AM1 - Anotações

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Conteúdo

Formulas de Recorrência

1. $\int \sin^n(a\,u)\,\mathrm{d}u = -\frac{\sin^{n-1}(a\,u)\,\cos(a\,u)}{a\,n} + (n-1)/n\,\int \sin^{n-2}(a\,u)\,\mathrm{d}u$
2. $\int \cos^n(a\,u)\,\mathrm{d}u = \quad \frac{\sin(a\,u)\,\cos^{n-1}(a\,u)}{a\,n} + (n-1)/n\,\int \cos^{n-2}(a\,u)\,\mathrm{d}u$
3. $\int \tan^n(a\,u)\,\mathrm{d}u = \quad \frac{\tan^{n-1}(a\,u)}{a(n-1)} - \int \tan^{n-2}(a\,u)\,\mathrm{d}u$
4. $\int \cot^n(a\,u)\,\mathrm{d}u = -\frac{\cot^{n-1}(a\,u)}{a(n-1)} - \int \cot^{n-2}(a\,u)\,\mathrm{d}u$
5. $\int \sec^n(a\,u)\,\mathrm{d}u = \quad \frac{\sec^{n-2}(a\,u)\tan(a\,u)}{a\,(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2}(a\,u)\,\mathrm{d}u$
6. $\int \csc^n(a\,u)\,\mathrm{d}u = -\frac{\csc^{n-2}(a\,u)\cot(a\,u)}{a\,(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2}(a\,u)\,\mathrm{d}u$

Identidades Trigonométricas

1. $\sin^2(x) + \cos^2(x) = 1$
2. $1 + \tan^2(x) = \sec^2(x)$
3. $1 + \cot^2(x) = \csc^2(x)$
4. $\sin^2(x) = \frac{1-\cos(2\,x)}{2}$
5. $\cos^2(x) = \frac{1+\cos(2\,x)}{2}$
6. $\sin(2x) = 2\sin(x)\,\cos(x)$
7. $2\,\sin(x)\,\cos(y) = \sin(x-y) + \sin(x+y)$
8. $2\,\sin(x)\,\sin(y) = \cos(x-y) - \cos(x+y)$
9. $\cos(x)\,\cos(y) = \cos(x-y) + \cos(x+1)$
10. $1 \pm \sin(x) = 1 \pm \cos(\pi/2 - x)$

Trigonometria Hiperbólica

1. $\sinh(x) = \frac{e^x - e^{-x}}{2}$
2. $\cosh(x) = \frac{e^x + e^{-x}}{2}$
3. $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth(x) = 1/\tanh(x)$
5. $\mathrm{sech}(x) = 1/\cosh(x)$
6. $\mathrm{csch}(x) = 1/\sinh(x)$

Tabela de Derivadas

1. $(u^n)' = n\, u^{n-1}\, u'$
2. $(u\, v)' = u'\, v + v'\, u$
3. $(u/v)' = \frac{u'\, v - v'\, u}{v^2}$
4. $(a^u)' = a^u\, \ln(a)\, u'$ . . . . $(a > 0, a \neq 1)$
5. $(e^u)' = e^u\, u'$
6. $\log_a'(u) = \frac{u'}{u}\, \log_a(e)$
7. $\ln'(u) = \frac{1}{u}\, u'$
8. $(u^v)' = v\, u^{v-1}\, u' + u^v\, \ln(u)\, v'$
9. $\sin'(u) = \quad u'\, \cos(u)$
10. $\cos'(u) = -u'\, \sin(u)$
11. $\tan'(u) = \quad u'\, \sec^2(u)$
12. $\cot'(u) = -u'\, \csc^2(u)$
13. $\sec'(u) = \quad u'\, \sec(u)\, \tan(u)$
14. $\csc'(u) = -u'\, \csc(u)\, \cot(u)$
15. $\sinh'(u) = \cosh(u)$
16. $\cosh'(u) = \sinh(u)$
17. $\tanh'(u) = 1 - \tanh^2(u)$
18. $\coth'(u) = 1 - \coth^2(u)$
19. $\operatorname{sech}'(u) = -\tanh(u)\, \operatorname{sech}(u)$
20. $\operatorname{csch}'(u) = -\coth(u)\, \operatorname{csch}(u)$
21. $\arcsin'(u) = \quad u'/\sqrt{1-u^2}$
22. $\arccos'(u) = -u'/\sqrt{1-u^2}$
23. $\arctan'(u) = \quad u'/1+u^2$
24. $\operatorname{arccot}'(u) = -u'/1+u^2$
25. $\operatorname{arcsec}'(u) = \quad u'/(|u|\, \sqrt{u^2-1})$ . . . $(|u| > 1)$
26. $\operatorname{arccsc}'(u) = -u'/(|u|\, \sqrt{u^2-1})$ . . . $(|u| > 1)$

Tabela de Integrais

1. $\int \mathrm{d}u = u + c$
2. $\int u^n \mathrm{d}u = \frac{u^{n+1}}{n+1} + c$ . . . . $(n \neq -1)$
3. $\int \frac{\mathrm{d}u}{u} = \ln|u| + c$
4. $\int a^u \mathrm{d}u = \frac{a^u}{\ln a} + c, a > 0$ . . . . $(a \neq 1)$
5. $\int e^u \mathrm{d}u = e^u + c$
6. $\int \sin(u)\,\mathrm{d}u = -\cos u + c$
7. $\int \cos(u)\,\mathrm{d}u = \quad \sin u + c$
8. $\int \tan(u)\,\mathrm{d}u = \ln|\sec(u)| + c$
9. $\int \cot(u)\,\mathrm{d}u = \ln|\sin(u)| + c$
10. $\int \sec(u)\,\mathrm{d}u = \ln|\sec(u) + \tan(u)| + c$
11. $\int \csc(u)\,\mathrm{d}u = \ln|\csc(u) - \cot(u)| + c$
12. $\int \sec(u)\,\tan(u)\,\mathrm{d}u = \quad \sec(u) + c$
13. $\int \csc(u)\,\cot(u)\,\mathrm{d}u = -\csc(u) + c$
14. $\int \sec^2(u)\,\mathrm{d}u = \quad \tan(u) + c$
15. $\int \csc^2(u)\,\mathrm{d}u = -\cot(u) + c$
16. $\int \mathrm{d}u/(u^2 + a^2) = \arctan(u/a)/a + c$
17. $\int \mathrm{d}u/(u^2 - a^2) = \ln\left|\frac{u-a}{u+a}\right|/2\,a + c \quad (u^2 > a^2)$
18. $\int \mathrm{d}u/\sqrt{u^2 + a^2} = \ln|u + \sqrt{u^2 + a^2}| + c$
19. $\int \mathrm{d}u/\sqrt{u^2 - a^2} = \arcsin(u/a) + c \quad (u^2 < a^2)$
20. $\int \mathrm{d}u/\sqrt{a^2 - u^2} = \arcsin(u/a) + c \quad (u^2 < a^2)$
21. $\int \mathrm{d}u/\left(u\sqrt{a^2 - u^2}\right) = \operatorname{arcsec}|u/a|/a + c$

I | 15/03 - Conceitos Básicos I

1 Majorante

$m \in \text{Majorante}(X) \iff$
$\iff m \in \mathbb{R} \wedge x \leq m \; \forall x \in X$

2 Minorante

$m \in \text{Minorante}(X) \iff$
$\iff m \in \mathbb{R} \wedge x \geq m \; \forall x \in X$

3 Infimo

(i) Standalone

$\text{Inf}(X) = i \iff$
$\iff i \in \mathbb{R} \wedge x \geq m \; \forall x \in X \wedge$
$\wedge \nexists y \in \mathbb{R} \backslash X : i < y < x \; \forall x \in X$

(ii) Usando Vizinhança

$\text{Inf}(X) = i \iff$
$\iff i \in \mathbb{R} \wedge x \geq i \; \forall x \in X \wedge V_\delta(i) \cap X \neq \emptyset$

4 Supremo

(i) Standalone

$\text{Sup}(X) = s \iff$
$\iff s \in \mathbb{R} \wedge x \leq s \; \forall x \in X \wedge$
$\wedge \nexists y \in \mathbb{R} \backslash X : x < y < s$

(ii) Usando Vizinhança

$\text{Sup}(X) = s \iff$
$\iff s \in \mathbb{R} \wedge x \leq s \; \forall x \in X \wedge V_\delta(s) \cap X \neq \emptyset$

5 Minimo

(i) Standalone

$\text{Min}(X) = m \iff$
$\iff m \in X \wedge m \leq y \; \forall y \in m$

(ii) Usando Minorante

$\text{Min}(X) = m \iff$
$\iff m \in X \wedge m \in \text{Minorante}(X)$

6 Maximo

(i) Standalone

$\text{Max}(X) = m \iff$
$\iff m \in X \wedge m \geq y \; \forall y \in m$

(ii) Usando Majorante

$= \text{Max}(X) = m \iff$
$\iff m \in X \wedge m \in \text{Majorante}(X)$

7 Conjunto Limitado

X é um conjunto limitado $\iff$
$\iff \{m_1 \leq x \leq m_2 \quad \forall x \in X : m_1 \in \text{Minorante}(X),\; m_2 \in \text{Majorante}(X)\}$

8 Vizinhanca

$V_\delta(x) = (x - \delta, x + \delta) \quad \delta \in \mathbb{R} \backslash \{0\}$

9 Interior

(i) Standalone

$$x \in \text{Int}(X) \iff$$
$$\iff \exists\, \delta > 0 : (x - \delta, x + \delta) \subseteq X$$

(ii) Usando Vizinhança

$$x \in \text{Int}(X) \iff$$
$$\iff V_\delta(x) \subseteq X$$

10 Exterior

(i) Standalone

$$x \in \text{Ext}(X) \iff$$
$$\iff \exists\, \delta \in \mathbb{R}\backslash\{0\} : (x - \delta, x + \delta) \cap X = \emptyset$$

(ii) Usando Vizinhança

$$x \in \text{Ext}(X) \iff$$
$$\iff V_\delta(x) \cap X = \emptyset$$

11 Fronteira

$$f \in \text{Fr}(X) \iff$$
$$\iff V_\delta(f) \cap X \neq \emptyset \wedge V_\delta(f) \cap \mathbb{R}\backslash X \neq \emptyset$$

