Ficha 3 - Integração Numérica

Soluções

Exercício 1

a) $I_T \approx 2.286557$

$|I - I_T| \leq 0.260136$

b) $I_{PM} \approx 2.489652$

$|I - I_{PM}| \leq 0.130068$

$I_S \approx 2.421954$

$|I - I_S| \leq 0.004755$

Exercício 2

a) $I_{T,4} \approx 4.197481$

$|I - I_{T,4}| \leq 0.047782$

b) $I_{S,2} \approx 4.169220$

$|I - I_{S,2}| \leq 0.000261$

c) Menor nº de subintervalos

i) 619

ii) 875

iii) 18

Exercício 3

a) $I_T = 94.4$

b) $I_{T,2} = 76.8$

c) $I = I_S = \frac{1064}{15}$

Exercício 4

a) $I_{T,3} = 194$

b) $I_{PM,2} = 12$

c) $w = 43$

Exercício 5

a) $\hat{I}_{S,2} \simeq 1.7701$

b) $|I - \hat{I}_{S,2}| \leq 0.0001$

Exercício 6

$I = 7.2$

Exercício 7

a) $\hat{I}_{PM,4} = 1.44875$ $\hat{I}_{T,4} = 1.49068$

b) $|I - \hat{I}_{PM}| \leq 0.0425$ $\hat{I}_{PM}$ tem 2 algarismos significativos

$|I - \hat{I}_{T,4}| \leq 0.089$ $I_{T,4}$ Tem 1 algarismo significativo

Exercício 8

18 Sub-intervalos

Exercício 9

Integral impróprio $\Rightarrow \hat{I}_G = -0.707957$ regra de Gauss com 2 pontos

Exercício 10

a) $\hat{I}_{T,4} = -8.827116$ $|I - \hat{I}_{T,4}| \leq 1.231509$

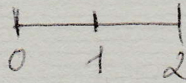
b) $\hat{I}_{PM,2} = -7.546884$ $|I - \hat{I}_{PM,2}| \leq 2.463019$

6. $I = \int_0^2 f(x)\,dx \qquad \bar{I} = 8$

Regra de Simpson com $h=1$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \overset{=3}{a_4} x^4 \qquad a_4 = 3$

$f'(x) = a_1 + 2a_2 x + 3a_3 x^3 + 12 x^3$



$f''(x) = 2a_2 x + 6a_3 x + 36x^2$

Se $h=1 \Rightarrow$ regra de simpson simples

$f'''(x) = 6a_3 + 72x$

$f^{(4)}(x) = 72$

$I - \bar{I} = -\frac{h^5}{90} f^{(4)}(\theta) \quad , \theta \in]0,2[$

$I - \bar{I} = -\frac{1}{90} \times 72 = -\frac{4}{5} = 0.8$

$I = -\frac{4}{5} + \bar{I} \iff I = -\frac{4}{5} + 8 = -\frac{4}{5} + \frac{40}{5} = \frac{36}{5}$

$\bar{I} = 8 \qquad\qquad = 7.2$

Exercício 11

a) $\hat{I}_{T,2} = 9$, $I = \frac{11}{3}$

b) $\hat{I}_{PM} = 2(\alpha-1)$, $I = \frac{8}{3} + 2(\alpha-1)$

c) $\alpha = \frac{3}{2}$

d) $\hat{I}_S = \frac{11}{3}$

Exercício 12

$\alpha = -4$

Exercício 13

a) $\hat{I}_{T,3} = 72$

b) $\hat{I}_{PM,3} = 54$

c) $I - \hat{I}_T = 12$

Exercício 14

a) $I_{G_2} \simeq 1.568544$

b) $I_{G_3} \simeq 1.573165$

Exercício 15

a) $\hat{I}_{S,2} \simeq 0.3414$

b) $|I - \hat{I}_{S,2}| \leq 0.26 \times 10^{-4}$

c) Erro efectivamente cometido

$|I - \hat{I}_{S,2}| \simeq 0.174179 \times 10^{-4} < 0.26 \times 10^{-4}$

Exercício 11.

