product space is a **Hilbert space**

All Hilbert spaces are Banach spaces, but the converse is not true.

Example

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f(x)| \mathrm{d}x \text{ is a Banach space and } \|f\|_{L^2(\Omega)} = \int_\Omega |f(x)|^2 \mathrm{d}x \text{ is a Hilbert space}$$

since

$$\langle f, g \rangle := \int_\Omega f(x)\overline{g(x)} \mathrm{d}x$$

Application to the Heat Equation

Let's apply this back to the heat equation. We know the solution to

$$(\partial_t - \Delta)\, u = 0$$

with $u(0, x) = g(x)$ and $t > 0$ is

$$u(t,x) = \int_t (x) g(x - z)\, \mathrm{d}^n z$$

where

$$H_t(z) = \frac{1}{(4\pi t)^{n/2}} e^{-|z|^2/4t}$$

and $\int H_t(z)\mathrm{d}z = 1$.

Our goal is to understand estimates of u at fixed t in terms of g:

$$\|u(t, \bullet)\|_{L^?} \leq ? \|g\|_{L^?}$$

Let's list all L^p spaces we have defined.

$$\begin{aligned} L^1(\Omega) &:= \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)|\, \mathrm{d}^n x < \infty \right\} \\ \|f\|_{L^1(\Omega)} &:= \int_\Omega |f(x)|\, \mathrm{d}^n x \end{aligned}$$

$$\begin{aligned} L^2(\Omega) &:= \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)|^2\, \mathrm{d}^n x < \infty \right\} \\ \|f\|_{L^2(\Omega)} &:= \left(\int_\Omega |f(x)|^2\, \mathrm{d}^n x \right)^{1/2} \end{aligned}$$

$$\begin{aligned} L^\infty(\Omega) &:= \{ f : \Omega \to \mathbb{C} \mid \|f\|_\infty < \infty \} \\ \|f\|_{L^\infty(\Omega)} &:= \inf\{ a \in \mathbb{R} \mid m(\{x \mid f(x) > a\}) = 0 \} \\ &\text{``} = \text{''} \sup |f(x)| \end{aligned}$$

The " = " is not a true equality but it holds in our case.

1. Find the bound in the form $\|u(t,\bullet)\|_{L^\infty} \leq ?$.

$$\begin{aligned}|u(t,x)| &= \left|\int H_t(z)g(x-z)\mathrm{d}z\right| \\ &\leq \int H_t(z)|g(x-z)|\mathrm{d}z \\ &\leq \int H_t(z)\|g\|_\infty \mathrm{d}z \\ &= \|g\|_\infty\end{aligned}$$

So, $\|u(t,\bullet)\|_{L^\infty} \leq \|g\|_\infty$. This is optimal if g is constant.

2. Find the bound in the form $\|u(t,\bullet)\|_{L^\infty} \leq ?\ \|g\|_{L^1}$.

$$\begin{aligned}|u(t,x)| &\left|\int H_t(z)g(x-z)\mathrm{d}z\right| \\ &\leq \int H_t(z)|g(x-z)|\mathrm{d}z \\ &\leq \sup_{z\in\mathbb{R}^n}(H_t(z)) \cdot \int |g(x-z)|\mathrm{d}z \\ &= \frac{1}{(4\pi t)^{n/2}}\|g\|_{L^1}\end{aligned}$$

So, $\|u(t,\bullet)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{n/2}}\|g\|_{L^1}$.

Example

Let $g(x) = H_s(x)$ for $s > 0$. H solves the heat equation, so

$$\begin{aligned}u(t,x) &= \int H_t(z)H_s(x-z)\mathrm{d}z \\ &= H_{t+s}(x) \\ &= \frac{e^{-|z|^2/4(t+s)}}{(4\pi(t+s))^{n/2}} \\ \|u(t,\bullet)\|_\infty &\leq (4\pi(t+s))^{-n/2}\|g\|_1 = 1 \\ \|u(t,\bullet)\|_{L^\infty} &\leq \frac{1}{(4\pi t)^{n/2}}\|g\|_{L^1}\end{aligned}$$

$\|u(t,\bullet)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{n/2}}\|g\|_{L^1}$ is nearly optimal at $g = H_2$ for $s > 0$.

3. Find the bound in the form $\|u(t,\bullet)\|_{L^1} \leq ?$.

$$\begin{aligned}\|u(t,x)\|_{L^1} &= \int |u(t,x)|\mathrm{d}x \\ &= \int \left|\int H_t(z)g(x-z)\mathrm{d}z\right|\mathrm{d}z \\ &\leq \int\int H_t(z)|g(x-z)|\mathrm{d}z\mathrm{d}x \\ &= \int H_t(z)\left(\int |g(x-z)|\mathrm{d}x\right)\mathrm{d}z \\ &= \int H_t(z)\left(\int |g(y)|\mathrm{d}y\right)\mathrm{d}z && y = x-z \\ &= \|g\|_{L^1}\end{aligned}$$

So, $\|u(t,\bullet)\|_{L^1} \leq \|g\|_{L^1}$. This is optimal if $g \geq 0$ or $g \leq 0$ for every x.

4. Find the bound in the form $\|u(t,\bullet)\|_{L^2} \leq ?$.

$$\begin{aligned}
\|u(t,x)\|_{L^2}^2 &= \int |u(t,x)|^2 \mathrm{d}x \\
&= \int \left| \int H_t(z) g(x-z) \mathrm{d}z \right|^2 \mathrm{d}x \\
&\leq \int \left| \int H_t(z) |g(x-z)| \mathrm{d}z \right|^2 \mathrm{d}x
\end{aligned}$$

Apply the Cauchy-Schwarz Inequality: $\left| \int f(z)\overline{h(z)} \mathrm{d}z \right|^2 \leq \int |f(z)|^2 \mathrm{d}z \int |h(z)|^2 \mathrm{d}z$. By applying this directly, we get

$$\leq \int \left(\int H_t(z)^2 \mathrm{d}z \right) \left(\int |g(x-z)|^2 \mathrm{d}z \right) \mathrm{d}x$$

but this does not help us. Instead, use

$$\begin{aligned}
\|u(t,x)\|_{L^2}^2 &\leq \int \left| \int H_t(z) |g(x-z)| \mathrm{d}z \right|^2 \mathrm{d}x \\
&= \int \left| \int H_t(z)^{1/2} \left(H_t(z)^{1/2} |g(x-z)| \right) \mathrm{d}z \right|^2 \mathrm{d}x
\end{aligned}$$

We have $f(z) = H_t(z)^{1/2}$ and $h(z) = H_t(z)^{1/2}|g(x-z)|$. So, by applying the Cauchy-Schwarz Inequality, we have

$$\leq \int \underbrace{\left(\int H_t(z) \mathrm{d}z \right)}_{=1} \left(\int H_t(z) |g(x-z)|^2 \mathrm{d}z \right) \mathrm{d}x$$

By Fubini's Theorem, we can switch the order of integration.

$$\begin{aligned}
&= \int H_t(z) \left(\int |g(x-z)|^2 \mathrm{d}x \right) \mathrm{d}z \\
&= \int H_t(z) \|g\|_{L^2}^2 \mathrm{d}z \\
&= \|g\|_{L^2}^2
\end{aligned}$$

So, $\|u(t,\bullet)\|_{L^2} \leq \|g\|_{L_2}$. This is nearly optimal.

