AM 1 - Resolução ficha 3

03/29

Contents

Exercício 1

1 - a)

1 - b) $\left|\frac{\cos n}{\sqrt{n+1}}\right| < \frac{1}{10^3}$

$$\implies \left|\frac{\cos n}{\sqrt{n+1}}\right| < \left|\frac{1}{\sqrt{n+1}}\right| < \frac{1}{10^3}$$

1 - c) $\left|e^{-n} + \frac{1}{n}\right| < 10^{-2}$

$$\left|e^{-n} + \frac{1}{n}\right| = \frac{1}{e^n} + \frac{1}{n} < \frac{1}{n} + \frac{1}{n} < 10^{-2}$$

Exercício 2

2 - a) $|u_n| < \sqrt{\epsilon};\ u_n = 1/n^2$

$$|u_n| < \sqrt{\epsilon} \implies \cdots$$

Nota: $\lfloor X \rfloor = \mathrm{Max}(k) \in \mathbb{Z} : k \leq x$

2 - b) $|v_n - 1| < \epsilon;\ v_n = \frac{n}{n+1}$

$$|v_n - 1| < \epsilon \cdots$$

2 - c) $|w_n - 2| < \epsilon/3;\ w_n = \frac{2n}{n-1/2}$

$$|w_n - 2| < \epsilon/3 \implies \left|\frac{2n}{n-1/2} - 2\right| < \epsilon/3 \cdots$$

Nota:

$$\exists\ L \in \mathbb{R} : \forall \epsilon > 0 \quad \exists\ p \in \mathbb{N} : n > p \implies |u_n - L < \epsilon|$$

Exercício 3

3 - a) $\lim_{n\to\infty} \frac{n^2+n\sqrt{n}}{3n^2+3}$

$$\frac{n^2+n\sqrt{n}}{3n^2+3} = \frac{1+1/\sqrt{n}}{3+3/n^2} = f_{a(n)} \implies \lim_{n\to\infty} f_{a(n)} = 1/3$$

3 - b) $\lim_{n\to\infty} \frac{n^3+n}{n^2+\ln n}$

$$\frac{n^3+n}{n^2+\ln n} = \frac{1+1/n^2}{1/n+\ln n/n^3} = f_{b(n)} \implies \lim_{n\to\infty} f_{b(n)} = \infty$$

3 - c) $\lim_{n\to\infty} \frac{n\sqrt[3]{8n+1}}{(n^{2/3}+1)^2}$

$$\frac{n\sqrt[3]{8n+1}}{(n^{2/3}+1)^2} = \frac{\sqrt[3]{8n^4+n^3}}{(n^{2/3}+1)^2} * \frac{n^{4/3}}{n^{4/3}} = \frac{\sqrt[3]{\frac{8n^4}{n^4}+\frac{n^3}{n^4}}}{\left(\frac{n^{2/3}}{n^{2/3}}+\frac{1}{n^{2/3}}\right)^2} = \frac{\sqrt[3]{8+n^{-1}}}{(1+n^{-2/3})} = f_{c(n)} \implies$$

$$\implies \lim_{n\to\infty} f_{c(n)} = \frac{\sqrt[3]{8}}{1} = 2$$

3 - d)

$$\lim_{n\to\infty} \frac{n\,\cos(n^2)}{n^2+1} = \lim_{n\to\infty} \frac{n^{-1}\cos(n^2)}{1+1/n^2} = 0$$

3 - e)

$$\lim_{n\to\infty} \frac{2\,n\,e^{1/n}}{\sqrt{n^2+5}} = \lim_{n\to\infty} \frac{2\,e^{1/n}}{\sqrt{1+5/n^2}} = 2$$

3 - f)

$$\lim_{n\to\infty} \frac{5^n + 4^n}{2\,5^{n+1} + 1} = \lim_{n\to\infty} \frac{1/5 + (4/5)^n\,1/5}{2 + 1/5^{n+1}} = \frac{1}{10}$$

