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– Apendice

1 Identidades Trigonométricas

$$\sin^2(x) + \cos^2(x) = 1$$
$$1 + \tan^2(x) = \sec^2(x)$$
$$1 + \cot^2(x) = \csc^2(x)$$

$$\sin^2(x) = \frac{1 - \cos(2\,x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2\,x)}{2}$$
$$\sin(2x) = 2\sin(x)\,\cos(x)$$

$$2\,\sin(x)\,\cos(y) = \sin(x - y) + \sin(x + y)$$
$$2\,\sin(x)\,\sin(y) = \cos(x - y) - \cos(x + y)$$
$$\cos(x)\,\cos(y) = \cos(x - y) + \cos(x + 1)$$
$$1 \pm \sin(x) = 1 \pm \cos(\pi/2 - x)$$

2 Trigonometria Hiperbólica

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth(x) = 1/\tanh(x)$$
$$\mathrm{sech}(x) = 1/\cosh(x)$$
$$\mathrm{csch}(x) = 1/\sinh(x)$$

3 Tabela de Derivadas

Basic

$$(u^n)' = n\,u^{n-1}\,u'$$
$$(u\;\;v)' = \;\;u'\,v + v'\,u$$
$$(u/v)' = (u'\,v - v'\,u)/v^2$$

Exponentials

$$(a^u)' = a^u\;\ln(a)\,u'; \qquad (a > 0 \land a \neq 1)$$

$$(e^u)' = e^u\,u'$$
$$\log_a'(u) = \frac{u'}{u}\;\log_a(e)$$
$$\ln'(u) = \frac{1}{u}\,u'$$
$$(u^v)' = v\,u^{v-1}\,u' + u^v\;\ln(u)\,v'$$

Trigonométric

$$\sin'(u) = \quad u'\;\cos(u)$$
$$\cos'(u) = -u'\;\sin(u)$$
$$\tan'(u) = \quad u'\;\sec^2(u)$$
$$\cot'(u) = -u'\;\csc^2(u)$$
$$\sec'(u) = \quad u'\;\sec(u)\;\tan(u)$$
$$\csc'(u) = -u'\;\csc(u)\;\cot(u)$$

Hyperbolic

$$\sinh'(u) = \cosh(u)$$
$$\cosh'(u) = \sinh(u)$$
$$\tanh'(u) = 1 - \tanh^2(u)$$
$$\coth'(u) = 1 - \coth^2(u)$$
$$\operatorname{sech}'(u) = -\tanh(u)\;\operatorname{sech}(u)$$
$$\operatorname{csch}'(u) = -\coth(u)\;\operatorname{csch}(u)$$

Arcs

$$\arcsin'(u) = \quad \frac{u'}{\sqrt{1-u^2}}$$
$$\arccos'(u) = -\frac{u'}{\sqrt{1-u^2}}$$
$$\arctan'(u) = \quad \frac{u'}{1+u^2}$$
$$\operatorname{arccot}'(u) = -\frac{u'}{1+u^2}$$
$$\operatorname{arcsec}'(u) = \quad \frac{u'}{|u|\,\sqrt{u^2-1}}; (|u| > 1)$$
$$\operatorname{arccsc}'(u) = -\frac{u'}{|u|\,\sqrt{u^2-1}}; (|u| > 1)$$

4 Tabela de Integrais

Basics

$$\int \mathrm{d}u = c + u$$

$$\int u^n \,\mathrm{d}u = c + \frac{u^{n+1}}{n+1}; \qquad (n \neq -1)$$

$$\int \mathrm{d}u/u = c + \ln|u|$$

$$\int a^u \,\mathrm{d}u = c + \frac{a^u}{\ln a}; \qquad (a > 0 \wedge a \neq 1)$$

$$\int e^u \,\mathrm{d}u = c + e^u$$

trigonometric

$$\int \sin(u)\,\mathrm{d}u = c - \cos u$$

$$\int \cos(u)\,\mathrm{d}u = c + \sin u$$

$$\int \tan(u)\,\mathrm{d}u = c + \ln|\sec(u)|$$

$$\int \cot(u)\,\mathrm{d}u = c + \ln|\sin(u)|$$

$$\int \sec(u)\,\mathrm{d}u = c + \ln|\sec(u) + \tan(u)|$$

$$\int \csc(u)\,\mathrm{d}u = c + \ln|\csc(u) - \cot(u)|$$

$$\int \sec(u)\,\tan(u)\,\mathrm{d}u = c + \sec(u)$$

$$\int \csc(u)\,\cot(u)\,\mathrm{d}u = c - \csc(u)$$

$$\int \sec^2(u)\,\mathrm{d}u = c + \tan(u)$$

$$\int \csc^2(u)\,\mathrm{d}u = c - \cot(u)$$

expressions

$$\int \mathrm{d}u/(u^2 + a^2) = \arctan(u/a)/a + c$$

$$\int \mathrm{d}u/(u^2 - a^2) = \ln\left|\frac{u-a}{u+a}\right|/2\,a + c; \qquad (u^2 > a^2)$$

$$\int \mathrm{d}u/\sqrt{u^2 + a^2} = \ln|u + \sqrt{u^2 + a^2}| + c$$

$$\int \mathrm{d}u/\sqrt{u^2 - a^2} = \arcsin(u/a) + c; \qquad (u^2 < a^2)$$

$$\int \mathrm{d}u/\sqrt{a^2 - u^2} = \arcsin(u/a) + c; \qquad (u^2 < a^2)$$

$$\int \mathrm{d}u/\left(u\sqrt{a^2 - u^2}\right) = \operatorname{arcsec}|u/a|/a + c$$

Uncommon Integrals

$$\int \sin^n(a\,u)\,\mathrm{d}u = -\frac{\sin^{n-1}(a\,u)\,\cos(a\,u)}{a\,n} + \frac{n-1}{n}\int \sin^{n-2}(a\,u)\,\mathrm{d}u$$

$$\int \cos^n(a\,u)\,\mathrm{d}u = \frac{\sin(a\,u)\,\cos^{n-1}(a\,u)}{a\,n} + \frac{n-1}{n}\int \cos^{n-2}(a\,u)\,\mathrm{d}u$$

$$\int \tan^n(a\,u)\,\mathrm{d}u = \frac{\tan^{n-1}(a\,u)}{a(n-1)} - \int \tan^{n-2}(a\,u)\,\mathrm{d}u$$

$$\int \cot^n(a\,u)\,\mathrm{d}u = -\frac{\cot^{n-1}(a\,u)}{a(n-1)} - \int \cot^{n-2}(a\,u)\,\mathrm{d}u$$

$$\int \sec^n(a\,u)\,\mathrm{d}u = \frac{\sec^{n-2}(a\,u)\tan(a\,u)}{a\,(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2}(a\,u)\,\mathrm{d}u$$

$$\int \csc^n(a\,u)\,\mathrm{d}u = -\frac{\csc^{n-2}(a\,u)\cot(a\,u)}{a\,(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2}(a\,u)\,\mathrm{d}u$$

Indefinite Integral Rules

$$P(u\,v') = u\,v - P(u'\,v)$$

$$P(f(g(x))\,g'(x)) = P(f(u)\,\mathrm{d}u); \quad u = g(x)$$

I – Equações Diferenciais Ordinárias (EDO)

All differential equation methods

Ordinary first order equations

$$y' + a(x)\,y = b(x) \qquad \text{EDO ord:1 (3);}$$
$$y' + a(x)\,y = b(x) \qquad \text{Method varying of arbitrary constants for edo ord:1 (4);}$$
$$y' + a(x)\,y = b(x)\,y^k \qquad \text{Bernoulli's equation (5.1);}$$
$$y' + a(x)\,y = b(x) + c(x)\,y^2 \qquad \text{Riccati's equation (5.2)}$$

General differential equations

$$P\,y = 0; \qquad y = \varphi(x)\ \mathrm{P}_x\,z \qquad \text{Order reduction of diffeq (8);}$$
$$P\,y = 0 \qquad \text{Method of variation of arbitrary constants for n (10);}$$
$$P\,y = e^{\alpha\,x}\,f(x); \qquad y = y_h + \bar{y}; \qquad \text{linear diffeq ord:n } \mathrm{D}_x^n \to r^i \text{ (11);}$$
$$P\,y = 0; \qquad y = y_h; \qquad \text{linear diffeq ord:n } \mathrm{D}_x^n \to r^i;\, f(x) = 0 \text{ (11.1);}$$
$$P\,y = e^{\alpha\,x}\,P_k(x); \qquad \bar{y} = x^p\,e^{\alpha\,x}\,Q_k(x); \qquad \text{linear diffeq ord:n } \mathrm{D}_x^n \to r^i;\, f = P_k(x) \text{ (11.2);}$$
$$P\,y = e^{\alpha\,x}\,(a\ \cos(w\,x) + b\ \sin(w\,x)); \qquad \bar{y} = x^p\,e^{\alpha\,x}\,(a_0\ \cos(w\,x) + b_0\ \sin(w\,x)); \qquad \text{linear diffeq ord:n } \mathrm{D}_x^n \to r^i \text{ (11.3);}$$
$$P\,y = y\sum_{i=0}^{n} a_i\,x^i\ \mathrm{D}_x^i = f(x) = \qquad =y\sum_{i=0}^{n} b_i\ \mathrm{D}_t^i = f(e^t);\, x = e^t; \qquad \text{Eulers equation (12)}$$

non-linear exact/non-exact equations

$$\begin{pmatrix} +u(x,y)\ \mathrm{d}x \\ +v(x,y)\ \mathrm{d}y \end{pmatrix} = 0; \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}; \qquad \text{exact (13);}$$
$$\phi(x,y)\begin{pmatrix} +u(x,y)\ \mathrm{d}x \\ +v(x,y)\ \mathrm{d}y \end{pmatrix} = 0; \qquad \frac{\partial(u\,\phi)}{\partial y} = \frac{\partial(v\,\phi)}{\partial x}; \qquad \text{non-exact w integrating factor (13.3);}$$
$$\begin{pmatrix} +u(x)\ \mathrm{d}x \\ +v(y)\ \mathrm{d}y \end{pmatrix} = 0; \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}; \qquad \mathrm{P}_x\,u(x) + \mathrm{P}_y\,v(y) = \text{const}; \qquad \text{exact w sep variables (13.4)}$$

First order equations not solved by the derivative

$$y = x\,\alpha(y') + \beta(y'); \qquad \alpha(y') \neq y'; \qquad y' = p; \qquad \text{Lagrange's equation (14);}$$
$$y = x\,y' + \beta(y') \qquad y' = p; \qquad \text{Clairaut's equation (15)}$$

linear eqsystem of constcoeff

$$\left\{\begin{array}{ccc} P_{x,0}\,x + P_{y,0}\,y & = & f_0(t) \\ P_{x,1}\,x + P_{y,1}\,y & = & f_1(t) \\ & \dots & \end{array}\right\}$$

1 EDO de Primeira Ordem

$$F(x, y(x), y'(x)) = 0$$

F é definida num conjunto aberto $D \subset \mathbb{R}^3$. Dado um intervalo aberto $I \subset \mathbb{R}$, Diz-se que uma função $\phi : I \to \mathbb{R}$ diferenciavel em I é uma solução da equação diferencial acima se:

1. $(x, \phi(x), \phi'(x)) \in D, \quad \forall x \in I$
2. $F(x, \phi(x), \phi'(x)) = 0, \quad \forall x \in I$

Ordem de uma equação diferencial é a ordem da derivada mais elevada referida na equação

Exemplo 1

A equação

$$y' - \frac{y}{x} = x\, e^x$$

é de primeira ordem e as funções

$$y(x) = c\, x + x\, e^x \quad c \in \mathbb{R}$$

são soluções em $]0, \infty[$ desta equação.

Exemplo 2

A equação

$$y'' + 4\, y = 0$$

é de segunda ordem e as funções

$$y(x) = c_1 \cos 2\, x + c_2 \sin 2\, x, \quad c_1, c_2 \in \mathbb{R}$$

São soluções em $\mathbb{R}$ desta equação

Forma normal

$$y'(x) = f(x, y(x))$$

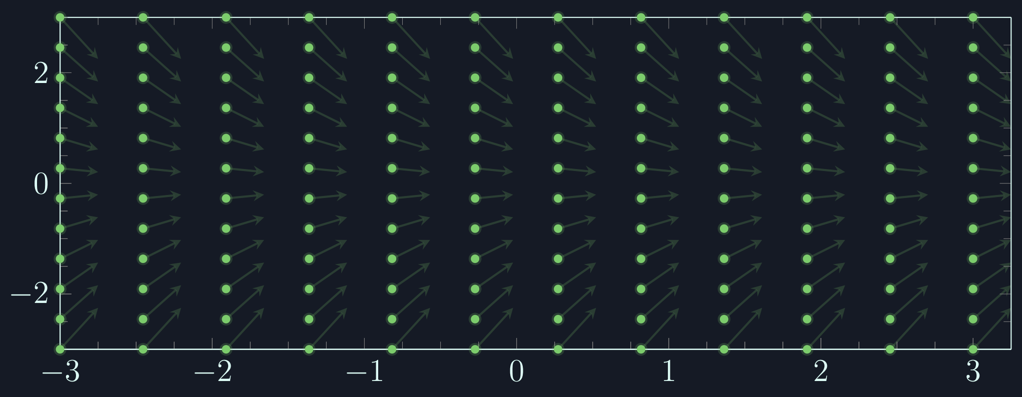
Com f definida no conjunto aberto $A \subset \mathbb{R}^2$. As equações de primeira ordem na forma normal admitem uma interpretação geométricas relativamente simples e que permite ter uma ideia aproxiamada dos gráficos das soluções destas esquações.

Campo de direções da equação

Com uma equação diferencial de primeira ordem na forma normal definida no conjunto aberto $A \subset \mathbb{R}^2$, se a cada ponto (x, y) de A se associar a direção das retas de declive igual a $f(x, y)$, se obtem aquilo a que usualmente se chama de campo de direções da equação.