In summary, we have the following:

1. $$\|u(t,\bullet)\|_{L^\infty} \leq \|g\|_\infty$$

   (optimal at g constant)

2. $$\|u(t,\bullet)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{n/2}} \|g\|_{L^1}$$

   (nearly optimal at $g = H_s$ for $s > 0$)

3. $$\|u(t,\bullet)\|_{L^1} \leq \|g\|_{L^1}$$

   (optimal if $g \leq 0$ or $g \geq 0$ for any x)

4. $$\|u(t,\bullet)\|_{L^2} \leq \|g\|_{L^2}$$

   (nearly optimal)

Hilbert Spaces

Let H be a Hilbert with inner product $\langle \bullet, \bullet \rangle$. $L^2(\Omega)$ for open $\Omega \subseteq \mathbb{R}^n$ is an example of a Hilbert space.

Take $u, v \in H$. u and v are **orthogonal** if $\langle u, v \rangle = 0$.

- A set $\{v_j\}_{j \in J}$ is **orthogonal** if $\langle v_i, v_j \rangle = 0$ for every $i \neq j$.
- A set $\{e_j\}_{j \in J}$ is **orthonormal** if $\langle e_i, e_j \rangle = 0$ for every $i \neq j$ and $\langle e_i, e_i \rangle = 1$ for every i.

A set $\{e_j\}_{j \in J}$ is a **basis** if and only if $\langle v, e_j \rangle = 0 \; \forall j$ implies $v = 0$. A basis is an **orthonormal basis** if and only if it is an orthonormal set and $\sum\limits_{j \in J} \langle v, e_j \rangle e_j = v$.

Gram-Schmidt Process: if $v = \sum\limits_{i=1}^{N} a_j e_j$, then $a_k = \langle v, e_k \rangle$ since

$$\langle v, e_k \rangle = \left\langle \sum a_j e_j, e_k \right\rangle = \sum a_j \langle e + j, e_k \rangle = a_k \langle e_k, e_k \rangle = a_k$$

Define $c_k[v] := \langle v, e_k \rangle$.

Theorem: Bessel's Inequality

Let $S_n v := \sum\limits_{j=1}^{n} c_j[v] e_j$.

$$\sum_{k=1}^{\infty} |c_j[v]|^2 \leq \|v\|^2$$

The equality holds if and only if $S_n v$ converges to v in H. In other words, $\|S_n v - v\| \to 0$ as $n \to \infty$.

- **Proof**:

$$\begin{aligned} 0 &\leq \|v - S_n v\|^2 \\ &= \langle v - S_n v, v - S_n v \rangle \\ &= \langle v, v \rangle - \langle S_n v, v \rangle - \langle v, S_n v \rangle + \langle S_n v, S_n v \rangle \end{aligned}$$

Evaluate each of these terms separately.

$$\langle S_n v, v \rangle = \sum_{j=1}^{n} \langle c_j[v] e_j, v \rangle = \sum_{j=1}^{n} c_j[v] \langle e_j, v \rangle = \sum_{j=1}^{n} c_j[v] \overline{\langle v, e_j \rangle} = \sum_{j=1}^{n} |c_j[v]|^2$$

By the same argument, note that $\langle v, S_n v \rangle$ is equal to the same.

$$\begin{aligned} \langle S_n v, S_n v \rangle &= \left\langle \sum_{j=1}^{n} c_j[v] e_j, \sum_{\ell=1}^{n} c_\ell[v] e_\ell \right\rangle \\ &= \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_j[v] \overline{c_\ell[v]} \langle e_j, e_\ell \rangle \\ &= \sum_{j=1}^{n} |c_j[v]|^2 \end{aligned}$$

Entering this back into the equation of $\|v - S_n v\|^2$, we have

$$\|v - S_n v\|^2 = \|v\|^2 - \sum |c_j[v]|^2 - \sum |c_j[v]|^2 + \sum |c_j[v]|^2 = \|v\|^2 - \sum |c_j[v]|^2$$

Since $\|v - S_n v\|^2 \geq 0$,

$$\sum_{j=1}^{n} |c_j[v]|^2 = \|v\|^2 - \|v - S_n v\|^2 \leq \|v\|^2$$

As $n \to \infty$, $\sum\limits_{j=1}^{n} |c_j[v]|^2$ is a non decreasing sequence. $\|v\|^2$ is an upper bound of $\sum\limits_{j=1}^{n} |c_j[v]|^2$, so

$$\sum_{n=1}^{\infty} |c_j[v]|^2 \leq \|v\|^2. \qquad \square$$

Interlude: Review of Linear Algebra

Let V be t finite dimensional vector space $V = \mathbb{C}^N$ where for $v \in V$, $v = (v_1 \cdots v_N)^\intercal$. For $v, w \in V$, we have

$$\langle v, w \rangle = \sum_{n=1}^{N} v_n \overline{w_n}$$

For $A : V \to V$ linear, $A(c_1 v_1 + c_2 v_2) = c_1 A V_1 + c_2 A V_2$. We say A is **self-adjoint** if

$$\langle Av, w \rangle = \langle v, Aw \rangle$$

i.e. for $A = (A_{ij})_{1 \le i,j \le N}$, $A_{ij} = \overline{A_{ji}}$.

If $A : V \to V$ is self-adjoint, then $\exists$ an orthonormal basis of V $\{\varphi_j\}_{j=1}^N$ such that $A\varphi_j = \lambda_j \varphi_j$. So, we have an orthonormal basis of eigenvectors of A.

We have two properties of the orthonormal basis.

1. All λ_j's are real.
   - **Proof**: We know $\langle A\varphi_j, \varphi_j \rangle = \lambda_j |\varphi_j|^2$.
   $$\begin{aligned} \langle A\varphi_j, \varphi_j \rangle &= \langle \varphi_j, A\varphi_j \rangle \\ &= \overline{\langle A\varphi_j, \varphi_j \rangle} \end{aligned}$$
   So, $\langle A\varphi_j, \varphi_j \rangle$ must be real. $|\varphi_j|^2$ is also real, so λ_j must be real. □
2. If $\lambda_i \ne \lambda_j$, then $\langle \varphi_i, \varphi_j \rangle = 0$.
   - **Proof**:
   $$\begin{aligned} \lambda_i \langle \varphi_i, \varphi_j \rangle &= \langle A\varphi_i, \varphi_j \rangle \\ &= \langle \varphi_i, A\varphi_j \rangle \\ &= \langle \varphi_i, \lambda_j \varphi_j \rangle \\ &= \lambda_j \langle \varphi_i, \varphi_j \rangle \end{aligned}$$
   $\lambda_i \ne \lambda_j$, so we must have $\langle \varphi_i, \varphi_j \rangle = 0$. □

