# CNA – Interpolation and Extrapolation

Felipe B. Pinto 71951 – EQB

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## Conteúdo

# 4 Cubic spline interpolation

A function composed of low-order piecewise polynomials is frequently referred to as a spline. A data point where two splines are joined is called a knot. Conventionally, a data interval is divided into more than one interval, where a linear, quadratic, or cubic polynomial is employed for each interval. In this respect, spline methods present a different approach than the polynomial interpolations covered so far.
The simplest spline method is a *linear spline*, where two data points are joined by a line
The use of cubic polynomials is the most common choice in literature, The reason for this is that cubic splines can be joined in different ways to produce an overall interpolating curve. At this stage, we shall consider non-uniformly spaced data and will employ a cubic polynomial $y_k(x)$, which has a different set of coefficients for each interval, $[x_k, x_{k+1}]$. Each cubic polynomial is then joined to its neighboring cubic polynomials at the knots by matching the slopes and curvatures $y', y''$. cubic polynomials for each interval is written as

$$y(x) = \begin{bmatrix} \sum_{i=0}^{3} c_{0,i}(x-x_i)^i \\ \sum_{i=0}^{3} c_{1,i}(x-x_i)^i \\ \vdots \\ \sum_{i=0}^{3} c_{n,i}(x-x_i)^i \end{bmatrix}$$

The following conditions will then be imposed to determine the unknows

| | | |
|---|---|---|
| Continuity of $y(x)$ | $y_{i-1}(x_i) = y_i(x_i) = f_i$ | $i \in [1, n-1]$ |
| Continuity of $y'(x)$ | $y'_{i-1}(x_i) = y'_i(x_i)$ | $i \in [1, n-1]$ |
| Continuity of $y''(x)$ | $y''_{i-1}(x_i) = y''_i(x_i)$ | $i \in [1, n-1]$ |

End conditions

From those conditions we can conclude

$$\begin{cases} y_i(x_{i+1}) = c_{i,3}\,\Delta x_i^3 + c_{i,2}\;\Delta x_i^2 + c_{i,1}\;\Delta x_i + c_0 = y_{i+1} = c_{i+1,0} \\ y_i'(x_{i+1}) = 3\,c_{i,3}\,\Delta x^2 + 2\,c_{i,2}\,\Delta x + c_{i,1} = y'_{i+1}(x_{i+1}) = c_{i+1,1} \\ y_i''(x_{i+1}) = 6\,c_{i,3}\,\Delta x + 2\,c_{i,2} = y''_{i+1}(x_{i+1}) = 2\,c_{i+1,2} \end{cases}$$

Merging the conditions and defining a few constructs we can organize all equations as follows

$$\left\{ \begin{array}{c} \begin{pmatrix} \Delta x_0\,S_0 + \Delta x_1\,S_{1+1} \\ +2\,(\Delta x_1 + \Delta x_0)\,S_1 \end{pmatrix} = 6\,(f[x_1,x_2] - f[x_0,x_1]) \\ \vdots \\ \begin{pmatrix} \Delta x_{k-1}\,S_{k-1} + \Delta x_k\,S_{k+1} \\ +2\,(\Delta x_k + \Delta x_{k-1})\,S_k \end{pmatrix} = 6\,(f[x_k,x_{k+1}] - f[x_{k-1},x_k]) \\ \vdots \\ \begin{pmatrix} \Delta x_{n-1}\,S_{n-1} + \Delta x_n\,S_{n+1} \\ +2\,(\Delta x_n + \Delta x_{n-1})\,S_n \end{pmatrix} = 6\,(f[x_n,x_{n+1}] - f[x_{n-1},x_n]) \end{array} \right\}$$

$$S_{k+a} = y_k''(x_{k+a}) \quad \Delta x_k = x_{k+1} - x_k \quad f[x_k, x_{k+1}] = \frac{f_{k+1} - f_k}{x_{k+1} - x_k}$$

Up to the third condition we can achieve $2(n-2)$ equations but we still have $3\,n-5$ variables, with two more equations we can compleete a system of linear equations, those are the end conditions and there are several spline formulations that are developed through the curvature estimations, we will consider only two cases: natural splines and linear extrapolation end conditions

## 4.1 Natural splines

$$S_0 = 0 \quad S_n = 0$$

Applying this two equations to the previous system gives

$$\begin{bmatrix} b_1 & c_1 & & & & & \\ a_2 & b_2 & c_2 & & & & \\ & a_3 & b_3 & c_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{n-2} & b_{n-2} & c_{n-2} & \\ & & & & a_{n-1} & b_{n-1} & c_{n-1} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{n-2} \\ S_{n-1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} \end{bmatrix}$$

Where

$$a_k = \Delta x_{k-1} \quad b_k = 2(\Delta x_{k-1} + \Delta x_k) \quad c_k = \Delta x_k$$
$$d_k = 6\,(f[x_k, x_{k+1}] - f[x_{k-1}, x_k])$$

## 4.2 Linear extrpolation end condition

This is one of the most frenquenty used end-conditions. the end point values are estimated by linear extrapolation.
linear extrapolation for the knots $1, n-1$ yelds

$$\frac{S_1 - S_0}{\Delta x_0} = \frac{S_2 - S_1}{\Delta x_1} \iff S_0 = S_1\left(1 + \frac{\Delta x_1}{\Delta x_2}\right) - S_2\,\frac{\Delta x_1}{\Delta x_2}$$
$$\frac{S_n - S_{n-1}}{\Delta x_{n-1}} = \frac{S_{n-1} - S_{n-2}}{\Delta x_{n-2}} \iff S_n = S_{n-1}\left(1 + \frac{\Delta x_{n-1}}{\Delta x_{n-2}}\right) - S_{n-2}\,\frac{\Delta x_{n-1}}{\Delta x_{n-2}}$$