AM3C – Laplace Transform

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Conteúdo

Tables

Proprierties

$$e^{a\,t}\,\sin w\,t \xrightarrow{\mathcal{L}} \frac{w}{(s-a)^2+w^2}$$

$$e^{a\,t}\,\cos w\,t \xrightarrow{\mathcal{L}} \frac{s-a}{(s-a)^2+w^2} \qquad t \xrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}$$

$$t^n\,e^{a\,t}, n\in\mathbb{N}^+ \xrightarrow{\mathcal{L}} \frac{n!}{(s-a)^{n+1}} \qquad t\,\sin w\,t \xrightarrow{\mathcal{L}} \frac{2\,s\,w}{(s^2+w^2)^2}$$

$$t\,\cos w\,t \xrightarrow{\mathcal{L}} \frac{s^2-w^2}{(s^2+w^2)^2} \qquad t\,\sinh w\,t \xrightarrow{\mathcal{L}} \frac{2\,s\,w}{(s^2-w^2)^2}$$

$$t\,\cosh w\,t \xrightarrow{\mathcal{L}} \frac{s^2+w^2}{(s^2-w^2)^2} \qquad \frac{\sin w\,t}{t} \xrightarrow{\mathcal{L}} \frac{\pi}{2}-\tan^{-1} s/w$$

$$a\,f+b\,g \xrightarrow{\mathcal{L}} a\,F(s)+b\,G(s) \qquad f(\lambda\,t) \xrightarrow{\mathcal{L}} \frac{1}{\lambda}F\left(\frac{s}{\lambda}\right)$$

$$\mathcal{H}(t-\tau)\,f(t-\tau) \xrightarrow{\mathcal{L}} e^{-s\,\tau}\,F(s) \qquad e^{-\lambda\,t}\,f(t) \xrightarrow{\mathcal{L}} F(s+\lambda)$$

$$f(t)/t \xrightarrow{\mathcal{L}} \int_s^\infty F(p)\,\mathrm{d}p \qquad (f\cdot g)(t) \xrightarrow{\mathcal{L}} F(s)\,G(s)$$

Derivative transform

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(\mathrm{D}_t^n f(t)) = s^n\,\mathcal{L}(f) - \sum_{k=0}^{n-1} s^{n-1-k}\,\mathrm{D}_t^k f(0)$$

Translations

$$\mathcal{L}(e^{a\,t}\,f(t)) = F(s-a) \qquad (2.23);$$

$$\mathcal{L}\,-1(F(s-a)) = e^{a\,t}\,f(t) \qquad (2.24);$$

$$\mathcal{L}(f(t-a)\,\mathcal{H}(t-a)) = e^{-a\,s}\,F(s)(2.30);$$

$$\mathcal{L}^{-1}(e^{-a\,s}\,F(s)) = f(t-a)\,\mathcal{H}(t-a)(2.31)$$

Basic transforms

$$\mathcal{L}(1) = 1/s, \quad s>0; \qquad \mathcal{L}(e^{a\,t}) = \frac{1}{s-a}, \quad s>a;$$

$$\mathcal{L}(\cos(w\,t)) = \frac{s}{s^2+w^2}, \quad s>0; \qquad \mathcal{L}(\cosh(w\,t)) = \frac{s}{s^2-w^2}, \quad s>\max(-w,w);$$

$$\mathcal{L}(\sin(w\,t)) = \frac{w}{s^2+w^2}, \quad s>0; \qquad \mathcal{L}(\sinh(w\,t)) = \frac{w}{s^2-w^2}, \quad s>\max(-w,w);$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \qquad s>0\wedge n\in\mathbb{N}^+;$$

Inverse Transforms

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) = \frac{e^{a\,t}-e^{b\,t}}{a-b}, \qquad a\neq b\wedge s>\max(a,b)$$

Derivative Transforms

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0); \qquad s>\rho : \rho \text{ is the exponential order 1 of } f(x)$$