Exemplo 3 Campo de direções da equação

$$y' = -y$$



2 Equação autonoma

Uma EDO em que não aparece explicitamente a variável independente. Se for y a função icógnita e x a variável independente, uma equa;cão diferencial autónoma de primeira ordem é uma equação da forma $F(y, y') = 0$ ou na forma normal:

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{y})$$

Pontos de equilíbrio (críticos ou estacionários) são os zeros da função

$$\boldsymbol{f}(\boldsymbol{c}) = \mathbf{0} \implies \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{c} \textbf{ é solução de } \boldsymbol{f}(\boldsymbol{x}) = \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}}$$

$y(x) = c$ chama-se solução de equilíbrio (ou estacionária)

Classificação dos pontos de equilíbrio (Eq autónomas)

Prestando atenção nos limites:

$$\boldsymbol{f}(\boldsymbol{c}) = \mathbf{0}$$

$x \to +\infty \implies y(x) \to c \implies c$ é um ponto de eq estável

$x \to -\infty \implies y(x) \to c \implies c$ é um ponto de eq instável

$x \to -\infty \wedge x \to +\infty \implies y(x) \to c \implies c$ é um ponto de eq semiestável

Exemplo 4 Pontos de equilíbrio

Considere-se a equação autónoma

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}\boldsymbol{x}} = \boldsymbol{y}(\boldsymbol{a} - \boldsymbol{b}\,\boldsymbol{y}); \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^+$$

Pontos de equilíbrio:

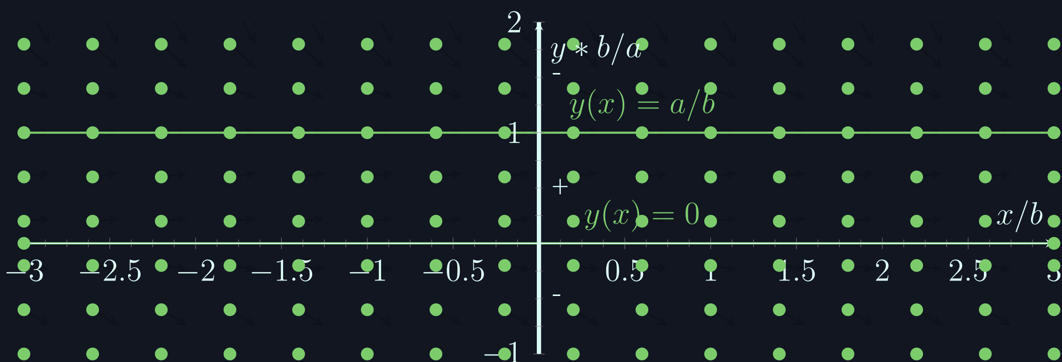
$$c = y : y(a - b\,y) = 0 \begin{cases} y = 0 \\ y = \frac{a}{b} \end{cases}$$

$$\therefore y(x) = 0 \vee y(x) = a/b$$

Podemos prever o comportamento da equalção pela seguinte tabela

y	*sign*		
	y	$a - b\,y$	$y(a - b\,y)$
$y < 0$	$-$	$+$	$-$
$0 < y < a/b$	$+$	$+$	$+$
$a/b < y$	$+$	$-$	$-$

Se desenharmos um grafico das soluções de equilíbrio



Podemos ver que as tres regiões divididas pelos dois pontos de equilíbrio tem um comportamento: R_1 Decrescente, R_2 Crescente e R_3 Decrescente
Seja $y(x) = 0$ a solução que verifica a condição inicial $y(0) = y_0$:

$$y_0 < 0 \qquad \begin{cases} x \to -\infty & \implies y(x) \to 0 \\ x \to +\infty & \implies y(x) \to -\infty \end{cases}$$

$$0 < y_0 < a/b \qquad \begin{cases} x \to -\infty & \implies y(x) \to 0 \\ x \to +\infty & \implies y(x) \to a/b \end{cases}$$

$$a/b < y_0 \qquad \begin{cases} x \to -\infty & \implies y(x) \to +\infty \\ x \to +\infty & \implies y(x) \to a/b \end{cases}$$

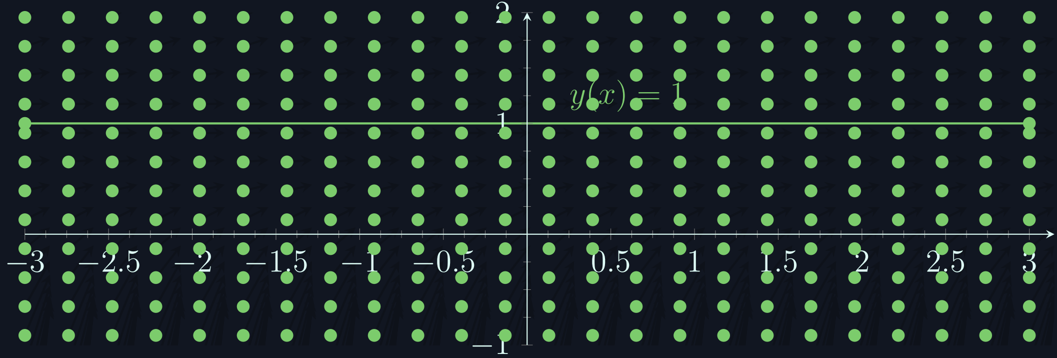
Podemos dizer que $y(x) = 0$ é um ponto de equilíbrio instável e que $y(x) = a/b$ é um ponto de equilíbrio estável

Exemplo 5 Equilíbrio semiestável

A equação autónoma

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (y-1)^2$$

tem $y = 1$ como único ponto de equilíbrio. Observando a reta fase, verifica-se que qualquer solução $y(x)$ em qualquer um dos intervalos $]-\infty, 1[$ e $]1, +\infty[$ é crescente



Podemos caracterizar esse ponto como ponto de equilíbrio semiestável

Soluções implicitas e explicitas

Soluções explicitas

$$y = f(x)$$

y isolado

Soluções implicitas

$$G(x, y) = 0$$

Define implicitamente uma função $y(x)$ solução da equação.

Exemplo 6

Da equação

$$y^2 \, y' = x^2$$

Podemos tirar a solução de duas formas

Da forma implicita

$$y^3 - x^3 - 8 = 0$$

Da forma explicita

$$y = \varphi(x) = \sqrt[3]{8 + x^3}$$

Famílias de Soluções

$$\boldsymbol{G(x, y, z) = 0}$$

Tal como sucede no cálculo da primitiva de uma função, em que aparece uma constante c de integração, **quando se resolve uma EDO de primeira ordem, geralmente obtém-se com solução uma expressão contendo uma constante (ou parâmetro) c, e que representa um conjunto de soluções** a que se chamará família de soluções a um parâmetro.

Soluções particulares são obtidas quando atribuimos valores ao parametro da familia de soluções

Solucções singulares nem sempre existem mas existem, não podem ser obtidas atribuindo um valor a constante c

Integral Geral Uma família de soluções que define todas as soluções de uma EDO para um intervalo I

3 Equação Linear de Primeira Ordem

$$\boldsymbol{y' = f(x,y) \iff y' + p(x)y = q(x)}$$

Com $p(x)$ e $q(x)$ funções contínuas num intervalo aberto $I \subseteq \mathbb{R}$

Exemplo

$$y' + 2\,x\,y = x^3 \begin{cases} p(x) = 2\,x \\ q(x) = x^3 \end{cases}$$

Equação linear homgénea

Equação linear em que $q(x) = 0$, quando em uma equação linear completa $(q(x) \neq 0 \wedge p(x) \neq 0)$ substituirmos $q(x)$ por 0, obtemos a equação linear homogénea associada.

Solução geral de equações lineáres de primeira ordem

$$\boldsymbol{y' + a(x)\, y = b(x)}$$

General solution

$$y = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int b(x)\, \varphi(x)\ \mathrm{d}x = \qquad \text{using } (1.0)\ (1.0)$$

$$= \dots$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\int a(x)\ \mathrm{d}x\right) = \dots \qquad (1.0)$$

Integrating

$$\mathrm{P}_x\left(b(x)\, \varphi(x)\right) = \qquad \text{using } (1.0)$$

$$= \dots \qquad (1.0)$$

Demonstração

$$y' + p(x)\, y = q(x) \implies (y' + p(x)\, y)\, \varphi(x) =$$

$$= y' \exp\left(\int p(x)\ \mathrm{d}x\right) + p(x)\, y \exp\left(\int p(x)\ \mathrm{d}x\right) = \left(y \exp \int p(x)\ \mathrm{d}x\right)' =$$

$$= q(x)\, \varphi(x) = q(x) \exp \int p(x)\ \mathrm{d}x \implies$$

$$\implies y \exp \int p(x)\ \mathrm{d}x = c + \int q(x) \exp\left(\int p(x)\ \mathrm{d}x\right) \mathrm{d}x \implies$$

$$\implies y = \frac{c}{\exp \int p(x)\ \mathrm{d}x} + \frac{1}{\exp \int p(x)\ \mathrm{d}x} \int q(x) \exp\left(\int p(x)\ \mathrm{d}x\right) \mathrm{d}x =$$

$$= \frac{c}{\varphi(x)} + \frac{1}{\varphi(x)} \int q(x)\, \varphi(x)\ \mathrm{d}x$$

Exemplo 7

Considere a equação

$$\boldsymbol{y' + (1 - 1/x)\, y = 2\, x, \quad x < 0}$$

Encontre a solução para a equação acima e a equação homgénea associada

Resposta

$$y = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int 2\, x\, \varphi(x)\ \mathrm{d}x =$$

using (1.0)

$$= \frac{c_0}{(e^x\, c_2/x)} + \frac{1}{(e^x\, c_2/x)} \int 2\, x\, (e^x\, c_2/x)\ \mathrm{d}x =$$

using (1.0)

$$= c_3\, x\, e^{-x} + \frac{x}{e^x\, c_2}\, 2\, c_2\, e^x = c_3\, x\, e^{-x} + 2\, x$$

$$\varphi(x) = \exp\left(\int (1 - 1/x)\ \mathrm{d}x\right) = \exp x - \ln x + c_1 = e^x\, c_2/x \tag{1.0}$$

$$P(2\, x\, \varphi(x)) =$$

using (1.0)

$$= P(2\, x\, e^x\, c_2/x) = 2\, c_2\, e^x \tag{1.0}$$

Exemplo 8

Na investigação de um homicídio, é, muitas vezes importante estimar o instante em que a morte ocorreu. A partir de observações experimentais, a lei de arrefecimento de Newton estabelece, com uma exatidão satisfatória, que a taxa de variação da temperatura $T(t)$ de um corpo em arrefecimento é proporcional à diferença entre a temperatura desse corpo e a temperatura constante T_a do meio ambiente, isto é:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\,(T - T_a) \tag{1.0}$$

Suponhamos que duas horas depois a temperatura é novamente medida e o valor encontrado é $T_1 = 23\,°\mathrm{C}$. O crime parece ter ocorrido durante a madrugada e corpo foi encontrado pela manhã bem cedo, pelas 6 horas e 17 minutos. A perícia então faz a suposição adicional de que a temperatura do meio ambiente entre a hora da morte e a hora em que o cadáver foi encontrado se manteve mais ou menos constante nos 20°C. A perícia sabe também que a temperatura normal de um ser humano vivo é de 37°C. Vejamos como, com os dados considerados, a perícia pode determinar a hora em que ocorreu o crime.

Resposta

Encontrando tempo de morte

$$t : T(t) = 37 \cong$$

using (1.0)

$$\cong 10\,e^{-t\,0.602} + 20 \implies t \cong -\ln(1.7)/0.602 \cong -0.881\,\mathrm{h} \cong -52.888\,\mathrm{min}$$

Desenvolvendo (1.0)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\,(T - T_a) = -k\,T + k\,T_a = -k\,T + k\,20 \implies \frac{\mathrm{d}T}{\mathrm{d}t} + k\,T = k\,20 \tag{1.0}$$

Solução geral $T(t)$ a partir de (1.0)

$$T = \frac{c_0}{\varphi(t)} + \frac{1}{\varphi(t)} \int k\,20\,\varphi(t)\ \mathrm{d}t =$$

using (1.0) (1.0)

$$= \frac{c_0}{c_1\,e^{k\,t}} + \frac{1}{c_1\,e^{k\,t}} k\,20\,c_1\left(c_2 + e^{k\,t}/k\right) = (c_0/c_1)\,e^{-k\,t} + c_2\,k\,20\,e^{-k\,t} + 20 =$$

$$= c_3\,e^{-k\,t} + 20 \cong \tag{1.0}$$

using (1.0)(1.0)

$$\cong 10\,e^{-t\,0.602} + 20 \tag{1.0}$$

Encontrando constantes k, c_3

$$T(2) = 23 =$$

using (1.0)(1.0)

$$= 10\,e^{-k\,2} + 20 \implies k = -\ln(0.3)/2 \cong 0.602; \tag{1.0}$$

$$T(0) = 30 =$$

using (1.0)

$$= c_3\,e^{-k*0} + 20 = c_3 + 20 \implies c_3 = 10 \tag{1.0}$$

Resolvendo $\varphi(x)$

$$\varphi(t) = \exp\left(\int k\ \mathrm{d}t\right) = c_1\,e^{k\,t} \tag{1.0}$$

Integrando

$$P(k\,20\,\varphi(t)) =$$

using (1.0)

$$= P\big(k\,20\,c_1\,e^{k\,t}\big) = k\,20\,c_1\left(c_2 + e^{k\,t}/k\right) \tag{1.0}$$

4 Método de Variação das constantes Eq Diff de ordem 1

Um metodo alternativo para resolver a mesma equação diferencial linear de primeira ordem

$$\boldsymbol{y' + a(x)\, y = b(x)}$$

Solução geral

$$y = \frac{C(x)}{\varphi(x)} = \tag{1.0}$$

using (1.0) (1.0)

$$= \dots$$

Finding $C(x)$

$$y' + a(x)\, y =$$

using (1.0)

$$= \mathrm{D}_x\left(\frac{C(x)}{\varphi(x)}\right) + a(x)\,\frac{C(x)}{\varphi(x)} = b(x) \implies C'(x) = \cdots \implies C(x) = \dots \tag{1.0}$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\left(a(x)\right)\right) = \dots \tag{1.0}$$

Podemos resolver a equação homogênea associada y_h substituir $c_0 \to c_0(x)$ e aplicar $y = c_0(x)/\varphi(x)$ na equação linear original, dessa forma podemos obter $c_0(x)$ e por sequencia $y = c_0(x)/\varphi x$

Método usando solução particular

$$\boldsymbol{y = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int q(x)\, \varphi(x)\ \mathrm{d}x = y_h + y_i}$$

- y_h é a solução da equação homogênea associada
- y_i é uma solução particular

Mesmo y_i aparecer como uma solução particular em que $c_0 = 1$, por estarmos trabalhando com uma solução arbitrária, isso não impede de ser qualquer solução particular, da no mesmo ao final das contas

Exemplo 9

$$\boldsymbol{y' - \frac{2x}{x^2+1}\,y = 1}$$

Encontre a solução geral usando o método de variação das constantes

Resposta

Solução geral

$$y = \frac{C(x)}{\varphi(x)} = \tag{1.0}$$

using (1.0) (1.0)

$$= \frac{c_1\left(\arctan x + c_2\right)}{\frac{c_1}{x^2+1}} = \left(x^2+1\right)\left(\arctan x + c_2\right)$$

Finding $C(x)$

$$y' - \frac{2\,x}{x^2+1} =$$

using (1.0) (1.0)

$$= \mathrm{D}_x\left(\frac{C(x)}{\frac{c_1}{x^2+1}}\right) - \frac{2\,x}{x^2+1}\,\frac{C(x)}{\frac{c_1}{x^2+1}} = \frac{1}{c_1}\left(C'(x)(x^2+1) + C(x)\,2\,x\right) - \frac{C(x)\,2\,x}{c_1} =$$

$$= C'(x)\,\frac{x^2+1}{c_1} = 1 \implies C'(x) = \frac{c_1}{x^2+1} \implies$$

$$\implies C(x) = \mathrm{P}_x\left(\frac{c_1}{x^2+1}\right) = c_1\left(\arctan x + c_2\right) \tag{1.0}$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\left(-\frac{2\,x}{x^2+1}\right)\right) = \exp\left(-\int\left(\frac{\mathrm{d}x^2+1}{x^2+1}\right)\right) =$$

$$= \exp\left(-(\ln(x^2+1) + c_0)\right) = \frac{c_1}{x^2+1} \tag{1.0}$$

5 Equação de Bernoulli e a equação de Riccati

São equações não lineares que, após mudanças de variáveis apropriadas, se transformam em equações lineares:

5.1 Eq de Bernoulli

A atribuição

$$\boldsymbol{y = z^{1/(1-k)}}$$

Transforma a eq diferencial

$$\boldsymbol{y' + a(x)\, y = b(x)\, y^k \implies z' + (1-k)\, a(x)\, z = (1-k)\, b(x)}$$

onde $\boldsymbol{z}$ pode ser encontrado por

$$\boldsymbol{z = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int (1-k)\, b(x)\, \varphi(x)\ \mathrm{d}x;}$$

$$\boldsymbol{\varphi(x) = \exp\left(\int (1-k)\, a(x)\ \mathrm{d}x\right)}$$

Quando encontramos uma EDO que possa ser escrita na forma acima, podemos realizar a substituiçãp de $z = y^{1-k}$ transformando a EDO em uma equação linear, assim podemos encontrar a solução geral para z que pode ser substituida para encontrar a solução de y que é a equação original.

workflow

$$\boldsymbol{y' + a(x)\, y = b(x)\, y^k}$$

Solução geral

$$y = z^{1/(1-k)} = \left(\frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int (1-k)\, b(x)\, \varphi(x)\ \mathrm{d}x\right)^{1/(1-k)} =$$

using (1.0) (1.0)

$$= \ldots$$

Encontrando $\varphi(x)$

$$\varphi(x) = \exp\left(\int (1-k)\, a(x)\ \mathrm{d}x\right) = \ldots \tag{1.0}$$

Resolvendo integral

$$\int (1-k)\, b(x)\, \varphi(x)\ \mathrm{d}x =$$

using (1.0)

$$= \ldots \tag{1.0}$$

5.2 Eq de Riccati

A substituição

$$\boldsymbol{y = y_1 + 1/z}$$

Transforma a eq diferencial

$$\boldsymbol{y' + a(x)\, y = b(x) + c(x)\, y^2 \implies}$$
$$\boldsymbol{\implies z' + (2\, c(x)\, y_1 - a(x))z = -c(x)}$$

onde $\boldsymbol{z}$ pode ser encontrado por

$$\boldsymbol{z = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int -c(x)\, \varphi(x)\ \mathrm{d}x;}$$

$$\boldsymbol{\varphi(x) = \exp\left(\int (2\, c(x)\, y_1 - a(x))\, \varphi(x)\ \mathrm{d}x\right)}$$

Exemplo 10   Eq de Bernoulli

Considere o problma de valores iniciais (PVI)

$$\boldsymbol{y' - x\,y = x\,y^3, \quad y(0) = 1}$$

Resposta (1.0)

Substituição de Bernoulli

$$y = z^{1/(1-3)} \implies y' - x\,y = x\,y^3 \implies z' + (1-k)\,(-x)\,z = (1-k)\,3$$

Solução geral

$$y = z^{1/(1-3)} = \left(\frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)} \int (1-3)\,x\,\varphi(x)\;\mathrm{d}x\right)^{-1/2} =$$

using (1.0) (1.0)

$$= \left(\frac{c_0}{c_2\,e^{x^2}} + \frac{1}{c_2\,e^{x^2}}(-c_2)\left(e^{x^2} + c_3\right)\right)^{-1/2} = \left(c_5\,e^{-x^2} - 1\right)^{-1/2} = \qquad (1.0)$$

using (1.0)

$$= \left(2\,e^{-x^2} - 1\right)^{-1/2} \qquad (1.0)$$

Encontrando c_5

$$y(0) = 1 =$$

using (1.0)

$$= \left(c_5\,e^{-0^2} - 1\right)^{-1/2} \implies$$

$$\implies c_5 = 1 + 1^2 = 2 \qquad (1.0)$$

Encontrando $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x((1-3)\,(-x))\right) = \exp\left(2(c_1 + x^2/2)\right) = c_2\,e^{x^2} \qquad (1.0)$$

Integrando

$$\int (1-3)\,x\,\varphi(x)\;\mathrm{d}x =$$

using (1.0)

$$= -\int 2\,x\,c_2\,e^{x^2}\;\mathrm{d}x = -c_2 \int e^{x^2}\;\mathrm{d}\left(x^2\right) = -c_2\left(e^{x^2} + c_3\right) \qquad (1.0)$$

Exemplo 11   Eq Bernoulli

Suponhamos que numa comunidade constituida por N individuos

- $y(t)$ representa o número de intectados pelo vírus da gripe A
- $x(t) = N - y(t)$ representa a população não infectada.

Considere-se que o vírus se propaga pelo contacto entre infectados e não infectados e que a propagação é proporcional ao número de contactos entre estes dois grupos. Suponhamos também que os elementos dos dois grupos se relacionam livremente entre si de modo que o número de contactos entre infectados e não infectados é proporcional ao produto de $x(t)$ por $y(t)$ isto é

$$\boldsymbol{k\, x(t) = k\,(N - y(t))\, y(t)}$$

em que k é a constante de proporcionalidade. se y_0 é o numero inicial de infectados, o número de infectados $y(t)$ no instante t é a solução PVI

$$\boldsymbol{y' = k\,(N - y)\, y; \qquad k > 0; \qquad y(0) = y_0}$$

Resposta

Incompleta:

$y : y' = k\,(N - y)\, y \implies y' - N\,k\,y = -k\,y^2;$

$y = z^{-1} = \left(c\,e^{-N\,k\,t} + \dfrac{1}{N\,t}\right)^{-1} = \cdots = \dfrac{N\,y_0}{(N - y_0)\,e^{-N\,k\,t} + y_0};$

$c : y(0)^{-1} = (z(0)) = c\,e^{-N\,k*0} + \dfrac{1}{N*0} = y_0^{-1};$

$z = y^{1-2} = 1/y \implies$

$\implies z' + N\,k\,z = kz = \dfrac{c_0}{\varphi(t)} + \dfrac{1}{\varphi(t)} \displaystyle\int k\,\varphi(t)\;\mathrm{d}t =$

$= \dfrac{c_0}{c_2\,e^{N\,k\,t}} + \dfrac{1}{c_2\,e^{N\,k\,t}} \displaystyle\int k\,c_2\,e^{N\,k\,t}\;\mathrm{d}t = e^{-N\,k\,t}\,\dfrac{c_0}{c_2} + e^{-N\,k\,t}\,\dfrac{k\,c_2}{c_2}\,\dfrac{e^{N\,k\,t}}{N\,k\,t} = c\,e^{-N\,k\,t} + \dfrac{1}{N\,t};$

$c = c_0/c_2;$

$\varphi(t) = \exp\left(\displaystyle\int N\,k\;\mathrm{d}t\right) = \exp\left(N\,k\,t + c_1\right) = c_2\,e^{N\,k\,t};$

$c_2 = e^{c_1}$

Exemplo 12 Eq Riccati

Determine a solução do PVI

$$\boldsymbol{y' - y = -2\,x + \frac{1}{2\,x^2}\,y^2, \quad y(1) = -2, \quad x > 0}$$

Sabendo que a equação admite a solução $y = 2\,x$

Resposta (1.0)

Riccati substitution

$$y = 2\,x + z^{-1} \implies y' + (-1)\,y = (-2\,x) + \frac{1}{2\,x^2}\,y^2 \implies$$
$$\implies z' + \left(2\,\frac{1}{2\,x^2}\,2\,x - (-1)\right) z = z' + (1 + 2/x)\,z = -\frac{1}{2\,x^2}$$

General solution

$$y = 2\,x + z^{-1} = 2\,x + \left(\frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\,\mathrm{P}_x\left(-\frac{1}{2\,x^2}\,\varphi(x)\right)\right)^{-1} =$$

using (1.0) (1.0)

$$= 2\,x + \left(\frac{c_0}{e^x\,x^2\,c_3} + \frac{1}{e^x\,x^2\,c_3}\,\frac{-c_3}{2}\,(e^x + c_4)\right)^{-1} = 2\,x + \left(\frac{c_6}{e^x\,x^2} - \frac{1}{2\,x^2}\right)^{-1} = \quad (1.0)$$

using (1.0)

$$= 2\,x + \left(\frac{e/4}{e^x\,x^2} - \frac{1}{2\,x^2}\right)^{-1} \quad (1.0)$$

Finding c_6

$$y(1) = -2 =$$

using (1.0)

$$= 2 * 1 + \left(\frac{c_6}{e^1\,1^2} - \frac{1}{2 * 1^2}\right)^{-1} = 2 + \left(\frac{c_6}{e} - \frac{1}{2}\right)^{-1} \implies$$
$$\implies c_6 = e\left((-2 - 2)^{-1} + \frac{1}{2}\right) = \frac{e}{4} \quad (1.0)$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\,(1 + 2/x)\right) = \exp\left(x + c_1 + 2\,(c_2 + \ln x)\right) = e^x\,x^2\,c_3 \quad (1.0)$$

Integrating

$$\mathrm{P}_x\left(-\frac{1}{2\,x^2}\,\varphi(x)\right) =$$

using (1.0)

$$= \mathrm{P}_x\left(-\frac{1}{2\,x^2}\,e^x\,x^2\,c_3\right) = -\frac{c_3}{2}\,\mathrm{P}_x\,(e^x) = -\frac{c_3}{2}\,(e^x + c_4) \quad (1.0)$$

6 Operador de Derivação

$$\mathrm{D}_x^n = \frac{\mathrm{d}^n}{\mathrm{d}x^n}$$

$$\mathrm{D}_x^k : C^n(I) \to C^{n-k}(I)$$

$$\mathrm{D}_x^k : y \to y^{(k)} = \frac{\mathrm{d}^k y}{\mathrm{d}x^k}$$

7 Equação Diferencial Linear de ordem n

$$\sum_{i=0}^{n} a_i \ \mathrm{D}_x^i(y) = \left(\sum_{i=0}^{n} a_i \ \mathrm{D}_x^i\right) y = P\, y = f(x)$$

- a_n é o Coeficiente lider
- Forma normal é quando esta escrita de forma que $a_n = 1$

Example

$$\mathrm{D}_x^3(y) + x^2 \ \mathrm{D}_x^2(y) - 5\, x \ \mathrm{D}_x(y) + y = x \ \cos(x)$$

está escrita na forma normal

Operador P

$$\boldsymbol{P = \mathrm{D}_x^n + \sum_{i=0}^{n-1} a_i\ \mathrm{D}_x^i}$$

Linearidade

Dadas duas funções $y_1, y_2 \in C^n(I)$ e α, β numeros reais

$$\boldsymbol{P(\alpha\, y_1 + \beta\, y_2) = \alpha\, P\, y_1 + \beta\, P\, y_2}$$

Espaço Solução da equação

$$\mathrm{nuc}(\boldsymbol{P}) : \boldsymbol{A = \{y \in C^n(I) : P\, y = 0\}}$$

O conjunto á é nucleo do operador P, sendo portanto um subespaço de $C^n(I)$. Este subespaço é designado por espaço solução da equação

Teorema: Solução que satisfaz $P\, y = 0$

$$\boldsymbol{y = \varphi(x) : \mathrm{D}_x^i \varphi(x_0) = \alpha_i}$$
$$\boldsymbol{x_0 \in I \wedge \alpha_i \in \mathbb{R} \quad \forall\, i}$$

Dado um x_0 no intervalo aberto I e constantes reais arbitrarias α, existe uma e só uma função que satisfaz $P\, y = 0$

Finidade da dimensão de $\mathrm{nuc}(P)$

$$\boldsymbol{\dim(\mathrm{nuc}(P)) = n \impliedby P = \mathrm{D}_x^n + \sum_{i=0}^{n-1} a_i\ \mathrm{D}_x^i}$$

m Sendo o espaço solução da equação $P\, y = 0$ ($\mathrm{nuc}(P)$) um subespaço do espaço liear $C^n(I)$, Não limitado a ter dimenção infinita, a dimensão do nucleo de P deve ser n (limitado).

Solução trivial

$$\boldsymbol{\alpha_i = 0 \quad \forall\, i \impliedby}$$

$$\boldsymbol{\impliedby \sum_{i=0}^{n} \alpha_i\, y_i(x) = 0 : \{y\}} \text{ é linearmente idependente}$$

Sistema fundamental de soluções de $P\, y = 0$

$$\boldsymbol{y = \sum_{i=1}^{n} c_i\, y_i}$$

- $\{y_i \,\forall\, i\}$ é um sistema fundamental de soluções de $P\, y = 0$
- $c_i \,\forall\, i$ são constantes arbitrárias que consituem a sua solução (ou integral) geral

Quaisquer n soluções linearmente idependentes de $P\, y = 0$ que constituem uma base de $\mathrm{nuc}(P)$

8 Abaixando a ordem de uma EDO

$$\boldsymbol{z(x) : y = \varphi(x) \int (z) \, \mathrm{d}x;}$$

$$\boldsymbol{P\,y = 0}$$

- $\varphi(x)$ é uma solução particular da equação linear homogenea de ordem n $(P\,y = 0)$

Exemplo 13 Baixamento de grau de uma Eq lin homogenea

Determine a solução geral da equação

$$\boldsymbol{y'' + y'/x - y/x^2 = 0, \quad x > 0}$$

Sabendo que $\varphi(x) = x$ é uma solução.

Resposta

Solução geral

$$y = \varphi(x)\ \mathrm{P}_x\left(z\right) = x\ \mathrm{P}_x\left(z\right) = \tag{1.0}$$

using (1.0)

$$= x\ \mathrm{P}_x\left(\frac{c_3}{x^3}\right) = x\, c_3\left(\frac{1}{2\, x^2} + c_4\right) = \frac{c_5}{x} + x\, c_6$$

Substitution $y \to z$

$$y'' + y'/x - y/x^2 = 0 \implies$$

using (1.0) (1.0) (1.0)

$$\implies (2\, z + x\, z') + \frac{1}{x}(\mathrm{P}_x\left(z\right) + x\, z) - \frac{1}{x^2}(x\ \mathrm{P}_x\left(z\right)) =$$

$$= 2\, z + x\, z' + \mathrm{P}_x\left(z\right)/x + z - \mathrm{P}_x\left(z\right)/x = x\, z' + 3\, z = 0 \implies z' + \frac{3}{x}\, z = 0 \tag{1.0}$$

Finding $\mathrm{D}_x y, \mathrm{D}_x^2 y$

$$\mathrm{D}_x y =$$

using (1.0)

$$= \mathrm{D}_x\varphi(x)\ \mathrm{P}_x\left(z\right) + \varphi(x)\, z = \mathrm{D}_x\varphi(x)\ \mathrm{P}_x\left(z\right) + \varphi(x)\, z = \mathrm{P}_x\left(z\right) + x\, z; \tag{1.0}$$

$$\mathrm{D}_x^2 y =$$

using (1.0)

$$= \mathrm{D}_x(\mathrm{P}_x\left(z\right) + x\, z) = 2\, z + x\, z' \tag{1.0}$$

Solving (1.0)

$$z = c_0\left(\varphi_z(x)\right)^{-1} = c_0\left(\exp\left(\int (3/x)\ \mathrm{d}x\right)\right)^{-1} = c_0\left(e^{3\left(\ln\left(x\right) + c_1\right)}\right)^{-1} = c_0\left(c_2\, x^3\right)^{-1} =$$

$$= \frac{c_3}{x^3} \tag{1.0}$$

9 Wronskiano: check dependencia linear

$$\boldsymbol{W}(\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n)(\boldsymbol{x}) = \det(\boldsymbol{w}); \qquad \boldsymbol{w} \in \boldsymbol{\mathcal{M}}_{n,m} : \boldsymbol{w}_{i,j} = \mathrm{D}_{\boldsymbol{x}}^{j}\, \boldsymbol{f}_i$$

$$\boldsymbol{W}(\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_n)(\boldsymbol{x}) \begin{cases} = \mathbf{0} & \text{Linear depedent} \\ \neq \mathbf{0} & \text{Linear independent} \end{cases}$$

