Ficha 2
Método de indução

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Contents

Exercício 1

1 - a) $\quad A = \left\{ \sum_{k=1}^{n} 1/2^k = 1 - 1/2^n \ \forall \ n \in \mathbb{N} \right\}$

$$1 \in A \iff \sum_{k=1}^{1} 1/2^k = 1 - 1/2^1 \implies 1/2 = 1/2$$

$$m+1 \in A \ \forall \ m \in \mathbb{N} \iff \sum_{k=1}^{m+1} 1/2^k = 1 - 1/2^{m+1} \implies$$

$$\implies \sum_{k=1}^{m} 1/2^k + 1/2^{m+1} = 1 - 1/2^{m+1}$$

1 - b) $\quad B = \left\{ \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \ \forall \ n \in \mathbb{N} \right\}$

$$1 \in B \iff \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1+1} \implies 1/2 = 1/2;$$

$$m+1 \in B \ \forall \ m \in \mathbb{N} \iff \sum_{k=1}^{m+1} \frac{1}{k(k+1)} = \frac{m+1}{m+1+1} \implies$$

$$\implies \sum_{k=1}^{m} \frac{1}{k(k+1)} + \frac{1}{(m+1)(m+2)} = \frac{m+1}{m+2} \implies$$

$$\implies \sum_{k=1}^{m} \frac{1}{k(k+1)} = \frac{(m+1)^2 - 1}{(m+1)(m+2)} = \frac{m^2 + 2\,m + 1 - 1}{(m+1)(m+2)} =$$

$$= \frac{m(m+2)}{(m+1)(m+2)} = \frac{m}{m+1}$$

1 - c) $C = \{\sum_{k=1}^{n} k\ k! = (n+1)! - 1\ \forall\ n \in \mathbb{N}\}$

$$1 \in C \iff \sum_{k=1}^{1} k\ k! = (1+1)! - 1 \implies 1\ 1! = 2 - 1 \implies 1 = 1$$

$$m+1 \in C\ \forall\ m \in \mathbb{N} \iff \sum_{k=1}^{m+1} k\ k! = (m+1+1)! - 1 \implies$$

$$\implies \sum_{k=1}^{m} k\ k! + (m+1)((m+1)!) = (m+2)((m+1)!) - 1 \implies$$

$$\implies \sum_{k=1}^{m} k\ k! = ((m+1)!)(m+2-m-1) - 1 = (m+1)! - 1$$

1 - d)

$$\boldsymbol{D = \{(\cos(x) + i\sin(x))^n = \cos(n\,x) + i\sin(n\,x)}$$
$$\boldsymbol{\forall\ n \in \mathbb{N},\ \forall\ x \in \mathbb{R},\ i = \sqrt{-1}\}}$$

$$1 \in D \iff (\cos(x) + i\sin(x))^1 = \cos(1\,x) + i\sin(1\,x)$$

$$m+1 \in D\ \forall\ m \in \mathbb{N} \iff (\cos(x) + i\sin(x))^{m+1} =$$

$$= \cos((m+1)x) + i\sin((m+1)x) \implies (\cos(x) + i\sin(x))^m =$$

$$= \frac{\cos(m\,x)\cos(x) - \sin(m\,x)\sin(x) + i(\sin(m\,x)\cos(x) + \sin(x)\cos(m\,x))}{\cos(x) + i\sin(x)} =$$

$$= \frac{\cos(m\,x)\cos(x) + i\cos(x)\sin(m\,x) + i\sin(x)\cos(m\,x) - \sin(x)\sin(m\,x)}{\cos(x) + i\sin(x)} =$$

$$= \frac{\cos(m\,x)\cos(x) + \cos(x)\ i\sin(m\,x) + i\sin(x)\ \cos(m\,x) + i^2\sin(x)\sin(m\,x)}{\cos(x) + i\sin(x)} =$$

$$= \frac{(\cos(x) + i\sin(x))(\cos(m\,x) + i\sin(m\,x))}{\cos(x) + i\sin(x)} = \cos(m\,x) + i\sin(m\,x)$$

Exercício 2

2 - a) $9^n - 3$ é multiplo de 6

$9^n - 3$ é multiplo de 6 $\iff \exists\, k \in \mathbb{N} : 9^n - 3 = 6\,k\ \forall\ n \in \mathbb{N}\backslash\{0\} \iff$
$\iff 9^n - 3 = 6\,k \iff (3^2)^n/3 = 3^{2\,n-1} = 2\,k + 1$