3 - g)

$$\lim_{n\to\infty} \frac{3^n + e^n}{3^n + \pi^n} = \lim_{n\to\infty} \frac{\dfrac{3^n}{\pi^n} + \left(\dfrac{e}{\pi}\right)^n}{\dfrac{3^n}{\pi^n} + 1} = 0$$

3 - h)

$$\lim_{n\to\infty} \frac{n^2\,9^n + n^3}{(n+2)^2\,3^{2n+1}} = \lim_{n\to\infty} \frac{n^2\,9^n + n^3}{(n+2)^2\,9^n\,3} = \lim_{n\to\infty} \frac{1 + n/9^n}{(1+2/n)^2\;3} = 1/3$$

Exercício 4

Exercício 5

5 - a)

$$\lim_{n\to\infty} \sqrt{n+5} - \sqrt{2n-1} = \lim_{n\to\infty} \frac{n+5-2n+1}{\sqrt{n+5}+\sqrt{2n-1}} =$$
$$= \lim_{n\to\infty} \frac{-1+6/n}{\sqrt{1/n+5/n^2}+\sqrt{2/n-1/n^2}} = -\infty$$

5 - b)

$$\lim_{n\to\infty} \sqrt{n^2+3\,n} - \sqrt{n^2-n+1} = \lim_{n\to\infty} \frac{n^2+3\,n-n^2+n-1}{\sqrt{n^2+3\,n}+\sqrt{n^2-n+1}} =$$
$$= \lim_{n\to\infty} \frac{4\,n-1}{\sqrt{n^2+3\,n}+\sqrt{n^2-n+1}} = \lim_{n\to\infty} \frac{4-1/n}{\sqrt{1+3/n}+\sqrt{1-1/n+1/n^2}} = 2$$

5 - c) incompleto

$$\lim_{n\to\infty} (1.1)^{n+1} - (1.05)^{2n-1} = \lim_{n\to\infty} \left(\frac{(1.1)^n (1.1)}{1.05^2} - (1.05)^{-1} \right) (1.05)^{2n}$$

5 - d)

$$\lim_{n\to\infty} \sqrt{\ln(e^4\, n+1)} - \sqrt{\ln(n+2)} = \lim_{n\to\infty} \frac{\ln(e^4\, n+1) - \ln(n+1)}{\sqrt{\ln(e^4\, n+1)} + \sqrt{\ln(n+2)}} =$$

$$= \lim_{n\to\infty} \frac{\ln\left(\dfrac{e^4\, n+1}{n+1}\right)}{\sqrt{\ln(e^4\, n+1)} + \sqrt{\ln(n+2)}} = \lim_{n\to\infty} \frac{\ln\left(\dfrac{e^4 + 1/n}{1+1/n}\right)}{\sqrt{\ln(e^4\, n+1)} + \sqrt{\ln(n+2)}} = 0$$

5 - e)

$$\lim_{n\to\infty} n\ln(n+1) - n\ln(n) = \lim_{n\to\infty} n\ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln(1+1/n)^n = \ln(e^1) = 1$$

5 - f)

5 - g)

5 - h)

5 - i)

5 - j)

$$\lim_{n\to\infty} (1+1/n)^{n^2} (1-1/n)^{n^2} = \lim_{n\to\infty} \left(1-1/n^2\right)^{n^2} = e^{-1}$$

Exercício 6 Extras

6 - a) $\lim_{n\to\infty} \frac{n\,3^n+e^n}{(n+\sqrt{n})\,3^{n+1}+n^{100}}$

$$\lim_{n\to\infty} \frac{n\,3^n+e^n}{(n+\sqrt{n})\,3^{n+1}+n^{100}} = \lim_{n\to\infty} \frac{1+\frac{e^n}{n\,3^n}}{3+\frac{3}{\sqrt{n}}+\frac{n^{99}}{3^n}} = \lim_{n\to\infty} \frac{1+n^{-1}\left(\frac{e}{3}\right)^n}{3+\frac{3}{\sqrt{n}}+\frac{n^{99}}{3^n}} = 1/3$$

