AM3C – Teste 2024.1 Resolução

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Conteúdo

Grupo I

Questão 1

A equação diferencial linear de primeira ordem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\cos(x)}{\sin(x)}\,y = -x; \quad x \in \,]0, \pi[$$

Tem como solução geral

- ■ $y = \frac{c}{\sin x} - x\,\frac{\cos x}{\sin x} + 1$
- □ $y = \frac{c}{\sin x} + x\,\frac{\sin x}{\cos x} - 1$
- □ $y = \frac{c}{\sin x} - x\,\frac{\cos x}{\sin x}$
- □ $y = \frac{c}{\cos x} - x\,\frac{\cos x}{\sin x} + 1$
- □ $y = \frac{c}{\cos x} + x\,\frac{\cos x}{\sin x} - 1$
- □ $y = \frac{c}{\cos x} + x\,\frac{\sin x}{\cos x}$

Resposta (1.1)

General solution

$$y = \frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\,\mathrm{P}_x\left(-x\,\varphi(x)\right) =$$

$$\text{using (1.2) (1.3)}$$

$$= \frac{c_0}{c_1\,\sin x} + \frac{1}{c_1\,\sin x}\left(-c_1\left(x\,\cos x - (c_2 + \sin x)\right)\right) =$$

$$= \frac{c_0}{c_1\,\sin x} + \frac{c_2}{\sin x} - x\,\frac{\cos x}{\sin x} + 1 = \frac{c_4}{\sin x} - x\,\frac{\cos x}{\sin x} + 1 \tag{1.1}$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\left(\frac{\cos x}{\sin x}\,\mathrm{d}x\right)\right) =$$

$$\mathrm{d}(\sin x) = \cos x\,\mathrm{d}x$$

$$= \exp\left(\int\left(\frac{\mathrm{d}(\sin x)}{\sin x}\right)\right) = \exp\left(c_0 + \ln\left(\sin x\right)\right) = c_1\,\sin x \tag{1.2}$$

Integrating

$$\mathrm{P}_x\left(-x\,\varphi(x)\right) =$$

$$\text{using (1.2)}$$

$$= \mathrm{P}_x\left(-x\,c_1\,\sin x\right) =$$

$$\mathrm{P}_x\left(u\,v'\right) = u\,v - \mathrm{P}_x\left(u'\,v\right)\begin{cases} u = x \\ v = \cos x \end{cases}$$

$$= x\,\cos x - \mathrm{P}_x\left(\cos x\right) = -c_1\left(x\,\cos x - (c_2 + \sin x)\right) \tag{1.3}$$

Questão 2

A solução da equação de Bernoulli

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{1}{y}$$

que satisfaz a condição $y(0) = 2$, é:

☐ $y = \sqrt{e^{+2\,x} + 3}$	☐ $y = \sqrt{3\,e^{+2\,x} + 1}$	☐ $y = \sqrt{2\,e^{+2\,x} + 2}$
☐ $y = \sqrt{e^{-2\,x} + 3}$	■ $y = \sqrt{3\,e^{-2\,x} + 1}$	☐ $y = \sqrt{2\,e^{-2\,x} + 2}$

Resposta (1.6)

General solution

$$y = z^{1/2} = \tag{1.4}$$

$$= \left(\frac{c_0}{\varphi(x)} + \frac{1}{\varphi(x)}\ \mathrm{P}_x\left(2 * 1\,\varphi(x)\right)\right)^{1/2} =$$

using (1.8) (1.9)

$$= \left(\frac{c_0}{c_1\,e^{2\,x}} + \frac{1}{c_1\,e^{2\,x}}\,c_1\,e^{2\,x}\right)^{1/2} = \left(\frac{c_0}{c_1\,e^{2\,x}} + 1\right)^{1/2} = \left(c_2\,e^{-2\,x} + 1\right)^{1/2} = \tag{1.5}$$

using (1.7)

$$= \left(3\,e^{-2\,x} + 1\right)^{1/2} \tag{1.6}$$

Finding c_2

$$y(0) = 2 =$$

using (1.5)

$$= \left(c_2\,e^{-2*0} + 1\right)^{1/2} \implies c_2 = 4 - 1 = 3 \tag{1.7}$$

Bernoulli's substitution

$$y' + y = y^{-1} \implies$$

using (1.4)

$$\implies z' + 2\,z = 2$$

Finding $\varphi(x)$

$$\varphi(x) = \exp\left(\mathrm{P}_x\left(2\right)\right) = c_1\,e^{2\,x} \tag{1.8}$$

