Introduction to Complex Analysis

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Course text: A First Course in Complex Analysis (Beck, Marchesi, Pixton, and Sabalka), Version 1.54

Disclaimer: I take full responsibility for any errors, typos, and other flaws in these notes. If you see any or have other suggestions for improvement, feel free to reach out to me at felicialim@berkeley.edu. My notes are a work in progress and I appreciate any ideas on how they can be improved.

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Contents

Overview

Complex analysis is about calculus with complex numbers. Calculus involves derivatives and integrals and complex numbers involve numbers in the form $a + bi$ with $a, b \in \mathbb{R}$ and $i^2 = -1$.

Power series and Taylor series: given $f : \mathbb{R} \to \mathbb{R}$, consider

$$f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}.$$

If we stop after n terms, we have the "best possible" approximation of $f(x)$ near 0 of degree $< n$.

Suppose we have a Taylor series

$$\sum_{n=0}^{\infty} a_n x^n$$

for $a_n \in \mathbb{R}$ which converges absolutely for $|x| < r$ where r is the radius of convergence. So,

$$\sum_{n=0}^{\infty} |a_n||x|^n < \infty \text{ when } |x| < r.$$

When r is real, $x \in (-r, r)$. For example, $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ converges absolutely for $|x| < 1$.

Now, consider the case where real x is replaced by complex z.

$$\sum_{n=0}^{\infty} a_n z^n$$

If $|z| < r$, then $\sum_{n=0}^{\infty} |a_n||z|^n < \infty$ meaning the sum still converges absolutely. Complex z allows for a larger range of convergence, which is $z \in D(0, r)$ which is the disc of radius r centered at 0. In this situation, our real-valued series can be extended to a complex-valued series.

Example

Let $f(z)$ be a function where

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

It turns out that $f(z)$ as a function $\mathbb{Z} \to \mathbb{Z}$ is infinitely differentiable at $z = 0$ and all derivatives of $f(z)$ are 0 at $z = 0$. So, the Taylor series is 0 meaning the Taylor series converges to 0 for any value of z. The Taylor series converges to a function different from $f(z)$.

Now, suppose $z = it$ where $t \in \mathbb{R}$. $i^2 = -1$, so $e^{-1/z^2} = e^{1/t^2}$.

$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Along the imaginary axis, $f(z)$ is not continuous. $f(z)$ is not complex-differentiable at $z = 0$.

Let $z = x + iy$ where $x, y \in \mathbb{Z}$. Here, $\mathbb{C}$ is identified with $\mathbb{R}^2$. Consider a suitable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

This can be viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$ instead of $\mathbb{C} \to \mathbb{C}$.

We want to differentiate this function with respect to x and y.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial x}{\partial x} + i\frac{\partial y}{\partial x} \\
&= 1 + i0 \\
&= 1 \\
\frac{\partial f(z)}{\partial x} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} \\
&= f'(z) \\
\frac{\partial^2 f(z)}{\partial x^2} &= \frac{\partial f'(z)}{\partial x} \\
&= \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} \\
&= f''(z)
\end{aligned}
\qquad
\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial x}{\partial y} + i\frac{\partial y}{\partial y} \\
&= 0 + i1 \\
&= i \\
\frac{\partial f(z)}{\partial y} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} \\
&= if'(z) \\
\frac{\partial^2 f(z)}{\partial y^2} &= i\frac{\partial f'(z)}{\partial y} \\
&= i\frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} \\
&= -f''(z)
\end{aligned}$$

So,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(z) = 0$$

which is the Laplace equation in two dimensions. So, the real and imaginary parts of $f(z)$ satisfy the two dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0.$$

Complex analysis is a very powerful tool for solving the 2D Laplace equation, which has applications in physics.

Example
Consider the integral.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x &= \arctan(\infty) - \arctan(-\infty) \\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\
&= \pi
\end{aligned}$$

Now, consider

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}\,\mathrm{d}x$$

where $a \in \mathbb{R}$ ($a = 0$ yields the previous example). It is quite tricky to find this with real-valued techniques, but very standard using complex techniques (contour integration). The answer is $\pi e^{-|a|}$. So, complex analysis is very powerful for evaluating integrals.

Continuity, Limits, & Derivatives

The **derivative** of a complex-valued function is

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case. Recall that for $f(x)$ real-valued ($f : \mathbb{R} \to \mathbb{R}$), $\lim\limits_{x\to a} f(x) = L$ means $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for any x obeying $0 < |x-a| < \delta$, $|f(x) - L| < \varepsilon$. In other words, for any "tolerance" ε, we can guarantee $f(x)$ is within ε of L by forcing x to be close enough to a. Note that $x = a$ does not satisfy $0 < |x-a|$, so the value of $f(x = a)$ does not affect whether $\lim\limits_{x\to a} f(x)$ exists.

Continuity

If $\lim_{x \to a} f(x) = f(a)$, then the function is **continuous** at a. Note that for $L = f(a)$, $|f(x) - f(a)| < \varepsilon$ holds automatically when $x = a$ even though $0 < |x - a|$ is not satisfied. So, when discussing continuity, we leave out $0 < |x - a|$.

For a function $f : \mathbb{C} \to \mathbb{C}$, $\lim_{z \to a} f(z) = L$ means $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $0 < |z - a| < \delta$, then $|f(z) - L| < \varepsilon$. Here, z, a, and $f(z)$ can all be in $\mathbb{C}$. For real numbers, $0 < |x - a| < \delta$ is an open interval on the number line centered at a, but for complex numbers, $0 < |z - a| < \delta$ is an open disc on the complex plane centered at a. Similarly, if $\lim_{z \to a} f(z) = f(a)$, then f is continuous at $z = a$.

Example

1. Show that $f(z) = z$ is continuous at any point $a \in \mathbb{C}$.
   For $\varepsilon > 0$, $|f(z) - f(a)| < \varepsilon$ is equivalent to $|z - a| < \varepsilon$. So, taking $\delta = \varepsilon$ satisfies the condition for continuity meaning $f(z)$ is continuous at a.

2. $$\lim_{z \to 0} \frac{\overline{z}}{z}$$

   Suppose $\lim_{z \to 0} \frac{\overline{z}}{z} = L$ for some L. Let's take $\varepsilon = 1$. We want to show that $\exists \delta > 0$ such that if $0 < |z - 0| < \delta$, then $|\overline{z}/z - L| < \varepsilon = 1$.

   Note that $z = \frac{\delta}{2}$ obeys $|z - 0| < \delta$, as does $z = i\frac{\delta}{2}$.

   When $z = \frac{\delta}{2}$, $\overline{z} = \frac{\delta}{2}$. $\frac{\overline{z}}{z} = 1$, so $|1 - L| < 1$.
   When $z = i\frac{\delta}{2}$, $\overline{z} = -i\frac{\delta}{2}$. $\frac{\overline{z}}{z} = -1$, so $|-1 - L| < 1$.

   The regions of L that satisfy both conditions are the open discs centered at ± 1 on the real axis of radius 1. Since these discs are open, there is no overlap between the two. Since there are no points in common, there is no such L.

   The function $f(z) = \frac{\overline{z}}{z}$ converges to different values on the real and imaginary axes. In conclusion, there is no way to extend $\frac{\overline{z}}{z}$ to a continuous function at $z = 0$

Limits

Limits of complex numbers obey the usual properties from real-valued analysis. Let $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$.

- $\lim_{x \to a} (f(x) + g(x)) = L_1 + L_2$
- $\lim_{x \to a} (f(x)g(x)) = L_1 L_2$
- $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L_1}{L_2}$ (provided $L_2 \neq 0$)

If f and g are continuous at a, then $L_1 = f(a)$ and $L_2 = g(a)$. If we plug these values into the rules above, we can see that if f and g are continuous, then $f + g$, fg, and f/g are also all continuous.

Additionally, if $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$. To show this, show that $|g(f(x)) - g(f(a))| < \varepsilon$.

> By continuity of g at $f(a)$, $|g(w) - g(f(a))| < \varepsilon$ when $|w - f(a)| < \delta_1$ for some $\delta_1 > 0$ where $w = f(x)$ for some x. By continuity of f at a, $|f(x) - f(a)| < \delta_1$ holds for $|x - a| < \delta_2$ for some $\delta_2 > 0$. Under this condition, $|g(f(x)) - g(f(a))| < \varepsilon$.

Derivatives

Recall the limit definition of the derivative:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

Equivalently, $f(z)$ is differentiable at $z = a$ if and only if the function

$$\frac{f(z) - f(a)}{z - a}$$

extends to a continuous function at $z = a$. This value is $f'(a)$. This can be seen by rewriting the function

$$\frac{f(a + (z-a)) - f(a)}{z - a}$$

where $h = z - a$.

Example

1. $f(z) = z$ is differentiable with $f'(z) = 1$.

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{z + h - z}{h} \\ &= \lim_{h \to 0} 1 \\ &= 1 \end{aligned}$$

2. $f(z) = \overline{z}$ is not differentiable (but is continuous).

$$\begin{aligned} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \to 0} \frac{\overline{z+h} - \overline{z}}{h} \\ &= \lim_{h \to 0} \frac{\overline{z} + \overline{h} - \overline{z}}{h} \\ &= \lim_{h \to 0} \frac{\overline{h}}{h} \end{aligned}$$

In the previous example, we showed that such a limit does not exist.

Properties of complex derivatives

Proposition

If $f(z)$ is differentiable at $z = a$, then $f(z)$ is continuous at $z = a$.

- **Proof**: we want to show that $\lim_{z \to a} f(z) = f(a)$.

$$\begin{aligned} \lim_{z \to a} f(z) - f(a) &= \lim_{z \to a} \frac{f(z) - f(a)}{z - a}(z - a) \\ &= \lim_{z \to a} \frac{f(z) - f(a)}{z - a} \cdot \lim_{z \to a}(z - a) \end{aligned}$$

The first limit exists since f is differentiable at a and the second can be computed directly.

$$\begin{aligned} &= f'(a) \cdot 0 \\ &= 0 \end{aligned}$$

$\lim_{z \to a} f(z) - f(a) = 0$, so $\lim_{z \to a} f(z) = f(a)$. $\square$

The same properties of real derivatives hold.

- $\frac{\mathrm{d}}{\mathrm{d}z} c \cdot f(z) = c \cdot f'(z)$ for constant $c \in \mathbb{Z}$
- $\frac{\mathrm{d}}{\mathrm{d}z} = f' + g'$
- $\frac{\mathrm{d}}{\mathrm{d}z} = f' \cdot g + f \cdot g'$
- $\frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{f}{g}\right) = \frac{f' \cdot g - f \cdot g'}{g^2}$
- $\frac{\mathrm{d}}{\mathrm{d}z} f(g(z)) = g'(z) f'(g(z))$

The proofs of all of these are the same as in the real-valued case.

Proposition

$$\frac{\mathrm{d}}{\mathrm{d}z} z^n = nz^{n-1}$$

for all integers n.

- **Proof**: use mathematical induction and the properties of complex derivatives.
  1. Base case: when $n = 0$, $\frac{\mathrm{d}}{\mathrm{d}z} z^0 = \frac{\mathrm{d}}{\mathrm{d}z} 1 = 0 = 0 \cdot z^{-1}$.
  2. Induction hypothesis: suppose $\frac{\mathrm{d}}{\mathrm{d}z} z^{n-1} = nz^{n-2}$ for some $n \geq 0$.
  3. Inductive step: by the product rule,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z} z^n &= \frac{\mathrm{d}}{\mathrm{d}z} \left(z \cdot z^{n-1}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}z}[z] \cdot z^{n-1} + z \cdot \frac{\mathrm{d}}{\mathrm{d}z} z^{n-1} \end{aligned}$$

By the induction hypothesis, $\frac{\mathrm{d}}{\mathrm{d}z} z^{n-1} = (n-1)z^{n-2}$.

$$\begin{aligned} &= 1 \cdot z^{n-1} + z \cdot (n-1) z^{n-2} \\ &= z^{n-1} + (n-1) z^{n-1} \\ &= nz^{n-1} \end{aligned}$$

For negative n, apply the quotient rule to $1/z^n$. $\square$

Being differentiable at a point says little about how "nice" it is.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

$f(x)$ is not continuous anywhere on $\mathbb{R}$, so it is nowhere differentiable.

Now, consider $x^2 f(x)$. This is differentiable at $x = 0$, which can be seen from the definition.

$$\begin{aligned} \lim_{h \to 0} \frac{h^2 f(h) - 0^2 f(0)}{h} &= \lim_{h \to 0} h f(h) \\ &= 0 \end{aligned}$$

We know this from the Squeeze Theorem since $h \to 0$ and because $f(h)$ is bounded.

Although $x^2 f(x)$ is differentiable at 0, it is still not a "nice" function and shares many characteristics of $f(x)$.

Holomorphic Functions on a Plane

A function $f : \mathbb{C} \to \mathbb{C}$ is **holomorphic** at a point a if it is differentiable at z for all z within distance r of a for some $r > 0$. In other words, the function $f(z)$ is differentiable everywhere sufficiently close to point a.

The **open disk** of radius r centered at $a \in \mathbb{C}$ is $D(a, r) := \{z \in \mathbb{C} \mid |z - a| < r\}$. The **closed disk** of radius r centered at $a \in \mathbb{C}$ is $\overline{D}(a, r) := \{z \in \mathbb{C} \mid |z - a| \leq r\}$.

So, we can say $f(z)$ is holomorphic at $a \in \mathbb{C}$ if $f(z)$ is differentiable on an open disc centered at a. If f is holomorphic (without specifying a point), then f is holomorphic at all points.

Example

1. Recall z^n is differentiable everywhere for $n \geq 0$. Multiplying by constants and adding yields linear combinations, so we get
$$a_0 z^0 + a_1 z^1 + \cdots + a_d z^d$$
which is a polynomial. Conversely, every polynomial arises this way, so all polynomials are differential everywhere. Thus, all polynomials are holomorphic.

2. Now, let $f(z) = |z|^2 = z\overline{z}$. $f(z)$ is differentiable at 0.
$$\begin{aligned}\lim_{h \to 0} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0} \frac{h\overline{h}}{h} \\ &= \lim_{h \to 0} \overline{h} \\ &= 0\end{aligned}$$
However, $f(z)$ is not differentiable anywhere else (proof is an exercise). If $f(z)$ were holomorphic, it would be differentiable on some disk $D(0, r)$ which contains infinitely many points. But since $f(z)$ is differentiable on just one point, it is not holomorphic.

Cauchy-Riemann Equations

How can we tell if a function is complex-differentiable? Lets reduce this to a question of real derivatives. Let $z = x + iy$ with $x, y \in \mathbb{R}$. If $f : \mathbb{C} \to \mathbb{C}$, we can define the derivative of f.
$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} && \text{(where } h \text{ is real)} \\ &= \lim_{h \to 0} \frac{f(x + h + iy) - f(x + iy)}{h} \\ \frac{\partial f}{\partial y} &= \lim_{h \to 0} \frac{f(z + i(y + h)) - f(z)}{h} && \text{(where } h \text{ is real)} \\ &= \lim_{h \to 0} \frac{f(x + iy + ih) - f(x + iy)}{h}\end{aligned}$$

Example

Let $f(z) = z^2$. $f(x + iy) = (x + iy)^2 = x^2 - y^2 + 2ixy$.
$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{(x + h)^2 - y^2 + 2i(x + h)y - (x^2 - y^2 + 2ixy)}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 2ihy}{h} \\ &= 2x + 2iy \\ &= 2z \\ &= f'(z)\end{aligned}$$

Now for y:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{x^2-(y+h)^2+2ix(y+h)-(x^2-y^2+2ixy)}{h}\\ &= \lim_{h\to 0}\frac{-2yh-h^2+2ixh}{h}\\ &= -2y+2ix\\ &= 2i(x+iy)\\ &= if'(z)\end{aligned}$$

So, $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$.

Theorem: Cauchy-Riemann Equations

(i) If $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and obey

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

(ii) If $f : \mathbb{C} \to \mathbb{C}$ is a function and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous on some open disc centered at z, if $\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$, then f is complex-differentiable at z.

- **Proof**

  (i) f is complex-differentiable, so

  $$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = f'(z).$$

  For every $\varepsilon > 0$, $\exists\delta > 0$ such that $\left|\frac{f(z+h)-f(z)}{h} - f'(z)\right| < \delta$ whenever $|h| < \delta$, or $h \in D(0,\delta)$.
  Suppose h is real. This means the limit is equivalent to $\frac{\partial f}{\partial x}$, so $\frac{\partial f}{\partial x} = f'(z)$.
  Now, let h be purely imaginary. $h = ik$, where $k \in \mathbb{R}$.

  $$\frac{f(z+h)-f(z)}{h} = \frac{f(x+iy+ik)-f(x+iy)}{ik}$$

  $\lim_{h\to 0}$ becomes $\lim_{k\to 0}$ since $|h| = |k|$, so

  $$\lim_{k\to 0}\frac{f(z+ik)-f(z)}{ik} = f'(z).$$

  The limit is equivalent to $\frac{1}{i}\frac{\partial f}{\partial y}$, so we have $\frac{\partial f}{\partial y} = if'(z)$. So,

  $$\frac{\partial f}{\partial x} = f'(z) = -i\frac{\partial f}{\partial y}.$$

  (ii) To show that f is complex-differentiable at z, we want to show

  $$\lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h} - \frac{\partial f}{\partial x}\right) = 0$$

  which would show that $f'(z)$ exists and equals $\frac{\partial f}{\partial x}$. Let $h = a + ib$ where $a, b \in \mathbb{R}$. Note that $\left|\frac{a}{h}\right|, \left|\frac{b}{h}\right| \leq 1$. This can be seen by expressing h in polar form:

  $$\begin{aligned}h &= re^{i\theta}\\ a &= r\cos(\theta)\\ b &= r\sin(\theta)\end{aligned}$$

So, if $h \to 0$, $a, b \to 0$ as well.

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(z+a+ib) - f(z+a) + f(z+a) - f(z)}{a+ib} - \frac{a+ib}{a+ib}\frac{\partial f}{\partial x} \\
&= \lim_{h\to 0} \frac{f(z+a+ib) - f(z+a)}{ib} \cdot \frac{ib}{a+ib} - \frac{\partial f}{\partial x}\frac{ib}{a+ib} \\
&\quad + \lim_{h\to 0} \frac{f(z+a) - f(z)}{a} \cdot \frac{a}{a+ib} - \frac{\partial f}{\partial x}\frac{a}{a+ib}
\end{aligned}$$

Show that the second limit goes to 0. Note that as $h \to 0$, a and b also $\to 0$.

$$\lim_{h\to 0} \underbrace{\left(\frac{f(z+a) - f(z)}{a} - \frac{\partial f}{\partial x} \right)}_{\to 0 \text{ as } h \to 0} \frac{a}{a+ib}$$

Note that $\left|\frac{a}{a+ib}\right| \le 1$, and for $a < \delta$, we have

$$\left| \frac{f(z+a) - f(z)}{a} - \frac{\partial f}{\partial x} \right| < \varepsilon$$

So,

$$\left| \frac{f(z+a) - f(z)}{a} - \frac{\partial f}{\partial x} \right| \left| \frac{a}{a+bi} \right| < \varepsilon$$

for $|h| < \delta$ (which implies $|a| < \delta$). So, the second limit is 0 as needed.

Now, for the imaginary axis (first limit):

$$\lim_{h\to 0} \left(\frac{f(z+a+ib) - f(z+a)}{ib} + i\frac{\partial f}{\partial y} \right) \frac{ib}{a+ib}$$

This is similar to the previous case, but as $b \to 0$, $\frac{f(z+a+ib)-f(z+a)}{ib} \to \frac{1}{i}\frac{\partial f}{\partial y}$. By continuity of $\frac{\partial f}{\partial y}$ as $a \to 0$, $\frac{\partial f}{\partial y} \to \frac{\partial f}{\partial y}$. Note once again that $\left|\frac{ib}{a+ib}\right| \le 1$. So, we have $\frac{1}{i}\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial y} = 0$, so the limit is as needed. $\square$

The Cauchy-Riemann equations are typically written in their real and imaginary parts:

$$f(x+iy) = u(x,y) + iv(x,y)$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$, or

$$\begin{aligned}
u(x,y) &= \mathrm{Re}(f(x+iy)) \\
v(x,y) &= \mathrm{Im}(f(x+iy))
\end{aligned}$$

So, we get

$$u_x + iv_y = -i(u_y + iv_y)$$

where the subscript denotes partial derivative: $u_x = \frac{\partial u}{\partial x}$. By taking the real and imaginary parts, we have

$$\begin{aligned}
u_x &= v_y \\
v_x &= -u_y
\end{aligned}$$

which is the normal way to write the Cauchy-Riemann equations.

Example

1. Let $f(z) = z^2$ where $z = x + iy$.

$$\begin{aligned} f(x+iy) &= x^2 - y^2 + 2ixy \\ u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \end{aligned}$$

From this, we can verify the Cauchy-Riemann equations.

$$\begin{aligned} u_x &= 2x & v_y &= 2x \\ u_y &= -2y & v_x &= 2y \end{aligned}$$

So, $u_x = v_y$ and $u_y = -v_x$ meaning the equations hold.

2. Now, let $f(z) = |z|^2$ where $z = x + iy$.

$$\begin{aligned} f(x+iy) &= x^2 + y^2 \\ u(x,y) &= x^2 + y^2 \\ v(x,y) &= 0 \end{aligned}$$

Find the partial derivatives:

$$\begin{aligned} u_x &= 2x & v_y &= 0 \\ u_y &= 2y & v_x &= 0 \end{aligned}$$

So, the Cauchy-Riemann equations do not hold unless $x = y = 0$, or $z = 0$. We know $f(z) = |z|^2$ is differentiable only at 0.

Suppose we have $f = u + iv$ where u and v are the real and imaginary components respectively. Then,

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial x} u_x \\ &= \frac{\partial}{\partial x} v_y && \text{(Cauchy-Riemann equations)} \\ &= \frac{\partial}{\partial x}\frac{\partial}{\partial y} v \\ &= \frac{\partial}{\partial y}\frac{\partial}{\partial x} v \end{aligned}$$

The order of derivatives can be swapped if the 2$^{\text{nd}}$ derivative exists and is continuous.

$$\begin{aligned} &= \frac{\partial}{\partial y} v_x \\ &= \frac{\partial}{\partial y}\left(-u_y\right) \\ &= -u_{yy} \end{aligned}$$

where the notation u_{xx} denotes $\frac{\partial}{\partial x}\frac{\partial}{\partial x} u = \frac{\partial^2 u}{\partial x^2}$. So, we have $u_{xx} + u_{yy} = 0$ which is Laplace's equation in 2 dimensions. Similarly,

$$v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = -(v_y)_y = -v_{yy}$$

so if $f = u + iv$, both u and v obey Laplace's equation.

For real-valued functions, we know that if $f'(x) = 0$, then f is a constant. In the complex case, the Cauchy-Riemann equations can be used to prove the same.

$f'(z) = \frac{\partial f}{\partial x}$, so $f'(z) = 0$ means $u_x + iv_x = 0$. Taking the real and imaginary parts separately yields $u_x = 0$ and $v_x = 0$, and the Cauchy-Riemann equations imply that $u_y = 0$ and $v_y = 0$ as well.

$u_x = 0$, so for a fixed y, $u(x,y)$ is constant in terms of x but may depend on y. So, $u(x,y) = g(y)$, but $u_y = 0$ so $g'(y) = 0$. This means g is independent of y, so u is globally constant. The same argument holds for v, so $f(z)$ is constant.

Special functions

Möbius Transforms

A **Möbius transform** (or linear fractional transform) is a function of form

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Note that if $ad = bc$, then $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$ so the rows are linearly dependent. So, $\exists$ some constants λ, μ such that

$$\lambda(a, b) + \mu(c, d) = (0, 0).$$

So, $a = -\frac{\mu}{\lambda}c$ and $b = -\frac{\mu}{\lambda}d$ meaning

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{-\frac{\mu}{\lambda}(cz+d)}{cz+d} \\ &= -\frac{\mu}{\lambda} \end{aligned}$$

which is independent of z, so $f(z)$ is constant.

Suppose we have two such functions

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \text{ and } f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Composing these function yields

$$\begin{aligned} f_1(f_2(z)) &= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\ &= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)} \end{aligned}$$

The composition of f_1 and f_2 is another Möbius transform. Note that if we multiply two matrices with entries of coefficients:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}$$

These entries coincide with the coefficients of the composition of the functions. In other words, if we have functions $f_M(z)$ and $f_N(z)$ where M and N correspond to the respective coefficient matrices, $f_M(f_N(z)) = f_{M \cdot N}(z)$ where $M \cdot N$ is the matrix product. We can describe this as a homomorphism from the general linear group of 2×2 matrices to the group of Möbius transforms.

Note that there is some redundancy in this notation. Let λ be a constant.

$$\begin{aligned} M &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} & f_M(z) &= \frac{az+b}{cz+d} \\ \lambda M &= \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix} & f_{\lambda M}(z) &= \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} \end{aligned}$$

The λ's in $f_{\lambda M}(z)$ cancel out, so $f_M(z) = f_{\lambda M}(z)$. So, scaling the matrix does not affect the resulting Möbius transform.

The matrix definition can also be used to find the inverse function to f_M, which is $f_{M^{-1}}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The fraction $\frac{1}{ad-bc}$ is an irrelevant scalar multiple, so the inverse function to $\frac{az+b}{cz+d}$ is $\frac{dz-b}{-cz+a}$, which is also a Möbius transform. Möbius transforms have inverses, so they should be bijections. However, we must specify a case. If $c \neq 0$, then $\frac{az+b}{cz+d}$ is undefined at $z = -\frac{d}{c}$ since this makes the denominator 0. Let's consider the value at $z = -\frac{d}{c}$ to be ∞. We can evaluate $\frac{az+b}{cz+d}$ at ∞. For our purpose, the complex variable z approaches ∞ when $\frac{1}{z} \to 0$.

$$\frac{az+b}{cz+d} = \frac{a+b/z}{c+d/z} \quad \underset{z\to\infty}{\longrightarrow} \quad \frac{a+0}{c+0} = \frac{a}{c}$$

So, we view Möbius transforms as functions from $\mathbb{C}\cup\{\infty\}$ to $\mathbb{C}\cup\{\infty\}$. $\mathbb{C}\cup\{\infty\}$ or $\hat{\mathbb{C}}$ is the extended complex plane, or the Riemann Sphere or 1-dimensional complex projective space ($\mathbb{P}^1_{\mathbb{C}}$). Note that for real functions, there are multiple notions of going to infinity: $x \to \pm\infty$. Here, we work with just one infinite point. When $c = 0$, we can view $\frac{a}{c}$ as ∞. This makes all Möbius transforms into bijections.

It turns out that if we apply a Möbius transform to a line or circle in the complex plane, the result is a line or a circle (circles can turn into lines and vice versa).

Example

Let $f(z) = \frac{z-1}{iz+i} = \frac{z-1}{i(z+1)}$. Apply this to the unit circle, so take $z = e^{i\theta}$.

$$\begin{aligned} f(e^{i\theta}) &= \frac{e^{i\theta}-1}{i(e^{i\theta}+1)} \\ &= \frac{\cos(\theta) - 1 + i\sin(\theta)}{i(\cos(\theta)+1+i\sin(\theta))} \end{aligned}$$

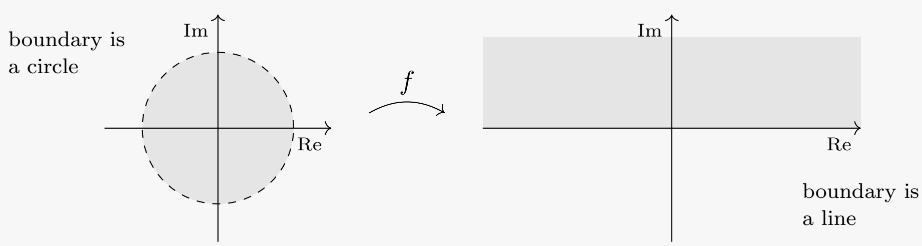
Use the following trig identities:

$$\begin{aligned} \cos(\theta) &= 2\cos(\theta/2)^2 - 1 \\ &= 1 - 2\sin(\theta/2)^2 \\ \sin(\theta) &= 2\sin(\theta/2)\cos(\theta/2) \end{aligned}$$

So, we have

$$\begin{aligned} f(e^{i\theta}) &= \frac{-2\sin\left(\frac{\theta}{2}\right)^2 + i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{i\left(2\cos\left(\frac{\theta}{2}\right)^2 + i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)} \\ &= \frac{2i\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)\sin\left(\frac{\theta}{2}\right)}{2i\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)\cos\left(\frac{\theta}{2}\right)} \\ &= \tan(\theta/2) \end{aligned}$$

$\tan(\theta/2) \in \mathbb{R}$ for $\theta \in [0, 2\pi)$, so we have turned a circle into a line. It turns out that f sends the interior of the unit disk to the interior of the upper half-plane.



If $g(z)$ is holomorphic on the upper half-plane, then $g(f(z))$ is a holomorphic function on the unit disk. Taking real and imaginary parts gives a solution to the Laplace equations. So, the solution to $\Delta u = 0$ on the upper half-plane yields $g(z)$. Applying f to $g(z)$ yields $g(f(z))$, which is the solution of $\Delta u = 0$ on the disk. This allows us to convert between solutions to the Laplace equations in different domains.

The textbook proves that Möbius transforms preserve the set of lines and circles by showing that $\frac{az+b}{cz+d}$ can be written in three steps:

1. A translation: $z \mapsto z + a$
2. A dilation: $z \mapsto az$
3. An inversion: $z \mapsto \frac{1}{z}$

It can be seen that 1. and 2. preserve lines and circles, and 3. can be checked directly.

Suppose $c = 0$. Then, $f(z) = \frac{a}{d}z + \frac{b}{c}$ and we have the following:

$$z \longrightarrow \tfrac{a}{d}z \longrightarrow \tfrac{a}{d}z + \tfrac{b}{d}$$

First, we have a rotation/dilation. This can be seen in terms of $re^{i\theta}$, where z is scaled by r and rotated by θ. This maintains lines and circles as lines and circles.

Next, we have a translation. This clearly also maintains lines and circles as lines and circles. So, when $c = 0$, $f(z)$ maintains lines and circles.

Now, suppose $n \neq 0$. We have

$$f(z) = \frac{az+b}{cz+d} = \frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}} + \frac{a}{c}.$$

So, we have the following:

$$z \longrightarrow z + \frac{d}{c} \longrightarrow \frac{1}{z+\frac{d}{c}} \longrightarrow \frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}} \longrightarrow \frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}} + \frac{a}{c}$$

In order, this is a translation, inversion, dilation/rotation, and a translation. The only difference between this and the case where $c = 0$ is the inversion, so this must be the step where the change between lines and circles occurs.

Recall the equation of a circle:

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{1}$$

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0 \tag{2}$$

Note that equation (2) has 4 parameters while (1) only has 3. This is because of the coefficient α, where in (1), the coefficient of x^2 and y^2 is both 1. Having α allows us to set $\alpha = 0$, which turns the equation into that of a line. We can find solutions to this equation by completing the square:

$$\alpha\left(x + \frac{\beta}{2\alpha}\right)^2 - \frac{\beta^2}{4\alpha} + \alpha\left(y + \frac{\gamma}{2\alpha}\right)^2 - \frac{\gamma^2}{4\alpha} + \gamma = 0$$

We only get a circle if $\beta^2 + \gamma^2 > 4\alpha\delta$. If $\alpha = 0$, then $\beta^2 + \gamma^2 > 0$ which is the condition for a line.

Now, we will observe the effect of inversion on solutions to the equation. Suppose $z = x + iy$ satisfies the equation.

$$\begin{aligned} z^{-1} &= \frac{\overline{z}}{|z|^2} \\ &= \frac{x - iy}{x^2 + y^2} \\ &= u + iv \end{aligned}$$

where $u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+iy}$.

$$\begin{aligned} \alpha + \beta\frac{x}{x^2+y^2} + \gamma\frac{y}{x^2+y^2} + \delta\frac{1}{x^2+y^2} &= 0 \\ \alpha + bu - \gamma v + \delta(u^2 + v^2) &= 0 \end{aligned}$$

Now, we have a similar equation but with constant term α and degree 2 coefficient δ.

- Having a degree 2 coefficient $= 0$ is equivalent to having a line.
- Having a constant term $= 0$ is equivalent to having the origin on the line/circle.

We have four possible outcomes.

1. $\alpha = 0$ and $\gamma = 0$

   Before inverting: line through the origin
   After inverting: line through the origin

2. $\alpha = 0$ and $\gamma \neq 0$

   Before inverting: line not through the origin
   After inverting: circle through the origin

3. $\alpha \neq 0$ and $\gamma = 0$

   Before inverting: circle through the origin
   After inverting: line not through the origin

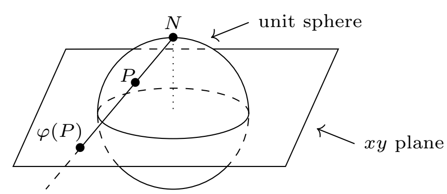
4. $\alpha \neq 0$ and $\gamma \neq 0$

   Before inverting: circle not through the origin
   After inverting: circle not through the origin

Stereographic Projections

We can use a geometric proof to show that Möbius transforms turn lines into circles and vice versa. In a 3D space, consider the unit sphere and $x-y$ plane. The equation of the unit circle is $x^2+y^2+z^2=1$ and the equation of the plane is $z=0$.

