Ficha 6

1. $\begin{cases} y'(t) = t\,y(t), & t \in [a,b] \\ y(a) = 1 \end{cases}$

$D = \{(t,y) \in \mathbb{R}^2 : a \leq t \leq b,\ y \in \mathbb{R}\}$

$f(t,y) = ty$ é contínua em D pois é uma função $C^\infty$ em $\mathbb{R}$

$\left|\frac{\partial f}{\partial y}(t,y)\right| = |t| = \begin{cases} t & \text{se } t \geq 0 \\ -t & \text{se } t < 0 \end{cases}$ $t \in [a,b]$

constante de Lipschitz $L > 0$

$L = \max\{|a|, |b|\}$

se $|b| > |a| \Rightarrow |t| \leq |b| = L > 0$

se $|b| < |a| \Rightarrow |t| \leq |a| = L > 0$

se $|b| = |a| \Rightarrow |t| \leq |a| = |b| > 0$

Logo pelo Teorema 6.3, como f é contínua e satisfaz a condição de Lipschitz em D relativamente a y o problema é bem posto.

$|f(t,y_1) - f(t,y_2)| = |ty_1 - ty_2| = |t|\,|y_1 - y_2| = \left|\frac{\partial f}{\partial y}(t,y)\right| |y_1 - y_2| \leq L$

$|t| = \begin{cases} t & \text{se } t \geq 0 \\ -t & \text{se } t < 0 \end{cases}$ $t \in [a,b]$

se $t \geq 0$ $|t| \leq b = L > 0$

se $t < 0$ $|t| \leq |a| = L > 0$

$L = \mathrm{Max}(|a|, |b|)$

$$2. \begin{cases} y'(t) = 1 - \dfrac{y(t)}{t} & t \in [2,3] \\ y(2) = 2 \end{cases}$$

a) $y(2.1)$? $\boxed{h = 0.1}$

$0.1 = h = \dfrac{3-2}{N} \Rightarrow N = \dfrac{1}{0.1} = 10$ $\quad t_i = a + hi$

$2.1 = 2 + 0.1i \Leftrightarrow i = \dfrac{2.1-2}{0.1} = 1$

Metodo de Euler Progressivo

$f(t, y) = 1 - \dfrac{y}{t}$

$$\begin{cases} w_0 = 2 \\ w_{i+1} = w_i + h f(t_i, w_i) \end{cases}$$

$t_i = 2 + 0.1i$

$i = 0, 1, \dots 9$

$$w_{i+1} = w_i + 0.1\left(1 - \frac{w_i}{t_i}\right)$$

$$w_1 = w_0 + 0.1\left(1 - \frac{w_0}{2+0.1\times 0}\right) = 2 + 0.1\cancel{\left(1 - \frac{2}{2}\right)} = 2$$

$y(2.1) \simeq w_1 = 2$ ✓

$\boxed{h = 0.05}$ $\quad t_i = 2 + 0.05 \times i = 2.1 \Leftrightarrow 0.05 \times i = 0.1 \Leftrightarrow i = \dfrac{0.1}{0.05} = 2$

$t_i = 2 + 0.05i$

$w_0 = 2$

$$w_1 = 2 + 0.05\left(1 - \frac{w_0}{2+0.05\times 0}\right) = 2 + 0.05\left(1 - \frac{2}{2}\right) = 2$$

$$w_2 = w_1 + 0.05\left(1 - \frac{w_1}{2+0.05\times 1}\right) = 2 + 0.05\left(1 - \frac{2}{2.05}\right) = 2 + 0.05 \times 0.02439$$

$= 2.00122$ ✓

b) Metodo de Taylor ordem 2

$\boxed{h = 0.1}$

$$\begin{cases} w_0 = 2 \\ w_{i+1} = w_i + h\Big(\overbrace{f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)}^{T^{(2)}(t_i, w_i)}\Big) \end{cases}$$

$y'(t) = f(t, y) = 1 - \dfrac{y}{t}$

$$f'(t, y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'(t) = \frac{y}{t^2} + \left(-\frac{1}{t}\right)\left(1 - \frac{y}{t}\right) = \frac{y}{t^2} - \frac{1}{t} + \frac{y}{t^2} = \frac{2y}{t^2} - \frac{1}{t}$$

