# AM3C – Teste 2024.1 Resolução

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10 de janeiro de 2025

## Conteúdo

# Grupo I

# Questão 1

A equação diferencial linear de primeira ordem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\cos(x)}{\sin(x)}\, y = -x; \quad x \in \left]0, \pi\right[$$

Tem como solução geral

- [x] $y = \frac{c}{\sin x} - x\,\frac{\cos x}{\sin x} + 1$
- [ ] $y = \frac{c}{\sin x} + x\,\frac{\sin x}{\cos x} - 1$
- [ ] $y = \frac{c}{\sin x} - x\,\frac{\cos x}{\sin x}$
- [ ] $y = \frac{c}{\cos x} - x\,\frac{\cos x}{\sin x} + 1$
- [ ] $y = \frac{c}{\cos x} + x\,\frac{\cos x}{\sin x} - 1$
- [ ] $y = \frac{c}{\cos x} + x\,\frac{\sin x}{\cos x}$

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## Resposta (1.1)

General solution

$$y = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\, \mathrm{P}_x\left(-x\,\varphi(x)\right) =$$

using (1.2) (1.3)

$$= \frac{c_0}{c_1\,\sin x} + \frac{1}{c_1\,\sin x}\left(-c_1\left(x\,\cos x - (c_2 + \sin x)\right)\right) =$$

$$= \frac{c_0}{c_1\,\sin x} + \frac{c_2}{\sin x} - x\,\frac{\cos x}{\sin x} + 1 = \frac{c_4}{\sin x} - x\,\frac{\cos x}{\sin x} + 1 \tag{1.1}$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\left(\frac{\cos x}{\sin x}\,\mathrm{d}x\right)\right) =$$

$\mathrm{d}(\sin x) = \cos x\,\mathrm{d}x$

$$= \exp\left(\int\left(\frac{\mathrm{d}(\sin x)}{\sin x}\right)\right) = \exp\left(c_0 + \ln\left(\sin x\right)\right) = c_1\,\sin x \tag{1.2}$$

Integrating

$$\mathrm{P}_x\left(-x\,\varphi(x)\right) =$$

using (1.2)

$$= \mathrm{P}_x\left(-x\,c_1\,\sin x\right) =$$

$$\mathrm{P}_x\left(u\,v'\right) = u\,v - \mathrm{P}_x\left(u'\,v\right)\begin{cases} u = x \\ v = \cos x \end{cases}$$

$$= x\,\cos x - \mathrm{P}_x\left(\cos x\right) = -c_1\left(x\,\cos x - (c_2 + \sin x)\right) \tag{1.3}$$

# Questão 2

A solução da equação de Bernoulli

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} + \boldsymbol{y} = \frac{\mathbf{1}}{\boldsymbol{y}}$$

que satisfaz a condição $y(0) = 2$, é:

- ☐ $y = \sqrt{e^{+2\,x} + 3}$
- ☐ $y = \sqrt{3\,e^{+2\,x} + 1}$
- ☐ $y = \sqrt{2\,e^{+2\,x} + 2}$
- ☐ $y = \sqrt{e^{-2\,x} + 3}$
- ■ $y = \sqrt{3\,e^{-2\,x} + 1}$
- ☐ $y = \sqrt{2\,e^{-2\,x} + 2}$

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## Resposta (1.6)

General solution

$$y = z^{1/2} = \tag{1.4}$$

$$= \left(\frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\ \mathrm{P}_x\left(2 * 1\,\varphi(x)\right)\right)^{1/2} =$$

using (1.8) (1.9)

$$= \left(\frac{c_0}{c_1\,e^{2\,x}} + \frac{1}{c_1\,e^{2\,x}}\,c_1\,e^{2\,x}\right)^{1/2} = \left(\frac{c_0}{c_1\,e^{2\,x}} + 1\right)^{1/2} = \left(c_2\,e^{-2\,x} + 1\right)^{1/2} = \tag{1.5}$$

using (1.7)

$$= \left(3\,e^{-2\,x} + 1\right)^{1/2} \tag{1.6}$$

Finding $c_2$

$$y(0) = 2 =$$

using (1.5)

$$= \left(c_2\,e^{-2*0} + 1\right)^{1/2} \implies c_2 = 4 - 1 = 3 \tag{1.7}$$

Bernoulli's substitution

$$y' + y = y^{-1} \implies$$

using (1.4)

$$\implies z' + 2\,z = 2$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\left(2\right)\right) = c_1\,e^{2\,x} \tag{1.8}$$

Integrating

$$\mathrm{P}_x\left(2 * 1\,\varphi(x)\right) =$$

using (1.8)

$$= 2\,c_1\,e^{2\,x}/2 = c_1\,e^{2\,x} \tag{1.9}$$

# Questão 3

A equação differencial

$$\boldsymbol{(5\,x\,y^2 - 2\,y)\ \mathrm{d}x + (3\,x^2\,y - x)\ \mathrm{d}y = 0}$$