10 Método de variação das constantes abitrárias para equação linear de ordem n

$$y : \begin{pmatrix} a_1(x) \\ +a_1(x)\ \mathrm{D}_x \\ +a_2(x)\ \mathrm{D}_x^2 \\ +a_3(x)\ \mathrm{D}_x^3 \end{pmatrix} y = f(x)$$

$$y = c_1(x)\, y_1(x) + c_2(x)\, y_2(x) + c_3(x)\, y_3(x)$$

$$\left\{ \begin{array}{c} c_1'(x)\ \mathrm{D}_x^0 y_1(x) + c_2'(x)\ \mathrm{D}_x^0 y_2(x) + c_3'(x)\ \mathrm{D}_x^0 y_3(x) = 0 \\ c_1'(x)\ \mathrm{D}_x\, y_1(x) + c_2'(x)\ \mathrm{D}_x\, y_2(x) + c_3'(x)\ \mathrm{D}_x\, y_3(x) = 0 \\ c_1'(x)\ \mathrm{D}_x^2\, y_1(x) + c_2'(x)\ \mathrm{D}_x^2\, y_2(x) + c_3'(x)\ \mathrm{D}_x^2\, y_3(x) = \frac{f(x)}{a_3(x)} \end{array} \right\}$$

Exemplo 14   Metodo das var const arb

Considere a equação

$$y'' + 9\,y = 1/\cos(3\,x); \quad x \in \left]-\pi/6, \pi/6\right[$$

As funções $\cos(3\,x)$ e $\sin(3\,x)$ são duas soluções linearmente idependentes da equação homogénea

$$y'' + 9\,y = 0$$

Pelo que seu integral geral será dado por

$$y = c_1\,\cos(3\,x) + c_2\,\sin(3\,x); \quad c_1, c_2 \in \mathbb{R} \tag{1.0}$$

Utilizemos o método da variação das constantes arbitrárias para determinar o integral geral da equação completa

---

Resposta (1.0)

General solution

$$
\begin{aligned}
& y = \\
& \qquad \text{using (1.0)} \\
& = C_1(x)\,y_1(x) + C_2(x)\,y_2(x) = \\
& \qquad \text{using (1.0) (1.0)} \\
& = (-\ln(\cos(3\,x)) - c_3)\cos(3\,x) + (x/3 + c_4)\sin(3\,x) \qquad (1.0)
\end{aligned}
$$

Finding $C_1(x), C_2(x)$

$$
\begin{aligned}
& C_1(x) = \mathrm{P}_x\left(C_1'(x)\right) = \\
& \qquad \text{using (1.0)} \\
& = \mathrm{P}_x\left(3\,\frac{\sin(3\,x)}{\cos(3\,x)}\right) = \\
& \qquad \text{using } \mathrm{d}(\cos(3\,x)) = -\sin(3\,x)\,3\,\mathrm{d}x \\
& = -\int\left(\frac{\mathrm{d}(\cos(3\,x))}{\cos(3\,x)}\right) = -\ln(\cos(3\,x)) - c_3; \qquad (1.0) \\
& C_2(x) = \mathrm{P}_x\left(C_2'(x)\right) = \\
& \qquad \text{using (1.0)} \\
& = \mathrm{P}_x(1/3) = x/3 + c_4 \qquad (1.0)
\end{aligned}
$$

Finding $C_1'(x), C_2'(x)$

$$
\begin{aligned}
& C_1'(x) = \\
& \qquad \text{using (1.0)} \\
& = (W_{y_1,y_2})^{-1}\begin{vmatrix} 0 & \mathrm{D}_x^0 y_2 \\ \frac{1}{\cos(3\,x)} & \mathrm{D}_x y_2 \end{vmatrix} = \\
& \qquad \text{using (1.0) (1.0)} \\
& = \frac{1}{3}\begin{vmatrix} 0 & \sin(3\,x) \\ \frac{1}{\cos(3\,x)} & 3\,\cos(3\,x) \end{vmatrix} = 3\,\frac{\sin(3\,x)}{\cos(3\,x)}; \qquad (1.0) \\
& C_2'(x) = \\
& \qquad \text{using (1.0)} \\
& = (W_{y_1,y_2})^{-1}\begin{vmatrix} \mathrm{D}_x^0 y_1 & 0 \\ \mathrm{D}_x y_1 & \frac{1}{\cos(3\,x)} \end{vmatrix} = \\
& \qquad \text{using (1.0) (1.0)} \\
& = \frac{1}{3}\begin{vmatrix} \cos(3\,x) & 0 \\ -3\,\sin(3\,x) & \frac{1}{\cos(3\,x)} \end{vmatrix} = 1/3 \qquad (1.0)
\end{aligned}
$$

Wronskiano

$$
\begin{aligned}
& W(y_1, y_2) = \det\begin{bmatrix} \mathrm{D}_x^0\,y_1 & \mathrm{D}_x^0\,y_2 \\ \mathrm{D}_x\,y_1 & \mathrm{D}_x\,y_2 \end{bmatrix} = \\
& \qquad \text{using (1.0) (1.0)} \\
& = \det\begin{bmatrix} \cos(3\,x) & \sin(3\,x) \\ -3\,\sin(3\,x) & +3\,\cos(3\,x) \end{bmatrix} = 3\,\cos^2(3\,x) + 3\,\sin^2(3\,x) = 3 \qquad (1.0)
\end{aligned}
$$

Equation system from Crammer's rule

$$
\left\{\begin{array}{rcl} c_1'(x)\ \mathrm{D}_x^0\,y_1(x) + c_2'(x)\ \mathrm{D}_x^0\,y_2(x) & = & 0 \\ c_1'(x)\ \mathrm{D}_x\,y_1(x) + c_2'(x)\ \mathrm{D}_x\,y_2(x) & = & \frac{1}{\cos(3\,x)} \end{array}\right\} \tag{1.0}
$$

Solving $\mathrm{D}_x(y_1, y_2)$

$$
\begin{aligned}
& \mathrm{D}_x y_1 = \mathrm{D}_x\,\cos(3\,x) = -3\,\sin(3\,x); \qquad (1.0) \\
& \mathrm{D}_x y_2 = \mathrm{D}_x\,\sin(3\,x) = +3\,\cos(3\,x) \qquad (1.0)
\end{aligned}
$$

11 A equação linear de ordem n de coeficientes consntantes

To find the solution for a diferential equation this method searches for the solutions for an associated polynom $P(x) \to r$

$$\boldsymbol{P(x)\, y = f(x)\, e^{\alpha\, x}}$$

General solution

$$y = y_h + \bar{y} = \qquad \text{using } (1.0) \wedge((1.0) \vee (1.0))$$
$$= \dots$$

Case 1: $f(x) = P_k(x)$ polynom of order k

Solving for $\bar{y}$

$$\bar{y} = x^p\, e^{\alpha\, x}\, Q_k(x) = x^p\, e^{\alpha\, x} \sum_{i=0}^{k} x^i\, \rho_i = x^p\, e^{\alpha\, x} \left(x^0\, \rho_0 x^1\, \rho_1 x^2\, \rho_2 x^3\, \rho_3 \right) = \tag{1.0}$$
$$\text{using } (1.0)$$
$$= \dots \tag{1.0}$$

Finding constants of (1.0)

$$\bar{y}\, P(x) = x^p\, e^{\alpha\, x}\, Q_k(x)\, P(x) = f(x)\, e^{\alpha\, x} \implies \begin{cases} \rho_0 = \dots \\ \rho_1 = \dots \\ \dots \end{cases} \tag{1.0}$$

Case 2: $f(x) = (a\, \cos(w\, x) + b\, \cos(w\, x))\, e^{\alpha\, x}$

Finding $\bar{y}$

$$\bar{y} = x^p\, (a_0\, \cos(w\, x) + b_0\, \sin(w\, x)) = \tag{1.0}$$
$$\text{using } (1.0)$$
$$= \dots \tag{1.0}$$

Finding constants of (1.0)

$$\bar{y}\, P(x) = x^p\, e^{\alpha\, x}\, (a_0\, \cos(w\, x) + b_0\, \sin(w\, x))\, P(x) =$$
$$= e^{\alpha\, x}\, (a\, \cos(w\, x) + b\, \cos(w\, x)) \implies$$
$$\begin{cases} x^p\, a_0 = a & \implies a_0 = \dots \\ x^p\, b_0 = b & \implies b_0 = \dots \end{cases} \tag{1.0}$$

Mapping (1.0) roots to solution of y_h

$$\begin{cases} r_i = \alpha_i & \to & (\mathrm{D}_x^i - \alpha_i)^{q_i} & \to & e^{r_i\, x} \sum_{j=0}^{i-1} a_{i,j}\, x^j \\ r_i = \alpha_i \pm i\, \beta_i & \to & ((\mathrm{D}_x^i - \alpha_i)^2 - \beta_i^2)^{q_i} & \to & e^{r_i\, x} \begin{pmatrix} \cos(\beta_i\, x) \sum_{j=0}^{i-1} a_{i,j,0}\, x^j \\ \sin(\beta_i\, x) \sum_{j=0}^{i-1} a_{i,j,1}\, x^j \end{pmatrix} \end{cases}$$

Examples

$$\begin{cases} r_0 = 2 & \to & (\mathrm{D}_x^i - 2)^1 & \to & e^{2\, x}\, a_{0,0} \\ r_1 = 3 & \to & (\mathrm{D}_x^i - 3)^4 & \to & e^{3\, x}\, (a_{1,0}\, x^0 + a_{1,1}\, x^1 + a_{1,2}\, x^2 + a_{1,3}\, x^3) \\ r_2 = 4 \pm i\, 1 & \to & ((\mathrm{D}_x^i - 4)^2 - 1^2)^1 & \to & e^{4\, x} \begin{pmatrix} \cos(1\, x) a_{2,0,0} \\ \sin(1\, x) a_{2,0,1} \end{pmatrix} \\ r_3 = 2 \pm i\, 2 & \to & ((\mathrm{D}_x^i - 2)^2 - 2^2)^2 & \to & e^{2\, x} \begin{pmatrix} \cos(2\, x)(a_{3,0,0}\, x^0 + a_{3,1,0}\, x^1) \\ \sin(2\, x)(a_{3,0,1}\, x^0 + a_{3,1,1}\, x^1) \end{pmatrix} \end{cases} \tag{1.0}$$

Associated polynom roots

$$P(x) = \mathrm{D}_x^n + \sum_{i=0}^{n-1} \left(a_i\, \mathrm{D}_x^i \right) \implies$$
$$\implies r^n + \sum_{i=0}^{n-1} \left(a_i\, r^i \right) \implies$$

Finding solutions for r

$$\implies r = \begin{cases} \alpha_1 \pm i\, \beta_1, \\ \alpha_2 \pm i\, \beta_2, \\ \dots, \\ \alpha_n \pm i\, \beta_n, \end{cases} \implies \tag{1.0}$$

The solutions for r allows to rewrite the differential equation like so

$$\implies P(x)\, y = y \prod_{i=0}^{n} (\mathrm{D}_i - \alpha_i) = f(x)\, e^{\alpha\, x}$$

In this format looking at α and $f(x)$ whe can find the general solution for y for specific cases, which include

- Homogeneous equation $P(x)\, y = 0$ 11.1

11.1 Quando $f(x) = 0$

$$\boldsymbol{P(x)\, y = 0}$$

General solution for y

$$P(x)\, y =$$

using (1.0) (1.0)

$$= y \prod_i \left(\mathrm{D}_x^i - \alpha_i\right)^{q_{r_i}} \prod_i \left(\left(\mathrm{D}_x^i - \alpha_i\right)^2 - \beta_i^2\right)^{q_{r_i}} \implies$$

using (1.0)

$$\implies \dots$$

Map for polynom root $\rightarrow$ solution for y
Finding solutions for (1.0)

$$r = \{\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \dots, \alpha_n \pm \beta_n, \} \tag{1.0}$$

Associated polynom

$$P(x) = \mathrm{D}_x^n + \sum_{i=0}^{n-1} a_i\, \mathrm{D}_x^i \implies r^n + \sum_{i=0}^{n-1} a_i\, r^i \tag{1.0}$$

Exemplo 15

A equação diferencial linear de coeficientes constantes

$$\mathbf{(D_x^2 - 2)((D_x - 2)^2 + 9)^2\, y = 0}$$

Encontre a solução geral

Resposta (1.0)

General solution

$$y = \qquad \text{using (1.0)}$$

$$= e^{+\sqrt{2}\,x}\,c_1 + e^{-\sqrt{2}\,x}\,c_2 + e^{2\,x}\begin{pmatrix} +\cos(3\,x)(c_{3,0,0} + c_{3,1,0}\,x) \\ +\sin(3\,x)(c_{3,0,1} + c_{3,1,1}\,x) \end{pmatrix} \tag{1.0}$$

Mapping roots (1.0) to solution

$$\begin{aligned} r_1 &= +\sqrt{2} \implies e^{+\sqrt{2}\,x}\,c_1; \\ r_2 &= -\sqrt{2} \implies e^{-\sqrt{2}\,x}\,c_2; \\ r_3 &= r_4 = 2 \pm i\,3 \implies e^{2\,x}\begin{pmatrix} +\cos(3\,x)(c_{3,0,0} + c_{3,1,0}\,x) \\ +\sin(3\,x)(c_{3,0,1} + c_{3,1,1}\,x) \end{pmatrix} \end{aligned} \tag{1.0}$$

Associated polynom

$$\begin{aligned} &P(x) = (\mathrm{D}_x^2 - 2)((\mathrm{D}_x - 2)^2 + 9)^2 \implies \\ &\implies (r^2 - 2)((r-2)^2 + 9)^2 = (r - \sqrt{2})(r + \sqrt{2})((r-2)^2 + 3^2)^2 \implies \\ &\implies r = \begin{cases} r_1 & = +\sqrt{2}, \\ r_2 & = -\sqrt{2}, \\ r_3 = r_4 & = 2 \pm i\,3 \end{cases} \end{aligned} \tag{1.0}$$

11.2 Quando $f(x) = P_k(x)$

In the case that $f(x)$ is a polynom of x with order k

$$\boldsymbol{y\, P(x) = P_k(x) \implies y\ \mathrm{D}_x^{k+1}\, P(x) = 0}$$

Solve y_h as in 11.1 from here we can expect two cases

- $r = 0$ is root of $P(x)$
- $r = 0$ is not root of $P(x)$

for both cases we just need to multiply x^q to $Q_k(x)$ where q is how many roots equal to zer are in y_h (homogeneous equation)

Solução geral

$$y = y_h + x^p\, Q_k(x)$$

Here p comes from the number of roots found in (1.0) that are equal to 0

Finding $Q_k(x)$

$$Q_k(x) = \sum_{i=0}^{1+k} c_{0,i}\, x^i$$

Mapping roots to solution

See 1.0

$$r_i \to \cdots \to \ldots$$

Associated polynom of homogeneous equation y_h

See 11.1

$$P(x) \implies \text{Polynom in } r \implies \text{roots} \tag{1.0}$$

Exemplo 16

$$\mathbf{D}_x^5 y - 3\,y''' - 2\,y'' = x^2 - 3\,x + 1$$

Resposta

General solution

$$\begin{aligned} & y = y_h + x^p\,Q_3(x) = \\ & \text{using } (1.0)\ (1.0) \\ & = e^{0\,x}\,(c_0 + c_1\,x) + e^{-1\,x}\,(c_2 + c_3\,x) + e^{-2\,x}\,(c_4) - 5/2 + x^1\,1/2 + x^2\,3 \end{aligned}$$