2.b) Continuação

$$w_{i+1} = w_i + h\left(\left(1-\frac{w_i}{t_i}\right) + \frac{h^2}{2}\left(\frac{2w_i}{t_i^2} - \frac{1}{t_i}\right)\right)$$

$$w_1 = w_0 + 0.1\left(\left(1-\frac{w_0}{t_0}\right) + \frac{0.1}{2}\left(\frac{2w_0}{t_0^2} - \frac{1}{t_0}\right)\right) = 2 + 0.1\left(\left(\cancel{1-\frac{2}{2}}\right) + \frac{0.1}{2}\left(\frac{2\times 2}{2^2} - \frac{1}{2}\right)\right)$$

$$= 2 + \frac{0.1}{2}\left(1-\frac{1}{2}\right) = 2 + \frac{0.01}{4}$$

$$y(2.1) \simeq w_1 = 2.0025 \checkmark$$

$t_1 = 2+0.05 = 2.05$

$\boxed{h=0.05} \Rightarrow i=2 \wedge t_i = 2+0.05i$

$w_0 = 2$

$$w_1 = 2 + 0.05\left(\cancel{\left(1-\frac{2}{2}\right)} + \frac{0.05}{2}\left(\frac{4}{4} - \frac{1}{2}\right)\right) = 2 + \frac{0.025}{4} = 2.00625$$

$$w_2 = w_1 + 0.05\left(\left(1-\frac{w_1}{t_1}\right) + \frac{0.05}{2}\left(\frac{2w_1}{t_1^2} - \frac{1}{t_1}\right)\right) =$$

$$= 2.0125 + 0.05\left(\left(1 - \frac{2.00625}{2.05}\right) + \frac{0.05}{2}\left(\frac{2\times 2.00625}{(2.05)^2} - \frac{1}{2.05}\right)\right)$$

$$= 2.00241$$

$$y(2.1) \simeq w_2 = 2.00241$$

c) $y(t) = \dfrac{4+t^2}{2t}$ $\quad y(2.1) = \dfrac{4+(2.1)^2}{2\times 2.1} = 2.002381$

Euler progressivo

erro absoluto $|2.002381 - 2.00122| = 0.001161 < 0.5\times 10^{-2}$ 2 c.d.s

Taylor ordem 2

erro absoluto $|2.002381 - 2.00241| = 0.000028 < 0.5\times 10^{-4}$ 4 c.d.s

Ficha 6 - suplementar

3. Metodo de Taylor de ordem 2 c/ $h = 0.25$

$$\begin{cases} y'(t) = \frac{1}{2} t \cos^2(y(t)), \; t \in [0,1] \\ y(0) = -1 \end{cases}$$

$t = 0.5$

$$\underset{t_0}{0} \quad \underset{t_1}{0.25} \quad \underset{t_2}{0.5}$$

$t_i = t_0 + hi$

$0.5 = 0 + 0.25 i \Rightarrow i = 2$

$$f(t,y) = \frac{1}{2} t \cos^2(y)$$

$$f'(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'(t) = \frac{1}{2}\cos^2(y) - t\cos(y)\,\text{sen}(y) \times \frac{1}{2} t \cos^2(y)$$

$$= \frac{1}{2}\cos^2(y)\left(1 - t^2 \cos(y)\,\text{sen}(y)\right)$$

$$\begin{cases} w_0 = -1 \\ w_{i+1} = w_i + h \times \frac{1}{2} t_i \cos^2(w_i) + \frac{h^2}{2} \times \frac{1}{2}\cos^2(w_i)\left(1 - t_i^2 \cos(w_i)\,\text{sen}(w_i)\right) \end{cases}$$

$$w_1 = w_0 + \frac{0.25}{2} \times \underbrace{t_0}_{=0} \times \cos^2(w_0) + \frac{(0.25)^2}{4}\cos^2(w_0)\left(1 - \underbrace{t_0^2}_{=0} \times \cos(w_0)\,\text{sen}(w_0)\right)$$

$$= -1 + \frac{0.25^2}{4}\cos^2(-1) \simeq -0.995439$$

$$w_2 = w_1 + \frac{0.25}{2} \times t_1 \times \cos^2(w_1) + \frac{(0.25)^2}{4}\cos^2(w_1)\left(1 - t_1^2 \cos(w_1)\,\text{sen}(w_1)\right)$$

$$= -0.995439 + \frac{0.25}{2} \times 0.25 \times \cos^2(-0.995439) + \frac{(0.25)^2}{4}\cos^2(-0.995439)\left(1 - (0.25)^2\cos(-0.995439)\,\text{sen}(-0.995439)\right)$$

$$\simeq -0.995439 + 0.009253 + 0.004758 \simeq -0.981428$$

$$y(0.5) \simeq -0.981428$$

# Ficha 6

6.1 $\begin{cases} y'(t) = t^2 y(t), & t \in [0,1] \\ y(0) = 5 \end{cases}$

é bem posto?

