AM 2C – Exame 2023.2 Resolução

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Conteúdo

Questão 1

Plano tangente a sup

$$\boldsymbol{\ln(y^3) + (x^2+1)\,e^z = 1 - x^3}$$
$$\boldsymbol{(-1, 1, 0)}$$

Resposta

$$f(x,y,z) = \ln(y^3) + (x^2+1)\,e^z - (1 - x^3) =$$
$$= \ln(y^3) + (x^2+1)\,e^z - 1 + x^3 = 0;$$

$$\begin{pmatrix} \dfrac{\partial f}{\partial x}(p_0)\,(x - x_0) & + \\ +\dfrac{\partial f}{\partial y}(p_0)\,(y - y_0) & + \\ +\dfrac{\partial f}{\partial z}(p_0)\,(z - z_0) & \end{pmatrix} = \begin{pmatrix} (e^z\,2\,x + 3\,x^2)(p_0)\,(x+1) & + \\ +(y^{-3}\,3\,y^2)(p_0)\,(y-1) & + \\ +((x^2+1)\,e^z)(p_0)\,(z-0) & \end{pmatrix} =$$
$$= \begin{pmatrix} e^0\,2 * (-1) + 3\,(-1)^2\,(x+1) & + \\ +((1)^{-3}\,3\,(1)^2)\,(y-1) & + \\ +(((-1)^2+1)\,e^(0))(p_0)\,(z-0) & \end{pmatrix} =$$
$$= x + 1 + 3\,(y-1) + 2\,z = x + 3\,y + 2\,z - 2 = 0$$

Questão 2

$$\boldsymbol{\mathcal{D}} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{\mathbf{2}} : \begin{pmatrix} \boldsymbol{x^2 + y^2 \leq 4} \\ \boldsymbol{x^2 + y^2/4 \geq 1} \\ \boldsymbol{0 \leq y \leq x\sqrt{3}/3} \end{pmatrix} \right\}$$

Escreva em coordenadas polares

Resposta

$$\begin{pmatrix} x^2 + y^2 \leq 4 \\ x^2 + y^2/4 \geq 1 \\ 0 \leq y \leq x\sqrt{3}/3 \end{pmatrix} =$$

$$= \begin{pmatrix} (r\, cos\theta)^2 + (r\ \sin\theta)^2 = r^2 \leq 4 \implies |r| \leq 2 \\ (r\ \cos\theta)^2 + (r\ \sin\theta)^2/4 = r^2\ \cos^2\theta + r^2\ \sin^2\theta/4 \geq 1 \\ 0 \leq r\ \sin\theta \leq (r\ \cos\theta)\sqrt{3}/3 \implies 0 \leq \tan\theta \leq \sqrt{3}/3 \end{pmatrix} =$$

$$= \begin{pmatrix} |r| \leq 2 \\ r \geq \sqrt{\dfrac{2}{\cos^2\theta 3 + 1}} \\ 0 \leq \theta \leq \pi/6 \end{pmatrix}$$

Questão 3

$$\boldsymbol{f(x,y)=\ln(x^2+y^2),\mathbb{R}^2\backslash\{(0,0)\}}$$

Tem-se:

Resposta

$$\nabla f(1,1)=\left((x^2+y^2)^{-1}\,2\,x,(x^2+y^2)^{-1}\,2\,y\right)(1,1)=(1,1)$$

$$D_{\vec{u}}f(1,1)=\lim_{t\to 0}\frac{f(1+t/\sqrt{2},1+t/\sqrt{2})+f(1,1)}{t}=$$
$$=\lim_{t\to 0}\frac{\ln\left((1+t/\sqrt{2})^2+(1+t/\sqrt{2})^2\right)+\ln(1^2+1^2)}{t}=$$
$$=\lim_{t\to 0}2\frac{\ln 2+\ln\left(1+t/\sqrt{2}\right)}{t}$$