Suppose $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = A(\varphi(t))$ with $\varphi(0) = \varphi$. φ_j's are a basis, so

$$\begin{aligned} \varphi &= \sum_{j=1}^{N} a_j \varphi_j \\ \langle \varphi, \varphi_k \rangle &= \left\langle \sum a_j \varphi_j, \varphi_k \right\rangle \\ &= \sum a_j \langle \varphi_j, \varphi_k \rangle \\ &= a_k \end{aligned}$$

since this is an orthonormal basis. So, for any a_j, $a_j = \langle \varphi, \varphi_j \rangle$.

$$\varphi(t) = \sum a_j e^{\lambda_j t} \varphi_j$$

We can check this:

$$\begin{aligned} \varphi(0) &= \sum a_j \varphi_j \\ \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) &= \sum a_j e^{\lambda_j t} \lambda_j \varphi_j \\ &= \sum a_j e^{\varphi_j t} A\varphi_j \\ &= A(\varphi(t)) \end{aligned}$$

This gives us a way of solving the equation by decomposing an arbitrary element of the vector space in terms of an orthonormal basis of the self-adjoint matrix A. We will observe how this can be done in the case of differential operators.

Fourier Series

Periodic Fourier Series

Example

This is an example. Let $\Omega \subset \mathbb{R}^n$ be open and $H = L^2(\Omega)$.

$$\langle v, u \rangle := \int_\Omega u(x)\overline{v(x)}\,\mathrm{d}^n x$$
$$\|v\| = \left(\int_\Omega |u(x)|^2\,\mathrm{d}^n x \right)^{1/2}$$

An important property of Hilbert spaces is the **approximation property**, which states that we can approximate any element of L^2 by nice functions (infinitely differentiable). We have

$$C_c^\infty(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \in C^\infty(\Omega, \operatorname{supp} f \subset \Omega \text{ is bounded}\}$$

where $\operatorname{supp} f := \overline{\{x \mid f(x) \neq 0\}}$ (closure of the set).

The approximation property states that if $f \in L^2(\Omega)$, then $\exists f_n \in C_c^\infty(\Omega)$ such that $\|f_n - f\| \to 0$ as $n \to \infty$.

For the heat equation, we have $(\partial_t - \Delta)\, u(t,x) = 0$ for $t > 0$ and $u(t,x) \to g(x)$ as $t \to 0^+$, we showed that we can take g continuous and bounded.

$$u(t,x) = \frac{1}{(4\pi t)^n} \int e^{-\frac{1}{4t}|x-z|^2} g(z)\mathrm{d}z$$

u is C^∞ once $t > 0$.

Operators are generalizations of matrices.

Proposition

Let $H = L^2(\Omega)$ and $f \in L^\infty(\Omega)$.

$$Au(x) = f(x)u(x)$$

If $u \in L^2(\Omega)$, then $Au \in L^2(\Omega)$.

- **Proof**:

$$\begin{aligned} \|Au\|^2 &= \int_\Omega |f(x)|^2 |u(x)|^2 \mathrm{d}x \\ &\leq \|f\|_\infty^2 \int |u(x)|^2 \mathrm{d}x \\ &= \|f\|_\infty^2 \|u\|_{L^2}^2 \end{aligned}$$

$\square$

Proposition

If $f : \Omega \to \mathbb{R}$ where $f \in L^\infty(\Omega)$, then A is self-adjoint.

- **Proof**: we want to show that $\langle Av, w \rangle = \langle v, Aw \rangle$. f is real valued, so $f(x) = \overline{f(x)}$.

$$\begin{aligned} \langle Au, v \rangle &= \int_\Omega f(x)u(x)\overline{v(x)}\mathrm{d}x \\ &= \int_\Omega u(x)\overline{f(x)v(x)}\mathrm{d}x \\ &= \langle u, Au \rangle \end{aligned}$$

$\square$

We want to consider the Laplacian as an operator $-\Delta : L^2(\Omega) \to L^2(\Omega)$.

However, we cannot apply the Laplacian to a function in $L^2(\Omega)$. So, we must think about the Laplacian on a smaller space which gives us both second derivatives and boundary conditions.

Let Ω be bounded where $\partial\Omega$ is piecewise C^1.

$$-\Delta : \left\{u \in C^2(\overline{\Omega}),\ u|_{\partial\Omega} = 0\right\} \to L^2(\Omega)$$

So, we have formal self-adjointness.

Theorem

If $u, v \in C^2$ and $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, then

$$\langle -\Delta u, v\rangle = \langle u, -\Delta v\rangle$$

- **Proof**: recall that $-\Delta u = -\nabla\cdot(\nabla u)$. The Divergence Theorem states that $\int_\Omega \nabla\cdot F\,\mathrm{d}^n x = \int_{\partial\Omega} \nu\cdot F\mathrm{d}S$.

$$\begin{aligned}\langle -\Delta u, v\rangle &= -\int_\Omega \nabla\cdot(\nabla u)\overline{v}\\ &= -\int_\Omega \nabla\cdot(\overline{v}\nabla u) + \int_\Omega \nabla u\cdot\nabla\overline{v}\\ &= -\int_{\partial\Omega} \nu\cdot\nabla u\overline{v}\mathrm{d}S + \int_\Omega \nabla u\cdot\nabla\overline{v} && \text{Divergence Theorem}\end{aligned}$$

$\overline{v} = 0$ on $\partial\Omega$. For any f and g, $\nabla\cdot(f\nabla g) = f\nabla\cdot(\nabla g) + \nabla f\cdot\nabla g$.

$$\begin{aligned}&= -\int_{\partial\Omega} \nu\cdot\nabla\overline{v}u\mathrm{d}S + \int_\Omega \nabla u\nabla\overline{v}\\ &= -\int_\Omega \nabla(u\nabla\overline{v}) + \int_\Omega \nabla u\nabla\overline{v}\\ &= \langle u, -\Delta v\rangle\end{aligned}$$

$\square$

Theorem

Let $\{\lambda_j\}_{j=1}^\infty \subset (0,\infty)$ be the set of all eigenvalues of $-\Delta$ with Dirichlet boundary conditions, i.e. there are nontrivial solutions to $-\Delta\varphi = \lambda\varphi$, $\varphi|_{\partial\Omega} = 0$. Then, $\exists$ an orthonormal basis of $L^2(\Omega)$ $\{\varphi_j\}_{j=1}^\infty$ such that $-\Delta\varphi_j = \lambda_j\varphi_j$ where $\varphi_j|_{\partial\Omega} = 0$ and

$$\langle\varphi_j,\varphi_k\rangle = \begin{cases}1 & j = k\\ 0 & j \neq k\end{cases}$$

We will not prove this theorem, but we will see it in an example.

Example

Let $\Omega = [0, L] \subset \mathbb{R}$. We have $\varphi_n = \sin\left(\frac{n\pi}{L}\right)x$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \ldots$. We use the fact that

$$\int_0^L \sin^2\left(\tfrac{n\pi}{L}x\right)\mathrm{d}x = \frac{L}{2}$$

to show that this is an orthonormal basis.