Solving notable transforms

Auxiliar demonstrations

$$\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right) = \frac{1}{b}\,\mathrm{P}_t\left(e^{a\,t}\,(b\,\cos(b\,t))\right) =$$

$$\mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \sin(b\,t) \end{cases}$$

$$= \frac{1}{b}\left(e^{a\,t}\,\sin(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right)\right) =$$
$$= \frac{1}{b}\left(e^{a\,t}\,\sin(b\,t) + \frac{a}{b}\,\mathrm{P}_t\left(e^{a\,t}\,(-b\,\sin(b\,t))\right)\right) =$$

$$\mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \cos(e^{b\,t}) \end{cases}$$

$$=$$
$$= \frac{1}{b}\left(e^{a\,t}\,\sin(b\,t) + \frac{a}{b}\left(e^{a\,t}\,\cos(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right)\right)\right) =$$
$$= \frac{1}{b}\,e^{a\,t}\,\sin(b\,t) + \frac{a}{b^2}\,e^{a\,t}\,\cos(b\,t) - \frac{a^2}{b^2}\,\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right) \implies$$
$$\implies \mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right) = e^{a\,t}\,\frac{a\,\cos(b\,t) + b\,\sin(b\,t)}{a^2+b^2} \tag{2.1}$$

$$\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right) = -\frac{1}{b}\,\mathrm{P}_t\left(e^{a\,t}\,(-b\,\sin(b\,t))\right) =$$

$$\mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \cos(b\,t) \end{cases}$$

$$= -\frac{1}{b}\left(e^{a\,t}\,\cos(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right)\right) =$$
$$= -\frac{1}{b}\left(e^{a\,t}\,\cos(b\,t) - \frac{a}{b}\,\mathrm{P}_t\left(e^{a\,t}\,(b\,\cos(b\,t))\right)\right) =$$

$$\mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \sin(b\,t) \end{cases}$$

$$= -\frac{1}{b}\left(e^{a\,t}\,\cos(b\,t) - \frac{a}{b}\left(e^{a\,t}\,\sin(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right)\right)\right) =$$
$$= -\frac{1}{b}\,e^{a\,t}\,\cos(b\,t) + \frac{a}{b^2}\,e^{a\,t}\,\sin(b\,t) - \frac{a^2}{b^2}\,\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right) \implies$$
$$\implies \mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right) = e^{a\,t}\,\frac{a\,\sin(b\,t) - b\,\cos(b\,t)}{a^2+b^2} \tag{2.2}$$

Transforms

$\mathcal{L}(1)$

$$\mathcal{L}(1) = \int_0^\infty \left(e^{-s\,t}\,1\;\mathrm{d}t\right) = \lim_{k\to\infty}\int_0^k \left(e^{-s\,t}\,1\;\mathrm{d}t\right) = \lim_{k\to\infty}\left(-\frac{1}{s}\,e^{-s\,t}\right)\Big|_0^k =$$
$$= \lim_{k\to\infty}\left(-\frac{1}{s}\,e^{-s\,k} + \frac{1}{s}\,e^{-s\,0}\right) = \frac{1}{s} \tag{2.3}$$

$\mathcal{L}(e^{a\,t})$

$$\mathcal{L}(e^{a\,t}) = \lim_{k\to\infty}\left(\int_0^k e^{-s\,t}\,e^{a\,t}\;\mathrm{d}t\right) = \lim_{k\to\infty}\left(\int_0^k e^{(a-s)\,t}\;\mathrm{d}t\right) = \lim_{k\to\infty}\left(\frac{e^{(a-s)\,t}}{a-s}\right)\Big|_0^k =$$
$$= \lim_{k\to\infty}\left(\frac{e^{(a-s)\,k}}{a-s} - \frac{e^{(a-s)\,0}}{a-s}\right) =$$

$$s > a$$

$$= -\frac{1}{a-s} = \frac{1}{s-a}; \tag{2.4}$$
$$\mathcal{L}^{-1}(1/(s-a)) = e^{a\,t} \tag{2.5}$$

