# Limiting Distribution of Self-Interacting Random Walks

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## Abstract

Classical simple random walks, after scaling of time and space, converge in distribution to Brownian motion. This project studies self-interacting random walks on the integer lattice, where the transition probabilities are determined by the weights of neighboring edges. The weight function is assumed to be monotonic and asymptotically polynomial. Our goal is to determine whether self-interacting random walks have a limiting distribution under proper scaling. To achieve this, we use numerical simulations for self-interacting random walks with a given weight function, and perform various analyses. We also develop an adjusted Kolmogorov-Smirnov test for testing limiting distribution of random walks. Finally, we establish convergence theorems for various test statistics under the assumption that the random walk converges. This provides numerical tools to investigate the convergence of self-repelling random walks.

## Background

### Simple Random Walks and Self-interacting Random Walks

A simple random walk is a random walk on $\mathbb{Z}$ starting from 0, where at each step the position changes by $+1$ or $-1$ with equal probability. This can be generalized to *self-interacting random walks (SIRW)*, where the transition probabilities depend on the history of the path. We call a monotone and asymptotically polynomial function $w : \mathbb{N} \to \mathbb{R}^+$ a *weight function*. At some time $T$, the weight of an edge on the integer lattice is defined to be $w(n)$, where $n$ is number of times that edge has been traversed before $t = T$. For the SIRW we are interested in, the transition probabilities are determined by ratio between weights of neighboring edges:

$$\mathbf{P}\left[X_{t+1} = X_t - 1 \middle| X_t = x_t, \ldots, X_0 = x_0\right] = \frac{w(n(\text{left edge}))}{w(n(\text{left edge})) + w(n(\text{right edge}))} \tag{1}$$

These SIRWs are then classified according to the asymptotic behavior of $w(n)$.

### Brownian Motion and Perturbed Brownian Motion

Brownian Motion is a continuous stochastic process $B_t$ with independent Gaussian increments. The simple random walk, under a scaled limit, converges to the Brownian Motion.

The Brownian Motion Perturbed at its Extrema (BMPE) is a Brownian Motion with an additional "perturbation" when it reaches its extrema. An $(\alpha, \beta)$-perturbed Brownian Motion is defined as

$$Y_t = B_t + \alpha M_t^Y + \beta I_t^Y \quad t \geqslant 0 \tag{2}$$

where $M_t^Y = \sup_{s \leqslant t} Y_s$ and $I_t^Y = \inf_{s \leqslant t}(Y_s)$. BMPE exists when both $\alpha < 1$ and $\beta < 1$.

### Limiting Distributions

Scaled limits can sometimes take a random walk, which is a discrete process, to some continuous process on $\mathbb{R}$. We are interested in the limiting behaviors of the self-interacting random walks.

| Type | Weight function $w(n)$ | Limiting Process | Fraction of time spent to the right of origin |
|---|---|---|---|
| Simple | constant | Brownian Motion | Arcsine distribution |
| Asymptotically Free | $1 - \frac{c}{n} + O(n^{-2})$ | BMPE$(\gamma, \gamma)$ (Kosygina et al. 2022) | Beta$\left(\frac{1-\gamma}{2}, \frac{1-\gamma}{2}\right)$ (Carmona et al. 1998) |
| Polynomially Self-repelling | $n^{-\alpha} - cn^{-\alpha-1} + O(n^{-\alpha-2})$ | Analytic form (Toth 1996) | unknown |

Table 1: Known and unknown results for limiting distributions of different types of SIRW. $c$ is an arbitrary constant, and $\alpha > 0$. While analytic form of PSR random walk is given in (Toth 1996), the existence of limit is still open.

