ALGA - Algebra Linear e Geometria Analítica

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1 Matrizes

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Lista 1 – Matrizes

Questão 1

Q1 - a)

B, E, F, H, I

Q1 - b)

B, E, F, H, I

Q1 - c)

B, E, F, I

Q1 - d)

B, E

Questão 2

Q2 - a)

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Q2 - b)

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

Q2 - c)

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Questão 3

Q3 - a)

$$= \begin{bmatrix} 4 & 1 & 5 \\ -2 & 1 & -1 \end{bmatrix}$$

Q3 - c)

$$= \begin{bmatrix} 2 & 1 & -4 \\ 2 & -1 & 0 \end{bmatrix}$$

Q3 - b)

$$= \begin{bmatrix} 8 & 2 & 10 \\ -2 & 2 & -2 \end{bmatrix}$$

Q3 - d)

$$= \begin{bmatrix} 0 & 2 & -15 \\ 11 & 2 & -2 \end{bmatrix}$$

Questão 4

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Questão 10

Q10 - c)

$$\boldsymbol{(AB)^{(i)} = (AB)^{(j)} \because A^{(j)} = A^{(i)} \quad \forall\, i \neq j}$$

$$\begin{aligned}
&A_i = A_j \wedge AB_{k_1,k_2} = \sum_{k=1}^{n} a_{k_1,k}\, b_{k,k_2} \implies \\
&\implies (AB)_{i,k_2} = \sum_{k=1}^{n} a_{i,k}\, b_{k,k_2} = \\
&= \sum_{k=1}^{n} a_{j,k}\, b_{k,k_2} = (AB)_{j,k_2} \implies \\
&\implies (AB)_i = (AB)_j
\end{aligned}$$

Q10 - d)

$$\boldsymbol{B^k = B^l : k \neq l \implies}$$
$$\boldsymbol{\implies (AB)^k = (AB)^l}$$

Questão 9

$$\boldsymbol{\{D, D'\} \in \mathcal{M}_{n\times n}(\mathrm{K}) :}$$
$$\boldsymbol{d_{i,j} = 0 \wedge d'_{i,j} \forall\, i \neq j \implies}$$
$$\boldsymbol{\implies (DD')_{i,j} = 0 \quad \forall\, i \neq j}$$

$$\{D, D'\} \in \mathcal{M}_{n\times n}(\mathrm{K}) : d_{i,j} = 0 \wedge d'_{i,j}\ \forall\, i \neq j;$$
$$(DD')_{i,j} = \sum_{k=1}^{n} d_{i,k}\, d'_{k,j} \implies$$
$$\implies (DD')_{i,j} = 0\ \forall\, \{i, j\} \in \mathbb{K} : i \neq j$$

Questão 10 Indique...

Q10 - a) Uma Condição para que uma matriz diag. seja invert.

$$A \in \mathcal{M}_{n\times n} : a_{i,j} = 0 \quad \forall\, i \neq j \wedge$$
$$\wedge\, \exists\, A^{-1} : AA^{-1} = I_n \impliedby$$
$$\impliedby a_{i,j} \neq 0 \quad \forall\, i = j$$

Q10 - b)

Questão 108

$$\boldsymbol{J_n \in \mathcal{M}_{n\times s}(\mathbb{K}) : (J_n)_{i,j} = 1} \\ \forall \boldsymbol{\{i, j\}} \in \mathbb{K}$$

Questão 22

$$\boldsymbol{A \in \mathcal{M}_{n\times n}(\mathbb{K})}$$

Q22 - a)

$$\boldsymbol{A^3 = I_n}$$

Q22 - b)

$$\boldsymbol{A^2 + 2\,A = I_n}$$

Q22 - c)

$$\boldsymbol{A^2 + \alpha\,A + \beta\,I_n = 0}$$
$$\boldsymbol{: \alpha \in \mathbb{K} \wedge \beta \in \mathbb{K} \backslash \{0\}}$$

Questão 34

Q34 - a)

A e C

Q34 - b)

E

Questão 37

Q37 - a)

$$A \in \mathcal{M}_{n\times n}(\mathbb{K}) : a_{i,j} = 0 \quad \forall\{i, j\}$$

Q37 - b)

...