$\mathcal{L}(\cos(w\,t))$

$$\mathcal{L}(\cos(w\,t)) = \lim_{k\to\infty}\left(\int_0^k e^{-s\,t}\,\cos(w\,t)\;\mathrm{d}t\right) =$$

using (2.1)

$$= \lim_{k\to\infty}\left(e^{-s\,t}\,\frac{-s\,\cos(w\,t) + w\,\sin(w\,t)}{s^2+w^2}\right)\Big|_0^k =$$
$$= \lim_{k\to\infty}\left(e^{-s\,k}\,\frac{-s\,\cos(w\,k) + w\,\sin(w\,k)}{s^2+w^2} - e^{-s\,0}\,\frac{-s\,\cos(w\,0) + w\,\sin(w\,0)}{s^2+w^2}\right) =$$

$$s > 0$$

$$= \frac{s}{s^2+w^2}; \tag{2.6}$$
$$\mathcal{L}^{-1}\left(\frac{s}{s^2+w^2}\right) = \cos w\,t \tag{2.7}$$

$\mathcal{L}(\sin(w\,t))$

$$\mathcal{L}(\sin(w\,t)) = \lim_{k\to\infty}\left(\int_0^k e^{-s\,t}\,\sin(w\,t)\;\mathrm{d}t\right) =$$

using (2.2)

$$= \lim_{k\to\infty}\left(e^{-s\,t}\,\frac{-s\,\sin(w\,t) - w\,\cos(w\,t)}{s^2+w^2}\right)\Big|_0^k =$$
$$= \lim_{k\to\infty}\left(e^{-s\,k}\,\frac{-s\,\sin(w\,k) - w\,\cos(w\,k)}{s^2+w^2} - e^{-s\,0}\,\frac{-s\,\sin(w\,0) - w\,\cos(w\,0)}{s^2+w^2}\right) =$$

$$s > 0$$

$$= \frac{w}{s^2+w^2} \tag{2.8}$$

$\mathcal{L}\left(\cosh(a\,t)\right)$

$$\mathcal{L}(\cosh(a\,t)) = \mathcal{L}\left(\frac{1}{2}(e^{a\,t} + e^{-a\,t})\right) = \frac{1}{2}\left(\mathcal{L}\left(e^{a\,t}\right) + \mathcal{L}\left(e^{-a\,t}\right)\right) =$$

using (2.4)

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2} \tag{2.9}$$

$\mathcal{L}\left(\sinh(a\,t)\right)$

$$\mathcal{L}(\cosh(a\,t)) = \mathcal{L}\left(\frac{1}{2}(e^{a\,t} - e^{-a\,t})\right) = \frac{1}{2}\left(\mathcal{L}\left(e^{a\,t}\right) - \mathcal{L}\left(e^{-a\,t}\right)\right) =$$

using (2.4)

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2} \tag{2.10}$$

Inverse Laplace transform

$\mathcal{L}^{-1}(1/(s-a)(s-b))$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) = \mathcal{L}^{-1}\left(\frac{1}{a-b}\left(\frac{a-b+s-s}{(s-a)(s-b)}\right)\right) =$$
$$= \frac{1}{a-b}\,\mathcal{L}^{-1}\left(\left(\frac{s-b}{(s-a)(s-b)} - \frac{s-a}{(s-a)(s-b)}\right)\right) =$$
$$= \frac{1}{a-b}\left(\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-b}\right)\right) =$$

using $(2.4) \land s > \max(a,b) \land a \neq b$

$$= \frac{e^{a\,t} - e^{b\,t}}{a-b} \tag{2.11}$$

$\mathcal{L}(t^n)$

$$\mathcal{L}(t^n) = \lim_{k=\infty}\int_0^k \left(e^{-s\,t}\,t^n\;\mathrm{d}t\right) =$$

$$\mathrm{P}(u\,v') = u\,v - \mathrm{P}(u'\,v)\begin{cases} u = t^n \\ v = -e^{-s\,t}/s \end{cases}$$

$$= \lim_{k=\infty}\left(-\left(\frac{t^n\,e^{-s\,t}}{s}\right)\Big|_0^k - \int_0^k\left(\frac{e^{-s\,t}}{-s}\,n\,t^{n-1}\;\mathrm{d}t\right)\right) =$$
$$= \lim_{k=\infty}\left(-\frac{k^n\,e^{-s\,k}}{s} + \frac{0^n\,e^{-s\,0}}{s} - \int_0^k\left(\frac{e^{-s\,t}}{-s}\,n\,t^{n-1}\;\mathrm{d}t\right)\right) =$$

