Exam 2024.3 Resolution

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Conteúdo

Grupo I

Questão 1

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2\ \sin(2\,x)\,y = \sin(2\,x); y(\pi/2) = 3/2$$

Resposta (1.2)

General solution

$$y = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\ \mathrm{P}_x\left(\sin(2\,x)\,\varphi(x)\right) =$$

using (1.4) (1.5)

$$= \frac{c_0}{c_2\,e^{(-\cos(2\,x))}} + \frac{1}{c_2\,e^{(-\cos(2\,x))}}\,c_2\left(c_3 + e^{-\cos(2\,x)}\right) =$$

$$= e^{\cos(2\,x)}\left(\frac{c_0}{c_2} + c_3\right) + 1 = c_4\,e^{\cos(2\,x)} + 1 = \tag{1.1}$$

using (1.3)

$$= (e/2)\,e^{\cos(2\,x)} + 1 = e^{1+\cos(2\,x)}/2 + 1;$$

Closest option:

$$e^{\cos(2\,x)+1} + 1 \tag{1.2}$$

Finding constants in (1.1)

$$y(\pi/2) = 3/2 =$$

using (1.1)

$$= c_4\,e^{-1} + 1 \implies c_4 = e^1\left(\frac{3}{2} - \frac{2}{2}\right) = e/2 \tag{1.3}$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\left(2\ \sin(2\,x)\right)\right) = \exp\left(c_1 - \cos(2\,x)\right) = c_2\,e^{(-\cos(2\,x))} \tag{1.4}$$

Integrating

$$\mathrm{P}_x\left(\sin(2\,x)\,\varphi(x)\right) =$$

using (1.4)

$$= \mathrm{P}_x\left(\sin(2\,x)c_2\ \exp\left(-\cos(2\,x)\right)\right) =$$

$$\mathrm{D}_x\left(e^{-\cos(2\,x)}\right) = e^{-\cos(2\,x)}\left(2\ \sin(2\,x)\right)$$

$$= c_2\left(c_3 + e^{-\cos(2\,x)}\right) \tag{1.5}$$

Questão 2

$$\left(\mathbf{D}_x^3 + \mathbf{D}_x^2\right) y = -4$$

Resposta (1.6)

General solution for y

$$y = y_h + \bar{y} =$$

using (1.8) (1.10)

$$= e^{+0\,x}\,(c_0 + c_1\,x) + e^{-1\,x}\,(c_2) - 2\,x^2 \tag{1.6}$$

Finding $\bar{y}$

$$\bar{y} = x^p\,Q_0(x) = x^p\,\sum_{i=0}^{0} \rho_i\,x^i = x^2\,\rho_0 = \tag{1.7}$$

using (1.9)

$$= -2\,x^2 \tag{1.8}$$

Finding constants of (1.7)

$$\bar{y}\,P = x^2\,\rho_0\left(\mathrm{D}_x^3 + \mathrm{D}_x^2\right) = 2\,\rho_0 =$$
$$= -4 \implies \rho_0 = -2 \tag{1.9}$$

Mapping roots of (1.11) to solution

$$\begin{cases} r_0 = r_1 = 0 \implies e^{+0\,x}\,(c_0 + c_1\,x) \\ r_2 = -1 \implies e^{-1\,x}\,(c_2) \end{cases} \tag{1.10}$$

Roots for characteristic equation for y_h

$$P = \mathrm{D}_x^3 + \mathrm{D}_x^2 \implies$$

$\mathrm{D}_x^i \to r^i$

$$\implies r^3 + r^2 = r^2(r+1) = 0 \implies \begin{cases} r_0 = r_1 = 0 \\ r_2 = -1 \end{cases} \tag{1.11}$$

Questão 3

$$\boldsymbol{y' + \frac{1}{x}\, y = -2\, x^5\, y^4, \quad x > 0}$$

Resposta (1.13)