Let N be the north pole of the sphere: $(0,0,1)$. We want to construct a bijection between all points on the sphere and points on the plane. Define φ : sphere $\to$ plane as follows: draw a line through N and the given point P on the sphere. $\varphi(P)$ is the intersection of this line and the plane. This can be described algebraically.



Let $P=(x,y,z)$. The line through N and P is parameterized as

$$\underbrace{(0,0,1)}_{N} + \underbrace{\lambda(x,y,z-1)}_{P-N}.$$

This intersects the plane when the z-coordinate is 0: $1+\lambda(z-1)=0$, or $\lambda = \frac{1}{1-z}$. So, the resulting point is $\left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$. The coordinate on the plane is then $\left(\frac{x}{1-z}, \frac{y}{1-z}\right)$.

This definition doesn't work at $P=N$, since then $z=1$ and the denominator is 0. The geometric definition also does not work when drawing a line from a point to itself. When we have $z\to 1$ (which is $P\to N$), we have the following:

$$\begin{aligned}
\left|\left(\frac{x}{1-z}, \frac{y}{1-z}\right)\right| &= |\varphi(P)|^2 \\
&= \frac{x^2}{(1-z)^2} + \frac{y^2}{(1-z)^2} \\
&= \frac{x^2+y^2}{(1-z)^2} \\
&= \frac{1-z^2}{(1-z)^2} && (x^2+y^2+z^2=1) \\
&= \frac{1+z}{1-z}
\end{aligned}$$

So, as $P\to N$, $z\to 1$ and $|\varphi(P)|^2\to\infty$. This means $\frac{1}{\varphi(P)}\to 0$ where $\varphi(P)$ is viewed as a complex number. To extend φ continuously to a function on the whole sphere (including $P=N$), take $\varphi(N)=\infty$ so φ is a function from the sphere to the extended complex plane.

If we have a circle on the sphere (an intersection between a sphere and a plane), the stereographic projection of this circle is a circle in the plane *unless* the circle goes through N. If so, the circle is the intersection of some plane containing N and P.

> $\varphi(P)$, which is on the line NP, is also on the plane which forms the circle. We know that $\varphi(P)$ lies on the horizontal plane. $\varphi(P)$ is on the intersection of the two planes, which is a line.
>
> So, circles on the sphere project to lines if they go through N and project to circles otherwise. In terms of the Riemann sphere, Möbius transforms send circles to other circles.
>
> Note that there are two proofs for showing that intersections of planes and spheres are circles. We can use the equations of planes and circles (which can get complicated with many variables), and we can use geometry (which can get complicated since it is in 3 dimensions).

Similarly, most properties of Möbius transforms are best understood in terms of the Riemann sphere.

Example

Using the same example from before, $f(z) = \frac{z-1}{i(z+1)}$ sends the unit sphere to the real axis. The unit circle is the "equator" of the unit sphere. The real axis is the "prime meridian" of the unit sphere.

Holomorphic Functions on a Sphere

We have described holomorphic functions on the plane. Holomorphic functions on the sphere can be defined in terms of f and φ.

$f \circ \varphi^{-1}$ is a function $\mathbb{C} \to \mathbb{C}$. We can define f to be holomorphic at P if $f \circ \varphi^{-1}$ is holomorphic at $\varphi(P)$.

$$\begin{tikzcd} \text{sphere} \arrow[r, "f"] \arrow[dr, "\varphi"'] & \mathbb{C} \arrow[d, "f\circ\varphi^{-1}"] \\ & \text{plane} \end{tikzcd}$$

Exponential and Trigonometric Functions

Exponential Functions

The exponential function is

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x \cos(y) + ie^x \sin(y) && \text{(Euler's Formula)} \end{aligned}$$

To show that e^z is holomorphic, we must show that the Cauchy-Riemann equations hold.

$$e^z = \underbrace{e^x \cos(y)}_{u(x,y)} + i\,\underbrace{e^x \sin(y)}_{v(x,y)}$$

Evaluate u_x, u_y, v_x, and v_y:

$$\begin{aligned} u_x &= e^x \cos(y) & v_x &= e^x \sin(y) \\ u_y &= -e^x \sin(y) & v_y &= e^x \cos(y) \end{aligned}$$

$u_x = v_y$ and $u_y = v_x$, so e^x is holomorphic.

Note that a more general definition of e^z is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This definition makes sense even if z is a square matrix, but it requires checking if the series converges and some properties are harder to see, so we won't use it.

$e^z = e^x(\cos(y) + i\sin(y))$ is never 0 since $e^x \neq 0$ for any x and $\cos^2(y) + \sin^2(y) = 1$, so $\cos(y) + i\sin(y)$ cannot be 0.

Trigonometric Functions

We define cos and sin in $\mathbb{C}$ as

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

for any $z \in \mathbb{C}$. Since e^{iz} and e^{-iz} are holomorphic, so are $\sin(z)$ and $\cos(z)$.

Example

1. $\sin(x + iy) = \sin(x) \cdot \cosh(y) + i\cos(x) \cdot \sinh(y)$, where

$$\cosh(y) = \frac{e^{y} + e^{-y}}{2} \text{ and } \sinh(y) = \frac{e^{y} - e^{-y}}{2}.$$

   Note that for $x \in \mathbb{R}$, $|\sin(x)| \leq 1$, but for $z = iy$, $\sin(iy) = i\sinh(y)$ which grows exponentially in y, so $\sin(z)$ is not bounded.

2. The double angle formula holds:

$$\begin{aligned} \sin(2z) &= \frac{e^{2iz} - e^{-2iz}}{2i} \\ &= \frac{2(e^{iz} + e^{-iz})}{2} \cdot \frac{e^{iz} - e^{-iz}}{2i} \\ &= 2\sin(z)\cos(z) \end{aligned}$$

3. It can also be shown directly that the Pythagorean identity holds.

Logarithmic Functions

We want $\log(z)$ to be the inverse function to e^z, meaning $e^{\log(z)} = z$, but if this holds, then

$$e^{\log(z) + 2\pi i k} = z$$

for $k = \ldots, -2, -1, 0, 1, 2, \ldots$ since $e^{2\pi i k} = 1$. There usually isn't a "correct" choice of k unless $z \in \mathbb{R}^+$, in which case we generally want $\log(z) \in \mathbb{R}$.

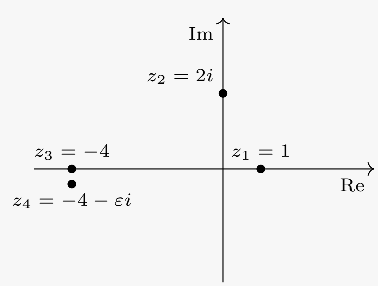
The **principal logarithm** is defined by $z = re^{i\theta}$ with $-\pi < \theta \leq \pi$. Then,

$$\log(z) = \log(r) + i\theta.$$

Example

Observe the following points on the complex plane. Note that ε is small and positive. We can calculate the log of each z value:

$$\begin{aligned} \log(z_1) &= 0 \\ \log(z_2) &= \log(2) + i\frac{\pi}{2} \\ \log(z_3) &= \log(4) + \pi i \\ \log(z_4) &\approx \log(4) - \pi i \end{aligned}$$



As $\varepsilon \to 0$, $\log(z_4) \to \log(4) - \pi i$.

$z_3 \in \mathbb{R}$, but $\log(z_3)$ has a complex component meaning log is not continuous on the negative real axis. This means that log is not differentiable on the negative real axis.

We want to check if $\log(z)$ is holomorphic away from the negative real axis. We have $z = x + iy = re^{i\theta}$ where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$.

$$\log(z) = \underbrace{\frac{1}{2}\log\left(x^2+y^2\right)}_{u(x,y)} + i\underbrace{\arctan\left(\frac{y}{x}\right)}_{v(x,y)}$$

$$\begin{aligned} u_x &= \frac{1}{2}\frac{2x}{x^2+y^2} & v_x &= -\frac{y}{x}\frac{1}{1+\left(\frac{y}{x}\right)^2} \\ &= \frac{x}{x^2+y^2} & &= -\frac{y}{x^2+y^2} \\ u_y &= \frac{1}{2}\frac{2y}{x^2+y^2} & v_y &= \frac{1}{x}\frac{1}{1+\left(\frac{y}{x}\right)^2} \\ &= \frac{y}{x^2+y^2} & &= \frac{x}{x^2+y^2} \end{aligned}$$

$u_x = v_y$ and $u_y = -v_x$, so the Cauchy-Riemann equations hold. $\log(z)$ is holomorphic where the derivatives "make sense", so we omit 0 and the negative real axis. Thus, $\log(z)$ is holomorphic in $\mathbb{C}$ other than the origin and negative real axis.

$$u_{xx} + u_{yy} = \frac{1\left(x^2+y^2\right) - 2x^2}{\left(x^2+y^2\right)^2} + \frac{1\left(x^2+y^2\right) - 2y^2}{\left(x^2+y^2\right)^2} = 0$$

So, $u(x,y)$ satisfies the Laplace equation on $\mathbb{C}\backslash\{0\}$.

We can solve the Cauchy-Riemann equations.

$$\begin{aligned} v_y &= \frac{x}{x^2+y^2} & &\longrightarrow & &\arctan\left(\frac{y}{x}\right) + C_1(x) \\ v_x &= -\frac{y}{x^2+y^2} & &\longrightarrow & &\arctan\left(\frac{y}{x}\right) + C_2(y) \end{aligned}$$

So, we get $\arctan\left(\frac{y}{x}\right) + C$ where C is independent of both x and y. $\theta = \arctan\left(\frac{y}{x}\right)$ does not extend to a continuous function on $\mathbb{C}\backslash\{0\}$ and for every "loop" around zero, θ increases by 2π. So, its value cannot be the same at both 0 and 2π.

There will not be a solution on a region that lets us enclose the origin in a circle. By cutting the negative real axis, we avoid this. $u(x,y)$ is not the real part of a holomorphic function on $\mathbb{C}\backslash\{0\}$, but it is on $\mathbb{C}\backslash\{$negative real axis$\}$.

Integration

For function $f : \mathbb{R} \to \mathbb{C}$, we define the integral on $[a,b]$ as

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b \mathrm{Re}(f(x))\,\mathrm{d}x + i\int_a^b \mathrm{Im}(f(x))\,\mathrm{d}x$$

where $f(x) = \mathrm{Re}(f(x)) + i\,\mathrm{Im}(f(x))$. This is automatic if we require $\int(f+g)\,\mathrm{d}x = \int f\,\mathrm{d}x + \int g\,\mathrm{d}x$ and $\int(cf)\,\mathrm{d}x = c\int f\,\mathrm{d}x$ where constant $c \in \mathbb{C}$.

When integrating a function $f : \mathbb{C} \to \mathbb{C}$, we must specify what path we are integrating along, not just endpoints. For us, a path is the image of a function $\gamma : [a,b] \to \mathbb{C}$ for continuous γ. The integral of a function $f : \mathbb{C} \to \mathbb{C}$ along the curve parameterized by $\gamma : [a,b] \to \mathbb{C}$ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\,.$$

Note that $f(\gamma(t))\gamma'(t)$ is a function $\mathbb{R} \to \mathbb{C}$, which we have already defined how to integrate.

Proposition

$\int_\gamma f(z)\,dz$ depends on the image of $\gamma(t)$, not the parameterization of $\gamma(t)$ itself.

- **Proof**: Suppose γ and β parameterize the same curve in different ways: $\gamma(t) = \beta(s)$, and at the endpoints, $\gamma(a) = \beta(a)$ and $\gamma(b) = \beta(b)$.

  Let
  $$\Theta(t) := \beta^{-1}(\gamma(t)) : [a,b] \to [a,b]$$
  where $\gamma, \beta : [a,b] \to \mathbb{C}$. We would expect the following properties from the reparameterization:

  - $\Theta(a) = a$
  - $\Theta(b) = b$
  - Θ is continuous
  - Θ is increasing

  We want to show that integration over γ and β is the same. Note that $\gamma(t) = \beta(\Theta(t))$ and $\gamma'(t) = \beta'(\Theta(t))\Theta'(t)$.

  $$\begin{aligned}\int_\gamma f(z)\mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_{a=\Theta(a)}^{b=\Theta(b)} f(\beta(\Theta(t)))\beta'(\Theta(t))\underbrace{\Theta'(t)\mathrm{d}t}_{\mathrm{d}\Theta} \\ &= \int_a^b f(\beta(\Theta))\beta'(\Theta)\mathrm{d}\Theta \\ &= \int_\beta f(z)\,\mathrm{d}z\end{aligned}$$

  So, the integral of $f(z)$ on a closed curve is independent on the parameterization of the curve. □

Example

Evaluate $\int_\gamma f(z)\mathrm{d}z$ for given γ on $[a,b]$.

1. Let $\gamma(t) = t$. $\gamma'(t) = 1$.
   $$\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t$$
   Note that $\int_a^b f(t)\,\mathrm{d}t$ is the usual integral of f restricted to the real axis.
2. Let $\gamma(t) = t + it^2$ and $f(z) = 1$.
   $$\begin{aligned}\int_\gamma 1\,\mathrm{d}z &= \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b 1(1+2it)\,\mathrm{d}t \\ &= \int_a^b 1\,\mathrm{d}t + i\int_a^b 2t\,\mathrm{d}t \\ &= (b-a) + i(b^2 - a^2) \\ &= (b + ib^2) - (a + ia^2) \\ &= \gamma(b) - \gamma(a)\end{aligned}$$

Example

Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$, and $f(z) = z^n$ where $n \in \mathbb{Z}$.

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} e^{int}\cdot ie^{it}\,\mathrm{d}t \\ &= i\int_0^{2\pi} e^{it(n+1)}\,\mathrm{d}t \\ &= i\int_0^{2\pi} \cos((n+1)t) + i\sin((n+1)t)\,\mathrm{d}t \\ &= i\left[\frac{\sin((n+1)t)}{n+1} - i\frac{\cos((n+1)t)}{n+1}\right]_0^{2\pi} \\ &= i\left[(0-i)-(0-i)\right] \\ &= i(0) \\ &= 0\end{aligned}$$

This assumes $n+1 \neq 0$. When $n+1 = 0$:

$$\begin{aligned}i\int_0^{2\pi} e^{it(n+1)}\,\mathrm{d}t &= i\int_0^{2\pi} 1\,\mathrm{d}t \\ &= 2\pi i \neq 0\end{aligned}$$

So, we have

$$\int x^n\,\mathrm{d}x = \begin{cases}\frac{x^{n+1}}{n+1} + c & n \neq -1 \\ \log(x) + c & n = -1\end{cases}$$

As we wrap around the origin, we end up with an extra $2\pi i$ to keep the function continuous. This is an important example.

Properties of Complex Integrals

Addition and constant multiplication

For real integrals, let f and g be real functions and $\alpha, \beta \in \mathbb{R}$.

$$\int_a^b [\alpha f(x) + \beta g(x)]\,\mathrm{d}x = \alpha\int_a^b f(x)\mathrm{d}x + \beta\int_a^b g(x)\mathrm{d}x$$

For complex integrals, let f and g be functions where $f, g : \mathbb{C} \to \mathbb{C}$ and $\lambda, \mu \in \mathbb{C}$.

$$\begin{aligned}\int_\gamma (\lambda f(z) + \mu g(z))\,\mathrm{d}z &= \int_a^b (\lambda f(\gamma(t)) + \mu g(\gamma(t)))\,\gamma'(t)\mathrm{d}t \\ &= \int_a^b \lambda f(\gamma(t))\gamma'(t) + \mu g(\gamma(t))\gamma'(t)\mathrm{d}t \\ &= \lambda\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t + \mu\int_a^b g(\gamma(t))\gamma'(t)\mathrm{d}t\end{aligned}$$

from properties of real integrals

$$= \lambda\int_\gamma f(z)\mathrm{d}z + \mu\int_\gamma g(z)\mathrm{d}z$$

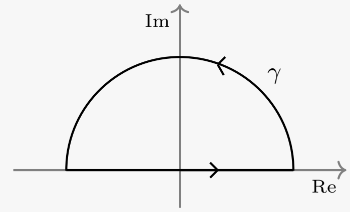
So, the same property for addition and constant multiplication holds for complex integrals.

Piecewise integrals

For real integrals, $\int_a^b g(x)\mathrm{d}x = \int_a^c g(x)\mathrm{d}x + \int_c^b g(x)\mathrm{d}x$. This is relevant when $g(x)$ is a piecewise defined function. For complex integrals $\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$, $f(z)$ is rarely piecewise but γ often is.

Example

Suppose γ is the northern semicircle of the unit circle on the complex plane.



$$\gamma(t) = \begin{cases} 2t-1 & 0 \leq t \leq 1 \\ e^{i(t-1)} & 1 \leq t \leq 1+\pi \end{cases}$$

where $\gamma : [0, 1+\pi] \to \mathbb{C}$.

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \int_0^{1+\pi} f(\gamma(t))\gamma'(t)\mathrm{d}t \\ &= \underbrace{\int_0^1 f(\gamma(t))\gamma'(t)\mathrm{d}t}_{\text{horizontal line segment}} + \underbrace{\int_1^{1+\pi} f(\gamma(t))\gamma'(t)\mathrm{d}t}_{\text{arc}} \end{aligned}$$

Here, we have split the integral into manageable chunks.

Reversing integral bounds

For real integrals:

$$\int_a^b g(x)\mathrm{d}x = -\int_b^a g(x)\mathrm{d}x$$

For complex integrals, the change of bounds corresponds to tracing out the same curve but in reverse.

Let $\delta(t) = \gamma(a+b-t)$. As t increases, we go backwards along γ.

$$\begin{aligned} \delta(a) &= \gamma(a+b-a) = \gamma(b) \\ \delta(b) &= \gamma(a+b-b) = \gamma(a) \end{aligned}$$

δ and γ have swapped endpoints. To show that $\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t = -\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$, we want to show that $\int_\gamma f(z)\mathrm{d}z = -\int_\delta f(z)\mathrm{d}z$.

$$\begin{aligned} \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t &\stackrel{?}{=} -\int_a^b f(\delta(t))\delta'(t)\mathrm{d}t \\ &= -\int_a^b f(\gamma(a+b-t))(-\gamma'(a+b-t))\mathrm{d}t \end{aligned}$$

The sign is from the chain rule. Using substitution, let $u = a+b-t$ where $\frac{\mathrm{d}u}{\mathrm{d}t} = -1$.

$$\begin{aligned} &= -\int_{t=a}^{t=b} f(\gamma(u))\gamma'(u)\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t \\ &= -\int_{u=b}^{u=a} f(\gamma(u))\gamma'(u)\mathrm{d}u \\ &= \int_a^b f(\gamma(u))\gamma'(u)\mathrm{d}u \\ &= \int_\gamma f(z)\mathrm{d}z \end{aligned}$$

So, the same property for complex integrals hold.

Bound of an integral (ML lemma)

For real integrals:

$$\left|\int_a^b g(x)\mathrm{d}x\right| \le (b-a)\max_{x\in[a,b]}|g(x)|$$

There is a similar property for complex integrals, called the ML lemma.

Theorem: ML Lemma

The integral of $f(z)$ on a closed curve $\gamma(t)$ is bounded:

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \le \max_{t\in[a,b]}|f(\gamma(t))| \cdot \text{ length}(\gamma) = ML$$

where $M = \max_{t\in[a,b]}|f(\gamma(t))||$ and L is $\text{length}(\gamma)$.

- **Proof**: $\text{length}(\gamma) = \int_a^b |\gamma'(t)|\mathrm{d}t$. If $\gamma(t)$ is the position of a particle at time t, $|\gamma'(t)|$ is its speed at time t.

  By the triangle inequality,

  $$\left|\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\,\mathrm{d}t$$

  This can be applied to integrals due to the following:

  $$\int_a^b g(x)\,\mathrm{d}x \text{ ``}=\text{''} \lim_{n\to\infty}\sum_{k=1}^n g\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}$$

  The triangle inequality can be applied to the sum.

  $$\begin{aligned}\left|\int_a^b g(x)\mathrm{d}x\right| &\text{ ``}=\text{''} \left|\lim_{n\to\infty}\sum_{k=1}^n g\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}\right| \\ &= \lim_{n\to\infty}\left|\sum_{k=1}^n g\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}\right| \\ &\le \lim_{n\to\infty}\sum_{k=1}^n \left|g\left(a+\frac{b-a}{n}k\right)\frac{b-a}{n}\right| \\ &\text{ ``}=\text{''} \int_a^b |g(x)|\mathrm{d}x\end{aligned}$$

  So, we have the following:

  $$\begin{aligned}\left|\int_\gamma f(z)\mathrm{d}z\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \\ &\le \int_a^b |f(\gamma(t))\gamma'(t)|\,\mathrm{d}t \\ &= \int_a^b |f(\gamma(t))|\cdot|\gamma'(t)|\,\mathrm{d}t \\ &\le \max_{t\in[a,b]}|f(\gamma(t))|\int_a^b|\gamma'(t)|\,\mathrm{d}t \\ &= \max_{t\in[a,b]}|f(\gamma(t))|\cdot\text{ length}(\gamma) \\ &= ML\end{aligned}$$

  □

Antiderivatives

For real integrals: if $F'(x) = f(x)$, then by the Fundamental Theorem of Calculus,

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

For complex integrals, suppose $F(z)$ is holomorphic in a region $G \subseteq \mathbb{C}$. We want to show that

$$\int_\gamma \frac{\mathrm{d}}{\mathrm{d}z} F(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

If $F'(\gamma(t))\gamma'(t)$ was $\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))$, we can appeal to the Fundamental Theorem of Calculus and be done.

Let $F(z) = u(x,y) + iv(x,y)$ where $z = x + iy$ and $\gamma(t) = \alpha(t) + i\beta(t)$ where $t \in [a,b]$.

$$F(\gamma(t)) = u(\alpha(t),\beta(t)) + iv(\alpha(t),\beta(t))$$

Use the multivariable chain rule.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t)) &= u_x(\alpha(t),\beta(t))\alpha'(t) + u_y(\alpha(t),\beta(t))\beta'(t)\\ &\quad + iv_x(\alpha(t),\beta(t))\alpha'(t) + iv_y(\alpha(t),\beta(t))\beta'(t)\end{aligned}$$

Apply the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

$$\begin{aligned}&= u_x(\alpha(t),\beta(t))(\alpha'(t) + i\beta'(t)) + iv_x(\alpha(t),\beta(t))(\alpha'(t) + i\beta'(t))\\ &= \underbrace{(u_x(\alpha(t),\beta(t)) + iv_x(\alpha(t),\beta(t)))}_{\frac{\partial}{\partial x}F(x+iy) = \frac{\mathrm{d}}{\mathrm{d}z}F(z)} \cdot \underbrace{(\alpha'(t) + i\beta'(t))}_{\gamma'(t)}\end{aligned}$$

This is evaluated at $(\alpha(t),\beta(t)) = \gamma(t)$.

A closed curve has the same start and end points, meaning $\gamma(a) = \gamma(b)$. If $f(z) = \mathrm{d}\,zF(z)$, the integral of $f(z)$ around a closed curve is 0:

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)) = 0$$

Example

This is a non-example. Let γ be the anticlockwise unit circle (which is a closed curve). As proven previously,

$$\int_\gamma \frac{1}{z}\mathrm{d}z = 2\pi i \neq 0.$$

So, there is no holomorphic function $F(z)$ defined on the whole unit circle having derivative $\frac{1}{z}$. However, consider the principal logarithm $\log\left(re^{i\theta}\right) = \log(r) + i\theta$ for $-\pi < \theta \leq \pi$. This is discontinuous on $\mathbb{R}_{\leq 0}$.

$$\frac{\mathrm{d}}{\mathrm{d}z}\log(x+iy) = \frac{x}{x+y^2} + \frac{-iy}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}$$

So, $\frac{1}{z}$ has an antiderivative on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$.

It turns out that if $f(z)$ is continuous and $\int_\gamma f(z) = 0$ for any closed curve, then $f(z)$ has an antiderivative (meaning $\exists F(z)$ such that $F'(z) = f(z)$).

By analogy, by the Fundamental Theorem of Calculus:

$$\mathrm{d}\,x\int_a^x f(t)\mathrm{d}t = f(z)$$

So, we want $F(z) = \int_\gamma f(z)\mathrm{d}z$ where γ is a curve from a fixed point q to w.

First, we must check that this does not depend on the path from q to w. Suppose δ_1 and δ_2 are two such paths from q to w. Observe that the reverse of δ_2 is a curve from w to q, so the path obtained by following δ_1 then the reverse of δ_2 goes q to w to q. So, it is a closed curve. Let this composition of paths be $\delta_1 - \delta_2$.

$$\begin{aligned}
\int_{\delta_1-\delta_2} f(z)\,\mathrm{d}z &= 0 && \text{(by assumption)}\\
\int_{\delta_1} f(z)\mathrm{d}z + \int_{-\delta_2} f(z)\mathrm{d}z &= 0\\
\int_{\delta_1} f(z)\,\mathrm{d}z - \int_{\delta_2} f(z)\,\mathrm{d}z &= 0\\
\int_{\delta_1} f(z)\,\mathrm{d}z &= \int_{\delta_2} f(z)\,\mathrm{d}z
\end{aligned}$$

So, it doesn't matter whether we integrate along δ_1 and δ_2. This means that the formula $F(w) = \int_\gamma f(z)\,\mathrm{d}z$ where γ is any path from q to w makes sense. Let's check $\mathrm{d}\,wF(w) = f(w)$.

$$\frac{\mathrm{d}}{\mathrm{d}w}F(w) = \lim_{h\to 0}\frac{F(w+h)-F(w)}{h}$$

To evaluate $F(w+h)$, we can choose the path of integration from q to $w+h$ arbitrarily. We choose one that goes from q to w via any path, then w to $w+h$ along a line segment ℓ. This line segment has length $|h|$.

Note that if our function is holomorphic at a point w, it is differentiable on a disk $D(w,\varepsilon)$ for some $\varepsilon > 0$. So, if $|h| < \varepsilon$, then the line segment ℓ is contained in $D(w,\varepsilon)$ and hence in a region where the function is differentiable.

$F(w+h)$ is the integral of $f(z)$ from q to w to h and $F(w)$ is the integral of $f(z)$ from q to w, so $F(w+h)-F(w)$ cancels out and we are left with the integral of $f(z)$ from w to h. This is the line segment ℓ.

We want to show

$$\lim_{h\to 0}\frac{1}{h}\int_\ell f(z)\mathrm{d}z - f(w) = 0.$$

Note that $f(w)$ is independent of z.

$$\begin{aligned}
\int_\ell f(w)\,\mathrm{d}z &= f(w)\int_\ell 1\,\mathrm{d}z\\
&= f(w)[z]_w^{w+h}\\
&= f(w)[(w+h)-w]\\
&= hf(w)\\
\lim_{h\to 0}\frac{1}{h}\int_\ell f(z)\,\mathrm{d}z - f(w) &= \lim_{h\to 0}\frac{\int_\ell f(z)\,\mathrm{d}z - \int_\ell f(w)\,\mathrm{d}z}{h}\\
&= \lim_{h\to 0}\frac{\int_\ell (f(z)-f(w))\,\mathrm{d}z}{h}
\end{aligned}$$

By the ML lemma:

$$\begin{aligned}
\left|\frac{\int_\ell f(z)-f(w)\mathrm{d}z}{h}\right| &= \frac{\left|\int_\ell f(z)-f(w)\,\mathrm{d}z\right|}{|h|}\\
&\leq \max_{z\in\ell}|f(z)-f(w)|\,\frac{\mathrm{length}(\ell)}{|h|}\\
&= \max_{z\in\ell}|f(z)-f(w)| && (\mathrm{length}(\ell) = |h|)
\end{aligned}$$

So, it is enough to show that

$$\lim_{h\to 0}\max_{z\in\ell}|f(z)-f(w)| = 0.$$

ℓ is a segment of length $|h|$ with one endpoint at w. Since $f(z)$ is continuous at w, for any $\varepsilon > 0$, $\exists\delta > 0$ such that if $|z-w| < \delta$, then $|f(z) - f(w)| < \varepsilon$. So when $|h| < \delta$, any $z \in \ell$ obeys $|z-w| < \delta$.

Additionally, $|f(z) - f(w)| < \varepsilon$, so for $|h| < \delta$,

$$\left|\frac{F(w+h) - F(w)}{h} - f(w)\right| < \varepsilon.$$

So, $\lim\limits_{h\to 0} \frac{F(w+h)-F(w)}{h} = f(w)$ as needed.

Cauchy's Theorem

We use Cauchy's Theorem to check that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any closed curve γ.

Theorem: Cauchy's Theorem

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a closed curve (meaning $\gamma(a) = \gamma(b)$) and $f(z)$ is holomorphic on γ and in the region enclosed by the curve γ. Then,

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Before proving this, note a few points:

- Recall the previous example $\int_\gamma \frac{1}{z}\mathrm{d}z = 2\pi i \neq 0$ where γ is the anticlockwise unit circle. $\frac{1}{z}$ is not holomorphic at $z = 0$, which is in the unit circle. So, the condition for Cauchy's Theorem is not satisfied.
- Checking that closed curves have well-defined interior regions is not trivial. It is the content of the Jordan Curve Theorem, which will not be covered in this course.
- There are several formulations of Cauchy's Theorem, we will assume that $f'(z)$ is continuous. Some formulas remove the concept of interior region and instead use the notion of a homotopy of 2 curves.

Proof of Cauchy's Theorem

Our proof for Cauchy's Theorem relies on Stokes' Theorem and the Divergence Theorem. Recall that a vector field $\vec{F}$ is a function $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\vec{F}(x,y) = (F_1(x,y), F_2(x,y))$.

$$\int_\gamma \vec{F}\cdot \mathrm{d}t = \int_a^b \vec{F}(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t$$

Note that $\vec{F}(\gamma(t))\cdot\gamma'(t)$ denotes the dot product. If we have $\gamma(t) = (\alpha(t), \beta(t))$ and $\gamma'(t) = (\alpha'(t), \beta'(t))$, the integral becomes

$$\int_a^b F_1(\alpha(t), \beta(t))\alpha'(t) + F_2(\alpha(t), \beta(t))\beta'(t)\,\mathrm{d}t\,.$$

Theorem: Stokes' Theorem

If F_1 and F_2 have continuous partial derivatives in the region enclosed by γ, then

$$\int_\gamma \vec{F}\cdot\mathrm{d}t = \int_A (\boldsymbol{\nabla}\times F)\,\mathrm{d}A$$

where A is the area enclosed by γ. The curl is

$$\boldsymbol{\nabla}\times\vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}.$$

Theorem: Divergence Theorem

If F_1 and F_2 have continuous partial derivatives, then the flux integral $\int_\gamma (\vec{F}\cdot\hat{n})\,\mathrm{d}\ell$, which is

$$\int_a^b \vec{F}(\gamma(t))\cdot\underbrace{(\beta'(t),-\alpha'(t))}_{\text{normal to }\gamma(t)}\mathrm{d}t\,,$$

is equal to

$$\int_\gamma (\vec{F}\cdot\hat{n})\,\mathrm{d}\ell = \int_A \boldsymbol{\nabla}\cdot\vec{F}\mathrm{d}A$$

where A is the area enclosed by γ. The divergence is

$$\boldsymbol{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

To prove Cauchy's Theorem, we want $\int_\gamma f(z)\mathrm{d}z = 0$, or $\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = 0$. Note that this involves the multiplication of complex numbers, not vectors. To use Stokes' and the Divergence Theorem, we must write these as vectors.

Let $f(x+iy) = u(x,y) + iv(x,y)$.

$$\int_\gamma \left(u(\alpha(t),\beta(t)) + iv(\alpha(t),\beta(t))\right)\left(\alpha'(t)+i\beta'(t)\right)\mathrm{d}t = \int_\gamma u\alpha'(t) - v\beta'(t)\mathrm{d}t + i\int_\gamma v\alpha'(t) + u\beta'(t)\mathrm{d}t.$$

Where u and v are calculated at $\gamma(t) = (\alpha(t),\beta(t))$.

Note that

$$\begin{aligned} u_{\alpha'} - v_{\beta'} &= (u,-v)\cdot(\alpha',\beta') \\ v\alpha' + u\beta' &= (u,-v)\cdot(\beta,-\alpha') \end{aligned}$$

So, we can define the vector field $\vec{F}(x,y) = (u(x,y), -v(x,y))$.

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma \vec{F}\cdot\mathrm{d}t + i\int_\gamma \vec{F}\cdot\hat{n}\mathrm{d}t$$

Compute the curl and divergence.

$$\begin{aligned} \boldsymbol{\nabla}\times\vec{F} &= \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} \\ &= -v_x - u_y \\ \boldsymbol{\nabla}\cdot\vec{F} &= \frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} \\ &= u_x - v_y \end{aligned}$$

By the Cauchy-Riemann equations, $u_y = -v_x$ and $u_x = v_y$ meaning $\boldsymbol{\nabla}\times\vec{F} = 0$ and $\boldsymbol{\nabla}\cdot\vec{F} = 0$. So,

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \int_\gamma \vec{F}\cdot\mathrm{d}t + i\int_\gamma \vec{F}\cdot\hat{n}\mathrm{d}t \\ &= \int_A \left(\boldsymbol{\nabla}\times\vec{F}\right)\mathrm{d}A + i\int_A \boldsymbol{\nabla}\cdot\vec{F}\mathrm{d}A \\ &= 0 + i0 = 0 \end{aligned}$$

So, $\int_\gamma f(z)\mathrm{d}z = 0$. $\square$

Note that Cauchy's Theorem and its variations are the most important theorems in this course.