Admite um fator integrante na forma $\phi(x, y) = x^m\,y^n$, com $m, n \in \mathbb{N}$. Então:

- ☐ $m = 3, n = 2$
- ☐ $m = 2, n = 2$
- ☐ $m = 1, n = 3$
- ☐ $m = 1, n = 1$
- ☐ $m = 2, n = 1$
- ■ $m = 3, n = 1$

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## Resposta (1.10)

Finding $m, n$

$$\begin{gathered}
\frac{\partial u\,\varphi(x)}{\partial y} = \frac{\partial}{\partial y}\big(x^m\,y^n\,(5\,x\,y^2 - 2\,y)\big) = \frac{\partial}{\partial y}\big(5\,x^{m+1}\,y^{n+2} - 2\,y^{n+1}\,x^m\big) = \\
= 5\,x^{m+1}\,(n+2)\,y^{n+1} - 2\,(n+1)\,y^n\,x^m = \\
= \frac{\partial v\,\varphi(x)}{\partial x} = \frac{\partial}{\partial x}\big(x^m\,y^n\,(3\,x^2\,y - x)\big) = \frac{\partial}{\partial x}\big(3\,x^{2+m}\,y^{n+1} - x^{m+1}\,y^n\big) = \\
= 3\,(2+m)\,x^{1+m}\,y^{n+1} - (m+1)\,x^m\,y^n \implies \\
\implies \begin{cases} 2(n+1) = m+1 \implies m = 2\,n+1 = 3 \\ 5\,(n+2) = 3(m+2) \implies n = 1 \end{cases}
\end{gathered} \tag{1.10}$$

# Questão 4

A equação diferencial linear homogénea

$$\boldsymbol{x\, y'' + x^2\, y' + 4\, y = 0, \quad x > 0}$$

Tem como solução geral a função $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$. Então a equação não homogénea

$$\boldsymbol{x\, y'' + x^2\, y' + 4\, y = x^3}$$

admite como solução geral a função $y(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$, onde as funções $c_1(x), c_2(x)$ são determinadas a partir do sistema

- ■ $\begin{cases} c_1'(x)\, y_1 + c_2'(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x^2 \end{cases}$
- □ $\begin{cases} c_1(x)\, y_1 + c_2(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x^2 \end{cases}$
- □ $\begin{cases} c_1'(x)\, y_1 + c_2'(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x \end{cases}$
- □ $\begin{cases} c_1(x)\, y_1 + c_2(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x \end{cases}$
- □ $\begin{cases} c_1'(x)\, y_1 + c_2'(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = 1 \end{cases}$
- □ $\begin{cases} c_1(x)\, y_1 + c_2(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = 1 \end{cases}$

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## Resposta (1.11)

Crammers equation system

$$\left\{\begin{array}{lcl} c_1'(x)\ \mathrm{D}_x^0\, y_1(x) + c_2'(x)\ \mathrm{D}_x^0\, y_2(x) & = & 0 \\ c_1'(x)\ \mathrm{D}_x\, y_1(x) + c_2'(x)\ \mathrm{D}_x\, y_2(x) & = & \frac{x^3}{x} = x^2 \end{array}\right\} \tag{1.11}$$

## Questão 5

Acerca de uma função $f(x)$ definida e com derivadas até à segunda ordem em $\mathbb{R}_0^+$ sabe-se que admite transformada de Laplace $F(s)$, que $f(0) = 1, f'(0) = -2.$ Então a trasnformada de Laplace da função

$$\boldsymbol{e^{-t}\, f''(t) + t\, f'(t)}$$

é:

- ☐ $(s+1)^2\, F(s+1) - s + 2 + s\, F'(s)$
- ☐ $s^2\, F(s) - s + 1 + s\, F'(s) - F(s)$
- ☐ $(s+1)^2\, F(s+1) - s + 1 - s\, F'(s) - F(s)$
- ☐ $s^2\, F(s) - s + 1 + s\, F'(s+1) - F(s+1)$
- ☐ $(s+1)^2\, F(s+1) - s + 1 + s\, F'(s) + F(s)$
- ☐ $s^2\, F(s) - s + 1 + s\, F'(s+1) + F(s+1)$

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## Resposta

$$\boldsymbol{(s+1)^2\, F(s+1) - s + 1 - s\, F'(s) - F(s)}$$

# Grupo II

Determine a solução geral da equação diferencial linear homogénea e de coeficientes constantes

$$\frac{\mathrm{d}^2 \boldsymbol{y}}{\mathrm{d}\boldsymbol{x}^2} + \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} - \boldsymbol{y}\, \mathbf{6} = \mathbf{0}$$