Questão 4

Seja a curva C

$$\begin{cases} \boldsymbol{C_1:} \quad \boldsymbol{z^2 = 7 - x^2 + 2\,x - 4\,y^2} \\ \boldsymbol{C_2:} \quad \boldsymbol{z = 2} \end{cases}$$

Parametrização regular

Resposta

$$\begin{cases} z^2 = 7 - x^2 + 2\,x - 4\,y^2 \\ z = 2 \end{cases}$$

$$\begin{cases} 2 = 7 - (r\,\cos\theta)^2 + 2\,(r\,\cos(\theta)) - 4\,(r\,\sin(\theta))^2 \\ z = 2 \end{cases}$$

$$-5 = -r^2\,3\,\sin^2\theta + 2\,r\,\cos\theta$$

$$\frac{((1 + 2\,\cos(t)) - 1)^2}{2^2} + \frac{(\sin(t))^2}{1^2} = \cos^2(t) + \sin^2(t) = 1$$

Questão 5

Considere a função $f : \mathbb{R}^2 \to \mathbb{R}$ definida por

$$\boldsymbol{f(x,y) = \begin{cases} \frac{y(x^2-y^2)}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}}$$

$$\boldsymbol{\vec{u} = (2/\sqrt{5}, 1/\sqrt{5})}$$

Tem-se:

Resposta

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0(h^2-0^2)}{\sqrt{h^2+0^2}} - 0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{h(0^2-h^2)}{\sqrt{0^2+h^2}} - 0}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{-h^3}{|h|}}{h} =$$

$$= \lim_{h \to 0} \frac{-h^1}{\operatorname{sgn} h} = 0$$

$$D_{\vec{u}}(0,0) = \lim_{t \to 0} \frac{f(0 + t\,2/\sqrt{5}, 0 + t/\sqrt{5}) - f(0,0)}{t} =$$

$$= \lim_{t \to 0} \frac{\frac{(t/\sqrt{5})((t\,2/\sqrt{5})^2-(t/\sqrt{5})^2)}{\sqrt{(t\,2/\sqrt{5})^2+(t/\sqrt{5})^2}} - 0}{t} =$$

$$= \lim_{t \to 0} \frac{3\,t/5\,\sqrt{5}}{\operatorname{sgn} t} = 0$$

Questão 6

Considere

$$\lim_{(x,y)\to(1,-1)} \frac{2(y+1)\ \cos(x-1)}{\sqrt{(x-1)^2+(y+1)^2}}$$

Resposta

$$\lim_{(x,y)\to(1,-1)} \frac{2(y+1)\ \cos(x-1)}{\sqrt{(x-1)^2+(y+1)^2}} = \lim_{x\to 1, y=-x} \frac{2(y+1)\ \cos(x-1)}{\sqrt{(x-1)^2+(y+1)^2}} =$$

$$= \lim_{x\to 1} \frac{2(-x+1)\ \cos(x-1)}{\sqrt{(x-1)^2+(-x+1)^2}} =$$

$$= \lim_{x\to 1} \frac{-2\ \cos(x-1)}{\sqrt{2}} = -\sqrt{2}$$

Questão 7

Designe por σ a sup em $\mathbb{R}^3$ def por:

$$\boldsymbol{\sigma} = \left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^{\mathbf{3}} : \begin{pmatrix} (\boldsymbol{x}, \boldsymbol{y}) \in [\mathbf{0}, \mathbf{1}] \times [\mathbf{0}, \mathbf{2}] \\ \boldsymbol{z} = \boldsymbol{x}^{\mathbf{2}} + \boldsymbol{y} \end{pmatrix} \right\}$$

Suponha σ orientada pela norma $\vec{n}$
O valor do fluxo do campo vetorial $\vec{F}(x, y, z) = -6\, x^2\, \hat{\jmath}$ através da superfície σ

