AM 2C – Teste Resolução 2022.1

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Conteúdo

Grupo I

Questão 1

A parábola com foco em (2,0) e recta directriz $x = -2$, e a elipse com centro em (0,1), foco em $(\sqrt{7}, 1)$ e vértice em (4,1) admitem, respectivamente, as seguintes equações:

Resposta

(i) Parabola

$$P = \left\{ (x,y) \in \mathbb{R}^2 : \begin{pmatrix} \quad (y - y') = (x - x')^2/(4\,a) & \wedge \\ \wedge\, f' = (x' + a, y') = (2,0) & \wedge \\ \wedge\, L \subset \mathbb{R}^2 : x = x' - a = -2 & \wedge \\ \wedge\, y' = 0 & \wedge \\ \wedge\, x' + a + x' - a = 2\,x' = 2 - 2 = 0 & \wedge \\ \wedge\, a = 2 + x' = 2 & \end{pmatrix} \right\} =$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : y = x^2/8 \right\}$$

(ii) Elipse

$$E = \left\{ (x,y) \in \mathbb{R}^2 : \begin{pmatrix} \quad \dfrac{(x - x')^2}{r_1^2} + \dfrac{(y - y')^2}{r_2^2} = 1 & \wedge \\ \wedge\, (x', y') = (0,1) & \wedge \\ \wedge\, f_2 = (x' + c, y') = (c, 1) = (\sqrt{7}, 1) & \wedge \\ \wedge\, P_x = (x' + r_1, y') = (r_1, y') = (4,1) & \wedge \\ \wedge\, r_2^2 = r_1^2 - c^2 = 16 - 7 = 9 & \end{pmatrix} \right\} =$$

$$= \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{16} + \frac{(y-1)^2}{9} = 9\,x^2 + 16(y-1)^2 = 1 * 9 * 16 = 144 \right\}$$

Questão 2

Considere-se as seguintes funções reais de variável real:

$$f(x,y) = \frac{y\,x^3}{y+x^3}; \qquad g(x,y) = \frac{\sin(x^2+y^2)}{e^{(x^2+y^2)^{-1}}};$$

$$h(x,y) = \frac{2\,x+y}{x-y}$$

Relativamente ao limite de cada uma delas no ponto (0,0) tem-se que:

a) f e g têm limite zero, e h não tem limite.

b) f e g têm limite zero, e h tem limite 2.

c) f e g têm limite zero, e h tem limite -1.

d) f têm limite zero, h tem limite 2 e o limite de g é infinito.

e) f e h não têm limite e o limite de g é infinito.

f) f e h não têm limite e o limite de g é zero.

(i) f

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y\,x^3}{y+x^3} = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{x^3} = 0 =$$
$$= \lim_{x\to 0} f(x, x^6 - x^3) = \lim_{x\to 0} \frac{(x^6-x^3)\,x^3}{(x^6-x^3)+x^3} = \lim_{x\to 0} x^3 - 1 = -1$$

(ii) g

$$-1 \leq \sin(x^2+y^2) \leq 1 \wedge \lim_{(x^2+y^2)\to 0} e^{(x^2+y^2)^{-1}} = +\infty \therefore \lim_{(x,y)\to(0,0)} g(x,y) = 0$$

(iii) h

$$\lim_{x\to 0}\left(\lim_{y\to 0} h(x,y)\right) = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{2\,x+y}{x-y}\right) = 2 \neq \lim_{y\to 0}\left(\lim_{x\to 0} \frac{2\,x+y}{x-y}\right) = -1$$