$I = \int_1^5 f(x)\,dx$ $\quad$ f é polinómio de grau 2 e $f''(x) = 4, \forall x \in \mathbb{R}$

	x_0	x_1	x_2	x_3	x_4
x	1	2	3	4	5
$f(x)$	-2	-1	1	α	9

a) Regra dos Trapézios com $n=2$

$h = \frac{b-a}{n} = \frac{4}{2} = 2$

$\hat{I}_T = \frac{h}{2}\left(f(x_0) + 2f(x_2) + f(x_4)\right) = \frac{1}{2}\left(-2 + 2\times 1 + 9\right) = 9$

$I - \hat{I}_T = -n\frac{h^3}{12}f''(\gamma) = -2 \times \frac{2^3}{12} \times 4 = -\frac{16}{3}$

$\Leftrightarrow I = 9 - \frac{16}{3} = \frac{11}{3}$

b) Regra do Ponto Médio com $n=2$

$\hat{I}_{PM} = h\left(f\left(\frac{x_0+x_2}{2}\right) + f\left(\frac{x_2+x_4}{2}\right)\right) = 2\left(f(x_1) + f(x_3)\right) = 2(\alpha - 1)$ $\quad$ $h = \frac{b-a}{n} = 2$

$I - \hat{I}_{PM} = \frac{nh^3}{24}f''(\gamma) = 2 \times \frac{2^3}{24} \times 4 = \frac{8}{3}$

$\Leftrightarrow I = \frac{8}{3} + 2(\alpha - 1)$

c) $\frac{8}{3} + 2(\alpha - 1) = \frac{11}{3} \Leftrightarrow 2(\alpha - 1) = \frac{11-8}{3} = 1 \Leftrightarrow \alpha = \frac{1}{2} + 1 = \frac{3}{2}$

d) Regra de Simpson $\quad n=2$

$h = \frac{b-a}{2n} = \frac{4}{4} = 1$

$\hat{I}_S = \frac{h}{3}\left(f(x_0) + 4\left(f(x_1) + f(x_3)\right) + 2f(x_2) + f(x_4)\right)$

$= \frac{1}{3}\left(-2 + 4\left(-1 + \frac{3}{2}\right) + 2\times 1 + 9\right) = \frac{1}{3}\left(-\cancel{2} + \cancel{2} + 2 + 9\right) = \frac{11}{3}$

Exercício 12

x_i	0	1	2	3	4	5	6
$f(x_i)$	a	-3	6	41	120	$-65a+1$	482

$$\hat{I} = 642$$

$$= \frac{1}{3}\left(-a + 4\times -3 + 2\times 6 + 4\times 41 + 2\times 120 + 4\times(-65a+1) + 482\right)$$

$$= \frac{1}{3}\left(-a - \cancel{12} + \cancel{12} + 164 + 240 - 260a + 4 + 482\right)$$

$$= \frac{1}{3}\left(-259a + 890\right)$$

$$(=) \quad -259a = 3\times 642 - 890 = 1036$$

$$(=) \quad a = -4$$

13 $\left[f(x) = 3x^2 - 4x + 1 \; ; \; I = \int_{-3}^{3} f(x)\,dx = 60 \right]$

x	-3	-2	-1	0	1	2	3
$f(x)$	40	21	8	1	0	5	16

a) Regra dos trapézios composta ($1 < n < 4$; $h < 3$)

- $h = \frac{b-a}{n} \Leftrightarrow \frac{3-(-3)}{n} < 3 \Leftrightarrow \frac{6}{n} < n \Leftrightarrow n > 2$

- $(n > 2 \wedge 1 < n < 4) \Rightarrow n = 3$

- $h = \frac{b-a}{n} = \frac{3-(-3)}{3} = \frac{6}{3} = 2$

- $x_0 = -3 \; ; \; y_0 = 40$
  $x_1 = -1 \; ; \; y_1 = 8$
  $x_2 = 1 \; ; \; y_2 = 0$
  $x_3 = 3 \; ; \; y_3 = 16$

- $\hat{I}_T = \frac{h}{2}\left(y_0 + 2(y_1 + y_2) + y_3\right) =$

$= \frac{2}{2}\left(40 + 2(8+0) + 16\right) = 72$

(b) Regra do ponto médio composta (h=2)