Integrating

$$\mathrm{P}_x\left(2 * 1\,\varphi(x)\right) =$$

using (1.8)

$$= 2\,c_1\,e^{2\,x}/2 = c_1\,e^{2\,x} \tag{1.9}$$

Questão 3

A equação differencial

$$\boldsymbol{(5\,x\,y^2 - 2\,y)\ \mathrm{d}x + (3\,x^2\,y - x)\ \mathrm{d}y = 0}$$

Admite um fator integrante na forma $\phi(x,y) = x^m\,y^n$, com $m,n \in \mathbb{N}$. Então:

☐ $m=3, n=2$	☐ $m=2, n=2$	☐ $m=1, n=3$
☐ $m=1, n=1$	☐ $m=2, n=1$	■ $m=3, n=1$

Resposta (1.10)

Finding m, n

$$\begin{gathered}
\frac{\partial u\,\varphi(x)}{\partial y} = \frac{\partial}{\partial y}\big(x^m\,y^n\,(5\,x\,y^2 - 2\,y)\big) = \frac{\partial}{\partial y}\big(5\,x^{m+1}\,y^{n+2} - 2\,y^{n+1}\,x^m\big) = \\
= 5\,x^{m+1}\,(n+2)\,y^{n+1} - 2\,(n+1)\,y^n\,x^m = \\
= \frac{\partial v\,\varphi(x)}{\partial x} = \frac{\partial}{\partial x}\big(x^m\,y^n\,(3\,x^2\,y - x)\big) = \frac{\partial}{\partial x}\big(3\,x^{2+m}\,y^{n+1} - x^{m+1}\,y^n\big) = \\
= 3\,(2+m)\,x^{1+m}\,y^{n+1} - (m+1)\,x^m\,y^n \implies \\
\implies \begin{cases} 2(n+1) = m+1 \implies m = 2\,n+1 = 3 \\ 5\,(n+2) = 3(m+2) \implies n = 1 \end{cases}
\end{gathered} \tag{1.10}$$

Questão 4

A equação diferencial linear homogénea

$$\boldsymbol{x\, y'' + x^2\, y' + 4\, y = 0, \quad x > 0}$$

Tem como solução geral a função $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$. Então a equação não homogénea

$$\boldsymbol{x\, y'' + x^2\, y' + 4\, y = x^3}$$

admite como solução geral a função $y(x) = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$, onde as funções $c_1(x), c_2(x)$ são determinadas a partir do sistema

- ■ $\begin{cases} c_1'(x)\, y_1 + c_2'(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x^2 \end{cases}$
- ☐ $\begin{cases} c_1(x)\, y_1 + c_2(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x^2 \end{cases}$
- ☐ $\begin{cases} c_1'(x)\, y_1 + c_2'(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x \end{cases}$
- ☐ $\begin{cases} c_1(x)\, y_1 + c_2(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = x \end{cases}$
- ☐ $\begin{cases} c_1'(x)\, y_1 + c_2'(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = 1 \end{cases}$
- ☐ $\begin{cases} c_1(x)\, y_1 + c_2(x) y_2 = 0 \\ c_1'(x)\, y_1' + c_2'(x) y_2' = 1 \end{cases}$

Resposta (1.11)

Crammers equation system

$$\left\{\begin{array}{lcl} c_1'(x)\ \mathrm{D}_x^0\, y_1(x) + c_2'(x)\ \mathrm{D}_x^0\, y_2(x) & = & 0 \\ c_1'(x)\ \mathrm{D}_x\, y_1(x) + c_2'(x)\ \mathrm{D}_x\, y_2(x) & = & \frac{x^3}{x} = x^2 \end{array}\right\} \tag{1.11}$$

Questão 5

Acerca de uma função $f(x)$ definida e com derivadas até à segunda ordem em $\mathbb{R}_0^+$ sabe-se que admite transformada de Laplace $F(s)$, que $f(0)=1, f'(0)=-2.$ Então a trasnformada de Laplace da função

$$\boldsymbol{e^{-t}\, f''(t) + t\, f'(t)}$$

é:

- ☐ $(s+1)^2\,F(s+1) - s + 2 + s\,F'(s)$
- ☐ $s^2\,F(s) - s + 1 + s\,F'(s) - F(s)$
- ☐ $(s+1)^2\,F(s+1) - s + 1 - s\,F'(s) - F(s)$
- ☐ $s^2\,F(s) - s + 1 + s\,F'(s+1) - F(s+1)$
- ☐ $(s+1)^2\,F(s+1) - s + 1 + s\,F'(s) + F(s)$
- ☐ $s^2\,F(s) - s + 1 + s\,F'(s+1) + F(s+1)$