Questão 8

Seja $f : \mathbb{R}^2 \mapsto \mathbb{R}$ uma função de classe C^1 em $\mathbb{R}^2$ tal que $\nabla f(2,1) = (-2,1)$.
Considere a função

$$\boldsymbol{g(x,y) = f\left(2\,x, \frac{2\,x}{y^2+1}\right)}$$

Tem-se

Resposta

$$\begin{cases} \phi(x,y) = 2\,x \\ \rho(x,y) = \frac{2\,x}{y^2+1} \end{cases}$$

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial \phi}(2,1)\,\frac{\partial \phi}{\partial x} = -2 * 2 = -4$$

Questão 9

Seja $\varphi : \mathbb{R}^3 \to \mathbb{R}$ uma função de classe C^1 em $\mathbb{R}$ e $c \in \mathbb{R}$. Considere a função $u(x,t) = x\,\varphi(x - c\,t)$. Para todo o $(x,t) \in \mathbb{R}^2$, Tem-se:

Resposta

$$c\,\frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t) = c\,\left(\varphi(x - c\,t) + x\,\varphi'(x - c\,t)\right)(x,t) + (x\,\varphi'(x - c\,t)($$

Questão 10

Seja $f : \mathbb{R}^3 \mapsto \mathbb{R}$ uma função da classe C^1 em $\mathbb{R}^3$ tal que $\nabla f(0,1,\ln 2) = (1,1,1)$.
Considere a função:

$$\boldsymbol{g : \mathbb{R}^2 \to \mathbb{R}^3}$$
$$\boldsymbol{g(x,y) = \big(\sin(x\,y), x - 2\,y, \ln(x^2+1)\big)}$$

Resposta

$$h = f \circ h = f \begin{bmatrix} \sin(1 * 0) \\ 1 - 2 * 0, \\ \ln(1^2+1) \end{bmatrix} = f(0,1,\ln 2)$$

$$J(h) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{bmatrix}_{(0,1)}$$

Questão 11

$$\boldsymbol{F(x, y, z) = x\ \cos(y\, z) - z\ \exp(x - y - z) + x^2 + 3\, y - 1}$$
$$\boldsymbol{P = (1, 0, 1)}$$

$F(x, y, z) = 0$ def x como func de y e z

Resposta

$$\frac{\partial x}{\partial y} = -\frac{\frac{\partial F}{\partial y}(1,0,1)}{\frac{\partial F}{\partial x}(1,0,1)} = -\frac{-(1)\ \sin(0*1)*1 - 1\ \exp(1-0-1)(-1) + 3}{\cos(0*1) - 1\ \exp(1-0-1) + 2\,1} =$$

Questão 12

Considere a função

$$\boldsymbol{(y-2)\,x^2 - y^2}$$

Tem-se:

Resposta

$$\det H_f = \begin{vmatrix} \frac{\partial^2 f}{\partial x\,\partial x} & \frac{\partial^2 f}{\partial x\,\partial y} \\ \frac{\partial^2 f}{\partial y\,\partial x} & \frac{\partial^2 f}{\partial y\,\partial y} \end{vmatrix} = \begin{vmatrix} 2\,y-4 & 2\,x \\ 2\,x & -2 \end{vmatrix} = -4\,y + 8 - 4\,x^2$$

$$\begin{cases} \det H_f(0,0) = 8, \frac{\partial^2 f}{\partial x^2} = -4 \therefore \text{Máximo local} \\ \det H_f(2,2) = \det H_f(-2,2) = -16 \quad \therefore \text{Sela} hessi \end{cases}$$

Questão 13

Seja D a região do plano definida pelas condições: $y \leq\leq 2-y^2, y \leq 0$.