- $h = 2 = \frac{b-a}{n} \Leftrightarrow n = \frac{b-a}{h} = \frac{3-(-3)}{2} = \frac{6}{2} = 3$

- $\begin{cases} x_0 = -3 \ ; \ y_0 = 40 \\ x_1 = -1 \ ; \ y_1 = 8 \\ x_2 = 1 \ ; \ y_2 = 0 \\ x_3 = 3 \ ; \ y_3 = 16 \end{cases}$

- $\widehat{I}_{PM} = h\left(f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right)\right) =$

$= 2\left(f\left(\frac{-3-1}{2}\right) + f\left(\frac{-1+1}{2}\right) + f\left(\frac{1+3}{2}\right)\right) =$

$= 2\left(f(-2) + f(0) + f(2)\right) =$

$= 2(21 + 1 + 5) = 54$

(c) • $I - \widehat{I}_{PM} = 6$

- $f''(x) = K$ (constante)

- $I - \widehat{I}_{PM} = n \frac{h^3}{24} \times f''(\theta) \qquad (\theta \in]-3,3[)$

$= 3 \times \frac{2^3}{24} \times K = K = 6$

(2)

Exercício 13 (continuação)

(c) $\bullet\ I - \hat{I}_T = -n \times \frac{h^3}{12} \times f''(\xi) \quad (\xi \in]-3,3[)$

$= -3 \times \frac{2^3}{12} \times k = -2k = -2(6) =$

$= -12$

Ficha 3

16.

- $\int_a^b f(x)\,dx \approx \alpha_0 f(a) + \alpha_1 f(x_1) + \alpha_2 f(b)$

- α_0, α_1 e $\alpha_2 \rightarrow$ pesos

- $x_1 \in \,]a,b[\, \rightarrow$ nodo

- Há 4 parâmetros desconhecidos: $\alpha_0, \alpha_1, \alpha_2$ e x_1

- Podemos estabelecer 4 condições para determinar esses parâmetros:

$$\begin{cases} \int_a^b 1\,dx = \alpha_0 \cdot 1 + \alpha_1 \cdot 1 + \alpha_2 \cdot 1 & (f(x)=1) \\ \int_a^b x\,dx = \alpha_0 \cdot a + \alpha_1 \cdot x_1 + \alpha_2 \cdot b & (f(x)=x) \\ \int_a^b x^2\,dx = \alpha_0 \cdot a^2 + \alpha_1 x_1^2 + \alpha_2 \cdot b^2 & (f(x)=x^2) \\ \int_a^b x^3\,dx = \alpha_0 \cdot a^3 + \alpha_1 \cdot x_1^3 + \alpha_2 \cdot b^3 & (f(x)=x^3) \end{cases} \Leftrightarrow$$

$$\begin{cases} \alpha_0 + \alpha_1 + \alpha_2 = b - a \\ \alpha_0 . a + \alpha_1 . x_1 + \alpha_2 . b = \frac{b^2 - a^2}{2} \\ \alpha_0 . a^2 + \alpha_1 . x_1^2 + \alpha_2 . b^2 = \frac{b^3 - a^3}{3} \\ \alpha_0 . a^3 + \alpha_1 . x_1^3 + \alpha_2 . b^3 = \frac{b^4 - a^4}{4} \end{cases} \Leftrightarrow$$

$$\ldots \Leftrightarrow \begin{cases} \alpha_0 = \frac{b-a}{6} \\ \alpha_1 = \frac{2b-2a}{3} = \frac{4b-4a}{6} \\ \alpha_2 = \frac{b-a}{6} \\ x_1 = \frac{a+b}{2} \end{cases}$$

- verifica-se assim que esta regra de quadratura tem grau de precisão 3 (é exata para polinómios de grau, no máximo, 3)

- $\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{4b-4a}{6} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b) \Leftrightarrow$

$\Leftrightarrow \int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \Leftrightarrow$

$\Leftrightarrow \int_a^b f(x)\,dx \approx \frac{\frac{b-a}{2}}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \Leftrightarrow$

$\Leftrightarrow \int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right],$

com $h = \frac{b-a}{2}$

- conclui-se que esta é a regra de Simpson simples

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ficha 3

17.

x	x_0	x_1	x_2	x_3	x_4
$f(x)$	f_0	f_1	f_2	f_3	f_4

  ($x_0 = a$, $x_4 = b$)