II | 16/03 - Conceitos Básicos II

1 Conjunto Aberto

X é um conjunto aberto $\iff$
$\iff X = \text{Int}(X)$

2 Conjunto Fechado

X é um conjunto fechado $\iff$
$\iff X = \text{Int}(X) \cup \text{Fr}(X)$

3 Feixe

$\bar{X} = \text{Int}(X) \cup \text{Fr}(X)$

4 Ponto de Acumulação

x é um ponto de acumulação de X $\iff$
$\iff V_\delta(x) \cap (X \backslash \{x\}) \neq \emptyset$

III | 19/03 - Indução por igualdade

1. Prove que algum numero pertence ao conjunto
2. Prove que o proximo pertence ao conjunto

Seja $V = \left\{P_{(n)} \quad \forall\, n \in \text{Dominio}\right\}$
$P_{(x)} \in V;\ P_{(x+1)} \in V$

1 Exemplo

$$\boldsymbol{\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n} \quad \forall\, n \in \mathbb{N}_0}$$

(i) $n = 0$

$$\sum_{i=0}^{0} \frac{1}{2^i} = 1 = 2 - \frac{1}{2^0} = 2 - 1 = 1$$

(ii) $n = m + 1$

$$\sum_{i=0}^{m+1} \frac{1}{2^i} = \sum_{i=0}^{m} \frac{1}{2^i} + \frac{1}{2^{m+1}} =$$
$$= 2 - \frac{1}{2^m} + \frac{1}{2^{m+1}} = 2 - \frac{1}{2^{m+1}}$$

IV | 22/03 - Indução por Desigualdade

1 Exemplo 1

$$\boldsymbol{n \leq 2^{n-1} \quad \forall\, n \in \mathbb{N}}$$

(i) $n = 1$

$1 \leq 2^{1-1} = 1$

(ii) $n = m + 1$

$$m + 1 \leq 2^{m-1} + 1 = \frac{2^m + 2}{2} \leq 2^{m+1-1} = 2^m \implies$$
$$\implies 2 \leq 2^{m+1} - 2^m = 2^m(2-1) = 2^m$$

2 Exemplo 2

$$\boldsymbol{n^2 \leq 2^n \quad \forall\, n \in \mathbb{N} \wedge n \geq 4}$$

(i) $n = 4$

$4^2 \leq 2^4 = 4^2$

(ii) $n = m + 1$

$$(m+1)^2 = m^2 + 2\,m + 1 \leq$$
$$\leq 2^m + 2\,m + 1 \leq 2 * 2^m = 2^{m+1} \implies$$
$$\implies 2\,m + 1 \leq 2^m;$$
$$\begin{cases} m = 4 \implies 2 * 4 + 1 = 9 \leq 2^4 = 16 \\ m = n + 1 \implies 2 * (n+1) + 1 = 2\,n + 1 + 2 \leq \\ \leq 2^n + 2 \leq 2 * 2^n = 2^{n+1} \implies 2 \leq 2^n \end{cases}$$

V | 23/03 - Sucessões

$$\boldsymbol{u_{(n)}} : \mathbb{N} \to \textbf{Imagem}$$

1 Sucessão monótona

1.1 Decrescente

$u_{(n)} \geq u_{(n+1)} \quad \forall n \in \mathbb{N}$

1.2 Crescente

$u_{(n)} \leq u_{(n+1)} \quad \forall n \in \mathbb{N}$

2 Sucessão limitada

$u_{(n)}$ é limitada $\iff$
$\iff m_1 \leq u_{(n)} \leq m_2 \quad \forall n \in \mathbb{N}$

3 Sucessão Convergente

$u_{(n)}$ converge para $l \in \mathbb{R} \iff$
$\iff \exists p \in \mathbb{N} : u_{(n)} \in V_\delta(l) \quad \forall n > p$

Nota:

$u_{(n)} \in V_\delta(l) \iff |u_{(n)} - l| < \delta \iff$
$\iff l - \delta < u_{(n)} < l + \delta$

3.1 Exemplo

$$\boldsymbol{u_{(n)} = 1/\sqrt{2\,n - 1};}$$
$$\boldsymbol{u_{(n)} > 0;\ \ \delta = 1/10}$$

$\iff 0 < \dfrac{1}{\sqrt{2\,n-1}} < \dfrac{1}{10} \iff$
$\iff 0 < \dfrac{1}{2\,n-1} < \dfrac{1}{100};\ 2\,n - 1 > 100 \implies$
$\implies \lfloor 101/2 \rfloor < n$

VI | 26/03 - Demonstrações para sucessões

1 Sucessão convergente $\implies$ Limitada

$u_{(n)}$ é uma sucessão convergente $\iff u_{(n)} \in V_\varepsilon(l) \quad \forall \varepsilon > 0 \iff l - \varepsilon < u_{(n)} < l + \varepsilon \iff$
$\iff \exists \{m_1, m_2\} \subset \mathbb{R} : m_1 < u_{(n)} < m_2 \quad \forall n \in \mathbb{N} \iff u_{(n)}$ é uma sucessão limitada

2 Sucessão é monótona e limitada $\implies$ convergente

$u_{(n)}$ é uma sucessão monotona e limitada $\iff$
$\iff (u_{(n)} < u_{(n+1)} \vee u_{(n)} > u_{(n+1)}) \wedge \exists \{m_1, m_2\} \subset \mathbb{R} : m_1 < u_{(n)} < m_2 \quad \forall n \in \mathbb{N} \implies$
$\implies \exists \, \mathrm{Sup}(u_{(n)}) \in \mathbb{R} \backslash u_{(n)} \iff u_{(n)}$ é uma sucessão convergente

VII | 29/03 - Lemas de Desigualdades

1 Lema $\lim_{n\to\infty} u_{(n)} \leq \lim_{n\to\infty} v_{(n)}$

$u_{(n)}$ e $v_{(n)}$ são convergentes $\wedge$
$\wedge\, u_{(n)} \leq v_{(n)} \quad \forall\, n \in \mathbb{N} \implies$
$\implies \lim_{n\to\infty} u_{(n)} \leq \lim_{n\to\infty} v_{(n)}$

2 Lema das sucessões enquadradas

$\{u_{(n)}, v_{(n)}, w_{(n)}\} : \mathbb{N} \to \mathbb{R} \wedge$
$\wedge\, v_{(n)} \leq u_{(n)} \leq w_{(n)} \quad \forall\, n \in \mathbb{N} \wedge$
$\wedge\, \exists\, l \in \mathbb{R} : \{v_{(n)}, w_{(n)}\} \xrightarrow{\text{converge}} l \implies$
$\implies u_{(n)} \xrightarrow{\text{converge}} l$

2.1 Exemplo: $u_{(n)} = \sin(n)/n$

$$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n};\ \left\{\frac{-1}{n}, \frac{1}{n}\right\} \to 0 \implies$$
$$\implies \frac{\sin(n)}{n} \to 0$$

3 Lema $u_{(n)} * v_{(n)} \to 0$

$u_{(n)}$ é limitada $\wedge\, v_{(n)} \to 0 \implies$
$\implies 0 \leq |u_{(n)} * v_{(n)}| = |u_{(n)}| * |v_{(n)}| \leq$
$\leq \text{Majorante}(u_n) * 0 = 0 \implies$
$\implies u_{(n)} * v_{(n)} \to 0$

3.1 Exemplo: $S_{(n)} = \sum_{k=1}^{n} \frac{\sqrt{k}}{n^2}$

$$0 \leq S_{(n)} = \sum_{k=1}^{n} \frac{\sqrt{k}}{n^2} \leq \frac{n\sqrt{n}}{n^2} = \frac{1}{\sqrt{n}} \to 0 \implies$$
$$\implies S_{(n)} \to 0$$

3.2 Exercicio: $\frac{2^n - n}{3^n + 1}$

$$0 \leq \frac{2^n - n}{3^n + 1} \leq \frac{2^n}{3^n} = (2/3)^n \to 0 \implies$$
$$\implies \frac{2^n - n}{3^n + 1} \to 0$$

4 Lema

$u_{(n)} \to l_1 \wedge v_{(n)} \to l_2 \implies$
$\implies u_{(n)} + v_{(n)} \to l_1 + l_2 \implies$
$\implies |u_{(n)} + v_{(n)} - (l_1 + l_2)| =$
$= |u_{(n)} - l_1 + v_{(n)} - l_2| \leq |u_{(n)} - l_1| + |v_{(n)} - l_2| < \varepsilon$

VIII | 30/03

1

$$u_{(n)} \to l_1 \wedge v_{(n)} \to l_2 \implies$$
$$\implies 0 \le |u_{(n)}\, v_{(n)} - l_1\, l_2| =$$
$$= |(u_{(n)} - l_1)\, v_{(n)} + v_{(n)}\, l_1 - l_1\, l_2| =$$
$$= |(u_{(n)} - l_1)\, v_{(n)} + l_1\, (v_{(n)} - l_2)| \le$$
$$\le |(u_{(n)} - l_1)|\, |v_{(n)}| + |l_1|\, |(v_{(n)} - l_2)| \to 0 \implies$$
$$\implies u_{(n)} * v_{(n)} \to l_1 * l_2$$

2 Sucessão que tende para o infinito

$$u_{(n)} \to \infty \iff$$
$$\iff u_{(n)} > m \quad \forall\, m \in \mathbb{R} \wedge \forall\, \{n, p\} \subset \mathbb{N} : n > p$$

3 Indeterminações

(i) $u_{(n)}\, v_{(n)} : u_{(n)} \to 0 \wedge v_{(n)} \to \infty$

(ii) $u_{(n)}/v_{(n)} : u_{(n)} \to \infty \wedge v_{(n)} \to \infty$

(iii) $u_{(n)} + v_{(n)} : u_{(n)} \to \infty \wedge v_{(n)} \to -\infty$

(iv) $(1 + 1/n)^n : n \to \infty$

4 Exercicios

(i) $u_{(n)} = \sqrt{n^2 + 2\, n} - n$

$$= \frac{n^2 + 2\, n - n^2}{\sqrt{n^2 + 2\, n} + n} = \frac{2}{\sqrt{1 + 2/n} + 1} \to$$
$$\to \frac{2}{\sqrt{1} + 1} = 1$$