Resposta

$$\boldsymbol{(s+1)^2\,F(s+1) - s + 1 - s\,F'(s) - F(s)}$$

Grupo II

Determine a solução geral da equação diferencial linear homogénea e de coeficientes constantes

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - y\,6 = 0$$

Resposta

Resposta (2.12)

General solution for y

$$\begin{aligned} y &= && \text{using (2.13)} \\ &= e^{+2\,x}\,c_0 + e^{-3\,x}\,c_1 && (2.12) \end{aligned}$$

Mapping roots of (2.14) to solution

$$\begin{cases} r_0 = +2 \implies e^{+2\,x}\,c_0; \\ r_1 = -3 \implies e^{-3\,x}\,c_1 \end{cases} \tag{2.13}$$

Roots for characteristic equation for y

$$\begin{aligned} & P = \mathrm{D}_x^2 + \mathrm{D}_x - 6 \implies && \mathrm{D}_x^i \to r^i \\ & \implies r^2 + r - 6 = 0 \implies r = \frac{-1 \pm \sqrt{1 - 4 * -6}}{2} = \frac{-1 \pm 5}{2} \implies \\ & \implies \begin{cases} r_0 = +2 \\ r_1 = -3 \end{cases} && (2.14) \end{aligned}$$

Utilizando o método da variação das constantes arbitrárias, determine a solução geral da equação não homogénea

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - y\,6 = -5\,e^{2\,x}\,\cos x$$

Resposta (2.15)

General solution

$$y =$$

using (2.12)

$$= c_1(x)\,e^{+2\,x} + c_2(x)\,e^{-3\,x} =$$

using (2.17) (2.18)

$$= (c_3 + \sin x)\,e^{+2\,x} + \left(\frac{e^{5\,x}}{26}\,(5\,\cos x + \sin x)\right)e^{-3\,x} =$$

$$= e^{2\,x}\left(\frac{5}{26}\,\cos x + \frac{27}{26}\sin x + c_3\right) \tag{2.15}$$

y_1, y_2 comes from (2.12) which is the homogeneous analogus equation for y

$$\begin{cases} y_1 = e^{+2\,x} \\ y_2 = e^{-3\,x} \end{cases} \tag{2.16}$$

Finding c_1, c_2

$$c_1(x) = \mathrm{P}_x\left(c_1'(x)\right) =$$

Using (2.19)

$$= \mathrm{P}_x\left(-\cos x\right) = c_3 + \sin x; \tag{2.17}$$

$$c_2(x) = \mathrm{P}_x\left(c_2'(x)\right) =$$

Using (2.20)

$$= \mathrm{P}_x\left(e^{5\,x}\,\cos x\right) =$$

$$\mathrm{P}_x\left(u\,v'\right) = u\,v - \mathrm{P}_x\left(u'\,v\right)\begin{cases} u = e^{5\,x} \\ v = \sin x \end{cases}$$

$$= e^{5\,x}\,\sin x - \mathrm{P}_x\left(5\,e^{x}\,\sin x\right) = e^{5\,x}\,\sin x + 5\,\mathrm{P}_x\left(e^{5\,x}\,(-\sin x)\right) =$$

$$\mathrm{P}_x\left(u\,v'\right) = u\,v - \mathrm{P}_x\left(u'\,v\right)\begin{cases} u = e^{5\,x} \\ v = \cos x \end{cases}$$

$$= e^{5\,x}\,\sin x + 5\left(e^{5\,x}\,\cos x - \mathrm{P}_x\left(5\,e^{5\,x}\,\cos x\right)\right) = e^{5\,x}\,\sin x + 5\,e^{5\,x}\,\cos x - 25\,\mathrm{P}_x\left(e^{5\,x}\,\cos\right.$$

$$\implies c_1 = \mathrm{P}_x\left(e^{5\,x}\,\cos x\right) = \frac{e^{5\,x}}{26}\,(5\,\cos x + \sin x) \tag{2.18}$$

solving $\mathrm{D}_x(c_1, c_2)$

Using (2.22)

$$c_1'(x) = \frac{1}{W(y_1, y_2)}\begin{vmatrix} 0 & \mathrm{D}_x^0 y_2 \\ \frac{-5\,e^{2\,x}\,\cos x}{1} & \mathrm{D}_x y_2 \end{vmatrix} =$$

using (2.21) (2.24)

$$= \frac{1}{-5\,e^{-x}}\begin{vmatrix} 0 & e^{-3\,x} \\ -5\,e^{2\,x}\,\cos x & -3\,e^{-3\,x} \end{vmatrix} = \frac{5\,e^{-x}\,\cos x}{-5\,e^{-x}} = -\cos x; \tag{2.19}$$

