AM 2C – Exame Epoca Especial 2023 Resolução

Felipe B. Pinto 61387 – MIEQB

12 de janeiro de 2024

Conteúdo

Questão 1

Considere a função vetorial

$$\boldsymbol{\vec{r}} : [\mathbf{0}, \mathbf{4}\,\boldsymbol{\pi}] \to \mathbb{R}^{\mathbf{3}},\, \boldsymbol{\vec{r}}(\boldsymbol{t}) = \begin{pmatrix} \quad \cos(\boldsymbol{\pi} + \boldsymbol{t})\,\boldsymbol{\hat{\imath}} & + \\ + \sin(\boldsymbol{\pi} + \boldsymbol{t})\,\boldsymbol{\hat{\jmath}} & + \\ + \sqrt{\mathbf{3}}\,\boldsymbol{t}\,\boldsymbol{\hat{k}} & \end{pmatrix}$$

e designe por C a curva orientada definida por $\vec{r}$. A equação vetorial da reta tangente à curva C no ponto $(1, 0, 3\sqrt{3}\,\pi)$ é:

$$\begin{cases} \cos(\pi + t_0) = 1 \\ \sin(\pi + t_0) = 0 \\ \sqrt{3}\, t_0 = 3\sqrt{3}\,\pi \implies t_0 = 3\,\pi \end{cases}$$

$$r'(t_0) = \frac{\mathrm{d}\cos(\pi + t)}{\mathrm{d}t}(t_0)\hat{\imath} + \frac{\mathrm{d}\sin(\pi + t)}{\mathrm{d}t}(t_0)\hat{\jmath} + \frac{\mathrm{d}\sqrt{3}\,t}{\mathrm{d}t}(t_0)\hat{k} =$$
$$= -\sin(\pi + 3\,\pi)\hat{\imath} + \cos(\pi + 3\,\pi)\hat{\jmath} + \sqrt{3}\hat{k} =$$
$$= \hat{\jmath} + \sqrt{3}\hat{k} \implies$$

$$\implies (x, y, z) = (1, 0, 3\sqrt{3}\pi) + t\,(0, 1, \sqrt{3})$$

Questão 2

A equação do plano tangente à superfície de nível

$$\boldsymbol{x^3 + y^3 + (x+1)\,e^z = 2}$$

no ponto $(1, -1, 0)$ é:

Resposta

$$\begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0)\,(x - x_0) & + \\ +\dfrac{\partial f}{\partial y}(y_0)\,(y - y_0) & + \\ +\dfrac{\partial f}{\partial z}(z_0)\,(z - z_0) & \end{pmatrix} = \begin{pmatrix} (3\,(1)^2 + e^0)\,(x-1) & + \\ +(3\,(-1)^2)\,(y+1) & + \\ +(1+1)e^(0)\,z & \end{pmatrix} =$$

$$= 4\,x + 3\,y + 2\,z - 1 = 0$$

Resposta D.

Questão 3

Considere o subconjunto de $\mathbb{R}^2$ definido por

$$\boldsymbol{C = \left\{(x, y) \in [-1, 1] \times [-1, 1] : x^2 + y^2 > 1\right\} \cup \{(0, 0)\}}$$

Tem-se (C' é o conjunto dos pontos de acumulação de C e $\bar{C}$ é a aderência de C)

Questão 4

Integrais de curva: Paremtrizar do segimento e multiplicar pela derivada da parametrização

$$\int_{C,t} \boldsymbol{f}(\boldsymbol{s})\,\mathrm{d}\boldsymbol{s} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{\phi}(\boldsymbol{t}))\,\|\boldsymbol{\phi}'(\boldsymbol{t})\|\,\mathrm{d}\boldsymbol{t}$$

$$\phi(t) = A + (B - A) = (0,0) + ((1,1) - (0,0))\,t = (1,1)\,t = (t,t) : t \in [0,1$$

$$\implies \int_{C,t} f(s)\,\mathrm{d}s = \int_a^b f(\phi(t))\,\|\phi'(t)\|\,\mathrm{d}t = \int_0^1 f((t,t))\,\|(1,1)\|\,\mathrm{d}t = \int_0^1 ($$

Questão 5

- limites direcionais:

Resposta

Q5 a.