(ii) $a_{(n)} = \frac{n + \cos(n)}{n\, \ln(n+1)}$

$$\lim_{n\to\infty} a_{(n)} = \lim_{n\to\infty} \frac{1}{\ln(n+1)} + \lim_{n\to\infty} \frac{\cos(n)}{n} *$$
$$* \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0 + 0 * 0 = 0$$

(iii) $b_{(n)} = \left(\frac{n^2 - 2\, n + 1}{n^2 + 2\, n + 1}\right)^n$

$$\lim_{n\to\infty} b_{(n)} = \lim_{n\to\infty} \left(\frac{(n-1)^2}{(n+1)^2}\right)^n = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{2\, n} =$$
$$= \left(\frac{\lim_{n\to\infty}(1 - 1/n)^n}{\lim_{n\to\infty}(1 + 1/n)^n}\right)^2 = \left(\frac{e^{-1}}{e^1}\right)^2 = e^{-4}$$

IX | 06/04 - Subsucessões

$$\boldsymbol{u \circ i_{(n)} = u_{(i_{(n)})}}$$

$$\boldsymbol{u : \mathbb{N} \to \mathbb{R} \wedge i : \mathbb{N} \to \mathbb{N} \wedge}$$
$$\boldsymbol{\wedge i_{(n)} < i_{(n+1)} \quad \forall\, n \in \mathbb{N}}$$

1 Exemplo: $u_{(n)} = (-1)^n$

(i) $i_{(n)} = 2\,n$

$u \circ i_{(n)} = (-1)^{2\,n} = 1$

(ii) $j_{(n)} = 2\,n - 1$

$u \circ j_{(n)} = (-1)^{2\,n-1} = -1$

2 Exemplo: $u_{(n)} = \sin(n\,\pi/4)$

(i) $i_{(n)} = 4\,n$

$u \circ i_{(n)} = \sin(4\,n\,\pi/4) = \sin(n\,\pi) = 0$

3 Subsusecção convergente

$u_{(n)}$ é uma sucessão convergente $\implies$
$\implies u_{(n)} \in V_\varepsilon(l) \quad \forall\,\{n,p\} \subset \mathbb{N} : n > p;$
$i_{(p)} \geq p \implies u \circ i_{(n)} \in V_\varepsilon(l)$
$\forall\,\{n,p\} \subset \mathbb{N} : n > p \implies$
$\implies u \circ i_{(n)}$ é uma sucessão convergente

4 Lema

$u \circ i_{(n)}$ é uma sucessão monotona $\implies$
$\implies u \circ i_{(n)} < u \circ i_{(m)} \quad \forall\,\{m,n\} \in \mathbb{N} : m > n$

5

“Qualquer Sucessão possui pelo menos uma subsucessão monótona”

6

“Qualquer sucessão limitada possui pelo menos uma subsucessão monótona convergente”

$u_{(n)}$ é limitada $\implies$
$\implies u \circ i_{(n)}$ é uma sucessão convergente

X | 09/04 - Limites

$$\boldsymbol{\limsup u_{(n)} = \text{Sup}\{l : l \text{ é sublimite de } u\}}$$
$$\boldsymbol{\liminf u_{(n)} = \text{Inf}\{l : l \text{ é sublimite de } u\}}$$

$u_{(n)} \to l \iff \limsup u_{(n)} = \liminf u_{(n)} = l$

1 Lema geral das funções enquadradas

$$\{u_{(n)}, v_{(n)}, w_{(n)}\} : \mathbb{N} \to \mathbb{R} : w_{(n)} \leq u_{(n)} \leq v_{(n)} \implies$$
$$\implies \liminf w_{(n)} \leq \liminf u_{(n)} \leq \limsup u_{(n)} \leq \limsup v_{(n)}$$

1.1 Exemplo

$$u_{(n)} = \left(1 + \frac{1 + (-1)^n}{n}\right)^n$$

$1^n = 1 \leq u_{(n)} \leq \left(1 + \dfrac{2}{n}\right)^n \to e^2$

2 Lema

"Se $u_{(n)} \to l$ então a sucessão $s_{(n)} = \frac{\sum_{i=0}^{n} u_{(n)}}{n} \to l$"

XI | 12/04 - Sucessão de Cauchy

$$\boldsymbol{u_{(n)} : |u_{(m)} - u_{(n)}| < \varepsilon \quad \forall \{m, n, p\} \in \mathbb{N} : m > p \wedge n > p}$$

1 Criterio suficiente para uma sucessão ser Cauchy

$|u_{(n+2)} - u_{(n+1)}| \leq \alpha |u_{(n+1)} - u_{(n)}|$
$\forall\, n \in \mathbb{N} : n > p \wedge \alpha \in (0, 1) \implies$
$\implies u_{(n)}$ é Cauchy

1.1 Exemplo

$$u_{(n)} = \begin{cases} 0 & n = 1 \\ (2/3)\, u_{(n-1)} + 1 & n > 1 \end{cases}$$

$|u_{(n+2)} - u_{(n+1)}| =$
$= |(2/3)\, u_{(n+1)} + 1 - (2/3)\, u_{(n)} - 1| =$
$= (2/3)\, |u_{n+1} - u_{(n)}|$
$\therefore u_{(n)}$ é Cauchy

2 Convergencia

$$u_n \begin{cases} 0 & n = 1 \\ u_{(n-1)}\, 2/3 + 1 & n > 1 \end{cases}$$

$u_n \to l;\ u_n\, 2/3 + 1 \to l\, 2/3 + 1 \quad \therefore l = l\, 2/3 + 1 \implies$
$\implies l = 3$

XII | 13/04 - Sucessões de Cauchy II

1

$$\boldsymbol{u_n = 1/2^n}$$

$$|u_m - u_n| = \left| u_m + \sum_{k=n+1}^{m-1} (-u_k + u_k) - u_n \right| =$$
$$= \left| \sum_{k=n}^{m-1} u_{k+1} - u_k \right| \leq \sum_{k=n}^{m-1} |u_{k+1} - u_k|$$

2

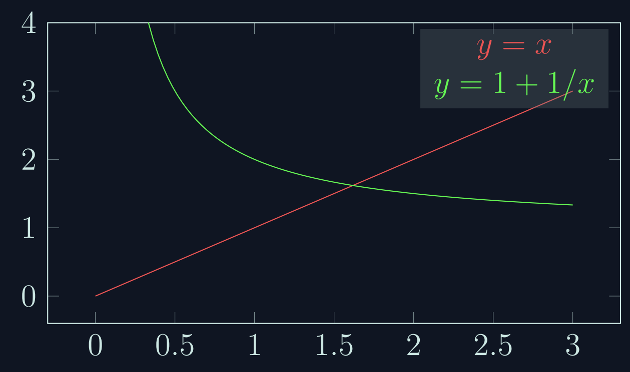
$$\boldsymbol{u_n = \begin{cases} 1 & n = 1 \\ 1 + 1/u_{n-1} & n > 1 \end{cases}}$$

$$1 \leq u_n \leq 2 \quad \forall n \in \mathbb{N} \implies 1 \leq u_{n+1} \leq 2 \quad \forall n \in \mathbb{N} \implies$$
$$\implies 2 \geq 1 + 1/u_n \geq 3/2 \geq 1$$

2.1 Limite

$$u_{n+1} = 1 + 1/u_n \implies l = 1 + 1/l \implies$$
$$\implies l^2 - l - 1 = 0 \implies l = \frac{1+\sqrt{5}}{2}$$

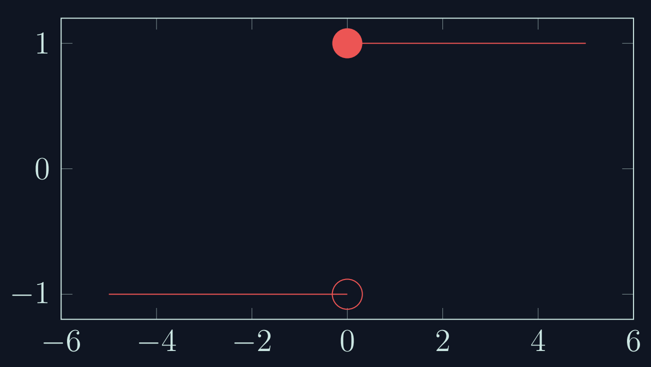
2.2 Visualização gráfica do limite



XIII | 16/04

1

$$H(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$



2 Definição derivada

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(i) f não é obrigada a estar definida em $x = a$

(ii) (x_n) é uma sucessão que aproxima de a por valores diferentes de a

3 Existencia de limite

$$\exists \lim_{x \to a} f(x) \quad \iff$$

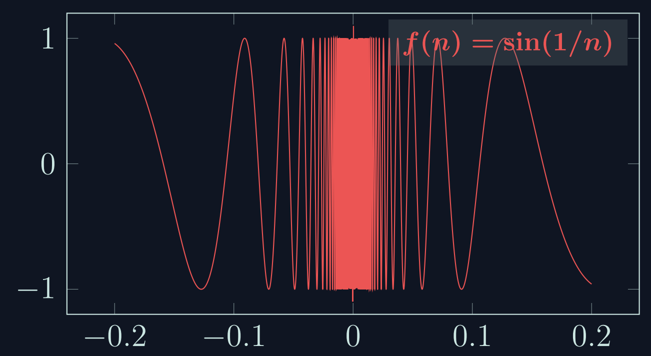
$$\iff \begin{cases} \exists \lim_{x \to a^-} f(x) \\ \exists \lim_{x \to a^+} f(x) \\ \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \end{cases}$$

4 Exemplo

$$g(a) = \begin{cases} x^2 & x < 0 \\ x & x \geq 0 \end{cases}$$

$$\exists \lim_{x \to 0} g(x) \iff \lim_{x \to 0^-} g(x) = 0; \ \lim_{x \to 0^+} g(x) = 0$$