Use this to solve the heat equation

$$(\partial_t - \Delta) = 0, \qquad u(0,x) = \varphi(x) = \sum_{n=1}^\infty a_n\varphi_n$$

where $a_n = \langle\varphi_1,\varphi_n\rangle$.

$$u(t,x) = \sum_{n=1}^\infty a_n e^{-\left(\frac{n\pi}{L}\right)^2 t}\varphi_n(x)$$

Convergence

We want to find the meaning of $\sum_{n=1}^{\infty}$.

Recall Bessel's inequality. Let H be a Hilbert space and $\{e_j\}_{j=1}^{\infty}$ is an orthonormal set. We have $c_j[u] := \langle u, c_j\rangle$. Let $S_n u := \sum_{j=1}^{n} c_j[u]e_j$.

$$\sum_{j=1}^{\infty} |c_j[u]|^2 \leq \|u\|^2 \qquad \text{Bessel's inequality}$$

$$\sum_{j=1}^{\infty} |c_j[u]|^2 = \|S_n u - u\| \to 0 \text{ as } n \to \infty$$

So, $\{e_j\}$ is an orthonormal basis if $\|S_n u - u\| \to 0$ for all $u \in H$.

Example

Let $H = L^2(D(0,1))$ where $D(0,1) \subset \mathbb{R}^2$ is a disk of radius 1 centered at the origin. This has eigenvalues $\left\{j_{n,k}^2\right\}_{n\in\mathbb{Z},k\in\mathbb{N}}$ where $j_{\ell,k}$ is the k^{th} zero of $J_\ell(z)$. The eigenfunction is

$$J_{|n|}(j_{|n|,k}r)e^{in\theta}$$

for $n \in \mathbb{Z}$, $k \in \mathbb{N}$.

$$\begin{aligned}
\int_{D(0,1)} f\mathrm{d}x &= \int_0^{2\pi}\int_0^1 fr\mathrm{d}r\mathrm{d}\theta \\
\langle \varphi_{n,k}, \varphi_{m,\ell}\rangle &= \int_0^{2\pi}\int_0^1 J_{|n|}(j_{|n|,k}r)J_{|m|}(j_{|m|,\ell}r)e^{i(n-m)\theta}r\mathrm{d}r\mathrm{d}\theta
\end{aligned}$$

We know $\int_0^{2\pi} e^{i(n-m)\theta}\mathrm{d}\theta = 2\pi$ when $n = m$ and 0 when $n \neq m$ for $n, m \in \mathbb{Z}$.

$$\begin{aligned}
&= \begin{cases} 0 & n \neq m \\ 2\pi\int_0^1 J_{|n|}(j_{|n|,k}r)J_{|m|}(j_{|n|,\ell}r)r\mathrm{d}r & n = m \end{cases} \\
&= 0 \text{ unless } \ell = k
\end{aligned}$$

The approximation theorem allows us to make smooth approximations. Suppose $\Omega = \mathbb{R}^n$ and $f \in L^2$, with

$$x = \begin{cases} 1 & x \in B(0,1) \\ 0 & x \in \mathbb{R}^n \backslash B(0,1) \end{cases}$$

Recall from the heat equation, we have

$$H_t(x) := \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/4t}$$

If we want $H_t * f(x) = \int f(x)H_t(x-z)\mathrm{d}z \to f(x)$ as $t \to 0^+$, we must have

$$f_n(x) = \underbrace{x\left(\tfrac{x}{n}\right)}_{=1 \text{ in } B(0,n)} \quad \underbrace{H_{\frac{1}{n}} * f(x)}_{\to f \text{ as } n\to\infty}$$

$\int |f - f_n|^2 \to 0$ as $n \to \infty$, so $\|f_n - f\|_{L^2} \to 0$ as $n \to \infty$.

Today, we will consider $\Omega = (-\pi, \pi) \subset \mathbb{R}$. $L^2((-\pi,\pi))$ contain functions on a circle of circumference 2π.

$$\begin{aligned}
L^2(-\pi,\pi)) &= L^2(\mathbb{T}) \\
&= \{f : \mathbb{R} \to \mathbb{C} \mid f(x + 2\pi) = f(x)\} \\
C^k(\mathbb{T}) &= \left\{f : \mathbb{R} \to \mathbb{C} \mid f \in C^k(\mathbb{R}),\ f(x+2\pi) = f(x)\right\}
\end{aligned}$$

For any $f \in C_c^\infty((-\pi,\pi))$, its periodic extension is $C^\infty(\mathbb{T})$.

$$e_k := \frac{1}{\sqrt{2\pi}} e^{ikx} \in L^2$$

We have $|e_k(x)| = \frac{1}{\sqrt{2\pi}}$ and $e_k \in C^\infty(\mathbb{T})$ since $e_k(x+2\pi) = e_k(x)$. We want to show that $\{e_k\}$ is an orthonormal basis.

$$\begin{aligned}
\langle e_k, e_\ell \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} \overline{e^{i\ell x}} \mathrm{d}x \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-\ell)x} \mathrm{d}x \\
&= \begin{cases} 1 & k = \ell \\ \frac{1}{2\pi} \frac{1}{i(k-\ell)} e^{i(k-\ell)x} \Big|_{x=-\pi}^{\pi} & k \neq \ell \end{cases} \\
&= \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}
\end{aligned}$$

We define

$$\langle f, e_k \rangle := c_k[f] = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx} \mathrm{d}x$$

By Bessel's inequality,

$$\begin{aligned}
\sum_{-\infty}^{\infty} |c_k[f]|^2 &\leq \|f\|^2 \\
&= \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x
\end{aligned}$$

We want to show that $\sum\limits_{k=-n}^{n} c_k[f] e_k \to f$ in L^2, i.e. $\{e_k\}_{k \in \mathbb{Z}}$ is an orthonormal basis.

$$f(x) \stackrel{?}{=} \sum_{-\infty}^{\infty} c_k[f] e^{ikx}$$

Can this be true as an infinite series?

Define

$$\begin{aligned}
\hat{f}(k) &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \langle f, e_k \rangle \\
&= \frac{1}{\sqrt{2\pi}} c_k[f]
\end{aligned}$$

We are asking if for some f,

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

where the RHS is the **Fourier series** of f. When does this hold?

Theorem

Suppose $f \in C^k(\mathbb{T})$ where $k \geq 0$.

$$\sum_{-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

converges absolutely and uniformly.

- **Proof**: we have $k \neq 0$.