B

$$|f(x,y)-0| = \left|\frac{x\,y^2}{x^4+y^2}\right| = \frac{y^2\ |x|}{x^4+y^2} \leq \|(x,y)\|\,\sqrt{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{x\,y^2}{x^4+y^2} = \lim_{r\to 0} \frac{r\ \cos(\theta)\,(r\ \sin(\theta))^2}{(r\ \cos(\theta))^4+(r\ \sin(\theta))^2} =$$

$$= \lim_{r\to 0} \frac{r\ \cos(\theta)\ \sin^2(\theta)}{r^2\ \cos^4(\theta)+\sin^2(\theta)} = \frac{0}{0+\sin^2(\theta)} = 0$$

$\therefore\ f$ é continua em $(0,0)$

Questão 6

$$\boldsymbol{\lim_{(x,y)\to(1,1)} \frac{3(y-1)\ \exp(-(x-1)^2)}{\sqrt{x-1}^2+(y-1)^2}}$$

Resposta

$$\lim_{(x,y)\to(1,1)} \frac{3(y-1)\ \exp(-(x-1)^2)}{\sqrt{(x-1)^2+(y-1)^2}} =$$

$$= \lim_{x\to1, y=m(x-1)+1} \frac{3(m(x-1)+1-1)\ \exp(-(x-1)^2)}{\sqrt{(x-1)^2+(m(x-1)+1-1)^2}} =$$

$$= \lim_{x\to1, y=m(x-1)+1} \frac{3(m(x-1))\ \exp(-(x-1)^2)}{\sqrt{(m^2+1)(x-1)^2}} =$$

$$= \lim_{x\to1, y=m(x-1)+1} \frac{3\,m\ \exp(-(x-1)^2)}{\frac{|x-1|}{x-1}\sqrt{(m^2+1)}} =$$

$$= \lim_{x\to1, y=m(x-1)+1} \frac{3\,m\ \exp(-(x-1)^2)}{\frac{|x-1|}{x-1}\sqrt{(m^2+1)}} \implies$$

$$\implies \lim_{x\to1^+, y=m(x-1)+1} \frac{3\,m\ \exp(-(x-1)^2)}{\frac{|x-1|}{x-1}\sqrt{(m^2+1)}} = \frac{3\,m\ \exp(0)}{\sqrt{(m^2+1)}} = \frac{3\,m}{\sqrt{(m^2+1)}};$$

$$\lim_{x\to1^-, y=m(x-1)+1} \frac{3\,m\ \exp(-(x-1)^2)}{\frac{|x-1|}{x-1}\sqrt{(m^2+1)}} = \frac{3\,m\ \exp(0)}{-\sqrt{(m^2+1)}} = \frac{3\,m}{-\sqrt{(m^2+1)}}$$

Questão 7

$$\boldsymbol{f(x,y) = \begin{cases} \frac{x\,y^2}{x^4+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h*0^2}{h^4+0^2}}{h} = 0;$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{0*k^2}{0^4+k^2}}{k} = 0;$$

$$D_{\vec{u}} f(0,0) = \lim_{t\to 0} \frac{f(0+t/\sqrt{2}, 0+t/\sqrt{2}) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{\frac{(t/\sqrt{2})*(t/\sqrt{2})^2}{(t/\sqrt{2})^4+(t/\sqrt{2})^2}}{t} = \lim_{t\to 0} \frac{1/2\sqrt{2}}{t^2/4+1/2} = \frac{1/2\,\sqrt{2}}{1/2} = 1/\sqrt{2}$$