Nota: funções exponenciais crescem sempre mais rápido que qualquer outra

6 - b) $\lim_{n\to\infty} \frac{2\,n\,e^{1/n}}{(-1)^n+\sqrt{n^2+5}}$

$$\lim_{n\to\infty} \frac{2\,n\,e^{1/n}}{(-1)^n+\sqrt{n^2+5}} = \lim_{n\to\infty} \frac{2\,e^{1/n}}{(-1)^n/n+\sqrt{1+5/n^2}} = 2$$

6 - c) $\lim_{n\to\infty} \frac{2^n+3^{n+1}+(-7)^n}{3^{n-2}+5^{n+1}}$

$$\lim_{n\to\infty} \frac{2^n+3^{n+1}+(-7)^n}{3^{n-2}+5^{n+1}} = \lim_{n\to\infty} \frac{2^n+3\,3^n+(-7)^n}{3^{-2}\,3^n+5\,5^n}$$
$$= \lim_{n\to\infty} \frac{(2/7)^n+3\,(3/7)^n+(-7/7)^n}{(3/7)^n/3^2+5(5/7)^n} = \{\infty\ \forall\ n \text{ par};\ -\infty\ \forall\ n\text{impar}\}$$

6 - d) $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n}) = \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

6 - e) $\lim_{n\to\infty} \sqrt{\log(n^2+1)}-\sqrt{\log(n^2)}$

$$\lim_{n\to\infty} \sqrt{\log(n^2+1)}-\sqrt{\log(n^2)} = \lim_{n\to\infty} \frac{\log(n^2+1)-\log(n^2)}{\sqrt{\log(n^2+1)}-\sqrt{\log(n^2)}} =$$
$$= \lim_{n\to\infty} \frac{\log(1+1/n^2)}{\sqrt{\log(n^2+1)}-\sqrt{\log(n^2)}} = 0$$

6 - f) $\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^{2n}$

$$\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^{2n} = \lim_{n\to\infty}\left(\frac{(1+3/n)^n}{(1+1/n)^n}\right)^2 = \lim_{n\to\infty}\left(\frac{e^3}{e^1}\right)^2 = e^4$$

6 - g) $\lim_{n\to\infty}\left(\frac{n+5}{2\,n+1}\right)^{n}$ **refazer**

$$\lim_{n\to\infty}\left(\frac{n+5}{2\,n+1}\right)^{n} = \lim_{n\to\infty}\frac{(1+5/n)^n}{(1+1/(2\,n))^n} = \lim_{n\to\infty}\frac{(1+5/n)^n}{((1+1/(2\,n))^{2\,n})^{1/2}} = \frac{e^5}{\sqrt{e^1}}$$

6 - h) $\lim_{n\to\infty}\left(1-3/n^2\right)^n$

$$\lim_{n\to\infty}\left(1-3/n^2\right)^n = \lim_{n\to\infty}\left(\left(1-3/n^2\right)^{n^2}\right)^{1/n} = (e^{-3})^{1/\infty} = 0$$

6 - i) $\lim_{n\to\infty}(1+1/\log(n))^n$

$$\lim_{n\to\infty}(1+1/\log(n))^n = \lim_{n\to\infty}\left((1+1/\log(n))^{\log(n)}\right)^{n/\log(n)} = (e^1)^{\infty} = \infty$$

6 - j) $\lim_{n\to\infty}\left(\frac{n^6-2}{n^6}\right)^{n^3+3}$

$$\lim_{n\to\infty}\left(\frac{n^6-2}{n^6}\right)^{n^3+3} = \lim_{n\to\infty}\left(\left(1-2/n^6\right)^{n^6}\right)^{\frac{n^3+3}{n^6}} = (e^{-2})^0 = 1$$