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\frac{1}{k^2}\left(-\partial_x^2 e^{ikx}\right)\mathrm{d}x
\end{aligned}$$

Apply integration by parts.

$$\begin{aligned}
&= \frac{1}{2\pi k^2}\int_{-\pi}^{\pi} \underbrace{(-\partial_x f(x))}_{\text{cts \& bdd}} e^{-ikx}\mathrm{d}x \\
\left|\hat{f}(k)\right| &\leq \frac{1}{2\pi k^2}\int_{-\pi}^{\pi}\left|\partial_x^2 f\right|\mathrm{d}x \\
&\leq \frac{1}{k^2}\max\left|\partial_x^2 f\right| \\
\sum_{-\infty}^{\infty}\left|\hat{f}(k)e^{ikx}\right| &= \sum_{-\infty}^{\infty}\left|\hat{f}(k)\right| \\
&= \left|\hat{f}(0)\right| + \sum_{k\neq 0}\left|\hat{f}(k)\right|
\end{aligned}$$

Let $M = \max \hat{f}$.

$$\begin{aligned}
&\leq \left|\hat{f}(0)\right| + \sum_{k\neq 0}\frac{M}{k^2} \\
&< \infty
\end{aligned}$$

□

So, we know that

$$\sum_{k\in\mathbb{Z}}\hat{f}(k)e^{ikx}$$

converges for $f \in C^k(\mathbb{T})$ and $k \geq 2$. We still want to show that it converges to $f(k)$.

Theorem

If $f \in C^k(\mathbb{T})$ and $k \geq 0$, then

$$f(x) = \sum_{k=-\infty}^{\infty}\hat{f}(k)e^{ikx}.$$

- **Proof**: we want to show that $f(a) = \sum \hat{f}(k)e^{ika}$.

$$\begin{aligned}
f_a(x) &:= f(x+a) \\
\hat{f}_a(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+a)e^{ixk}\mathrm{d}x
\end{aligned}$$

Substitute $x = y - a$.

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{iak}e^{-iyk} \\
&= e^{aik}\hat{f}(k)
\end{aligned}$$

So, we have $f_a(0) = \sum \hat{f}_a(k)$. It is enough to show that for any $f \in C^k(\mathbb{T})$, $f(0) = \sum \hat{f}(k)$.

Let $h(k) := f(x) - f(0)$.

$$\hat{h}(k) = \begin{cases} \hat{f}(0) - f(0) & k = 0 \\ \hat{f}(k) & k \neq 0 \end{cases}$$

So, we want to show that if $h(0) = 0$, then $\sum \hat{h}(k) = 0$.

$$\hat{h}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) e^{-ixk} \mathrm{d}x$$

If $h(0) = 0$, then $h(x) = \left(e^{ix} - 1\right) g(x)$ for $g \in C^2(\mathbb{T})$ since $e^{ix} - 1$ has simple zeros at $x = 2\pi k$.

$$\begin{aligned} \hat{h}(k) &= \frac{1}{2\pi}(e^{ix} - 1) g(x) e^{ixk} \mathrm{d}x \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \left(e^{-i(k-1)x} - e^{-ikx}\right) \mathrm{d}x \\ &= \hat{g}(k-1) - \hat{g}(k) \end{aligned}$$

$g \in C^k(\mathbb{T})$ for $k \geq 2$, so $\hat{g}_k(z)$ converges absolutely, i.e. $\sum |\hat{g}(k)| < \infty$.

$$\begin{aligned} \sum_{-\infty}^{\infty} \hat{h}(k) &= \sum_{-\infty}^{\infty} [\hat{g}(k-1) - \hat{g}(k)] \\ &= \sum_{-\infty}^{\infty} \hat{g}(k-1) - \sum_{-\infty}^{\infty} \hat{g}(k) \\ &= \sum_{-\infty}^{\infty} \hat{g}(k) - \sum_{-\infty}^{\infty} \hat{g}(k) \\ &= 0 \end{aligned}$$

□

So, for $e_j \in L^2(\mathbb{T})$ where $e_j(x) := \frac{e^{ijx}}{\sqrt{2\pi}}$ where $\langle e_j, e_k \rangle = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$, if $f \in C^k(\mathbb{T})$ with $k \geq 0$, then

$$f(x) = \sum_{j \in \mathbb{Z}} \langle f, e_j \rangle \, e_j = \sum_{j \in \mathbb{Z}} \hat{f}(j) e^{ijx}$$

where

$$\hat{f}(j) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ijx} \mathrm{d}x.$$

We proved this using two facts:

1. If $f \in C^k$ and $k \geq 2$, then $\sum \left| \hat{f}(j) \right| < \infty$
2. For $f \in C^k$ and $k \geq 2$, if $f(0) = 0$, then $\sum \hat{f}(j) = 0$.

We define $\hat{f}(j) e^{ij}$ as the **Fourier Series**.

Convergence in L^2

For $g \in C^k(\mathbb{T})$ and $k \geq 2$, $g(x) = \sum_{j\in\mathbb{Z}} \langle g, e_j \rangle e_j$ converges as an infinite series and uniformly in x.

For $f \in L^2(\mathbb{T})$, $f = \sum_{j\in\mathbb{Z}} \langle f, e_j \rangle e_j$ in a weaker sense: as $N \to \infty$,

$$\left\| f - \sum_{|j|\leq N, j\in\mathbb{Z}} \langle f, e_j \rangle e_j \right\| \to 0$$

i.e.

$$\int \left| f(x) - \sum_{|j|\leq N} \langle f, e_j \rangle \frac{e^{ijx}}{\sqrt{2\pi}} \right|^2 \mathrm{d}x \to 0$$

Theorem

$\{e_j\}_{j\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$ where $e_j(x) := \frac{e^{ijx}}{\sqrt{2\pi}}$. This means we can write any element in $L^2(\mathbb{T})$ as a linear combination of $\{e_j\}_{j\in\mathbb{Z}}$.

- **Proof**: we need to check that if $f \in L^2(\mathbb{T})$ and $\langle f, e_j \rangle = 0 \ \forall j$, then $f = 0$. Take $f_n \in C^\infty(\mathbb{T})$ such that $\|f_n - f\|_{L^2} = 0$ as $n \to \infty$. We know

$$f_n(x) = \sum_j \langle f_n, e_j \rangle e_j(x).$$

For any g_j,

$$\left\langle \sum g_j, f \right\rangle = \sum_j \langle g_j, f \rangle .$$

Let $g_j = f_n = \langle f_n, e_j \rangle e_j$.

$$\begin{aligned} \langle f_n, f \rangle &= \left\langle \sum_j \langle f_n, e_j \rangle e_j, f \right\rangle \\ &= \sum_j \langle f_n, e_j \rangle \langle e_j, f \rangle \\ &= 0 \end{aligned}$$

since $\langle f, e_j \rangle = 0$. So, $\langle f, f_n \rangle = 0$.

$$\begin{aligned} \langle f, f \rangle &= \langle f, f \rangle - \langle f, f_n \rangle \\ &= \langle f, f - f_n \rangle \\ |\langle f, f - f_n \rangle| &\leq \|f\| \|f - f_n\| && \text{(Cauchy-Schwartz Ineq.)} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

So, $\langle f, f \rangle = 0$ meaning $f = 0$. So, $\langle f, e_j \rangle = 0$ implies $f = 0$ meaning $\{e_j\}_{j\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$. $\square$

Parseval's formula states that

$$\|f\|^2 = \sum |\langle f, e_j \rangle|^2 ,$$

or

$$\int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x = \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} \left| \int_{-\pi}^{\pi} f(x) e^{-ijx} \right|^2 .$$

Summary of Hilbert Spaces

A Hilbert space H is a vector space with defined inner produce $\langle v, w\rangle = \overline{\langle w, v\rangle}$. For Hilbert spaces, we have $\|f\| = \langle f, f\rangle^{1/2}$. The Cauchy-Schwartz Inequality states that $|\langle f, g\rangle| \le \|f\|\|g\|$.