Questão 3

Considere a função $f : \mathbb{R}^2 \to \mathbb{R}^2$ definida por

$$\boldsymbol{f(x,y) = (3\,x - y^2, x^3 - 3\,y^2) f(u,v)}$$

Que é invertivel numa vizinhança do ponto (1,1). Tem-se que:

a) $\frac{\partial x}{\partial u}(2,-2) = \frac{\partial x}{\partial v}(2,-2)$

b) $\frac{\partial x}{\partial u}(2,-2) = -\frac{\partial x}{\partial v}(2,-2)$

c) $\frac{\partial y}{\partial u}(2,-2) = -\frac{\partial y}{\partial v}(2,-2)$

d) $\frac{\partial x}{\partial u}(2,-2) = \frac{\partial y}{\partial u}(2,-2)$

e) $\frac{\partial y}{\partial u}(2,-2) = \frac{\partial y}{\partial v}(2,-2)$

f) $\frac{\partial x}{\partial v}(2,-2) = \frac{\partial y}{\partial v}(2,-2)$

$$f(1,1) = (3 * 1 - 1^2, 1^3 - 3 * 1^2) = (2,-2)$$

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}_{(2,2)} = \mathrm{J}_{f^{-1}}\,(2,-2) = \mathrm{J}_{f^{-1}}\,(f(1,1)) = \mathrm{J}_{f}\,(f(1,1))^{-1} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}^{-1}_{(1,1)}$$

$$= \begin{bmatrix} 3 & -2\,y \\ 3\,x^2 & -6\,y \end{bmatrix}^{-1}_{(1,1)} = \begin{bmatrix} 3 & -2 \\ 3 & -6 \end{bmatrix}^{-1} = \begin{vmatrix} 3 & -2 \\ 3 & -6 \end{vmatrix}^{-1} \mathrm{adj} \begin{bmatrix} 3 & -2 \\ 3 & -6 \end{bmatrix} = -12^{-1} \begin{bmatrix} -6 & \\ -3 & \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/6 \\ 1/4 & -1/4 \end{bmatrix}$$

Questão 4

A equação define z como função de x e de y numa vizinhança do ponto $(x_0, y_0, z_0) = (1, -1, 0)$. Então:

$$\boldsymbol{x^4 + y^4 + (x+1)\, e^z + 8\, x\ \sin(z) - 4 = 0}$$

a) $\frac{\partial z}{\partial x}(1,-1) = +1/2, \frac{\partial z}{\partial y}(1,-1) = +2/5$

b) $\frac{\partial z}{\partial x}(1,-1) = -1/2, \frac{\partial z}{\partial y}(1,-1) = -2/5$

c) $\frac{\partial z}{\partial x}(1,-1) = -1/2, \frac{\partial z}{\partial y}(1,-1) = +2/5$

d) $\frac{\partial z}{\partial x}(1,-1) = +1/2, \frac{\partial z}{\partial y}(1,-1) = +2$

e) $\frac{\partial z}{\partial x}(1,-1) = -1/2, \frac{\partial z}{\partial y}(1,-1) = +2$

f) $\frac{\partial z}{\partial x}(1,-1) = -1/2, \frac{\partial z}{\partial y}(1,-1) = -2$

(i) $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x}(1,-1) = -\frac{\frac{\partial f}{\partial x}(1,-1,0)}{\frac{\partial f}{\partial z}(1,-1,0)} = -\frac{\left(4\,x^3 + e^z + 8\ \sin(z)\right)(1,-1,0)}{((x+1)\,e^z + 8\,x\ \cos(z))\,(1,-1,0)} = -\frac{4}{(1}$$

(ii) $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y}(1,-1) = -\frac{\frac{\partial f}{\partial y}(1,-1,0)}{\frac{\partial f}{\partial z}(1,-1,0)} = -\frac{\left(4\,y^3\right)(1,-1,0)}{((x+1)\,e^z + 8\,x\ \cos(z))\,(1,-1,0)} =$$

$$= -\frac{4\,(-1)^3}{(1+1) * e^{(0)} + 8 * 1 * \cos(0)} = \frac{2}{5}$$

Questão 5

Sejam g e h duas funções reais e de classe C^1 em $\mathbb{R}$. Considere a função $f : \mathbb{R}^3 \to \mathbb{R}^2$ definida por

$$\boldsymbol{f(x, y, z) = \left(x\, g\, \left(\frac{x}{z}\right), h\, (x^2\, y)\right)}$$