General solution

$$y = z^{-1/3} = \tag{1.12}$$

$$= \left(\frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\ \mathrm{P}_x\left(-3\,(-2)\,x^5\,\varphi(x)\right)\right)^{-1/3} =$$

using (1.14) (1.15)

$$= \left(\frac{c_0}{c_1\,x^{-3}} + \frac{1}{c_1\,x^{-3}}\,6\,c_1(c_2 + x^3/3)\right)^{-1/3} =$$

$$= \left(x^3\left(\frac{c_0}{c_1} + 6\,c_2\right) + 2\right)^{-1/3} =$$

$$= \left(x^3\,c_3 + 2\right)^{-1/3};$$

Closest option

$$\frac{1}{\sqrt[3]{2\,x^6 + c\,x^3}} \tag{1.13}$$

Bernoulli's substitution

$$y' + \frac{1}{x}\,y = (-2)\,x^5\,y^4 \implies$$

using (1.12)

$$\implies z' + -3\frac{1}{x}\,z = -3\,(-2)\,x^5$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\,(-3\,\frac{1}{x})\right) = \exp\left(-3\,(c_0 + \ln x)\right) = c_1\,x^{-3} \tag{1.14}$$

Integrating

$$\mathrm{P}_x\left(-3\,(-2)\,x^5\,\varphi(x)\right) =$$

using (1.14)

$$= \mathrm{P}_x\left(-3\,(-2)\,x^5 c_1\,x^{-3}\right) = 6\,c_1\,\mathrm{P}_x\left(x^2\right) = 6\,c_1(c_2 + x^3/3) \tag{1.15}$$

Resposta (1.17)

General solution

$$y = z^{-1/3} = \tag{1.16}$$

$$= \left(\frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\ \mathrm{P}_x\left((-3)\,(-2)\,x^5\,\varphi(x)\right)\right)^{-1/3} =$$

using (1.18) (1.19)

$$= \left(\frac{c_0}{c_1\,x^{-3}} + \frac{1}{c_1\,x^{-3}}6\,c_1(c_2 + x^3/3)\right)^{-1/3} = \left(x^3\left(\frac{c_0}{c_1} + 6\,c_2\right) + 2\right)^{-1/3} = \left(x^3\,c_3 + 2\right)^{-1/3} \tag{1.17}$$

Bernoulli's substitution

$$y' + \frac{1}{x}\,y = -2\,x^5\,y^4 \implies$$

using (1.16)

$$\implies z' + -3\frac{1}{x}\,z = (-3)\,(-2)\,x^5$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\,(-3\,\frac{1}{x})\right) = \exp\left(-3\ \mathrm{P}_x\,(\frac{1}{x})\right) = \exp\left(-3\,(c_0 + \ln x)\right) =$$

$$= c_1\,x^{-3} \tag{1.18}$$

Integrating

$$\mathrm{P}_x\left((-3) * (-2)\,x^5\,\varphi(x)\right) =$$

using (1.18)

$$= \mathrm{P}_x\left(6\,x^5\,c_1\,x^{-3}\right) = 6\,c_1(c_2 + x^3/3) \tag{1.19}$$

Questão 4

As series converge?

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2}+\sqrt[3]{n^5}}{\sqrt{n^3}+\sqrt{n^5}}; \tag{1.20}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}; \tag{1.21}$$

$$\sum_{n=1}^{\infty} \frac{1 * 3 * 5 * \cdots * (2\,n-1)}{n!} \tag{1.22}$$

Resposta Todas convergem

Finding convergence for (1.20)