O valor do integral

$$\iint_{D} \boldsymbol{y} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y}$$

Resposta

$$\begin{cases} y + y^2 - 2 = (y-1)(2+y) = 0 \\ y = 0 \implies x = 2 \\ x = 0 \implies |y| = \sqrt{2} \end{cases}$$

$$\iint_D y \, \mathrm{d}x \, \mathrm{d}y = \int_0^2 \int_y^{2-y^2} y \, \mathrm{d}x \, \mathrm{d}y = \int_0^2 ((2-y^2)-y)y \, \mathrm{d}y = \int_0^2 (2\,y - y$$

Questão 14

Considere o seguinte integral triplo

$$\boldsymbol{I = \int_0^1 \int_{x^2}^{3-2\,x} x\,y\ \mathrm{d}y\,\mathrm{d}x}$$

Inverta a integração

Resposta

$$\left\{ x^2 = 3 - 2\,x \implies x^2 - 3 + 2\,x = (x-3)(x-1) = 0 \right.$$

$$I = \int_0^1 \int_{x^2}^{3-2\,x} x\,y\ \mathrm{d}y\,\mathrm{d}x = \int_1^3 \int_0^{(3-y)/2} x\,y\ \mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^{\sqrt{x}} x\,y\ \mathrm{d}x\,\mathrm{d}y$$

Questão 15

Volume do sólido

$$\boldsymbol{S = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{pmatrix} 0 \le z \le 2 - x^2 - y^2 \\ \wedge 0 \le y \le -x \end{pmatrix} \right\}}$$

Resposta

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ x^2 + y^2 = r^2 \\ \tan\theta = y/x \\ |detJ| = r \end{cases}$$

$$\begin{cases} 0 \le z \le 2 - (r\ \cos\theta)^2 - (r\ \sin\theta)^2 = 2 - r^2 \\ 0 \le r\ \sin\theta \le -r\ \cos\theta \implies 0 \le \tan\theta \le -1 \implies \pi \ge \theta \ge 3\pi/4 \end{cases}$$

$$\int_0$$

Questão 16

Considere a função $f : \mathbb{R}^2 \to \mathbb{R}$

$$\boldsymbol{f(x,y) = \begin{cases} \frac{x^2\, e^x + y^2}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}}$$

Tem-se:

Resposta

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2\, e^h + 0^2}{h^2 + 0^2} - 1}{h} = \lim_{h\to 0} \frac{e^h - 1}{h}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^2\, e^0 + h^2}{0^2 + h^2} - 1}{h} = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0, y = m\, x} \frac{x^2\, e^x + y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2\, e^x + (m\, x)^2}{x^2 + (m\, x)^2} =$$

$$= \lim_{x\to 0} \frac{e^x + m^2}{(m^2 + 1)} = 1$$

Questão 17

Considere o sólido

$$\boldsymbol{\mathcal{E} = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} x^2 + y^2 + z^2 \leq 9 \\ z \geq \sqrt{x^2 + y^2} \\ y \geq 0 \end{pmatrix} \right\}}$$

Resposta

$$\begin{cases} x = r \sin\phi \, \cos\theta \\ y = r \sin\phi \, \sin\theta \\ z = \cos\phi \\ |etJ| = r^2 \sin\phi \end{cases}$$

$$\begin{cases} (r \, \sin\phi \, \cos\theta)^2 + (r \, \sin\phi \, \sin\theta)^2 + (r \, \cos\phi)^2 = r^2 \leq 9 \\ r \, \cos\phi \geq \sqrt{(r \, \sin\phi \, \cos\theta)^2 + (r \, \sin\phi \, \sin\theta)^2} = |r \, \sin\phi| \implies \\ \implies 1 \geq \tan\phi \\ r \, \sin\phi \, \sin\theta \geq 0 \implies \sin\phi \, \sin\theta \geq 0 \implies \theta \in [0, \pi] \end{cases}$$

Questão 18

Teorema de green

$$\boldsymbol{\Sigma = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, 0 \leq y \leq x\right\}}$$

Resposta

Questão 20

Integral seg de reta entre 2 pontos

$$\boldsymbol{p_0 = (0, 1), p_1 = (2, 0)}$$
$$\boldsymbol{I = \int_C y \; \mathrm{d}x 2 \, x \; \mathrm{d}y}$$

Resposta

$$\phi(t) = A + (B - A) = (0, 1) + ((2, 0) - (0, 1))\, t = (0, 1) + (2, -1)\, t = (2\, t,$$

$$I = \int_0^1 (1 - t)\, 2 \; \mathrm{d}t$$