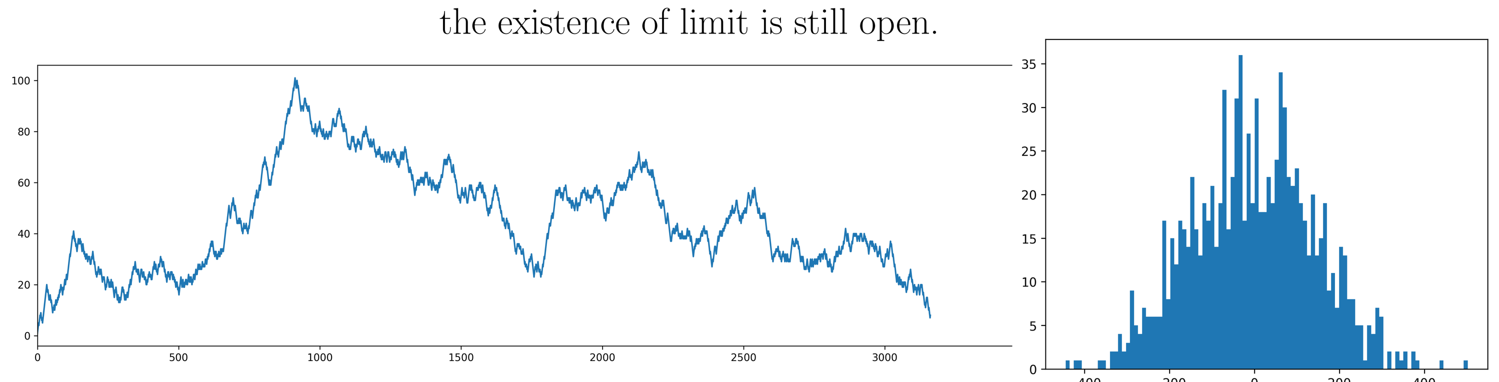


Figure 1: Simulation of PSR-RW with $\alpha = 0.8$, and empirical distribution at step 3000 with 1000 samples.

## Numerical Results

### Adaptation of Kolmogorov-Smirnov Test for Limiting Distributions

Our null hypothesis in the one-sample test is convergence of empirical distributions $H_0 : F_m \Rightarrow F_0$. We define our test statistic $D_{m,n}$ to be the K-S statistic between the limit and the empirical distribution at path length $m$, with sample size $n$. Note that we are testing the null hypothesis over a sequence of empirical distributions. For two-sample test, the null hypothesis is that the two sequences of distributions converge to some identical limit.

The classical K-S test will always fail if we take enough samples, so we need to relax the decision criterion. We do so by adding an error term. The decision criterion for one-sample test is relaxed to

$$D_{m,n} > \frac{v^*}{\sqrt{n}} = \frac{L^{-1}(1-\alpha)}{\sqrt{n}} + C\sqrt{\frac{1}{m}} \tag{3}$$

$L$ is the Kolmogorov-Smirnov Distribution. $C = 0.4748$ is chosen as in the Berry-Esseen Theorem. Decision criterion for two-sample test is developed in a similar way.

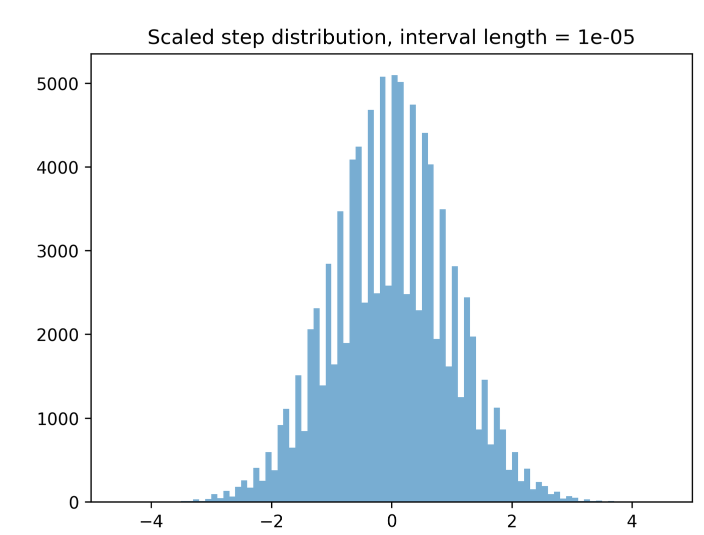



Figure 2: Oversampling Effect. Scaled distribution of simple RW at 1000-th step, while taking 100000 samples.