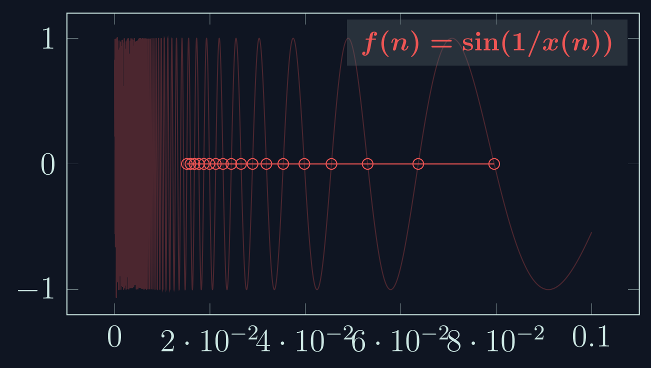
5 Exemplo



5.1 Usando sucessões para encontrar limites

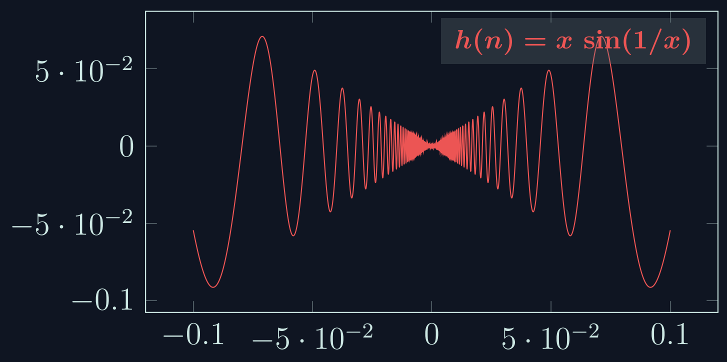
$$x(n) = 1/(n\,\pi) \quad n \in \mathbb{N} \backslash \{0\}$$

$$f(x(n)) = \sin\left(\frac{1}{1/(n\,\pi)}\right) = \sin(n\,\pi) \quad n \in \mathbb{N} \backslash \{0\}$$



6

$h(x) = x\ \sin(1/x) \quad x \in \mathbb{R}\backslash\{0\}$



$$h(x) = x\ \sin(1/x) = \frac{\sin(1/x)}{1/x}$$

XIV | 19/04 - Limites

$$\boldsymbol{f : \mathrm{D} \subseteq \mathbb{R} \to \mathbb{R} \qquad f \text{ está definida em } V_\delta(a)\backslash\{a\}}$$

$$\lim_{x\to a} f(x) = L \in \mathbb{R} \iff \lim f(x(n)) = L \quad \forall\, x(n) : x(n) \to a$$

1 Limite notável: $\sin(x)/x$

graph missing

area triangulo menor $= x/2 <$

$< $ area triangulo maior $= \tan(x)/2 \implies$

$\implies x < \dfrac{\sin(x)}{\cos(x)} \implies \cos(x) < \dfrac{\sin(x)}{x} < 1 \implies$

$\implies \lim\limits_{x\to 0} \cos(x) = 1 < \lim\limits_{x\to 0} \dfrac{\sin(x)}{x} < \lim\limits_{x\to 0} 1 = 1$

$\therefore \lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$

1.1 Exemplo

$$\lim_{x\to 0} \sin(x^2)/x^2$$

$= 1$

1.2 Exemplo

$$\lim_{x\to 0} \sin(x^2)/x$$

$$= \lim_{x\to 0} x \left(\frac{\sin(x^2)}{x^2}\right) = 0$$

1.3 Exemplo

$$\lim_{x\to 0} \sin(2\,x)/x$$

$$= \lim_{x\to 0} 2 \left(\frac{\sin(2\,x)}{2\,x}\right) = 2$$

1.4 Exemplo

$$\lim_{x\to 0} \frac{1-\cos(x)}{x}$$

$$= \lim_{x\to 0} \frac{1-\cos^2(x)}{x\,(1+\cos(x))} = \lim_{x\to 0} \frac{x}{1+\cos(x)} \frac{\sin^2(x)}{x^2} = 0$$

2 Limite notável: $(e^x - 1)/x$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

$$= \lim_{x\to 0} \frac{\left(1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\right) - 1}{x} =$$

$$= \lim_{x\to 0} \sum_{n=1}^{\infty} \frac{x^{(n-1)}}{n!} = \lim_{x\to 0} 1 + \sum_{n=2}^{\infty} \frac{x^{(n-1)}}{n!} = 1$$

XV | 20/04 - Funções continuas

1

$$\boldsymbol{f : \mathrm{I} \to \mathbb{R} \quad a \in \mathrm{int}(\mathrm{I})}$$

$$f \text{ é continua em } x = a \iff \exists \lim_{x \to a} f(x) = L \wedge f(a) = L$$

2 Continuidade

$$\boldsymbol{f \text{ é continua em } x = a}$$

$$\iff \exists \lim_{x \to a^+} = L^+ \wedge \exists \lim_{x \to a^-} = L^- \wedge L^+ = L^- = f(a)$$

$$\boldsymbol{\exists \lim_{x \to a} f(x) \wedge \nexists f(a)}$$

$$\overline{f} = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$$

2.1 Exemplo

$$\boldsymbol{h(x) = \sin(x)/x}$$

$$\overline{h} = \begin{cases} \sin(x)/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

2.2 Exemplo

$$\boldsymbol{g(x) = e^{-1/|x|}}$$

$$\overline{g} = \begin{cases} e^{-1/|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

3 Continuidade segundo Cauchy

$$\boldsymbol{f : \mathrm{I} \to \mathbb{R} \wedge a \in \mathrm{int}(\mathrm{I})}$$

$\therefore f$ é continua em $x = a$ segundo Cauchy $\iff$
$\iff \exists \delta_{(\varepsilon)} > 0 \quad \forall \varepsilon > 0 : |f(a) - f(x)| < \varepsilon \quad \forall |x - a| < \delta$

XVI | 23/04 - Continuidade: Teoremas

$$\boldsymbol{f \text{ é continua em } [a,b]}$$

$$\Longleftarrow \; f \text{ é continua em } x \in (a,b) \wedge$$
$$\wedge \lim_{x\to a^-} f(x) = f(a) \wedge \lim_{x\to b^+} f(x) = f(b)$$

1 Teorema de Bolzano

$$\boldsymbol{f : [a,b] \to \mathbb{R} \wedge f \text{ é continua em } [a,b]}$$
$$\boldsymbol{: f(a) < f(b) \quad \forall\, k : f(a) < k < f(b)}$$
$$\boldsymbol{\Longrightarrow \exists\, c \in (a,b) : f(c) = k}$$

2 Teorema de Weierstrass

$$\boldsymbol{f : [a,b] \to \mathbb{R} \wedge f \text{ é continua em } [a,b]}$$
$$\boldsymbol{\therefore \exists \{x_{\max}, x_{\min}\} \subset [a,b] :}$$
$$\boldsymbol{: f(x_{\min}) \leq f(x) \leq f(x_{\max})}$$

3 Corolário de Bolzano e Weierstrass

$$\boldsymbol{f : [a,b] \to \mathbb{R} \wedge f \text{ é continua em } [a,b]}$$
$$\boldsymbol{\therefore f([a,b]) = [x_{\max}, x_{\min}]}$$

3.1 Exemplo Aplicação

$$\boldsymbol{e^{-x} = x}$$

$f : [0,1] \to \mathbb{R}; \quad f(x) = e^{-x} - x \implies$
$\implies f(0) = 1; \; f(1) = 1/e - 1 < 0$
$\therefore \exists\, c \in [0,1] : e^{-c} = c$

3.2 Exemplo Aplicação

$$\boldsymbol{g : [0,\pi] \to \mathbb{R}; \quad g(x) = x \sin(x)}$$

$g(0) = g(\pi) = 0 \therefore \exists\, x_{\max} \geq 0$

XVII | 26/04 - Funções Inversas

$$\boldsymbol{f : \mathrm{I} \to \mathrm{J} \wedge \mathrm{I}, \mathrm{J} \subset \mathbb{R} \wedge f} \textbf{ é injetiva e subjetiva (bijetiva)}$$

$$f^{-1} : \mathrm{J} \to \mathrm{I} \quad \begin{cases} f^{-1} \circ f(x) = x & \forall\, x \in \mathrm{I} \\ f \circ f^{-1}(y) = y & \forall\, y \in \mathrm{J} \end{cases}$$

1 Exemplo

$$\begin{aligned} f &: [0, +\infty] \to [0, +\infty] \quad f(x) &&= x^2 \\ f^{-1} &: [0, +\infty] \to [0, +\infty] \quad f^{-1}(x) &&= \sqrt{x} \end{aligned}$$

2 Exemplo

$$f : [0, 1] \to \mathbb{R} \quad f(x) = 2x + 1$$

(i) f **é injetiva?**

$\Longleftarrow$ f é estritamente monotona

(ii) Contradomínio de f

$= [1, 3]$

(iii) f^{-1}

$$f^{-1} : [1, 3] \to [0, 1] \quad f^{-1}(y) = \frac{y - 1}{2}$$

3 Exemplo

$$\begin{aligned} \sin &: [-\pi/2, \pi/2] \to [-1, 1] \\ \arcsin &: [-1, 1] \to [-\pi/2, \pi/2] \end{aligned}$$

4 Exemplo

$$\begin{aligned} \tan &: [-\pi/2, \pi/2] \to [-\infty, \infty] \\ \arctan &: [-\infty, \infty] \to [-\pi/2, \pi/2] \end{aligned}$$

XVIII | 27/04 - Derivadas

1 Definição

$$f : \mathrm{I} \to \mathbb{R} \wedge a \in \mathrm{int}(\mathrm{I})$$

$$D \in \mathbb{R} : D = f'(a) = \frac{\mathrm{d}f(a)}{\mathrm{d}x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2 Definição Alternativa

$$\begin{aligned} \exists\, f'(a) \;\Longleftarrow\; \exists\, \mathrm{D} \in \mathbb{R}, \\ Z : V_\delta(0) \to \mathbb{R} : \\ f(x) = f(a) + D\,(x-a) + \\ +(x-a)\, Z(x-a) \end{aligned}$$