$$s > 0$$

$$= \frac{n}{s}\lim_{k=\infty}\left(\int_0^k \left(e^{-s\,t}\,t^{n-1}\;\mathrm{d}t\right)\right) = \frac{n}{s}\,\mathcal{L}(t^{n-1}) = \frac{n}{s}\,\frac{n-1}{s}\,\mathcal{L}(t^{n-2}) =$$
$$= \prod_{i=0}^{n-1}\left(\frac{n-i}{s}\right)\mathcal{L}(t^{n-n}) = \frac{n!}{s^n}\,\mathcal{L}(1) =$$

using (2.3)

$$= \frac{n!}{s^n}\,\frac{1}{s} = \frac{n!}{s^{n+1}} \tag{2.12}$$

1 Introduction

$$\mathcal{L}\, f(x) = F(x)$$

Let $f(t)$ be a function of the real variable t, for all $t \in \mathbb{R}$; the values of $f(t)$ may be either real or complex, although in our applications they will be real. The function f is said to be differetiable at tpnly finitely many points of I, and all its points of discotinuity are jumps (i.e. there are right and left limits of the function at those points).

Exploring the existence of the transform

We now introduce a class of functions for which the transformatio will be defined. We assume that the following three conditions are satisfied:

(1) $t = 0 \implies f(t) = 0$

(2) f is piecewise differentialbe

(3) there exist real numbers M, ρ such that

$$|f(t)| \leq M\, e^{\rho\, t} \quad \forall\, t \in \mathbb{R}$$

note: here ρ is said to be the *exponential order* of f

Checking if transform exists

$\cosh t;$ $\qquad$ $\sinh t;$ $\qquad$ t^n

Resposta

$$|\cosh t| = \left|\frac{e^t + e^{-t}}{2}\right| = \frac{e^t + e^{-t}}{2} \leq \frac{e^t + e^t}{2} = e^t;$$

$$|\sinh t| = \left|\frac{e^t - e^{-t}}{2}\right| \leq \frac{1}{2}\left(|e^t| + |e^{-t}|\right) = \frac{1}{2}\left(e^t + e^{-t}\right) \leq \frac{1}{2}\left(e^t + e^t\right) = e^t;$$

$$|t^n| = n!\,\frac{t^n}{n!} \leq n! \sum_{i=0}^{\infty} \frac{t^i}{i!} = n!\, e^t$$

1.1 The Heaviside function

$$\mathcal{H}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{2.13}$$

Like any bounded function, this satisfies condition (3) with $\rho = 0$. Any function $\phi(t)$ that fails to satisfy conditions (1), but does satisfy conditions (2) and (3), then the function $f(t) = \mathcal{H}(t)\,\phi(t)$ will satisfy all three conditions. for Example

$\mathcal{H}(t)\ \sin w\, t,$ $\qquad$ $\mathcal{H}(t)\, t^n,$ $\qquad$ $\mathcal{H}(t)\, e^{a\, t}$

For simplicity we usually omit the factor $\mathcal{H}(t)$

$$\tilde{f}(t) = f(t-a)\ \mathcal{H}(t-a)$$

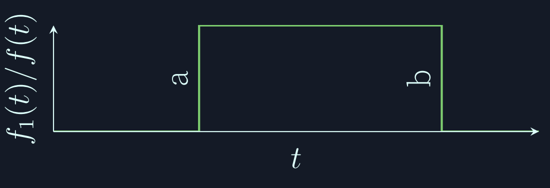
Uses for the Heaviside function

Let $f(t)$ be a function on the interval $t \geq 0$, and let $f_1(t)$ be a "piece" of $f(t)$ on the interval $[a, b[, a \geq 0$, that is

$$f_1(t) \begin{cases} f(t), & t \in [a, b[\\ 0, & c.c. \end{cases}$$

To set the value of $f_1(t)$ to zero for $t < 0$, we multiply $f(t)$ by $\mathcal{H}(t-a)$. To get zero for $t \geq b$ we can substrac from $f(t)$ the values $f(t)$ as $t \geq b$, that is subtract $\mathcal{H}(t-b)\, f(t)$. thus

$$f_1(t) = (\mathcal{H}(t-a) - \mathcal{H}(t-b))f(t)$$



Exemplo 1

Using the Heaviside function, write down the piecewise definition of the function

$$\boldsymbol{f(t)\begin{cases} 0, & 0 \leq t < 2 \\ 3\,t & 2 \leq t < 4, \\ 2, & t \geq 4 \end{cases}}$$