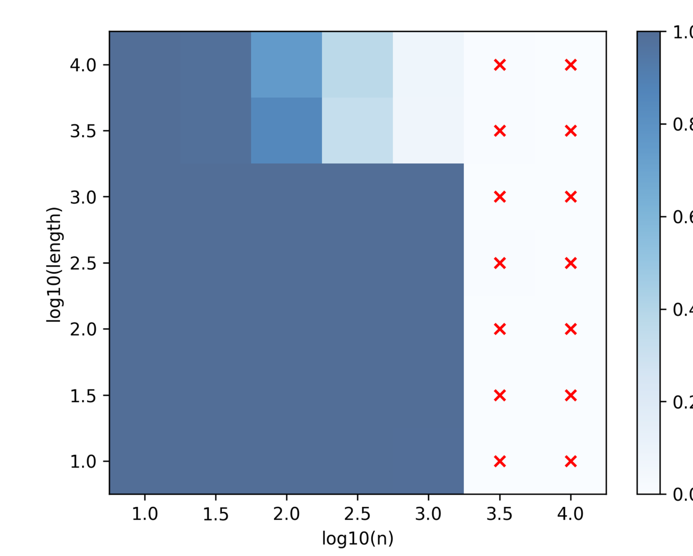
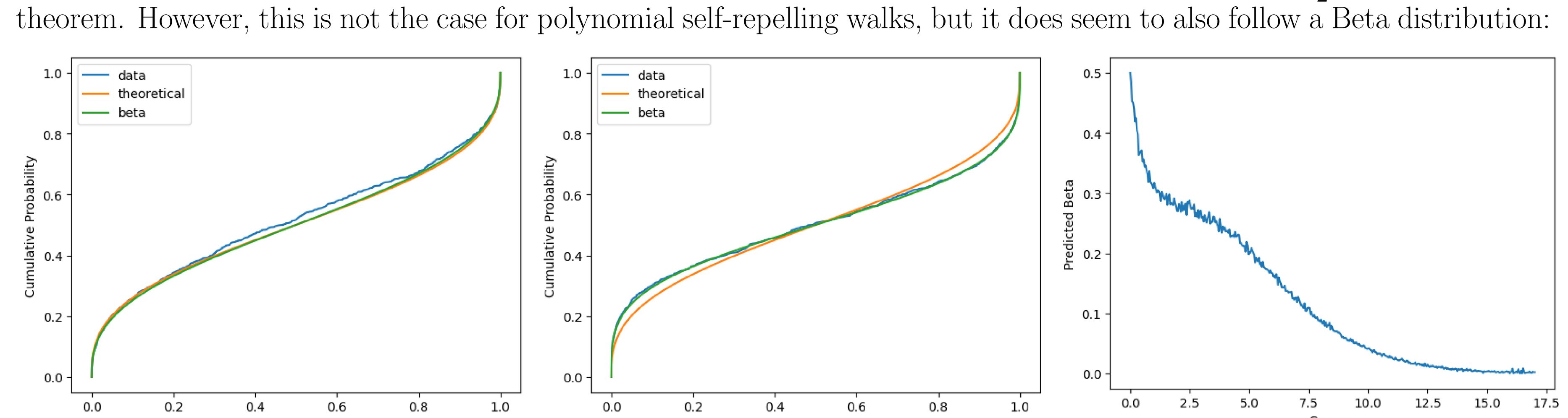



Figure 3: 2-sample test between Polynomially Self-Repelling RW with $\alpha = 0.8$ and BMPE$(\gamma, \gamma)$. Red cross means rejection at confidence level 0.05.