3 Definição derivada para funções de imagens abertas

$$f : \mathrm{I} \to \mathbb{R} \wedge \mathrm{I} \neq \bar{\mathrm{I}}$$

$$f' : \mathrm{I} \to \mathbb{R} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h)\, f(h)}{h}$$

XIX | 30/04 - Regras de Derivação

1 Derivação da Soma

$$(f+g)'(a) = f'(a) + g'(a)$$

$$(f+g)'(a) = \lim_{x\to 0} \frac{f(x)+g(x)-f(a)-g(a)}{x-a} =$$
$$= \lim_{x\to 0}\left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right) =$$
$$= f'(a) + g'(a)$$

2 Derivação do Produto

$$(f\,g)'(a) = f'(a)\,g(a) + f(a)\,g'(a)$$

$$(f\,g)'(a) = \lim_{x\to a} \frac{f(x)\,g(x) - f(a)\,g(a)}{x-a} =$$
$$= \lim_{x\to a} \frac{(f(x)-f(a))\;g(x) + f(a)\,(g(x)-g(a))}{x-a} =$$
$$= \lim_{x\to a}\left(g(x)\frac{f(x)-f(a)}{x-a} + f(x)\frac{g(x)-g(a)}{x-a}\right) =$$
$$= g(x)\,f'(x) + f(x)\,g'(x)$$

3 Derivada da Divisão

3.1 Previa

$$(1/f(x))' = -f'(a)/f^2(a)$$

$$= \lim_{x\to a} \frac{1/f(x) - 1/f(a)}{x-a} = \lim_{x\to a} \frac{f(a)-f(x)}{(x-a)\,f(x)\,f(a)} =$$
$$= -f'(a)/f^2(a)$$

4 Derivada de Conjugada

$$(f\circ u)'(a) = f'(u(a))\,u'(a)$$

$$(f\circ u)'(a) = \lim_{x\to a} \frac{f(u(x)) - f(u(a))}{x-a} =$$
$$= \lim_{x\to a} \frac{f(u(x)) - f(u(a))}{u(x)-u(a)}\,\frac{u(x)-u(a)}{x-a} = f'(u(a))\,u'(a)$$

4.1 Exemplo

$$\left(e^{u(x)}\right)' = e^{u(x)}\,u'(x) \qquad \left(e^{x^3}\right)' = e^{x^3}\,3\,x^2$$

4.2 Exemplo

Nota: $(\ln)'(u) = u'/u$

$$(\ln'(1/\cos(x))) = \frac{-\sin(x)/x^2}{\cos(x)} = \frac{1}{\tan(x)\,x^2}$$

5 Tecnica de encontrar derivações

5.1 $\ln'(x) = 1/x$

$$\left(e^{\ln(x)}\right)' = (x)' \implies e^{\ln(x)}\,\ln'(x) = x\,\ln'(x) = 1 \implies$$
$$\implies (\ln)'(x) = 1/x$$

5.2 $\arctan'(x) = 1/(1+x^2)$

$$(\tan(\arctan(x)))' = (x)' \implies$$
$$\implies \tan'(\arctan(x))\,\arctan'(x) =$$
$$= \left(1+\tan^2(\arctan(x))\right)\,\arctan'(x) =$$
$$= \left(1+x^2\right)\,\arctan'(x) = 1 \implies$$
$$\implies \arctan'(x) = \frac{1}{1+x^2}$$

5.3 $\arcsin'(x) = 1/\sqrt{1-x^2}$

$$(\sin(\arcsin(x)))' = (x)' \implies \cos(\arcsin(x)) \arcsin'(x) =$$
$$= \sqrt{1-\sin^2(\arcsin(x))} \arcsin'(x) =$$
$$= \sqrt{1-x^2} \arcsin'(x) = 1 \implies \arcsin'(x) = 1/\sqrt{1-x^2}$$

XX | 03/05 - Teoremas da diferenciabilidade

1 Teorema de Rolle

$$f \begin{cases} f : [a,b] \to \mathbb{R} \wedge \\ f \text{ é continua em } [a,b] \wedge \\ \exists\, f'(x) \quad \forall\, x \in (a,b) \wedge \\ f(a) = f(b) \end{cases}$$

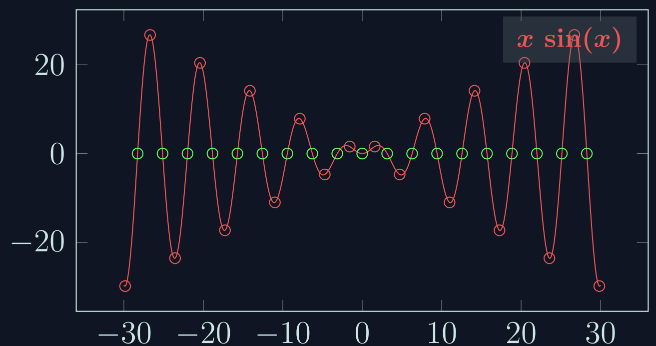
$$\therefore \begin{array}{l} \exists\, c \in [a,b] : f'(c) = 0 \wedge \\ \max(f), \min(f) \subset f(c) \end{array}$$

$$\min(f) = x_0 \in (a,b) : \\ f(x_0) \leq f(x) \quad \forall\, x \in (a,b)$$

$\lim\limits_{x \to x_0^+} \dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$; $\lim\limits_{x \to x_0^-} \dfrac{f(x) - f(x_0)}{x - x_0} \leq 0$
$\therefore f'(x_0) = 0$

1.1 Demonstração

$$h \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \to x \sin(x) \end{cases}$$



2 Teorema de Lagrange

$$f \begin{cases} f : [a,b] \to \mathbb{R} \\ f \text{ é continua em } [a,b] \\ \exists\, f'(x)\ \forall\, x \in (a,b) \end{cases}$$

$$\therefore \exists\, c \in (a,b) : f'(c) = \frac{f(b) - f(a)}{b - a}, \\ f(b) = f(a) + f'(c)\,(b - a)$$

2.1 Corolário

$$f \begin{cases} f : [a,b] \to \mathbb{R} \wedge \\ f \text{ é continua em } [a,b] \wedge \\ \exists\, f'(x) \quad \forall\, x \in (a,b) \wedge \\ f'(x) > 0 \quad \forall\, x \in (a,b) \end{cases}$$

$$\therefore f \text{ é estritamente crescente em } (a,b)$$

$\{x, y\} \in (a,b) : f(x) < f(y) \implies$
$\implies f(y) = f(x) + f'(c)\,(y - x)$

2.2 Ideia de Demonstração do teorema de Lagrange

Nota: REVER

$$h(x) = (b - a)\,f(x) - (f(b) - f(a))(x - a)$$

$\begin{cases} h(a) = (b-a)\,f(a) \\ h(b) = (b-a)\,f(b) - (f(b) - f(a))(b - a) = (b-a)\,f(a) \end{cases}$
$\implies h'(x) = (b-a)f'(x) - (f(b) - f(a)) = 0 \implies$
$\implies f(b) = f(a) + f'(x)(b - a)$

3 Regra de L'Hospital-Cauchy

$$a \in \mathbb{R} \cup \{-\infty\}; \quad \varepsilon > 0;$$

$$\{f, g\} \begin{cases} \text{continua em } (a, \infty) \land \\ ((\text{diferenciaveis em } (a, a+\varepsilon) \land a \in \mathbb{R}) \lor \\ \text{diferenciaveis em } (-\infty, -\varepsilon) \land a = -\infty) \end{cases}$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = \begin{cases} 0 \\ +\infty \end{cases}$$

$$\exists L \in \mathbb{R} \cup \{-\infty, +\infty\} : L = \lim_{x \to a^+} f'(x)/g'(x) \implies$$
$$\implies \lim_{x \to a^+} f(x)/g(x) = \lim_{x \to a^+} f'(x)/g'(x) = L$$

XXI | 04/05 - Formula de Taylor

1 Aproximação com funções polinomiais

$$\boldsymbol{f(x) = e^x \begin{cases} f(0) = 1 \\ f'(0) = 1 \\ f''(0) = 1 \end{cases}}$$

1.1 Polinômio de grau 1

$f_1(x) = f'(0)(x-0) + f(0) = x + 1$

1.2 Polinômio de grau 2

$f_2(x) = f''(0)\dfrac{(x-0)^2}{2} + f'(x-0) + f(0) = x^2/2 + x + 1$

1.3 Polinômio de grau 3

$f_2(x) = f'''(0)\dfrac{(x-0)^3}{6} + f''(0)\dfrac{(x-0)^2}{2} +$
$+ f'(x-0) + f(0) = x^3/6 + x^2/2 + x + 1$

2 Formula de Taylor com grau n e resto de lagrange

$$\boldsymbol{f :\ \exists \frac{\mathrm{d}^{n+1} f}{\mathrm{d}x^{n+1}} \quad \forall\, x \in \mathrm{I} : a \in \mathrm{I} \implies}$$
$$\boldsymbol{\implies f_{(\mathrm{ord}=n,\ \mathrm{em}=a)} =}$$
$$\boldsymbol{= \sum_{k=0}^{n} \frac{\mathrm{d}^k f}{\mathrm{d}x^k} \frac{(x-a)^k}{k!} + r_n(x-a)}$$
$$\boldsymbol{\forall\, x \in \mathrm{I} \backslash \{a\}}$$

$$r_n(x-a) = \frac{\mathrm{d}^{n+1} f(c)}{\mathrm{d}x^{n+1}} \frac{(x-a)^{n+1}}{(n+1)!} \qquad c \in [x, a]$$

2.1 Exemplo: Ordem 3 em 0

$$\boldsymbol{f(x) = \sin(x)}$$

$f_3(x) = \sin(0) + \cos(0)\, x - \sin(0)\, x^2/2 +$
$- \cos(0)\, x^3/6 + \sin(c)\, x^4/24 =$
$= x - x^3/6 + \sin(c)\, x^4/24$