Questão 8

$$\boldsymbol{f : \mathbb{R}^2 \to \mathbb{R}; C^1 \in \mathbb{R}^2 : \nabla f(1/2, 0) = (-1, 1)}$$

$$\boldsymbol{H(x,y) = f\left(\frac{\sin y}{1+x^2}, x + \cos(2\,y)\right)}$$

Resposta

$$H(x,y) = f\left(\frac{\sin y}{1+x^2}, x + \cos(2\,y)\right) = f\left(\phi(x,y), \rho(x,y)\right);$$

$$\frac{\mathrm{d}H}{\mathrm{d}x}(1,\pi/2) = \frac{\mathrm{d}f}{\mathrm{d}\phi(x,y)}\left(\frac{\sin\pi/2}{1+1^2}, 1+\cos(2\,\pi/2)\right)\frac{\mathrm{d}\phi}{\mathrm{d}x}(1,\pi/2) =$$
$$= -1\,\frac{\sin\pi/2}{(1+1^2)^2}\,2(1+1^2) = -1;$$

$$\frac{\mathrm{d}H}{\mathrm{d}x}(1,\pi/2) = \frac{\mathrm{d}f}{\mathrm{d}x}\left(\frac{\sin\pi/2}{1+1^2}, 1+\cos(2\,\pi/2)\right)\frac{\mathrm{d}\frac{\sin y}{1+x^2}}{\mathrm{d}x}(1,\pi/2) =$$
$$= \frac{\mathrm{d}f}{\mathrm{d}x}\left(\frac{1}{2}, 0\right)\frac{\sin\pi/2}{(1+1^2)^2}\,2*1 = \frac{\mathrm{d}f}{\mathrm{d}x}\left(\frac{1}{2}, 0\right)\frac{\sin\pi/2}{(1+1^2)^2}\,2*1 = -2$$

Questão 9

Seja $\varphi : \mathbb{R} \to \mathbb{R}$ uma função de classe C^2 em $\mathbb{R}$ tal que $\varphi'(0) = 5, \varphi"(0) = -1$. Considere $u(x,t) = \varphi(x^2 - 2\,t)$. Tem-se:

Resposta

$$\frac{\partial^2 u}{\partial x^2}(2,2) = \frac{\partial^2(\varphi(x^2 - 2\,t))}{\partial x^2}(2,2) = \frac{\partial(\varphi'(x^2 - 2\,t)\,2\,x)}{\partial x}(2,2) =$$
$$= 2\,\left(\varphi"(2^2 - 2 * 2)\,2 + \varphi'(2^2 - 2 * 2)\right) =$$
$$= 2\,(-1 * 2 + 5) = 6$$

Questão 10

Seja

$$\int_0^2 \int_y^2 \exp(x^2)\ \mathrm{d}x\,\mathrm{d}y$$

Tem-se:

Sugestão: Troque a ordem de integração

Resposta

$$\begin{cases} y \in [0,2] \\ x \in [y,2] \end{cases}$$

$$\begin{cases} y \in [0,x] \\ x \in [0,2] \end{cases} \quad ;$$

$$\int_0^2 \int_y^2 \exp(x^2)\ \mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_0^x \exp(x^2)\ \mathrm{d}y\,\mathrm{d}x = \int_0^2 \exp(x^2)\,(x-0)\,\mathrm{d}x =$$

$$= \int_0^2 \exp(x^2)\,2\,x\,\mathrm{d}x/2 = (\exp 2^2 - \exp(0^2))/2 = (\exp 4 - 1)/2$$

Questão 11

A equação

$$\exp(x\,z) + y\,\sin x - y^2 + z^3 + 2\,x = 2\,\pi$$

define implicitamente x como função de y e z numa vizinhança do ponto $(x_0, y_0, z_0) = (\pi, 1, 0)$. Para essa função tem-se:

Resposta

$$\frac{\partial x}{\partial y}(1,0) = -\frac{\frac{\partial f}{\partial y}(\pi,1,0)}{\frac{\partial f}{\partial x}(\pi,1,0)} = -\frac{\sin(\pi) - 2 * 1}{\exp(\pi * 0)\,0 + 1\,\cos\pi + 2} = 2$$

$$\frac{\partial x}{\partial y}(\pi,1,0) : \exp(x\,z) + y\,\sin x - y^2 + z^3 + 2\,x = 2\,\pi \implies$$

$$\implies \frac{\partial(\exp(x\,z) + y\,\sin x - y^2 + z^3 + 2\,x)}{\partial y}(\pi,1,0) =$$

$$= \begin{pmatrix} \frac{\partial \exp(x\,z)}{\partial y}(\pi,1,0) & + \\ +\frac{\partial y\,\sin x}{\partial y}(\pi,1,0) & + \\ -\frac{\partial y^2}{\partial y}(\pi,1,0) & + \\ +\frac{\partial z^3}{\partial y}(\pi,1,0) & + \\ +\frac{\partial 2\,x}{\partial y}(\pi,1,0) & \end{pmatrix} = \begin{pmatrix} 0 & + \\ +\sin\pi 1\,\cos\pi\,\frac{\partial x}{\partial y}(1,0) & + \\ -2\,(1) & + \\ +3\,(0)^2\,\frac{\partial z}{\partial y}(\pi,1) & + \\ +2\,\frac{\partial x}{\partial y}(1,0) & \end{pmatrix} =$$

$$= -2 + \frac{\partial x}{\partial y}(1,0) =$$

$$= \frac{\partial(2\,\pi)}{\partial x} = 0 \implies$$

$$\implies \frac{\partial x}{\partial y}(1,0) = 2$$

Questão 12

Considere a função

$$\boldsymbol{f(x,y) = y^4/2 - x\,y^2 + x^2 - 4\,x}$$

Escolha a affirmativa correta

Resposta

$$\det H_f = \begin{vmatrix} \frac{\partial^2 f}{\partial x\,\partial x} & \frac{\partial^2 f}{\partial x\,\partial y} \\ \frac{\partial^2 f}{\partial y\,\partial x} & \frac{\partial^2 f}{\partial y\,\partial y} \end{vmatrix} = \begin{vmatrix} 2 & 2\,y \\ -2\,y & 6\,y^2 - x\,2 \end{vmatrix} =$$
$$= 12\,y^2 - 4\,x + 4\,y^2 = 16\,y^2 - 4\,x;$$

$$\det H_f(4,-2) = 16\,(-2)^2 - 4 * 4 = 48$$
$\therefore$ Crítico mínimo Local;

$$\det H_f(2,0) = 16\,(0)^2 - 4 * 2 = -8$$
$\therefore$ Ponto de sela

Questão 13

Cnsidere a superfície

$$\rho = \left\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{3}\,x, (x, y) \in [0, 1] \times [0, 2]\right\}$$

orientada com a terceira componente do campo vetorial normal não negativa, e o campo vetorial

$$\vec{F}(x, y, z) = -3\left(x^2 + \sqrt{z^2}4\hat{\jmath}\right)$$

Seja $\mathcal{L}$ o bordo de σ orientado positivamente de acordo com σ. O valor do integral curvilíneo

$$\int_{\mathcal{L}} -\frac{z^3}{4}\,\mathrm{d}x + x^3\ \mathrm{d}z$$

body

Questão 14

Seja $f(x,y)$ uma função contínua em $\mathbb{R}^2$. Considere a igualdade

$$\iint_{\boldsymbol{\mathcal{R}}} \boldsymbol{f(x,y)}\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} = \int_{\mathbf{0}}^{\mathbf{1}} \int_{\boldsymbol{y^2}}^{\sqrt{\mathbf{2}-\boldsymbol{y^2}}} \boldsymbol{f(x,y)}\ \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$$

Tem se:

Resposta

$$\begin{cases} x = \sqrt{2-y^2} \implies |y| = \sqrt{2-x^2} \\ x = y^2 \implies \sqrt{x} = |y| \\ \text{Integra com} \end{cases}$$

$$\int_0^1 \int_{y^2}^{\sqrt{2-y^2}} f(x,y)\ \mathrm{d}x\,\mathrm{d}y =$$

$$= \int_0^1 \int_0^{\sqrt{x}} f(x,y)\ \mathrm{d}y\,\mathrm{d}x + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} f(x,y)\ \mathrm{d}y\,\mathrm{d}x$$

