Ficha 1: Noções básicas de topologia na recta real

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March 29, 2021

Contents

Questão 1

$$\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\} \in \mathbb{R}$$

1-a) $A =]-1, \sqrt{2}\,]$

$\text{int}(A) = (-1, \sqrt{2});$

$\text{Ext}(A) = \mathbb{R} \backslash [\,1, \sqrt{2}\,];$

$\text{Front}(A) = \{-1, \sqrt{2}\}.$

Nota:

$\lim_{n \to \infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha$

1-b) $B = \{e^{-n} : n \in \mathbb{N}\} \cup \{(1 + 3/n)^n : n \in \mathbb{N}\}$

$\text{Int}(B) = \emptyset;$

$\text{Ext}(B) = \mathbb{R} \backslash B - \{e^3, 0\};$

$\text{Font}(B) = \{0, e^3\}$

1-c) $C = (0, 1) \cap \mathbb{Q}$

$\text{Int}(C) = \emptyset$

$\text{Ext}(C) = \mathbb{R} \backslash [0, 1]$

$\text{Font}(C) = [0, 1]$

1-d) $D = \{1/(n + \sqrt{n}) \ : \ n \in \mathbb{N}\} \cap \mathbb{Q}$

$$\frac{1}{n + \sqrt{n}} = \frac{1}{m^2 + m} \ \forall \{\{m, n\} \in \mathbb{N} : n = m^2\} \implies$$

$\text{Int}(D) = \emptyset$

$\text{Ext}(D) = \mathbb{R} \backslash D - \{0\}$

$\text{Font}(D) = D + \{0\}$

Questão 2

2-a) $f(x) = \ln(-x^2 + 2x)$

$$D = \{x \in \mathbb{R} : -x^2 + 2x > 0\}; \ -x^2 + 2x = x(-x + 2) > 0 \implies$$
$$\implies D = \{x \in \mathbb{R} : 0 < x < 2\} \implies$$
$$\implies \text{Sup}(D) = 2; \ \text{Inf}(D) = 0; \ \text{Max}(D) = \text{Min}(D) = \emptyset$$

2-b) $g(x) = \sqrt[6]{\pi^2 - x^2}\ \tan(x)$

$$D = \{x \in \mathbb{R} : \pi^2 - x^2 \geq 0\};\ \pi^2 - x^2 \geq 0 \implies x^2 \leq \pi^2 \implies$$
$$\implies D = \{x \in \mathbb{R} : |x| \leq \pi\} \implies$$
$$\implies \mathrm{Sup}(D) = \mathrm{Max}(D) = \pi;\ \mathrm{Inf}(D) = \mathrm{Min}(D) = -\pi$$

Questão 3

$$\boldsymbol{A = \{(1/n) : n \in \mathbb{N}\}}$$

3-a)

$$1 \in A \iff \exists n \in \mathbb{N} : 1/n = 1 \iff n = 1$$

$$1 \notin \mathrm{Acum}(A) \iff \nexists \epsilon \in \mathbb{R} : V_\epsilon(1) \cap A - \{1\} \neq \emptyset \iff$$
$$\iff V_\epsilon(1) \cap A - \{1\} = \emptyset\ \forall \epsilon \in \mathbb{R} : \epsilon < 0.5$$
$$0 \in \mathrm{Acum}(A) \iff \nexists \epsilon \in \mathbb{N} : V_\epsilon(0) \cap A - \{0\} = \emptyset \iff$$
$$\iff V_\epsilon(0) \cap A - \{0\} = [0, \epsilon] \cap \{1/n : n \in \mathbb{N}\} \neq \emptyset\ \forall \epsilon \in \mathbb{R}, \forall n \in \mathbb{N} \quad \iff$$
$$\iff \nexists \epsilon \in \mathbb{R} : \epsilon > 0 \wedge \epsilon < 1/n\ \forall n \in \mathbb{N}$$

3-b)

$A' = [-1, \sqrt{2}]$ $\qquad$ $B' = \{0, \epsilon^3\}$ $\qquad$ $C' = [0, 1]$ $\qquad$ $D' = \{0\}$

Questão 4

$$\boldsymbol{X} \subset \mathbb{R}$$

4-a)

$$x \in \mathrm{Fr}(X) \iff \{V_\epsilon(x) \cap X \neq \emptyset \wedge V_\epsilon(x) \not\subset X\}\ \forall\, \epsilon \in \mathbb{R} \implies$$
$$\implies \exists\, x \in \mathrm{Fr}(X) : x = V_\epsilon(x) \cap X;\ \ X' = \{x \in \mathbb{R} : V_\epsilon(x) \cap X - \{x\} \neq \emptyset\} \implies$$
$$\implies \exists\, x \in \mathrm{Fr}(X) : x \notin X'$$

4-b)

$$V_\epsilon(x) \cap X = \{x\} \implies V_\epsilon(x) \cap X \neq \emptyset \wedge V_\epsilon(x) \not\subset X \iff x \in \mathrm{Fr}(X)$$

4-c)

$$\mathrm{Fr}(\mathrm{Ext}(X)) = \{x \in \mathbb{R} : V_\epsilon(x) \cap \mathrm{Ext}(X) \neq \emptyset \wedge V_\epsilon(x) \not\subset \mathrm{Ext}(X)\};$$
$$\forall\ x \in \mathbb{R} : V_\epsilon(x) \cap X = V_\epsilon(x) - \{x\} \implies x \in \mathrm{Fr}(X) \wedge x \notin \mathrm{Fr}(\mathrm{Ext}(X));$$
$$\forall\ x \in \mathbb{R} : V_\epsilon(x) \cap X = \{x\} \implies x \in \mathrm{Fr}(X) \wedge x \notin \mathrm{Fr}(\mathrm{Int}(X)) \wedge x \in \mathrm{Fr}(\mathrm{Ext}(X));$$
$$\therefore \mathrm{Fr}(\mathrm{Ext}(X)) \neq \mathrm{Fr}(\mathrm{Int}(X)) \neq \mathrm{Fr}(X) \neq \mathrm{Fr}(\mathrm{Ext}(X))$$