$9^n - 3$ é multiplo de 6 $\iff \exists\, k \in \mathbb{N} : 9^n - 3 = 6\,k\ \forall\ n \in \mathbb{N}\backslash\{0\} \iff$

$$\iff \begin{cases} n = 1 \implies 9^1 - 3 = 6 \\ n = m + 1 \implies 9^{m+1} - 3 = 9\,9^m - 3 = 9\,(6\,k + 3) - 3 = \\ = 6\,(9\,k + 4);\ (9\,k + 4) \in \mathbb{N}\ \forall\ k \in \mathbb{N} \end{cases}$$

2 - b) $6^n - 1$ é múltiplo de 5

$6^n - 1$ é múltiplo de 5 $\iff \exists\, k \in \mathbb{N} : 6^n - 1 = 5\,k\ \forall\ n \in \mathbb{N} \iff$

$$\iff \begin{cases} n = 1 \implies 6^1 - 1 = 5 \\ n = m + 1 \implies 6^{m+1} - 1 = 6\,6^m - 1 = 6\,(5\,k + 1) - 1 = \\ = 5\,(6\,k + 1);\ (6\,k + 1) \in \mathbb{N}\ \forall k \in \mathbb{N} \end{cases}$$

2 - c) $3\,n^2 + 3\,n$ é múltiplo de 6

$3\,n^2 + 3\,n$ é múltiplo de 6 $\iff \exists\, k \in \mathbb{N} : 3\,n^2 + 3\,n = 6\,k\ \forall\ n \in \mathbb{N} \iff$

$$\iff \begin{cases} n = 1 \implies 3 * 1^2 + 3 * 1 = 6 \\ n = m + 1 \implies 3\,(m + 1)^2 + 3\,(m + 1) = 3\,(m + 1)\,(m + 1 + 1) = \\ = 3\,m^2 + 3\,m + 2\,(3\,m^2 + 3\,m)/m = 6\,k + 2\,(6\,k)/m = 6\,(k + 2k/m) = \\ = 6\,(k + m + 1);\ (k + m + 1) \in \mathbb{N}\ \forall\ \{m, k\} \subset \mathbb{N} \end{cases}$$

2 - d) Extra: $5^n - 1$ é multiplo de 4

$5^n - 1$ é multiplo de 4 $\iff \exists\, k \in \mathbb{N} : 5^n - 1 = 4\,k\ \forall\ n \in \mathbb{N} \iff$

$$\iff \begin{cases} n = 1 \implies 5^1 - 1 = 4\,k \implies k = 0; \\ n = m + 1 \implies 5^{m+1} - 1 = 5\,5^m - 1 = 5\,(5^m - 1) + 4 = \\ = 5\,(4\,k) + 4 = 4\,(5\,k + 1) \end{cases}$$

Exercício 3

$$\boldsymbol{I_i} \text{ é um intervalo aberto } \forall\, \boldsymbol{i} \in \mathbb{N} \cap [\mathbf{1}, \boldsymbol{n}];$$
$$\bigcap_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{I_i} \neq \emptyset$$

$$A = \bigcup_{i=1}^{n} I_i \text{ é um intervalo aberto } \iff A = \mathrm{Int}(A) \iff V_\epsilon a \subset A\ \forall\, a \in A \iff$$
$$\iff \begin{cases} \exists B \subset A : B \subset I_i\ \forall\, i \in \mathbb{N} \cap [1,n] \impliedby \bigcap_{i=1}^{n} I_i \neq \emptyset \\ V_\epsilon b \subset I_i\ \forall\, b \in I_i\ \forall\, i \in \mathbb{N} \cap [1,n] \iff I_i = \mathrm{Int}(I_i)\ \forall\, i \in \mathbb{N} \cap [1,n] \iff \\ \iff I_i \text{ é um intervalo aberto } \forall\, i \in \mathbb{N} \cap [1,n] \end{cases}$$