Questão 15

Seja

$$I = \iint_{\mathcal{R}} \exp\left(\frac{y-x}{y+x}\right) \mathrm{d}x \, \mathrm{d}y$$

Onde $\mathcal{R}$ é o triangulo definido pelas retas de equações

$$x = 0, y = 0 \text{ e } x + y = 4.$$

Tem se:

Sugestão: Considere a mudança de variáveis $y = x - y, v = x + y$

Questão 18

$$\begin{aligned} &\boldsymbol{C_1 : y = 1, 0 \leq x \leq 2} \\ &\boldsymbol{C_2 : x^2 + y^2 = 4, 1 \leq x \leq 2 \wedge y \geq 0} \\ &\boldsymbol{C_3 : y = \sqrt{3}\, x, 0 \leq x \leq 1} \end{aligned}$$

Resposta

$$\begin{cases} \phi_1(t) = (t, 0), & 0 \leq t \leq 2 \\ \phi_2(t) = (2\,\cos t, 2\,\sin t), & 0 \leq t \leq \pi/3 \\ \phi_3(t) = (t, \sqrt{3}\, t) & 0 \leq t \leq 1 \\ \phi_1'(t) = (1, 0) \\ \phi_2'(t) = (-2\,\sin t, 2\,\cos t) \\ \phi_3'(t) = (1, \sqrt{3}) \end{cases}$$

$$\oint_C 0\ \mathrm{d}x + x\,y\ \mathrm{d}y = \iint_{C_{int}} y\ \mathrm{d}A =$$

$$= \iint_{C_1} y\ \mathrm{d}A + \iint_{C_2} y\ \mathrm{d}A + \iint_{C_3} y\ \mathrm{d}A =$$

$$= \left(\begin{array}{lr} \quad \displaystyle\int_0^2 0 * \|1, 0\|\ \mathrm{d}t & + \\ + \displaystyle\int_0^{\pi/3} 2\,\sin t\ \|(-2\sin t, 2\,\cos t)\|\ \mathrm{d}t & + \\ + \displaystyle\int_0^1 \sqrt{3}\, t\ \left\|(1, \sqrt{3})\right\|\ \mathrm{d}t & \end{array} \right)$$

Questão 21

$$\boldsymbol{0 \leq z \leq 3, 1 \leq x^2 + y^2, x^2 + y^2 \leq z + 1}$$

Resposta

$$\begin{cases} 0 \leq z \leq 3 \\ 1 \leq r^2 \\ r^2 \leq z + 1 \end{cases}$$

$$\iiint_{\varepsilon} \mathrm{d}V = \int_0^3 \int_0^{2\,\pi} \int_1^{\sqrt{z+1}} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z = \cdots = 2\,\pi\,(3^2/2)/2 = \pi\,9/2$$

Questão 22

Considere o sólido $\mathcal{D}$ de $\mathbb{R}^2$ definido por

$$\boldsymbol{\mathcal{D} = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 \leq y \leq 1 \wedge 0 \leq z \leq 2\right\}}$$

$$\boldsymbol{\vec{F}(x,y,z) = \left(\cos y + x^3, y+1, 2\,x^3 - z\right)}$$

O valor do fluxo $\iint_\sigma \vec{F} \cdot \vec{n}\,\mathrm{d}S$

Resposta

$$\iint_\sigma \vec{F} \cdot \vec{n}\,\mathrm{d}S = \iiint divF\ \mathrm{d}V = \iiint \left(3\,x^2 + 1 - 1\right)\ \mathrm{d}V =$$

$$= \int_0^2 \int_0^1 \int_{x^2}^1 3\,x^2\ \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z =$$

$$= \int_0^2 \int_0^1 3\,x^2\left(1 - x^2\right)\ \mathrm{d}x\,\mathrm{d}z =$$

$$= \int_0^2 3\left(1^3 - 0^3\right)/3 - 3\left(1^5 - 0^5\right)/5\ \mathrm{d}z =$$

$$= 2(1 - 3/5) = 4/5$$