Finding $Q_2(x)$

$$Q_2(x) = \sum_{i=0}^{2} \rho_i\,x^i = \rho_0 + \rho_1\,x + \rho_2\,x^2 = \tag{1.0}$$

using (1.0)

$$= -5/2 + x^1\,1/2 + x^2\,3 \tag{1.0}$$

Finding coeficients of $Q_3(x)$

$$\begin{aligned} & x^2\,Q_2(x)\,P(x) = \\ & \text{using } (1.0) \\ & = x^2\left(\rho_0 + \rho_1\,x + \rho_2\,x^2\right)\left(\mathrm{D}_x^5 - 3\ \mathrm{D}_x^3 - 2\ \mathrm{D}_x^2\right) = \\ & = \begin{pmatrix} -3\,(\rho_1\,3*2 + \rho_2\,4*3*2\,x^1) \\ -2\,(\rho_0\,2 + \rho_1\,3*2\,x + \rho_2\,4*3\,x^2) \end{pmatrix} = \begin{pmatrix} -\rho_0\,4 - \rho_1\,18 \\ -\rho_1\,12\,x - \rho_2\,72\,x \\ -\rho_2\,24\,x^2 \end{pmatrix} = \\ & = P_k(x) = x^2 - 3\,x + 1 \implies \\ & \begin{cases} -\rho_2\,24 = 1 & \implies \rho_2 = -1/24; \\ -\rho_1\,12 - (-1/24)\,72 = -3 & \implies \rho_1 = (3 + 72/24)/12 = 1/2; \\ -\rho_0\,4 - (1/2)\,18 = 1 & \implies \rho_0 = (-1 - 18/2)/4 = -5/2 \end{cases} \end{aligned} \tag{1.0}$$

Mapping (1.0) for solution

$$\begin{cases} r_1 = r_2 = 0 & \implies e^{+0\,x}\,(c_0 + c_1\,x) \\ r_3 = r_4 = -1 & \implies e^{-1\,x}\,(c_2 + c_3\,x) \\ r_5 = 2 & \implies e^{-2\,x}\,(c_4) \end{cases} \tag{1.0}$$

Roots for characteristic linear equation for y_h

$$P(x) = \mathrm{D}_x^5 - 3\ \mathrm{D}_x^3 - 2\ \mathrm{D}_x^2 \implies r^5 - 3\,r^3 - 2\,r^2 \implies r = \begin{cases} r_1 = r_2 & = 0 \\ r_3 = r_4 & = -1 \\ r_5 & = 2 \end{cases} \tag{1.0}$$

11.3 Quando $f(x) = a\ \cos(w\,x) + b\ \sin(w\,x)$

$$\boldsymbol{P\,y = e^{\alpha\,x}\,f(x) = e^{\alpha\,x}\,(a\ \cos(w\,x) + b\ \sin(w\,x))}$$

Resposta (1.0)

General solution for y

$$y = y_h + \bar{y} =$$

using (1.0) (1.0)

$$= y_h + \bar{y} \qquad (1.0)$$

Finding $\bar{y}$

$$\bar{y} = x^p\,e^{\alpha\,x}\,(a_0\ \cos(w\,x) + b_0\ \cos(w\,x)) = \qquad (1.0)$$

using (1.0)

$$(a_0\ \cos(w\,x) + b_0\ \cos(w\,x)) \qquad (1.0)$$

Finding constants of (1.0)

$$\begin{gathered}\bar{y}\,P = x^p\,(a_0\ \cos(w\,x) + b_0\ \cos(w\,x))\,P = x^p\,(a_0\ \cos(w\,x) + b_0\ \cos(w\,x))\,P = \\ = a\ \cos(w\,x) + b\ \cos(w\,x) \implies \\ \begin{cases} \implies a_0 = \\ \implies b_0 = \end{cases}\end{gathered} \qquad (1.0)$$

Mapping roots of (1.0) to solution

$$\begin{cases} r_i = \alpha_i \implies e^{\alpha_i\,x}\,(c_0 + c_1\,x + \dots); \\ r_i = \alpha_i \pm i\,\beta_i \implies e^{r_i\,x}\begin{pmatrix}\cos(\beta_i\,x)\,(c_0 + c_1\,x + \dots) \\ \sin(\beta_i\,x)\,(c_2 + c_3\,x + \dots)\end{pmatrix}\end{cases} \qquad (1.0)$$

Roots for characteristic equation for y_h

$$P = P \implies$$

$$\mathrm{D}_x^i \to r^i$$

$$P \implies \left\{ r_0 = \dots \right. \qquad (1.0)$$

Exemplo 17

Encontre a solução geral da equação

$$\boldsymbol{y' + 2\,y = \cos(x)}$$

Resposta (1.0)

General solution for y

$$y = y_h + \bar{y} =$$

using (1.0) (1.0)

$$= e^{-2\,x}\,c_0 + 1/5(2\,\cos(x) + \sin(x)) \tag{1.0}$$

Finding $\bar{y}$

$$\bar{y} = x^p\,(a_0\,\cos(x) + b_0\,\sin(x)) = \tag{1.0}$$

using (1.0)

$$= 1/5(2\,\cos(x) + \sin(x)) \tag{1.0}$$

Finding constants of (1.0)

$$\begin{gathered}\bar{y}\,P = \,(a_0\,\cos(1\,x) + b_0\,\sin(1\,x))\,(\mathrm{D}_x + 2) = \\ = -a_0\,\sin(x) + b_0\,\cos(x) + 2\,a_0\,\cos(x) + 2\,b_0\,\sin(x) = \\ = \cos(x) \implies \\ \begin{cases} a_0 = 2\,b_0 = 2/5 \\ 2\,(2\,b_0) + b_0 = 1 \implies b_0 = 1/5 \end{cases}\end{gathered} \tag{1.0}$$

Mapping roots of (1.0) to solution

$$\begin{cases} r_i = -2 \implies e^{-2\,x}\,c_0 \end{cases} \tag{1.0}$$

Roots for characteristic equation for y_h

$$P = \mathrm{D}_x + 2 \implies$$

$$\mathrm{D}_x^i \to r^i$$

$$r + 2 = 0 \implies \begin{cases} r_0 = -2 \end{cases} \tag{1.0}$$

Exemplo 18

Considere a equação

$$\boldsymbol{y'' + y' = 1 + \cos(2\,x)}$$

Encontre a solução geral

Resposta (1.0)

General solution for y

$$y = y_h + \bar{y}_1 + \bar{y}_2 =$$
$$\text{using (1.0) (1.0) (1.0)}$$
$$= e^{+0\,x}\,c_0 + e^{-1\,x}\,c_1 + x + (-1/2)\,\cos(2\,x) + (+1/4)\,\sin(2\,x) =$$
$$= c_0 + e^{-1\,x}\,c_1 + x - \cos(2\,x)/2 + \sin(2\,x)/4 \qquad (1.0)$$

Finding $\bar{y}_1$

$$\bar{y}_1 = x^1\,Q_0(x) = x\sum_{i=0}^{0}\rho_i\,x^i = x\,\rho_0 = \qquad (1.0)$$
$$\text{using (1.0)}$$
$$= x \qquad (1.0)$$

Finding constants of (1.0)

$$\bar{y}_1\,P = x\,\rho_0\left(\mathrm{D}_x^2 + \mathrm{D}_x\right) = \rho_0 = 1 \qquad (1.0)$$

Finding $\bar{y}_2$

$$\bar{y}_2 = a_0\,\cos(2\,x) + b_0\,\sin(2\,x) = \qquad (1.0)$$
$$\text{using (1.0)}$$
$$= (-1/2)\,\cos(2\,x) + (+1/4)\,\sin(2\,x) \qquad (1.0)$$

Finding constants of (1.0)

$$\bar{y}_2\,P = (a_0\,\cos(2\,x) + b_0\,\sin(2\,x))\,P = (a_0\,\cos(2\,x) + b_0\,\sin(2\,x))\left(\mathrm{D}_x^2 + \mathrm{D}_x\right) =$$
$$= (-a_0\,2*2\,\cos(2\,x) - b_0\,2*2\,\sin(2\,x)) + (-a_0\,2\,\sin(2\,x) + b_0\,2\,\cos(2\,x)) =$$
$$= (-a_0\,4 + b_0\,2)\,\cos(2\,x) + (-b_0\,4 - a_0\,2)\,\sin(2\,x) =$$
$$= 1\,\cos(2\,x) + 0\,\sin(2\,x) \implies$$
$$\begin{cases} -b_0\,4 - a_0\,2 = 0 & \implies a_0 = -b_0\,2 = -(1/4)\,2 = -1/2 \\ -(-b_0/2)\,4 + b_0\,2 = 1 & \implies b_0 = 1/4 \end{cases} \qquad (1.0)$$

Mapping roots of (1.0) to solution

$$\begin{cases} r_0 = 0 \implies e^{0\,x}\,c_0 \\ r_1 = -1 \implies e^{-1\,x}\,c_1 \end{cases} \qquad (1.0)$$

Roots for characteristic equation for y_h

$$P = \mathrm{D}_x^2 + \mathrm{D}_x \implies$$
$$\mathrm{D}_x^i \to r^i$$
$$\implies r^2 + r = r(r+1) = 0 \implies \begin{cases} r_0 = 0 \\ r_1 = -1 \end{cases} \qquad (1.0)$$

12 Equação de Euler

Its called an Euler's equation the following:

$$\sum_{i=0}^{n} a_i\, x^i\ \mathrm{D}_x^i y = f(x) =$$

$$x \to e^t ; \mathrm{D}_x y = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{x}\,\mathrm{D}_t y$$

$$= \sum_{i=0}^{n} b_i\ \mathrm{D}_t^i y = f(e^t)$$

Exemplo 19

Encontre a solução geral da equação:

$$\frac{x^3}{4}\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}+\frac{x^2}{4}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{4}$$

Resposta (1.0)

General solution for (1.0)

$$\begin{aligned}
&y = y_h + \bar{y} = \\
&\hspace{20em}\text{using (1.0) (1.0)}\\
&= e^{0\,t}\,c_0 + e^{3\,t}\,c_1 + e^{-1\,t}\,c_2 - t/3 = \\
&\hspace{20em} t=\ln(x)\\
&= e^{0\,\ln(x)}\,c_0 + e^{3\,\ln(x)}\,c_1 + e^{-1\,\ln(x)}\,c_2 - \ln(x)/3 = c_0 + x^3\,c_1 + x^{-1}c_2 - \ln(x)/3 \quad (1.0)
\end{aligned}$$

Finding $\bar{y}$

$$\begin{aligned}
&\bar{y} = t^1\,Q_0(t) = t^1\sum_{i=0}^{0}\rho_i\,t^i = t\,\rho_0 = \quad (1.0)\\
&\hspace{20em}\text{using (1.0)}\\
&= -t/3 \quad (1.0)
\end{aligned}$$

Finding constants of (1.0)

$$\bar{y}\,P = t^1\,\rho_0\left(\mathrm{D}_t^3 - 2\,\mathrm{D}_t^2 - 3\,\mathrm{D}_t\right) = -3\,\rho_0 = 1 \implies \rho_0 = -1/3 \quad (1.0)$$

Mapping roots of (1.0) to solution

$$\begin{cases}
r_0 = 0 \implies e^{0\,t}\,c_0\\
r_1 = 3 \implies e^{3\,t}\,c_1\\
r_2 = -1 \implies e^{-1\,t}\,c_2
\end{cases} \quad (1.0)$$

Roots for characteristic equation for y_h

$$\begin{aligned}
&P = \mathrm{D}_t^3 - 2\,\mathrm{D}_t^2 - 3\ \mathrm{D}_t \implies \\
&\hspace{20em}\mathrm{D}_t^i \to r^i\\
&\implies r^3 - 2\,r^2 - 3\,r = r(r^2 - 2\,r^1 - 3) = 0 \implies \\
&\begin{cases}
r_0 = 0\\
r = \frac{-(-2)\pm\sqrt{(-2)^2 - 4\,1\,-3}}{2\,1} = 1 \pm 2
\end{cases} \quad (1.0)
\end{aligned}$$

Finding linear equation of constant coefficients using euler's equation

$$\begin{aligned}
&\frac{x^3}{4}\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}+\frac{x^2}{4}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x} = 1/4 \implies x^3\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}+x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-4\,x\frac{\mathrm{d}y}{\mathrm{d}x} = 1 = \\
&\hspace{20em}\text{using (1.0)}\ ,x \to e^t\\
&= x^3\,\frac{1}{x^2}\left(+2\ \mathrm{D}_t y - 3\ \mathrm{D}_t^2 y + \mathrm{D}_t^3 y\right) + x^2\,\frac{1}{x^2}(-\mathrm{D}_t y + \mathrm{D}_t^2 y) - 4\,x\,\frac{1}{x}\,\mathrm{D}_t y = \\
&= -3\ \mathrm{D}_t y - 2\ \mathrm{D}_t^2 y + \mathrm{D}_t^3 y = 1 \quad (1.0)
\end{aligned}$$

Finding $\mathrm{D}_x y, \mathrm{D}_x^2 y, \mathrm{D}_x^3 y$

$$\begin{cases}
\mathrm{D}_x y = \frac{1}{x}\,\mathrm{D}_t y;\\
\mathrm{D}_x^2 y = \mathrm{D}_x\left(\frac{1}{x}\,\mathrm{D}_t y\right) = -\frac{1}{x^2}\,\mathrm{D}_t y + \frac{1}{x}\,\mathrm{D}_t^2 y\,\mathrm{D}_x t = \frac{1}{x^2}(-\mathrm{D}_t y + \mathrm{D}_t^2 y);\\
\mathrm{D}_x^3 y = \mathrm{D}_x\left(\frac{1}{x^2}(-\mathrm{D}_t y + \mathrm{D}_t^2 y)\right) = \\
= -\frac{2}{x^3}(-\mathrm{D}_t y + \mathrm{D}_t^2 y) + \frac{1}{x^2}(-\mathrm{D}_t^2 y\,\mathrm{D}_x t + \mathrm{D}_t^3 y\,\mathrm{D}_x t) = \\
= \frac{1}{x^3}(+2\ \mathrm{D}_t y - 3\ \mathrm{D}_t^2 y + \mathrm{D}_t^3 y)
\end{cases} \quad (1.0)$$

Equações diferenciais não lineares

13 Equações diferenciais Exatas

$$u(x,y) + v(x,y)y' = 0 \iff \text{Forma normal}$$

$$\iff u(x,y)\ \mathrm{d}x + v(x,y)\ \mathrm{d}y = 0 \quad \text{Forma diferencial}$$

An differential equation is said to be exact in if the pair of functions (u, v) is a gradient of some function continiously derivable in , that is

$$\exists\, f(x,y) : \frac{\partial f}{\partial x} = u(x,y) \wedge \frac{\partial f}{\partial y} = v(x,y)$$

13.1 Theorem

Be $u(x,y), v(x,y)$ two functions continiously derivable in the rectangle $R = \{(x,y) \in \mathbb{R}^2 : |x-a| < \alpha \wedge |y-b| < \beta\}$ Is a necessary and sufficient condition for the equation $u(x,y)\ \mathrm{d}x + v(x,y)\ \mathrm{d}y = 0$ be exact and:

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

13.2 Theorem

...