Exercício 4

4 - a) $(1+k)^n \geq 1 + n\,k\ \forall\, n \in \mathbb{N}$ para $k > -1$ fixado

$$(1+k)^n \geq 1 + n\,k\ \forall\, n \in \mathbb{N},\ \forall\, k \in \mathbb{R} : k > -1 \iff$$
$$\iff \begin{cases} n = 0 \implies (1+k)^0 = 1 \geq 1 + 0\,k = 1; \\ n = m+1 \implies (1+k)^{m+1} = (1+k)\,(1+k)^m;\ (1+k) > 0\ \forall\, k > -1 \implies \\ \implies (1+k)\,(1+k)^m \geq (1+k)\,(1+m\,k) = \\ = 1 + m\,k + k + m\,k^2 \geq 1 + (m+1)\,k = k + 1 + m\,k \implies \\ \implies m\,k^2 \geq 0 \end{cases}$$

4 - b) $\sum_{k=1}^{n} k^2 < (n+1)^3\ \forall\, n \in \mathbb{N}$

$$\sum_{k=1}^{n} k^2 < (n+1)^3\ \forall\, n \in \mathbb{N} \iff$$
$$\iff \begin{cases} n = 0 \implies \sum_{k=1}^{0} k^2 = 0 < (0+1)^3 = 1 \\ n = m+1 \implies \sum_{k=1}^{m+1} k^2 = \sum_{k=1}^{m} k^2 + (m+1)^2; \\ \sum_{k=1}^{m} k^2 > 0 : k^2 > 0\ \forall\, k \in \mathbb{N} \implies \sum_{k=1}^{m} k^2 + (m+1)^2 < \\ < (m+1)^3 + (m+1)^2 = (m+1+1)(m+1)^2 < (m+1+1)^3 \implies \\ \implies (m+1)^2 < (m+2)^2 \implies |m+1| < |m+2|; m \geq 0\ \forall\, m \in \mathbb{N} \implies \\ \implies 0 < 1 \end{cases}$$

4 - c) $\sum_{k=1}^{n} 1/(2^k+1) < 1 - 1/2^n \; \forall\, n \in \mathbb{N}\backslash\{0\}$

$$\sum_{k=1}^{n} 1/(2^k+1) < 1 - 1/2^n \; \forall\, n \in \mathbb{N}\backslash\{0\} \implies$$

$$\implies \begin{cases} n=1 \implies \sum_{k=1}^{1} 1/(2^k+1) = 1/3 < 1 - 1/2^1 = 1/2 \\ n=m+1 \implies \sum_{k=1}^{m+1} 1/(2^k+1) = \sum_{k=1}^{m} 1/(2^k+1) + 1/(2*2^m+1) < \\ < 1 - 1/2^m + 1/(2*2^m+1) = 1 - 1/(2*2^m) - 1/(2*2^m) + \\ +1/(2*2^m+1) < 1 - 1/2^{m+1} = 1 - 1/(2*2^m) \implies \\ \implies 1/(2*2^m+1) < 1/(2*2^m) \implies 1 > 0 \end{cases}$$

4 - d) $\sum_{k=1}^{n} 1/k^2 \leq 2 - 1/n \; \forall\, n \in \mathbb{N}$

$$\sum_{k=1}^{n} 1/k^2 \leq 2 - 1/n \; \forall\, n \in \mathbb{N}\backslash\{0\} \implies$$

$$\implies \begin{cases} n=1 \implies \sum_{k=1}^{1} 1/k^2 = 1 \leq 2 - 1/1 = 1 \\ n=m+1 \implies \sum_{k=1}^{m+1} 1/k^2 = \sum_{k=1}^{m} 1/k^2 + 1/(m+1)^2 \leq \\ \leq 2 - 1/m + 1/(m+1)^2 \leq 2 - 1/(m+1) = 2 - (m+1)/(m+1)^2 \implies \\ \implies (m+1+1)/(m+1)^2 \leq 1/m \implies m^2 + 2\,m \leq m^2 + 2\,m + 1 \implies \\ \implies 0 \leq 1 \end{cases}$$

Exercício 5

$$\boldsymbol{u_1 = -1, \qquad u_{n+1} = \frac{u_n}{1 - 2\,u_n}, \qquad \forall\, n \in \mathbb{N}}$$

$$u_n = 1/(1-2\,n) \; \forall\, n \in \mathbb{N} \iff$$

$$\iff \begin{cases} n=1 \implies 1/(1-2) = -1/1 = u_1 \\ n=m+1 \implies \frac{1}{1-2\,(m+1)} = \frac{1}{-1-2\,m} = \frac{1}{1-2\,m} * \frac{1-2\,m}{1-2\,m-2} = \\ = \frac{1/(1-2\,m)}{(1-2\,(1/(1-2\,m))} = u_m/(1-2\,u_m) = u_{m+1} \end{cases}$$