Cauchy's Integral Formula

We will explore multiple versions of Cauchy's Integral Formula.

First version

The first version of Cauchy's Integral Formula has the most specific conditions.

Theorem: Cauchy's Integral Formula (1st version)

Let $f(z)$ be holomorphic on the closed disk $|z-a| \leq R$ (on and inside a circle of radius R centered at a). Then,

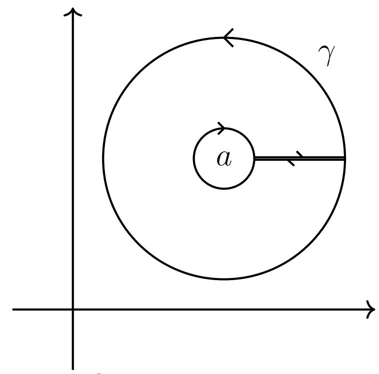
$$\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z = 2\pi i f(a)$$

where γ is the circle of radius R centered at a in the anticlockwise direction.

- **Proof**: note that at $z=a$, $\frac{f(z)}{z-a}$ is not holomorphic, so Cauchy's Theorem doesn't apply directly. Instead, to prove this, we integrate along the pictured curve.

  The region enclosed does not contain a. This curve has 4 parts:

  1. Big circle of radius R (anticlockwise)
  2. Line segment connecting the two curves (inward)
  3. Small circle of radius r (clockwise)
  4. Line segment connecting the two curves (outward)



  Note that the curve part 4 is the reverse of curve part 2, so their integrals are negatives:

$$\int_{\text{part 2}} = -\int_{\text{part 4}}$$

  From Cauchy's Theorem, the total integral is 0.

$$\int_{\text{part 1}} + \int_{\text{part 2}} + \int_{\text{part 3}} + \int_{\text{part 4}} = 0$$

  We know $\int_{\text{part 1}} + \int_{\text{part 3}} = 0$, so $\int_{\text{part 2}} + \int_{\text{part 4}} = 0$. Let $C_R = \gamma$ denote the path of the large circle and C_r denote the path of the small circle. The arrows note whether the circles are clockwise or anticlockwise.

$$\begin{aligned}\int_{C_R\circlearrowleft} \frac{f(z)}{z-a}\mathrm{d}z &= -\int_{C_r\circlearrowright} \frac{f(z)}{z-a}\mathrm{d}z \\ &= \int_{C_r\circlearrowleft} \frac{f(z)}{z-a}\mathrm{d}z\end{aligned}$$

  So, the integral of $\frac{f(z)}{z-a}$ around the circle of radius r is the same for any $0 < r \leq R$. Let $C_r = C_r \circlearrowleft$. It is enough to compute

$$\int_{C_r} \frac{f(z)}{z-a}\mathrm{d}z$$

  in the limit $z \to 0$. We want to show

$$\int_{C_r} \frac{f(z)}{z-a}\mathrm{d}z = 2\pi i f(a).$$

Note that

$$\int_{C_r} \frac{f(a)}{z-a} \mathrm{d}z = 2\pi i f(a).$$

We parameterize $\gamma = C_r = a + re^{it}$ where $0 \le t \le 2\pi$.

$$\begin{aligned}
\int_\gamma \frac{f(a)}{z-a} \mathrm{d}z &= f(a) \int_\gamma \frac{1}{z-a} \mathrm{d}z \\
&= f(a) \int_0^{2\pi} \frac{1}{\gamma(t)-a} \gamma'(t) \mathrm{d}t \\
&= f(a) \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} \mathrm{d}t \\
&= if(a) \int_0^{2\pi} 1 \mathrm{d}t \\
&= 2\pi i f(a)
\end{aligned}$$

So, it suffices to show that $\int_\gamma \frac{f(z)}{z-a} \mathrm{d}z = \int_\gamma \frac{f(a)}{z-a} \mathrm{d}z$.

$$\int_\gamma \frac{f(z)-f(a)}{z-a} \mathrm{d}z \stackrel{?}{=} 0$$

Although γ depends on radius r, the values of the integral do not. So, consider the limit $r \to 0$. Apply the ML lemma:

$$\left| \int_\gamma \frac{f(z)-f(a)}{z-a} \mathrm{d}z \right| \le \max_{z \in \gamma} \left| \frac{f(z)-f(a)}{z-a} \right| \mathrm{length}(\gamma)$$

$|z-a| = r$ since $z \in \gamma$ and γ is a circle of radius r centered at a. $\mathrm{length}(\gamma) = 2\pi r$.

$$\begin{aligned}
&= \max_{z \in \gamma} \left| \frac{f(z)-f(a)}{r} \right| (2\pi r) \\
&= 2\pi \max_{x \in \gamma} |f(z) - f(a)|
\end{aligned}$$

$f(z)$ is differentiable, so it must be continuous. By definition of continuity, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|z-a| < \delta$, then $|f(z) - f(a)| < \varepsilon$. Let $r < \delta$.

$$\left| \int_\gamma \frac{f(z)-f(a)}{z-a} \mathrm{d}z \right| \le 2\pi\varepsilon$$

This holds for any $\varepsilon > 0$, so the LHS is either 0 or positive. If it is positive, then we can take $2\pi\varepsilon$ to be smaller which is a contradiction. So,

$$\begin{aligned}
\left| \int_\gamma \frac{f(z)-f(a)}{z-a} \mathrm{d}z \right| &= 0 \\
\int_\gamma \frac{f(z)-f(a)}{z-a} \mathrm{d}z &= 0
\end{aligned}$$

So, we have

$$\begin{aligned}
\int_\gamma \frac{f(z)}{z-a} \mathrm{d}z &= \int_\gamma \frac{f(a)}{z-a} \mathrm{d}z \\
&= 2\pi i f(a)
\end{aligned}$$

This proves the first (simplest) version of Cauchy's Integral Formula. $\square$

Second version

The second version of Cauchy's Integral Formula is the general version.

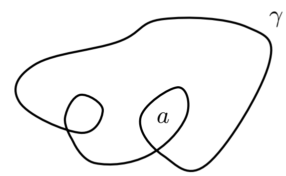
Theorem: Cauchy's Integral Formula (2nd version)

Let $f(z)$ be holomorphic on and inside a closed curve γ in the anticlockwise direction, which encloses the point $a \in \mathbb{C}$ exactly once. Then,

$$\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z = 2\pi i f(a).$$

This has the same statement as the 1st version, but with a more general condition for γ. Proving this version requires defining how a point is "enclosed once". For example, observe the pictured curve γ.

To integrate over γ, we split the curve into three parts: the outer curve and the two inner loops. Note that the outer curve and one of the inner loops contain a, but the other inner loop does not.



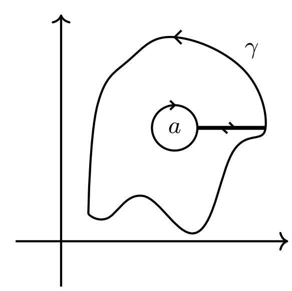
By Cauchy's Theorem and the 1st version of Cauchy's Integral Formula, the integral over γ is

$$0 + 2\pi i f(a) + 2\pi i f(a) = 2 \cdot 2\pi i f(a).$$

The Jordan Curve Theorem gives us a way to tell how many times a curve wraps around a point. If the curve has no self-intersections other than the start/end point, then it encloses each point in its interior exactly once. So, for a general curve, we can break it into parts that don't intersect themselves.

- **Proof**: the proof of the 2nd version of Cauchy's Integral Formula is similar to the 1st. We integrate along the pictured curve.

  Like before, the integrals of the two line segments connecting the curves cancel out. By Cauchy's Theorem, the sum of the integrals is 0.



$$\begin{aligned}\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z &= -\int_{C_r\circlearrowright} \frac{f(z)}{z-a}\mathrm{d}z \\ &= \int_{C_r\circlearrowleft} \frac{f(z)}{z-a}\mathrm{d}z\end{aligned}$$

  This equals $2\pi i f(a)$ by the 1st version, so $\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z = 2\pi i f(a)$. □

Example

Let $\omega \in \mathbb{R}_{\geq 0}$. Show that

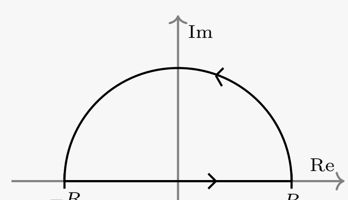
$$\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2+1}\mathrm{d}x = \pi e^{-\omega}.$$

To show this, write it as a subset of an integral $\int_\gamma \cdots \mathrm{d}z$. Recall that $e^{i\omega z} = \cos(\omega z) + i\sin(\omega z)$. Consider $f(z)$ where

$$f(z) = \frac{e^{i\omega z}}{z^2+1} = \frac{g(z)}{z-i}$$

where $g(z) = \frac{e^{i\omega z}}{z+i}$.

We want to integrate the function along the real axis, but we must have a closed curve. Choose the pictured curve.



This curve has a line segment $[-R, R]$ and a semicircle (top half) of radius R centered at (0, 0). We will take $R \to \infty$. Let the semicircle arc be denoted by the symbol $\curvearrowleft$.

We want to evaluate the integral $\int_\gamma f(z)\mathrm{d}z$.

$$\int_\gamma \frac{g(z)}{z-i}\mathrm{d}z = \int_{-R}^{R} \frac{g(z)}{z-i}\mathrm{d}z + \int_{\frown} \frac{g(z)}{z-i}\mathrm{d}z$$

Observe each integral separately. For the first integral, take $\delta : [-R, R] \to \mathbb{C}$ where $\delta(t) = t$. As $R \to \infty$,

$$\begin{aligned}
\int_\delta \frac{g(z)}{z-i}\mathrm{d}z &= \int_{-R}^{R} \frac{g(\delta(t))}{\delta(t)-i}\delta'(t)\mathrm{d}t \\
&= \int_{-R}^{R} \frac{g(t)}{t-i}\mathrm{d}i \\
&= \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{x^2+1}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{\cos(\omega x) + i\sin(\omega x)}{x^2+1}\mathrm{d}x \\
&= \underbrace{\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2+1}\mathrm{d}x}_{\text{what we want}} + i \underbrace{\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x^2+1}\mathrm{d}x}_{= 0 \text{ since sin is even}}
\end{aligned}$$

Now for the second integral, use the ML lemma.

$$\left|\int_{\frown} \frac{g(z)}{z-i}\mathrm{d}z\right| = \left|\int_{\frown} \frac{e^{i\omega z}}{z^2+1}\mathrm{d}z\right| \leq \max_{z\in\frown}\left|\frac{e^{i\omega z}}{z^2+1}\right| \text{ length}(\frown)$$

The length of the semicircle arc is πR. $z \in \frown$ is in the upper half-plane, so $\mathrm{Im}(z) \geq 0$. $\omega \geq 0$, so $\mathrm{Im}(\omega z) \geq 0$. Let $\omega z = a + ib$ where $b \geq 0$. $i\omega z = -b + ia$, so $\mathrm{Re}(i\omega z) \leq 0$ meaning

$$\left|e^{i\omega z}\right| = e^{\mathrm{Re}(i\omega z)} \leq e^0 = 1.$$

$|z^2+1| \geq |z|^2 - 1 = R^2 - 1$ by the reverse triangle inequality, so $\left|\frac{1}{z^2+1}\right| \leq \frac{1}{R^2-1}$. So, we have $\left|\frac{e^{i\omega z}}{z^2+1}\right| = |e^{i\omega z}| \cdot \left|\frac{1}{z^2+1}\right| \leq 1 \cdot \frac{1}{R^2-1} = \frac{1}{R^2-1}$. Take $R \to \infty$.

$$\begin{aligned}
\left|\int_{\frown} \frac{g(z)}{z-i}\mathrm{d}z\right| &\leq \frac{1}{R^2-1}\pi R \\
&= \frac{\pi R}{R^2-1} \\
&= 0 \text{ as } R \to \infty
\end{aligned}$$

$\left|\int_{\frown} \frac{g(z)}{z-i}\mathrm{d}z\right| \geq 0$, so it must equal 0. For any a, if $|a| = 0$, then $a = 0$. So,

$$\int_{\frown} \frac{g(z)}{z-i}\mathrm{d}z = 0.$$

The integral $\int_\gamma \frac{g(z)}{z-i}\mathrm{d}z$ can be evaluated using the Cauchy Integral Formula. Here, $a = i$ since $\frac{g(z)}{z-i}$ is undefined at $z = i$.

$$\begin{aligned}
\int_\gamma \frac{g(z)}{z-i}\mathrm{d}z &= 2\pi i g(i) \\
&= 2\pi i \frac{e^{i\omega i}}{i+i} \\
&= \pi e^{-\omega}
\end{aligned}$$

Put all parts together.

$$\begin{aligned}\int_\gamma \frac{g(z)}{z-i}\mathrm{d}z &= \int_{-R}^{R} \frac{g(z)}{z-i}\mathrm{d}z + \int_{\frown} \frac{g(z)}{z-i}\mathrm{d}z \\ &= \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2+1}\mathrm{d}x + i\int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x^2+1}\mathrm{d}x + 0 \\ &= \pi e^{-\omega}\end{aligned}$$

Compare the real and imaginary parts.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2+1}\mathrm{d}x &= \pi e^{-\omega} \\ \int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x^2+1}\mathrm{d}x &= 0\end{aligned}$$

We confirm that the sin integral is 0 since sin is an even function. So, we have proved the desired integral using Cauchy's Integral Formula.

Consequences of Cauchy's Integral Formula

We can generalize the cases for Cauchy's Theorem and Cauchy's Integral Formula.

$$\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z = \begin{cases} 2\pi i f(a) & a \text{ is enclosed by } \gamma \\ 0 & a \text{ is outside } \gamma \end{cases}$$

We will not consider the case that a lies on γ. Let $\gamma : [a', b']$ be the curve.

$$\begin{aligned}f(a) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{a'}^{b'} \frac{f(\gamma(t))}{\gamma(t)-a}\gamma'(t)\mathrm{d}t\end{aligned}$$

The only values of f that appear in the integral are on the curve: $f(\gamma(t))$. So, the values of f on γ determine the values of f inside γ.

Example

Let $f(z) = 0$ on the unit circle: $f(e^{i\theta}) = 0$. For $|a| < 1$ (in the circle), we have

$$\begin{aligned}f(a) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}-a} ie^{i\theta}\mathrm{d}\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} 0\mathrm{d}\theta \\ &= 0\end{aligned}$$

So, if $f(z)$ is 0 on the circle, it is also 0 inside the circle. Note that this works for any curve γ where $f(z) = 0$ on γ.

For comparison, take $f(z)$ where $z = x + iy$,

$$f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

$f(x+iy)$ is real differentiable and is 0 on the unit circle (when $x^2+y^2 = 1$) but not inside, so it is not holomorphic.

Proposition

If $f(z)$ is holomorphic on and inside γ, then for a inside γ,

$$f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\mathrm{d}z.$$

- **Proof**: use the definition of the limit.

$$\begin{aligned} 2\pi i f'(a) &= \lim_{h\to 0} 2\pi i\left(\frac{f(a+h)-f(a)}{h}\right) \\ &= \lim_{h\to 0}\frac{\int_\gamma \frac{f(z)}{z-(a+h)}\mathrm{d}z - \int_\gamma \frac{f(z)}{z-a}\mathrm{d}z}{h} \end{aligned}$$

We assume a is inside γ, but we must also check that $a+h$ is.

Let $\gamma : [a', b'] \to \mathbb{C}$ be the curve. Consider $|\gamma(t) - a|$, which is the distance from $\gamma(t)$ to a. This is continuous since it is a composition of continuous functions.

The domain of $|\gamma(t)-a|$ is $[a', b']$ which is a compact set. By the Heine-Borel Theorem, a set is compact if and only if it is closed and bounded. The continuous image of a compact set is also compact, so it is closed. This means its complement is open.

Note that $|\gamma(t) - a|$ is never 0, since otherwise $\gamma(t) = a$ meaning a is on γ, which we don't allow. So, 0 is in the complement of the image of $|\gamma(t) - a|$. This complement set is open, so it must also contain a neighborhood of 0 $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

So, not only does $\gamma(t)$ avoid a, it never comes within ε of it: $|\gamma(t)-a| \geq \varepsilon$ since its complement contains $(0, \varepsilon)$.

If $|a+h-a| = |h| < \varepsilon$, then $a+h$ is inside γ. For h small enough (which we can assume since $h \to 0$):

$$\begin{aligned} 2\pi i\frac{f(a+h)-f(a)}{h} &= \frac{\int_\gamma \frac{f(z)}{z-(a+h)} - \frac{f(z)}{z-a}\mathrm{d}z}{h} \\ &= \frac{1}{h}\int_\gamma \frac{(z-a)f(z)-(z-a-h)f(z)}{(z-a)(z-a-h)}\mathrm{d}z \\ &= \frac{1}{h}\int_\gamma \frac{hf(z)}{(z-a)(z-a-h)}\mathrm{d}z \\ &= \int_\gamma \frac{f(z)}{(z-a)(z-a-h)}\mathrm{d}z \end{aligned}$$

We want to show

$$\lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)(z-a-h)}\mathrm{d}z = \int_\gamma \frac{f(z)}{(z-a)^2}\mathrm{d}z.$$

Alternatively, we want to show

$$\lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2}\mathrm{d}z = 0.$$

$$\begin{aligned} \lim_{h\to 0}\int_\gamma \frac{f(z)(z-a)-(z-a)(z-a-h)f(z)}{(z-a)^2(z-a-h)}\mathrm{d}z &= \lim_{h\to 0}\int_\gamma \frac{hf(z)}{(z-a)^2(z-a-h)}\mathrm{d}z \\ &= \lim_{h\to 0} h\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)}\mathrm{d}z \end{aligned}$$

To prove that this equals 0, since h goes to 0 and the integral does not contain h, it suffices to prove that the integral is bounded. To do this, use the ML lemma.

$$\left|\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)}\mathrm{d}z\right| \leq \max_{z\in\gamma}\frac{|f(z)|}{|z-a|^2|z-a-h|}\ \text{length}(\gamma)$$

$|z-a| \geq \varepsilon$ since $|\gamma(t)-a| \geq \varepsilon$ and $z \in \gamma$. So, for $|a+h-a| = |h| \leq \varepsilon/2$:

$$\begin{aligned}|z-a-h| &\geq |z-a| - |h| \\ &\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}\end{aligned}$$

Take the denominator of the bound.

$$\frac{1}{|z-a|^2|z-a-h|} \leq \frac{1}{\varepsilon^2 \cdot \frac{\varepsilon}{2}} = \frac{1}{\varepsilon^3/2}$$

We can rewrite the bound:

$$\left|\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)} \mathrm{d}z\right| \leq \max_{z\in\gamma} \frac{|f(z)|}{\varepsilon^3/2} \text{ length}(\gamma)$$

So, we have found a bound for the integral that is independent of h. Thus, we have

$$\begin{aligned}\lim_{h\to 0} \int_\gamma \frac{f(z)(z-a) - (z-a)(z-a-h)f(z)}{(z-a)^2(z-a-h)} &= 0 \\ \lim_{h\to 0} \int_\gamma \frac{f(z)}{(z-a)(z-a-h)} \mathrm{d}z &= \frac{f(z)}{(z-a)^2}\end{aligned}$$

Thus, $2\pi i f'(a) = \int_\gamma \frac{f(z)}{(z-a)^2} \mathrm{d}z$, or

$$f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2} \mathrm{d}z.$$

This proves that our claim is true. □

Theorem: Liouville's Theorem

If $f(z)$ is holomorphic on $\mathbb{C}$ and is bounded (meaning $|f(z)| \leq M$ for some fixed M), then $f(z)$ is constant.

- **Proof**: use the formula we just proved to show that $f'(z) = 0$. Let γ be the circle of radius R centered at a. We will take $R \to \infty$. Apply the ML lemma:

  $$|f'(a)| \leq \frac{1}{2\pi} \max_{z\in\gamma} \left|\frac{f(z)}{(z-a)^2}\right| \text{ length}(\gamma)$$

  γ is a circle of radius R centered at a and $z \in \gamma$, so $|z-a| = R$ and length $(\gamma) = 2\pi R$. $f(z)$ is bounded, so $|f(z)| \leq M$ everywhere, including on γ.

  $$\begin{aligned}|f'(a)| &\leq \frac{1}{2\pi} \frac{M}{R^2} 2\pi R \\ &= \frac{M}{R}\end{aligned}$$

  This goes to 0 as $R \to \infty$, so $|f'(a)| \leq 0$ meaning $f'(a) = 0$. This is true for every a, so $f'(z) = 0$ meaning $f(z)$ is constant. □

Once again, consider $f(z)$ where $z = x + iy$:

$$\begin{aligned}f(x+iy) &= \frac{x^2+y^2-1}{x^2+y^2+1} \\ &= 1 - \frac{2}{1-x^2+y^2}\end{aligned}$$

This function is real differentiable and bounded, but it isn't constant. However, this is not a contradiction since as discussed previously, $f(x+iy)$ is not holomorphic.

Proposition

If $f(z)$ is holomorphic on and inside γ, then for a inside γ, $f''(a)$ exists and equals

$$f''(a) = \frac{1}{\pi i}\int_\gamma \frac{f(z)}{(z-a)^3}\mathrm{d}z.$$

Note that we must specify that $f''(a)$ exists since $f(z)$ being holomorphic is not enough to assume this.

- **Proof**: like with the previous claim, use the definition of the limit along with the formula for $f'(a)$.

$$\begin{aligned}
2\pi i f''(a) &= \lim_{h\to 0} 2\pi i \frac{f'(a+h)-f'(a)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_\gamma \frac{f(z)}{(z-a-h)^2}\mathrm{d}z - \int_\gamma \frac{f(z)}{(z-a)^2}\mathrm{d}z\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int_\gamma \frac{f(z)\left[(z-a)^2-(z-a-h)^2\right]}{(z-a-h)^2(z-a)^2}\mathrm{d}z\right] \\
&= \lim_{h\to 0}\frac{1}{h}\int \frac{2h(z-a)-h^2}{(z-a)^2(z-a-h)^2}f(z)\mathrm{d}z \\
&= \lim_{h\to 0}\int_\gamma \frac{2(z-a)-h}{(z-a)^2(z-a-h)^2}f(z)\mathrm{d}z
\end{aligned}$$

Remember that the equation we want to prove has πi, not $2\pi i$. We want to show

$$\lim_{h\to 0}\int_\gamma \frac{2(z-a)-h}{(z-a)^2(z-a-h)^2}f(z)\mathrm{d}z - 2\int_\gamma \frac{f(z)}{(z-a)^3}\mathrm{d}z = 0.$$

$$\begin{aligned}
&\lim_{h\to 0}\int_\gamma \frac{(2(z-a)-h)(z-a)-2(z-a-h)^2}{(z-a)^3(z-a-h)^2}f(z)\mathrm{d}z \\
&= \lim_{h\to 0}\int_\gamma \frac{2(z-a)^2-h(z-a)-2(z-a)^2+4h(z-a)-2h}{(z-a)^3(z-a-h)^2}f(z)\mathrm{d}z \\
&= \lim_{h\to 0}\int_\gamma \frac{3h(z-a)-2h^2}{(z-a)^3(z-a-h)^2}f(z)\mathrm{d}z \\
&= \lim_{h\to 0}\left[3h\int_\gamma \frac{(z-a)f(z)}{(z-a)^3(z-a-h)^2}\mathrm{d}z - 2h^2\int_\gamma \frac{f(z)}{(z-a)^3(z-a-h)^2}\mathrm{d}z\right] \\
&= \lim_{h\to 0}\left[3h\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)^2}\mathrm{d}z - 2h^2\int_\gamma \frac{f(z)}{(z-a)^3(z-a-h)^2}\mathrm{d}z\right]
\end{aligned}$$

Apply the ML lemma to both integrals to find bounds independent of h. Use the same relation to ε used in the proof of the previous claim.

$$\left|\int_\gamma \frac{1}{(z-a)^2(z-a-h)^2}\mathrm{d}z\right| \le \max_{z\in\gamma}\frac{|f(z)|}{\varepsilon^2\left(\frac{\varepsilon}{2}\right)^2}\ \mathrm{length}(\gamma)$$

$$\left|\int_\gamma \frac{f(z)}{(z-a)^3(z-a-h)^2}\mathrm{d}z\right| \le \max_{z\in\gamma}\frac{|f(z)|}{\varepsilon^3\left(\frac{\varepsilon}{2}\right)^2}\ \mathrm{length}(\gamma)$$

Both have bounds that are independent of h. So,

$$\lim_{h\to 0}\left[h(\text{bounded function}) + h^2(\text{bounded function})\right] = 0$$

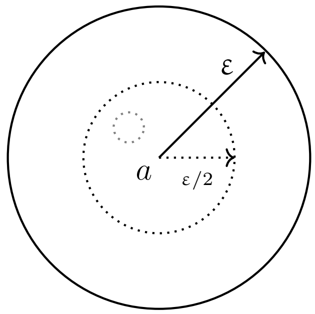
So, we have $2\pi i f''(a) = 2\int_\gamma \frac{f(z)}{(z-a)^3}\mathrm{d}z$, or

$$f''(a) = \frac{1}{\pi i}\int_\gamma \frac{f(z)}{(z-a)^3}\mathrm{d}z.$$

This proves that our claim is true. □

Consequences of the Existence of the Second Derivative

Suppose $f(z)$ is holomorphic at the point a. This means $f'(z)$ exists on a disk centered at a. Suppose this disk has radius $\varepsilon > 0$.



Observe the smaller dotted disk inside the larger disk of radius ε. $f(z)$ is differentiable not just at every point on this smaller disk, but on a (possibly smaller) disk centered at any point on this disk. Let this smaller disk have radius $\varepsilon/2$.

So, $f(z)$ is holomorphic on and inside a circle centered at a with radius $\varepsilon/2$.

Let's call this curve γ. Apply the previous result. $f''(w)$ exists for any w inside the circle of radius $\varepsilon/2$.

The derivative of $f'(z)$ exists on this smaller disk. By definition, this means $f'(z)$ is holomorphic at a. So, if $f(z)$ is holomorphic at a, then $f'(z)$ is also holomorphic at a.

This process can be repeated to show that $f'(z)$ is also holomorphic at a. This can continue to be repeated, meaning if $f(z)$ is holomorphic at a, then it is infinitely differentiable at a, i.e. $\frac{\partial^n}{\partial z^n} f(z)$ exists at $z = a$ and on a small disk centered at a.

Note that this shows a way that complex differentiability is stronger than real differentiability. Real-differentiable functions are not always infinitely differentiable. For example, $f(x) = x|x|$ is real differentiable, but $f'(x) = 2|x|$ is not at $x = 0$.

Fundamental Theorem of Algebra

We can use Liouville's Theorem to prove the Fundamental Theorem of Algebra.

Theorem: Fundamental Theorem of Algebra (general version)

If
$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$
is a polynomial with complex coefficients, then either the polynomial is constant or it can be written as a product of linear factors in the form $az + b$ where $a, b \in \mathbb{C}$.

Note that even if the coefficients of $p(z)$ are real or rational, it does not necessarily mean that the linear factors will have real or rational coefficients. For example, the coefficients of $z^2 + 1$ are real, but it factors into $(z - i)(z + i)$.

Theorem: Fundamental Theorem of Algebra (alternate version)

If $p(z)$ is a nonconstant polynomial (degree > 0), then there is some w such that $p(w) = 0$, meaning any nonconstant polynomial has a root.

Before proving the Fundamental Theorem of Algebra, first show that these two versions are equivalent.

1. General $\implies$ alternate: if some $az + b$ is a factor of $p(z)$, then $w = -\frac{b}{a}$ yields $az + b = 0$ meaning $p(w) = 0$.

2. Alternate $\longrightarrow$ general: it turns out that if $p(w) = 0$, then $z - w$ divides $p(z)$. The proof of this relies on abstract algebra. This results in another polynomial $\frac{p(z)}{z-w}$, which we can repeat the process with until $p(z)$ is fully factored.

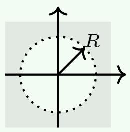
We will prove the alternate version, which will imply the general version.

Lemma

If $p(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$ is nonconstant (meaning $a_d \neq 0$ and $d \geq 1$), then there is a real number R such that if $|z| \geq R$, then

$$\frac{1}{2}\left|a_d z^d\right| \leq |p(z)| \leq \frac{3}{2}\left|a_d z^d\right|.$$

The area of $|z| \geq R$ can be illustrated by



$|z| \geq R$ refers to all z outside the circle of radius R centered at the origin, which is the area shaded grey in the diagram.

- **Proof** of lemma: this is equivalent to
$$\frac{1}{2} \leq \left|\frac{p(z)}{a_d z^d}\right| \leq \frac{3}{2} \quad \Longleftrightarrow \quad \left|\left|\frac{p(z)}{a_d z^d}\right| - 1\right| \leq \frac{1}{2}.$$
So, we have
$$\begin{aligned}\frac{p(z)}{a_d z^d} - 1 &= \frac{a_d z^d}{a_d z^d} + \frac{a_{d-1}z^{d-1}}{a_d z^d} + \cdots + \frac{a_0}{a_d z^d} - 1 \\ &= \frac{a_{d-1}}{a_d}z^{-1} + \frac{a_{d-2}}{a_d}z^{-2} + \cdots + \frac{a_0}{a_d}z^{-d}\end{aligned}$$
We want to show that as $|z| \to \infty$, $|z|^{-r} \to 0$ for any $r > 0$. In other words, we want to show that for any $\varepsilon > 0$, $\exists R$ such that if $|z| \geq R$, then $||z|^{-r} - 0| < \varepsilon$ ($|z|^r$ is the function and 0 is the limit). So, we want R such that $|z| \geq R$ implies $|z|^{-r} < \varepsilon$.
$$\begin{aligned}|z|^{-r} &< \varepsilon \\ \log\left(|z|^{-r}\right) &< \log(\varepsilon) \\ -r\log(|z|) &< \log(\varepsilon) \\ \log(|z|) &> -\frac{1}{r}\log(\varepsilon) \\ |z| &> \varepsilon^{-1/r}\end{aligned}$$
So, the limit exists for $R > \varepsilon^{-1/r}$. This proves that
$$\frac{a_{d-1}}{a_d}z^{-1} + \frac{a_{d-2}}{a_d}z^{-2} + \cdots + \frac{a_0}{a_d}z^{-d} \to 0 \text{ as } R \to \infty.$$
In other words, for any $|z| \geq 0$, then
$$\left|\frac{a_{d-1}}{a_d}z^{-1} + \frac{a_{d-2}}{a_d}z^{-2} + \cdots + \frac{a_0}{a_d}z^{-d}\right| < \varepsilon.$$
We take $\varepsilon = \frac{1}{2}$. For the resulting R, $|z| \geq R$ implies $\left|\frac{p(z)}{a_d z^d} - 1\right| < \varepsilon = \frac{1}{2}$. Thus, $\left|\left|\frac{p(z)}{a_d z^2}\right| - 1\right| \leq \frac{1}{2}$. $\square$

- **Proof** of the Fundamental Theorem of Algebra: we cannot apply Liouville's Theorem directly to $p(z)$ since $p(z)$ is not bounded. Instead, we apply it to $\frac{1}{p(z)}$. If $p(z) \neq 0$ for any z, then $\frac{1}{p(z)}$ is holomorphic since it is the composition of holomorphic functions $\frac{1}{z}$ and $p(z)$. Recall from the lemma that for $|z| \geq R$:
$$\begin{aligned}\frac{1}{2}\left|a_d z^d\right| &\leq |p(z)| \\ \left|\frac{1}{p(z)}\right| &\leq \frac{2}{|a_d|}|z|^{-d} \\ &\leq \frac{2}{|a_d|}R^{-d} \qquad (|z| \geq R)\end{aligned}$$

So, we have a constant bound independent of $|z|$ for $|z| \geq R$. We also need a bound for $|z| \leq R$. In other words, we want to find a bound for $\frac{1}{p(z)}$ on the disk of radius R centered at 0. Note that this region is compact, meaning it is closed and bounded. Additionally, $\left|\frac{1}{p(z)}\right|$ is the composition of two holomorphic functions, $\frac{1}{p(z)}$ and the absolute value. This means $\left|\frac{1}{p(z)}\right|$ is holomorphic, which implies that it is continuous.

Recall that the continuous image if a compact set is also compact, so the image of $\left|\frac{1}{p(z)}\right|$ on $|z| \leq R$ is compact, and in particular, bounded. So, we have $\left|\frac{1}{p(z)}\right| \leq M$ when $|z| \leq R$. Now, we can combine the two bounds:

$$\left|\frac{1}{p(z)}\right| \leq \max\left(\frac{z}{|a_d|}R^{-d}, M\right)$$

The first is valid when $|z| \geq R$ and the second is valid when $|z| \leq R$. Since for any choice of z, either $|z| \geq R$ or $|z| \leq R$. so the inequality holds for any z. This means that $\frac{1}{p(z)}$ is bounded. Liouville's Theorem can be applied, so $\frac{1}{p(z)}$ is constant.