XXII | 07/05 - Resto Lagrange

1 Resto de Lagrange

$$\boldsymbol{f_{(\text{ord}=1,\ \text{em}=0)} = f(0) + f'(0)\,x + M_x\,x^2;}$$

$$\begin{aligned} &h(x) = f(x) - f(0) + f'(0)\,x + M_x\,x^2; && h(0) = 0; \\ &h'(x) = f'(x) - f'(0) + M_x\,2\,x; && h'(0) = 0; \\ &h''(x) = f''(x) - M_x\,2; && h'(c) = 0 \\ &\implies M_x = f''(c)/2 \end{aligned}$$

1.1 Exemplo

$$\boldsymbol{f(x) = \ln(x); \quad a = 1}$$

$$f_{(\text{ord}=1,\text{em}=1)} = \ln(1) + c^{-1}\,(x-1) = (x-1)/c$$

$$\implies \begin{cases} x > 1 \implies f_{(1,1)}(x) < x - 1 > 0 \\ x < 1 \implies f_{(1,1)}(x) < x - 1 < 0 \end{cases}$$

1.2 Exemplo

$$\boldsymbol{f(x) = \text{arctan}(x)}$$

$$\begin{aligned} &f_{(\text{ord}=1,\ \text{em}=0)} = \arctan(0) + (1+0^2)^{-1}(x-0) + \\ &-(1+c^2)^{-2}\,2 * c\,(x-0)^2/2 = x - \frac{c\,x^2}{(1+c^2)^2} \end{aligned}$$

$$0 < x < n \implies 0 < c < x \therefore x \to 0^+ \implies c \to 0$$

(i) $\lim_{x\to 0^+}(\arctan(x) - x)/x^2$

$$\begin{aligned} &= \lim_{x\to 0^+} x^{-2}\left(x - \frac{c\,x^2}{(1+c^2)^2} - x\right) = \\ &= \lim_{x\to 0^+} \frac{c}{(1+c^2)^2} = \frac{0}{(1+0^2)^2} = 0 \end{aligned}$$

XXIII | 10/05 - 1º Teste: Revisão

1 Não foi visto

XXIV | 11/05 - Primitivação

$$f : \mathrm{I} \to \mathbb{R} \quad F \text{ é primitiva de } f \iff F' = f$$

$$\int f(x)\,\mathrm{d}x = F(x) + c$$

1 Exercício

$$F : \begin{cases} F(0) = 0 \\ f = \sin(x) \end{cases}$$

$F = 1 - \cos(x)$

2 Propriedades de Primitivas

$$\int (k\,f(x) + g(x))\,\mathrm{d}x =$$
$$= \int k\,f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x =$$
$$= k\,F(x) + C_f + G(x) + C_g$$

2.1 Exercicio

$$f(x) = 1 + x + x^2/2! + x^3/3!$$

$F(x) = x + x^2/2 + x^3/6! + x^4/4! + C =$
$= x + x^2/2 + x^3/6! + x^4/4! + 1$

3 Primitivas Imediatas

$$f(x) = e^{2\,x}$$

$F(x) = \int e^{2\,x}\,\mathrm{d}x = e^{2\,x}/2 + c \impliedby$
$\impliedby F'(x) = 2\,e^{2\,x}/2 = e^{2\,x}$

3.1 Exercicio

$$f(x) = \cos(2\,x)$$

$F(x) = \int \cos(2\,x)\,\mathrm{d}x = \sin(2\,x)/2 + C \impliedby$
$\impliedby F'(x) = 2\,\cos(2\,x)/2 = f(x)$

3.2 Exercicio

$$f(x) = 2\,x/(1 + x^2)$$

$F(x) = \int 2\,x\,\mathrm{d}x/(1 + x^2) = \ln(1 + x^2) + c \impliedby$
$\impliedby F'(x) = \dfrac{2\,x}{1 + x^2}$

XXV | 14/05 - Primitivação II

1 Exercicio: $f(x) = x(x^2+1)^3$

$$F(x) = \int x(x^2+1)^3\,\mathrm{d}x = 0.5\int u'(u)^3\,\mathrm{d}x =$$
$$= u^4/4 + c \impliedby F'(x) = 3\,u^3\,u' = f(x)$$

2 Primitivas por substituição

$$\boldsymbol{F'(x(t)) = f(x(t))\,\frac{\mathrm{d}x}{\mathrm{d}t}}$$

2.1 Exemplo: $f(x) = e^x/(1+e^{2\,x})$

$$F'(x) = \int \frac{e^x}{1+e^{2\,x}}\,\mathrm{d}x = \int \frac{t}{1+t^2}\frac{\mathrm{d}\ln(t)}{\mathrm{d}t}\,\mathrm{d}t =$$
$$= \int (1+t^2)^{-1}\,\mathrm{d}t = \arctan(t) + c = \arctan(e^x) + c$$

2.2 Exemplo: $f(x) = e^{\sqrt{x}}/\sqrt{x}$

$$F(x) = \int e^{\sqrt{x}}\,\sqrt{x}^{-1}\,\mathrm{d}x = \int e^t\,t^{-1}\,\frac{\mathrm{d}t^2}{\mathrm{d}t}\,\mathrm{d}t =$$
$$= \int 2\,e^t\,\mathrm{d}t = 2\,e^t + c = 2\,e^{\sqrt{x}} + c$$

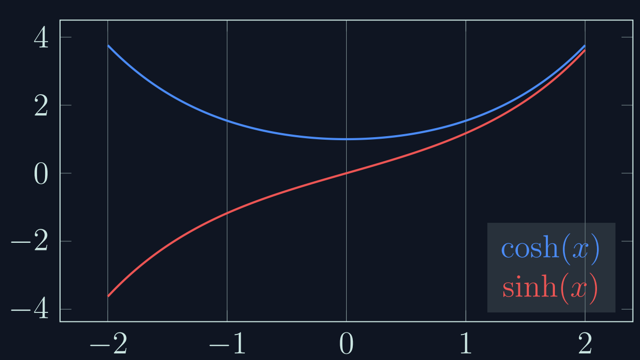
2.3 Exemplo: $f(x) = \sqrt{x-1}/x$

$$F(x) = \int \sqrt{x-1}\,x^{-1}\,\mathrm{d}x = \int \frac{t}{t^2+1}\,\frac{\mathrm{d}(t^2+1)}{\mathrm{d}t}\,\mathrm{d}t =$$
$$= \int \frac{2\,(t^2+1-1)}{t^2+1}\,\mathrm{d}t = \int 2\,\mathrm{d}t - \int \frac{2}{t^2+1}\,\mathrm{d}t =$$
$$= 2\,t - 2\arctan(t) + c =$$
$$= 2(\sqrt{x-1} - \arctan(\sqrt{x-1})) + c$$

XXVI | 17/05 - Introdução a Trigonometria Hiperbólica

1 Funções Trigonométricas Hiperbólicas

$$\boldsymbol{\cosh(x) = \frac{e^x + e^{-x}}{2}}$$
$$\boldsymbol{\sinh(x) = \frac{e^x - e^{-x}}{2}}$$
$$\boldsymbol{\cosh'(x) = \sinh(x)}$$
$$\boldsymbol{\sinh'(x) = \cosh(x)}$$



1.1 Relação fundamental da trigonometria hiperbólica clássica

$$\boldsymbol{\cosh^2(x) - \sinh^2(x) = 1}$$

$$\begin{aligned}
&\cosh^2(x) - \sinh^2(x) \;= \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \\
&= (1/4)\left(e^{2\,x} + 2 + e^{-2\,x} - e^{2\,x} + 2 - e^{-2\,x}\right) = 1
\end{aligned}$$

2 Exemplo: $f(x) = \sqrt{1 + x^2}$

$$\begin{aligned}
&F(x) = \int \sqrt{1 + x^2}\,\mathrm{d}x \;= \\
&= \int \sqrt{1 + \sinh^2(t)}\,\frac{\mathrm{d}\sinh(t)}{\mathrm{d}t}\,\mathrm{d}t = \int \cosh^2(t)\,\mathrm{d}t \;= \\
&= \int \left(\frac{e^t + e^{-t}}{2}\right)^2 \mathrm{d}t = \int \frac{e^{2\,t} + 2 + e^{-2\,t}}{4}\,\mathrm{d}t \;= \\
&= (1/4)\left(e^{2\,t}/2 + 2\,t - e^{-2\,t}/2\right) + c \;= \\
&= \left(\frac{e^t - e^{-t}}{2}\right)^2 (1/4) + t/4 + c \;= \\
&= \frac{\sinh^2 t + t}{4} + c = \frac{x^2 + \operatorname{arcsinh}(x)}{4} + c
\end{aligned}$$

3 Exemplo: $f(x) = \sqrt{1 - x^2}$

$$\begin{aligned}
&F(x) = \int \sqrt{1 - x^2}\,\mathrm{d}x \;= \\
&= \int \sqrt{1 - \sin^2(t)}\,\frac{\mathrm{d}\sin(t)}{\mathrm{d}t}\,\mathrm{d}t = \int \cos^2(t)\,\mathrm{d}t \;= \\
&= \int (\cos(2\,t)/2 + 1/2)\,\mathrm{d}t = \sin(2\,t)/2 + t/2 + c \;= \\
&= \sin(2\,\arcsin(x))/4 + \arcsin(x)/2 + c \;= \\
&= (2/4)(\sin(\arcsin(x)\,\cos(\arcsin(x))))+ \\
&+ \arcsin(x)/2 + c = (x\sqrt{1 - x^2} + \arcsin(x))/2 + c
\end{aligned}$$

4 Desafio: $f(x) = 1/\sqrt{1 + x^2}$

$$\begin{aligned}
&F(x) = \int \sqrt{1 + x^2}^{-1}\,\mathrm{d}x \;= \\
&= \int \sqrt{1 + \sinh^2(t)}^{-1}\,\frac{\mathrm{d}\sinh(x)}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{\cosh(t)}{\cosh(t)}\,\mathrm{d}t \;= \\
&= t + c = \operatorname{arcsinh}(x) + c
\end{aligned}$$