Using (2.22)

$$c_2'(x) = \frac{1}{W(y_1, y_2)}\begin{vmatrix} \mathrm{D}_x^0 y_1 & 0 \\ \mathrm{D}_x y_1 & \frac{-5\,e^{2\,x}\,\cos x}{1} \end{vmatrix} =$$

using (2.21) (2.23)

$$= \frac{1}{-5\,e^{-x}}\begin{vmatrix} e^{+2\,x} & 0 \\ 2\,e^{+2\,x} & -5\,e^{2\,x}\,\cos x \end{vmatrix} = \frac{-5\,e^{4\,x}\,\cos x}{-5\,e^{-x}} = e^{5\,x}\,\cos x \tag{2.20}$$

Solving Wronskiano

$$W(y_1, y_2) = \det\begin{bmatrix} y_1 & y_2 \\ \mathrm{D}_x\,y_1 & \mathrm{D}_x\,y_2 \end{bmatrix} =$$

using (2.23) (2.24)

$$= \det\begin{bmatrix} +e^{+2\,x} & +e^{-3\,x} \\ +2\,e^{+2\,x} & -3\,e^{-3\,x} \end{bmatrix} = -3\,e^{-x} - 2\,e^{-x} = -5\,e^{-x} \tag{2.21}$$

Crammers equation system

$$\left\{\begin{matrix} c_1'(x)\,\mathrm{D}_x^0\,y_1(x) + c_2'(x)\,\mathrm{D}_x^0\,y_2(x) & = & 0 \\ c_1'(x)\,\mathrm{D}_x\,y_1(x) + c_2'(x)\,\mathrm{D}_x\,y_2(x) & = & \frac{-5\,e^{2\,x}\,\cos x}{1} \end{matrix}\right\} \tag{2.22}$$

Solving $\mathrm{D}_x(y_1, y_2)$

$$\mathrm{D}_x\,y_1 = \mathrm{D}_x\left(e^{+2\,x}\right) = +2\,e^{+2\,x}; \tag{2.23}$$

$$\mathrm{D}_x\,y_2 = \mathrm{D}_x\left(e^{-3\,x}\right) = -3\,e^{-3\,x} \tag{2.24}$$

Grupo III

Q1 a.

Determine todas as soluções da equação de Clairaut

$$\boldsymbol{y = x\,\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3}$$

Resposta (3.25)

General solution

$$y = \sum_i y_i =$$

using (3.26) (3.27)

$$= \begin{cases} x\,c - c^3 & \text{General solution} \\ \pm 2\,(x/3)^{3/2} & \text{Singular solutions} \end{cases} \tag{3.25}$$

Finding y_i

$$y_0 =$$

using (3.29) (3.28) $p = c$

$$= x\,c - c^3; \tag{3.26}$$

$$y_1 =$$

using (3.29) (3.28) $p = \pm\sqrt{x/3}$

$$= x\,(\pm\sqrt{x/3}) - (\pm\sqrt{x/3})^3 = \pm x\,(x/3)^{1/2} \pm (-(x/3)^{3/2}) =$$

$$= \pm(x/3)^{1/2}(x - x/3) = \pm x\,(x/3)^{1/2}(3/3 - 1/3) = \pm 2\,(x/3)^{3/2} \tag{3.27}$$

Finding p

$$y' = \mathrm{D}_x y = p =$$

using (3.29)

$$= \mathrm{D}_x\big(x\,p - p^3\big) = p + x\;\mathrm{D}_x p - 3\,p^2\;\mathrm{D}_x p \implies (x - 3\,p^2)\;\mathrm{D}_x p = 0 \implies$$

$$\implies \begin{cases} \mathrm{D}_x p = 0 \implies p = c \\ p = \pm\sqrt{x/3} \end{cases} \tag{3.28}$$

Clairut's substitution

$$y = x\,\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3 =$$

$\mathrm{D}_x^i y = \mathrm{D}_x^{i-1} p$

$$= x\,p - p^3 \tag{3.29}$$

Q1 b.

Utilizando a mudança de variável definida por $x = 1/t$, resolva a equação

$$\boldsymbol{y = -x\,\frac{\mathrm{d}y}{\mathrm{d}x} + x^6 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3, \quad x > 0}$$

Sug: Após a mudança de variável utilize (Q1 a.).