Exemplo 20

Verifique que a equação diferencial em baixo é exata

$$\left(6\,x+2\,y^{2}\right)\,\mathrm{d}x+4\,x\,y\,\,\mathrm{d}y=0$$

Resposta

Finding $f(x)$

$$f(x)=\mathrm{P}_x(u)=\mathrm{P}_y(v)=$$

$$\text{using }(1.0)\ (1.0)\ (1.0)$$

$$=6\,x^{2}+2\,x\,y^{2}$$

Solving $\mathrm{P}_x(u),\mathrm{P}_y(v)$

$$\mathrm{P}_x\,u=\mathrm{P}_x\left(6\,x+2\,y^{2}\right)=6\left(c_0+x^{2}\right)+2\,y^{2}(c_1+x)= \quad (1.0)$$

$$\text{using }(1.0)$$

$$=6\left(c_0+x^{2}\right)+2\,y^{2}\,x; \quad (1.0)$$

$$\mathrm{P}_y\,v=\mathrm{P}_y\left(4\,x\,y\right)=4\,x\left(c_2+y^{2}/2\right)= \quad (1.0)$$

$$\text{using }(1.0)$$

$$=4\,x\left(3\,x/2+y^{2}/2\right)=2\,x^{2}\,3+2\,x\,y^{2} \quad (1.0)$$

Finding constants in (1.0) (1.0)

$$u=6\,x+2\,y^{2}=\frac{\partial}{\partial x}\mathrm{P}_y\,v=\frac{\partial}{\partial x}4\,x\left(c_2+y^{2}/2\right)=4\left(c_2+y^{2}/2\right)\implies c_2=3\,x/2; \quad (1.0)$$

$$v=4\,x\,y=\frac{\partial}{\partial y}\mathrm{P}_x\,u=\frac{\partial}{\partial y}\left(6\left(c_0+x^{2}\right)+2\,y^{2}(c_1+x)\right)=4\,y\left(c_1+x\right)\implies c_1=0; \quad (1.0)$$

$$f(x)=\mathrm{P}_x\,u=$$

$$\text{using }(1.0)$$

$$=6\left(c_0+x^{2}\right)+2\,y^{2}\,x=\mathrm{P}_y\,v=$$

$$\text{using }(1.0)$$

$$=2\,x^{2}\,3+2\,x\,y^{2}\implies c_0=\left(2\,x^{2}\,3-6\,x^{2}\right)/6=0 \quad (1.0)$$

Exemplo 21

Consider the differential equation

$$\boldsymbol{u(x,y)\ \mathrm{d}x + v(x,y)\,\mathrm{d}y =}$$
$$\boldsymbol{= (2\,e^{2\,x}\,y + 2\,x\,y^2)\,\mathrm{d}x + (e^{2\,x} + 2\,x^2\,y)\,\mathrm{d}y = 0}$$

Check if its exact and find the implicit solution of the equation

---

Resposta (1.0)

Finding implicit solution $f(x)$

$$f(x) = \mathrm{P}_x(u) = \mathrm{P}_y(v) =$$

using (1.0) (1.0)

$$= 2\,c_3\,(x + x^2) + e^{2\,x}\,y + x^2\,y^2 \quad (1.0)$$

Solving $\mathrm{P}_x\,u, \mathrm{P}_y\,v$

$$\mathrm{P}_x\,u = \mathrm{P}_x\,(2\,e^{2\,x}\,y + 2\,x\,y^2) = 2\,y\,\left(c_0 + e^{2\,x}/2\right) + 2\,y^2(c_1 + x^2/2) = \quad (1.0)$$

using (1.0) (1.0)

$$= 2\,y\,\left(\left(c_3\,\frac{(x + x^2)\,2}{y}\right) + e^{2\,x}/2\right) + 2\,y^2\left(\left(-c_3\,\frac{(x + x^2)\,2}{2\,y^2}\right) + x^2/2\right) =$$

$$= y\,e^{2\,x} + 4\,c_3\,(x + x^2) - c_3\,(x + x^2)\,2 + y^2\,x^2 = y\,e^{2\,x} + 2\,c_3\,(x + x^2) + y^2\,x^2; \quad (1.0)$$

$$\mathrm{P}_y\,v = \mathrm{P}_y\,(e^{2\,x} + 2\,x^2\,y) = e^{2\,x}(c_2 + y) + 2\,x^2\,(c_3 + y^2/2) = \quad (1.0)$$

using (1.0)

$$= e^{2\,x}((c_3\,2\,x\,e^{-2\,x}) + y) + 2\,x^2\,(c_3 + y^2/2) = 2\,c_3\,(x + x^2) + e^{2\,x}\,y + x^2\,y^2 \quad (1.0)$$

Finding constants in (1.0) (1.0)

$$u = 2\,e^{2\,x}\,y + 2\,x\,y^2 = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}_y\,v =$$

using (1.0)

$$= \frac{\mathrm{d}}{\mathrm{d}x}(e^{2\,x}(c_2 + y) + 2\,x^2\,(c_3 + y^2/2)) = 2\,e^{2\,x}(c_2 + y) + 4\,x\,(c_3 + y^2/2) \implies$$

$$\implies 2\,e^{2\,x}\,c_2 + 4\,x\,c_3 = 0 \implies c_2 = c_3\,2\,x\,e^{-2\,x}; \quad (1.0)$$

$$v = e^{2\,x} + 2\,x^2\,y = \frac{\mathrm{d}}{\mathrm{d}y}\mathrm{P}_x\,u =$$

using (1.0)

$$= \frac{\mathrm{d}}{\mathrm{d}y}\left(2\,y\,\left(c_0 + e^{2\,x}/2\right) + 2\,y^2(c_1 + x^2/2)\right) = 2\,\left(c_0 + e^{2\,x}/2\right) + 4\,y(c_1 + x^2/2) \implies$$

$$\implies 0 = 2\,c_0 + 4\,y\,c_1 \implies c_0 = -2\,y\,c_1 = \quad (1.0)$$

using (1.0)

$$= -2\,y\,\left(-c_3\,\frac{(x + x^2)\,2}{2\,y^2}\right) = c_3\,\frac{(x + x^2)\,2}{y}; \quad (1.0)$$

$$f(x) = \mathrm{P}_x\,u =$$

using (1.0) (1.0)

$$= 2\,y\,\left((-2\,y\,c_1) + e^{2\,x}/2\right) + 2\,y^2(c_1 + x^2/2) = \mathrm{P}_y\,u =$$

using (1.0) (1.0)

$$= e^{2\,x}\left((c_3\,2\,x\,e^{-2\,x}) + y\right) + 2\,x^2\,(c_3 + y^2/2) \implies$$

$$\implies -4\,y^2\,c_1 + y\,e^{2\,x} + 2\,y^2\,c_1 + y^2\,x^2 = e^{2\,x}\,c_3\,2\,x\,e^{-2\,x} + e^{2\,x}\,y + 2\,x^2\,c_3 + x^2\,y^2 \implies$$

$$\implies -2\,y^2\,c_1 = c_3\,2\,x + 2\,x^2\,c_3 \implies c_1 = -c_3\,\frac{(x + x^2)\,2}{2\,y^2} \quad (1.0)$$

13.3 Fator integrante

Its a function that multiplies a non exact function making it exact

$$\boldsymbol{\phi(x, y) : \begin{pmatrix} \frac{\partial u(x,y)}{\partial y} & \neq & \frac{\partial v(x,y)}{\partial x} \wedge \\ \wedge \frac{\partial \phi(x,y)\, u(x,y)}{\partial y} & = & \frac{\partial \phi(x,y)\, v(x,y)}{\partial x} \end{pmatrix}}$$

finding the integrating factor

Its not always possible but when its possible:

$$\phi(x,y)_1 = \exp\left(\mathrm{P}_y\left(\frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{u}\right)\right); \quad \phi(x,y)_2 = \exp\left(\mathrm{P}_x\left(\frac{\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}}{v}\right)\right)$$

Exemplo 22

The equation

$$\boldsymbol{u(x,y)\ \mathrm{d}x + v(x,y)\ \mathrm{d}y =}$$
$$\boldsymbol{= (e^{x} + 5\,e^{-y})\,\mathrm{d}x + (e^{x} - 4\,e^{-y})\,\mathrm{d}y = 0} \tag{1.0}$$

Check if its exact and find the Integral factor if needed

Resposta

Checking if its exact

$$\frac{\partial u}{\partial y} =$$

using (1.0)

$$= -5\,e^{-y} \neq$$
$$\neq \frac{\partial v}{\partial x} =$$

using (1.0)

$$= e^{x}$$

(1.0) is not exact

Finding the integral factor $\phi(x,y)$

$$\phi_1(x,y) = \exp\left(\mathrm{P}_y\left(\frac{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{u}\right)\right) =$$

using (1.0) (1.0)

$$= \exp\left(\mathrm{P}_y\left(\frac{e^{x} + 5\,e^{-y}}{e^{x} + 5\,e^{-y}}\right)\right) = e^{y+c_0}$$

Solving $\frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(e^{x} + 5\,e^{-y}\right) = -5\,e^{-y}; \tag{1.0}$$
$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(e^{x} - 4\,e^{-y}) = e^{x} \tag{1.0}$$

13.4 Equations with separable variables

$$y' = f(x) \iff u(x)\,\mathrm{d}x + v(y)\,\mathrm{d}y$$
$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$$

Finding general solution

$$\mathrm{P}_x\, u(x) + \mathrm{P}_y\, v(y) = \mathrm{const}$$

Exemplo 23

$$\boldsymbol{y' = \frac{2\,x}{3\,y^2 + 4}}$$

Write in the $u(x,y)\,\mathrm{d}x + v(x,y)\,\mathrm{d}y$ form and find the general integral of the equation

---

Resposta (1.0)

Writting in the exact differential equation form

$$-2\,x\;\mathrm{d}x + (3\,y^2 + 4)\,\mathrm{d}y = 0 \tag{1.0}$$

Finding general solution

$$f(x,y) = \mathrm{P}_x\,(u(x)) + \mathrm{P}_y\,(v(y)) =$$

using (1.0)

$$= \mathrm{P}_x\,(-2\,x) + \mathrm{P}_y\,(3\,y^2 + 4) = -2\,(c_0 + x^2/2) + 3\,(c_1 + y^3/3) + 4\,(c_2 + y) =$$
$$= -2\,c_0 - x^2 + 3\,c_1 + y^3 + 4\,c_2 + 4\,y = 0 \implies$$
$$\implies -x^2 + y^3 + 4\,y = +2\,c_0 - 4\,c_2 - 3\,c_1 = c_3$$

First order equations not solved by the derivative

14 Lagrange's equation

$$\boldsymbol{y = x\,\alpha(y') + \beta(y'); \alpha(y') \neq y'}$$

Method

General solution for y

$$y = x\,\alpha(y') + \beta(y') =$$

using (1.0)

$$= g(p,c)\,\alpha(p) + \beta(p)$$

General solution for (1.0)

$$x = \cdots = g(p,c) \tag{1.0}$$

Linear differential equation for x

$$\mathrm{D}_p x - \frac{\alpha'(p)}{p-\alpha(p)}\,x = \frac{\beta'(p)}{p-\alpha(p)} \tag{1.0}$$

Substitution $y' = p$

$$y = x\,\alpha(y') + \beta(y') =$$

y'=p

$$= x\,\alpha(p) + \beta(p) =$$

using (1.0)

Finding p

$$y' = \mathrm{D}_x(x\,\alpha(p) + \beta(p)) = \alpha(p) + \mathrm{D}_x p\,(x\,\alpha'(p) + \beta'(p)) \tag{1.0}$$

Exemplo 24

Considere o PVI

$$\boldsymbol{y = x\left(1 + y'\right) + y', \quad y(0) = 1}$$

Solving general solutio for y

$$y = x\left(1 + y'\right) + y' =$$
$$\text{using } (1.0)$$
$$= x\left(1 + p\right) + p =$$
$$\text{using } (1.0)$$
$$= x\left(1 + \left(\ln \frac{c_0}{x+1}\right)\right) + \ln \frac{c_0}{x+1} = x\left(1 + \left(\ln c_0 - \ln x + 1\right)\right) + \ln \frac{c_0}{x+1} = \qquad (1.0)$$
$$\text{using } (1.0)$$
$$= \left(e^{-1}\, c_0 - 1\right) x\left(1 + 1\right) + 1 = c_1\, x - 1$$

Finding constants in (1.0)

$$y(0) = 1 =$$
$$\text{using } (1.0)$$
$$= \left(e^{-0}\, c_0 - 1\right)\left(1 + 0\right) + 0 = c_0 - 1 \implies c_0 = 2 \qquad (1.0)$$

Solving general solution for (1.0)

$$x = x_h + \bar{x} =$$
$$\text{using } (1.0)\ (1.0)$$
$$= e^{-p}\, c_0 - 1 \iff p = \ln \frac{c_0}{x+1} \qquad (1.0)$$

Finding $\bar{x}$

$$\bar{x} = Q_0(p) = \rho_0 = \qquad (1.0)$$
$$\text{using } (1.0)$$
$$= -1 \qquad (1.0)$$

Finding constants of (1.0)

$$\bar{x}\, P = \left(\rho_0\right)\left(\mathrm{D}_p + 1\right) = \rho_0 = -1 \qquad (1.0)$$

Mapping roots of (1.0) to solution

$$\left\{ r_i = -1 \implies e^{-1\,p}\, c_0 \right. \qquad (1.0)$$

Roots for characteristic equation for x_h

$$P = \mathrm{D}_p + 1 \implies$$
$$\mathrm{D}_p^i \to r^i$$
$$\implies r + 1 = 0 \implies \left\{ r_0 = -1 \right. \qquad (1.0)$$

Finding diff equation for x

$$y' = p =$$
$$\text{using } (1.0)$$
$$= \mathrm{D}_x(x\left(1 + p\right) + p) = \left(1 + p\right) + x\ \mathrm{D}_x p + \mathrm{D}_x p = 1 + p + \left(x + 1\right)\ \mathrm{D}_x p \implies$$
$$\implies \mathrm{D}_p x + x = -1 \qquad (1.0)$$

Subsitution $y' = p$

$$y = x\left(1 + y'\right) + y' =$$
$$y' = p$$
$$= x\left(1 + p\right) + p \qquad (1.0)$$

15 Clairut's equation

$$y = x\,y' + \beta(y')$$

Method

General solution for y

$$y = x\,y' + \beta(y') =$$

$$\text{using (1.0)}$$

$$= x\,p + \beta(p) =$$

$$\text{using (1.1)}$$

$$= \dots$$

Substitution $y' = p$

$$y = x\,y' + \beta(y') =$$

$$y' = p$$

$$= x\,p + \beta(p) \tag{1.0}$$

Finding differential equation for x

$$y' = p =$$

$$\text{using (1.0)}$$

$$= \mathrm{D}_x(x\,p + \beta(p)) = p + x\,p' + \beta'(p)\,p' \implies p'(x + \beta'(p)) = 0 \implies$$

$$\implies \begin{cases} p' = 0 \\ x = -\beta'(p) \end{cases} \tag{1.1}$$

Exemplo 25

Find the singular and general solutions for the following Clairut's equation

$$\boldsymbol{y = x\,y' + (y')^2}$$

Resposta (1.1)

General and singluar solutions for y

$$\begin{aligned}
&y = x\,y' + (y')^2 = && y' = p \\
&= x\,p + p^2 = && (1.1) \\
& && \text{using } (1.1) \\
&= \begin{cases} y = x\,c + c^2 & \text{general solution} \\ \begin{cases} x = -2\,p \\ y = -p^2 \end{cases} & \text{singular solution parametric} \\ y = -x^2/4 & \text{singular solution cartesian} \end{cases} && (1.1)
\end{aligned}$$