Resposta

$$f(t) = \begin{pmatrix} +3\,t(\mathcal{H}(t-2) - \mathcal{H}(t-4)) \\ +2\,(\mathcal{H}(t-4)) \end{pmatrix}$$

2 Laplace Transform of the Derivative

For the first derivative

Suppose that $f(x)$ follows all three laplace conditions 1 and has exponential order 1 γ

$$\mathcal{L}(f') = s\ \mathcal{L}(f) - f(0), \quad s > \gamma$$

For the n-th derivative

Suppose that $\mathrm{D}_t^i f\ \forall\, i$ follows all three laplace conditions 1 and has exponential order 1 γ

$$\boldsymbol{\mathcal{L}\left(\mathrm{D}_t^n f\right) = s^n\ \mathcal{L}(f) - \sum_{i=1}^{n} s^{n-i}\ \mathrm{D}_t^{i-1} f(0)} \tag{2.14}$$

Exemplo 2

Find the transforms using the derivative method

$$\boldsymbol{t^n}; \qquad \mathbf{sin}\,\boldsymbol{w\,t} \qquad \mathbf{sin}^2\,\boldsymbol{t}$$

Resposta

Solving for t^n

$$\mathcal{L}(\mathrm{D}_t^{n+1} t^n) = \mathcal{L}(0) = 0 =$$

using (2.14)

$$= s^{n+1}\ \mathcal{L}(t^n) - \sum_{i=1}^{n+1} s^{n-i}\ \mathrm{D}_t^{i-1} t^n(0) = s^{n+1}\ \mathcal{L}(t^n) - n! \implies \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

Solving for $\sin w\,t$

$$\mathcal{L}(\mathrm{D}_t^2 \sin w\,t) = \mathcal{L}(-w^2\ \sin(w\,t)) = -w^2\ \mathcal{L}(\sin(w\,t)) =$$

using (2.14)

$$= s^2\ \mathcal{L}(\sin w\,t) - s\ \sin w * 0 - w\ \cos w * 0 \implies \mathcal{L}(\sin w\,t) = \frac{w}{s^2 + w^2}$$

Solving for $\sin^2(t)$

$$\mathcal{L}(\mathrm{D}_t \sin^2 t) = \mathcal{L}(2\ \sin(t)\ \cos(t)) = \mathcal{L}(\sin(2\,t)) =$$

using (2.8)

$$= \frac{2}{s^2 + 4} =$$

using (2.14)

$$= s\ \mathcal{L}(\sin^2 t) - \sin(2 * 0) = s\ \mathcal{L}(\sin^2 t) \implies \mathcal{L}(\sin^2 t) = \frac{2/s}{s^2 + 4}$$

Exemplo 3 Applying to differential equations

Considere o PVI

$$\boldsymbol{y'' + 4\,y' + 3\,y = 0, \quad y(0) = 3, y'(0) = 1}$$

Resposta

Finding general solution

$$y = \mathcal{L}^{-1}\,Y = \tag{2.15}$$

using (2.19)

$$= \mathcal{L}^{-1}\left(\frac{5}{s+1} + \frac{-2}{s+3}\right) = 5\,\mathcal{L}(1/(s+1)) - 2\,\mathcal{L}(1/(s+3)) = \tag{2.16}$$

using (2.4)

$$= 5\,e^{-1\,t} - 2\,e^{-3\,t} \tag{2.17}$$

Checking existence of Laplace transform

$$y_1 = 5\,e^{-t}$$
$$y_2 = -2\,e^{-3\,t}$$
$$s > \max(-1, -3) = -1$$
$$\lambda = \max(-1, -3) = -1$$