Questão 129

Q129 - a)

(i)

$$(A + A^T) = ((A + A^T)^T)^T = (A^T + A)^T = (A + A^T)^T$$
$$\therefore\ (A + A^T) \text{ é simétrica}$$

(ii)

Q129 - b)

Questão 42

a = s, III b = s, II c = s, I d = n e = s, II

Questão 43

Q43 - a)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Q43 - b)

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q43 - c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/5 & 1 \end{bmatrix}$$

Questão 45

Q45 - a)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Questão 48

A = s B = n C = s D = n

Questão 51

A = s B = s C = n D = s E = s

Questão 49

Q49 - a)

$$A' = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 2 & 2 \end{bmatrix} \xleftarrow[l_2 \to l_2 - 2\,l_1]{} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 2 & 2 \end{bmatrix} \xleftarrow[\substack{l_2 \to l_2 - 2\,l_1 \\ l_3 \to l_3 + l_1}]{} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Questão 171

$$[A|I] = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[l2 += -2\,l1]{}$$

Lista 2 – Sistemas de Equações Lineares

Questão 2

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{2} & \mathbf{-2} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{6} & \mathbf{-4} \end{bmatrix} \in \boldsymbol{\mathcal{M}}_{\mathbf{3}\times\mathbf{4}}(\mathbb{R})$$

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{-1} \\ \mathbf{4} \\ \mathbf{-6} \end{bmatrix} \in \boldsymbol{\mathcal{M}}_{\mathbf{3}\times\mathbf{1}}(\mathbb{R})$$

Lista 3 – Determinantes

Questão 72

$$\boldsymbol{A \in \mathcal{M}_{n\times n} : A^2 = -A}$$

$\det(-A) = \det A(-1)^n = \det(A^2) \implies \det A(\det A - (-1)^n) = 0 \implies$
$\implies \det A = 0 \vee \det A = (-1)^n$

Questão 73

$$\boldsymbol{A \in \mathcal{M}_{n\times n} : A\,A^* = I_n}$$

$|\det A| = 1 = \cdots = \det(A)\ \det \overline{A}^T = \det(A)\,\overline{\det A^T} = \det(A\,A^*) = \det(I_n) = 1$

Questão 28

Q28 - b)

$$V_\alpha = \begin{vmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) - (-\sin^2(x)) = 1 \neq 0$$
$\therefore \exists\, V_\alpha^{-1}$

$$\boldsymbol{\widehat{a_{i\,j}} = (-1)^{i+j} \det(A - A_i - A_j)}$$

$$V_\alpha^{-1} = \frac{\operatorname{adj} V_\alpha}{\det V_\alpha} = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

$$\operatorname{adj} V_\alpha = \widehat{V_\alpha^T} = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}^T = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}^T$$

Questão 29

$$\begin{bmatrix} \boldsymbol{m} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{m} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \boldsymbol{m} \end{bmatrix}$$

Q29 - a)

$$\operatorname{adj} M = \begin{bmatrix} m^2-1 & 1-m & 1-m \\ 1-m & m^2-1 & 1-m \\ 1-m & 1-m & m^2-1 \end{bmatrix}$$

Q29 - b)

$$\begin{aligned} &\det M = m(m^2-1) + 1(1-m) + 1(1-m) = \\ &= m(m+1)(m-1) - 2(m-1) = (m-1)(m^2+m-2) = \\ &= (m-1)(m(m-1) + 2(m-1)) = (m-1)^2(m+2) \\ &\therefore \exists M^{-1} \forall\, M : m \in \mathbb{R}\backslash\{1,-2\} \end{aligned}$$

Q29 - c)

$$M^{-1} = \frac{\operatorname{adj} M}{\det M} = \frac{1}{(m-1)^2(m+2)} \begin{bmatrix} m^2-1 & 1-m & 1-m \\ 1-m & m^2-1 & 1-m \\ 1-m & 1-m & m^2-1 \end{bmatrix} =$$

$$= \frac{1}{(m-1)^2(m+2)} \begin{bmatrix} m+1 & -1 & -1 \\ -1 & m+1 & -1 \\ -1 & -1 & m+1 \end{bmatrix}$$

Questão 31

Q31 - a)

$$\exists(\operatorname{adj} A)^{-1} \because (\frac{A}{\det A}) \operatorname{adj} A = I_n : \det A \neq 0 \land \exists A^{-1}$$

Q31 - c)

$$\det(\operatorname{adj} A) = \det(\det A\, A^{-1}) = (\det A)^n \det A^{-1} = (\det A)^n / \det A =$$
$$= (\det A)^{n-1})$$

$$\mathbf{adj(\boldsymbol{AB}) = (adj\ \boldsymbol{A})(adj\ \boldsymbol{B})}$$

$$A\, B \operatorname{adj}(A\, B) = \det(A\, B)\, I_n \implies \operatorname{adj}(A\, B) = (A\, B)^{-1} \det(A\, B)\, I_n =$$
$$= \det A \det B\, B^{-1}\, A^{-1}\, I_n = (\operatorname{adj} A)(\operatorname{adj} B)$$

Questão 25

$$\boldsymbol{A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 4 & 4 \\ 1 & 3 & 1 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}}$$

$$\det A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 4 & 4 \\ 1 & 3 & 1 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 4 \\ 1 & 3 & 1 \end{pmatrix} =$$
$$= 2 * 1 * (2 * 1 - 3 * 4) + 2 * 1 * (-1 * 4 - 2 * 1) = -32$$

Questão 33

$$\boldsymbol{A_k = \begin{bmatrix} 1 & -k & 10 & k & kk & k & -k \end{bmatrix}}$$