Solutions for y'

$$\begin{aligned}
&y' = p = && \\
& && \text{using } (1.1) \\
&= \mathrm{D}_x\big(x\,p + p^2\big) \implies x\,p' + 2\,p\,p' = 0 \implies \begin{cases} p' = 0 \\ x = -2\,p \end{cases} && (1.1)
\end{aligned}$$

II – Laplace Transform

Tables

Proprierties

$$e^{a\,t}\,\sin w\,t \xrightarrow{\mathcal{L}} \frac{w}{(s-a)^2+w^2}$$

$$e^{a\,t}\,\cos w\,t \xrightarrow{\mathcal{L}} \frac{s-a}{(s-a)^2+w^2} \qquad t \xrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}}$$

$$t^n\,e^{a\,t}, n\in\mathbb{N}^+ \xrightarrow{\mathcal{L}} \frac{n!}{(s-a)^{n+1}} \qquad t\,\sin w\,t \xrightarrow{\mathcal{L}} \frac{2\,s\,w}{(s^2+w^2)^2}$$

$$t\,\cos w\,t \xrightarrow{\mathcal{L}} \frac{s^2-w^2}{(s^2+w^2)^2} \qquad t\,\sinh w\,t \xrightarrow{\mathcal{L}} \frac{2\,s\,w}{(s^2-w^2)^2}$$

$$t\,\cosh w\,t \xrightarrow{\mathcal{L}} \frac{s^2+w^2}{(s^2-w^2)^2} \qquad \frac{\sin w\,t}{t} \xrightarrow{\mathcal{L}} \frac{\pi}{2} - \tan^{-1} s/w$$

$$a\,f+b\,g \xrightarrow{\mathcal{L}} a\,F(s)+b\,G(s) \qquad f(\lambda\,t) \xrightarrow{\mathcal{L}} \frac{1}{\lambda}F\left(\frac{s}{\lambda}\right)$$

$$\mathcal{H}(t-\tau)\,f(t-\tau) \xrightarrow{\mathcal{L}} e^{-s\,\tau}\,F(s) \qquad e^{-\lambda\,t}\,f(t) \xrightarrow{\mathcal{L}} F(s+\lambda)$$

$$f(t)/t \xrightarrow{\mathcal{L}} \int_s^\infty F(p)\;\mathrm{d}p \qquad (f\cdot g)(t) \xrightarrow{\mathcal{L}} F(s)\,G(s)$$

Derivative transform

$$\mathcal{L}(f') = s\;\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(\mathrm{D}_t^n f(t)) = s^n\;\mathcal{L}(f) - \sum_{k=0}^{n-1} s^{n-1-k}\;\mathrm{D}_t^k f(0)$$

Translations

$$\mathcal{L}(e^{a\,t}\,f(t)) = F(s-a) \qquad (2.1);$$

$$\mathcal{L}\,{-1}(F(s-a)) = e^{a\,t}\,f(t) \qquad (2.1);$$

$$\mathcal{L}(f(t-a)\;\mathcal{H}(t-a)) = e^{-a\,s}\,F(s)(2.1);$$

$$\mathcal{L}^{-1}(e^{-a\,s}\,F(s)) = f(t-a)\;\mathcal{H}(t-a)(2.1)$$

Basic transforms

$$\mathcal{L}(1) = 1/s, \quad s>0; \qquad \mathcal{L}(e^{a\,t}) = \frac{1}{s-a}, \quad s>a;$$

$$\mathcal{L}(\cos(w\,t)) = \frac{s}{s^2+w^2}, \quad s>0; \qquad \mathcal{L}(\cosh(w\,t)) = \frac{s}{s^2-w^2}, \quad s>\max(-w,w);$$

$$\mathcal{L}(\sin(w\,t)) = \frac{w}{s^2+w^2}, \quad s>0; \qquad \mathcal{L}(\sinh(w\,t)) = \frac{w}{s^2-w^2}, \quad s>\max(-w,w);$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \qquad s>0\wedge n\in\mathbb{N}^+;$$

Inverse Transforms

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) = \frac{e^{a\,t}-e^{b\,t}}{a-b}, \qquad a\neq b\wedge s>\max(a,b)$$

Derivative Transforms

$$\mathcal{L}(f') = s\;\mathcal{L}(f) - f(0); \qquad s>\rho : \rho \text{ is the exponential order } 1 \text{ of } f(x)$$

Solving notable transforms

Auxiliar demonstrations

$$
\begin{aligned}
&\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right) = \frac{1}{b}\,\mathrm{P}_t\left(e^{a\,t}\left(b\,\cos(b\,t)\right)\right) = \\
&\hfill \mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \sin(b\,t) \end{cases} \\
&= \frac{1}{b}(e^{a\,t}\,\sin(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right)) = \\
&= \frac{1}{b}\Big(e^{a\,t}\,\sin(b\,t) + \frac{a}{b}\,\mathrm{P}_t\left(e^{a\,t}\left(-b\,\sin(b\,t)\right)\right)\Big) = \\
&\hfill \mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \cos(e^{b\,t}) \end{cases} \\
&= \\
&= \frac{1}{b}\Big(e^{a\,t}\,\sin(b\,t) + \frac{a}{b}\left(e^{a\,t}\,\cos(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right)\right)\Big) = \\
&= \frac{1}{b}\,e^{a\,t}\,\sin(b\,t) + \frac{a}{b^2}\,e^{a\,t}\,\cos(b\,t) - \frac{a^2}{b^2}\,\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right) \implies \\
&\implies \mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right) = e^{a\,t}\,\frac{a\,\cos(b\,t) + b\,\sin(b\,t)}{a^2+b^2} \qquad (2.1)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right) = -\frac{1}{b}\,\mathrm{P}_t\left(e^{a\,t}\left(-b\,\sin(b\,t)\right)\right) = \\
&\hfill \mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \cos(b\,t) \end{cases} \\
&= -\frac{1}{b}(e^{a\,t}\,\cos(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\cos(b\,t)\right)) = \\
&= -\frac{1}{b}\Big(e^{a\,t}\,\cos(b\,t) - \frac{a}{b}\,\mathrm{P}_t\left(e^{a\,t}\left(b\,\cos(b\,t)\right)\right)\Big) = \\
&\hfill \mathrm{P}_t(u\,v') = u\,v - \mathrm{P}_t(u'\,v)\begin{cases} u = e^{a\,t} \\ v = \sin(b\,t) \end{cases} \\
&= -\frac{1}{b}\Big(e^{a\,t}\,\cos(b\,t) - \frac{a}{b}\left(e^{a\,t}\,\sin(b\,t) - a\,\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right)\right)\Big) = \\
&= -\frac{1}{b}\,e^{a\,t}\,\cos(b\,t) + \frac{a}{b^2}\,e^{a\,t}\,\sin(b\,t) - \frac{a^2}{b^2}\,\mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right) \implies \\
&\implies \mathrm{P}_t\left(e^{a\,t}\,\sin(b\,t)\right) = e^{a\,t}\,\frac{a\,\sin(b\,t) - b\,\cos(b\,t)}{a^2+b^2} \qquad (2.1)
\end{aligned}
$$

Transforms

$\mathcal{L}(1)$

$$
\begin{aligned}
&\mathcal{L}(1) = \int_0^\infty (e^{-s\,t}\,1\ \mathrm{d}t) = \lim_{k\to\infty}\int_0^k (e^{-s\,t}\,1\ \mathrm{d}t) = \lim_{k\to\infty}\left(-\frac{1}{s}\,e^{-s\,t}\right)\Big|_0^k = \\
&= \lim_{k\to\infty}\left(-\frac{1}{s}\,e^{-s\,k} + \frac{1}{s}\,e^{-s\,0}\right) = \frac{1}{s} \qquad (2.1)
\end{aligned}
$$

$\mathcal{L}(e^{a\,t})$

$$
\begin{aligned}
&\mathcal{L}(e^{a\,t}) = \lim_{k\to\infty}\left(\int_0^k e^{-s\,t}\,e^{a\,t}\ \mathrm{d}t\right) = \lim_{k\to\infty}\left(\int_0^k e^{(a-s)\,t}\ \mathrm{d}t\right) = \lim_{k\to\infty}\left(\frac{e^{(a-s)\,t}}{a-s}\right)\Big|_0^k = \\
&= \lim_{k\to\infty}\left(\frac{e^{(a-s)\,k}}{a-s} - \frac{e^{(a-s)\,0}}{a-s}\right) = \\
&\hfill s > a \\
&= -\frac{1}{a-s} = \frac{1}{s-a}; \qquad (2.1) \\
&\mathcal{L}^{-1}(1/(s-a)) = e^{a\,t} \qquad (2.1)
\end{aligned}
$$

$\mathcal{L}(\cos(w\,t))$

$$
\begin{aligned}
&\mathcal{L}(\cos(w\,t)) = \lim_{k\to\infty}\left(\int_0^k e^{-s\,t}\,\cos(w\,t)\ \mathrm{d}t\right) = \\
&\hfill \text{using } (2.1) \\
&= \lim_{k\to\infty}\left(e^{-s\,t}\,\frac{-s\,\cos(w\,t) + w\,\sin(w\,t)}{s^2+w^2}\right)\Big|_0^k = \\
&= \lim_{k\to\infty}\left(e^{-s\,k}\,\frac{-s\,\cos(w\,k) + w\,\sin(w\,k)}{s^2+w^2} - e^{-s\,0}\,\frac{-s\,\cos(w\,0) + w\,\sin(w\,0)}{s^2+w^2}\right) = \\
&\hfill s > 0 \\
&= \frac{s}{s^2+w^2}; \\
&\mathcal{L}^{-1}\left(\frac{s}{s^2+w^2}\right) = \cos w\,t \qquad (2.1)
\end{aligned}
$$

$\mathcal{L}(\sin(w\,t))$

$$
\begin{aligned}
&\mathcal{L}(\sin(w\,t)) = \lim_{k\to\infty}\left(\int_0^k e^{-s\,t}\,\sin(w\,t)\ \mathrm{d}t\right) = \\
&\hfill \text{using } (2.1) \\
&= \lim_{k\to\infty}\left(e^{-s\,t}\,\frac{-s\,\sin(w\,t) - w\,\cos(w\,t)}{s^2+w^2}\right)\Big|_0^k = \\
&= \lim_{k\to\infty}\left(e^{-s\,k}\,\frac{-s\,\sin(w\,k) - w\,\cos(w\,k)}{s^2+w^2} - e^{-s\,0}\,\frac{-s\,\sin(w\,0) - w\,\cos(w\,0)}{s^2+w^2}\right) = \\
&\hfill s > 0 \\
&= \frac{w}{s^2+w^2} \qquad (2.1)
\end{aligned}
$$

$\mathcal{L}\left(\cosh(a\,t)\right)$

$$
\begin{aligned}
&\mathcal{L}(\cosh(a\,t)) = \mathcal{L}\,(\frac{1}{2}(e^{a\,t} + e^{-a\,t})) = \frac{1}{2}(\mathcal{L}\left(e^{a\,t}\right) + \mathcal{L}\left(e^{-a\,t}\right)) = \\
&\hfill \text{using } (2.1) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}
\end{aligned}
$$

$\mathcal{L}\left(\sinh(a\,t)\right)$

$$
\begin{aligned}
&\mathcal{L}(\cosh(a\,t)) = \mathcal{L}\,(\frac{1}{2}(e^{a\,t} - e^{-a\,t})) = \frac{1}{2}(\mathcal{L}\left(e^{a\,t}\right) - \mathcal{L}\left(e^{-a\,t}\right)) = \\
&\hfill \text{using } (2.1) \\
&= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2}
\end{aligned}
$$

Inverse Laplace transform

$\mathcal{L}^{-1}(1/(s-a)(s-b))$

$$
\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right) = \mathcal{L}^{-1}\left(\frac{1}{a-b}\left(\frac{a-b+s-s}{(s-a)(s-b)}\right)\right) = \\
&= \frac{1}{a-b}\,\mathcal{L}^{-1}\left(\left(\frac{s-b}{(s-a)(s-b)} - \frac{s-a}{(s-a)(s-b)}\right)\right) = \\
&= \frac{1}{a-b}\left(\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-b}\right)\right) = \\
&\hfill \text{using } (2.1) \land s > \max(a,b) \land a \neq b \\
&= \frac{e^{a\,t} - e^{b\,t}}{a-b}
\end{aligned}
$$

$\mathcal{L}(t^n)$

$$
\begin{aligned}
&\mathcal{L}(t^n) = \lim_{k=\infty}\int_0^k \left(e^{-s\,t}\,t^n\ \mathrm{d}t\right) = \\
&\hfill \mathrm{P}(u\,v') = u\,v - \mathrm{P}(u'\,v)\begin{cases} u = t^n \\ v = -e^{-s\,t}/s \end{cases} \\
&= \lim_{k=\infty}\left(-\left(\frac{t^n\,e^{-s\,t}}{s}\right)\Big|_0^k - \int_0^k \left(\frac{e^{-s\,t}}{-s}\,n\,t^{n-1}\ \mathrm{d}t\right)\right) = \\
&= \lim_{k=\infty}\left(-\frac{k^n\,e^{-s\,k}}{s} + \frac{0^n\,e^{-s\,0}}{s} - \int_0^k \left(\frac{e^{-s\,t}}{-s}\,n\,t^{n-1}\ \mathrm{d}t\right)\right) = \\
&\hfill s > 0 \\
&= \frac{n}{s}\lim_{k=\infty}\left(\int_0^k \left(e^{-s\,t}\,t^{n-1}\ \mathrm{d}t\right)\right) = \frac{n}{s}\,\mathcal{L}(t^{n-1}) = \frac{n}{s}\,\frac{n-1}{s}\,\mathcal{L}(t^{n-2}) = \\
&= \prod_{i=0}^{n-1}\left(\frac{n-i}{s}\right)\mathcal{L}(t^{n-n}) = \frac{n!}{s^n}\,\mathcal{L}(1) = \\
&\hfill \text{using } (2.1) \\
&= \frac{n!}{s^n}\,\frac{1}{s} = \frac{n!}{s^{n+1}} \qquad (2.1)
\end{aligned}
$$

1 Introduction

$$\mathcal{L}\, f(x) = F(x)$$

Let $f(t)$ be a function of the real variable t, for all $t \in \mathbb{R}$; the values of $f(t)$ may be either real or complex, although in our applications they will be real. The function f is said to be differetiable at tpnly finitely many points of I, and all its points of discotinuity are jumps (i.e. there are right and left limits of the function at those points).