$$\frac{\sqrt[3]{n^2}+\sqrt[3]{n^5}}{\sqrt{n^3}+\sqrt{n^5}} = \left(\frac{\sqrt[3]{n^2}}{\sqrt{n^3}+\sqrt{n^5}}\right) + \left(\frac{\sqrt[3]{n^5}}{\sqrt{n^3}+\sqrt{n^5}}\right) =$$
$$= \left(\frac{\sqrt{n^3}+\sqrt{n^5}}{\sqrt[3]{n^2}}\right)^{-1} + \left(\frac{\sqrt{n^3}+\sqrt{n^5}}{\sqrt[3]{n^5}}\right)^{-1} =$$
$$= \left(n^{\frac{3}{2}-\frac{2}{3}} + n^{\frac{5}{2}-\frac{2}{3}}\right)^{-1} + \left(n^{\frac{3}{2}-\frac{5}{3}} + n^{\frac{5}{2}-\frac{5}{3}}\right)^{-1} =$$
$$= \left(n^{\frac{5}{6}} + n^{\frac{11}{6}}\right)^{-1} + \left(n^{\frac{-1}{6}} + n^{\frac{5}{6}}\right)^{-1} =$$
$$= \frac{1}{n^{\frac{6}{6}}\,n^{-1/6} + n^{\frac{12}{6}}\,n^{-1/6}} + \frac{1}{n^{\frac{-1}{6}} + n^{\frac{6}{6}}\,n^{-1/6}} =$$
$$= \frac{n^{1/6-1}}{1+n} + \frac{n^{1/6}}{1+n} =$$
$$= \frac{n^{1/6}}{n+1}(n^{-1}+1)$$

Converge

Verificando convergencia de (1.21)

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

Converge

Verificando convergencia de (1.22)

$$\frac{1 * 3 * 5 * \cdots * (2\,n-1)}{n!} = \frac{\prod_{i=0}^{n} 2\,i+1}{\prod_{i=0}^{n} i} = \prod_{i=0}^{n} 2+1/i$$

converge

Questão 5

$$\begin{cases} \left(\mathrm{D}_y - 2\right) x + \left(\mathrm{D}_x^2 + 3\ \mathrm{D}_x\right) y = e^{2\,t} \\ \left(5\ \mathrm{D}_y^2 - 12\ \mathrm{D}_y + 4\right) x + \left(5\ \mathrm{D}_x^3 + 13\ \mathrm{D}_x^2 - 7\ \mathrm{D}_x - 3\right) y = 8\, e^{2\,t} \end{cases}$$

Resposta

$$\begin{cases} (\mathrm{D}_y - 2)x + (\mathrm{D}_x^2 + 3\,\mathrm{D}_x)y = e^{2\,t} \\ (\mathrm{D}_x + 3)y = 8\, e^{2\,t} \end{cases}$$

Questão 6 Laplace

$$\boldsymbol{f(t) = t\,e^{-t}; \qquad g(t) = \mathcal{H}(t-1)\,e^{-t}; \qquad h(t) = e^{-2\,t}\,\cos(2\,t)}$$

Resposta

Solving f

$$\mathcal{L}(f(t)) = \mathcal{L}\left(t\,e^{-t}\right) = \frac{1}{(s+1)^2}$$

Solving g

$$\mathcal{L}(g) = \mathcal{L}\left(\mathcal{H}(t-1)\,e^{-t}\right) = \frac{e^{-(s+1)}}{s+1}$$

Solving h

$$\mathcal{L}(h) = \mathcal{L}(e^{-2\,t}\,\cos(2\,t)) = \frac{s+2}{(s+2)^2+2^2}$$

Questão 7

Resposta

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2\,n-1}\sin((2\,n-1)\,\pi\,x)$$

Questão 8

$$\boldsymbol{\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x\,\partial t}; y = t; z = x + t}$$

Resposta

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y\,\partial z}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x\,\partial t} = \frac{\partial^2 u}{\partial z^2}\frac{\partial^2 z}{\partial x^2} - \left(\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)\frac{\partial}{\partial z}\frac{\partial z}{\partial t}$$

Grupo II

Questão 1

Det a sol geral da eq lin hom de coef const

$$\frac{d^4 y}{dx^4} - 4\,\frac{d^3 y}{dx^3} + 13\,\frac{d^2 y}{dx^2} = 0$$

Sabendo

$$\frac{d^4 y}{dx^4} - 4\,\frac{d^3 y}{dx^3} + 13\,\frac{d^2 y}{dx^2} = x^4 + x - 2$$