1. Finite dimension Hilbert spaces

   For a Hilbert space H of dimension d, $\{e_j\}_{j=1}^d$ forms an orthonormal basis of H where

   $$\langle e_i, e_j\rangle = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$$

   (meaning the set is orthonormal) and for $f \in H$ where $f = \sum \langle f, e_j\rangle e_j$, if $\langle f, e_j\rangle = 0$, then $f = 0$.

   $c_j[f] := \langle f, e_j\rangle$ and $\langle f, g\rangle = \sum c_j[f]\overline{c_j[g]}$.

2. Infinite dimension Hilbert spaces

   For $f, f_n \in H$, $f_n \to f$ when $\|f - f_n\| \to 0$ as $n \to \infty$.

   If $\{e_j\}_{j=1}^\infty$ is an orthonormal set, then it satisfies Bessel's inequality $\sum_{j=1}^N |c_j[f]|^2 \le \|f\|^2$.

   $\forall f \in H$, $\sum |c_j[f]|^2 = \|f\|^2$ if and only if the following statement is true: $\forall f$, $\sum c_j[f]e_j \to f$ if and only if $\forall e_j$, if $\langle f, e_j\rangle \to 0$, then $f = 0$.

   $\langle f, g\rangle = \sum c_j[f]\overline{c_j[g]}$ still holds in infinite dimensions.

3. $H = L^2((-\pi, \pi)) = L^2(\mathbb{T}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid f(x + 2\pi) = f(x),\ \int_{-\pi}^{\pi} |f(x)|^2 \mathrm{d}x < \infty \right\}$

   Here, we define $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\mathrm{d}x$. We proved the theorem that for $e_j(x) := \frac{e^{ijx}}{\sqrt{2\pi}}$, $\{e_j\}_{j\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$. Moreover, $\|f\|^2 = \sum_{j\in\mathbb{Z}} |c_j[f]|^2$ and $\langle f, g\rangle = \sum_{j\in\mathbb{Z}} c_j[f]c_j[g]$ where $c_j[f] = \langle f, e_j\rangle$. We proved this based on 3 main facts.

   1. For $f \in C^k(\mathbb{T})$,

      $$|c_j(f)| \le \frac{2\sqrt{2\pi}\|f\|C^k(\mathbb{T})}{1 + j^k}$$

      since for $j \ne 0$,

      $$\begin{aligned} |c_j[f]| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ijx} f(x)\mathrm{d}x \right| \\ &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \frac{1}{j^k} \left(-\frac{1}{i}\partial_x \right)^k \left(e^{-ijx}\right) f(x)\mathrm{d}x \right| \\ &= \left| \frac{1}{\sqrt{2\pi}} \frac{1}{j^k} \int_{-\pi}^{\pi} e^{-ijx}\partial_x^k f(x)\mathrm{d}x \right| \end{aligned}$$

   2. If $f \in C^k(\mathbb{T})$ and $k \ge 2$, then $f(x) = \sum_{j\in\mathbb{Z}} c_j[f]e_j(x)$, i.e.

      $$\max_x \left| f(x) - \sum_{|j|\le N} c_j[f]e_j(x) \right| \to 0 \text{ as } N \to \infty$$

   3. $\forall f \in L^2$, $\exists f_n \in C^\infty(\mathbb{T})$ such that $\|f_n - f\| \to 0$ as $n \to \infty$.

Application to PDEs

We will observe PDEs on $\mathbb{T}$, or PDEs with periodic boundary conditions $f(x+2\pi) = f(x)$.

Consider $\left(\partial_t^2 - \partial_x^2\right) u = 0$ with $u(0,x) = g(x)$, $\partial_t u(0,x) = h(x)$, and $u(t, x+2\pi) = u(t,x)$. We solved this by demanding that periodic extensions of g and h are in C^2 and C^1 respectively.

Now, suppose we simply take $g, h \in L^2$, which is a weaker statement.

$$g(x) = \sum_{n\in\mathbb{Z}} g_n e^{inx} \qquad h(x) = \sum_{n\in\mathbb{Z}} h_n e^{inx}$$

where

$$g_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx} = \frac{1}{\sqrt{2\pi}} c_n[g]$$

with convergence in L^2. We conjecture that

$$u(t,x) = \sum_{n\in\mathbb{Z}} u_n(t) e_n, \qquad e_n := \frac{e^{inx}}{\sqrt{2\pi}}$$

solves the wave equation.

$$\begin{aligned}\left(\partial_t^2 - \partial_x^2\right) u &= \sum \left(\partial_t^2 - \partial_x^2\right)\left(u_j(t) e_j(t)\right) \\ &= \sum \left(\partial_t^2 u_j(t) + j^2 u_j(x)\right) e_j(x)\end{aligned}$$

If this solves the wave equation, then it equals zero. So, to solve the wave equation with periodic boundary equation, it suffices to solve the system of equations

$$\begin{aligned}\partial_t^2 u_n(t) + j^2 u_n(t) &= 0 \\ u_n(0) &= g_n \\ \partial_t u_n(0) &= h_n\end{aligned}$$

This is a system of ODEs that we know how to solve.

$$u_n(t) = \cos(nt) g_n + \frac{1}{n}\sin(nt) h_n$$

So, we have

$$u(t,x) = \sum \left(g_n \cos(nt) + h_n \frac{\sin(nt)}{n}\right) e^{inx}$$

with convergence in L^2.

An alarming fact is that we don't know the meaning of $\left(\partial_t^2 - \partial_x^2\right) u = 0$, since convergence is not "nice" enough to differentiate.

How do we define $\left(\partial_t^2 - \partial_x^2\right) u$? u solves the wave equation in the weak sense. For $u \in L^2$, define "$\partial_x u = f$" (although we don't know what $\partial_x u$ means exactly).

For $\varphi \in C^\infty(\mathbb{T})$, $\langle \partial_x u, \varphi\rangle = -\langle u, \partial_x \varphi\rangle$. By formal integration by parts, for $u \in C^1(\mathbb{T})$,

$$\begin{aligned}\int_{-\pi}^{\pi} \partial_x u \varphi(x) \mathrm{d}x &= \underbrace{u(x)\varphi(x)|_{-\pi}^{\pi}}_{=0} - \int_{-\pi}^{\pi} u(x)\partial_x\varphi(x)\mathrm{d}x \\ \langle \partial_x u, \varphi\rangle &= -\langle u, \partial_x\varphi\rangle\end{aligned}$$

For $u \in L^2$ and $f \in L^2$, we say $\partial_x u = f$ in the weak sense if and only if

$$\langle f, \varphi\rangle = -\langle u, \partial_x \varphi\rangle$$

for every $\varphi \in C^\infty(\mathbb{T})$

Lemma

Suppose $g \in L^2$ and $\langle g, \varphi \rangle = 0$ for every $\varphi \in C^\infty(\mathbb{T})$. Then, $g = 0$ in L^2.