Both follow the conditions 1 with greater exponential being -1

Finding Y

$$\mathcal{L}(y'' + 4\,y' + 3\,y) = \mathcal{L}(y'') + 4\ \mathcal{L}(y') + 3\ \mathcal{L}(y) =$$
$$= s^2\ \mathcal{L}(y) - s\,y(0) - y'(0) + 4\,(s\ \mathcal{L}(y) - y(0)) + 3\ \mathcal{L}(y) =$$
$$= s^2\ \mathcal{L}(y) + 4\,s\ \mathcal{L}(y) - 13 - s\,3 + 3\ \mathcal{L}(y) = 0 \implies$$
$$\implies \mathcal{L}(y) = \frac{13 + s\,3}{s^2 + 4\,s + 3} =$$

$$s = \tfrac{-4 \pm \sqrt{4^2 - 4*1*3}}{2\,1} = -2 \pm 1$$

$$= \frac{13 + s\,3}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \tag{2.18}$$

using (2.20)

$$= \frac{5}{s+1} + \frac{-2}{s+3} \tag{2.19}$$

Finding constants in (2.18)

$$13 + s\,3 = A(s+3) + B(s+1) = (A+B)s + 3\,A + B \implies$$

$$\implies \begin{cases} B = 13 - 3\,A = 13 - 15 = -2 \\ A + (13 - 3\,A) = 3 \implies A = 10/2 = 5 \end{cases} \tag{2.20}$$

3 Laplace transform of an Integral

$$\mathcal{L}\left(\int_0^t f(x)\ \mathrm{d}x\right) = \frac{1}{s}\ \mathcal{L}(f(t)); \tag{2.21}$$

$$\mathcal{L}^{-1}(f(t)/s) = \int_0^t (f(x)\ \mathrm{d}x) \tag{2.22}$$

Exemplo 4

Find the inverse Laplace transform of

$$\boldsymbol{F(s) = \frac{1}{s(s^2+w^2)}}$$

Resposta

$$\mathcal{L}^{-1}\left(F(s)\right) = \mathcal{L}^{-1}\left(\frac{1}{s(s^2+w^2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{1}{w}\,\frac{w}{s^2+w^2}\right) =$$

using (2.8)

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{w}\,\mathcal{L}\left(\sin w\,t\right)\right) =$$

using (2.21)

$$= \frac{1}{w}\int_0^t \left(\sin w\,x\,\mathrm{d}x\right) = \frac{1}{w}\left(-\cos(w\,x)/w\right)\Big|_0^t =$$

$$= \frac{1}{w^2}\left(-\cos(w\,t)+\cos(w\,0)\right) = \frac{1-\cos w\,t}{w^2}$$

4 Translação da variavel s

$$\mathcal{L}(f(t)) = F(s), \quad s \in \,]\gamma, \infty[\implies \mathcal{L}(e^{a\,t} f(t)) = F(s-a), \quad s \in \,]a \tag{2.23}$$

$$\mathcal{L}^{-1}(F(s-a)) = e^{a\,t} f(t) \tag{2.24}$$

Exemplo 5

Consider the problem with inital values

$$\boldsymbol{y'' + 2\,y' + 5\,y = 0, \quad y(0) = 2, \quad y'(0) = -4}$$

Find the general solution

Resposta

General solution for y

$$y = \mathcal{L}^{-1}\,Y = \tag{2.25}$$

using (2.29)

$$= \mathcal{L}^{-1}\left(2\,\frac{s+1}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}\right) = \tag{2.26}$$

$$= 2\;\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) - \mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+2^2}\right) = \tag{2.27}$$

using (2.7)

$$= 2\,e^{-t}\,\cos(2\,t) - e^{-t}\,\sin(2\,t) \tag{2.28}$$

Finding Y

$$\mathcal{L}(y'' + 2\,y' + 5\,y) = \mathcal{L}(y'') + 2\;\mathcal{L}(y') + 5\;\mathcal{L}(y) =$$

using (2.14)

$$= s^2\,Y - s\,y(0) - y'(0) + 2\,(s\,Y - y(0)) + 5\,Y =$$

$$= s^2\,Y - s\,(2) - (-4) + 2\,s\,Y - 2\,(2) + 5\,Y = 0 \implies$$

$$\implies = Y = \frac{s\,2}{s^2+2\,s+5} = \frac{2\,(s+1-1)}{(s+1)^2+4} = 2\,\frac{s+1}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2} \tag{2.29}$$

