AM 2C – Teste 2 2022.1 Resolução

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Conteúdo

Grupo I

Questão 1

Considere o conjunto

$$\boldsymbol{A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 1 \wedge (x-1)^2+y^2 \leq 1\}}$$

E seja L a fronteira de A percorrida no sentido direto. O integral de linha

$$\int_L \boldsymbol{(x^3 - y^2/2)\,\mathrm{d}x + (y^5-1)\,\mathrm{d}y}$$

pode ser calculado, utilizando coordenadas polares, a partir do seguinte integral repetido:

$$(x,y) : x^2+y^2 = (x-1)^2+y^2 \implies (x,y) : |x| = |(x-1)| \implies$$
$$\implies (x,y) = (0.5, \pm\sqrt{3}/2) \implies$$
$$\implies (\rho,\theta) = \left(\sqrt{(0.5)^2 + (\sqrt{3}/2)^2}, \arccos\left(\frac{\sqrt{3}/2}{\rho}\right)\right) = (1, \pm\pi/3)$$

$$\begin{cases} 1 = x^2+y^2 = (\rho\,\cos(\theta))^2 + (\rho\,\sin(\theta))^2 = \rho^2 \\ 1 = (x-1)^2+y^2 = \rho^2 - 2\,\rho\,\cos(\theta) + 1 \implies \rho = 2\,\cos(\theta) \end{cases}$$

$$(x-1)^2 + y^2 = (\rho\,\cos(\theta) - 1)^2 + (\rho\,\sin(\theta))^2 =$$
$$= \rho^2\,\cos^2(\theta) - 2\rho\,\cos(\theta) + 1 + \rho^2\,\sin^2(\theta) = \rho^2 - 2\rho\,\cos(\theta) + 1 = 1 \implies$$
$$\implies \rho = 2\cos\theta$$

$$|\mathrm{J}| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2(\theta) + \rho\sin^2(\theta) = \rho$$

$$\int_L (x^3 - y^2/2)\,\mathrm{d}x + (y^5-1)\,\mathrm{d}y \overset{Riemmann-Green}{=}$$
$$= \iint_A \left(\frac{\partial\psi(x,y)}{\partial x} - \frac{\partial\varphi(x,y)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y =$$
$$= \iint_A \left(\frac{\partial}{\partial x}\left(y^5-1\right) - \frac{\partial}{\partial y}\left(x^3 - y^2/2\right)\right)\mathrm{d}x\,\mathrm{d}y =$$
$$= \iint_A y\,\mathrm{d}x\,\mathrm{d}y =$$
$$= \int_{-\pi/3}^{\pi/3}\int_1^{2\,\cos\theta} (\rho\,\sin\theta)\,(\rho)\,\mathrm{d}\rho\,\mathrm{d}\theta$$

Questão 2

Utilizando coordenadas polares, o volume de um domínio fechado $\mathbb{D}$, limitado superiormente pela semisuperfície esférica $z = \sqrt{1 - x^2 - y^2}$, inferiormente pela superfície cónica $z = 1 - \sqrt{x^2 + y^2}$ e compreendida entre os planos $y = 0$ e $y = x$ pode ser calculado, utilizando coordenadas polares, a partir do seguinte integral repetido:

$$z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - (\rho \cos\theta)^2 - \rho \sin\theta^2} = \sqrt{1 - \rho^2}$$

$$z = 1 - \sqrt{x^2 + y^2} = 1 - \sqrt{(\rho \cos\theta)^2 + (\rho \sin\theta)^2} = 1 - \rho$$

$$z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - (1 - z)^2} = \sqrt{2z - z^2} \implies z^2 = 2\,z - z^2 \implies$$
$$z - 1 = 0 = -\sqrt{x^2 + y^2} \implies (x, y) = (0, 0) \implies$$
$$\implies (0, 0, 1) \to () \implies$$

$$\implies \int_0^{\pi/4} \int_0^1 (\sqrt{1 - \rho^2} - (1 - \rho))\, \rho \;\mathrm{d}\rho \;\mathrm{d}\theta = \int_0^{\pi/4} \int_0^1 (\sqrt{1 - \rho^2} - 1 + \rho)$$

Questão 3

O integral repetido

$$\int_{-2}^{0}\int_{0}^{x^2} x^2 \,\mathrm{d}y\,\mathrm{d}x + \int_{0}^{2}\int_{0}^{x^2} x^2 \,\mathrm{d}y\,\mathrm{d}x$$

Ultilizando a ordem de integração inversa da apresentada, pode ser calculado a partir de:

$$\left\{\begin{array}{l} y_1 : 0 \to x^2 \\ x_1 : -2 \to 0 \\ y_2 : 0 \to x^2 \\ x_2 : 0 \to 2 \end{array}\right\} \implies \left\{\begin{array}{l} x_1 : -2 \to -\sqrt{y} \\ y_1 : 0 \to (-2)^2 \\ x_2 : \sqrt{y} \to 2 \\ y_2 : 0 \to 2^2 \end{array}\right\}$$

$$\left\{(x,y)\in\mathbb{R}^2 : \begin{pmatrix} 0 \le y \le 4 & \wedge \\ \wedge -2 \le x \le -\sqrt{y} & \end{pmatrix}\right\} \cup \left\{(x,y)\in\mathbb{R}^2 : \Bigg(\begin{array}{l} 0 \le y \le 4 \\ \wedge \sqrt{y} \le x \le \end{array}\right.$$

$$\therefore \int_{0}^{4}\int_{-2}^{-\sqrt{y}} x^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{0}^{4}\int_{\sqrt{y}}^{2} x^2 \,\mathrm{d}x\,\mathrm{d}y$$