We have shown that if $p(z)$ is never zero, then $\frac{1}{p(z)}$ = constant meaning $p(z)$ is also constant. By taking the contrapositive, if $p(z)$ is nonconstant, it must have a zero. This proves the alternate version of the Fundamental Theorem of Algebra. □

Note that this is not the only proof of the Fundamental Theorem of Algebra, but it is a very typical application of Liouville's Theorem, which involves showing that some condition implies a function is bounded then deducing that it is constant.

Harmonic Functions

Recall the Cauchy-Riemann Equations for $f(x+iy) = u(x,y) + iv(x,y)$. If f is holomorphic, then

$$u_x = v_y \text{ and } u_y = -v_x.$$

Conversely, if the 1$^{\text{st}}$ derivatives exist and the Cauchy-Riemann Equations hold, then f is holomorphic.

Suppose also that 2$^{\text{nd}}$ derivatives exist and are continuous (which implies $u_{xy} = u_{yx}$). Then,

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}.$$

So, we have the **Laplace Equation**:

$$u_{xx} + u_{yy} = 0$$

We will explore when we can be sure a solution of the Laplace Equation comes from a holomorphic function. Note that we know that holomorphic functions are infinitely differentiable, so the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ derivatives of holomorphic functions exist. So, further conditions are not needed (existence of 3$^{\text{rd}}$ derivative implies continuity of 2$^{\text{nd}}$ derivative).

What does it mean for $u : G \to \mathbb{R}$ for $G \subseteq \mathbb{R}^n$ to obey the Laplace Equation? We require u_{xx}, u_{xy}, u_{yx}, and u_{yy} to exist and be continuous, and for $u_{xx} + u_{yy} = 0$.

Theorem

Suppose $u(x,y)$ is a solution to the Laplace Equation, also known as a **harmonic function**, defined on an open set $G \subseteq \mathbb{R}^n$ with no "holes". This means that if $\forall$ closed curves γ in G, the region enclosed by γ is also contained in G.

Then, $\exists$ a function $f : G \to \mathbb{C}$ such that $\text{Re}(f(x+iy)) = u(x,y)$ and f is holomorphic on G. Equivalently, $\exists$ a harmonic function $v : G \to \mathbb{R}$ such that $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic. Such a $v(x,y)$ depends on $u(x,y)$ and is called a **harmonic conjugate** of $u(x,y)$.

- **Proof**: we don't know anything about $v(x, y)$, but we know

$$f'(x + iy) = u_x(x, y) + iv_x(x, y).$$

f is holomorphic, so we can apply the Cauchy-Riemann Equations. We have $v_x = -u_y$, so

$$f'(x + iy) = u_x(x, y) - iu_y(x, y).$$

So, we have f expressed in terms of u alone. Let

$$g(x + iy) = u_x(x, y) - iu_y(x, y).$$

Check that g is holomorphic using the Cauchy-Riemann Equations.

$$(u_x)_x = \underbrace{u_{xx} = -u_{yy}}_{u \text{ is harmonic}} = (-u_y)_y$$
$$(u_x)_y = \underbrace{u_{xy} = u_{yx}}_{\text{cont. } 2^{\text{nd}} \text{ deri.}} = -(-u_y)_x$$

Since u has continuous 2^{nd} derivatives, $g(x + iy)$ has continuous 1^{st} derivatives. So, $g(x + iy)$ is holomorphic.

Now, we find and antiderivative of $g(z)$: $f(z)$ such that $f'(z) = g(z)$. This requires two conditions:

1. $g(z)$ is continuous.
2. $\int_\gamma g(z)\mathrm{d}z = 0$ for any closed curve γ on G (to construct the integral, find $f(w) = \int_\delta g(z)\mathrm{d}z$ where δ is any path from a fixed base point to w).

The first condition is satisfied since $g(z)$ is holomorphic which implies differentiability which implies continuity. For the second condition, we want to show

$$\int_\gamma g(z)\mathrm{d}z = 0$$

for any closed curve γ on G. We want to use Cauchy's Theorem, but we need $g(z)$ to be holomorphic in the region enclosed by γ. We already know that $g(z)$ is holomorphic on γ.

Note that if $\gamma \subseteq G$, then the region enclosed by γ is also $\subseteq G$ by the "no holes" assumption of G. So, if $g(z)$ is holomorphic on G, then $g(z)$ is holomorphic on the region enclosed by γ for any closed curve $\gamma \subseteq G$. By Cauchy's Theorem, the second condition is satisfied meaning we have $f(z)$ such that $f'(z) = g(z)$.

$$\begin{aligned} f(x + iy) &= a(x, y) + ib(x, y) \\ f'(x + iy) &= a_x(x, y) + ib_x(x, y) \\ &= a_x(x, y) - ia_y(x, y) \\ &= g(x + iy) \\ &= u_x(x, y) - iu_y(x, y) \end{aligned}$$

Take the real and imaginary parts of $f'(x + iy)$ and $g(x + iy)$:

$$\begin{aligned} a_x(x, y) &= u_x(x, y) & -a_y(x, y) &= -u_y(x, y) \\ a(x, y) &= u(x, y) + C(y) & a(x, y) &= u(x, y) + D(x) \end{aligned}$$

By subtracting both sides, we have $C(y) = D(x)$. We can fix y and vary x, which shows us that $D(x)$ is independent of x. Similarly, we have $C(y)$ independent of y. Let $C(y) = D(x) = k$ for constant k. So,

$$\begin{aligned} a(x, y) &= u(x, y) + k \\ f(x + iy) - k &= a(x, y) - k + ib(x, y) \\ &= u(x, y) + ib(x, y) \end{aligned}$$

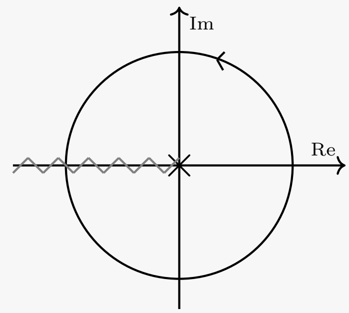
which is holomorphic and has real part $u(x, y)$. □

Example

A nonexample is $G = \mathbb{C}\backslash\{(0,0)\}$ since it has a "hole".

The circle encloses the point $(0,0)$ which is not in G, so G does not satisfy the Laplace Equation. For example, $\log(r) = \log\left(\sqrt{x^2+y^2}\right)$ is a harmonic function, but it is not the real part of a holomorphic function on all of G.

Instead, we can use $G = \mathbb{C}\backslash\mathbb{R}_{\geq 0}$. Now, the theorem does apply since $\log(z) = \log(r) + i\theta$, so $\operatorname{Re}\log(z) = \log(r)$ which doesn't extend to a holomorphic function on G.

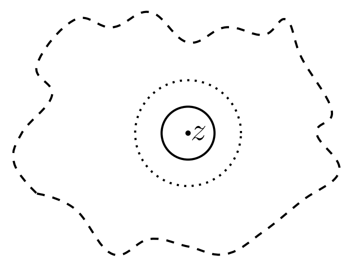


Corollary

Harmonic functions are infinitely differentiable.

- **Proof**: check differentiability at point z.

  If u is defined on an open set, then u is defined on an open disk centered at z of radius ε. So, it contains a closed disk of radius $\varepsilon/2$. We want to check that this closed disk has no "holes", meaning if γ is a closed curve inside the disk, then the area enclosed by γ is also inside the disk.

  Recall that the Jordan Curve Theorem states that a closed curve divides the plane into a bounded enclosed region and an unbounded exterior.
  Suppose there is a point z enclosed by γ and outside the disk. We draw a line that is tangent to the disk such that the point is on the other side of the line. Since the curve does not cross the tangent line, the half of the plane on the side of the tangent line must be either inside or outside the curve. The line is unbounded, so it cannot be in the bounded enclosed region. So, the theorem on harmonic functions applies and $a(z) = \operatorname{Re}(f(z))$ for some f holomorphic on the disk. Holomorphic functions are infinitely differentiable, so they are infinitely real differentiable. So, the real part is infinitely differentiable. □

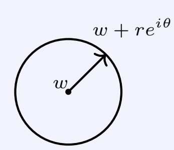
Maximum Modulus Principle

Proposition

Suppose $u : G \to \mathbb{R}$ is a harmonic function on an open set containing a closed disk centered at w with radius r. Then,

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})\mathrm{d}\theta.$$

This states that the value of a harmonic function at the center is the average of the values on the boundary.



- **Proof**: use the Cauchy Integral Formula. Let $f(z)$ be a holomorphic function where $u(z) = \operatorname{Re}(f(z))$.

  $$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\mathrm{d}z$$

  Choose γ to be the circle centered at w with radius r.

  $$\begin{aligned} \gamma(\theta) = w + re^{i\theta} \quad &\Rightarrow \quad \gamma'(\theta) = ire^{i\theta} \\ f(w) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(theta))}{\gamma(\theta)-w}\gamma'(\theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f(\gamma(\theta))i\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\gamma(\theta))\mathrm{d}\theta \end{aligned}$$

Take the real parts to find u.

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(\gamma(\theta))\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})\mathrm{d}\theta$$

So, $u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})\mathrm{d}\theta$. $\square$

Recall that harmonic functions are solutions to the Laplace Equation, which can be written as $\nabla \cdot \nabla u = 0$ where ∇ is the divergence which describes the net "flow" out and ∇u is the gradient which describes the change of u in all directions. For $\nabla \cdot \nabla u = 0$, the "net charge" of u in all directions is zero. So, the average in all directions is just the value at the point.

Proposition

Suppose $u : G \to \mathbb{R}$ is a harmonic function where $G \subseteq \mathbb{C}$ is an open disk containing the closed disk centered at w with radius r. Suppose that $\forall z \in G$, $u(z) \le u(w)$ meaning $u(w)$ is maximal. Then for z in the closed disk centered at w with radius r, $u(z) = u(w)$.

- **Proof**: note that to deduce the result for z in the disk, it is enough to prove for z on the circle centered at w with radius r since we can apply the same result for a smaller circle which also contains z.

  From the previous proposition, we have $u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w + re^{i\theta})\mathrm{d}\theta$. We also know that $u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(w)\mathrm{d}\theta$. To show that $u(w) = u(z) = u(w + re^{i\theta})$, it suffices to show

  $$\int_0^{2\pi} \underbrace{u(w) = u(w + re^{i\theta})}_{g(\theta)} \mathrm{d}\theta = 0.$$

  To show $u(w + re^{i\theta}) = u(w)$, show that $g(\theta) = 0$.

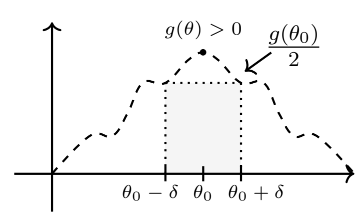
  Since u is harmonic, it is infinitely differentiable, so it is continuous. So, $u(w + re^{i\theta})$ is a continuous function of θ since it is a composition of continuous functions $\theta \mapsto w + re^{i\theta} \mapsto u(w + re^{i\theta})$. This also means $g(\theta)$ is continuous. Since $u(w)$ is maximal, $u(w + re^{i\theta}) \le u(w)$ meaning $g(\theta)$ is nonnegative.

  By contradiction, suppose $g(\theta_0) \ne 0$ for some θ_0. g is nonnegative, so $g(\theta_0) > 0$. g is continuous, and by definition of continuous functions, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|\theta - \theta_0| < \delta$, then $|g(\theta) - g(\theta_0)| < \varepsilon$. Pick $\varepsilon = g(\theta_0)/2 > 0$. So, we have

  $$\begin{aligned} g(\theta_0) - g(\theta) &\le |g(\theta) - g(\theta_0)| \\ &< \varepsilon = \frac{1}{2}g(\theta_0) \\ \frac{g(\theta_0)}{2} &< g(\theta) \end{aligned}$$

  By observing the plot, we can see that the integral from $\theta_0 - \delta$ to $\theta_0 + \delta$ of $\frac{g(\theta_0)}{2}$, shown in grey, must be less than the integral of $g(\theta)$ over that same curve since $\frac{g(\theta_0)}{2} < g(\theta)$.

  $$\begin{aligned} 0 = \int_0^{2\pi} g(\theta)\mathrm{d}\theta &\ge \int_{theta_0 - \delta}^{\theta_0 + \delta} g(\theta)\mathrm{d}\theta \\ &\ge \int_{\theta_0 - \delta}^{\theta_0 + \delta} \frac{g(\theta_0)}{2}\mathrm{d}\theta \\ &= 2\delta \cdot \frac{g(\theta_0)}{2} > 0 \end{aligned}$$

  We have $0 > 0$ which is a contradiction, so our assumption is incorrect and $u(z) = u(w)$ for any $z = w + re^{i\theta}$ on the disk. $\square$

So, for an open set G, $u(w)$ is constant on the disk centered at w meaning $u(z) = u(w)\ \forall z$ on the disk. u is constant on the disk centered at z as well, so we want to keep adding disks to show that u is constant on all of G. We will show that if G is an open, path connected region, then a harmonic function $u : G \to \mathbb{R}$ such that $\exists w \in G$ with $u(w) \geq u(z)\ \forall z \in G$ must be a constant function.

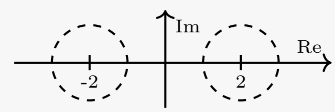
A set G is **path connected** if for any $p, q \in G$, there is a path with p and q as endpoints

$$\gamma : [a, b] \to G \text{ with } \gamma(a) = p,\ \gamma(b) = q.$$

Example

1. Any disk (open or closed) is path connected in u since it is convex. If we have two points in a convex set, the line segment joining them is also in the set. This line segment is the path that we need. This works for any convex set.

2. Non-example: let G be the union of open disks of radius 1 centered at 2 and -2.



   Suppose $\gamma : [a, b] \to G$ where $\gamma(a) = -2$ and $\gamma(b) = 2$. We will use the IVT to show that at some point $g \in G$, $\gamma(g) \notin G$.

   Consider $\mathrm{Re}(\gamma(t))$, a real-valued continuous function. By the IVT, $\exists t \in [a, b]$ such that $\mathrm{Re}(\gamma(t)) = 0$. But there are no points in G with real part zero. So, $\gamma(t) \notin G$ which is a contradiction. So, $\gamma(t)$ is not path connected.

   Now, consider

$$g(p) = \begin{cases} 1 & p \in \text{ right circle} \\ 0 & p \in \text{ left circle} \end{cases}$$

   Note that $g(z)$ is holomorphic on G since $\lim\limits_{h\to 0} \frac{g(p+h)-g(p)}{h} = \lim\limits_{h\to 0} \frac{0}{h} = 0$ since for h small enough, $p+h$ is in the same circle as p meaning $g(p) = g(p+h)$.

   $g(p) = \mathrm{Re}(g(p))$, so g may also be viewed as a harmonic function on G. $\max(g) = 1$ on the right disk, but it is not constant since it is it has a different value on the left disk.

For a path connected set G, we want to find a sequence of disks along the curve connecting w and z. This would allow us to conclude that the function is constant on each disk, eventually covering the entire curve.

Consider a path $\gamma : [a, b] \to G$ with $\gamma(a) = w$ and $\gamma(b) = z$. We want to show that u is constant on this path. We argue as follows: let ζ be the point on γ which is length $\varepsilon/2$ along the curve from w. ε is the radius of a disk centered at w contained in g. So, ζ is within this disk meaning $u(\zeta) = u(w)$.

Now, let's try this again with γ replaced by the remainding part from ζ to z, which has length $\text{length}(\gamma) - \varepsilon/2$. If we do this n times, we will get a curve of length $\text{length}(\gamma) - n\varepsilon/2$. Eventually, n is large enough such that this is less than ε. So, the endpoints are contained in a disk of radius ε centered at the first endpoint. We conclude that u is constant along γ, which we showed step by step.

We still need to show that we can use the same ε for all these points, i.e. if $z \in \mathbb{C}\backslash G$, then $|\gamma(t) - z| \geq \varepsilon > 0$. To do this, we define function $d : [a, b] \to \mathbb{R}$ where $[a, b]$ is the domain of γ:

$$d(t) := \inf_{z \in \mathbb{C}\backslash G} |\gamma(t) - z|$$

$d(t)$ is the size of the largest disk we can draw at $\gamma(t)$.

Recall the definition of the infimum: if $S \subseteq \mathbb{R}$ is a nonempty set, a lower bound for S is $L \in \mathbb{R}$ such that $\forall s \in S$, $L \leq s$. For $\mathbb{R}$, if S has a lower bound, then it has a greatest lower bound, known as $\inf S$. So, $\inf S \leq s\ \forall s \in S$ and $L \leq \inf S$ for any lower bound L of S.

We want to show two things:

1. $d(t) > t$ for every t
2. $d(t)$ is continuous

First, show that $d(t) > 0\ \forall t$.

Note that G is open. Since $\gamma(t) \in G$, G contains a disk centered at $\gamma(t)$ of some radius $\varepsilon(t) > 0$. If $z \in \mathbb{C}\backslash G$, then $|\gamma(t) - z| \geq \varepsilon(t)$ since if $|\gamma(t) - z| < \varepsilon(t)$, then z is on this disk and hence in G. So, $\varepsilon(t)$ is a lower bound for $\gamma(t) - z|$ meaning

$$d(t) = \inf_{z \in \mathbb{C}\backslash G} |\gamma(t) - z| \geq \varepsilon(t) > 0.$$

Now, show that $d(t)$ is continuous.

$d(t)$ measures the distance from $\gamma(t)$ to $\mathbb{C}\backslash G$, so we will use the Triangle Inequality.

We need $\lim_{h\to 0} d(t+h) = d(t)$, or $\lim_{h \to 0} |d(t+h) - d(h)| = 0$. For $z \in \mathbb{C}\backslash G$, since the infimum is a lower bound:

$$\begin{aligned}
|\gamma(t) - z| &\geq d(t) \\
|\gamma(t+h) - z| &\geq d(t+h) \\
d(t) \leq |\gamma(t) - z| &= |\gamma(t) - \gamma(t+h) + \gamma(t+h) - z| \\
&\leq |\gamma(t) - \gamma(t+h)| + |\gamma(t+h) - z| \\
\underbrace{d(t) - |\gamma(t) - \gamma(t+h)|}_{\text{lower bound of } |\gamma(t+h)-z|} &\leq |\gamma(t+h) - z|
\end{aligned}$$

This lower bound is independent of z. Any lower bound is $\leq$ than the greatest lower bound which is the infimum, so:

$$\begin{aligned}
d(t) - |\gamma(t) - \gamma(t+h)| &\leq d(t+h) \\
d(t) - d(t+h) &\leq |\gamma(t) - \gamma(t+h)|
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
d(t+h) \leq |\gamma(t+h) - z| &= |\gamma(t+h) - \gamma(t) + \gamma(t) - z| \\
&\leq |\gamma(t+h) - \gamma(t)| + |\gamma(t) - z| \\
\underbrace{d(t+h) - |\gamma(t+h) - \gamma(t)|}_{\text{lower bound of } |\gamma(t)-z|} &\leq |\gamma(t) - z|
\end{aligned}$$

This lower bound is also independent of z. Again, by definition of the infimum,

$$\begin{aligned}
d(t+h) - |\gamma(t+h) - \gamma(t)| &\leq d(t) \\
-|\gamma(t+h) - \gamma(t)| &\leq d(t) - d(t+h)
\end{aligned}$$

So, we have

$$0 \leq |d(t+h) - d(t)| \leq |\gamma(t+h) - \gamma(t)|.$$

We know γ is continuous, so by definition of continuity, $|\gamma(t+h) - \gamma(t)| \to 0$ as $h \to 0$. So, by the squeeze theorem, as $h \to 0$, $|d(t+h) - d(t)| \to 0$ meaning d is continuous.

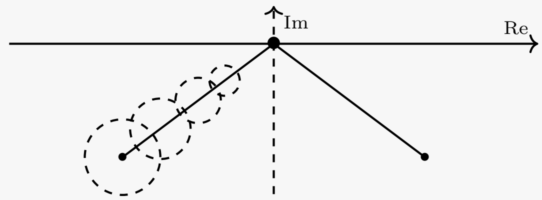
So, we have $d(t) > 0$ and d continuous. d is defined on a compact set $[a, b]$. By the Weierstrass Theorem, a continuous function on a compact set attains a minimum, so $\exists$ some $t' \in [a, b]$ such that $d(t') \leq d(t)$ $\forall t \in [a, b]$. So, d is bounded below by this $\varepsilon = d(t') > 0$. We conclude that $d(t) \geq \varepsilon > 0$ for some ε, in particular $|\gamma(t) - z| \geq \varepsilon\ \forall z \in \mathbb{C}\backslash G$.

So, a disk centered at radius $\gamma(t)$ with radius ε is constant on G, where ε is independent of $t \in [a, b]$.

We conclude that if u is a harmonic function on $G \subseteq \mathbb{C}$ where G is open and path connected and u attains some maximum, then u is constant on all of G.

Example

This is a non-example, which will show that G must be open. Let G be the plane $\mathbb{C}$ without the imaginary axis but with the origin added back on. G is path continuous, but a sequence of circles won't fit from one side to the other. To prove this, show that the circles have some minimum size.



However, we have proved that $d(t) \geq \varepsilon > 0$ for some ε where d is the size of the largest disk we can draw at $\gamma(t)$. So, the circles cannot fit through the point $(0,0)$ meaning u may not be constant on G. This is not a contradiction since G is not an open set in this case.

Theorem: Maximum Modulus Principle

If $G \subseteq \mathbb{F}$ is open and path connected, f is holomorphic on G, and $|f|$ attains a maximum in G, then f is constant.

Note that the name "modulus" refers to the absolute value.

To prove this, not that we cannot use what we proved previously since $|f|$ may not be harmonic. Instead, we will consider $\ln(|f(z)|) = \operatorname{Re}(\ln(f(z)))$. This turns out to be harmonic. Then, we can apply our result for harmonic functions.

Power Series

Our aim is to understand power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad a_n \in \mathbb{C}$$

which are functions of z. Examples of power series include Taylor series:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2} + \cdots$$

If f is holomorphic, then it is infinitely differentiable, so each summand makes sense. But, we still need to address convergence. It turns out to be more convienient to talk about convergence of sequences rather than series.

Note that for a series $\sum\limits_{n=0}^{\infty} f_n(z)$ to converge, it is equivalent for the sequence of partial sums of the series $\sum\limits_{n=0}^{k} f_n(z)$ to converge.

Convergence of Sequences

Suppose $f_n : G \to \mathbb{C}$ where $n \in \mathbb{Z}_{\geq 0}$ is a sequence of functions and $G \subseteq \mathbb{C}$. We say $f_n(z)$ converges to $f(z)$ **pointwise** on G if for each $z \in G$, $f_n(z) \to f(z)$ where $f_n(z)$ is a sequence of numbers. So, for each $z \in G$, $\forall \varepsilon > 0$, $\exists N > 0$ such that if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$. This N can depend on z.

Example

Let $f_n(z) = z^n$ and $G = [0,1] \subseteq \mathbb{R}$. We claim that $f_n(z)$ converges to $f(z)$ pointwise where

$$f(z) = \begin{cases} 1 & z = 1 \\ 0 & 0 \leq z < 1 \end{cases}$$

Note that $f(z)$ is not continuous.

When $z = 1$, $f_n(z) = 1$ is a constant sequence which converges to 1.

When $0 \leq z < 1$, we want N such that $\forall \varepsilon > 0$, if $n \geq N$, then

$$\begin{aligned} |f_n(z) - f(z)| &< \varepsilon \\ |z^n - 0| &< \varepsilon \\ z^n &< \varepsilon && (z \in \mathbb{R}_{\geq 0}) \\ n\ln(z) &< \ln(\varepsilon) \\ n &> \frac{\ln(\varepsilon)}{\ln(z)} && (z < 1, \text{ so } \ln(z) < 0) \end{aligned}$$

Take $N = \frac{\ln(\varepsilon)}{\ln(z)}$. For $n \geq N$, $|f_n(z) - f(z)| < \varepsilon$ meaning $f_n(z)$ converges to $f(z)$ for $0 \leq z < 1$. So, $f_n(z)$ converges to $f(z)$ pointwise.

$f_n(z)$ converges to $f(z)$ **uniformly** on G if $\forall \varepsilon > 0$, $\exists N > 0$ such that $\forall z \in G$, if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon$. Here, N cannot depend on z.

Example

Let $f_n(z) = z^n$ and $G = [0, r]$ for $0 < r < 1$. We claim that this converges uniformly to $f(z) = 0$.

$$\begin{aligned} |f_n(z) - f(z)| &< \varepsilon \\ z^n &< \varepsilon && \forall z \in G \\ n &> \frac{\ln(\varepsilon)}{\ln(z)} && \forall z \in G \\ \frac{\ln(\varepsilon)}{\ln(z)} &\leq \frac{\ln(\varepsilon)}{\ln(r)} \end{aligned}$$

So, we can take $N = \frac{\ln(\varepsilon)}{\ln(r)}$ which is independent of z, meaning $f_n(z)$ converges to $f(z)$ uniformly.

Note that this would not work for $G = [0, 1)$ since then, $\sup_r \frac{\ln(\varepsilon)}{\ln(r)} = \infty$, so no such N can exist. So, convergence would no longer be uniform.

Uniform convergence has the following (desirable) properties.

Proposition

1. Suppose $f_n(z)$ are continuous functions converging to $f(z)$ uniformly on G. Then, $f(z)$ is continuous.
2. If $f_n(z)$ converges to $f(z)$ uniformly in G and γ is a curve contained in G, then

$$\lim_{n\to\infty} \int_\gamma f_n(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z.$$

   So, we can interchange the order of limits and integration in this case.

- **Proof**:
  1. Fix some $z_0 \in G$. We need to show that for any $\varepsilon > 0$, $\exists \delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$.

     By uniform convergence, we can pick N such that for $n \geq N$, $|f_n(z) - f(z)| < \frac{\varepsilon}{3}$. Fix such $n \geq N$. Use the fact that $f_n(z)$ are continuous. $\exists \delta > 0$ such that if $|z - z_0| < \delta$, then $|f_n(z) - f_n(z_0)| < \frac{\varepsilon}{3}$.

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \\ &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \\ &\leq \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon \end{aligned}$$

     So, $f(z)$ is continuous on $z_0 \in G$ which is an arbitrary point, so $f(z)$ is continuous in z. □

2. For $\varepsilon > 0$, pick N such that for $n > N$,

$$|f_n(z) - f(z)| < \varepsilon / \text{ length}(\gamma).$$

We want to show that $\left|\int_\gamma f_n(z)\mathrm{d}z - \int_\gamma f(z)\mathrm{d}z\right| < \varepsilon$ for $n > N$.

$$\left|\int_\gamma f_n(z)\mathrm{d}z - \int_\gamma f(z)\mathrm{d}z\right| = \left|\int_\gamma \left[f_n(z) - f(z)\right]\mathrm{d}z\right|$$

Apply the ML lemma.

$$\begin{aligned} &\leq \max_{z\in\gamma} |f_n(z) - f(z)| \cdot \text{ length}(\gamma) \\ &< \frac{\varepsilon}{\text{length}(\gamma)} \cdot \text{ length}(\gamma) = \varepsilon \end{aligned}$$

So, $\int_\gamma f_n(z)\mathrm{d}z$ converges to $\int_\gamma f(z)\mathrm{d}z$. Ultimately, we can use this to control power series. □

Weierstrass M-test

The Weierstrass M-test allows us to determine if a sequence of functions is uniformly convergent.

Theorem: Weierstrass M-test

Suppose $G \subseteq \mathbb{C}$, $f_k : G \to \mathbb{C}$, and $M_k \in \mathbb{R}$ with $|f_k(z)| \leq M_k \ \forall z \in G$ and $k \geq 0$. If $\sum_{k\geq 0} M_k$ converges, then $\sum_{k\geq 0} f_k(z)$ converges uniformly on G.

- **Proof**: $0 \leq |f_k(z)| \leq M_k$, so M_k must be nonnegative. Since $\sum_{k=0}^{\infty} M_k$ converges to some M, $\forall \varepsilon > 0$, $\exists N > 0$ such that for $n \geq N$, $\left|\sum_{k=0}^{n} M_k - M\right| < \varepsilon$.

$$\left|\sum_{k=0}^{n} M_k - \sum_{k=0}^{\infty} M_k\right| < \varepsilon$$

$$\left|-\sum_{k=n+1}^{\infty} M_k\right| = \sum_{k=n+1}^{\infty} M_k < \varepsilon$$

We need

$$\left|\sum_{k=0}^{n} f_k(z) - \underbrace{\sum_{k=0}^{\infty} f_k(z)}_{\text{limit of sum}}\right| < \varepsilon$$

We know that the limit of the sum $\sum_{k=0}^{\infty} f_k(z)$ exists since $\sum_{k=0}^{\infty} |f_k(z)| \leq \sum_{k=0}^{\infty} M_k < \infty$. So, $\sum_{k=0}^{\infty} f_n(z)$ converges absolutely. Additionally, we have

$$\begin{aligned} \left|\sum_{k=n+1}^{\infty} f_k(z)\right| &\leq \sum_{k=n+1}^{\infty} |f_k(z)| \\ &\leq \sum_{k=n+1}^{\infty} M_k && \text{(independent of } z\text{)} \\ &< \varepsilon \end{aligned}$$

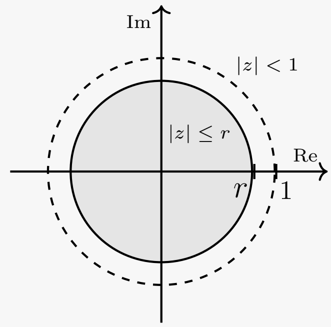
The N from $\sum M_k$ can be used, meaning $\sum_{k=0}^{\infty} f_k(z)$ converges uniformly. □

Example

Let $f_k(z) = z^k$ where $|z| < 1$.

$$\sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} z^k \qquad \text{(geometric series with } |z| < 1\text{)}$$
$$= \frac{1}{1-z}$$

We want to determine if it converges uniformly.



Let's view $f_k(z)$ as functions on the closed disk $|z| \le r$ where $0 \le r < 1$ is fixed. We claim that convergence on $|z| \le r$ is uniform. We observe the defined area of $|z| < 1$, which is an open disk of radius 1 containing the shaded closed disk of radius $r < 1$. We want to show uniform convergence for the shaded area.

To prove uniform convergence on $|z| \le r$, use the Weierstrass M-text.

We need $M_k \ge |f_k(z)|$.

$$|f_k(z)| = |z^k| = |z|^k \le r^k$$

since $|z| < r$. Set $M_k = r^k$.

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} r^k \qquad \text{(geometric series with } r < 1\text{)}$$
$$= \frac{1}{1-r}$$

So, the M-test applies and we have uniform convergence.

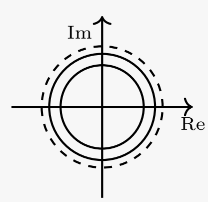
What about the original set $|z| < 1$? Convergence is not uniform. If it was, then $\forall \varepsilon > 0$, $\exists N > 0$ such that if $n \ge N$, then

$$\left| \sum_{k=0}^{n} z^k - \frac{1}{1-z} \right| < \varepsilon.$$

$\frac{1}{1-z}$ is the limit of the series.

$$\left| \frac{1 - z^{n+1}}{1-z} - \frac{1}{1-z} \right| < \varepsilon$$
$$\frac{|z|^{n+1}}{|1-z|} < \varepsilon$$

This holds for any $n \ge N$. As $z \to 1$, this goes to ∞ so the bound $\frac{|z|^{n+1}}{|1-z|} < \varepsilon$ cannot hold $\forall |z| < 1$. So, convergence is not uniform.



Even though $\{z \mid |z| < 1\}$ is the union of all $\{z \mid |z| \le r\}$ for $r > 1$, we don't have uniform convergence on the open unit disk.

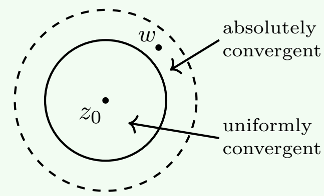
Convergence of Power Series

In the previous example, we observed the special case where $a_k = 1$ and $z_0 = 0$.

Lemma

If $\sum\limits_{k\geq 0} a_k(z-z_0)^k$ converges for some $w \in \mathbb{C}$, then $\sum\limits_{k\geq 0} a_k(z-z_0)^k$ converges absolutely when $|z-z_0| < |w-z_0|$.

Morever, if $0 \leq \ell < |w-z_0|$, then convergence is uniform on $|z-z_0| \leq \ell$.



- **Proof**: $\sum\limits_{k\geq 0} a_k(w-z_0)^k$ converges, so its terms must go to zero. If $\sum\limits_{k\geq 0} c_k$ converges,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \underbrace{\left[\sum_{k=0}^{n} c_k - \sum_{k=0}^{n-1} c_k\right]}_{\text{same limit}} = 0.$$

So, $\lim\limits_{k\to 0} a_k(w-z_0)^k = 0$, i.e. for any $\varepsilon > 0$, $\exists N > 0$ such that if $n \geq N$, then $|a_n(w-z_0)^n - 0| < \varepsilon$ where 0 is the limiting value. So, we have $|a_n(w-z_0)^n| < \varepsilon$.