5 Translation of the variable t

$$\mathcal{L}(f(t)) = F(s), s \in]\gamma, \infty[\implies \mathcal{L}(f(t-a)\ \mathcal{H}(t-a)) = e^{-a\,s}\, F(s); \tag{2.30}$$

$$\mathcal{L}^{-1}(e^{-a\,s}\, F(s)) = f(t-a)\ \mathcal{H}(t-a) \tag{2.31}$$

Exemplo 6

$$\boldsymbol{F(s) = \frac{e^{-3\,s}}{s^3}}$$

Resposta

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{e^{-3\,s}}{s^3}\right) = \mathcal{L}^{-1}\left(\frac{e^{-3\,s}}{2}\,\frac{2}{s^3}\right) =$$

using (2.12)

$$= \frac{1}{2}\,\mathcal{L}^{-1}\left(e^{-3\,s}\,\mathcal{L}(t^2)\right) =$$

using (2.31)

$$= \frac{(t-3)^2\;\mathcal{H}(t-3)}{2}$$

Exemplo 7

Find the laplace transform of the function

$$\boldsymbol{f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\,\pi \\ \sin t, & t > 2\,pi \end{cases}}$$

Resposta

Solving lagplace transform of f

$$\begin{aligned} &\mathcal{L}(f(t)) = \mathcal{L}(1(\mathcal{H}(t-0) - \mathcal{H}(t-\pi)) + \sin(t)\,\mathcal{H}(t-2\,\pi)) = \\ &= \mathcal{L}(1) - \mathcal{L}(\mathcal{H}(t-\pi)) + \mathcal{L}(\sin(t)\,\mathcal{H}(t-2\,\pi)) = \\ &= \frac{1}{s} - \frac{e^{-\pi\,s}}{s} + \frac{e^{-2\,\pi\,s}}{s^2+1} \end{aligned} \qquad \text{using (2.8)}$$

Exemplo 8

Consider the problem of initial values

$$\boldsymbol{y'' + 3\,y' + 2\,y = r(t), \quad y(0) = 0, y'(0) = 0;}$$

$$\boldsymbol{r(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}}$$

Resposta

Finding y

$$y = \mathcal{L}^{-1}\,Y =$$

using (2.33)

$$= \mathcal{L}^{-1}\left((1 - e^{-s})\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right)\right) =$$

$$= \mathcal{L}^{-1}\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2} - e^{-s}\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right)\right) =$$

using (2.5) (2.31)

$$= \frac{1}{2} - e^{-1\,t} + \frac{1}{2}\,e^{-2\,t} - \mathcal{H}(t-1)\left(\frac{1}{2} - e^{-1\,t} + \frac{1}{2}\,e^{-2\,t}\right)$$

$$\mathcal{L}(y'' + 3\,y' + 2\,y) = \mathcal{L}(y'') + 3\;\mathcal{L}(y') + 2\;\mathcal{L}(y) =$$

using (2.14)

$$= s^2\,Y - s\,y(0) - y'(0) + 3\,(s\,Y - y(0)) + 2\,Y = Y(s^2 + 3\,s + 2) =$$

$$= \mathcal{L}(r(t)) = \mathcal{L}(\mathcal{H}(t-0) - \mathcal{H}(t-1)) = \mathcal{L}(1 - \mathcal{H}(t-1)) = \frac{1}{s} - \frac{e^{-1\,s}}{s} \implies$$

$$\implies Y = \frac{1 - e^{-s}}{s(s^2 + 3\,s + 2)} = \frac{1 - e^{-s}}{s(s(s+1) + 2(s+1))} =$$

$$= \frac{1 - e^{-s}}{s(s+2)(s+1)} = (1 - e^{-s})\left(\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right) = \tag{2.32}$$

using (2.34)

$$= (1 - e^{-s})\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right) \tag{2.33}$$

Finding constants in (2.32)

$$1 = (A + C + B)\,s^2 + s\,(3\,A + 2\,B + C) + A\,2 \implies$$

$$\begin{cases} A = 1/2 \\ C = -B - A = -B - 1/2 = -(-1) - 1/2 = 1/2 \\ 3\,(1/2) + 2\,B + (-B - 1/2) = 0 \implies B = -1 \end{cases} \tag{2.34}$$