REU in Random Walk

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Contents

Note 4: Concepts of Continuity

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1 Continuity and Hölder continuity test

1.1 Definitions and Examples

The main reference in this note is Durrett, 2010.

Definition 1 (Lipschitz continuous)**.** f is said to be *Lipschitz continuous* if there is a constant C so that $|f(x) - f(y)| \leq C\rho(x, y)$.

Lipschitz continuity is a stronger notion of continuity than classical continuity. Lipschitz continuity implies continuity.

Geometrically, Lipchitz condition puts a finite bound on the slope of any secant line one can get from the graph of the function.

Definition 2. A real or complex valued function f on d-dimensional Euclidean space satisfies a Hölder condition with exponent α, or is α-Hölder continuous, when there are nonnegative real constants $C, a > 0$, such that

$$|f(x) - f(y)| \leq C\|x - y\|^\alpha$$

for all x and y in the domain of f.

- When $\alpha > 1$, an α-Hölder continuous function is *constant.*
- When $\alpha = 1$, an α-Hölder continuous function is *Lipchitz continuous.*
- When $\alpha > 0$, an α-Hölder continuous function is *uniformly continuous.*

- Whenver $0 < \alpha \leq \alpha'$, α'-Hölder continuity implies α-Hölder continuity.

Remark. (from wikipedia) We have the following chain of strict inclusions for functions over a closed and bounded non-trivial interval of the real line:

Continuously differentiable $\subset$ Lipschitz continuous $\subset$ α-Hölder continuous $\subset$ uniformly continuous $\subset$ continuous

where $\alpha \in (0, 1]$.

Theorem 1. 8.1.5. Brownian paths are Hölder continuous for any exponent $\gamma < 1/2$.

Theorem 2. 8.1.6. With probability one, Brownian paths are not Lipschitz continuous (and hence not differentiable) at any point.

1.2 Test for hölder continuity exponent

From the definition of hölder continuity:

$$|f(x) - f(y)| \leq C\|x - y\|^\alpha$$

Our goal is to find smallest possible α such that the equation above holds true empirically.

We take logarithm on both sides (using the fact that log is monotone increasing):

$$\log |f(x) - f(y)| \leq \log C + \alpha \log \|x - y\|$$

rearrange and assuming $\log \|x - y\| < 0$ for small $\|x - y\|$:

$$\alpha \leq \frac{\log |f(x) - f(y)| - \log C}{\log \|x - y\|} \sim \frac{\log |f(x) - f(y)|}{\log \|x - y\|}. \tag{1}$$

The smallest such α is then

$$\alpha = \inf_{x,y} \frac{\log |f(x) - f(y)| - \log C}{\log \|x - y\|}. \tag{2}$$

The term involving $\log C$ should fade away when $\|x - y\|$ is sufficiently small.

1.2.1 Compute α by finding a limit

Let $x_1, \ldots, x_i, \ldots, x_n$ be random points in the domain of f. Let $\varepsilon_1, \ldots, \varepsilon_j, \ldots, \varepsilon_m$ be small changes.

Denote

$$\hat{a}_i^{(j)} := \frac{\log |f(x_i) - f(x_i + \varepsilon_j)| - \log C}{\log \varepsilon_j}.$$

We know $\hat{a}_i^{(j)}$ is always an **upper estimate** of α. Thus a reasonable guess of α would be:

$$\hat{\alpha} = \inf_{i,j} \hat{a}_i^{(j)}.$$

Since $\hat{a}_i^{(j)} \sim a_i^{(j)}$ as $j \to \infty$, we can replace our estimate of α by

$$\hat{\alpha} = \lim_{j\to\infty} \inf_i \hat{a}_i^{(j)}.$$

The limit, however, converges very slowly. As ε_j decreases exponentially, $|\log \varepsilon_j|$ only increases linearly, so the error terms decreases very slowly.

1.2.2 Find α and C by Curve Fitting

Rewrite equation (2) as

$$\alpha = \inf_{x,\varepsilon} \frac{\log|f(x) - f(x+\varepsilon)| - \log C}{\log \varepsilon} \quad \text{where } \varepsilon > 0.$$

Hence

$$\alpha = \inf_\varepsilon \left(\inf_x \frac{\log|f(x) - f(x+\varepsilon)|}{\log \varepsilon} - \frac{\log C}{\log \varepsilon} \right).$$

The next claim justifies us to take smaller and smaller ε, and implies that α can be bounded from below arbitrarily closely in finitely many steps.