XXVII | 18/05 - Primitivação III

1 Primitivação por partes

$$\boldsymbol{u\,v = \int u\,v'\,\mathrm{d}x + \int u'\,v\,\mathrm{d}x \iff \int u\,v'\,\mathrm{d}x = u\,v - \int u'\,v\,\mathrm{d}x}$$

$$\int (u\,v)'\,\mathrm{d}x = u\,v = \int u\,v'\,\mathrm{d}x + \int u'\,v\,\mathrm{d}x$$

1.1 Exemplo: $f(x) = x\,e^x$

$$F(x) = \int x\,e^x\,\mathrm{d}x = x\,e^x - \int e^x\,\mathrm{d}x =$$
$$= x\,e^x - e^x + c = e^x(x-1) + c$$

1.2 Exercício: $f(x) = x\,\cos(x)$

$$F(x) = \int x\,\cos(x)\,\mathrm{d}x = x\,\sin(x) - \int \sin(x)\,\mathrm{d}x =$$
$$= x\,\sin(x) + \cos(x) + c$$

1.3 Exercício: $f(x) = \ln(x)$

$$F(x) = \int \ln(x)\,\mathrm{d}x = x\,\ln(x) - \int x\,x^{-1}\,\mathrm{d}x =$$
$$= x\,\ln(x) - x + c = x(\ln(x) - 1) + c$$

1.4 Exercício: $f(x) = \arctan(x)$

$$F(x) = \int \arctan(x)\,\mathrm{d}x =$$
$$= x\,\arctan(x) - (1/2)\int 2\,x\,(1+x^2)^{-1}\,\mathrm{d}x =$$
$$= x\,\arctan(x) - 0.5\,\ln(|1+x^2|) + c$$

1.5 Exercício: $f(x) = e^x\,\sin(x)$

Rever!

$$F(x) = \int e^x\,\sin(x)\,\mathrm{d}x =$$
$$= -e^x\,\cos(x) - \int e^x\,\sin(x)\,\mathrm{d}x \implies$$
$$\implies F(x) = -e^x\,\cos(x)/2 + c$$

$$F(x) = \int e^x\,\sin(x)\,\mathrm{d}x =$$
$$= e^x\,\sin(x) - \int e^x\,\cos(x)\,\mathrm{d}x =$$
$$= e^x\,\sin(x) - e^x\,\cos(x) - \int e^x\,\sin(x)\,\mathrm{d}x \implies$$
$$\implies F(x) = \frac{e^x(\sin(x) - \cos(x))}{2} + c$$

1.6 Exercicio: $f(x) = e^x\,\cos(x)$

$$F(x) = \int e^x\,\cos(x)\,\mathrm{d}x =$$
$$= e^x\,\cos(x) + \int e^x\,\sin(x)\,\mathrm{d}x =$$
$$= e^x\,\cos(x) + e^x\,\sin(x) - \int e^x\,\cos(x)\,\mathrm{d}x \implies$$
$$\implies F(x) = \frac{e^x(\sin(x) + \cos(x))}{2} + c$$

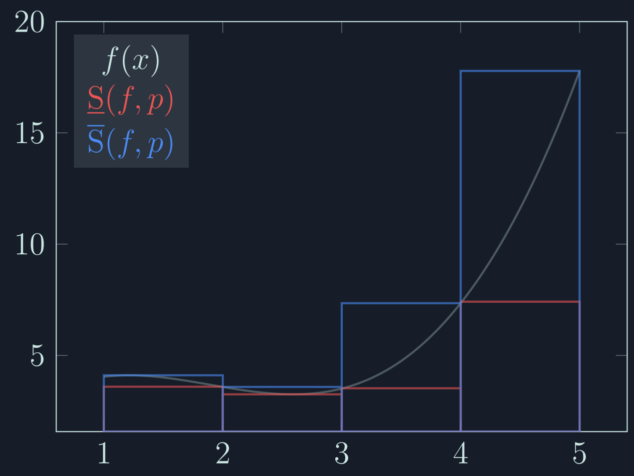
XXVIII | 21/05 - Revisões Primitivisação (Aula vazia)

XXIX | 24/05 - Integração: Soma de Darboux

1 Introdução: Cálculo de Areas

$$\text{Area}_{\min}(f(x)) = \underline{\text{S}}(f(x), p) = \sum_{k=0}^{n} \inf(f([x_k,\ x_k + 1]))$$

$$\text{Area}_{\max}(f(x)) = \overline{\text{S}}(f(x), p) = \sum_{k=0}^{n} \sup(f([x_k,\ x_k + 1]))$$



$$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\left(\underline{\text{S}}(f,p) : p \text{ partição de } [a,b]\right)$$

$$\overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\left(\overline{\text{S}}(f,p) : p \text{ partição de } [a,b]\right)$$

$$\overline{\int_a^b} f(x)\,\mathrm{d}x \geq \underline{\int_a^b} f(x)\,\mathrm{d}x$$

2 Definição: Integração de Rieman

$$f\begin{cases} \boldsymbol{f : [a,b] \to \mathbb{R} \land} \\ \textbf{limitada} \end{cases}$$

f é integrável em Rieman $\Longleftarrow$

$$\Longleftarrow \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$$

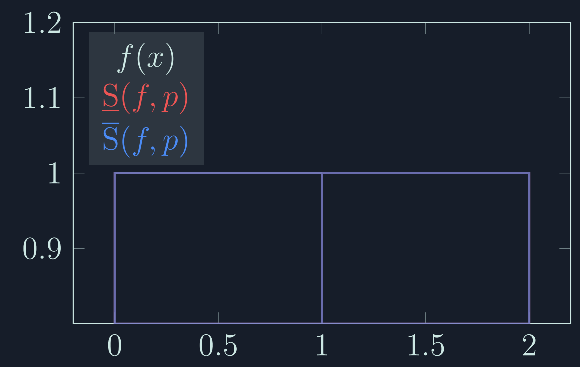
3 Critério necessário e suficiente para a integralidade de uma função

$$f\begin{cases} \boldsymbol{f : [a,b] \to \mathbb{R} \land} \\ \textbf{Limitada} \end{cases}$$

$$\exists\, P_n \subset [a,b] : \lim_{n\to\infty} \left(\overline{\mathrm{S}}(f,P_n) - \underline{\mathrm{S}}(f,P_n)\right) = 0$$

4 Exemplo

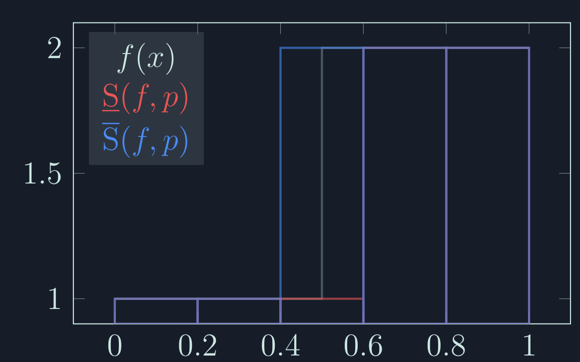
$$f\begin{cases} \boldsymbol{f : [0,1] \to \mathbb{R} \land} \\ \boldsymbol{f(x) = 1} \end{cases}$$



$$= \underline{\mathrm{S}}(f,P) = \overline{\mathrm{S}}(f,P) = 1 * \frac{1}{3} + 1 * \frac{1}{3} + 1 * \frac{1}{3} = 1$$

5 Exemplo

$$f\begin{cases} \boldsymbol{f : [0,1] \to \mathbb{R}} \\ \boldsymbol{f(x) =} \begin{cases} \boldsymbol{1} & \boldsymbol{x < 0.5} \\ \boldsymbol{2} & \boldsymbol{x \geq 0.5} \end{cases} \end{cases}$$

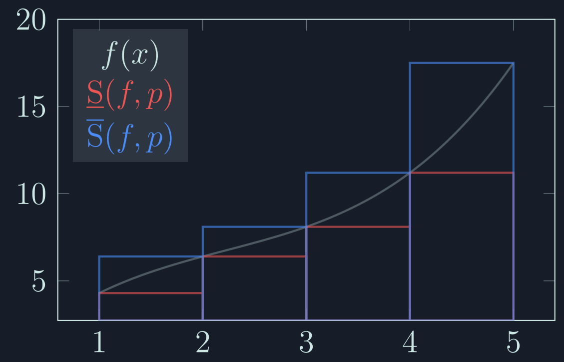


$$\underline{\mathrm{S}}(f,P) - \overline{\mathrm{S}}(f,P) = (2-1)\,\varepsilon;$$

$$\lim_{n\to\infty} \underline{\mathrm{S}}(f,P) - \overline{\mathrm{S}}(f,P) = (2-1)\,0 = 0$$

XXX | 25/05 - Integração II

1 Função monotona



(i) $\overline{S}(f, P_n)$

$$= \sum_{k=0}^{n-1} \sup(f([k,k+1]))\frac{b-a}{n} =$$
$$= \sum_{k=0}^{n-1} f(k+1)\frac{b-a}{n}$$

(ii) $\underline{S}(f, P_n)$

$$= \sum_{k=0}^{n-1} \inf(f([k,k+1]))\frac{b-a}{n} = \sum_{k=0}^{n-1} f(k)\frac{b-a}{n}$$

$$\overline{S}(f,P_n) - \underline{S}(f,P_n) = \frac{b-a}{n}\sum_{k=1}^{n}(f(k+1)-f(k)) =$$
$$= \frac{b-a}{n}(f(b)-f(a))$$

(iii) Limite

$$\lim_{n\to\infty} \frac{b-a}{n}(f(b)-f(a)) = 0$$
$f(x)$ é integrável por Rieman

2 Contra Exemplo

$$f \begin{cases} f : [0,1] \to \mathbb{R} \land \\ f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \end{cases}$$

$\sup(f([k,k+1])) = 1 \land \inf(f([k,k+1])) = 0$
$\forall\, k \in P_n$ Partição de $[a,b] \therefore \overline{S}(f,P_n) - \underline{S}(f,P_n) \neq 0$

3 Diâmetro de uma partição

$$\delta(P) = \max(|x_{k+1} - x_k|)$$
$$\forall\, k \in [0, n-1]$$

4 Soma de Riemam

$$\int_a^b f(x)\,\mathrm{d}x =$$
$$= \lim_{\delta(P)\to 0} \sum_{k=0}^{n-1} f(x'_k)(f(x_{k+1}) - f(x_k))$$
$x'_k \in [x_k, x_{k+1}]$