Let $M = \max\left\{|a_0(w-z_0)^0|, \left|a_1(w-z_0)^1\right|, \ldots, |a_N(w-z_0)^N|, \varepsilon\right\}$. So, $M \geq \left|a_k(w-z_0)^k\right|$ $\forall k$. Then,

$$\begin{aligned}\sum_{k\geq 0} |a_k(z-z_-)^k| &= \sum_{k\geq 0} |a_k(w-z_0)^k| \cdot \left|\frac{z-z_0}{w-z_0}\right|^k \\ &\leq \sum_{k\geq 0} M \left|\frac{z-z_0}{w-z_0}\right|^k\end{aligned}$$

This is a geometric series with common ration $\left|\frac{z-z_0}{w-z_0}\right| < 1$ since $|z-z_0| < |w-z_0|$. This proves absolute convergence.

To prove uniform convergence for on $|z-z_0| \leq \ell$ for $0 \leq \ell < |w-z_0|$, use the Weierstrass M-test.

$$\begin{aligned}\left|a_k(z-z_0)^k\right| &\leq M \left|\frac{z-z_0}{w-z_0}\right|^k \\ &\leq M \left|\frac{\ell}{w-z_0}\right|^k\end{aligned}$$

Let

$$M_k = M \left|\frac{\ell}{w-z_0}\right|^k.$$

We need the sum of M_k to converge.

$$\sum_{k\geq 0} M_k = \sum_{k\geq 0} M \left|\frac{\ell}{w-z_0}\right|^k$$

This is a geometric series with common ration $\left|\frac{\ell}{w-z_0}\right| < 1$. $\sum\limits_{k\geq 0} M_k$ converges, so by the Weierstrass M-test, $\sum\limits_{k\geq 0} a_k(z-z_0)^k$ converges on $|z-z_0| \leq \ell$. $\square$

Radii of convergence

> **Theorem**
>
> For a power series $\sum_{k\geq 0} a_k(z-z_0)^k$, $\exists R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that
>
> - If $|z-z_0| < R$, the series converges absolutely.
> - If $|z-z_0| > R$, the series diverges.
>
> If $\ell < R$, convergence is uniform on $|z-z_0| \leq \ell$.

- **Proof**: let $S = \left\{x \in \mathbb{R}_{\geq 0} \mid \sum_{k\geq 0} a_k x^k \text{ converges}\right\}$. Note that if $x \in S$ and $0 \leq y < x$, then $y \in S$. Apply the lemma for $w = z_0 + x$ and $z = z_0 + y$. We have $y = |z-z_0| < |w-z_0| = x$, so if $x \in S$, $[0, x) \subseteq S$ (we also have $[0, x] \subseteq S$).

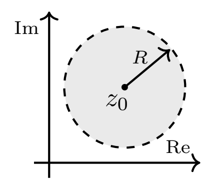
  If S is unbounded, then for any $t \in \mathbb{R}_{\geq 0}$, $\exists x \in S$ with $t < x$ since otherwise, t is an upper bound. $t \in [0, x)$ so $t \in S$. So, $S = \mathbb{R}_{\geq 0}$. We conclude that the series converges absolutely $\forall z$ since the series converges when $w = z_0 + x$ where $|x| > |z-z_0|$. So, it converges absolutely for $|z-z_0| < |x|$.

  If S is unbounded, let R be the supremum $R = \sup(S)$. The supremum is the least upper bound. If $|z-z_0| < R = \sup(S)$, then $\exists t \in S$ such that $|z-z_0| < t \leq R$ since otherwise, $|z-z_0|$ is an upper bound for S which contradicts R being the least upper bound since $|z-z_0| < R$. Now, apply the lemma. Convergence at t implies convergence when $|z-z_0| < t$. If $|z-z_0| > R$, if the series converges at t, then S contains all t with $0 \leq t < |z-z_0|$ such as $t = \frac{R+|z-z_0|}{2} > R$. This is a contradiction to R being an upper bound to S. So, the series must diverge if $|z-z_0| > R$.

  To address uniform convergence if $\ell < R$, then $\frac{\ell+R}{2} \in S$ and the lemma implies uniform convergence when $|z-z_0| \leq \ell$. $\square$

This R is called the **radius of convergence**. We cannot immediately say what happens when $|z-z_0| = R$.

How can we find R without already knowing where the series converges?



> **Lemma**
>
> The radius of convergence obeys
> $$\frac{1}{R} = \limsup \sqrt[k]{|a_k|}.$$

- **Proof**: if $L = \limsup \sqrt[k]{|a_k|}$, then for any $\varepsilon > 0$, $\exists N > 0$ such that if $k \geq N$, then $\sqrt[k]{|a_k|} < L + \varepsilon$. Additionally, for any $\varepsilon > 0$, there are infinitely many $\sqrt[k]{|a_k|} > L - \varepsilon$. If $k \geq N$,

  $$\begin{aligned} \sqrt[k]{|a_k|} &> L + \varepsilon \\ |a_k| &< (L+\varepsilon)^k \end{aligned}$$

  So,

  $$\begin{aligned} \sum_{k\geq 0} \left|a_k(z-z_0)^k\right| &= \sum_{k=0}^{N-1} \left|a_k(z-z_0)^k\right| + \sum_{k=N}^{\infty} \left|a_k(z-z_0)^k\right| \\ &\leq [\text{finite constant}] + \sum_{k=N}^{\infty} \underbrace{(L+\varepsilon)^k |z-z_0|^k}_{\text{geometric series}} \end{aligned}$$

  This converges when $(L+\varepsilon)|z-z_0| < 1$. This implies that the power series converges absolutely.

So, we have absolute convergence when $|z-z_0| < \frac{1}{L+\varepsilon}$ for some $\varepsilon > 0$. $L+\varepsilon$ can be any number longer than L meaning $\frac{1}{L+\varepsilon}$ can be any element in $\left(0, \frac{1}{L}\right)$. So, it is enough that $|z-z_0| < \frac{1}{L}$.

$|z-z_0| < \frac{1}{L}$ implies absolute convergence. We must have $\frac{1}{L} \leq R$ since if $R > \frac{1}{L}$, then for $R < |z-z_0| < \frac{1}{L}$, $R < |z-z_0|$ implies that the power series diverges while $|z-z_0| < \frac{1}{L}$ implies that the power series converges, which is a contradiction.

We want to show that $\frac{1}{L} = R$, so we want to show that if $|z-z_0| > \frac{1}{L}$, the series diverges. If $L = 0$, $|z-z_0| < \frac{1}{\varepsilon}$ for some $\varepsilon > 0$ implies convergence. But, we can make $\frac{1}{\varepsilon}$ arbitrarily large, so we will always be able to do this if $L = 0$. We have convergence $\forall z$, so $R = \infty$.

For any $\varepsilon > 0$, there are infinitely many k such that $\sqrt[k]{|a_k|} > L - \varepsilon$. To raise this to the k^{th} power, assume $L - \varepsilon > 0$ meaning $L \neq 0$. $|a_k| > (L-\varepsilon)^k$ for infinitely many k. If $\sum a_k(z-z_0)^k$ converges, then $\left|a_k(z-z_0)^k\right| \to 0$ as $k \to \infty$.

For infinitely many k,

$$0 \leq (L-\varepsilon)^k |z-z_0|^k < \underbrace{\left|a_k(z-z_0)^k\right|}_{\to 0}.$$

So, we must have $(L-\varepsilon)^k|z-z_0|^k \to 0$, meaning $(L-\varepsilon)|z-z_0| < 1$. So, convergence of the power series implies $|z-z_0| < \frac{1}{L-\varepsilon}$ for some $\varepsilon > 0$.

$L-\varepsilon$ can be anything smaller than L, so $\frac{1}{L-\varepsilon}$ can be anything larger than L. This is satisfied $\forall \varepsilon > 0$ with $L - \varepsilon > 0$ when $|z-z_0| \leq \frac{1}{L}$. So, if $|z-z_0| > \frac{1}{L}$, the series must diverge.

So, $\frac{1}{L} \geq R$ since if $\frac{1}{L} < R$, then for $\frac{1}{L} < |z-z_0| < R$, $\frac{1}{L} < |z-z_0|$ implies that the power series diverges while $|z-z_0| < R$ implies that the power series convergence, which is a contradiction.

So, we have $\frac{1}{L} \leq R$ and $\frac{1}{L} \geq R$. Thus, $\frac{1}{L} = R$ meaning $\frac{1}{R} = L = \limsup \sqrt[k]{|a_k|}$. $\square$

Example

Let $z_0 = 0$ and $a_k = \frac{1}{k!}$. So, the power series is

$$\sum_{k \geq 0} a_k(z-z_0)^k = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

This is secretly equal to e^z.

To know that the series makes sense $\forall z$, we must show that it converges everywhere ($R = \infty$).

$$\frac{1}{R} = \limsup \sqrt[k]{|a_k|} = \limsup \sqrt[k]{1/k!}$$

One method of solving this applies Stirling's Formula:

$$n! \approx \sqrt{2\pi n}\left(\tfrac{n}{e}\right)^n$$

This ratio goes to 1 as $n \to \infty$. So, for n large, $n! > \left(\frac{n}{e}\right)^n$ or $\sqrt[n]{n!} > \frac{n}{e}$.

$$\sqrt[n]{1/n!} < \tfrac{e}{n}$$

As $n \to \infty$, this goes to 0 so $L = 0$ and $R = \infty$.

We can also prove this using the ratio test, which is the more standard approach. $\sum\limits_{k \geq 0} c_k$ where $c_k = \frac{z^k}{k!}$ converges if $\lim \left|\frac{c_{k+1}}{c_k}\right| < 1$.

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{z^{k+1}/(k+1)!}{z^k/k!}\right| = \left|\frac{z}{k+1}\right| = \frac{|z|}{k+1} \to 0 \text{ as } k \to \infty$$

By the ratio test, $\sum \frac{z^k}{k!}$ converges $\forall z \in \mathbb{C}$, hence $R = \infty$. If $R < \infty$, then it would diverge somewhere.

Example

Let $z_0 = 0$ and

$$a_k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k-1)/2}}{k} & k \text{ odd} \end{cases}.$$

So, the power series is

$$\sum_{k\geq 0} a_k(z-z_0)^k = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

This is secretly equal to $\arctan(z)$.

We claim that $R = 1$, so $\limsup \sqrt[k]{|a_k|} = \frac{1}{R} = 1$. It suffices to show that

$$\limsup_{k \text{ odd}} \sqrt[k]{|a_k|} = \limsup_{k \text{ odd}} \sqrt[k]{1/k} = 1.$$

Take the log of both sides.

$$\ln\left(\sqrt[k]{1/k}\right) = \frac{1}{k}\left(-\ln(k)\right) \to 0 \text{ as } k \to \infty$$

since polynomials grow faster than logarithms.

If sequence $x_n \to x$ and function g is continuous at x, then $g(x_n) \to g(x)$. Let $g(t) = e^t$.

$$-\frac{\ln(k)}{k} \to 0, \text{ so } \underbrace{e^{-\ln(k)/k}}_{\sqrt[k]{1/k}} \to e^0 \to 1.$$

Thus, the radius of convergence is $R = 1$.

Holomorphic Functions as Power Series

Proposition

If $\sum\limits_{k\geq 0} a_k(z-z_0)^k$ has radius of convergence R, then it defines a continuous function on the open disk of radius R centered at z_0.

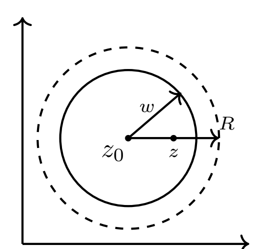
- **Proof**: it suffices to verify continuity at every z where $|z - z_0| < R$.

  Consider the power series

  $$\sum_{k\geq 0} a_k(w-z_0)^k$$

  for

  $$|w - z_0| \leq \frac{|z-z_0| + R}{2}.$$

  This disk contains z but is smaller, so we have uniform convergence.

  On this smaller disk, we have uniform convergence of

  $$\sum_{k=0}^{n} a_k(z-z_0)^k$$

  which is a sum of polynomials, which is continuous. So, the limit function is continuous on this region where we have uniform convergence, particularly at z. We can do this for any z, so we have continuity for the disk. ∎

Proposition

Suppose $\sum_{k\geq 0} a_k(z-z_0)^k$ has radius of convergence R. If γ is a curve inside the disk $|z-z_0|<R$, then

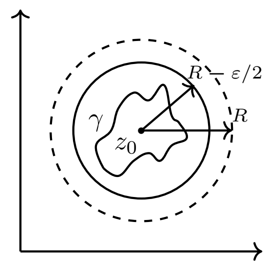
$$\int_\gamma \sum_{k\geq 0} a_k(z-z_0)^k \mathrm{d}z = \sum_{k\geq 0} a_k \int_\gamma (z-z_0)^k \mathrm{d}z.$$

If γ is closed, then this is zero.

- **Proof**: if we have uniform convergence, then

$$\begin{aligned}\int_\gamma \sum_{k\geq 0} a_k(z-z_0)^k \mathrm{d}z &= \lim_{n\to\infty} \int_\gamma \sum_{k=0}^{n} a_k(z-z_0)^k \mathrm{d}z \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_\gamma (z-z_0)^k \mathrm{d}z \\ &= \sum_{k=0}^{\infty} a_k \int_\gamma (z-z_0)^k \mathrm{d}z\end{aligned}$$

So, it would be enough for the series to converge uniformly on a set containing the curve γ.



We want to find a disk of radius $< R$ still containing γ. When discussing harmonic functions, we considered

$$d(t) = \inf_{z\in\mathbb{C}\setminus G} |\gamma(t)-z|$$

where G is the open disk of radius R. We showed $d(t)\geq\varepsilon>0$. This allows us to draw a closed disk of radius $R-\varepsilon/2$ with radius $< R$. We claim that the closed disk of radius $R-\varepsilon/2$ contains γ. If there was a point on γ outside this region, then it would lie within $\varepsilon/2$ of a point outside the disk of radius R. This would contradict $\inf|\gamma(t)-z|\geq\varepsilon$ since we have produced a value of $\gamma(t)$ that is at most $\varepsilon/2$ from z. So, we can find such a disk which implies uniform convergence.

If γ is closed, then by Cauchy's Theorem,

$$\int_\gamma (z-z_0)^k \mathrm{d}z = 0.$$

So,

$$\int_\gamma \sum_{k\geq 0} a_k(z-z_0)^k \mathrm{d}z = \sum_{k\geq 0} a_k \cdot 0 = 0.$$

So, the statement holds on such a curve and if the curve is closed, then it is zero. □

However, this is still not enough to prove that power series are holomorphic. For this, we will need one more theorem.

Theorem: Morera's Theorem

Suppose $f(z)$ is continuous on an open set G and for any closed curve γ in G,

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Then, $f(z)$ is holomorphic on G.

- **Proof**: recall that if $f(z)$ is continuous and $\int_\gamma f(z)\mathrm{d}z = 0$ for any closed curve γ, then it has an antiderivative $F(z)$ such that $F'(z) = f(z)$.

  Note that $F(z)$ is differentiable on G so it is holomorphic on G since G is open. Holomorphic functions are infinitely differentiable, so F is twice differentiable. So, $F'(z) = f(z)$ is differentiable on G, so it is holomorphic. Thus, $f(z)$ is holomorphic on G. $\square$

Now, we have enough information to make a conclusion about holomorphic functions as power series.

Theorem

If $\sum_{k\geq 0} a_k(z-z_0)^k$ has radius of convergence R, then on $|z-z_0| < R$, it defines a holomorphic function.

- **Proof**: we saw that $\sum_{k\geq 0} a_k(z-z_0)^k$ is continuous and its integrals along closed curves are 0 for $|z-z_0| < R$. So, Morera's Theorem applies and the power series is holomorphic on $|z-z_0| < R$. $\square$

Derivatives of Power Series

Now, let's understand derivatives of power series.

Lemma

If $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$ for $|z-z_0| < R$, then for such z,

$$f'(z) = \sum_{k\geq 0} k a_k (z-z_0)^{k-1}.$$

So, we can differentiate power series termwise inside the disk of convergence.

- **Proof**: recall

  $$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}\mathrm{d}w$$

  where γ is a closed curve enclosing z. So, we have

  $$f'(z) = \frac{1}{2\pi i}\int_\gamma \sum_{k\geq 0} \frac{a_k(w-z_0)^k}{(w-z)^2}\mathrm{d}w.$$

  We want this series to converge uniformly on a set containing γ. We will consider the set γ. Note that $\gamma(t)$ is continuous, so $\left|\frac{1}{(\gamma(t)-z)^2}\right|$ is continuous since z cannot lie on γ meaning $\gamma(t) - z \neq 0$.

  So, $\left|\frac{1}{(\gamma(t)-z)^2}\right|$ is a continuous function $[a,b] \to \mathbb{R}$ where $[a,b]$ is compact. So, its values are compact.

  We claim that if the series $\sum_{n\geq 0} g_n(z)$ converges uniformly to $g(z)$ and $h(z)$ is bounded, then $\sum_{n\geq 0} g_n(z)h(z) \to g(z)h(z)$ uniformly.

  For any $\varepsilon > 0$, $\exists N > 0$ such that if $k \geq N$, $\left|\sum_{n=0}^{k} g_n(z) - g(z)\right| < \varepsilon \ \forall z$.

  $$\begin{aligned}\left|\sum_{n=0}^{k} g_n(z)h(z) - g(z)h(z)\right| &= \left|\sum_{n=0}^{k} g_n(z) - g(z)\right| |h(z)| \\ &\leq \varepsilon \sup_z |h(z)|\end{aligned}$$

  $\sup|h(z)|$ is finite since h is bounded. So, $\sum_{n\geq 0} g_n(z)h(z) \to g(z)h(z)$ uniformly.

For us, $g_n(w) = a_n(w-z_0)^n$ and $h(w) = \frac{1}{(w-z)^2}$. So,

$$\begin{aligned}
f'(z) &= \frac{1}{2\pi i}\sum_{k\geq 0}\int_\gamma \frac{a_k(w-z_0)^k}{(w-z)^2}\mathrm{d}w \\
&= \frac{1}{2\pi i}\sum_{k\geq 0} a_k \int_\gamma \frac{(w-z_0)^k}{(w-z)^k}\mathrm{d}w \\
&= \frac{1}{2\pi i}\sum_{k\geq 0} a_k \left(2\pi i \frac{\mathrm{d}}{\mathrm{d}w}(w-z_0)^k\Big|_{w=z}\right) \\
&= \sum_{k\geq 0} a_k k(z-z_0)^{k-1}
\end{aligned}$$

□

This has the consequence that for $f(z) = \sum\limits_{k\geq 0} a_k(z-z_0)^k$,

$$\begin{aligned}
f(z_0) &= a_0 \\
f'(z_0) &= 1\cdot a_1 \\
f''(z_0) &= 2\cdot 1\cdot a_2 \\
&\vdots \\
f^{(k)}(z_0) &= k!a_k
\end{aligned}$$

So,

$$\begin{aligned}
f(z) &= \sum_{k\geq 0} a_k(z-z_0)^k \\
&= \sum_{k\geq 0} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k
\end{aligned}$$

This is exactly a Taylor Series.

Example

Let $z_0 = 0$ and $a_k = \frac{1}{k!}$. The power series is

$$f(z) = \sum_{k\geq 0} a_k(z-z_0)^k = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots$$

We claimed that this is equal to e^z. To prove this, we will first calculate its derivative.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}f(z) &= \sum_{k\geq 0}\frac{k}{k!}(z-z_0)^{k-1} \\
&= f(z)
\end{aligned}$$

Now, consider $f(z)e^{-z}$. Taking its derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}f(z)e^{-z} = f'(z)e^{-z} + f(z)\left(-e^{-z}\right) = 0$$

since $f(z) = f'(z)$. Since its derivative is 0, $f(z)e^{-z}$ is constant. To find this constant, evaluate $f(z)e^{-z}$ at $z = 0$.

$$\begin{aligned}
f(0) &= 1 + 0 + 0 + \cdots = 1 \\
e^{-0} &= 1
\end{aligned}$$

So, $f(z)e^{-z} = 1$ meaning $f(z) = e^z$.

Now, we will see that any holomorphic function can be written as a power series on some disk.

Theorem

Suppose $f(z)$ is holomorphic on a disk $|z-z_0|<R$. Then, $\sum\limits_{k\geq 0} a_k(z-z_0)^k$ converges to $f(z)$ where

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\mathrm{d}w$$

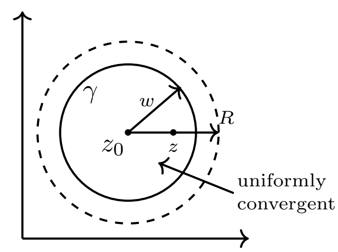
for $|z-z_0|<R$ where γ is a closed curve enclosing z_0.

- **Proof**: we have

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w.$$

Pick γ to be the circle

$$|w-z_0| = \frac{|z-z_0|+R}{z}.$$



So, we have

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)}\\ &= \frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}\end{aligned}$$

$w\neq z_0$ since w is on γ and z_0 is the center of γ. We want to show that this is the uniform limit of the geometric series $\sum\limits_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k$ for $|w-z_0| = \frac{|z-z_0|+R}{2}$, which is the curve γ along which we integrate.

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w\\ &= \frac{1}{2\pi i}\int_\gamma \sum_{k\geq 0}\underbrace{\frac{f(w)}{w-z_0}\left(\frac{z-z_0}{w-z_0}\right)^k}\mathrm{d}w\end{aligned}$$

We want the specified part to converge uniformly for $w\in\gamma$. Observe that it is the uniformly convergent series

$$\sum_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k$$

multiplied by the function

$$\frac{f(w)}{w-z_0}$$

which is a continuous function on the compact set γ. Hence, this function is bounded. So, we have

$$[\text{uniformly convergent series}]\times[\text{bounded function}] = \text{uniformly convergent series}$$

So, we can swap the order of the sum and the integral.

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\sum_{k\geq 0}\int_\gamma \frac{f(w)}{w-z_0}\frac{(z-z_0)^k}{(w-z_0)^k}\mathrm{d}w\\ &= \sum_{k\geq 0}\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\mathrm{d}w}_{a_k}(z-z_0)^k\\ &= \sum_{k\geq 0}a_k(z-z_0)^k\end{aligned}$$

$\square$

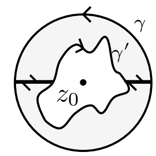
We still want to know two things:

1. What curve γ are we allowed to integrate on?
2. What is the radius of convergence?

To address the curve γ, we want to show that the integral over γ and some γ' is the same. Note that

$$\frac{f(w)}{(w-z_0)^{n+1}}$$

is holomorphic away from z_0 meaning Cauchy's theorem applies to the two pictured regions. The line segment integrals cancel out, so $\int_\gamma - \int_{\gamma'} = 0$ or $\int_\gamma = \int_{\gamma'}$. So, γ can be replaced by any curve enclosing z_0 without changing a_k.



Since we had uniform convergence, we in particular had convergence at our given value of z where $|z-z_0| < R$. So, the radius of convergence is at least R and could possibly larger than R.

Theorem

The radius of convergence R of $f(z) = \sum_{k \geq 0} a_k(z-z_0)^k$ is the largest number R' such that $f(z)$ extends to a holomorphic function on $|z-z_0| < R'$.

- **Proof**: we just showed that if $f(z)$ is holomorphic on $|z-z_0| < R'$, then radius of convergence is at least R', so $R \geq R'$.

  If the power series for $f(z)$ has radius of convergence R, then it defines a holomorphic function inside $|z-z_0| < R$ (and this extends $f(z)$). So, $R' \geq R$ meaning $R = R'$. $\square$

Example

Recall the power series

$$f(z) = \sum_{k \geq 0} a_k(z-z_0) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

which is secretly equal to $\arctan(z)$. We computed $R = 1$.

It turns out that we cannot extend $\arctan(z)$ holomorphically on a disk centered at 0 beyond $R = 1$. Roughly speaking, $\arctan(\pm i)$ is undefined. We can show this two ways.

If $\arctan(\pm i) = z$, we have $\tan(z) = i$ and $\sin(z) = i\cos(z)$ meaning $\sin^2(z) + \cos^2(z) = 0$. But, $\sin^2(z) + \cos^2(z) = 1$ for any z, which is a contradiction.

Alternatively, we have

$$\arctan(z) = \int_0^z \frac{1}{1+w^2} \mathrm{d}w$$

which forms a singularity at $w = \pm i$.

So, although $\arctan(z)$ and e^z are both infinitely differentiable as functions $\mathbb{R} \to \mathbb{R}$, their radii of convergence of their Taylor Series are 1 and ∞ respectively. We had to look at complex numbers to see why.

Theorem

If $f(z)$ is holomorphic at z_0, then

$$\frac{\mathrm{d}^n f}{\mathrm{d}z^n}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}} \mathrm{d}w$$

So, we can take higher derivatives of power series termwise inside the disk of convergence.

- **Proof**: we have already proved this for $k = 0, 1$, and 2. For $|z - z_0| < \varepsilon$ where ε is small enough to be contained in the open set on which f is holomorphic, $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$. Differentiate k times and set $z = z_0$:

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}z^k} f(z_0) &= k! a_k \\ &= \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\mathrm{d}w\end{aligned}$$

  using the formula for a_k. $\square$

Observe that this also shows that f determines a_k, so there is only one power series converging to $f(z)$ centered at z_0.

Additionally, we can see that if $f(z)$ is holomorphic at z_0, then $f(z)$ can be written as a power series centered at z_0. The converse is also true, so we have

$$\left\{\ f(z) \text{ holomorphic at } z_0\ \right\} \Longleftrightarrow \left\{\begin{array}{l} f(z) \text{ can be written as a} \\ \text{power series around } z_0\end{array}\right\}$$

If $f(z)$ can be written as a power series around z_0, then $f(z)$ is **analytic** at z_0. In complex analysis, “holomorphic” and “analytic” mean the same thing. In other areas of mathematics, the word “analytic” may make sense even when “holomorphic” may not.

Factorization of Holomorphic Functions

For nonconstant polynomial $p(z)$, if $p(z_0) = 0$, then

$$p(z) = (z-z_0)\frac{p(z)}{z-z_0}$$

where $\frac{p(z)}{z-z_0}$ is also a polynomial.

Example

Let $p(z) = z^2 - 1$. $p(-1) = 0$, so we can factor

$$p(z) = z^2 - 1 = (z+1)\frac{z^2-1}{z+1} = (z+1)(z-1)$$

where $\frac{z^2-1}{z+1} = z - 1$ is still a polynomial.

We want to show that we can do the same for holomorphic functions.

Theorem

If $f(z)$ is holomorphic at z_0, then either $f(z)$ is constant on some neighborhood of z_0 or there is some $m \in \mathbb{Z}_{\geq 0}$ and function $g(z)$ holomorphic at z_0 such that

1. $f(z) = (z-z_0)^m g(z)$
2. $g(z_0) \neq 0$

This m is called the **order of the zero** z_0 for $f(z)$ or the **order of vanishing** of $f(z)$ at z_0.

Note that this does not hold for real-valued functions. For example, $\sqrt{|x|^3}$ as a function $\mathbb{R} \to \mathbb{R}$ cannot be written in this way.

- **Proof**: let $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$. If $a_k = 0\ \forall k$, then $f(z) = \sum_{k\geq 0} 0 = 0$ meaning $f(z)$ is the constant function zero at a disk centered at z_0.

Otherwise, if not all a_k are zero, let m be the least number such that $a_m \neq 0$ but $a_k = 0$ for $k < m$. So,

$$f(z) = \cancel{a_0} + \cancel{a_1(z - z_0)} + \cancel{\cdots} + a_m(z - z_0)^m + \cdots$$

We can pull out a factor of $(z - z_0)^m$ from all these remaining terms.

$$\begin{aligned} f(z) &= (z - z_0)^m \left(a_m + a_{m+1}(z - z_0) + \cdots\right) \\ &= (z - z_0)^m \underbrace{\sum_{k \geq 0} a_{k+m}(z - z_0)^k}_{g(z)} \\ &= (z - z_0)^m g(z) \end{aligned}$$

We must still check that g satisfies both properties. We have $g(z_0) = a_m + 0 + \cdots = a_m \neq 0$ by definition. We must still check that $g(z)$ is holomorphic at z_0.

Since convergent power series are holomorphic, it is enough to know that $g(z)$ equals a power series with positive radius of convergence. Since $f(z)$ is holomorphic at z_0, $\sum\limits_{k \geq 0} a_k(z - z_0)^k$ has positive radius of convergence. So,

$$\frac{1}{R} = \limsup \sqrt[k]{|a_k|} < \infty.$$

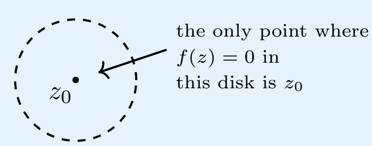
Let $\alpha \in \mathbb{R}_{\geq 0}$ satisfy $\limsup \sqrt[k]{|a_k|} < \alpha$. So,

$$\begin{aligned} \sqrt[k]{|a_k|} &< \alpha && \text{for } k \text{ sufficiently large} \\ |a_k| &< \alpha^k \\ |a_{k+m}| &< \alpha^{k+m} && \text{for } k \text{ sufficiently large} \\ \sqrt[k]{|a_{k+m}|} &< \alpha^{\frac{k+m}{k}} \\ &= \alpha^{1+m/k} \\ &\leq \alpha^{1+m} \end{aligned}$$

This can actually be reduced to α since as $k \to \infty$, $m/k \to 0$. So, $\limsup \sqrt[k]{|a_{k+m}|} < \infty$ meaning $g(z) = \sum\limits_{k \geq 0} a_{k+m}(z - z_0)^k$ converges on some disk centered at z_0. This implies that $g(z)$ is holomorphic at z_0. $\square$

Corollary

If $f(z)$ is a nonconstant function holomorphic at z_0 and $f(z_0) = 0$, then there is an open disk centered at z_0 on which f is 0 only at z_0.



- **Proof**: from the theorem, we can write $f(z) = (z - z_0)^m g(z)$ with $g(z_0) \neq 0$. $(z - z_0)^m \neq 0$ away from z_0, so it suffices to show that $g(z) \neq 0$ away from z_0.

  $g(z)$ is holomorphic at z_0, so it is continuous at z_0. By definition of continuity, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|z - z_0| < \delta$, then
  $$|g(z) - g(z_0)| < \varepsilon.$$
  From the proof of the theorem, we have $g(z_0) = a_m \neq 0$. Take $\varepsilon = |a_m|$.
  $$|g(z) - a_m| < |a_m|$$
  so $g(z) \neq 0$ for $|z - z_0| < \delta$ since $|a_m| < |a_m|$ which is a contradiction. So, $f(z) \neq 0$ where $z \neq z_0$ on some open disk $|z - z_0| < \delta$ centered at z_0. $\square$

So, the zeros of holomorphic functions are isolated. Each are contained in an open set containing no other.

Identity Principle

Theorem: Identity Principle

If $f(z)$ is holomorphic on an open connected set G and

$$z_1,\ z_2,\ z_3,\ \ldots$$

are points in G with $f(z_i) = 0$, if the sequence (z_i) has an accumulation point in G, then $f(z)$ is the zero function.

- **Proof**: let w be an accumulation point of $z_i \in G$. So, there is a subsequence z_{i_j} converging to w. Since $f(z)$ is holomorphic in G, it is continuous at w.

$$\begin{aligned} f(w) &= f\big(\lim_j z_{i_j}\big) \\ &= \lim_j f(z_{i_j}) && (f \text{ is continuous at } w) \\ &= \lim_j 0 \\ &= 0 \end{aligned}$$

By the previous corollary, if f is a nonconstant function, then $f(z)$ must be nonzero on some disk centered at w except at w itself. But, the subsequence (z_{i_j}) must get arbitrarily close to w, so it would intersect any disk centered at w and give a point where f vanishes.

This is a contradiction, so f cannot be nonconstant. So, $f(z) = 0$ since it is the constant function equal to 0 at some point, hence it is 0 at all points. $\square$

Corollary

If $f(z)$ and $g(z)$ are holomorphic on G and agree on a set $\{z_1, z_2, \ldots\}$ with an accumulation point in G (meaning $f(z_i) = g(z_i)$ for each i), then $f(z) = g(z)$.

- **Proof**: apply the theorem to $f(z) - g(z)$, which is holomorphic on G and vanishes at z_i. So, $f(z) - g(z)$ must be the constant function equal to zero, meaning $f(z) = g(z)$. $\square$

If we write $f(z) = \sum\limits_{k \geq 0} a_k (z - z_0)^k$, we have

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \mathrm{d}w.$$

$f(z)$ is determined by its power series, hence by a_k. a_k is determined by values of $f(z)$ along γ, a curve around z_0. So, $f(z)$ is determined by its behaviour along an open set containing z_0.

We have shown that $f(z)$ is determined by its value on a set having z_0 as an accumulation point. This could be a discrete set:

$$\underbrace{\bullet\quad\bullet\ \ \bullet\ \bullet\bullet\bullet}_{\text{set}}\, z_0$$

Although we know that if $g(z)$ is another holomorphic function agreeing with $f(z)$ on this set, then $f(z) = g(z)$ (meaning a set of values determines f), it is not clear how to find values of $f(z)$ at other points from this information.