Proposition 1.

$$\alpha = \lim_{\varepsilon'\to 0} \inf_{\varepsilon<\varepsilon'} \left(\inf_x \frac{\log|f(x) - f(x+\varepsilon)|}{\log \varepsilon} - \frac{\log C}{\log \varepsilon} \right).$$

Proof. Fix some ε' such that for some $\delta > 0$ and all $\varepsilon < \varepsilon'$,

$$\inf_x \frac{\log|f(x) - f(x+\varepsilon)|}{\log \varepsilon} - \frac{\log C}{\log \varepsilon} > \alpha + \delta.$$

We claim that f is actually $(\alpha + \delta)$-Hölder continuous, which results in a contradiction. It suffices to show that there exists some constant C such that for all ε where $\varepsilon \geq \varepsilon'$,

$$|f(x) - f(x+\varepsilon)| \leq C\varepsilon^{\alpha+\delta}.$$

But since f is α-Hölder continuous, there exists some $C' > 0$ such that

$$|f(x) - f(x+\varepsilon)| \leq C'\varepsilon^{\alpha}.$$

Take $C = \frac{C'}{(\varepsilon')^{-\delta}}$, and we have

$$|f(x) - f(x+\varepsilon)| \leq C(\varepsilon')^{\delta}\varepsilon^{\alpha} \leq C\varepsilon^{\delta}\varepsilon^{\alpha} = C\varepsilon^{\alpha+\delta}.$$

□

To get a even faster approximation, we want to fix some particular ε and take infimum only over x. We want to be assured that dropping the infimum over ε when calculating α gives a good enough approximation. This motivates the next proposition:

Proposition 2. If f is a random process with independent increment, and is scale invariant in the sense that for any increments ε and ε', the scaled increments are equal in distribution, i.e. $\frac{f(x+\varepsilon)-f(x)}{\varepsilon} \overset{d}{=} \frac{f(x'+\varepsilon')-f(x')}{\varepsilon'}$, then for all $\delta > 0$, we have the bound

$$\Pr\left[\left|\inf_x \frac{\log|f(x)-f(x+\varepsilon)|}{\log\varepsilon} - \frac{\log C}{\log\varepsilon} - \alpha\right| > \delta\right] < O(\varepsilon)$$

Proof. We assume $\alpha + \delta < 1$, and $\varepsilon < 1$.

$$\begin{aligned}
&\Pr\left[\inf_x \frac{\log|f(x)-f(x+\varepsilon)|}{\log\varepsilon} - \frac{\log C}{\log\varepsilon} - \alpha < \delta\right] \\
&\geq \Pr\left[\frac{\log|f(x)-f(x+\varepsilon)|}{\log\varepsilon} - \frac{\log C}{\log\varepsilon} - \alpha < \delta\right] \\
&= \Pr\left[|f(x)-f(x+\varepsilon)| > C\varepsilon^{\alpha+\delta}\right]
\end{aligned}$$

Now we choose some $\varepsilon' < \varepsilon$, and note that this implies $C\varepsilon'^{\alpha+\delta}\frac{\varepsilon}{\varepsilon'} > C\varepsilon^{\alpha+\delta}$, so

$$\begin{aligned}
&\geq \Pr\left[|f(x)-f(x+\varepsilon)| > C\varepsilon'^{\alpha+\delta}\frac{\varepsilon}{\varepsilon'}\right] \\
&= \Pr\left[|f(x')-f(x'+\varepsilon')| > C\varepsilon'^{\alpha+\delta}\right] && \text{using scale invariance} \\
&\geq \Pr\left[\frac{\log|f(x')-f(x'+\varepsilon')|}{\log\varepsilon'} - \frac{\log C}{\log\varepsilon'} - \alpha < \delta\right].
\end{aligned}$$

Using the previous proposition, we can choose small enough ε' to make this probability arbitrarily close to 1. Thus for all $\delta > 0$,

$$\Pr\left[\left|\inf_x \frac{\log|f(x)-f(x+\varepsilon)|}{\log\varepsilon} - \frac{\log C}{\log\varepsilon} - \alpha\right| > \delta\right] = 0$$

□

Using the approximation above, for some specific ε, we can approximate α by

$$\alpha = \inf_x \frac{\log|f(x)-f(x+\varepsilon)|}{\log\varepsilon} - \frac{\log C}{\log\varepsilon} + O(\varepsilon).$$

We then use $\inf_i a_i^{(j)}$ as an upper estimation:

Theorem 3. Assume (some condition). Then for any specific ε_j,

$$\inf_{i\in[n]} a_i^{(j)} < \inf_x \frac{\log|f(x)-f(x+\varepsilon_j)|}{\log\varepsilon_j} + O\left(\frac{1}{\log n}\right).$$

Proof. To be done. Need to work out what assumptions I need. □

Put all these together, we have a curve to fit, with error terms bounded and going to zero:

$$\alpha + \frac{\log C}{\log \varepsilon_j} = \inf_i a_i^{(j)} + O(\varepsilon) + O\left(\frac{1}{\log n}\right). \tag{3}$$

More work needed to turn this into an error bound for α, by considering the curve regression error.

Thus our testing strategy is:

1. Pick some small ε_1 and randomly pick $x_1^{(1)}, \dots, x_n^{(1)}$ in the domain of interested function.
2. Calculate
$$a_i^{(j)} := \frac{\log \left| f(x_i^{(j)}) - f(x_i^{(j)} + \varepsilon_j) \right|}{\log \varepsilon_j}.$$
3. Calculate $\inf_i a_i^{(j)}$
4. Do the same for successively smaller $\varepsilon_2, \dots, \varepsilon_m, \dots$ that approach 0.
5. Fit the curve (3). Then read the α and C.

References

Durrett, Richard (2010). *Probability: Theory and Examples*. 4th ed. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge ; New York: Cambridge University Press. 428 pp. ISBN: 978-0-521-76539-8.