AM3C – Exam 2023.1.3 Resolution

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Conteúdo

Grupo I

Questão 1

A equação diferencial linear de primeira ordem

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} + \frac{\mathbf{1}}{\mathbf{4}}\,\boldsymbol{x}\,\boldsymbol{y} = \frac{\mathbf{1}}{\mathbf{4}}\,\boldsymbol{x}^{\mathbf{3}}$$

Com acondição $y(0) = -4$ tem como solução:

Resposta

$$y = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int \left(\frac{1}{4}\,x^3\right) \varphi(x)\ \mathrm{d}x =$$

Using (1.3)

$$= \frac{c_0}{\left(c_1\,e^{x^2/8}\right)} + \frac{1}{\left(c_1\,e^{x^2/8}\right)} \int \left(\frac{1}{4}\,x^3\right) \left(c_1\,e^{x^2/8}\right)\ \mathrm{d}x =$$

$$= \frac{c_0}{\left(c_1\,e^{x^2/8}\right)} + \frac{1}{e^{x^2/8}} \int \left(\frac{1}{4}\,x^3\right) \left(e^{x^2/8}\right)\ \mathrm{d}x =$$

Using (1.4)

$$= c_2\,e^{-x^2/8} + \frac{1}{e^{x^2/8}} \left(\left(x^2 - 8\right) e^{x^2/8}\right) =$$

$$= c_2\,e^{-x^2/8} + x^2 - 8 = \tag{1.1}$$

Using (1.2)

$$= 4\,e^{-x^2/8} + x^2 - 8$$

$$c_2 = c_0/c_1$$

$$y(0) = c_2\,e^{-0^2/8} + 0^2 - 8 = c_2 - 8 = -4 \implies c_2 = 4 \tag{1.2}$$

$$\varphi(x) = \exp\left(\int \frac{1}{4}\,x\ \mathrm{d}x\right) = \exp\left(\frac{1}{4}\left(\frac{x^2}{2} + c\right)\right) = \exp\left(\frac{c}{4}\right)\ \exp\left(\frac{x^2}{8}\right) = c_1\,e^{\frac{x^2}{8}}; \tag{1.3}$$

$$c_1 = e^{c/4}$$

$$P\left(\left(\frac{1}{4}\,x^3\right) \left(e^{x^2/8}\right)\right) = P\left(\left(x^2\right) \left(e^{x^2/8}\,\frac{x}{4}\right)\right) = P\left(\left(x^2\right) \left(e^{x^2/8}\right)'\right) =$$

$$= x^2\,P\left(\left(e^{x^2/8}\right)'\right) - P\left(P\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2/8}\right)\right) \frac{\mathrm{d}x^2}{\mathrm{d}x}\right) =$$

$$= x^2\,e^{x^2/8} - P\left(e^{x^2/8}\,2\,x\right) = x^2\,e^{x^2/8} - 8\,P\left(e^{x^2/8}\,x/4\right) =$$

$$= \left(x^2 - 8\right) e^{x^2/8} \tag{1.4}$$

Questão 2

A equação diferencial

$$\boldsymbol{3\,x\,y^2\ \mathrm{d}x + 4\,x^2\,y\ \mathrm{d}y = 0}$$

admite um fator integrante da forma $\varphi(x,y) = x\,y^k$, em que k é uma constante real. Encontre k

Resposta

$$k : \varphi(x,y) = x\,y^k \implies$$

$$\implies (x\,y^k)\,3\,x\,y^2\ \mathrm{d}x + (x\,y^k)\,4\,x^2\,y\ \mathrm{d}y = 0 \implies$$

$$\implies \frac{\partial}{\partial y}((x\,y^k)\,3\,x\,y^2) = \frac{\partial}{\partial y}(3\,x^2\,y^{2+k}) = (2+k)\big(3\,x^2\,y^{1+k}\big) =$$

$$= \frac{\partial}{\partial x}((x\,y^k)\,4\,x^2\,y) = \frac{\partial}{\partial x}(4\,x^3\,y^{1+k}) = 3\big(4\,x^2\,y^{1+k}\big) \implies$$

$$\implies k = (12-6)/3 = 2$$

Questão 3

Designando $\frac{\mathrm{d}y}{\mathrm{d}x}$ por p a solução geral da equação de Lagrange

$$\boldsymbol{y = -2\,x\,\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

na forma paramétrica é...

Resposta

Equações paramétricas: $\begin{cases} x(y') = x(p) \\ y(y') = y(p) = -2\,x\,p + \frac{1}{2}\,p^2 \end{cases}$

Using (1.5)

$x(p) = \dfrac{c_0}{p^{2/3}} + \dfrac{1}{5}\,p$

Solving (1.8)

$$x = \frac{c_0}{\varphi(p)} + \frac{1}{\varphi(p)} \int \frac{1}{3}\,\varphi(p)\;\mathrm{d}p =$$

Using (1.6)

$$= \frac{c_0}{p^{2/3}} + \frac{1}{p^{2/3}} \int \frac{1}{3}\,p^{2/3}\;\mathrm{d}p =$$

Using (1.7)

$$= \frac{c_0}{p^{2/3}} + \frac{1}{p^{2/3}}\,\frac{1}{5}\,p^{5/3} = \frac{c_0}{p^{2/3}} + \frac{1}{5}\,p \tag{1.5}$$

$$\varphi(p) = \exp\left(\int \frac{2}{3\,p}\;\mathrm{d}p\right) = \exp\left(\frac{2}{3}\ln p\right) = p^{2/3} \tag{1.6}$$

$$P\left(\frac{1}{3}\,p^{2/3}\right) = \frac{1}{3}\,\frac{3}{5}\,p^{5/3} = \frac{1}{5}\,p^{5/3} \tag{1.7}$$

$$p = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(-2\,x\,p + \frac{1}{2}\,p^2\right) = -2\left(x\,\frac{\mathrm{d}p}{\mathrm{d}x} + p\right) + \frac{1}{2}\,2\,p\,\frac{\mathrm{d}p}{\mathrm{d}x} \implies$$

$$\implies 3\,p = (-2\,x + p)\,\frac{\mathrm{d}p}{\mathrm{d}x} \implies 3\,p\,\frac{\mathrm{d}x}{\mathrm{d}p} = -2\,x + p \implies$$

$$\implies \frac{\mathrm{d}x}{\mathrm{d}p} + \frac{2\,x}{3\,p} = \frac{1}{3} \tag{1.8}$$

Questão 4

Um sistema equivalente ao seguinte sistema de equações diferenciais lineares de coeficientes constantes é...

$$\begin{cases} (D-2)\,x + (D^2+3\,D)\,y = t+1 \\ \begin{pmatrix} (5\,D^2-12\,D+4)\,x \\ +(5\,D^3+13\,D^2-7\,D-3)\,y \end{pmatrix} = -2\,t+4 \end{cases}$$

(D designa o operador de derivação a ordem t)

Resposta

$(D-2)\,x + (D^2+3\,D)\,y = t+1$

$\begin{pmatrix} (5\,D^2-12\,D+4)\,x \\ +(5\,D^3+13\,D^2-7\,D-3)\,y \end{pmatrix}$