Example

$\sin\left(\frac{1}{z}\right)$ is holomorphic for $z \neq 0$ since it is a composition of holomorphic functions. $\sin\left(\frac{1}{z}\right) = 0$ at $z_n = \frac{1}{\pi n}$ at $n \in \mathbb{N}$. This sequence (z_n) converges to zero, but $\sin\left(\frac{1}{2}\right)$ is not the constant zero function. This is not a contradiction since the accumulation point is outside the set G where the function is holomorphic.

Laurent Series

A **Laurent series** is a series of the form

$$\sum_{-\infty}^{\infty} a_k(z-z_0)^k.$$

It is like a power series, but with negative indices allowed. When expanded, the Laurent series is

$$\underbrace{\cdots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1}}_{\substack{\text{singular when } z=z_0 \text{ unless} \\ \text{all coefficients are 0} \\ \text{and we just have a power series}}} + \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{\substack{\text{defines a holomorphic function inside} \\ \text{a radius of convergence,} \\ \text{provided the series converges}}}$$

We can define Laurent series about singularities of $f(z)$ instead of having to expand around a point where it is holomorphic. This can be useful when computing integrals, where we will get a better version of the Cauchy Integral Formula.

Convergence of Laurent Series

Laurent series converge when taking limits $b \to -\infty$ and $c \to \infty$ independently makes

$$\sum_{k=b}^{c} a_k(z-z_0)^k.$$

In other words,

$$\sum_{k=-\infty}^{-1} a_k(z-z_0)^k \text{ and } \sum_{k=0}^{\infty} a_k(z-z_0)^k$$

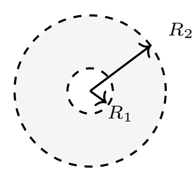
both converge. We can observe these two sums independently.

We know $\sum\limits_{k=0}^{\infty} a_k(z-z_0)^k$ converges when $|z-z_0| < R_2$ for radius of convergence R_2. We will not consider points where $|z-z_0| = R_2$ since these points may or may not converge.

For the first sum, we can change the parameters to be $\sum\limits_{k=-\infty}^{-1} a_k(z-z_0)^k = \sum\limits_{k=1}^{\infty} a_{-k}\left((z-z_0)^{-1}\right)^k$. This is the power series in the variable $(z-z_0)^{-1}$. So, it converges when $\left|(z-z_0)^{-1}\right| < \frac{1}{R_1}$, or $R_1 < |z-z_0|$.

Both conditions are satisfied when

$$R_1 < |z-z_0| < R_2.$$



This region is called an **annulus**.

Note that it might happen that $R_1 \geq R_2$, in which case the series might never converge.

Example

Let $a_k = 1$ for every k and $z_0 = 0$. So, the Laurent series is

$$\begin{aligned} \sum_{k=-\infty}^{\infty} z^k &= \sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} z^k \\ &= \sum_{k=1}^{\infty} z^{-k} + \sum_{k=0}^{\infty} z^k \end{aligned}$$

Both of these are geometric series. The sum on the left converges only when $|z^{-1}| < 1$ and the right converges only when $|z| < 1$. Both cannot be true, so the two series will never both converge. Thus, the series does not converge for any z.

Theorem

Suppose $f(z)$ is holomorphic in the annulus $R_1 < |z - z_0| < R_2$. Then, $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$ where

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \mathrm{d}w$$

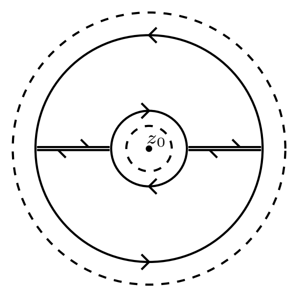
for $R_1 < |z - z_0| < R_2$ where γ is any curve inside the annulus enclosing z_0, usually a curve centered at z_0 with radius between R_1 and R_2.

- **Proof**: modify the argument for power series. Apply the Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \int_{\gamma'} \frac{f(w)}{w - z} \mathrm{d}w$$

where γ' is a curve around z inside of which $f(z)$ is holomorphic.

Choose γ' to consist of two parts:



1. Anticlockwise big circle C enclosing z but with a smaller radius than R_2
2. Clockwise small circle c not enclosing z but with a larger radius than R_1

The line segments justify applying the Cauchy Integral Formula, but they cancel out.

$$f(z) = \frac{1}{2\pi i} \left[\int_C \frac{f(w)}{w - z} \mathrm{d}w - \int_c \frac{f(w)}{w - z} \mathrm{d}w \right]$$

The minus sign is to account for the orientation of the small circle.

Note that if f is holomorphic for every $|z - z_0| < R_2$, then the second integral is 0 by Cauchy's Theorem and we are in the same situation as we were when discussing power series.

For the first integral (over the big circle), we have

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{(w - z_0) - (z - z_0)} \\ &= \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \frac{1}{w - z_0} \sum_{k \geq 0} \left(\frac{z - z_0}{w - z_0} \right)^k \end{aligned}$$

This is a geometric series which converges when $\left| \frac{z - z_0}{w - z_0} \right| < 1$, or $|z - z_0| < |w - z_0|$.

For the second integral (over the small circle), we have

$$\begin{aligned} -\frac{1}{w - z} &= -\frac{1}{(w - z_0) - (z - z_0)} \\ &= -\frac{1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}} \\ &= -\frac{1}{z - z_0} \sum_{k \geq 0} \left(\frac{w - z_0}{z - z_0} \right)^k \end{aligned}$$

This is also a geometric series which converges when $\left| \frac{w - z_0}{z - z_0} \right| < 1$, or $|w - z_0| < |z - z_0|$.

So, we can evaluate the two integrals. Applying uniform convergence of geometric series allows us to interchange the order of summation and integration.

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\mathrm{d}w &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\sum_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k \mathrm{d}w \\
&= \sum_{k\geq 0}\underbrace{\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{k+1}}\mathrm{d}w}_{a_k}(z-z_0)^k \\
&= \sum_{k\geq 0} a_k(z-z_0)^k \\
-\frac{1}{2\pi i}\int_c \frac{f(w)}{w-z}\mathrm{d}w &= -\frac{1}{2\pi i}\int_c -\frac{f(w)}{z-z_0}\sum_{k\geq 0}\left(\frac{w-z_0}{z-z_0}\right)^k \mathrm{d}w \\
&= \sum_{k\geq 0}\frac{1}{2\pi i}\int_c f(w)(w-z_0)^k\mathrm{d}w\cdot\frac{1}{(z-z_0)^{k+1}} \\
&= \sum_{k=-\infty}^{-1}\frac{1}{2\pi i}\int_c f(w)(w-z_0)^{-(k+1)}\mathrm{d}w\cdot\frac{1}{(z-z_0)^{-k}} \\
&= \sum_{k=-\infty}^{-1}\underbrace{\frac{1}{2\pi i}\int_c \frac{f(w)}{(w-z_0)^{k+1}}\mathrm{d}w}_{a_k}(z-z_0)^k \\
&= \sum_{k=-\infty}^{-1} a_k(z-z_0)^k \\
f(z) &= \sum_{k\geq 0} a_k(z-z_0)^k + \sum_{k=-\infty}^{-1} a_k(z-z_0)^k \\
&= \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k
\end{aligned}$$

So, $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ for the given a_k. □

Note that $\frac{f(w)}{(w-z_0)^{k+1}}$ is holomorphic in the annulus, so we can compute a_k with integrals over the curves other than the big and small circles.

Functions as Laurent Series

We define Laurent series as two parts:

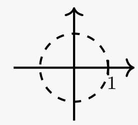
$$\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k = \underbrace{\sum_{k=-\infty}^{-1} a_k(z-z_0)^k}_{\substack{\text{“singular part”}\\ \text{power series in } (z-z_0)^{-1}}} + \underbrace{\sum_{k=0}^{\infty} a_k(z-z_0)^k}_{\substack{\text{“regular part”}\\ \text{power series in } (z-z_0)}}$$

Corollary

There is only one Laurent series equal to a given function on a given annulus.

Note that we need to fix the annulus because we can have different series for the same function, but valid in different places.

Example



$f(z) = \frac{1}{1-z}$ is singular at $z = 1$. Observe the two intervals that $f(z)$ is defined on. We have $|z| < 1$ and $1 < |z| < \infty$.

If $|z| < 1$, we can write

$$\begin{aligned} f(z) &= \frac{1}{1-z} \\ &= \sum_{k \geq 0} z^k && \text{(geometric series)} \end{aligned}$$

If $1 < |z| < \infty$, we can write

$$\begin{aligned} f(z) &= \frac{1}{1-z} \\ &= -\frac{1}{z}\frac{1}{1-z^{-1}} \\ &= -z^{-1}\sum_{k \geq 0}\left(z^{-1}\right)^k && \text{(geometric series)} \\ &= -\sum_{k \geq 1} z^{-k} \end{aligned}$$

So, we have 2 different Laurent series for the two different regions.

Now, let's prove the corollary.

- **Proof**: fix the annulus where we are considering $f(z)$:

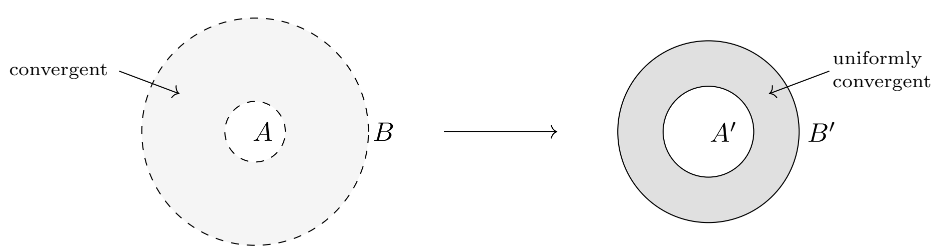
  $$A < |z - z_0| < B$$

  In this region, write

  $$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=-\infty}^{-1} a_k (z - z_0)^k.$$

  The first term converges on some disk of radius of at most B. So, it converges uniformly when $|z - z_0| < B'$ where B' is any fixed number less than B.

  The second term converges for $\left|\frac{1}{z-z_0}\right| < \frac{1}{A}$ or $A < |z - z_0|$ or a larger set, so we can shrink this region to $A' \leq |z - z_0|$ for any $A' > 0$ to get uniform convergence.

  Consider

  $$\int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w$$

  for $n \in \mathbb{Z}$ where γ is a closed curve in the uniformly convergent annulus going around z_0.

By uniform convergence, we can swap the order of integration and summation.

$$\int_\gamma \frac{\sum_{k=-\infty}^{\infty} a_k (w-z_0)^k}{(w-z_0)^{n+1}} \mathrm{d}w = \sum_{k=-\infty}^{\infty} a_k \int_\gamma (w-z_0)^{k-(n+1)} \mathrm{d}w$$

Recall that if $N \neq -1$, $\int z^N \mathrm{d}z = \frac{z^{N+1}}{N+1}$, so

$$\begin{aligned} \int_\gamma z^N \mathrm{d}z &= \frac{\gamma(b)^{N+1}}{N+1} - \frac{\gamma(a)^{N+1}}{N+1} \\ &= 0 \end{aligned}$$

if $\gamma(a) = \gamma(b)$, or the curve is closed. So, all the terms of the sum are zero except when $k-(n+1) = -1$, or $k = n$. In this case, we get

$$\begin{aligned} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w &= a_n \int_\gamma (w-z_0)^{n-(n+1)} \mathrm{d}w \\ &= a_n \int_\gamma \frac{1}{w-z_0} \mathrm{d}w \\ &= a_n (2\pi i) && \text{(Cauchy Integral Formula)} \\ a_n &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{n+1}} \mathrm{d}w \end{aligned}$$

In particular, once $f(z)$ is fixed along with the annulus containing γ, then a_n is determined. So, only one power series defines $f(z)$. □

If we find a Laurent series by any method, it must agree with the formula

$$2\pi i a_{-1} = \int_\gamma f(w) \mathrm{d}w.$$

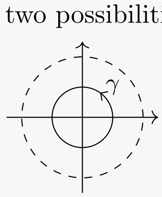
So, we can use Laurent series to compute integrals.

Example

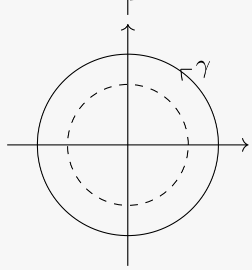
Once again, for $f(z) = \frac{1}{1-z}$, we have

$$f(z) = \frac{1}{1-z} = \begin{cases} \sum\limits_{k \geq 0} z^k & |z| < 1 \\ -\sum\limits_{k \geq 1} z^{-k} & |z| > 1 \end{cases}$$

We have two possibilities for γ.



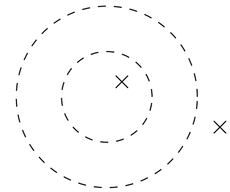
$\int_\gamma f(z)\mathrm{d}z = 0$ by Cauchy's Theorem since $f(z)$ is holomorphic inside γ. We can also observe that $a_{-1} = a_{-2} = \cdots = 0$ and in particular, $a_{-1} = 0$ meaning $\int_\gamma f(w)\mathrm{d}w = 0$.



$\int_\gamma f(z)\mathrm{d}z = \int_\gamma \frac{-1}{z-1}\mathrm{d}z = -2\pi i$ by the Cauchy Integral Formula since $f(z)$ is not holomorphic at $z = 1$ inside γ.
We can also observe that $a_{-1} = a_{-2} = \cdots = -1$ and in particular, $a_{-1} = -1$ meaning $\int_\gamma f(w)\mathrm{d}w = -2\pi i$.

Note that if $f(z) = \sum\limits_{k \geq 0} a_k (z - z_0)^k$, the radius of convergence is the distance from z_0 to the nearest singularity of $f(z)$. We get a similar result from splitting a Laurent series into two power series.

If $f(z)$ is holomorphic on this annulus but singular at the points marked by $\times$, then the Laurent series of $f(z)$ on this annulus cannot be extended past the singularities.



So, the singularities of $f(z)$ control which annulus we consider when writing Laurent series for $f(z)$.

Example

$f(z) = \frac{1}{(z-1)(z-2)}$ has singularities at $z = 1$ and $z = 2$. We define 3 annuli: $|z| < 1$, $1 < |z| < 2$, and $2 < |z| < \infty$. Observe that

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

by partial fractions.

$$\text{If } |z| < 2, \qquad \frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k \geq 0}\left(\frac{z}{2}\right)^k$$

$$\text{If } |z| > 2, \qquad \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{k \geq 0}\left(\frac{2}{z}\right)^k$$

$$\text{If } |z| < 1, \qquad \frac{1}{z-1} = -\sum_{k \geq 0} z^k$$

$$\text{If } |z| > 1, \qquad \frac{1}{z-1} = \sum_{k \geq 1} z^{-k}$$

Only one of each pair is valid in a given annulus. This determines what the Laurent series is.

$$\text{On } |z| < 1, \qquad f(z) = -\frac{1}{2}\sum_{k \geq 0}\left(\frac{z}{2}\right)^k + \sum_{k \geq 0} z^k$$

$$\text{On } 1 < |z| < 2, \qquad f(z) = -\frac{1}{2}\sum_{k \geq 0}\left(\frac{z}{2}\right)^k - \sum_{k \geq 1} z^{-k}$$

$$\text{On } 2 < |z| < \infty, \qquad f(z) = \frac{1}{z}\sum_{k \geq 0}\left(\frac{2}{z}\right)^k - \sum_{k \geq 2} z^{-k}$$

Isolated Singularities

We know that nonconstant holomorphic functions have isolated zeros, meaning if $f(z_0) = 0$, $\exists \varepsilon > 0$ such that $f(z) \neq 0$ for $0 < |z - z_0| < \varepsilon$. A singularity z_0 of a function $f(z)$ is **isolated** if $\exists \varepsilon > 0$ such that $f(z)$ is holomorphic on $0 < |z - z_0| < \varepsilon$. So, there are no singularities within ε of z_0.

If z_0 is an isolated singularity of $f(z)$:

1. it is a **removable** singularity if there is $g(z)$ holomorphic on $|z - z_0| < \varepsilon$ with $f(z) = g(z)$ on $0 < |z - z_0| < \varepsilon$ (i.e. $f(z)$ extends to a holomorphic function at z_0).
2. it is a **pole** if $\lim\limits_{z \to z_0} |f(z)| = \infty$.
3. it is an **essential** singularity otherwise.

Example

1. $f(z) = \frac{e^z - 1}{z}$ is a removable singularity.

   We know $e^z - 1 = z^m \cdot g(z)$ where m is the order of vanishing, $g(0) \neq 0$, and $g(z)$ is holomorphic at 0. In this case, $m = 1$ so $g(z)$ is the holomorphic extension of $f(z)$.

2. $f(z) = \frac{1}{z^2}$ at $z = 0$ is a pole.

   $$\lim_{z \to 0} \left|\frac{1}{z^2}\right| = \infty$$

   The same holds for $f(z) = \frac{1}{z^n}$ for any $n \in \mathbb{N}$.

3. $f(z) = e^{1/z}$ at $z = 0$ is an essential singularity.

   Along the positive and negative real axes respectively, we have

   $$\lim_{z \to 0^+} e^{1/z} = e^{\infty} = \infty \text{ and } \lim_{z \to 0^-} e^{1/z} = e^{-\infty} = -\infty.$$

   If $z = 0$ was removable, $\lim_{z \to z_0} f(z)$ would equal the value of the holomorphic (hence continuous) extension of $f(z)$ at z_0, but the limit does not exist. $\lim_{z \to z_0} |f(z)|$ does not exist either, so it is not a pole. Hence, it is an essential singularity.

Proposition

If z_0 is an isolated singularity of $f(z)$:

1. it is removable if and only if $\lim_{z \to z_0} (z - z_0) f(z) = 0$.
2. it is a pole if and only if it is not removable and $\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$ for $n \in \mathbb{Z}_{>0}$ where n is the **order of the pole** z_0.

- **Proof**:

1. $\Longrightarrow$ show that if z_0 is a removable singularity of $f(z)$, then $\lim_{z \to z_0} (z - z_0) f(z) = 0$. Let $g(z)$ be a holomorphic extension of $f(z)$ to z_0.

   $$\begin{aligned} \lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} (z - z_0) g(z) \\ &= \lim_{z \to z_0} (z - z_0) \cdot \lim_{z \to z_0} g(z) \\ &= 0 \cdot g(z_0) \\ &= 0 \end{aligned}$$

   $\Longleftarrow$ show that if $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then z_0 is a removable singularity of $f(z)$. Let

   $$h(z) = \begin{cases} (z - z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases}$$

   We claim that this is holomorphic at z_0. We know that it is differentiable near z_0 since $f(z)$ is. At z_0:

   $$\begin{aligned} h'(z_0) &= \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} \\ &= \lim_{z \to z_0} \frac{(z - z_0)^2 f(z) - 0}{z - z_0} \\ &= \lim_{z \to z_0} (z - z_0) f(z) \\ &= 0 && \text{(by assumption)} \end{aligned}$$

So, $h(z)$ is holomorphic at z_0. Write

$$h(z) = (z - z_0)^m k(z)$$

where $k(z)$ is holomorphic at z_0 and $k(z_0) \neq 0$. We want $m \geq 2$. We have $h(z_0) = 0$, so $m \geq 1$ since otherwise, $h(z_0) = k(z_0) \neq 0$ which is a contradiction. Then,

$$\begin{aligned} h'(z) &= m(z - z_0)^{m-1} k(z) + (z - z_0)^m k'(z) \\ 0 = h'(z_0) &= \underbrace{m}_{\neq 0} (z - z_0)^{m-1} \underbrace{k(z_0)}_{\neq 0} + \underbrace{(z - z_0)^m k'(z_0)}_{=0} \end{aligned}$$

This forces $m - 1 \geq 1$, or $m \geq 2$. Then, $(z - z_0)^{m-2} k(z)$ where $(z - z_0)^{m-2} = \frac{h(z)}{(z-z_0)^2} = f(z)$ is holomorphic at $z \neq z_0$, and we have a holomorphic extension to z_0. So, z_0 is a removable singularity of $f(z)$.

2. $\Longrightarrow$ show that if z_0 is a pole of $f(z)$, then $\lim\limits_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$ for $n \in \mathbb{Z}_{>0}$. We know $\lim\limits_{z \to z_0} |f(z)| = \infty$. So, for some $\varepsilon > 0$, $f(z) \neq 0$ when $0 < |z - z_0| < \varepsilon$.
$\frac{1}{f(z)}$ is holomorphic for $0 < |z - z_0| < \varepsilon$. $|f(z)| \to \infty$ as $z \to z_0$, so $\frac{1}{f(z)} \to 0$ as $z \to z_0$. So,

$$\begin{aligned} \lim_{z \to z_0} (z - z_0) \frac{1}{f(z)} &= \lim_{z \to z_0} (z - z_0) \cdot \lim_{z \to z_0} \frac{1}{f(z)} \\ &= 0 \cdot 0 \\ &= 0 \end{aligned}$$

So, z_0 is a removable singularity of $\frac{1}{f(z)}$.

$$\frac{1}{f(z)} = (z - z_0)^n \ell(z)$$

where n is the order of vanishing when ℓ is holomorphic at z_0 and $\ell(z_0) \neq 0$.

$$f(z) = (z - z_0)^{-n} \ell(z)^{-1}$$

So, we have

$$\begin{aligned} \lim_{z \to z_0} (z - z_0)^{n+1} f(z) &= \lim_{z \to z_0} (z - z_0) \ell(z) \\ &= \lim_{z \to z_0} (z - z_0) \cdot \lim_{z \to z_0} \ell(z) \\ &= 0 \cdot \ell(z_0)^{-1} \\ &= 0 \end{aligned}$$

$\Longleftarrow$ show that if $\lim\limits_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$ for $n \in \mathbb{Z}_{>0}$, then z_0 is a pole of $f(z)$. Assume n is as small as possible such that $\lim\limits_{z \to z_0} (z - z_0)^{n+1} f(z) = 0$.
Let $p(z) = (z - z_0)^n f(z)$. Then, $\lim\limits_{z \to z_0} (z - z_0) p(z) = 0$, meaning $p(z)$ has a removable singularity at z_0. In particular, $\lim\limits_{z \to z_0} p(z)$ exists and is nonzero, otherwise we could have chosen n to be smaller.

$$\begin{aligned} f(z) &= p'(z)(z - z_0)^{-n} \\ \lim_{z \to z_0} |f(z)| &= \lim_{z \to z_0} |p(z)| \left|(z - z_0)^{-n}\right| \\ &= |p(z_0)| \cdot \infty \\ &= \infty \end{aligned}$$

since $p(z_0)$ is some nonzero complex number. So, z_0 is a pole of $f(z)$.

Thus, both directions of both statements have been proved. $\square$

Proposition

Suppose z_0 is an isolated singularity of $f(z)$ with Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

Then,

1. z_0 is removable if and only if $a_k = 0$ for all $k < 0$.
2. z_0 is a pole of order n if and only if $a_{-n} \neq 0$ and $a_{-m} = 0$ for $m > n$. In particular, only finitely many a_k with $k < 0$ are nonzero.
3. z_0 is an essential singularity if and only if infinitely many a_k with $k < 0$ are nonzero.

- **Proof**:

1. $\implies$ show that if z_0 is a removable singularity of $f(z)$, then $a_k = 0$ for all $k < 0$. If z_0 is a removable singularity, $f(z)$ extends to a function holomorphic at z_0. We can expand this to a power series about z_0, which is a Laurent series with all negative terms being zero.

   $\impliedby$ show that if $a_k = 0$ for all $k < 0$ meaning the Laurent series is a power series, then z_0 is a removable singularity. This follows from the fact that power series converge to a holomorphic function on some disk including the center.

2. $\implies$ show that if z_0 is a pole of order n of $f(z)$, then there are only finitely many nonzero a_k with $k < 0$. Let $g(z) = (z - z_0)^n f(z)$.

$$\lim_{z \to z_0} (z - z_0) g(z) = 0 \cdot g(z_0) = 0$$

   So, $g(z)$ has a removable singularity at z_0. By minimality of n, we have $g(z_0) \neq 0$.

   Since $g(z)$ is holomorphic at z_0, we can expand it as a power series where $a_0 \neq 0$.

$$\begin{aligned} g(z) &= \sum_{k \geq 0} a_k (z - z_0)^k \\ f(z) &= (z - z_0)^{-n} g(z) \\ &= \sum_{k \geq 0} a_k (z - z_0)^{k-n} \end{aligned}$$

   The $-n^{\text{th}}$ term is $a_0 (z - z_0)^{-n} \neq 0$ and the exponents of $z - z_0$ are all $\geq n$.

   $\impliedby$ show that if only finitely many a_k with $k < 0$, then z_0 is a pole of order n of $f(z)$. If

$$f(z) = \sum_{k \geq -n} a_k (z - z_0)^k$$

   then

$$\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = \lim_{z \to z_0} \sum_{k \geq -n} a_k (z - z_0)^{n+k+1}$$

   where the exponent is ≥ 1. So, as $z \to z_0$, we have $(z - z_0)^{n+k+1} \to 0$ meaning the entire term goes to 0. By the previous proposition, this means that $f(z)$ has a pole of order n at z_0.

   Note that n is minimal since $\lim_{z \to z_0} (z - z_0)^n f(z) = a_{-n} \neq 0$.

3. z_0 is an essential singularity if and only if z_0 is not removable and is not a pole. From 1. and 2., it follows directly that this can only be true if and only if some negative index term in the Laurent series is nonzero and there are infinitely many nonzero negative index terms.

Residues and the Residue Theorem

Recall that if $f(z)$ has an isolated singularity at z_0 and it is holomorphic on a set of the form $0 < |z - z_0| < \varepsilon$, then we can write

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \text{ where } a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \mathrm{d}w$$

where γ is a curve enclosing z_0 but no other singularities of $f(z)$. For $k = -1$,

$$2\pi i a_{-1} = \int_\gamma f(w) \mathrm{d}w.$$

So, if we can compute a_{-1}, we can find integrals of $f(w)$.

The coefficient a_{-1} of the Laurent series of $f(z)$ is the **residue** of $f(z)$ at z_0, denoted as $\operatorname*{Res}_{z=z_0} f(z)$. Using this, we can state the "best" version of the Cauchy Integral Formula.

Theorem: Residue Theorem

Let γ be a closed curve and $f(z)$ be a function holomorphic on γ and the region enclosed by γ except for isolated singularities. Then, there are finitely many singularities enclosed by γ and

$$\int_\gamma f(z) \mathrm{d}z = 2\pi i \sum_k \operatorname*{Res}_{z=z_0} f(z)$$

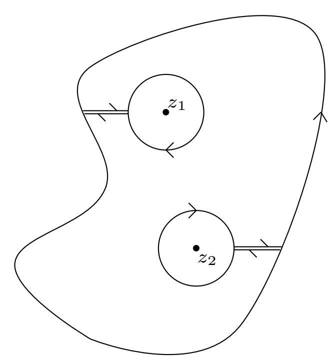
where we take the sum over enclosed singularities.

- **Proof**: let S be the set of singularities of $f(z)$ enclosed by γ. First, show that S is finite. S is centered in the region enclosed by γ, which is bounded by the Jordan Curve Theorem. So, S is bounded.

  We can also show that S is closed. The complement of S is all points where $f(z)$ is holomorphic. $f(z)$ is holomorphic at z if and only if f is differentiable on a disk centered at z. So, f is differentiable on another disk centered at a point on the first disk, meaning f is holomorphic at any point on the first disk. So, $f(z)$ is holomorphic on an open disk centered at z. The set of points where $f(z)$ is holomorphic is open, so its complement S is closed.

  By contradiction, suppose S is infinite. Let $z_1, z_2, \dots$ be an infinite series of distinct points in S. The Bolazano-Weierstrass Theorem states that a bounded sequence has a convergent subsequence.Let $z_1, \dots, z_n$ be a convergent subsequence with limit w. S is closed, so the limit point w must be in S. Then, w is a singularity of $f(z)$ which cannot be isolated since $\forall \varepsilon > 0$, $\exists N$ such that if $j > N$, then $|z_j - w| < \varepsilon$. But then, $0 < |z - w| < \varepsilon$ contains z_j, which is a singularity of $f(z)$. This is a contradiction since we assumed all singularities are isolated. So, S is finite.

  If γ is the curve surrounding the singularities of $f(z)$, we can draw a small circle surrounding each singularity with line segments to obtain a new curve which has no singularities. By Cauchy's Theorem, the integral of $f(z)$ over this new line segment is 0.

  $$\int_{\text{new curve}} f(z) \mathrm{d}z = \int_\gamma + \sum \int_{\substack{\text{line}\\ \text{segments}}} + \sum \int_{\circlearrowright}$$

  where $\circlearrowright$ are the clockwise circles around the singularities. The integrals over the line segments sum to 0, so so we have

  $$\int_\gamma = \sum \int_{\circlearrowleft} = \sum_k 2\pi i \operatorname*{Res}_{z=z_k} f(z)$$

  where $\circlearrowleft$ are the anticlockwise circles around the singularities. We use the fact that for any small circle $\circlearrowleft$ around z_k,

  $$\int_{\circlearrowleft} f(z) \mathrm{d}z = 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$ □

Corollary

1. If $f(z)$ is holomorphic at z_0 and is nonconstant, then it has of the zero m at z_0 if and only if $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is holomorphic at z_0 and $g(z_0) \neq 0$.
2. $f(z)$ has a pole at z_0 of order n if and only if $f(z) = (z - z_0)^{-n} g(z)$ where $g(z)$ is holomorphic at z_0 and $g(z_0) \neq 0$.

- **Proof**: The reverse direction of 1. follows from the fact that any $(z - z_0)^m g(z)$ can be written as $\sum\limits_{k \geq 0} a_k (z - z_0)^{m+k}$, which is a power series meaning it is holomorphic at z_0.

  2. follows from the fact that z_0 is a pole of order n if and only if $f(z) = \sum\limits_{k \geq -n} a_k (z - z_0)^k$. So, we can write $f(z) = \sum\limits_{k \geq 0} a_k (z - z_0)^k (z - z_0)^{-n} = (z - z_0)^{-n} g(z)$ with $g(z) = \sum\limits_{k \geq 0} a_k (z - z_0)^k$. This means $g(z) = (z - z_0)^n f(z)$ meaning $g(z)$ is holomorphic at z_0 and $g(z_0) \neq 0$ since n is minimal. $\square$

Proposition

If $f(z)$ has a pole of order n at z_0, then the residue of $f(z)$ at z_0 is given by

$$a_{-1} = \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z)\right] \Big|_{z=z_0}.$$

- **Proof**: $f(z)$ has a pole of order n at z_0, so we can write

$$\begin{aligned} f(z) &= \sum_{k \geq -n} a_k (z - z_0)^k \\ &= a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1} + \cdots \\ (z - z_0)^n f(z) &= a_{-n} + \cdots + a_{-1}(z - z_0)^{n-1} + \cdots \\ \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z)\right] \Big|_{z=z_0} &= \underbrace{\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} [a_n + \cdots + a_{-2}(z - z_0)^{n-2}] \Big|_{z=z_0}}_{=0 \text{ (from } n-1^{\text{th}} \text{ derivative)}} + (n-1)! a_{-1} + \underbrace{\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} [a_0 (z - z_0)^n + \cdots] \Big|_{z=z_0}}_{=0 \text{ (from factor of } z - z_0)} \\ a_{-1} &= \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z)\right] \Big|_{z=z_0} \end{aligned}$$

$\square$

Corollary

If $f(z)$ and $g(z)$ are holomorphic at z_0 where $f(z_0) \neq 0$ and $g(z)$ has a zero of order 1 at z_0, then $\frac{f(z)}{g(z)}$ has a pole of order 1 at z_0 with residue $\frac{f(z_0)}{g'(z_0)}$.

- **Proof**: $g(z)$ has a zero of order 1 at z_0, so $g(z) = (z - z_0) h(z)$ for some $h(z)$ holomorphic at z_0 and $h(z_0) \neq 0$.

$$\begin{aligned} \frac{f(z)}{g(z)} &= \frac{f(z)}{(z - z_0) h(z)} \\ &= (z - z_0)^{-1} \tfrac{f(z)}{h(z)} \end{aligned}$$

  We have $\frac{f(z)}{h(z)}$ holomorphic at z_0 and $\frac{f(z_0)}{h(z_0)} \neq 0$, so $\frac{f(z)}{g(z)}$ has a pole of order 1. Now, by applying the previous formula, we have

$$a_{-1} = \frac{1}{0!} \frac{\mathrm{d}^0}{\mathrm{d}z^0} \left[(z - z_0)^1 \frac{f(z)}{g(z)}\right] \Bigg|_{z=z_0} = \frac{f(z_0)}{h(z_0)}.$$

  We have $g'(z) = (z - z_0) h'(z) + h(z)$, so $g'(z_0) = 0 + h(z_0)$. So we have residue $a_{-1} = \frac{f(z_0)}{g'(z_0)}$. $\square$

Evaluating Integrals with the Residue Theorem

The Cauchy Integral Formula can only handle poles of order 1, but the Residue Theorem can handle poles of any order. This can make the Residue Theorem more useful in certain applications.