4-d) Duvida

X é um conjunto fechado $\implies$ $\mathrm{Fr}(X) \subset X$;
$\mathbb{R}$ é um conjunto fechado $\implies$ $\mathrm{Fr}(\mathbb{R}) = \{-\infty, \infty\} \not\subset \mathbb{R}$

4-e)

$$X' = \{x \in \mathbb{R} : V_\epsilon(x) \cap X - \{x\} \neq \emptyset\} \implies (V_\epsilon(x) - \{x\}) \cap X = \emptyset\ \forall\, x \in \mathbb{R}\backslash X' \implies$$
$\implies \mathbb{R}\backslash X'$ é um grupo aberto $\implies$ X' é um grupo fechado

Questão 5

$$\boldsymbol{A = \left\{x \in \mathbb{R} : \frac{\ln(x^2+1)}{x^2-16} \geq 0\right\};\ B = \{x \in \mathbb{R} : |x^2 - 18| \leq 18\}}$$

5-a)

$$\begin{aligned} A &= \left\{x \in \mathbb{R} : \frac{\ln(x^2+1)}{x^2-16} \geq 0\right\} = \\ &= \{x \in \mathbb{R} : x^2+1 > 0 \wedge x^2 - 16 \neq 0 \wedge x^2 - 16 \geq 0\} = \{x \in \mathbb{R} : |x| > 4\} = \\ &= \{x \in \mathbb{R} \cap ((-\infty, -4) \cup (4, \infty))\} \\ B &= \{x \in \mathbb{R} : |x^2 - 18| \leq 18\} = \{x \in \mathbb{R} : x^2 \leq 36 \wedge x^2 \geq 0\} = \{x \in \mathbb{R} : |x| \leq 6\} \\ &= \{x \in \mathbb{R} \cap [-6, 6]\} \end{aligned}$$

5-b)

$A \cap B = [-6, -4) \cup (4, 6];$

$$\mathrm{Inf}(A \cap B) = \{4\};\ \mathrm{Min}(A \cap B) = \emptyset;\ \mathrm{Sup}(A \cap B) = \mathrm{Max}(A \cap B) = \{6\}$$

Questão 6

6-a) $\mathbf{Int}(X) = (0,1) \wedge X' = [0,1] \cup \{e\}$

$X = [0,1] \cup \{(1 + 1/x)^x : x \in \mathbb{N}\}$

6-b) $\mathbf{Ext}(X) = (-\infty, 0) \wedge \mathbf{Int}(X) = \emptyset$

$X = \{x \in \mathbb{Q} : x < 0\}$

6-c) $X' = \mathbb{Z}$

$$X = \left\{x + \frac{1}{y} : x \in \mathbb{Z} \wedge y \in \mathbb{N}\right\}$$

6-d) $X' = (0,1)$

$\nexists\, X : X'$ é um conjunto aberto

6-e) $\mathbf{Fr}(X) = [0,1]$

$X = [0,1] \cap \mathbb{Q}$

Questão 7

$$\boldsymbol{f(x) = \frac{\sqrt{2 - x^2}\ \ln(x+1)}{\sin(x)}}$$

$D = \{x \in \mathbb{R} : 2 - x^2 \geq 0 \wedge x + 1 \geq 0 \wedge \sin(x) \neq 0\} =$
$= \{x \in \mathbb{R} : |x| \leq \sqrt{2} \wedge x \geq -1 \wedge x \neq \pi\, n\ \forall n \in \mathbb{Z}\} =$
$= \{x \in \mathbb{R} : -1 \leq x \leq \sqrt{2} \wedge x = \pi\, n\ \forall n \in \mathbb{Z}\} = \left\{x \in \mathbb{R} \cap [-1, 0) \cap \left(0, \sqrt{2}\right]\right\}$

$(A \cup D)' = \left[-1, \sqrt{2}\right] \cup \{x \in \mathbb{R} : V_\epsilon(x) \cap A - \{x\} \neq \emptyset\} =$
$= \left[-1, \sqrt{2}\right] \cup \left[-\frac{4}{3}, \frac{4}{3}\right] = \left[-\frac{4}{3}, \sqrt{2}\right]$

Questão 8

$$\boldsymbol{f(x) = \sin(x)/x;\ f : (0, \infty) \to \mathbb{R}}$$

$f(x) = \{y \in \mathbb{R} : y = \sin(x)/x \;\forall\, x \in \mathbb{R} \cap (0, \infty)\};\ x > 0 \wedge -1 \leq \sin(x) \leq 1 \implies$
$\implies -1 < f(x) < 1$

$\mathrm{Fr}(f(x)) \not\subset f(x) \iff$
$\iff \exists\, y \in \mathbb{R} : V_\epsilon(y) \cap f(x) - \{y\} \neq \emptyset \wedge V_\epsilon(y) \not\subset f(x) \wedge y \notin f(x) \iff$
$\iff 1 \in \mathrm{Fr}(f(x))$

$\mathrm{Int}(f(x)) \neq f(x) \iff \exists\, y \in f(x) : V_\epsilon(y) \not\subset f(x) \iff$
$\iff g(x) = \{y \in \mathbb{R} : y = \sin(x)/x \;\forall\, x \in [\pi, 2\pi]\} \subset f(x); g(x) = [m, 0] \implies$
$\implies \exists\, m \in f(x) \cap (-1, 0] : f(x) = [m, 1)$