Capítulo 4 - Teorema Limite Central

Resolução de alguns exercícios

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4.2 X_i = nº de sismos no Japão no mês i, $\quad i = 1, \ldots, 480$

$\underset{iid}{\sim} X: \quad E(X) = 5$ e $V(X) = 2^2 = 4$

pelo que se tem

- $E(X_i) = \mu = 5, \;\; i = 1, \ldots, 480$
- $V(X_i) = \sigma^2 = 2^2, \;\; i = 1, \ldots, 480$

Nestas condições

$$S_{480} = \sum_{i=1}^{480} X_i = \text{nº total de sismos no Japão em 40 anos} \underset{TLC}{\overset{a}{\sim}} N(E(S_{480}), V(S_{480}))$$

onde

$$E(S_{480}) = E\Big(\sum_{i=1}^{480} X_i\Big) = \sum_{i=1}^{480} E(X_i) \underset{X_i ident.dists.}{=} \sum_{i=1}^{480} E(X)$$
$$= 480 \times E(X) = 480 \times 5 = 2400$$

e

$$V(S_{480}) = V\Big(\sum_{i=1}^{480} X_i\Big) \underset{X_i indeps.}{=} \sum_{i=1}^{480} V(X_i) \underset{X_i ident.dists.}{=} \sum_{i=1}^{480} V(X)$$
$$= 480 \times V(X) = 480 \times 4 = 1920$$

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Pelo Teorema Limite Central, sabemos que

$$Z = \frac{S_{480} - 2400}{\sqrt{1920}} \overset{a}{\sim} N(0,1)\,.$$

Assim,

$$P(S_{480} \leq 2300) = P\left(\frac{S_{480} - 2400}{\sqrt{1920}} \leq \frac{2300 - 2400}{\sqrt{1920}}\right) \simeq P(Z \leq -2.28)$$

$$\underset{TLC}{\overset{}{\underset{a}{\approx}}} \; \Phi(-2.28) = 1 - \Phi(2.28) \simeq 0.0113$$

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4.5 X = peso de um envelope (gr.) v.a. : $E(X) = 1$ e $V(X) = 0.05^2$

a) X_i = peso do envelope i, $\quad i = 1, \ldots, 100$

$\underset{iid}{\sim} X$

e portanto temos

- $E(X_i) = \mu = 1\, g, \;\; i = 1, \ldots, 100$
- $V(X_i) = \sigma^2 = 0.05^2\, g^2, \;\; i = 1, \ldots, 100$

Nestas condições

$$S_{100} = \sum_{i=1}^{100} X_i = \text{peso total dos 100 envelopes} \underset{TLC}{\overset{a}{\sim}} N(E(S_{100}), V(S_{100}))$$

onde

$$E(S_{100}) = E\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} E(X_i) \underset{X_i ident.dists.}{=} \sum_{i=1}^{100} E(X)$$
$$= 100 \times E(X) = 100 \times 1 = 100$$

e

$$V(S_{100}) = V\left(\sum_{i=1}^{100} X_i\right) \underset{X_i indeps.}{=} \sum_{i=1}^{100} V(X_i) \underset{X_i ident.dists.}{=} \sum_{i=1}^{100} V(X)$$
$$= 100 \times V(X) = 100 \times 0.05^2 = 0.25$$

Pelo Teorema Limite Central, sabemos que

$$Z = \frac{S_{100} - 100}{\sqrt{0.25}} \overset{a}{\sim} N(0,1)\,.$$

Assim,

$$P(S_{100} > 100.5) = P\left(\frac{S_{100} - 100}{\sqrt{0.25}} > \frac{100.5 - 100}{\sqrt{0.25}}\right) = P(Z > 1)$$

$$\underset{TLC}{\underset{a}{\approx}} \; 1 - \Phi(1) \simeq 0.1587$$

4.5 b) Sejam

$$\overline{X} = \frac{1}{100}\sum_{i=1}^{100} X_i \underset{TLC}{\underset{a}{\sim}} N(E(\bar{X}), V(\bar{X})) \equiv N(1, 0.05^2/100) \text{ e } Z = \frac{\overline{X} - 1}{\sqrt{0.05^2/100}} \underset{TLC}{\underset{a}{\sim}} N(0,1)$$

então

$$\begin{aligned} P\left(|\overline{X} - 1| > 0.01\right) &= P\left(\left|\frac{\overline{X} - 1}{\sqrt{0.05^2/100}}\right| > \frac{0.01}{\sqrt{0.05^2/100}}\right) \\ &= P(|Z| > 2) = P(Z > 2) + P(Z < -2) \\ &\underset{TLC}{\underset{a}{\approx}} 2(1 - \Phi(2)) \simeq 0.0456. \end{aligned}$$

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4.6 $X_i =$ nº de flores produzidas pelo bolbo i, $\quad i = 1, \ldots, 240$

$$\underset{iid}{\sim} X : \quad E(X) = 4 \text{ e } V(X) = 2^2 = 4$$

pelo que se tem

- $E(X_i) = \mu = 4, \; i = 1, \ldots, 240$
- $V(X_i) = \sigma^2 = 4,, \; i = 1, \ldots, 240$

Nestas condições

$$S_{240} = \sum_{i=1}^{240} X_i = \text{nº total de flores produzidas pelos 240 bolbos} \underset{TLC}{\overset{a}{\sim}} N(E(S_{240}), V(S_{240}))$$

onde

$$E(S_{240}) = E(\sum_{i=1}^{240} X_i) = \sum_{i=1}^{240} E(X_i) \underset{X_i ident.dists.}{=} \sum_{i=1}^{240} E(X)$$
$$= 240 \times E(X) = 240 \times 4 = 960$$

e

$$V(S_{240}) = V(\sum_{i=1}^{240} X_i) \underset{X_i indeps.}{=} \sum_{i=1}^{240} V(X_i) \underset{X_i ident.dists.}{=} \sum_{i=1}^{240} V(X)$$
$$= 240 \times V(X) = 240 \times 4 = 960$$

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Pelo Teorema Limite Central, sabemos que

$$Z = \frac{S_{240} - 960}{\sqrt{960}} \overset{a}{\sim} N(0,1)\,.$$

Assim,

$$P(S_{240} > 1000) = P\left(\frac{S_{240} - 960}{\sqrt{960}} > \frac{1000 - 960}{\sqrt{960}}\right) \simeq 1 - P(Z \leq 1.29)$$

$$\underset{TLC}{\overset{a}{\approx}} 1 - \Phi(1.29) \simeq 0.0985$$