$D = \{(t,y) \in \mathbb{R}^2 : 0 \leq t \leq 1 \text{ e } y \in \mathbb{R}\}$

$f(t,y) = t^2 y$ é contínua em D

sejam $(t,y_1)$ e $(t,y_2) \in D$

$$|f(t,y_1) - f(t,y_2)| = |t^2 y_1 - t^2 y_2| = \underbrace{t^2}_{\leq 1} |y_1 - y_2| \leq L |y_1 - y_2|$$

com $L = 1$
$\forall (t,y_1),(t,y_2) \in D$

ou então basta provar que

$$\left|\frac{\partial f(t,y)}{\partial y}\right| = \left|\frac{\partial f}{\partial y}(t^2 y)\right| = |t^2| \leq 1 = L \Rightarrow$$ o problema

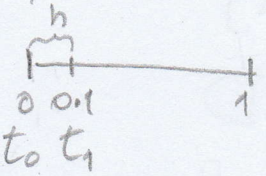
satisfaz a condição de Lipschitz em D, logo o problema é bem posto.

6.2 $\begin{cases} y'(t) = t^2 - y(t), & t \in [0,1] \\ y(0) = 1 \end{cases}$

a) $y(0.1)$ pelo metodo de Euler progressivo com $h = 0.1$ e $h = 0.05$

$f(t,y) = t^2 - y$

<u>$h = 0.1$</u>




$t_i = a + ih$

$0.1 = 0 + i \times 0.1 \iff i = 1$

Método de Euler progressivo

$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h f(t_i, w_i) \end{cases}$

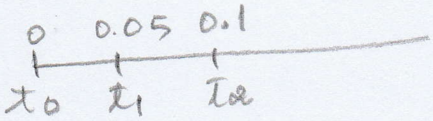
$\begin{cases} w_0 = 1 = y(0) \\ w_{i+1} = w_i + 0.1 (t_i^2 - w_i) \end{cases}$

$w_1 = w_0 + h f(t_0, w_0)$

$w_1 = 1 + 0.1 f(0,1) = 1 + 0.1(0-1) = 0.9$

$y(0.1) \simeq w_1 = 0.9$

<u>$h = 0.05$</u>




$t_i = a + ih$

$0.1 = 0 + i \times 0.05$

$i = 2$

$w_1 = w_0 + h f(t_0, w_0)$

$w_1 = 1 + 0.05 (t_0^2 - w_0) = 1 + 0.05(0-1) = 0.95$

$w_2 = w_1 + h f(t_1, w_1)$

$w_2 = 0.95 + 0.05(0.05^2 - 0.95) \simeq 0.902\,625$

$y(0.1) \simeq w_2 = 0.902625$

6.2

b) Metodo de Taylor de ordem 2 com $h = 0.05$

$y(0.1)$?

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\, f(t_i, w_i) + \frac{h^2}{2} f'(t_i, w_i) \end{cases}$$

$0.1 = 0 + i \times 0.05 \Rightarrow i = 2$

| | | |
|---|---|---|
| $0$ | $0.05$ | $0.1$ |
| $t_0$ | $t_1$ | $t_2$ |

$f(t, y) = t^2 - y$

$f'(t, y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'(t) = 2t - 1(t^2 - y) = 2t - t^2 + y$

$$w_{i+1} = w_i + 0.05(t_i^2 - w_i) + \frac{0.05^2}{2}(2t_i - t_i^2 + w_i)$$

$$= w_i + 0.05(t_i^2 - w_i) + 0.00125(2t_i - t_i^2 + w_i)$$

$i = 1$, $w_1 = w_0 + 0.05(t_0^2 - w_0) + 0.00125(2t_0 - t_0^2 + w_0)$

$= 1 + 0.05(0 - 1) + 0.00125(2 \times 0 - 0 + 1)$

$= 1 - 0.05 + 0.00125 = 0.95125$

$i = 2$, $w_2 = w_1 + 0.05(t_1^2 - w_1) + 0.00125(2t_1 - t_1^2 + w_1)$

$= 0.95125 + 0.05((0.05)^2 - 0.95125) + 0.00125(2 \times 0.05 - (0.05)^2 + 0.95125)$

$\simeq 0.95125 - 0.047438 + 0.001311 \simeq 0.905123$

$y(0.1) \simeq w_2 \simeq 0.905123$

c) $y(t) = -e^{-t} + t^2 - 2t + 2$

$y(0.1) = -e^{-0.1} + (0.1)^2 - 2 \times 0.1 + 2 = 0.905163$

$|y(0.1) - w_2| = |0.905163 - 0.905123| \leq 0.00004 < 0.5 \times 10^{-4}$

Podemos garantir pelo menos 4 c.d.s.