Note that if $f(z) = \frac{g(z)}{h(z)}$ with g and h holomorphic, then $f(z)$ is singular when $h(z) = 0$. We know that zeroes of a nonconstant holomorphic function are isolated, so singularities of $f(z)$ are also isolated. So, we can apply the residue theorem to $f(z)$.

Near a point z_0, we have

$$\begin{aligned} g(z) &= (z - z_0)^a \alpha(z) && (\alpha(z_0) \neq 0) \\ h(z) &= (z - z_0)^b \beta(z) && (\beta(z_0) \neq 0) \\ f(z) &= \frac{g(z)}{h(z)} = (z - z_0)^{a-b} \frac{\alpha(z)}{\beta(z)} \end{aligned}$$

We know $\frac{\alpha(z)}{\beta(z)}$ is both holomorphic and nonzero at z_0.

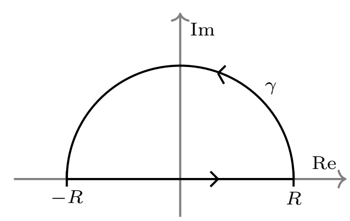
- If $a > b$, $f(z)$ has a zero of order $a - b$.
- If $a < b$, $f(z)$ has a pole of order $b - a$.

In particular, all singularities are poles, not essential singularities. If a function $f(z)$ has only isolated singularities, all of which are poles, then it is **meromorphic**. It turns out that all meromorphic functions can be written as an example of two holomorphic functions.

We will compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} \mathrm{d}x$$

using the Residue Theorem.



Let γ be the semicircular curve pictured. $\frac{1}{(z^2+1)^n}$ has singularities at $z = \pm i$, but only i is enclosed in the curve γ.

$$\int_{-R}^{R} \frac{1}{(x^2+1)^n} \mathrm{d}x + \int_{\text{arc}} \frac{1}{(z^2+1)^n} \mathrm{d}z = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^n}$$

As $R \to \infty$, the first integral goes to $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} \mathrm{d}x$, which is what we want to compute. We want to show that as $R \to \infty$, the integral over the arc goes to zero. Apply the ML lemma.

$$\left| \int_{\text{arc}} \frac{1}{(z^2+1)^n} \mathrm{d}z \right| \leq \max_{z \in \text{arc}} \left| \frac{1}{(z^2+1)^n} \right| \cdot \text{ length(arc)}$$

We know length(arc) $= \pi R$. By the reverse triangle inequality, $|z^2 + 1| \geq |z|^2 - 1 = R^2 - 1$.

$$\begin{aligned} &\leq \frac{1}{\min |z^2+1|^n} \cdot \pi R \\ &\leq \frac{1}{(R^2-1)^n} \cdot \pi R \end{aligned}$$

This goes to 0 as $R \to \infty$ since the power of R in the denominator is $2n$ which is greater than the power of R in the numerator, which is 1. So, we have $0 \leq \left| \int_{\text{arc}} \frac{1}{(z^2+1)^n} \mathrm{d}z \right| \leq 0$ meaning $\left| \int_{\text{arc}} \frac{1}{(z^2+1)^n} \mathrm{d}z \right| = 0$, or $\int_{\text{arc}} \frac{1}{(z^2+1)^n} \mathrm{d}z = 0$.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} \mathrm{d}x = 2\pi i \operatorname*{Res}_{x=i} \frac{1}{(z^2+1)^n}$$

To compute the integral, it suffices to find the residue.

If $n = 1$, we want to find $\operatorname*{Res}_{z=i} \frac{1}{z^2+1}$. The residue is the coefficient of $(z-i)^{-1}$. Expand $\frac{1}{z^2+1}$ as a Laurent series in $z - i$.

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{z-i} \cdot \frac{1}{z+i} \\ \frac{1}{z+i} &= \frac{1}{z-i+2i} \\ &= \frac{1}{2i}\frac{1}{1+\frac{z-i}{2i}}\end{aligned}$$

$\frac{1}{1+\frac{z-i}{2i}}$ is a geometric series for $\left|\frac{z-i}{2i}\right| < 1$, or $|z-i| < 2$.

$$\begin{aligned} &= \frac{1}{2i}\sum_{k\geq 0}\left(-\frac{z-i}{2i}\right)^k \\ \frac{1}{z^2+1} &= \frac{1}{z-i} \cdot \frac{1 - \frac{z-i}{2i} + \cdots}{2i} \\ &= \frac{1}{2i}(z-i)^{-1} - \frac{1}{2i} \cdot \frac{1}{2i} + \cdots\end{aligned}$$

The coefficient of $\frac{1}{z-i}$ is $\frac{1}{2i}$, so $\operatorname*{Res}_{z=i} \frac{1}{z^2+1} = \frac{1}{2i}$.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \mathrm{d}x = 2\pi i \cdot \frac{1}{2i} = \pi$$

If $n = 2$, we want to find $\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2}$. To find the residue, we can expand $\frac{1}{(z^2+1)^2}$ as a Laurent series in $z - i$ and find the coefficient of $(z-i)^{-1}$. We can square the Laurent expansion of $\frac{1}{z^2+1}$.

$$\begin{aligned}\frac{1}{(z^2+1)^2} &= \left(\frac{1}{z-i} \cdot \frac{1 - \frac{z-i}{2i} + \cdots}{2i}\right)^2 \\ &= \left[\frac{\frac{1}{z-i} - \frac{1}{2i} + \text{ higher order terms}}{2i}\right]^2 \\ &= \frac{\frac{1}{(z-i)^2} + 2 \cdot \frac{1}{z-i}\left(-\frac{1}{zi}\right) + \text{ higher order terms}}{(2i)^2}\end{aligned}$$

So, the coefficient of $\frac{1}{z-i}$ is

$$\begin{aligned}\frac{2\left(-\frac{1}{2i}\right)}{(2i)^2} &= \frac{-\frac{1}{i}}{-4} \\ &= \frac{1}{4i}\end{aligned}$$

So, we have $\operatorname*{Res}_{z=i} \frac{1}{(z^2+1)^2} = \frac{1}{4i}$.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} \mathrm{d}x = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$$

This process can easily become very complicated for higher n. To do the calculation for the general n, we need the Binomial Theorem

$$(1+z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

However, we want $(1+z)^{-1}$ but this only holds for $n \geq 1$. So, we need a different form of this.

Proposition

For any $\alpha \in \mathbb{C}$, the Binomial Theorem holds, meaning

$$(1+z)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} z^k.$$

Here, we define

$$\binom{\alpha}{z} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k(k-1)\cdots 1}.$$

If $\alpha = n \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}
\binom{\alpha}{z} &= \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \\
&= \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \cdot \frac{(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots 1} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

- **Proof**: compute the power series (Taylor series). Differentiate the LHS, which is $(1+z)^\alpha$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left[(1+z)^\alpha\right] &= \alpha(1+z)^{\alpha-1} \\
\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[(1+z)^\alpha\right] &= \alpha(\alpha-1)(1+z)^{\alpha-2} \\
&\vdots \\
\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left[(1+z)^\alpha\right] &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+z)^{\alpha-k}
\end{aligned}$$

Plug this into the formula for Taylor series.

$$\begin{aligned}
f(z) &= \sum_{k \geq 0} \frac{\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\Big|_{z=z_0}}{k!} z^k \\
(1+z)^\alpha &= \sum_{k \geq 0} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} z^k \\
&= \sum_{k \geq 0} \binom{\alpha}{k} z^k
\end{aligned}$$

Note that $(1+z)^\alpha = e^{\alpha \log(1+z)}$. $\log(z)$ is holomorphic everywhere away from the negative real axis. So, we take $|z| < 1$ so that $z+1$ is not on the negative real axis.

The radius of convergence is the distance to the nearest singularity. The nearest singularity is at zero and the distance from 1 to zero is 1, so the radius of convergence is 1. So, for $|z| < 1$,

$$(1+z)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} z^k. \qquad \square$$

Note that if $\alpha = n$, $\binom{\alpha}{k} = \binom{n}{k}$. If $k > n$, this is zero since the numerator will contain some $n - n$ before getting to $n - k$ meaning the numerator is zero.

We can apply this back to our problem. We want to find the Laurent series of $\frac{1}{(z^2+1)^n}$ about $z = i$.

$$\frac{1}{(z^2+1)^n} = \frac{1}{(z-i)^n} \cdot \frac{1}{(z+i)^n}$$
$$\frac{1}{(z+i)^n} = \frac{1}{(z-i+2i)^n}$$
$$= \frac{1}{(2i)^n} \cdot \frac{1}{\left(1+\frac{z-i}{2i}\right)^n}$$
$$= \frac{1}{(2i)^n} \left(1 + \frac{z-i}{2i}\right)^{-n}$$

We can apply the Binomial Theorem for $\alpha = -n$. This is valid for $\left|\frac{z-i}{2i}\right| < 1$, or $|z-i| < 2$.

$$= \frac{1}{(2i)^n} \sum_{k\geq 0} \binom{-n}{k} \left(\frac{z-i}{2i}\right)^k$$
$$= \sum_{k\geq 0} \frac{1}{(2i)^n} \frac{1}{(2i)^k} \binom{-n}{k} (z-i)^k$$
$$= \sum_{k\geq 0} \frac{1}{(2i)^{n+k}} \binom{-n}{k} (z-i)^k$$
$$\frac{1}{(z^2+1)^n} = \sum_{k\geq 0} \frac{1}{(2i)^{n+k}} \binom{-n}{k} (z-i)^{k-n}$$

This is our Laurent series. The residue of $(z-i)^{-1}$, which is at $k = n-1$.

$$\frac{1}{(2i)^{n+(n-1)}} \binom{-n}{n-1} = \frac{1}{(2i)^{2n-1}} \binom{-n}{n-1}$$
$$\binom{-n}{n-1} = \frac{(-n)(-n-1)\cdots(-n-(n-1)+1)}{(n-1)!}$$
$$= (-1)^{n-1} \frac{(n)(n+1)\cdots(2n-1)}{(n-1)!}$$
$$= (-1)^{n-1} \frac{(2n-2)(2n-3)\cdots(n)}{(n-1)!} \cdot \frac{(n-1)(n-2)\cdots 1}{(n-1)!}$$
$$= (-1)^{n-1} \binom{2n-2}{n-1}$$

So, we can use this to evaluate the integral. Note that $(-1)^{n-1} = (i^2)^{n-1} = i^{2n-2}$.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} \mathrm{d}x = 2\pi i \frac{1}{(2i)^{n-1}} i^{2n-2} \binom{2n-1}{n-1}$$
$$= \pi \frac{i^{2n-2}}{(2i)^{2n-2}} \binom{2n-2}{n-1}$$
$$= \frac{\pi}{2^{2n-2}} \binom{2n-2}{n-1}$$
$$= \frac{\pi}{4^{n-1}} \binom{2n-2}{n-1}$$

We can check our values for $n = 1$ and $n = 2$.

when $n = 1$, $$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \mathrm{d}x = \frac{\pi}{4^0} \binom{0}{0} = \pi \qquad \checkmark$$

when $n = 2$, $$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} \mathrm{d}x = \frac{\pi}{4^1} \binom{2}{1} = \frac{\pi}{2} \qquad \checkmark$$

Other Applications of the Residue Theorem

Laplace Transforms

Given $f : (0, \infty) \to \mathbb{C}$, the **Laplace transform** of f is

$$\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t.$$

Here, t is a positive real number (as shown by the interval of integration) but s is a complex number. The Laplace transform may not converge for every s.

Example

Let $f(t) = \cos(\omega t)$ with $\omega \in \mathbb{R}_{>0}$.

$$\begin{aligned}
f(t) &= \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \\
\mathcal{L}(f)(s) &= \int_0^\infty \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)e^{-st}\mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty e^{-(s-i\omega)t}\mathrm{d}t + \frac{1}{2}\int_0^\infty e^{-(s+i\omega)t}\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{1}{-(s-i\omega)}e^{-(s-i\omega)t}\right]_{t=0}^\infty + \frac{1}{2}\left[\frac{1}{-(s-i\omega)}e^{-(s+i\omega)t}\right]_{t=0}^\infty
\end{aligned}$$

Note that $\left|e^{-(t\pm i\omega)}\right| = e^{-\operatorname{Re}(s)t}$, so $\lim\limits_{t\to\infty} e^{-(s\pm i\omega)t} = 0$ only when $\operatorname{Re}(s) > 0$. So, if $\operatorname{Re}(s) > 0$, we get

$$\begin{aligned}
&= \frac{1}{2}\left[0 - \frac{1}{-(s-i\omega)}\right] + \frac{1}{2}\left[0 - \frac{1}{-(s+i\omega)}\right] \\
&= \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) \\
&= \frac{1}{2}\left(\frac{s+i\omega+s-i\omega}{(s+i\omega)(s-i\omega)}\right) \\
&= \frac{1}{2}\left(\frac{2s}{s^2+\omega^2}\right) \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}$$

So, $\mathcal{L}(f)(s) = \frac{s}{s^2+\omega^2}$ if $\operatorname{Re}(s) > 0$. Otherwise, $\mathcal{L}(f)(s)$ is not defined.

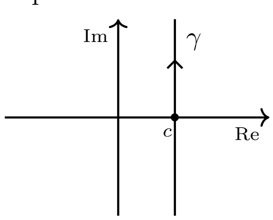
Note that $\frac{s}{s^2+\omega^2}$ still makes sense for $s \neq \pm i\omega$. We will make use of this.

Inverse Laplace Transforms

Given $f : (0, \infty) \to \mathbb{C}$ and $F(s) = \mathcal{L}(f)(s)$, the **inverse Laplace transform** of f is

$$\mathcal{L}^{-1}(f)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\mathrm{d}s$$

where t is a positive real number and s is a complex number.



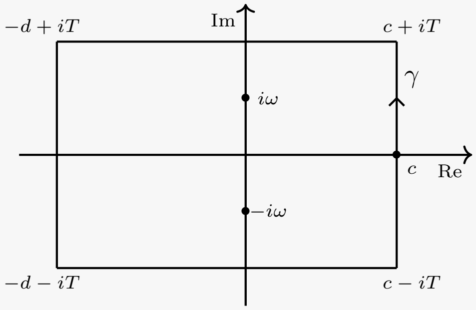
We integrate over a line parallel to the imaginary axis from $c - i\infty$ to $c + i\infty$ where c is any real number large enough such that all singularities of $F(s)$ are to the left of this line.

Example

Going back to the previous example, we want to show that for $f(t) = \cos(\omega t)$ with $\omega \in \mathbb{R}_{>0}$, $\mathcal{L}^{-1}(f)(t) = f$ where $F(s) = \frac{s}{s^2+\omega^2}$. So, we want to compute

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{s}{s^2+\omega^2}e^{st}\mathrm{d}s.$$

To do this, we want to use the residue theorem, but the curve is not closed.



We will integrate around a rectangle with the following vertices:

$$c-iT,\ c+iT,\ -d+iT,\ -d-iT$$

Observe that the curve γ contains both singularities $s = i\omega$ and $s = -i\omega$. As $T \to \infty$, the right side of the rectangle becomes the integral that we want.

We want both $d \to \infty$ and $T \to \infty$, so we will take $d = \sqrt{T}$. This way, as $T \to \infty$, $d \to \infty$ but $\frac{d}{T} \to 0$.

First, observe the top of the rectangle. We want to show that $\int_{c+iT}^{-d+iT} \frac{s}{s^2+\omega^2}e^{st}\mathrm{d}s$ goes to 0 as $T \to \infty$. Apply the ML lemma.

$$\left|\int_{c+iT}^{-d+iT} \frac{s}{s^2+\omega^2}e^{st}\mathrm{d}s\right| \leq \text{ length(curve)} \max_{s\in\text{curve}}\left|\frac{s}{s^2+\omega^2}e^{st}\right|$$

The length of the curve is $c+d$. Use change of variables $s = r+iT$ with $-d \leq r \leq c$.

$$= (c+d)\max_{r\in[-d,c]}\left|\frac{r+iT}{(r+iT)^2+\omega^2}e^{(r+iT)t}\right|$$

Find upper bounds for every term.

$$\begin{aligned}
|r+iT| &\leq |d+iT| \\
&= \sqrt{d^2+T^2} \\
&= \sqrt{T^2+T} \\
\frac{1}{|(r+it)^2+\omega^2|} &= \frac{1}{|r^2-T^2+\omega^2+2riT|} \\
&\leq \frac{1}{|r^2-T^2+\omega^2|} \\
&\leq \frac{1}{|T^2-T+\omega^2|}
\end{aligned}$$

since $r^2 \geq \max(c^2, d^2) = d^2 = T$. Note that ω is a constant.

$$\begin{aligned}
\left|e^{(r+iT)t}\right| &= e^{\mathrm{Re}((r+iT)t)} \\
&= e^{rT} \\
&\leq e^{cT}
\end{aligned}$$

Plug this back into the inequality.

$$\begin{aligned}
\left|\int_{c+iT}^{-d+iT} \frac{s}{s^2+\omega^2}e^{st}\mathrm{d}s\right| &\leq \frac{(c+\sqrt{T})\sqrt{T^2+T}}{T^2-T+\omega^2}e^{ct} \\
&= \frac{\sqrt{T}T}{T^2}\underbrace{\frac{\left(1+c/\sqrt{T}\right)\sqrt{1+1/T}}{1-1/T+\omega^2/T^2}}_{\to\text{ const. as }T\to\infty}e^{ct} \to 0
\end{aligned}$$

So, $\int_{c+iT}^{-d+iT} \frac{s}{s^2+\omega^2}e^{st}\mathrm{d}s = 0$ when $T \to \infty$.

The integral along the bottom side of the rectangle since we can swap T and $-T$. The same argument applies since every relevent T is squared.

Now, address the left side of the rectangle. we want to show that $\int_{-d+iT}^{-d-iT} \frac{s}{s^2+\omega^2} e^{st} \mathrm{d}s$ goes to 0 as $T \to \infty$. Apply the ML lemma.

$$\left| \int_{-d+iT}^{-d-iT} \frac{s}{s^2+\omega^2} e^{st} \mathrm{d}s \right| \leq \text{ length(curve)} \max_{s \in \text{curve}} \left| \frac{s}{s^2+\omega^2} e^{st} \right|$$

The length of the curve is $2T$. Use change of variables $s = -d + ir$ with $-T \leq r \leq T$.

$$= 2T \max_{r \in [-T,T]} \left| \frac{-d+ir}{(-d+ir)^2+\omega^2} e^{(-d+ir)t} \right|$$

Find upper bounds for every term.

$$\begin{aligned}
|-d+ir| &= \sqrt{d^2+r^2} \\
&\leq \sqrt{d^2+T^2} \\
&= \sqrt{T^2+T} \\
\frac{1}{|(-d+ir)^2+\omega^2|} &\leq \frac{1}{|(-d+ir)^2|-\omega^2} \\
&\leq \frac{1}{d^2-\omega^2} \\
&= \frac{1}{T-\omega^2} \\
\left| e^{(-d+ir)t} \right| &= e^{\mathrm{Re}((-d+ir)t)} \\
&= e^{-dt} \\
&= e^{t\sqrt{T}}
\end{aligned}$$

Plug this back into the inequality.

$$\begin{aligned}
\left| \int_{-d+iT}^{-d-iT} \frac{s}{s^2+\omega^2} e^{st} \mathrm{d}s \right| &\leq \frac{2T\sqrt{T^2+T}}{T-\omega^2} e^{-t\sqrt{T}} \\
&= \frac{T \cdot T}{T} e^{-tT} \underbrace{\frac{2\sqrt{1+1/T}}{1-\omega^2/T}}_{\to 2 \text{ as } T \to \infty} \to 0
\end{aligned}$$

So, we want to show that $T = e^{t\sqrt{T}} \to 0$ as $T \to \infty$. We claim that $e^{t\sqrt{T}}$ grows faster than any polynomial of T. For any $x > 0$,

$$e^x = \sum_{k \geq 0} \frac{x^k}{k!} > x^n$$

for a single term n in the sum.

$$e^{t\sqrt{T}} \geq \frac{\left(t\sqrt{T}\right)^4}{4!} = \frac{t^4}{24} T^2$$

So,

$$\frac{T}{e^{t\sqrt{T}}} \leq \frac{T}{\frac{t^4}{24} T^2} \to 0 \text{ as } T \to \infty.$$

So, $\int_{-d+iT}^{-d+iT} \frac{s}{s^2+\omega^2} e^{st} \mathrm{d}s = 0$ when $T \to \infty$.

Now, as $T \to \infty$, we have

$$\int_{\text{rectangle}} = \int_{c-iT}^{c+iT} + \int_{c+iT}^{-d+iT} + \int_{-d+iT}^{-d-iT} + \int_{-d-iT}^{c-iT} = \int_{c-i\infty}^{c+i\infty}$$

Going back to the rectangle, we want to use the Residue Theorem to calculate the integral.

$$\int_{\text{rectangle}} \frac{s}{s^2+\omega^2} e^{st} \mathrm{d}s = 2\pi i \left(\operatorname*{Res}_{s=i\omega} \frac{s}{s^2+\omega^2} e^{st} + \operatorname*{Res}_{s=-i\omega} \frac{s}{s^2+\omega^2} e^{st} \right)$$

Note that our function is $\frac{s}{(s-i\omega)(s+i\omega)} e^{st}$. Each singularity is a pole of order 1, so the residues are

$$\begin{aligned}
\operatorname*{Res}_{s=i\omega} \frac{s}{s^2+\omega^2} e^{st} &= \lim_{s\to i\omega} (s-i\omega) \frac{s}{(s-i\omega)(s+i\omega)} e^{st} \\
&= \frac{i\omega}{i\omega+i\omega} e^{i\omega t} \\
&= \frac{1}{2} e^{i\omega t} \\
\operatorname*{Res}_{s=i\omega} \frac{s}{s^2+\omega^2} e^{st} &= \lim_{s\to -i\omega} (s+i\omega) \frac{s}{(s-i\omega)(s+i\omega)} e^{st} \\
&= \frac{-i\omega}{-i\omega-i\omega} e^{-i\omega t} \\
&= \frac{1}{2} e^{-i\omega t}
\end{aligned}$$

So,

$$\int_{\text{rectangle}} = 2\pi i \left(\frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t} \right).$$

As $T \to \infty$, $\int_{\text{rectangle}} \to \int_{c-i\infty}^{c+i\infty}$, so

$$\begin{aligned}
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{s}{s^2+\omega^2} e^{st} \mathrm{d}s &= \frac{1}{2} \left(e^{i\omega t} + e^{-i\omega t} \right) \\
&= \cos(\omega t)
\end{aligned}$$

So, the inverse Laplace Transform recovered the original function.

Discrete Application of the Residue Theorem

The residue theorem can be used to solve some problems in discrete math.

Question: in how many ways can $k \in \mathbb{Z}_{>0}$ be written as a sum using only the numbers a and b for $a, b \in \mathbb{Z}_{\geq 0}$? In other words, we want to compute the number of solutions to

$$ax + by = k$$

where a, b, k are fixed and $x, y \in \mathbb{Z}_{\geq 0}$.

Example

Let $a = 5$ and $b = 7$. What numbers can be written as $ax + by$?

$$0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24, 25, 26, \ldots$$

Note the absence of 23. It turns out that the largest number that cannot be written using a and b is $ab - a - b$. In this case,

$$\begin{aligned}
a \cdot b - a - b &= 5 \cdot 7 - 5 - 7 \\
&= 23
\end{aligned}$$

We will not be proving this in this course.

To use complex analysis, we will write down a power series whose coefficients are the numbers we want to calculate. This is a **generating function**.

$$\sum_{k\geq 0} a_k z^k$$

where a_k = number of solutions to $ax+by=k$ for $x,y\in\mathbb{Z}_{\geq 0}$. The key point is that we can express this function in a more manageable way.

How many ways are there to express k as a sum of a's? We can count the solutions to $ax=k$ for $x\in\mathbb{Z}_{\geq 0}$, which is $x=\frac{k}{a}$ (nonnegative integer). So, there is one solution if k is divisible by a and zero otherwise. The associated generating function is

$$\begin{aligned}1z^0+0z^1+\cdots+0z^{a-1}+1z^a+0z^{a+1}+\cdots &= 1+z^a+z^{2a}+z^{3a}+\cdots\\ &= \frac{1}{1-z^a}\\ &= \sum_{x\geq 0} z^{ax}\end{aligned}$$

since it is a geometric series for $|z^a|<1$, or $|z|<1$. Note that similarly,

$$\sum_{y\geq 0} z^{by} = \frac{1}{1-z^b}.$$

So, we have

$$\begin{aligned}\frac{1}{1-z^a}\cdot\frac{1}{1-z^b} &= \sum_{x\geq 0} z^{ax}\sum_{y\geq 0} z^{by}\\ &= \sum_{x,y\geq 0} z^{ax+by}\end{aligned}$$

The number of times z^k appears in this sum is the number of solutions to $ax+by=k$ with $x,y\in\mathbb{Z}_{\geq 0}$, i.e. the number we are trying to compute.

By the Residue Theorem, if

$$\frac{1}{1-z^a}\cdot\frac{1}{1-z^b} = \sum_{k\geq 0} a_k z^k$$

then

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}\mathrm{d}z$$

where γ is a small curve enclosing $z=0$.

This is because we are integrating

$$\sum_{k\geq 0} a_n z^{n-(k+1)}$$

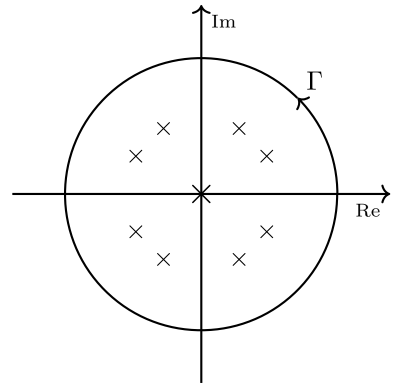
where the residue is the coefficient of z^{-1}. This appears when $n-(k+1)=-1$, or $n=k$. So, the coefficient of z^{-1} is a_k meaning

$$\int_\gamma \sum_{k\geq 0} a_n z^{n-(k+1)}\mathrm{d}z = 2\pi i a_k$$

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{1}{(1-z^a)(1-z^b)z^{k-1}}\mathrm{d}z$$

What we will actually do is compute

$$\int_\Gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}\mathrm{d}z$$



where Γ is the circle entered at 0 with radius R, which we will take the limit to ∞.

The function

$$f(z) = \frac{1}{1-z^a}\frac{1}{1-z^b}\frac{1}{z^{k+1}}$$

has singularities at $z = 0$ and when $z^a = 1$ and $z^b = 1$, which are roots of unity which lie on the unit circle.
So, we have

$$\frac{1}{2\pi i}\int_\Gamma f(z)\mathrm{d}z = \underbrace{\operatorname*{Res}_{z=0} f(z)}_{\text{what we want}} + \sum_{\substack{w^a = 1 \\ \text{or } w^b = 1}} \underbrace{\operatorname*{Res}_{z=w} f(z)}_{\text{singularities on the unit circle}}$$

Apply the ML lemma to the LHS.

$$\begin{aligned}\left|\int_\Gamma \frac{1}{(1-z^a)(1-z^b)z^{k+1}}\mathrm{d}z\right| &\le 2\pi R \max_{|z|=R}\left|\frac{1}{(1-z^a)(1-z^b)z^{k+1}}\right| \\ &\le 2\pi R \max_{|z|=R}\frac{1}{|1-z^a|}\max_{|z|=R}\frac{1}{|1-z^b|}\max_{|z|=R}\frac{1}{|z^{k+1}|}\end{aligned}$$

By the reverse triangle inequality, $|1-z^a| \ge |z^a| - |1| = R^a - 1$, so $\frac{1}{|1-z^a|} \le \frac{1}{R^a-1}$.

$$\le \frac{2\pi R}{(R^a-1)(R^b-1)R^{k+1}}$$

We assume $a, b > 0$ since otherwise the function does not make sense, so this goes to 0 as $R \to \infty$ since the denominator has higher degree than the numerator. So, the sum of the residues is zero.

$$\operatorname*{Res}_{z=0} f(z) = -\sum_{\substack{w^a = 1 \\ \text{or } w^b = 1}} \operatorname*{Res}_{z=w} f(z)$$

From now, we assume that a and b are coprime, meaning $\gcd(a, b) = 1$. This implies that the only solution to $w^a = 1$ and $w^b = 1$ is $w = 1$. That way, $f(z)$ turns out to have poles of order 1 at $e^{2\pi i\frac{r}{a}}$ $(r = 1, 2, \ldots, a-1)$ and at $e^{2\pi i\frac{s}{b}}$ $(s = 1, 2, \ldots, b-1)$ and a pole of order 2 at $z = 1$.

It is easier to compute the residue at the poles of order 1 (although the answer is messy), but it is harder to compute the residue at the pole of order 2. It is

$$-\frac{2k+a+b}{2ab}.$$

Upon computing the other residues, we get

$$a_k = \frac{2k+a+b}{2ab} + \frac{1}{a}\sum_{r=1}^{a-1}\frac{1}{e^{2\pi i\frac{r}{a}k}\left(1-e^{2\pi i\frac{r}{a}b}\right)} + \frac{1}{b}\sum_{s=1}^{b-1}\frac{1}{e^{2\pi i\frac{s}{b}k}\left(1-e^{2\pi i\frac{s}{b}a}\right)}$$

The two sums are periodic in k with periods of a and b respectively:

$$\begin{aligned} e^{2\pi i\frac{r}{a}(k+a)} &= e^{2\pi i\frac{r}{a}k + 2\pi i\frac{r}{a}a} \\ &= e^{2\pi i\frac{r}{a}k}\cdot \cancelto{1}{e^{2\pi i r}}\end{aligned}$$

Since the sums are periodic, they only take finitely many values. If k is large enough, $\frac{k}{ab}$ is larger than all the other terms, so $a_k > 0$ for k large enough.

Special Functions

Instead of talking about things that apply to all holomorphic functions, we will take a look at a particular function that comes up somewhat often and cannot be expressed using more familiar functions, such as polynomials and exponents.

Such functions are called **special functions**. These include Gamma functions, elliptic functions, Bessel functions, and Painlevé transcendents to list a few.

The Gamma Function

The **gamma function** is

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \mathrm{d}x$$

where $x^{s-1} = e^{(s-1)\ln(x)}$ is a holomorphic function of s. The Gamma function converges for $\mathrm{Re}(s) > 0$. We can see this by writing it as

$$\Gamma(s) = \int_0^1 x^{s-1} e^{-x} \mathrm{d}s + \int_1^\infty x^{s-1} e^{-x} \mathrm{d}s.$$

The right integral converges for any value of s since exponentials grow much faster than polynomials. However, the left integral only converges when $\mathrm{Re}(s) > 0$:

$$\begin{aligned} \int_\varepsilon^1 x^{s-1} \mathrm{d}x &= \left[\frac{1}{s} x^s\right]_{x=\varepsilon}^1 \\ &= \frac{1}{s}\left(1 - \varepsilon^s\right) \end{aligned}$$

The limit $\varepsilon \to 0$ only exists when $\mathrm{Re}(s) > 0$.

A key fact is that for $\mathrm{Re}(s) > 0$, $\Gamma(s)$ is holomorphic. The idea is the same as for Laplace transforms, which uses facts from measure theory. We know $\Gamma(s)$ is continuous. Let ζ be a closed curve. Fubini's theorem states that we can swap the order of integration.

$$\begin{aligned} \int_\zeta \Gamma(s) \mathrm{d}s &= \int_\zeta \int_0^\infty x^{s-1} e^{-x} \mathrm{d}x \mathrm{d}s \\ &= \int_0^\infty \underbrace{\int_\zeta x^{s-1} e^{-x} \mathrm{d}s}_{=0 \text{ (Cauchy's Thm)}} \ \mathrm{d}x && \text{(Fubini's Thm)} \\ &= 0 \end{aligned}$$

Morera's Theorem applies, so $\Gamma(s)$ is holomorphic.

Now, let's integrate by parts.

$$\int_0^\infty x^{s-1} e^{-x} \mathrm{d}x = \left[\frac{x^s}{s} e^{-x}\right]_{x=0}^\infty - \int_0^\infty \frac{x^s}{s}\left(-e^{-x}\right) \mathrm{d}x$$

$$\begin{aligned} &\lim_{x\to\infty} \frac{x^s}{s} e^{-x} = 0 \text{ since exponentials grow faster than polynomials} \\ &\lim_{x\to 0} \frac{x^s}{s} e^{-x} = 0 \text{ since } \mathrm{Re}(s) > 0, \text{ so } x^s \to 0 \text{ as } x \to 0 \end{aligned}$$

So, we have

$$\begin{aligned} \Gamma(s) &= \int_0^\infty \frac{x^s}{s} e^{-x} \mathrm{d}x \\ &= \frac{1}{s}\Gamma(s+1) \end{aligned}$$

So, given any $\Gamma(s)$, we can compute $\Gamma(s+1)$:

$$\Gamma(s+1) = s\Gamma(s)$$

Let's compute some values of $\Gamma(s)$.