Admite sol part $\bar{y} = x^p\,Q_k(x)$ diga just. os val de p e k

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Resposta $p = 2, k = 4$

General solution for y

$$y = y_h + \bar{y} =$$

$$\text{using } (2.25)\ (2.27)$$

$$= c_0 + c_1\,x + e^{2\,x}\begin{pmatrix}+\cos(3\,x)\,c_2\\+\sin(3\,x)\,c_3\end{pmatrix} + \begin{pmatrix}x^2\,\rho_0\\+x^3\,\rho_1\\+x^4\,\rho_2\\+x^5\,\rho_3\\+x^6\,\rho_4\end{pmatrix} \tag{2.23}$$

Finding $\bar{y}$

$$\bar{y} = x^2\,Q_4(x) = x^2\,\sum_{i=0}^{4}\rho_i\,x^i = x^2\begin{pmatrix}x^0\,\rho_0\\+x^1\,\rho_1\\+x^2\,\rho_2\\+x^3\,\rho_3\\+x^4\,\rho_4\end{pmatrix} = \tag{2.24}$$

$$\text{using } (2.26)$$

$$= \begin{pmatrix}x^2\,\rho_0\\+x^3\,\rho_1\\+x^4\,\rho_2\\+x^5\,\rho_3\\+x^6\,\rho_4\end{pmatrix} \tag{2.25}$$

Finding constants of (2.24)

$$\bar{y}\,P = x^2\begin{pmatrix}x^0\,\rho_0\\+x^1\,\rho_1\\+x^2\,\rho_2\\+x^3\,\rho_3\\+x^4\,\rho_4\end{pmatrix}\left(D_x^4 - 4\,D_x^3 + 13\,D_x^2\right) =$$

$$= \begin{pmatrix}13*2*1\,\rho_0\\+-4*3*2*1\,\rho_1 + 13*3*2\,x^1\,\rho_1\\+4*3*2*1\,\rho_2 - 4*4*3*2\,x^1\,\rho_2 + 13*4*3\,x^2\,\rho_2\\+5*4*3*2\,x^1\,\rho_3 - 4*5*4*3\,x^2\,\rho_3 + 13*5*4\,x^3\,\rho_3\\+6*5*4*3\,x^2\,\rho_4 - 4*6*5*4\,x^3\,\rho_4 + 13*6*5\,x^4\,\rho_4\end{pmatrix} =$$

$$= \begin{pmatrix}+13*2*1\,\rho_0 - 4*3*2*1\,\rho_1 + 4*3*2*1\,\rho_2\\x(+13*3*2\,\rho_1 - 4*4*3*2\,\rho_2 + 5*4*3*2\,\rho_3)\\x^2(+13*4*3\,\rho_2 - 4*5*4*3\,\rho_3 + 6*5*4*3\,\rho_4)\\x^3(+13*5*4\,\rho_3 - 4*6*5*4\,\rho_4)\\+x^4\,13*6*5\,\rho_4\end{pmatrix} =$$

$$= x^4 + x - 2 \implies$$

$$\begin{cases}\rho_4 = 1/13*6*5)\\\rho_3 = \frac{4*6\,\rho_4}{13} = \frac{4*6/13*6*5}{13} = 4/13*13*5\\+13*4*3\,\rho_2 = \frac{1}{13}(1-(1/13)+(4*4/13*13))\ldots\end{cases} \tag{2.26}$$

Mapping roots of (2.28) to solution

$$\begin{cases}r_0 = r_1 = 0 \implies c_0 + c_1\,x;\\r_3 = 2 \pm i\,3 \implies e^{2\,x}\begin{pmatrix}+\cos(3\,x)\,c_2\\+\sin(3\,x)\,c_3\end{pmatrix}\end{cases} \tag{2.27}$$