Resposta (3.30)

General solution

$$
\begin{aligned}
y &= \\
&& \text{using } (3.31) \\
&= \frac{\mathrm{d}y}{\mathrm{d}t}\,t - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^3 = \\
&& \text{using } (3.25) \\
&= \begin{cases} t\,c - c^3 \\ \pm 2(t/3)^{3/2} \end{cases} && (3.30)
\end{aligned}
$$

Variable change $x = 1/t$

$$
\begin{aligned}
y &= -x\,\frac{\mathrm{d}y}{\mathrm{d}x} + x^6 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 = \\
&& x = 1/t \\
&= -(1/t)\,\frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}x} + (1/t)^6 \left(\frac{\mathrm{d}y}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}x}\right)^3 = \\
&& \tfrac{\mathrm{d}t}{\mathrm{d}x} = -1/x^2 = -1/(1/t)^2 = -t^2 \\
&= -(1/t)\,\frac{\mathrm{d}y}{\mathrm{d}t}\,(-t^2) + (1/t)^6 \left(\frac{\mathrm{d}y}{\mathrm{d}t}(-t^2)\right)^3 = \frac{\mathrm{d}y}{\mathrm{d}t}\,t - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^3 && (3.31)
\end{aligned}
$$

Grupo IV

Utilize a transformada de Laplace para resolver o problema de valores iniciais

$$\boldsymbol{y'' + y' + y\,5/2 = \delta(t-2), \quad y(0) = 0, \quad y'(0) = 1}$$

Sug: tenha em contra que $s^2 + s + 5/2 = (s + 1/2)^2 + 9/4$.

Grupo V

Considere a equação diferencial lienar de ordem n e coeficientes constantes

$$\left(\mathbf{D}_x^n + \sum_{k=0}^{n-1} \alpha_k\ \mathbf{D}_x^k\right) y = e^{\alpha\, x}, \alpha \in \mathbb{R} \tag{5.32}$$

Seja $P(r) = r^n + \sum_{k=0}^{n-1} a_k\, r^k$. admitimos que $r = \alpha$ é raiz da equação $P(r) = 0$ com grau de multiplicidade um. Justifique $P'(\alpha) = 0$.

Resposta

Solving $P'(\alpha) = 0$

$$P'(\alpha) =$$

Finding $P'(r)$

$$P'(r) =$$

using (??)

$$= \mathrm{D}_r((r-\alpha)) = 1$$

Finding $P(r)$

$$P = \mathrm{D}_x^n + \sum_{k=0}^{n-1} a_k\ \mathrm{D}_x^k \implies$$

$\mathrm{D}_x^n \to r^n$

$$\implies P(r) = r^n + \sum_{k=0}^{n-1} a_k\, r^k = r^n + a_0 + \sum_{k=1}^{n-1} a_k\, r^k = \tag{5.33}$$

Since α is the only root with multiplicity 1 we can write $P(r)$ as follows

$$= r - \alpha \implies \tag{5.34}$$

$$\implies \begin{cases} a_0 = -\alpha \\ n = 1 \end{cases} \tag{5.35}$$

Mapping $P(r)$ roots to general solution of y_h

$$r_0 = 0 \implies e^{0\,x}\, c_0 \tag{5.36}$$

Resposta

Finding $\bar{y}$

$p = 1$ from given roots of $P(r)$

$$\bar{y} = x^p\, e^{\alpha\, x}\, Q_0(x) = x^1\, e^{\alpha\, x} \sum_{i=0}^{0} \rho_i\, x^i = x^1\, e^{\alpha\, x}\, \rho_0 = \tag{5.37}$$

using (5.39)

$$= x^p\, e^{\alpha\, x}\, \rho_0 \tag{5.38}$$

Finding constants of (5.37)

$$\bar{y}\, P = x^1\, \rho_0 \left(\mathrm{D}_x^n + \sum_{k=0}^{n-1} a_k\ \mathrm{D}_x^k\right) = a_0\, x + a_1\, \rho_0 =$$

$$= 1 \implies$$

$$\begin{cases} \implies \rho_0 = \\ \implies \rho_1 = \end{cases} \tag{5.39}$$

A equação (5.32) tem uma solução particular da forma

$$\bar{y} = \frac{c}{2\,P'(\alpha)}\,x\,e^{\alpha\,x}, \quad c \in \mathbb{R}$$

Sabendo que

$$\mathrm{D}_x^k\left(x\,e^{\alpha\,x}\right) = k\,\alpha^{k-1}\,e^{\alpha\,x} + \alpha^k\,x\,e^{\alpha\,x}, \quad \forall\,k \in \mathbb{N}, \quad \mathrm{D}_x^k = \frac{\mathrm{d}^k}{\mathrm{d}x^k}$$

Determine, justificando detalhadamente, o valor de c.