- **Proof**: by the approximation property, $\exists g_n \in C^\infty$ such that $g_n \to g$ in L^2. So,

$$\begin{aligned} \langle g, g \rangle &= \langle g - g_n, g \rangle \\ |\langle g, g \rangle| &= |\langle g - g_n, g \rangle| \\ &\leq \|g - g_n\| \|g\| \to 0 \end{aligned}$$

So, $\langle g, g \rangle = 0$ meaning $g = 0$. $\square$

Lemma

Suppose $u \in C^1$ and $f \in C^0$. Then, $\partial_x u = f$ is equivalent to the statement $\langle u, -\partial_x \varphi \rangle = \langle f, \varphi \rangle$ for every $\varphi \in C^\infty(\mathbb{T})$.

- **Proof**:

$\Longrightarrow$ Suppose $\partial_x u = f$. Show that $\langle u, -\partial_x \varphi \rangle = \langle f, \varphi \rangle$.

$$\begin{aligned} \langle u, -\partial_x \varphi \rangle &= -\int_{-\pi}^{\pi} u(x) \partial_x \overline{\varphi(x)} \\ &= \underbrace{-u \cdot \overline{\varphi}\big|_{-\pi}^{\pi}}_{=0} + \underbrace{\int_{-\pi}^{\pi} \partial_x^n \overline{\varphi}}_{=\langle \partial x, \varphi \rangle} \\ &= \int_{-\pi}^{\pi} f \overline{\varphi} \\ &= \langle f, \varphi \rangle \end{aligned}$$

$\Longleftarrow$ Suppose $\langle u, -\partial_x \varphi \rangle = \langle f, \varphi \rangle$ for every $\varphi \in C^\infty(\mathbb{T})$. Show that $\partial_x u = f$. Since $u \in C^1$, we can integrate by parts. For every $\varphi \in C^\infty(\mathbb{T})$,

$$\langle f, \varphi \rangle = \langle u, -\partial_x \varphi \rangle = \langle \partial_x u, \varphi \rangle$$

So, $\langle \partial_x u - f, \varphi \rangle = 0$. From the previous lemma, $\partial_x u = f$ in L^2 meaning $\partial_x u = f$ as functions. $\square$

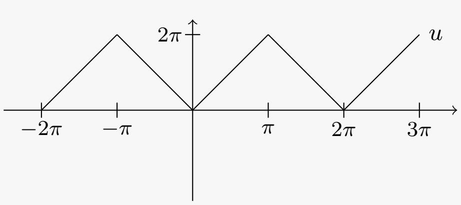
This holds in higher dimensions. $\partial_x^m u = f$ weakly if and only if $\langle u, (-\partial_x)^m \varphi \rangle = \langle f, \varphi \rangle$ for every $\varphi \in C^\infty(\mathbb{T})$. The lemma holds in this case: for $u \in C^m(\mathbb{T})$ and $f \in C^{m-1}(\mathbb{T})$, $\partial_x^m u = f$ if and only if $\langle u, (-\partial_x)^m \varphi \rangle = \langle f, \varphi \rangle$ for every $\varphi \in C^\infty(\mathbb{T})$.

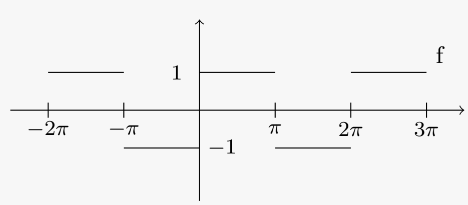
Example

Let $u(x)$ be the 2π periodic function given by

$$u(x) = \begin{cases} x & 0 \leq x \leq \pi \\ -x + 2\pi & \pi \leq x \leq 2\pi \end{cases}$$

on $[0, 2\pi]$.





We claim that $\partial_x u = f$ weakly where f is the 2π periodic function given by

$$f(x) = \begin{cases} 1 & 0 \le x \le \pi \\ -1 & \pi \le x \le 2\pi \end{cases}$$

To prove this, we want to show that $\langle u, -\partial_x \varphi \rangle = \langle f, \varphi \rangle$ for every $\varphi \in C^\infty(\mathbb{T})$. Suppose φ is real (since we forgot to write the bars over φ).

$$\begin{aligned}
\langle u, -\partial_x \varphi \rangle &= \int_0^\pi x\,(-\partial_x \varphi)\,\mathrm{d}x + \int_\pi^{2\pi} (-x+2\pi)\,(-\partial_x\varphi)\,\mathrm{d}x \\
&= -x \cdot \varphi\big|_0^\pi + \int_0^\pi \varphi(x)\mathrm{d}x + x \cdot \varphi\big|_\pi^{2\pi} + \int_\pi^{2\pi} (-1)\varphi(x)\mathrm{d}x + 2\pi(-\varphi)\big|_\pi^{2\pi} \\
&= -\pi\varphi(\pi) + \int_0^\pi \varphi(x)\mathrm{d}x + 2\pi\varphi(2\pi) - \pi\varphi(\pi) + \int_\pi^{2\pi} (-1)\varphi(x)\mathrm{d}x - 2\pi(\varphi(2\pi)) + 2\pi(\varphi(\pi)) \\
&= \int_0^\pi \varphi(x)\mathrm{d}x + \int_\pi^{2\pi} (-1)\varphi(x)\mathrm{d}x \\
&= \int_0^{2\pi} f(x)\varphi(x) \\
&= \langle f, \varphi \rangle
\end{aligned}$$

□

Theorem

u given by $\langle u, -\partial_x\varphi\rangle = \langle f, \varphi\rangle$ for every $\varphi \in C^\infty(\mathbb{T})$ solves $\left(\partial_t^2 - \partial_x^2\right) u = 0$ weakly in the following sense: for every $\varphi \in C^\infty(\mathbb{T})$,

$$\partial_t^2 \underbrace{\langle u(t, \bullet), \varphi\rangle}_{C^\infty(\mathbb{R})} - \left\langle u(t,\bullet), \partial_x^2 \varphi\right\rangle = 0.$$

- We know that

  $$c_j[f] = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ijx} = \sqrt{2\pi} f_j$$

  and same holds for $c_j[g]$. So,

  $$\begin{aligned}
  \langle f, g\rangle &= \sum c_j[f]\overline{c_j[g]} \\
  &= \sum \sqrt{2\pi} f_j \sqrt{2\pi}\overline{g_j} \\
  &= 2\pi \sum f_j \overline{g_j}
  \end{aligned}$$

  So, we have

  $$\begin{aligned}
  \langle u(t,\bullet), \varphi\rangle &= 2\pi \sum u_n(t)\overline{\varphi_n} \\
  &= 2\pi \sum \left(g_n \cos(nt) + h_n \frac{\sin(nt)}{n}\right) \overline{\varphi_n}
  \end{aligned}$$