Determine a matriz jacobiana de f no ponto (-1,0,-1) em função das derivadas de g e de h.

$$J_f(-1,0,-1) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}_{(-1,0,-1)} =$$

$$= \begin{bmatrix} g\left(\frac{x}{z}\right) + x\, g'\left(\frac{x}{z}\right)\, z^{-1} & 0 & -g'\left(\frac{x}{z}\right)\, z^{-2} \\ h'(x^2\, y)\, y\, 2\, x & h'(x^2\, y)\, x^2 & 0 \end{bmatrix}_{(-1,0,-1)} =$$

$$= \begin{bmatrix} g\left(\frac{-1}{-1}\right) + (-1)\, g'\left(\frac{-1}{-1}\right)(-1)^{-1} & 0 & -g'\left(\frac{-1}{-1}\right)\, (-1)^{-2} \\ h'((-1)^2\, 0)\, (0)\, 2\, (-1) & h'((-1)^2\, 0)\, (-1)^2 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} g(1) + g'(1) & 0 & -g'(1) \\ 0 & h'(0) & 0 \end{bmatrix}$$

Grupo II

Questão 1

Considere a função real g, de duas variáveis reais, definida por

$$g(x,y) = \begin{cases} \frac{x^3 - 3\,x\,y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Q1 a.

Estude, por definição, a continuidade de *g(x,y)* em (0,0).

$$\forall\,\delta>0\,\exists\,\varepsilon>0:\left(\left(\forall(x,y)\neq(0,0)\wedge\left\|\sqrt{x^2+y^2}\right\|<\varepsilon\right)\implies|g(x,y)-0|\right.$$
$$\implies\left|\frac{x^3-3\,x\,y^2}{x^2+y^2}\right|\leq|x|\,\frac{(x^2+3\,y^2)}{x^2+y^2}\leq|x|\,\frac{(3\,x^2+3\,y^2)}{x^2+y^2}\leq$$
$$\leq 3\,|x|\leq 3\sqrt{x^2+y^2}\leq 3\,\varepsilon=\delta\implies\varepsilon=\delta/3$$

Q1 b.

Determine $\frac{\partial g}{\partial x}(0,0) e \frac{\partial g}{\partial y}(0,0)$

(i)

$$\frac{\partial g}{\partial x}(0,0) = \lim_{h \to 0} \frac{g(h,0) - g(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3 - 3*h*0^2}{h^2 + 0^2}}{h} = \lim_{h \to 0} 1 = 1$$

(ii)

$$\frac{\partial g}{\partial y}(0,0) = \lim_{h \to 0} \frac{g(0,h) - g(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0^3 - 3*0*h^2}{0^2 + h^2}}{h} = \lim_{h \to 0} 0 = 0$$

Q1 c.

Estude a diferenciabilidade de g no ponto (0,0).

$$g(a,b) - g(0,0) = \frac{a^3 - 3\,a\,b^2}{a^2 + b^2} = \frac{\partial g}{\partial x}(0,0)\,a + \frac{\partial g}{\partial y}(0,0)\,b + \varepsilon(a,b)\,\sqrt{a^2+b^2}$$

$$= 1\,a + \varepsilon(a,b)\,\sqrt{a^2+b^2} \implies \varepsilon(a,b) = \frac{-4\,a\,b^2}{(a^2+b^2)^{3/2}} \implies$$

$$\implies \lim_{a\to 0^+} \varepsilon(a,a) = \lim_{a\to 0^+} \frac{-4 * a * a^2}{(a^2+a^2)^{3/2}} = \lim_{a\to 0^+} -2^{1/2} = -\sqrt{2} \neq 0$$