### Fraction of time to the right of origin

As proven later, if a walk converges to BMPE$(\alpha, \beta)$, then the (TSRO) fraction of time spent at the right of the origin is Beta$\left(\frac{1-\alpha}{2}, \frac{1-\beta}{2}\right)$ distributed. We generate 1,000 asymptotically free and polynomial self-repelling random walks of length 10,000 with weight function $w(n) = 1 + \frac{1}{1+n}$ and $w(n) = \frac{1}{1+n}^{\gamma}$ (with varying gamma) respectively, and obtain their empirical CDFs for TSRO. The asymptotically free case with that specific weight function is known to converge to BMPE$(1 - ln(2), 1 - ln(2))$ and Beta$\left(\frac{ln(2)}{2}, \frac{ln(2)}{2}\right)$ for TSRO. Then we fit a Beta$(\hat{\theta}, \hat{\theta})$ distribution, such that $\hat{\theta}$ is the least squares estimate of the true Beta parameter. We find that for the asymptotically free case, $\hat{\theta} \approx \frac{ln(2)}{2}$, confirming the theorem. However, this is not the case for polynomial self-repelling walks, but it does seem to also follow a Beta distribution:




| Type | Limiting Process (BMPE) | Fraction of time spent to the right of origin (Beta Distribution) |
|---|---|---|
| Asymptotically Free | Adapted K-S test fails to reject | Chi-squared test fails to reject |
| Polynomially Self-repelling | Adapted K-S test rejects for large $m$ and $n$ | Chi-squared test rejects |

Table 2: Summary of results from statistical tests.

## Theory

**Convergence of test statistics.** By a continuity theorem, if $F$ is continuous with respect to the Skorohod topology of cadlag functions, then whenever the paths $x_n \to x$, we have $F(x_n) \to F(x)$.

**Statistical Proof for Nonconvergence.** For a continuous process $Z$ we can calculate distribution of $F(Z)$. Consider any random walk $X$. Then whenever we have statistical evidence that $F(X)$ does not converge to $F(Z)$, it automatically becomes evidence that the limiting distribution of $X$ is not $Z$. This is useful, since the statistical test on $F(X)$ is much simpler than the K-S test on $X$ itself.

**Theorem 1.** *If our random walk has a scaled limit of brownian motion perturbed at extrema:*

$$\left(\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{t \in [0,1]} \implies (\alpha, \beta)\text{-}BMPE$$

*then the fraction of time spent to the right of origin converges to a specific Beta distribution:*

$$\frac{1}{n}\sum_{k=1}^{n} 1_{X_k > 0} \implies Beta\left(\frac{1-\alpha}{2}, \frac{1-\beta}{2}\right).$$

*Proof.* (Sketch) It suffices to show that $F(x) = \frac{1}{L}\int_0^L 1_{x(t)>0}dt$ is continuous with respect to the Skorohod Topology on a set $C$ with probability 1. From (Carmona et al. 1998) we know that BMPE has a continuous pdf, thus the zero set of a realized path of BMPE has measure zero with probability 1. Now take any such path $x$, using the uniform continuity property of the path, we can bound $x$ away from zero outside a set with small measure. Outside the exceptional set, small perturbation of $x$ does not affect $1_{x(t)>0}$. This is enough to show that $F$ is continuous. $\square$

## Future Work

- Improve the strength of adapted KS test and make it applicable to more general scenarios.
- Investigate the interesting behaviors regarding convergence PSR random walk.
- Numerically calculate the limiting distribution in (Toth 1996) and compare to empirical simulation.
- Theoretically establish the existence of limit of PSR random walk.

## Conclusion

- In this study, we studied limiting distributions of random walks.
- We empirically confirmed known results of limiting behavior of AF random walks and PSR random walks.
- The numerical methods we developed can be applied to provide empirical evidences to conjectures about limiting behavior of PSR random walks..

## References

Carmona, Philippe, et al. "Beta Variables as Times Spent in $[0, \infty[$ By Certain Perturbed Brownian Motions." Journal of the London Mathematical Society, vol. 58, no. 1, Aug. 1998, pp. 239–56.

Kosygina, Elena, et al. Convergence and Non-Convergence of Scaled Self-Interacting Random Walks to Brownian Motion Perturbed at Extrema. arXiv:2208.02589, arXiv, 4 Aug. 2022.

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