  For $\varphi \in C^\infty(\mathbb{T})$, then $|\varphi_n| \le \frac{C_k}{1+n^k}$. This converges absolutely for $k \ge 4$.

  $$\left| \partial_t^2 \left[\left(g_n \cos(nt) + h_n \frac{\sin(nt)}{n} \right) \overline{\varphi_n} \right] \right| \le \frac{C_n n^2}{1 + |n|^k}$$

  for every k. So,

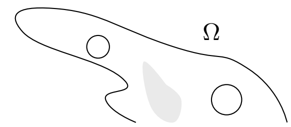
  $$\begin{aligned}
  \partial_t^2 \langle u(t,\bullet), \varphi\rangle &= 2\pi \sum u_n''(t)\overline{\varphi_n} \\
  \left\langle u(t,\bullet), \partial_x^2\varphi\right\rangle &= 2\pi \sum \left(-n^2 u_n(t)\right)\overline{\varphi_n} \\
  \partial_t^2 \langle u(t,\bullet), \varphi\rangle - \left\langle u(t,\bullet), \partial_x^2\varphi\right\rangle &= 0
  \end{aligned}$$

□

Higher Dimension Cases

Suppose $\Omega \in \mathbb{R}^n$ is open and $u, f \in L^2(\Omega)$. We say that $\partial_{x_j} u = f$ weakly if and only if $\langle u, -\partial_{x_j}\varphi \rangle = \langle f, \varphi \rangle$ for every $\varphi \in C_c^\infty(\Omega)$.

$$\operatorname{supp}(\varphi) = \overline{\{x \mid \varphi(x) \neq 0\}}$$



$\varphi \in C_c^\infty(\Omega)$ means that $\varphi \in C^\infty(\Omega)$, $\operatorname{supp}(\varphi) \subset \Omega$, and $\operatorname{supp}(\varphi) \subset B(0, R)$ for some R.

Re call that $\left(\partial_t^2 - \partial_x^2\right) u = 0$ for $u \in C^2(\mathbb{R}^2)$ if and only if $\exists f, g \in C^2(\mathbb{R})$ such that

$$u(t, x) = f(x - t) + g(x + t).$$

Note that formula $u(t, x)$ is valid for any functions f and g.

We say that $u \in L^2_{\text{loc}}(\mathbb{R})$ (local $L^2(\mathbb{R})$) if for any closed bounded $K \subset \Omega$, if $\int_K |u(x)|^2 \mathrm{d}x < \infty$, then $\forall \psi \in C_c^\infty(\Omega)$, $\langle u, \psi \rangle$ is well defined.

$$\left| \int_\Omega u\overline{\psi} \right| = \left| \int_K u\overline{\psi} \right| \leq \left(\int_K |u|^2 \right)^{1/2} \left(\int_K |\psi|^2 \right)^{1/2}$$

where $K = \operatorname{supp}\psi$.

Theorem

Suppose $f, g \in L^2_{\text{loc}}(\mathbb{R})$. Then,

$$u(t, x) = f(x - t) + g(x + t)$$

satisfies

$$\left(\partial_t^2 - \partial_x^2\right) u = 0$$

weakly on $\mathbb{R}^2$.

- **Proof**: we need to prove that $\forall \varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\int\int u(t, x) \left(\partial_t^2 - \partial_x^2\right) \varphi(t, x) \mathrm{d}t\mathrm{d}x = 0$$

i.e.

$$\int\int \left(f(x - t) + g(x - t)\right) \left(\partial_t^2 - \partial_x^2\right) \varphi(t, x) \mathrm{d}t\mathrm{d}x = 0$$

for every $\varphi \in C_c^\infty$. Let $s = x + t$ and $r = x - t$.

$$\begin{aligned} \partial_x &= \frac{\partial s}{\partial x}\partial_s + \frac{\partial r}{\partial x}\partial_r \\ &= \partial_s + \partial_r \end{aligned} \qquad \begin{aligned} \partial_t &= \frac{\partial s}{\partial t}\partial_s + \frac{\partial r}{\partial t}\partial_r \\ &= \partial_s - \partial_r \end{aligned}$$

So, we can write $\partial_t^2 - \partial_x^2$ in terms of s and r.

$$\begin{aligned} \partial_t^2 - \partial_x^2 &= (\partial_s - \partial_r)^2 - (\partial_s + \partial_r)^2 \\ &= \left(\partial_s^2 - 2\partial_s\partial_r + \partial_r^2\right) - \left(\partial_s^2 + 2\partial_s\partial_r + \partial_r^2\right) \\ &= -4\partial_s\partial_r \end{aligned}$$

$x = \frac{s+r}{2}$ and $t = \frac{s-r}{2}$. Let $\tilde{\varphi}(r, s) := \varphi\left(\frac{s-r}{2}, \frac{s+r}{2}\right)$.

$$\left[\left(\partial_t^2 - \partial_x^2\right)\varphi\right]\left(\tfrac{s-r}{2}, \tfrac{s+r}{2}\right) = -4\partial_s\partial_r\tilde{\varphi}(r, s)$$

We have $\mathrm{d}t\mathrm{d}x = \frac{\partial(t,x)}{\partial(s,r)}\mathrm{d}s\mathrm{d}r$. We must calculate the Jacobian.

$$\begin{pmatrix} \partial_s t & \partial_r t \\ \partial_s x & \partial_r x \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \frac{1}{2}$$

So, $\mathrm{d}t\mathrm{d}x = \frac{1}{2}\mathrm{d}s\mathrm{d}r$.

$$
\begin{aligned}
&\int\int\left(f(x-t)+g(x+t)\right)\left(\partial_t^2-\partial_x^2\right)\varphi(t,x)\mathrm{d}t\mathrm{d}x\\
=&-2\int\int\left(f(r)+g(s)\right)\partial_s\partial_r\tilde{\varphi}(r,s)\mathrm{d}r\mathrm{d}s\\
=&-2\int\int f(r)\partial_s\partial_r\tilde{\varphi}(r,s)\mathrm{d}r\mathrm{d}s-2\int\int g(s)\partial_r\partial_s\tilde{\varphi}(r,s)\mathrm{d}s\mathrm{d}r\\
=&-2\int\int\partial_s\left(f(r)\partial_r\tilde{\varphi}(r,s)\right)\mathrm{d}r\mathrm{d}s-2\int\int\partial_r\left(g(s)\partial_s\tilde{\varphi}(r,s)\right)\mathrm{d}s\mathrm{d}r\\
=&-2\int\partial_s\underbrace{\left(\int f(r)\partial_r\tilde{\varphi}(r,s)\mathrm{d}r\right)}_{C_c^\infty(\mathbb{R})}\mathrm{d}s-2\int\partial_r\underbrace{\left(\int g(s)\partial_s\tilde{\varphi}(r,s)\mathrm{d}s\right)}_{C_c^\infty(\mathbb{R})}\mathrm{d}r\\
=&0+0\\
=&0
\end{aligned}
$$

□