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## Resposta

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## Resposta (2.12)

General solution for $y$

$$\begin{aligned} y &= \\ &\qquad \text{using (2.13)} \\ &= e^{+2\,x}\, c_0 + e^{-3\,x}\, c_1 \end{aligned} \tag{2.12}$$

Mapping roots of (2.14) to solution

$$\begin{cases} r_0 = +2 \implies e^{+2\,x}\, c_0; \\ r_1 = -3 \implies e^{-3\,x}\, c_1 \end{cases} \tag{2.13}$$

Roots for characteristic equation for $y$

$$\begin{aligned} & P = \mathrm{D}_x^2 + \mathrm{D}_x - 6 \implies \\ & \qquad \mathrm{D}_x^i \to r^i \\ & \implies r^2 + r - 6 = 0 \implies r = \frac{-1 \pm \sqrt{1 - 4 * -6}}{2} = \frac{-1 \pm 5}{2} \implies \\ & \implies \begin{cases} r_0 = +2 \\ r_1 = -3 \end{cases} \end{aligned} \tag{2.14}$$

Utilizando o método da variação das constantes arbitrárias, determine a solução geral da equação não homogénea

$$\frac{\mathrm{d}^2 \boldsymbol{y}}{\mathrm{d}\boldsymbol{x}^2} + \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} - \boldsymbol{y}\,\mathbf{6} = -\mathbf{5}\,\boldsymbol{e}^{\mathbf{2}\,\boldsymbol{x}}\ \cos\boldsymbol{x}$$

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**Resposta** (2.15)

General solution

$$y = \qquad \text{using (2.12)}$$
$$= c_1(x)\,e^{+2\,x} + c_2(x)\,e^{-3\,x} = \qquad \text{using (2.17) (2.18)}$$
$$= (c_3 + \sin x)\,e^{+2\,x} + \left(\frac{e^{5\,x}}{26}\,(5\,\cos x + \sin x)\right)e^{-3\,x} =$$
$$= e^{2\,x}\left(\frac{5}{26}\,\cos x + \frac{27}{26}\sin x + c_3\right) \tag{2.15}$$

$y_1, y_2$ comes from (2.12) which is the homogeneous analogus equation for $y$

$$\begin{cases} y_1 = e^{+2\,x} \\ y_2 = e^{-3\,x} \end{cases} \tag{2.16}$$

Finding $c_1, c_2$

$$c_1(x) = \mathrm{P}_x\left(c_1'(x)\right) = \qquad \text{Using (2.19)}$$
$$= \mathrm{P}_x\left(-\cos x\right) = c_3 + \sin x; \tag{2.17}$$
$$c_2(x) = \mathrm{P}_x\left(c_2'(x)\right) = \qquad \text{Using (2.20)}$$
$$= \mathrm{P}_x\left(e^{5\,x}\ \cos x\right) = \qquad \mathrm{P}_x\left(u\,v'\right) = u\,v - \mathrm{P}_x\left(u'\,v\right)\begin{cases} u = e^{5\,x} \\ v = \sin x \end{cases}$$
$$= e^{5\,x}\ \sin x - \mathrm{P}_x\left(5\,e^{x}\ \sin x\right) = e^{5\,x}\ \sin x + 5\ \mathrm{P}_x\left(e^{5\,x}\left(-\sin x\right)\right) = \qquad \mathrm{P}_x\left(u\,v'\right) = u\,v - \mathrm{P}_x\left(u'\,v\right)\begin{cases} u = e^{5\,x} \\ v = \cos x \end{cases}$$
$$= e^{5\,x}\ \sin x + 5\left(e^{5\,x}\ \cos x - \mathrm{P}_x\left(5\,e^{5\,x}\ \cos x\right)\right) = e^{5\,x}\ \sin x + 5\,e^{5\,x}\ \cos x - 25\ \mathrm{P}_x\left(e^{5\,x}\ \cos\right.$$
$$\implies c_1 = \mathrm{P}_x\left(e^{5\,x}\ \cos x\right) = \frac{e^{5\,x}}{26}\,(5\,\cos x + \sin x) \tag{2.18}$$