Roots for characteristic equation for y_h

$$P = D_x^4 - 4\,D_x^3 + 13\,D_x^2 \implies$$

$$D_x^i \to r^i$$

$$\implies r^4 - 4\,r^3 + 13\,r^2 = r^2(r^2 - 4\,r + 13) = 0 \implies$$

$$\implies \begin{cases}r_0 = r_1 = 0\\p = 2\\r_3 = \frac{-(-4)\pm\sqrt{-4^2-4*1*13}}{2*1} = 2 \pm i\,3\end{cases} \tag{2.28}$$

Questão 2

$$(3\,y + 20\,x/y)\,\mathrm{d}x + (2\,x - 6\,y/x^2)\,\mathrm{d}y = 0$$
$$\phi(x,y) = x^2\,y$$
$$x = 1 \implies y = 2$$

Resposta (2.30)

Transformando em equação exata

$$\phi(x,y)(u(x,y)dx + v(x,y)dy) = x^2\,y\big((3\,y + 20\,x/y)\,\mathrm{d}x + (2\,x - 6\,y/x^2)\,\mathrm{d}y\big) =$$
$$= (3\,x^2\,y^2 + 20\,x^2)\,\mathrm{d}x + (2\,x^3\,y - 6\,y^2)\,\mathrm{d}y$$

Resposta (2.30)

Finding general solution $f(x)$

$$f(x) = \mathrm{P}_x\,u(x) + \mathrm{P}_y\,v(y) = \mathrm{P}_x\left(3\,x^2\,y^2 + 20\,x^2\right) + \mathrm{P}_y\left(2\,x^3\,y - 6\,y^2\right) =$$
$$= 3\,y^2\,(c_0 + x^3/3) + 20\,(c_1 + x^3/3) + 2\,x^3\,(c_2 + y^2/2) - 6\,(c_3 + y^3/3) = 0 \implies$$
$$\implies 3\,y^2\,c_0 + y^2\,x^3(4) + x^3(20/3 + 2\,c_2) - 2\,y^3 = 6\,c_3 - 20\,c_1 = \tag{2.29}$$

using (2.31)

$$3\,y^2\left(\frac{1}{12}(-20/3 + 6\,c_3 - 20\,c_1 - 2\,c_2)\right) + y^2\,x^3(4) + x^3(20/3 + 2\,c_2) - 2\,y^3 =$$
$$= 6\,c_3 - 20\,c_1 \tag{2.30}$$

finding constants in (2.29)

$$3\,(2)^2\,c_0 + (2)^2\,(1)^3(4) + (1)^3(20/3 + 2\,c_2) - 2\,(2)^3 = 6\,c_3 - 20\,c_1 \implies$$
$$\implies c_0 = \frac{1}{12}(-20/3 + 6\,c_3 - 20\,c_1 - 2\,c_2) \tag{2.31}$$

Grupo IV

Questão 1 laplace

$$\boldsymbol{y'' + 36\,y = \delta(t - \pi/6); y(0) = 1, y'(0) = 1}$$

Resposta

solving for y

$$y = \mathcal{L}^{-1}\,Y =$$

using (4.32)

$$= \mathcal{L}^{-1}\,(\frac{1}{s^2+6^2}\big(e^{\pi/6} + 1 + s\big)) = \frac{1}{6}\,\mathcal{L}^{-1}\,(\frac{6}{s^2+6^2}\big(e^{\pi/6} + 1 + s\big)) =$$
$$= \frac{1}{6}\sin w\,t\,\mathcal{L}^{-1}\,(\big(e^{\pi/6} + 1 + s\big)) = \ldots$$

Finding Y

$$\mathcal{L}(y'') + 36\ \mathcal{L}(y) = s^2\,Y - s\,y(0) - y'(0) + 36\,Y = s^2\,Y - s\,1 - 1 + 36\,Y =$$
$$= \mathcal{L}(\delta(t - \pi/6)) = e^{\pi/6} \implies = Y = \frac{1}{s^2+6^2}\big(e^{\pi/6} + 1 + s\big) \tag{4.32}$$