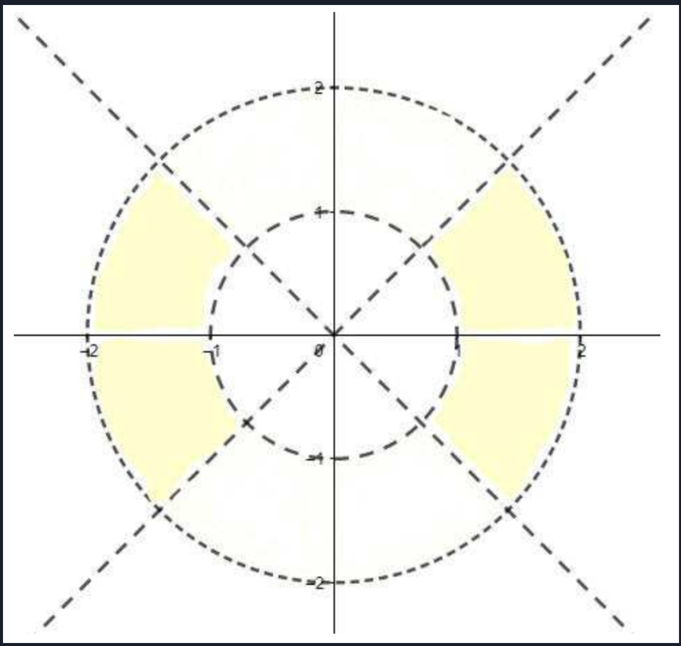
Questão 2

Considere a função real f de duas variáveis reais, definida por

$$f(x,y) = \frac{\log(x^2+y^2-1)\,\log(x^2-y^2)}{\sqrt{4-x^2-y^2}}$$

Indique o seu domínio D e esboce-o. Caracterize $\operatorname{int}\mathrm{D}$ usando coordenadas polares. Diga, justificando, se D é um conjunto aberto ou fechado.

$$\mathrm{D}(f) = \left\{ (x,y)\in\mathbb{R}^2 : \begin{pmatrix} 4-x^2-y^2 \geq 0 & \wedge \\ \wedge\sqrt{4-x^2-y^2} \neq 0 & \wedge \\ \wedge\, x^2+y^2-1>0 & \wedge \\ \wedge\, x^2-y^2>0 & \end{pmatrix} \right\} =$$
$$= \left\{ (x,y)\in\mathbb{R}^2 : 1 < x^2+y^2 < 4 \wedge |x| > |y| \right\}$$



$$\operatorname{int}(\mathrm{D}) = \mathrm{D} = \{(\rho,\theta)\in\,]1,4[\,\times\,]-\pi/4,\pi/4[\} \cup \{(\rho,\theta)\in\,]1,4[\,\times\,]3\pi/4,5\pi$$

Grupo III

Questão 1

Considere a função:

$$\begin{aligned} f : \mathrm{D} \subset \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x, y) &\to x^2 + x\,y + y^2 \end{aligned}$$

Q1 a.

Calcule os extremos locais de $f(x,y)$ quando $\mathrm{D}=\mathbb{R}^2$.

$$\left\{(x,y)\in\mathbb{R}^2 : \left\{\begin{matrix}\dfrac{\partial f}{\partial x}=2\,x+y=0\\ \dfrac{\partial f}{\partial y}=x+2\,y=0\end{matrix}\right\}\right\}=$$

$$=\left\{(x,y)\in\mathbb{R}^2 : \left\{\begin{matrix}-3\,y=0\\ x=-2\,y\end{matrix}\right\}\right\}=\{(0,0)\}$$

$$\det \mathrm{H}(f(x,y))=\det\begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\,\partial y}\\ \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y\,\partial x}\end{bmatrix}_{(0,0)}=\det\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}_{(0,0)}=4-1=3$$

$\therefore\ (0,0)$ é um mínimo local

Q1 b.