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty x^{1-1}e^{-x}\mathrm{d}x \\
&= \int_0^\infty e^{-x}\mathrm{d}x \\
&= 1 \\
\Gamma(2) &= 1\cdot\Gamma(1) = 1\cdot 1 = 1 \\
\Gamma(3) &= 2\cdot\Gamma(2) = 2\cdot 1\cdot\Gamma(1) = 2 \\
\Gamma(4) &= 3\cdot\Gamma(3) = 3\cdot 2\cdot 1\cdot\Gamma(1) = 6
\end{aligned}$$

In general, if $n \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
\Gamma(n) &= (n-1)\Gamma(n-1) \\
&= (n-1)(n-2)\Gamma(n-2) \\
&\vdots \\
&= (n-1)(n-2)\cdots(1)\Gamma(1) \\
&= (n-1)!
\end{aligned}$$

So, we can think of $\Gamma(s)$ as an extension of the factorial to complex numbers.

The bad news is that for most values of s, the value of $\Gamma(s)$ cannot be expressed in terms of simpler functions. For example, $\Gamma\left(\frac{1}{3}\right) = 2.6789\ldots$ cannot be simplified. This is similar as to how $\sin(1)$ cannot be expressed more explicitly.

The good news is that at least we can find $\Gamma\left(\frac{1}{2}\right)$.

$$\begin{aligned}
\Gamma\left(\tfrac{1}{2}\right) &= \int_0^\infty x^{\frac{1}{2}-1}e^{-x}\mathrm{d}x \\
&= \int_0^\infty x^{-1/2}e^{-x}\mathrm{d}x
\end{aligned}$$

Substitute $x = y^2$, so $\mathrm{d}x = 2y\mathrm{d}y$. The bounds still hold.

$$\begin{aligned}
&= \int_0^\infty \frac{1}{y}e^{-y^2}2y\mathrm{d}y \\
&= 2\int_0^\infty e^{-y^2}\mathrm{d}y \\
&= \sqrt{\pi}
\end{aligned}$$

So, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. We also get

$$\begin{aligned}
\Gamma\left(\tfrac{2n+1}{2}\right) &= \frac{2n-1}{2}\Gamma\left(\tfrac{2n-1}{2}\right) \\
&= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\Gamma\left(\tfrac{2n-3}{2}\right) \\
&\vdots \\
&= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) \\
&= \frac{(2n-1)(2n-3)\cdots 1}{2^n}\sqrt{\pi}
\end{aligned}$$

We can rewrite this to get

$$\Gamma\left(\tfrac{2n+1}{2}\right) = \frac{(2n)!}{n!4^n}\sqrt{\pi}.$$

Going back to the previous property of the gamma function,

$$\Gamma(s) = \tfrac{1}{s}\Gamma(s+1)$$

where the RHS makes sense when $\mathrm{Re}(s+1) > 0$, or $\mathrm{Re}(s) > -1$, and $s \neq 0$. This serves as a definition of $\Gamma(s)$ on a larger set.

$$\begin{aligned}\Gamma(s) &= \frac{1}{s}\Gamma(s+1) \\ &= \frac{1}{s}\cdot\frac{1}{s+1}\Gamma(s+2)\end{aligned}$$

The RHS makes sense when $\mathrm{Re}(s) > -2$ and $s \neq 0, -1$. In general, we have

$$\Gamma(s) = \frac{1}{s}\cdot\frac{1}{s+1}\cdots\frac{1}{s+(n-1)}\Gamma(s+n).$$

So, $\Gamma(s)$ extends to a function for $\mathrm{Re}(s) > -n$ and $s \neq 0, -1, -2, \ldots, -(n-1)$.

For any $s \in \mathbb{C}$ that is not a nonpositive integer,

$$\Gamma(s) = \frac{1}{s(s+1)\cdots(s+n-1)}\Gamma(s+n)$$

for any n with $\mathrm{Re}(s) > -n$.

It turns out that $\Gamma(s)$ is holomorphic on $\mathbb{C}\backslash\{0, -1, -2, \ldots\}$.

Proposition

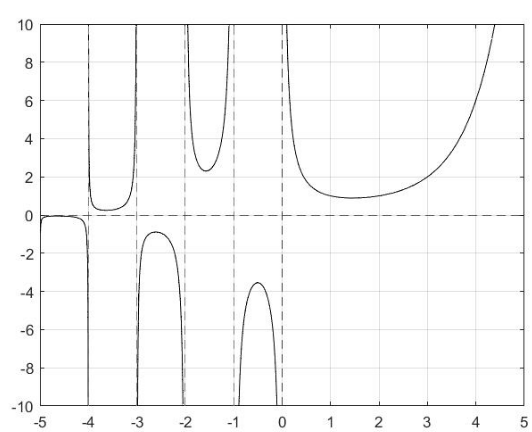
For $k \in \mathbb{Z}_{\geq 0}$, $\Gamma(s)$ has a pole of order 1 at $-k$ with residue $\frac{(-1)^k}{k!}$.

- **Proof**: to find the residue of function $f(z)$ with a pole of order $n = 1$ at z_0, apply the formula $a_{-1} = \frac{1}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}[(z-z_0)^n f(z)]\big|_{z=z_0}$ for $n = 1$. We have residue $a_{-1} = \lim\limits_{z\to z_0}(z-z_0)f(z)$. Here, $f(z) = \Gamma(s)$ and $z_0 = -k$.

$$\begin{aligned}\lim_{s\to -k}(s-(-k))\Gamma(s) &= \lim_{s\to -k}(s+k)\frac{1}{s(s+1)\cdots(s+k)}\Gamma(s+k+1) \\ &= \lim_{s\to -k}\frac{1}{s(s+1)\cdots(s+k-1)}\Gamma(s+k+1) \\ &= \frac{1}{-k(-k+1)\cdots(-1)}\Gamma(-k+k+1) \\ &= \frac{(-1)^k}{k!}\end{aligned}$$

□

We can plot the general shape of the gamma function.



The gamma function has been very well studied, and we can observe several more of its interesting properties:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Note that the RHS has poles at $s \in \mathbb{Z}$, and the LHS has poles at $s \in \mathbb{Z}_{\leq 0}$ and $\mathbb{Z}_{>0}$, meaning they match.

We also have property

$$\Gamma(s)\Gamma\left(s+\tfrac{1}{2}\right) = \Gamma(2s)2^{1-2s}.$$

This is not an exhaustive list by any means.

The Riemann Zeta Function

When talking about the **Riemann zeta function**, we will skip the technical details. The zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

This does not converge for every s, but it converges for $\operatorname{Re}(s) > 1$ since $|n^{-s}| = n^{-\operatorname{Re}(s)}$, and when s is real, $\sum n^{-s}$ converges for $s > 1$ by the integral test.

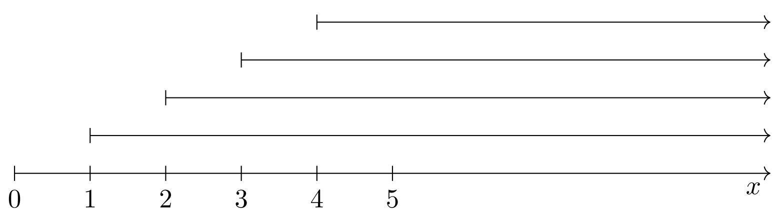
Note that for any $\varepsilon > 0$, we can show $\sum\limits_{n \geq 1} n^{-s}$ converges uniformly on $\operatorname{Re}(s) \geq 1 + \varepsilon$ by the Weierstrass M-test. This can be used to show continuity. Applying Morera's Theorem shows that $\zeta(s)$ is holomorphic for $\operatorname{Re}(s) > 1$.

$$\begin{aligned} n^{-s} &= s \int_n^{\infty} x^{-(s+1)} \mathrm{d}x \\ &= s \left[\frac{x^{-s}}{-s} \right]_{x=n}^{\infty} \\ &= (-\infty^{-s}) - (-n^{-s}) \\ &= n^{-s} \text{ for } \operatorname{Re}(s) > 0 \end{aligned}$$

So, we can write

$$\zeta(s) = s \left[\int_1^{\infty} + \int_2^{\infty} + \int_3^{\infty} + \cdots x^{-(s+1)} \mathrm{d}x \right].$$

We can visualize the bounds of these integrals.



Observe that the integral over [1, 2] is counted once, the integral over [2, 3] is counted twice, and so on. So, we can write

$$\zeta(s) = s \int_1^{\infty} \lfloor x \rfloor x^{-(s+1)} \mathrm{d}x$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x (floor function). $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x, where $0 \leq \{x\} < 1$.

$$\begin{aligned} \zeta(s) &= s \int_1^{\infty} \frac{x - \{x\}}{x^{s+1}} \mathrm{d}x \\ &= s \int_1^{\infty} \frac{x}{x^{s+1}} \mathrm{d}x - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} \mathrm{d}x \\ &= \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} \mathrm{d}x \end{aligned}$$

which converges for $\operatorname{Re}(s) > 0$. Note that $\frac{s}{s-1}$ makes sense for $s \neq 1$ and $\int_1^{\infty} \frac{\{x\}}{x^{s+1}} \mathrm{d}x$ is bounded by $\int_1^{\infty} \left| \frac{1}{x^{s+1}} \right| \mathrm{d}x$. This allows us to write $\zeta(s)$ as a function for $\operatorname{Re}(s) > 0$ and $s \neq 1$ (pole of order 1 with residue 1 at $s = 1$). It turns out that $\zeta(s)$ extends to $\mathbb{C} \backslash \{1\}$, but showing this requires more work.

The zeta function also has a version of the reflection formula:

$$\zeta(s) = \frac{(2\pi)^s}{\pi} \sin\!\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

This is tricky to prove and is more complicated than the gamma function.

Application to Prime Numbers

The zeta function is famous for its connection to prime numbers. The fundamental theorem of arithmetic states that any $n \in \mathbb{Z}_{>1}$ can be written as a product of primes:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

So, we can write

$$\begin{aligned}
n^{-s} &= p_1^{-a_1 s} p_2^{-a_2 s} \cdots p_k^{-a_k s} \\
\zeta(s) &= 1^{-s} + 2^{-s} + 3^{-s} + \cdots \\
&= \prod_{\text{primes } p} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots\right)
\end{aligned}$$

This is a sum of terms

$$p_1^{-a_1 s} p_2^{-a_2 s} \cdots$$

if we choose the $a_1{}^{\text{th}}$ term from $(1 + p_1^{-s} + p_1^{-2s} + \cdots)$

if we choose the $a_2{}^{\text{th}}$ term from $(1 + p_2^{-s} + p_2^{-2s} + \cdots)$

By uniqueness of prime factorization, each n^{-s} arises as $p_1^{-a_1 s} p_2^{-a_2 s} \cdots$ in exactly one way.

$$\begin{aligned}
\zeta(s) &= \prod_{\text{primes } p} \left(1 - p^{-s}\right)^{-1} \\
\ln(\zeta(s)) &= \sum_{\text{primes } p} -\ln\left(1 - p^{-s}\right) \\
\frac{\zeta'(s)}{\zeta(s)} &= \sum_{\text{primes } p} -\frac{\ln(p) p^{-s}}{1 - p^{-s}} \\
&= \sum_{\text{primes } p} -\left(\frac{\ln(p)}{p^s} + \frac{\ln(p)}{p^{2s}} + \frac{\ln(p)}{p^{2s}} + \cdots\right)
\end{aligned}$$

Let $\Lambda(n) = \begin{cases} \ln(p) & n \text{ is a power of the prime } p \\ 0 & n \text{ is not a power of a prime} \end{cases}$.

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Let $x > 0$.

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^{-s} \frac{x^s}{s} \mathrm{d}s = \begin{cases} 1 & x > n \\ 0 & x < n \end{cases}$$

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \mathrm{d}s = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n \geq 1} \frac{\Lambda(n)}{n^s} \frac{x^s}{s} \mathrm{d}s$$

Switching the order of the sum and integral requires some properties, but we won't worry about that.

$$\begin{aligned}
&= \sum_{n \geq 1} \Lambda(n) \underbrace{\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} n^{-s} \frac{x^s}{s} \mathrm{d}s}_{\substack{= 0 \text{ if } n > x \\ = 1 \text{ if } n < x}} \\
&= \sum_{n \leq x} \Lambda(n)
\end{aligned}$$

So, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\mathrm{d}s=\sum_{n\le x}\Lambda(n).$$

Let $\pi(x)$ be the number of primes less than or equal to x. Then,

$$\begin{aligned}\sum_{n\le x}\Lambda(n)&=\sum_{p^k\le x}\Lambda(p^k)\\&=\sum_{p^k\le x}\ln(p)\\&=\sum_{p\le x}\ln(p)\left\lfloor\log_p(x)\right\rfloor\\&=\sum_{p\le x}\ln(p)\left\lfloor\frac{\ln(x)}{\ln(p)}\right\rfloor\\&\le\sum_{p\le x}\frac{\ln(p)\ln(x)}{\ln(p)}\\&=\ln(x)\sum_{p\le x}(1)\\&=\ln(x)\pi(x)\\\frac{\sum_{n\le x}\Lambda(n)}{\ln(x)}&\le\pi(x)\end{aligned}$$

It turns out that

$$\lim_{n\to\infty}\frac{\frac{\sum_{n\le x}\Lambda(n)}{\ln(x)}}{\pi(x)}=1$$

which means that as $n\to\infty$,

$$\frac{\sum_{n\le x}\Lambda(n)}{\ln(x)}\to\pi(x).$$

Additionally, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\mathrm{d}s=\underbrace{x}_{\text{from pole at } x=1}-\underbrace{\sum_{\mathscr{S}}\frac{x^{\mathscr{S}}}{\mathscr{S}}}_{\substack{\text{zeros } \mathscr{S} \text{ of } \zeta(s)\\ 0\le\operatorname{Re}(\mathscr{S})\le 1}}-\underbrace{\ln(2\pi)}_{\text{when } s=0}-\underbrace{\frac{1}{2}\ln\left(1-x^{-2}\right)}_{\substack{\text{other zeros of } \zeta(s)\\ -2,-4,-6,-8,\ldots}}.$$

It turns out that $\operatorname{Re}(\mathscr{S})<1$, so x is the dominant term. $\sum_{n\ge x}\Lambda(n)\approx x$, so

$$\pi(x)\approx\frac{x}{\ln(x)}.$$

So, the number of primes $\le x$ is roughly $\frac{x}{\ln(x)}$. This is the prime number theorem.

Better understanding of $\operatorname{Re}(\mathscr{S})$ allows for better understanding of how good the approximation is. The strongest case is the Riemann hypothesis, $\operatorname{Re}(\mathscr{S})=\frac{1}{2}$.

The Riemann hypothesis states that the real part of every nontrivial zero of $\zeta(s)$ is $\frac{1}{2}$, and it remains unproven.

Summary

Complex Derivatives

The **complex derivative** of function $f : \mathbb{C} \to \mathbb{C}$ is defined as

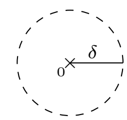
$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

By definition of the limit, this is equivalent to $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $0 < |h| < \delta$, then

$$\left| \frac{f(z+h) - f(z)}{h} - L \right| < \varepsilon$$

where $L = f'(z)$.

The set of values of h that we consider are on a punctured open disk of radius δ and excludes the point 0.



This turned out to be a more stringent requirement than being real-differentiable.

If $z = x + iy$, $f(z) = f(x+iy)$ can be thought of as a function $\mathbb{R}^2 \to \mathbb{C}$.

If $u(x,y) = \mathrm{Re}(f(x+iy))$ and $v(x,y) = \mathrm{Im}(f(x+iy))$, then $f : \mathbb{C} \to \mathbb{C}$ becomes two functions $\mathbb{R}^2 \to \mathbb{R}$.

The **Cauchy-Riemann equations** express differentiability of f in terms of u and v:

> f is complex differentiable if and only if u and v have continuous derivatives and satisfy the Cauchy-Riemann equations
>
> $$u_x = v_y \text{ and } u_y = -v_x$$
>
> where the subscripts indicate partial derivatives.

This implies that u and v satisfy the 2D Laplace equation:

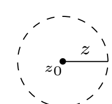
$$u_{xx} = (u_x)_x = {}^{[1]}(v_y)_x = v_{yx} = {}^{[2]}v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}$$

This can be rewritten as the Laplace equation $u_{xx} + u_{yy} = 0$.

Most rules of ordinary calculus apply to complex derivatives, including the product and quotient rules and the chain rule, which states that $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = g'(x) f'(g(x))$. In particular, if f and g are differentiable, then so is $f(g(x))$.

f is **holomorphic** at z_0 for some $\varepsilon > 0$ if f is differentiable at z for $|z - z_0| < \varepsilon$.

One consequence is that f is holomorphic at z_0, then it is also holomorphic nearby z_0.

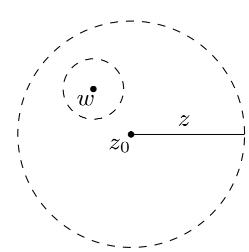


We can show this using the fact that $f(z)$ is differentiable for $|z - z_0| < \varepsilon$. Pick w such that $|w - z_0| < \varepsilon$. For z obeying

$$|z - w| < \underbrace{\varepsilon - |w - z_0|}_{>0},$$

show that $f(z)$ is differentiable by showing that it satisfies $(z - z_0) < \varepsilon$.

$$\begin{aligned} \varepsilon - |w - z_0| &> |z - w| \\ &= |(z - z_0) - (w - z_0)| \\ &\geq |z - z_0| - |w - z_0| \qquad\qquad \text{(rev } \triangle \text{ ineq.)} \\ \varepsilon &> |z - z_0| \end{aligned}$$



So, z satisfying $|z - w| < \varepsilon - |w - z_0|$ is in the region where we know f is differentiable meaning f is holomorphic at w.

[1] This follows from the Cauchy-Riemann Equations.

[2] This uses commutativity of mixed partial derivatives, which follows from existence and continuity of second derivatives.

Special functions

Möbius Transforms and the Stenographic Projection

Möbius transforms are functions of the form

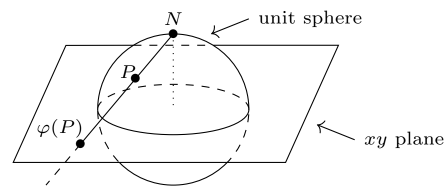
$$f(z) = \frac{az+b}{cz+d}. \qquad (ad - bc \neq 0)$$

Note that the denominator is zero at $z = -\frac{d}{c}$. To remedy this, we considered the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

We define $f\left(-\frac{d}{c}\right) = \infty$. We also define $f(\infty) = \frac{a+b/\infty}{c+d/\infty} = \frac{a}{c}$. So, f is a function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. This makes f into a bijection.

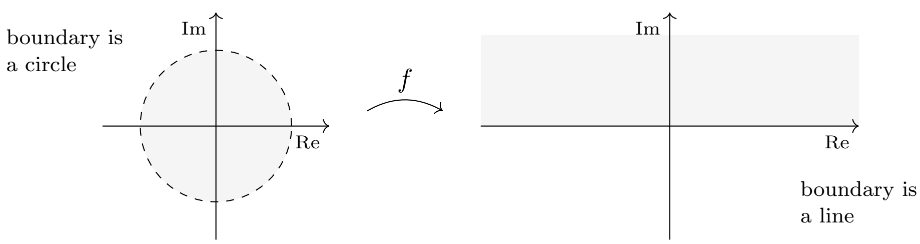
Additionally, the inverse of a Möbius transform and a composition of two Möbius transforms are both also Möbius transforms.

The **stenographic projection** identifies $\hat{\mathbb{C}}$ with the sphere. For the unit sphere with north pole N and point $P \neq N$ on the sphere, the stenographic projection φ of P is the point where the line passing through P and N passes through the xy plane.



∞ corresponds with the north pole N.

Möbius transforms sends lines and circles to lines and circles, but do not necessarily conserve lines and circles. For example, $f(z) = \frac{z-1}{i(z+1)}$ maps the unit circle to the upper half plane.

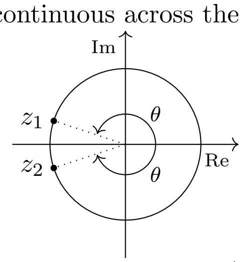


The Logarithm

The **logarithm** considers solutions to $z = e^w$ where $z \neq 0$. There are infinitely many solutions since if $e^w = z$, then $e^{w+2\pi ik} = z$ where $k \in \mathbb{Z}$.

If z is a positive real number, we can take w to also be real. This is the usual logarithm. Otherwise, there is no "best" solution w.

We could ask for $-\pi < \operatorname{Im}(\log(z)) \leq \pi$. There is always a unique solution landing in this range. The upshot is that this defines a function $\mathbb{C}\backslash\{0\} \to \mathbb{R}$, which is the principal logarithm. The drawback is that this function is not continuous across the negative real axis.



$$\log(z_1) = i\theta \approx i\pi$$
$$\log(z_2) = -i\theta \approx -i\pi$$

As we approach -1 from above, $\log(z) \to \pi i$ and as we approach -1 from below, $\log(z) \to -\pi i$. So, $\log(z)$ is not continuous at -1. This holds for all other negative real numbers.

If we choose the constraints on $\operatorname{Im}(\log(z))$ differently, we will get a discontinuity elsewhere.

Conceptually, $\log(z)$ takes infinitely many values. This stops $\log(z)$ from being a function in the usual sense.

Complex Integration

For $f : \mathbb{R} \to \mathbb{C}$, we define the **complex integral** as

$$\int_a^b f(x)\mathrm{d}x = \int_a^b \mathrm{Re}(f(x))\mathrm{d}x + i\int_a^b \mathrm{Im}(f(x))\mathrm{d}x.$$

We also have

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

where γ is a curve in $\mathbb{C}$ with parameterization $\gamma : [a, b] \to \mathbb{C}$ for a parameterization of this curve. Note that both $\gamma(t)$ and $\gamma'(t)$ can be complex numbers.

We proved several facts about complex integrals.

- The value of $\int_\gamma f(z)\mathrm{d}z$ does not depend on the choice of paramterization.
- If $\overline{\gamma}$ is γ with orientation reversed, then

$$\int_{\overline{\gamma}} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z.$$

- **ML lemma**:

$$\left|\int_\gamma f(z)\mathrm{d}z\right| \leq ML$$

where $M = \max_{z\in\gamma} |f(z)|$ and $L =$ length(γ).

Fundamental Theorem of Calculus

Suppose $F(z)$ obeys $F'(z) = f(z)$. Then,

$$\int_\gamma f(z)\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

where a and b are the endpoints of γ.

Suppose $\int_\gamma f(z)\mathrm{d}z$ depends only on the endpoints of γ, not the curve joining them. This is equivalent to the integral of $f(z)$ around any closed curve being zero. Then, we may write $\int_w^z f(\zeta)\mathrm{d}\zeta$ to mean the integral of f along any path from w to z:

$$\frac{\mathrm{d}}{\mathrm{d}z}\int_w^z f(\zeta)\mathrm{d}\zeta = f(z).$$

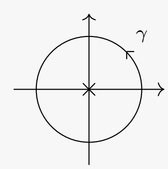
Given $f(z)$, when does $F(z)$ exist with $F'(z) = f(z)$? If $f(z)$ is continuous and $\int_\gamma f(z)\mathrm{d}z = 0$ for closed curve γ, then such F exists.

- **Morera's Theorem**: if $f(z)$ is continuous and $\int_\gamma f(z)\mathrm{d}z = 0$ for every closed curve γ, then $f(z)$ is holomorphic. Proving this uses the fact that holomorphic functions are infinitely differentiable.

Example

Let $f(z) = \frac{1}{z}$.

$$\int_\gamma \frac{1}{z}\mathrm{d}z = 2\pi i \neq 0$$



where γ is a closed curve enclosing zero.

$f(z)$ is defined on $\mathbb{C}\backslash\{0\}$. Indeed, $\frac{1}{z}$ does not have an antiderivative on $\mathbb{C}\backslash\{0\}$. But, if we restrict to the region $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$, then any closed curve will not enclose zero, and $\int_\gamma \frac{1}{z}\mathrm{d}z = 0$. So, there is an antiderivative which is the principal logarithm.

Restricting a different region may lead to a different version of the logarithm.

Cauchy's Theorem and Integral Formula

Cauchy's Theorem states that if γ is a closed curve and $f(z)$ is holomorphic on and inside γ, then

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

The Cauchy Integral Formula states that if γ is a closed curve enclosing a and $f(z)$ is holomorphic on and inside γ, then

$$\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z = 2\pi i f(a).$$

Note that we may have to consider how many times γ encloses a. If it encloses a multiple times, then we will need to multiply $2\pi i f(a)$ by that number of times.

Cauchy's Theorem and the Cauchy Integral Formula has several consequences.

- **Liouville's Theorem**: if $f(z)$ is holomorphic on all of $\mathbb{C}$ nand is bounded, then f is constant.
- Holomorphic functions are infinitely differentiable.
- If $u(x, y)$ is a harmonic function on a set with no holes, then there is a holomorphic function $f(x+iy)$ where $\mathrm{Re}(f(x+iy)) = u(x, y)$. So, we can interconvert between harmonic and holomorphic functions.
- **Maximum modulus principle**: if $f(z)$ is a nonconstant holomorphic function, then $|f(z)|$ has no local maximum.

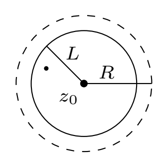
Power Series

A **power series**

$$\sum_{k\geq 0} a_k(z-z_0)^k$$

has radius of convergence R where the power series converges on $|z-z_0| < R$ and diverges on $|z-z_0| > R$.

Uniform convergence is determined by the Weierstrass M-test. It turns out that power series converge uniformly on $|z-z_0| \leq L$ where $L < R$.



This lets us use the convinient properties of uniform convergence, especially swapping the order of integration and summation. As a consequence, power series define holomorphic functions.

$$\underbrace{f'(a)}_{\text{derivative}} = \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\mathrm{d}z}_{\text{integral}}$$

We can differentiate power series termwise.

$$\frac{\mathrm{d}}{\mathrm{d}z}f(z) = \sum_{k\geq 1} k a_k (z-z_0)^{k-1}$$

Secretly, a power series is a limit of partial sums, so this result is saying we can swap the order of the limit and derivative.

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\mathrm{d}z$$

Expand $\frac{1}{z-a}$ as a uniformly convergent geometric series and swap the order of summation and integration.

$$f(z) = \sum_{k\geq 0} a_k (z-z_0)^k$$

where a_k's are given by some integral.

General properties of uniform convergence state that this sum converges, and it converges to $f(z)$. Differentiating termwise and evaluating at z_0 yields

$$a_k = \frac{1}{k!}\frac{\mathrm{d}^k f}{\mathrm{d}z^k}.$$

So, power series expansions are the same thing as Taylor series.

Holomorphic Functions as Power Series

If $f(z)$ is holomorphic at z_0, then $f(z)$ can be written as a power series

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k.$$

If $a_0,\ a_1,\ \ldots,\ a_{m-1}$ are all zero and $a_m \neq 0$, then $f(z)$ can be written as

$$f(z) = (z - z_0)^m \sum_{k \geq 0} a_k (z - z_0)^{k-m}.$$

The coefficients of $k = 0, 1, \ldots, m-1$ are zero, so we are left with

$$\begin{aligned} f(z) &= (z - z_0)^m \underbrace{\sum_{k \geq m} a_k (z - z_0)^{k-m}}_{\substack{\text{power series for} \\ \text{some holomorphic } g(z)}} \\ &= (z - z_0)^m g(z) \end{aligned}$$

$g(z_0) = a_m \neq 0$, and m is the order of the zero at z_0.

Identity Principle

If $f(z_0) = 0$, let $f(z) = (z - z_0)^m g(z)$ where we assume that f is not the zero function. $g(z_0) \neq 0$, and since $g(z)$ is continuous, $g(z) \neq 0$ on some small disk centered at z_0. On this small disk, $f(z)$ is zero only at z_0. So, zeros of holomorphic functions are isolated.

If $f(z)$ has a zero that is not isolated (i.e. the zero set has an accumulation point), then $f(z)$ must be the zero function.

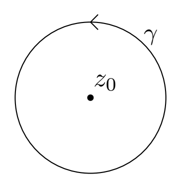
If f and g are holomorphic, $s = \{z \in \mathbb{C} \mid f(z) = g(z)\}$ is the zero set of $f - g$. If s contains an accumulation point, then $f - g$ is the zero function and so $f = g$ as functions.

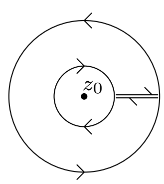
Laurent Series

For a function holomorphic inside a curve γ,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \mathrm{d}w$$

by the Cauchy Integral Formula.





Now, if the function is not holomorphic at some point inside the curve, then

$$f(z) = \frac{1}{2\pi i}\left[\int_{\text{big circle}} - \int_{\text{small circle}}\right].$$

Expanding $\frac{1}{w-z}$ as a uniformly convergent power series and swap the order of summation and integration.

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

This can be expanded as a geometric series when $|z - z_0| < |w - z_0|$, which is valid on the big circle. So, $\int_{\text{big circle}}$ gives us a power series in $(z - z_0)$.

Similarly, we can expand $\frac{1}{w-z}$ as

$$\frac{1}{w - z} = \frac{1}{-((z - z_0) - (w - z_0))} = -\frac{1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}}.$$

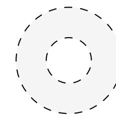
This can be expanded as a geometric series when $|w - z_0| < |z - z_0|$, which is valid on the small circle. So, $\int_{\text{small circle}}$ gives us a power series in $(z - z_0)^{-1}$.

So, we get

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_k(z-z_0)^k}_{\text{converges for } |z-z_0|<R_2} + \underbrace{\sum_{k=-\infty}^{-1} a_k(z-z_0)^k}_{\text{converges for } \frac{1}{|z-z_0|}<\frac{1}{R_1}} .$$

The entire sum only converges on

$$R_1 < |z-z_0| < R_2$$



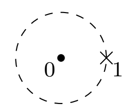
which is an annulus.

Note that a function can have different Laurent series that converge on different regions. For example, let $f(z) = \frac{1}{1-z}$.

$$\frac{1}{1-z} = \sum_{k\geq 0} z^k \qquad (0 < |z| < 1)$$

$$\begin{aligned}\frac{1}{1-z} &= -\frac{1}{z}\frac{1}{1-z^{-1}} \\ &= -\sum_{k=-\infty}^{-1} z^k \qquad (|z|^{-1} < 1, \text{ or } |z| > 1)\end{aligned}$$

$f(z)$ has a singularity only at $z = 1$. One Laurent series for $f(z) = \frac{1}{1-z}$ about $z_0 = 0$ converges inside the circle centered at z_0 passing through the singularity, while another Laurent series converges outside the circle.



Isolated Singularities

If z_0 is an isolates singularity, then f is holomorphic near z_0 (except at z_0 itself). So, the Cauchy Integral Formula can be applied and we are guaranteed to get a Laurent series converging for $0 < |z-z_0| < \varepsilon$.

We identified 3 types of **isolated singularities**.

1. **Removable singularity**: f extends to a function holomorphic at z_0.
2. **Pole**: $\lim_{z\to z_0} |f(z)| = \infty$.
3. **Essential singularity**: any other case.

We can interpret these cases in terms of the Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k.$$

1. z_0 is a **removable singularity** if and only if $a_{-1}, a_{-2}, a_{-3}, \ldots$ are all zero, so the Laurent series is actually a power series.
2. z_0 is a **pole of order** m if and only if $a_{-m} \neq 0$, but $a_{-(m+1)}, a_{-(m+2)}, a_{-(m+3)}, \ldots$ are all zero. Then,

   $$\begin{aligned}(z-z_0)^m f(z) &= (z-z_0)^m \sum_{k\geq -m} a_k(z-z_0)^k \\ &= \sum_{k\geq -m} a_k(z-z_0)^{k+m}\end{aligned}$$

   This is the power series for some $g(x)$ with $g(z_0) = a_m \neq 0$. So,

   $$f(z) = \frac{g(z)}{(z-z_0)^m}$$

   with $g(z)$ holomorphic at z_0.
3. z_0 is an **essential singularity** if and only if it satisfies all other cases, meaning there are infinitely many nonzero negative index coefficients.

Residue Theorem

Suppose $f(z)$ has an isolated singularity at z_0.

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma \sum_{k=-\infty}^{\infty} a_k(z-z_0)\mathrm{d}z$$

To sqap the order of the sum and integral, γ must be contained in the annulus of convergence.

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} a_k \int_\gamma (z-z_0)^k \mathrm{d}z \\ &= 2\pi i a_{-1} \end{aligned}$$

since

$$\int_\gamma (z-z_0)^k \mathrm{d}z = \begin{cases} 2\pi i & k=-1 \\ 0 & k \neq -1 \end{cases}$$

This a_{-1} is the residue of $f(z)$ at z_0. So, the Residue Theorem states that

$$\int_\gamma f(z)\mathrm{d}z = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} f(z)$$

where we take the sum over singular points z_k enclosed by γ.

Special Functions

Special functions are functions that cannot be expressed in more simple terms.

The Gamma Function

The **gamma function** is given by

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x$$

and is holomorphic on $\mathbb{C}\backslash\{0,-1,-2,\dots\}$. An interesting property of the gamma function is

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

The Riemann Zeta Function

The **zeta function** is given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

and converges for $\mathrm{Re}(s) > 1$. It has a version of the reflection formula:

$$\zeta(s) = \frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

In addition, by using the fact that any number can be written as a product of primes, we have

$$\zeta(s) = \prod_{\text{primes } p} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots\right)$$

which eventually yields the prime number theorem. This states that the number of primes $\leq x$ is approximately $\frac{x}{\ln(x)}$.