- $x_{i+1} - x_i = h, \quad i=0,1,2,3$

- $$I_{T,4} = \frac{\frac{b-a}{4}}{2}\left[f_0 + 2(f_1 + f_2 + f_3) + f_4\right] =$$

  $$= \frac{4h}{8}\left(f_0 + f_4 + 2f_1 + 2f_2 + 2f_3\right) =$$

  $$= \frac{h}{2}\left(f_0 + f_4 + 2f_1 + 2f_2 + 2f_3\right)$$

- $$I_{S,2} = \frac{\frac{b-a}{2\times 2}}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + f_4\right] =$$

  $$= \frac{4h}{12}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4\right) =$$

  $$= \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4\right)$$

$\bullet\ 3 I_{S,2} - 2 I_{T,4} =$

$$= \cancel{3}\left[\frac{h}{\cancel{3}}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4)\right] -$$

$$- \cancel{2}\left[\frac{h}{\cancel{2}}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4)\right] =$$

$$= h(\cancel{f_0} + 4f_1 + \cancel{2f_2} + 4f_3 + \cancel{f_4}) -$$

$$- h(\cancel{f_0} + 2f_1 + \cancel{2f_2} + 2f_3 + \cancel{f_4}) =$$

$$= h(4f_1 - 2f_1 + 4f_3 - 2f_3) =$$

$$= h(2f_1 + 2f_3) =$$

$$= 2h(f_1 + f_3).$$

18. $\bullet\ I = \int_a^b \frac{1}{x^3}\, dx \quad ; \quad f(x) = \frac{1}{x^3}$

$\bullet\ a < b \ , \ a, b \in \mathbb{R}$

$\bullet\ I - I_{PM} = \frac{h^3}{24} f''(\theta) \quad , \quad \theta \in]a, b[$

$\bullet\ f''(x) = \frac{12}{x^5} \Rightarrow$

$\Rightarrow I - I_{PM} = \frac{(b-a)^3}{24} \times \frac{12}{\theta^5} = \frac{2(b-a)^3}{\theta^5} \ , \ \theta \in]a,b[$

$\bullet$ se $b<0$, tem-se $\begin{cases} a < 0 \\ \theta < 0 \end{cases}$.

$\bullet$ Assim

$$\frac{2\boxed{(b-a)^3}^{\,+}}{\boxed{\theta^5}_{\,-}} < 0 \Rightarrow I - I_{PM} < 0$$

$\therefore$ I_{PM} é uma aproximação <u>por excesso</u>

Ficha 3

- $I - I_S = -\frac{h^5}{90} f^{(4)}(\xi)$, $\xi \in]a,b[$

- $f^{(4)}(x) = \frac{360}{x^7}$ $\Rightarrow$

$\Rightarrow I - I_S = -\frac{\left(\frac{b-a}{2}\right)^5}{90} \times \frac{360}{\xi^7} =$

$= \frac{\boxed{-(b-a)^5}\,-}{\boxed{8\xi^7}\,-} > 0$

$b<0 \Longrightarrow \begin{cases} a<0 \\ b<0 \end{cases}$

- Assim

$-\frac{(b-a)^5}{8\xi^7} > 0 \Rightarrow I - I_S > 0$

$\therefore$ I_S é uma aproximação por defeito

Ex. 22 – Ficha 3

$I = \int_a^b f(x)\,dx$ tal que $a=-b$ e $f(-x)+f(x)=2$, $\forall x \in \mathbb{R}$

Regra de Gauss com 2 pontos

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}y+\frac{b+a}{2}\right)dy = b\int_{-1}^{1} f(by)\,dy =$$

$\frac{b-a}{2} = \frac{2b}{2} = b$

$\frac{b+a}{2} = 0$

$$\simeq b\left(f\left(b\times\left(-\frac{1}{\sqrt{3}}\right)\right)+f\left(b\times\frac{1}{\sqrt{3}}\right)\right) = b\left(f\left(-\frac{b}{\sqrt{3}}\right)+f\left(\frac{b}{\sqrt{3}}\right)\right) = 2b$$

$f(-x)+f(x)=2$
$\forall x \in \mathbb{R}$