No conjunto $D = (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1$ a função $f(x, y)$ admite mínimo e máximo absolutos. Escreva a função lagrangiana associada a este problema e determine os referidos extremos.

$$\mathrm{L} = f + \lambda\, g = x^2 + x\, y + y^2 + \lambda(x^2 + y^2 - 1) \implies$$

$$\implies \left\{\begin{aligned} \frac{\partial\, \mathrm{L}}{\partial x} &= 0 \\ \frac{\partial\, \mathrm{L}}{\partial y} &= 0 \\ \frac{\partial\, \mathrm{L}}{\partial \lambda} &= 0 \end{aligned}\right\} = \left\{\begin{aligned} 2\,x + y + \lambda(2\,x) &= 0 \\ x + 2\,y + \lambda(2\,y) &= 0 \\ x^2 + y^2 - 1 &= 0 \end{aligned}\right\} = \left\{\begin{aligned} 2\,x(\lambda + 1) + y &= 0 \\ 2(\lambda + 1)(x - y) &= x - \\ x^2 + y^2 &= 1 \end{aligned}\right.$$

$$= \left\{\begin{aligned} &\left(\begin{aligned} (\lambda = -1/2 \wedge x = y) \quad &\vee \\ \vee\, (x = -y \wedge \lambda = -3/2) \quad & \end{aligned}\right) \\ &\qquad\qquad\qquad x^2 + y^2 = 1 \end{aligned}\right\} \implies C_f = \left\{\begin{aligned} &(+2^{-1/2}, +2^{-1/2}), \\ &(+2^{-1/2}, -2^{-1/2}), \\ &(-2^{-1/2}, +2^{-1/2}), \\ &(-2^{-1/2}, -2^{-1/2}) \end{aligned}\right\} =$$

$$\implies \left\{\begin{aligned} f(+2^{-1/2}, +2^{-1/2}) &= 3/2 \\ f(+2^{-1/2}, -2^{-1/2}) &= 1/2 \\ f(-2^{-1/2}, +2^{-1/2}) &= 1/2 \\ f(-2^{-1/2}, -2^{-1/2}) &= 3/2 \end{aligned}\right.$$

$$\therefore \left\{\begin{aligned} &\text{Minimizantes:}\{(+2^{-1/2}, -2^{-1/2}), (-2^{-1/2}, +2^{-1/2})\} \\ &\text{Maximizantes:}\{(+2^{-1/2}, +2^{-1/2}), (-2^{-1/2}, -2^{-1/2})\} \end{aligned}\right.$$

Grupo IV

Questão 1

Determine os pontos da superfície de equação $x^2 + 2\,y^2 - 3\,z^2 = 1$ nos quais o plano tangente é paralelo ao plano de equação $3\,x - 2\,y + 3\,z = 1$

$$(2\,x_0)(x - x_0) + (4\,y_0)(y - y_0) + (-6\,z_0)(z - z_0) =$$
$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} -2\,x_0 \\ 4\,y_0 \\ -6\,z_0 \end{bmatrix} - 2\,x_0^2 - 4\,y_0^2 + 6\,z_0^2 = 0$$

$$3\,x - 2\,y + 3\,z - 1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} - 1 = 0 \implies$$

$$\implies \begin{bmatrix} -2\,x_0 \\ 4\,y_0 \\ -6\,z_0 \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} \implies \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 3\,\alpha/2 \\ -\alpha/2 \\ -\alpha/2 \end{bmatrix} \implies$$
$$\implies (3\,\alpha/2)^2 + 2\,(-\alpha/2)^2 - 3\,(-\alpha/2)^2 - 1 =$$
$$= 9\,\alpha^2 + 2\,\alpha^2 - 3\,\alpha^2 - 4 = 8\,\alpha^2 - 4 = 0 \implies$$
$$\implies \alpha = \pm\sqrt{1/2}$$

$$\therefore \left\{ \begin{matrix} (+3/2^{3/2}, -2^{-3/2}, -2^{-3/2}), \\ (-3/2^{3/2}, +2^{-3/2}, +2^{-3/2}) \end{matrix} \right\}$$