Exercício 6

$$\boldsymbol{e_n := (1 + 1/n)^n} \qquad\qquad (\boldsymbol{n} \in \mathbb{N})$$

6 - a) $\binom{n}{k}\frac{1}{n^k} \leq \frac{1}{k!} \quad \forall\, n \in \mathbb{N},\ \forall\, k \in \mathbb{N} \cap [0, n]$

$$\binom{n}{k}\frac{1}{n^k} \leq \frac{1}{k!} \quad \forall\, n \in \mathbb{N},\ \forall\, k \in \mathbb{N} \cap [0, n] \iff$$
$$\iff \binom{n}{k}\frac{1}{n^k} = \frac{1}{k!}\,\frac{n!}{(n-k)!n^k} \leq \frac{1}{k!} \implies n! \leq (n-k)!n^k \implies$$
$$\implies \prod_{i=1}^{k-1}(n-i) \leq n^k \implies \log_n\left(\prod_{i=1}^{k-1}(n-i)\right) \leq k \implies \sum_{i=1}^{k-1}\log_n(n-i) \leq$$
$$\leq \sum_{i=1}^{k-1}\log_n(n) = k - 1 \leq k$$

6 - b) $\sum_{k=0}^{n} \frac{1}{k!} \leq 3 - 1/n \quad \forall\, n \in \mathbb{N}\backslash\{0\}$

$$\sum_{k=0}^{n} \frac{1}{k!} \leq 3 - 1/n \quad \forall\, n \in \mathbb{N}\backslash\{0\} \implies$$
$$\implies \sum_{k=2}^{n} \frac{1}{k!} \leq 1 - 1/n = (n-1)!(n-1)/n! \implies$$
$$\implies \sum_{k=2}^{n} \frac{n!}{k!} = \sum_{k=0}^{n-2} \frac{n!}{(n-k)!} = \sum_{k=0}^{n-2} \frac{n!}{(n-k)!} \leq n! - (n-1)! \cdots$$

$$\sum_{k=0}^{n} \frac{1}{k!} \leq 3 - 1/n \quad \forall\, n \in \mathbb{N}\backslash\{0\} \iff$$
$$\iff \begin{cases} n = 1 \implies \sum_{k=0}^{1} 1/k! = 1 \leq 3 - 1/1 = 2 \\ n = m+1 \implies \sum_{k=0}^{m+1} 1/k! = 1/(m+1)! + \sum_{k=0}^{m} 1/k! \leq \\ \leq m^{-1}/(m+1) + 3 - 1/m = (1+m^{-1})/(m+1) - 1/(m+1) + \\ +3 - 1/m \leq 3 - 1/(m+1) \implies \frac{1}{m+1} + \frac{1}{m(m+1)} - \frac{1}{m} \leq 0 \implies \\ \implies 1 + 1/m \leq 1 + 1/m \end{cases}$$

6 - c) $e_m \leq 3 \quad \forall\, n \in \mathbb{N}$ **Duvida**

$$e_m \leq 3 \quad \forall\, n \in \mathbb{N} \iff (1+1/n)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n^i} 1^{n-i} = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} n^{-i} =$$
$$= 1 + \sum_{i=1}^{n} \frac{n!\, n^{-i}}{i!(n-i)!} \leq 3 \implies \sum_{i=1}^{n} \frac{n!\, n^{-i}}{i!(n-i)!} \leq 2 \iff$$
$$\iff \begin{cases} n = 0 \implies \sum_{i=1}^{0} \frac{0!\, 0^{-i}}{i!(0-i)!} = 0 \leq 2 \\ n = m+1 \implies \sum_{i=1}^{m+1} \frac{(m+1)!\, (m+1)^{-i}}{i!(m+1-i)!} = \sum_{i=1}^{m+1} \frac{(m+1)\, (m!)\, (m+1)^{-i}}{i!\, (m+1-i)\, (m-i)!} = \\ = (m+1) \sum_{i=1}^{m+1} \frac{(m!)\, (m+1)^{-i}}{(m+1-i)\, i!\, (m-i)!} \leq (m+1) \sum_{i=1}^{m+1} \frac{(m!)\, (m)^{-i}}{i!\, (m-i)!} \leq \frac{(m+1)}{(m+1-i)} 2 \leq \\ \leq 2 \cdots \end{cases}$$