$\mathrm{D}_x(c_1, c_2)$

$$\text{Using (2.22)}$$
$$c_1'(x) = \frac{1}{W(y_1, y_2)}\begin{vmatrix} 0 & \mathrm{D}_x^0 y_2 \\ \frac{-5\,e^{2\,x}\ \cos x}{1} & \mathrm{D}_x y_2 \end{vmatrix} = \qquad \text{using (2.21) (2.24)}$$
$$= \frac{1}{-5\,e^{-x}}\begin{vmatrix} 0 & e^{-3\,x} \\ -5\,e^{2\,x}\ \cos x & -3\,e^{-3\,x} \end{vmatrix} = \frac{5\,e^{-x}\ \cos x}{-5\,e^{-x}} = -\cos x; \tag{2.19}$$

$$\text{Using (2.22)}$$
$$c_2'(x) = \frac{1}{W(y_1, y_2)}\begin{vmatrix} \mathrm{D}_x^0 y_1 & 0 \\ \mathrm{D}_x y_1 & \frac{-5\,e^{2\,x}\ \cos x}{1} \end{vmatrix} = \qquad \text{using (2.21) (2.23)}$$
$$= \frac{1}{-5\,e^{-x}}\begin{vmatrix} e^{+2\,x} & 0 \\ 2\,e^{+2\,x} & -5\,e^{2\,x}\ \cos x \end{vmatrix} = \frac{-5\,e^{4\,x}\ \cos x}{-5\,e^{-x}} = e^{5\,x}\ \cos x \tag{2.20}$$

Solving Wronskiano

$$W(y_1, y_2) = \det\begin{bmatrix} y_1 & y_2 \\ \mathrm{D}_x\,y_1 & \mathrm{D}_x\,y_2 \end{bmatrix} = \qquad \text{using (2.23) (2.24)}$$
$$= \det\begin{bmatrix} +e^{+2\,x} & +e^{-3\,x} \\ +2\,e^{+2\,x} & -3\,e^{-3\,x} \end{bmatrix} = -3\,e^{-x} - 2\,e^{-x} = -5\,e^{-x} \tag{2.21}$$

Crammers equation system

$$\left\{\begin{matrix} c_1'(x)\ \mathrm{D}_x^0\,y_1(x) + c_2'(x)\ \mathrm{D}_x^0\,y_2(x) & = & 0 \\ c_1'(x)\ \mathrm{D}_x\,y_1(x) + c_2'(x)\ \mathrm{D}_x\,y_2(x) & = & \frac{-5\,e^{2\,x}\ \cos x}{1} \end{matrix}\right\} \tag{2.22}$$

Solving $\mathrm{D}_x(y_1, y_2)$

$$\mathrm{D}_x\,y_1 = \mathrm{D}_x\left(e^{+2\,x}\right) = +2\,e^{+2\,x}; \tag{2.23}$$
$$\mathrm{D}_x\,y_2 = \mathrm{D}_x\left(e^{-3\,x}\right) = -3\,e^{-3\,x} \tag{2.24}$$

# Grupo III

## Q1 a.

Determine todas as soluções da equação de Clairaut

$$y = x\,\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{3}$$

## Q1 b.

Utilizando a mudança de variável definida por $x = 1/t$, resolva a equação

$$\boldsymbol{y = -x\,\frac{\mathrm{d}y}{\mathrm{d}x} + x^6 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3, \quad x > 0}$$

**Sug:** Após a mudança de variável utilize (Q1 a.).

# Grupo IV

Utilize a transformada de Laplace para resolver o problema de valores iniciais

$$y'' + y' + y\,5/2 = \delta(t-2), \quad y(0) = 0, \quad y'(0) = 1$$

**Sug:** tenha em contra que $s^2 + s + 5/2 = (s+1/2)^2 + 9/4$.

# Grupo V

Considere a equação diferencial lienar de ordem $n$ e coeficientes constantes

$$\left(\mathrm{D}_{\boldsymbol{x}}^{\boldsymbol{n}} + \sum_{\boldsymbol{k=0}}^{\boldsymbol{n-1}} \boldsymbol{\alpha_k}\ \mathrm{D}_{\boldsymbol{x}}^{\boldsymbol{k}}\right) \boldsymbol{y} = \boldsymbol{e^{\alpha\, x}}, \boldsymbol{\alpha} \in \mathbb{R} \tag{5.25}$$

Seja $P(r) = r_n + \sum_{k=0}^{n-1} a_k\, r^k$. admitimos que $r = \alpha$ é raiz da equação $P(r) = 0$ com grau de multiplicidade um. Justifique $P'(\alpha) = 0$.

A equação (5.25) tem uma solução particular da forma

$$\bar{y} = \frac{c}{2\,P'(\alpha)}\,x\,e^{\alpha\,x}, \quad c \in \mathbb{R}$$

Sabendo que

$$\mathrm{D}_x^k\,(x\,e^{\alpha\,x}) = k\,\alpha^{k-1}\,e^{\alpha\,x} + \alpha^k\,x\,e^{\alpha\,x}, \quad \forall\,k \in \mathbb{N}, \quad \mathrm{D}_x^k = \frac{\mathrm{d}^k}{\mathrm{d}x^k}$$

Determine, justificando detalhadamente, o valor de c.