Exploring the existence of the transform

We now introduce a class of functions for which the transformatio will be defined. We assume that the following three conditions are satisfied:

(1) $t = 0 \implies f(t) = 0$

(2) f is piecewise differentialbe

(3) there exist real numbers M, ρ such that

$$|f(t)| \leq M\, e^{\rho\, t} \quad \forall\, t \in \mathbb{R}$$

note: here ρ is said to be the *exponential order* of f

Checking if transform exists

$\cosh t;$ $\quad$ $\sinh t;$ $\quad$ t^n

Resposta

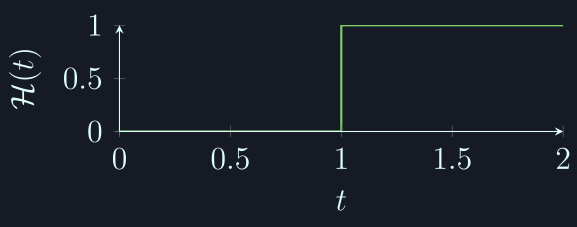
$$|\cosh t| = \left|\frac{e^t + e^{-t}}{2}\right| = \frac{e^t + e^{-t}}{2} \leq \frac{e^t + e^t}{2} = e^t;$$

$$|\sinh t| = \left|\frac{e^t - e^{-t}}{2}\right| \leq \frac{1}{2}(|e^t| + |e^{-t}|) = \frac{1}{2}(e^t + e^{-t}) \leq \frac{1}{2}(e^t + e^t) = e^t;$$

$$|t^n| = n!\,\frac{t^n}{n!} \leq n! \sum_{i=0}^{\infty} \frac{t^i}{i!} = n!\, e^t$$

1.1 The Heaviside function

$$\mathcal{H}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Like any bounded function, this satisfies condition (3) with $\rho = 0$. Any function $\phi(t)$ that fails to satisfy conditions (1), but does satisfy conditions (2) and (3), then the function $f(t) = \mathcal{H}(t)\,\phi(t)$ will satisfy all three conditions. for Example

$\mathcal{H}(t)\ \sin w\, t,$ $\quad$ $\mathcal{H}(t)\, t^n,$ $\quad$ $\mathcal{H}(t)\, e^{a\, t}$

For simplicity we usually omit the factor $\mathcal{H}(t)$

$$\tilde{f}(t) = f(t-a)\ \mathcal{H}(t-a)$$

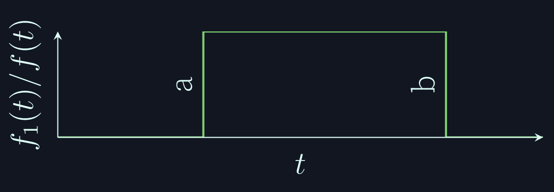
Uses for the Heaviside function

Let $f(t)$ be a function on the interval $t \geq 0$, and let $f_1(t)$ be a "piece" of $f(t)$ on the interval $[a, b[, a \geq 0$, that is

$$f_1(t) \begin{cases} f(t), & t \in [a, b[\\ 0, & c.c. \end{cases}$$

To set the value of $f_1(t)$ to zero for $t < 0$, we multiply $f(t)$ by $\mathcal{H}(t-a)$. To get zero for $t \geq b$ we can substrac from $f(t)$ the values $f(t)$ as $t \geq b$, that is subtract $\mathcal{H}(t-b)\, f(t)$. thus

$$f_1(t) = (\mathcal{H}(t-a) - \mathcal{H}(t-b))f(t)$$



Exemplo 1

Using the Heaviside function, write down the piecewise definition of the function

$$\boldsymbol{f(t)\begin{cases} 0, & 0 \leq t < 2 \\ 3\,t & 2 \leq t < 4, \\ 2, & t \geq 4 \end{cases}}$$

Resposta

$$f(t) = \begin{pmatrix} +3\,t(\mathcal{H}(t-2) - \mathcal{H}(t-4)) \\ +2\,(\mathcal{H}(t-4)) \end{pmatrix}$$

2 Laplace Transform of the Derivative

For the first derivative

Suppose that $f(x)$ follows all three laplace conditions 1 and has exponential order 1 γ

$$\mathcal{L}(f') = s\ \mathcal{L}(f) - f(0), \quad s > \gamma$$

For the n-th derivative

Suppose that $\mathrm{D}_t^i f \; \forall \, i$ follows all three laplace conditions 1 and has exponential order 1 γ

$$\mathcal{L}\left(\mathrm{D}_t^n f\right) = s^n\ \mathcal{L}(f) - \sum_{i=1}^{n} s^{n-i}\ \mathrm{D}_t^{i-1} f(0) \tag{2.1}$$

Exemplo 2

Find the transforms using the derivative method

$$\boldsymbol{t^n}; \qquad \textbf{sin}\,\boldsymbol{w\,t} \qquad \textbf{sin}^{\mathbf{2}}\,\boldsymbol{t}$$

Resposta

Solving for t^n

$$\mathcal{L}(\mathrm{D}_t^{n+1} t^n) = \mathcal{L}(0) = 0 =$$

using (2.1)

$$= s^{n+1}\ \mathcal{L}(t^n) - \sum_{i=1}^{n+1} s^{n-i}\ \mathrm{D}_t^{i-1} t^n(0) = s^{n+1}\ \mathcal{L}(t^n) - n! \implies \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

Solving for $\sin w\,t$

$$\mathcal{L}(\mathrm{D}_t^2 \sin w\,t) = \mathcal{L}(-w^2\ \sin(w\,t)) = -w^2\ \mathcal{L}(\sin(w\,t)) =$$

using (2.1)

$$= s^2\ \mathcal{L}(\sin w\,t) - s\ \sin w * 0 - w\ \cos w * 0 \implies \mathcal{L}(\sin w\,t) = \frac{w}{s^2 + w^2}$$

Solving for $\sin^2(t)$

$$\mathcal{L}(\mathrm{D}_t \sin^2 t) = \mathcal{L}(2\ \sin(t)\ \cos(t)) = \mathcal{L}(\sin(2\,t)) =$$

using (2.1)

$$= \frac{2}{s^2 + 4} =$$

using (2.1)

$$= s\ \mathcal{L}(\sin^2 t) - \sin(2 * 0) = s\ \mathcal{L}(\sin^2 t) \implies \mathcal{L}(\sin^2 t) = \frac{2/s}{s^2 + 4}$$

Exemplo 3 Applying to differential equations

Considere o PVI

$$\boldsymbol{y'' + 4\,y' + 3\,y = 0, \quad y(0) = 3, y'(0) = 1}$$

Resposta

Finding general solution

$$y = \mathcal{L}^{-1}\,Y =$$

using (2.1)

$$= \mathcal{L}^{-1}\left(\frac{5}{s+1} + \frac{-2}{s+3}\right) = 5\,\mathcal{L}(1/(s+1)) - 2\,\mathcal{L}(1/(s+3)) =$$

using (2.1)

$$= 5\,e^{-1\,t} - 2\,e^{-3\,t}$$

Checking existence of Laplace transform

$$y_1 = 5\,e^{-t}$$
$$y_2 = -2\,e^{-3\,t}$$
$$s > \max(-1, -3) = -1$$
$$\lambda = \max(-1, -3) = -1$$

Both follow the conditions 1 with greater exponential being -1

Finding Y

$$\mathcal{L}(y'' + 4\,y' + 3\,y) = \mathcal{L}(y'') + 4\;\mathcal{L}(y') + 3\;\mathcal{L}(y) =$$
$$= s^2\;\mathcal{L}(y) - s\,y(0) - y'(0) + 4\,(s\;\mathcal{L}(y) - y(0)) + 3\;\mathcal{L}(y) =$$
$$= s^2\;\mathcal{L}(y) + 4\,s\;\mathcal{L}(y) - 13 - s\,3 + 3\;\mathcal{L}(y) = 0 \implies$$
$$\implies \mathcal{L}(y) = \frac{13 + s\,3}{s^2 + 4\,s + 3} =$$

$$s = \frac{-4 \pm \sqrt{4^2 - 4*1*3}}{2\,1} = -2 \pm 1$$

$$= \frac{13 + s\,3}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \tag{2.1}$$

using (2.1)

$$= \frac{5}{s+1} + \frac{-2}{s+3} \tag{2.1}$$

Finding constants in (2.1)

$$13 + s\,3 = A(s+3) + B(s+1) = (A+B)s + 3\,A + B \implies$$

$$\implies \begin{cases} B = 13 - 3\,A = 13 - 15 = -2 \\ A + (13 - 3\,A) = 3 \implies A = 10/2 = 5 \end{cases} \tag{2.1}$$

3 Laplace transform of an Integral

$$\mathcal{L}\left(\int_0^t f(x)\ \mathrm{d}x\right) = \frac{1}{s}\ \mathcal{L}(f(t)); \tag{2.1}$$

$$\mathcal{L}^{-1}(f(t)/s) = \int_0^t (f(x)\ \mathrm{d}x)$$

Exemplo 4

Find the inverse Laplace transform of

$$\boldsymbol{F(s) = \frac{1}{s(s^2 + w^2)}}$$

Resposta

$$\mathcal{L}^{-1}\left(F(s)\right) = \mathcal{L}^{-1}\left(\frac{1}{s(s^2+w^2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{1}{w}\,\frac{w}{s^2+w^2}\right) =$$

using (2.1)

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{w}\,\mathcal{L}\left(\sin w\,t\right)\right) =$$

using (2.1)

$$= \frac{1}{w}\int_0^t \left(\sin w\,x\,\mathrm{d}x\right) = \frac{1}{w}\left(-\cos(w\,x)/w\right)\Big|_0^t =$$

$$= \frac{1}{w^2}\left(-\cos(w\,t) + \cos(w\,0)\right) = \frac{1 - \cos w\,t}{w^2}$$

4 Translação da variavel s

$$\boldsymbol{\mathcal{L}(f(t)) = F(s), \quad s \in \,]\gamma, \infty[\; \Longrightarrow \; \mathcal{L}(e^{a\,t}\,f(t)) = F(s-a), \quad s \in \,]a} \tag{2.1}$$

$$\boldsymbol{\mathcal{L}^{-1}(F(s-a)) = e^{a\,t}\,f(t)} \tag{2.1}$$

Exemplo 5

Consider the problem with inital values

$$\boldsymbol{y'' + 2\,y' + 5\,y = 0, \quad y(0) = 2, \quad y'(0) = -4}$$

Find the general solution

Resposta

General solution for y

$$y = \mathcal{L}^{-1}\,Y =$$

using (2.1)

$$= \mathcal{L}^{-1}\left(2\,\frac{s+1}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}\right) =$$
$$= 2\;\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) - \mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+2^2}\right) =$$

using (2.1)

$$= 2\,e^{-t}\,\cos(2\,t) - e^{-t}\,\sin(2\,t)$$

Finding Y

$$\mathcal{L}(y'' + 2\,y' + 5\,y) = \mathcal{L}(y'') + 2\;\mathcal{L}(y') + 5\;\mathcal{L}(y) =$$

using (2.1)

$$= s^2\,Y - s\,y(0) - y'(0) + 2\,(s\,Y - y(0)) + 5\,Y =$$
$$= s^2\,Y - s\,(2) - (-4) + 2\,s\,Y - 2\,(2) + 5\,Y = 0 \implies$$
$$\implies = Y = \frac{s\,2}{s^2 + 2\,s + 5} = \frac{2\,(s+1-1)}{(s+1)^2+4} = 2\,\frac{s+1}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2} \tag{2.1}$$

5 Translation of the variable t

$$\mathcal{L}(f(t)) = F(s), s \in]\gamma, \infty[\implies \mathcal{L}(f(t-a)\ \mathcal{H}(t-a)) = e^{-a\,s}\,F(s) \tag{2.1}$$

$$\mathcal{L}^{-1}(e^{-a\,s}\,F(s)) = f(t-a)\ \mathcal{H}(t-a) \tag{2.1}$$

Exemplo 6

$$\boldsymbol{F(s) = \frac{e^{-3\,s}}{s^3}}$$

Resposta

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{e^{-3\,s}}{s^3}\right) = \mathcal{L}^{-1}\left(\frac{e^{-3\,s}}{2}\,\frac{2}{s^3}\right) =$$

using (2.1)

$$= \frac{1}{2}\,\mathcal{L}^{-1}\left(e^{-3\,s}\,\mathcal{L}(t^2)\right) =$$

using (2.1)

$$= \frac{(t-3)^2\ \mathcal{H}(t-3)}{2}$$

Exemplo 7

Find the laplace transform of the function

$$\boldsymbol{f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\,\pi \\ \sin t, & t > 2\,pi \end{cases}}$$

Resposta

Solving lagplace transform of f

$$\mathcal{L}(f(t)) = \mathcal{L}(1(\mathcal{H}(t-0) - \mathcal{H}(t-\pi)) + \sin(t)\,\mathcal{H}(t-2\,\pi)) =$$
$$= \mathcal{L}(1) - \mathcal{L}(\mathcal{H}(t-\pi)) + \mathcal{L}(\sin(t)\,\mathcal{H}(t-2\,\pi)) =$$

using (2.1)

$$= \frac{1}{s} - \frac{e^{-\pi\,s}}{s} + \frac{e^{-2\,\pi\,s}}{s^2+1}$$

Exemplo 8

Consider the problem of initial values

$$\boldsymbol{y'' + 3\,y' + 2\,y = r(t), \quad y(0) = 0, y'(0) = 0;}$$
$$\boldsymbol{r(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases}}$$

Resposta

Finding y

$$y = \mathcal{L}^{-1}\,Y =$$

using (2.1)

$$= \mathcal{L}^{-1}\left((1 - e^{-s})\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right)\right) =$$
$$= \mathcal{L}^{-1}\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2} - e^{-s}\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right)\right) =$$

using (2.1) (2.1)

$$= \frac{1}{2} - e^{-1\,t} + \frac{1}{2}\,e^{-2\,t} - \mathcal{H}(t-1)\left(\frac{1}{2} - e^{-1\,t} + \frac{1}{2}\,e^{-2\,t}\right)$$

$$\mathcal{L}(y'' + 3\,y' + 2\,y) = \mathcal{L}(y'') + 3\ \mathcal{L}(y') + 2\ \mathcal{L}(y) =$$

using (2.1)

$$= s^2\,Y - s\,y(0) - y'(0) + 3\,(s\,Y - y(0)) + 2\,Y = Y\,(s^2 + 3\,s + 2) =$$
$$= \mathcal{L}(r(t)) = \mathcal{L}(\mathcal{H}(t-0) - \mathcal{H}(t-1)) = \mathcal{L}(1 - \mathcal{H}(t-1)) = \frac{1}{s} - \frac{e^{-1\,s}}{s} \implies$$
$$\implies Y = \frac{1 - e^{-s}}{s(s^2 + 3\,s + 2)} = \frac{1 - e^{-s}}{s(s(s+1) + 2(s+1))} =$$
$$= \frac{1 - e^{-s}}{s(s+2)(s+1)} = (1 - e^{-s})\left(\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right) = \tag{2.1}$$

using (2.1)

$$= (1 - e^{-s})\left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right) \tag{2.1}$$

Finding constants in (2.1)

$$1 = (A + C + B)\,s^2 + s\,(3\,A + 2\,B + C) + A\,2 \implies$$
$$\begin{cases} A = 1/2 \\ C = -B - A = -B - 1/2 = -(-1) - 1/2 = 1/2 \\ 3\,(1/2) + 2\,B + (-B - 1/2) = 0 \implies B = -1 \end{cases} \tag{2.1}$$

III – Series

1 Notas:

Inicialmente queremos saber se as series convergem ou divergem, aprender os diversos casos espeçificos que nos provam limites espeçificos

Critérios	Teste	Casos	
		Divergente	Convergente
Comparação	$\lim_{n\to\infty} \frac{b_n}{a_n}$	> 1	< 1
A'lamberk	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$	> 1	< 1
Raiz	$\limsup_{n\to\infty} \sqrt[n]{a_n}$	> 1	< 1
Integral	$\int_1^\infty f(x)\ \mathrm{d}x : f(n) = a_n$	> 1	< 1

Critério do Integral

$$\int_1^\infty \boldsymbol{f(x)}\ \mathrm{d}\boldsymbol{x} : \boldsymbol{f(n) = n}$$

- $f(x)$ é continua em $[1, \infty[$
- Testar f para saber se converge ou diverge, a_n segue o mesmo comportamento
- $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ Aponta con/divergencia de f