Propriedades da soma de Rieman

(i)

$$\int_a^b (k\,f(x) + g(x))\,\mathrm{d}x = \\ = k\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

(ii)

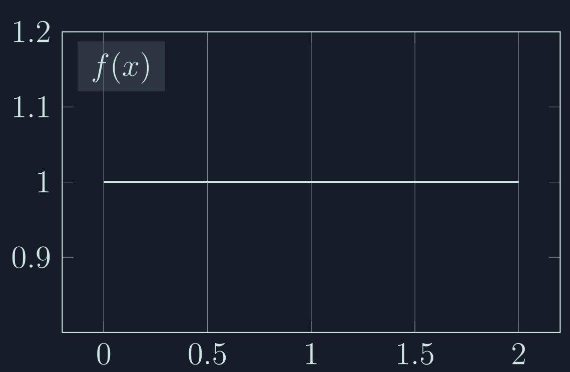
$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x \\ \forall\,\{f, g\} : f(x) \leq g(x) \quad \forall\, x \in [a, b]$$

(iii)

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)\,\mathrm{d}x|$$

5 Exercicio

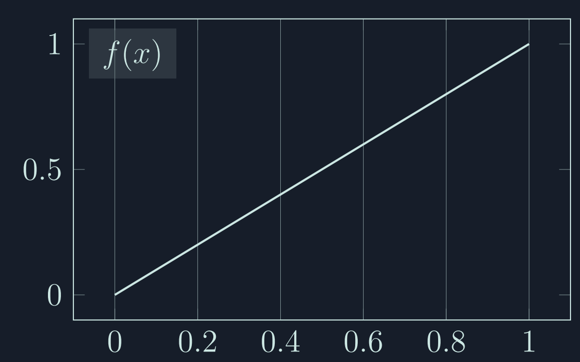
$$\int_0^2 1\,\mathrm{d}x$$



$= 2$

6 Exercicio

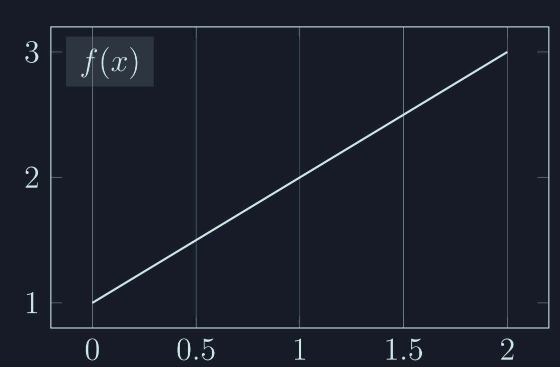
$$\int_0^1 x\,\mathrm{d}x$$



$0.5\Delta(x^2)\big|_0^1 = 0.5$

7 Exercicio

$$\int_0^1 (1 + x)\,\mathrm{d}x$$



$\Delta\left(x + x^2/2\right)\big|_0^1 = 1.5$

XXXI | 28/05 - Integrais Indefinidas

1 Propriedade de Integração

Por convenção

$$\int_a^b f(x)\,\mathrm{d}x = \int_b^a f(x)\,\mathrm{d}x$$

2 Propriedade de Integração

$$f\begin{cases} f : \mathrm{I} \to \mathbb{R} \land \\ \text{integrável em I} \end{cases}$$

$$\{a, b, c\} \in \mathrm{I}$$

$$\int_a^b f(x)\,\mathrm{d}x =$$

$$= \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

3 Integral Indefinida

$$f\begin{cases} f : \mathrm{I} \to \mathbb{R} \\ f \text{ integrável em todo subintervalo} \\ \text{compacto de I} \end{cases}$$

Definição:

$$F_a : x \to \int_a^x f(t)\,\mathrm{d}t$$

4 Teorema fundamental do cálculo

$$f : \mathrm{I} \to \mathbb{R}$$

$$F_a(x) = \int_a^x f(t)\,\mathrm{d}t$$

F_a é Primitiva de f que se anula em $x = a$

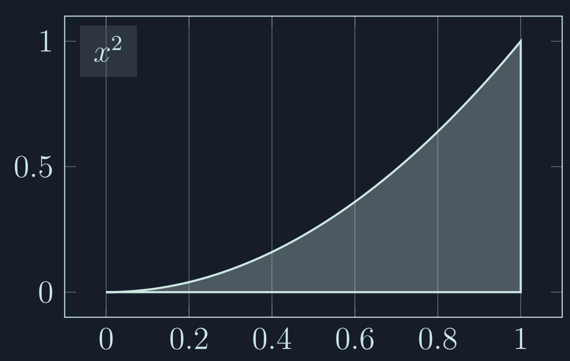
5 Regra de Barrow

$$f\begin{cases} \text{continua em } [a, b] \\ F' = f \end{cases}$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Delta F(x)\Big|_a^b$$

6 Exercicio

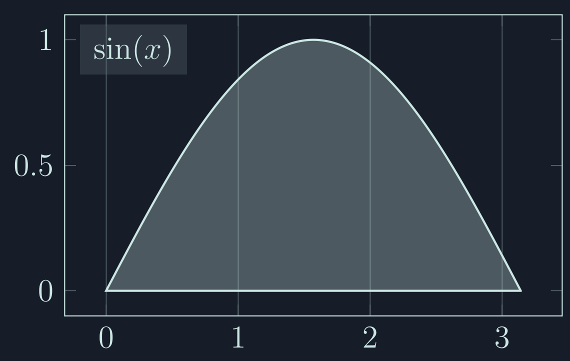
$$\int_0^1 \boldsymbol{x^2\,\mathrm{d}x}$$



$$\int_0^1 x^2\,\mathrm{d}x = \Delta F(x)\Big|_0^1 = (1/3)(1^3 - 0^3) = 1/3$$

7 Exercicio

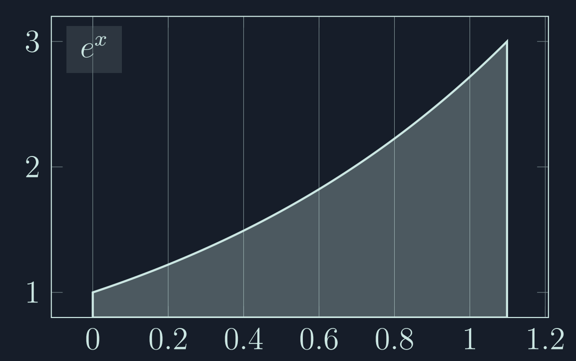
$$\int_0^{\pi} \boldsymbol{\sin(x)\,\mathrm{d}x}$$



$$\int_0^{\pi} \sin(x)\,\mathrm{d}x = \Delta(-\cos(x))\Big|_0^{\pi} = 2$$

8 Exercicio

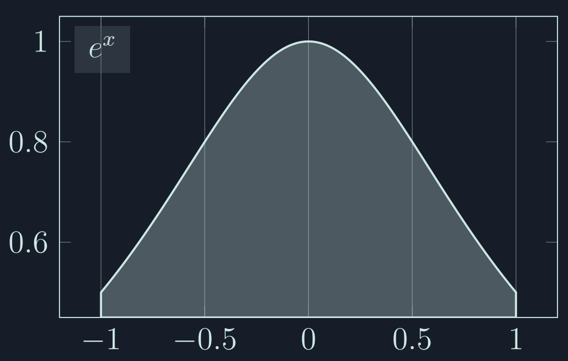
$$\int_0^{\ln(3)} \boldsymbol{e^x\,\mathrm{d}x}$$



$$\int_0^{\ln(3)} e^x\,\mathrm{d}x = \Delta(e^x)\Big|_0^{\ln(3)} = 2$$

9 Exercicio

$$\int_{-1}^{1} \boldsymbol{\mathrm{d}x/(1+x^2)}$$



$$\int_{-1}^{1} \mathrm{d}x/(1+x^2) = \Delta\left(\arctan(x)\right)\Big|_{-1}^{1} = \pi/2$$

XXXII | 32/05 - Teorema Fundamental do Cálculo

Area dentre funções

1 Exercicio

$$\boldsymbol{f(x) = x + 6;\ g(x) = x^2}$$

(i) $\{a, b\}$

$$x = \frac{-1 \pm \sqrt{1 - 4 * 6 * (-1)}}{-1 * 2} = \frac{1 \mp 5}{2}$$
$\therefore a = -2; b = 3$

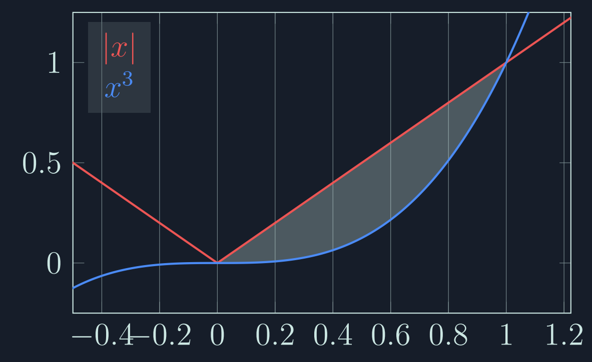
$$\int_{-2}^{3} (x + 6 - x^2)\, \mathrm{d}x \Delta \left(x^2/2 + 6\,x - x^3/3\right) \Big|_{-2}^{3} =$$
$$= 5 + 30 - 11/3 = 116/3$$

2 Exercicio

$$\boldsymbol{f(x) = |x|;\ g(x) = x^3}$$

(i) $\{a, b\}$

$x : f(x) - g(x) = |x| - x^3 =$
$= x(1 - x^2) = 0 \quad \forall\, x \geq 0$
$\therefore a = 0;\ b = 1$



$$\int_{0}^{1} (x - x^3)\, \mathrm{d}x = \Delta \left(x^2/2 - x^4/4\right) \Big|_{0}^{1} = 1/4$$

Integrais Indefinidas Compostas

3 Exemplo

$$\boldsymbol{H(x) = \int_0^{\ln(x)} \sin(e^t)\,\mathrm{d}t}$$

$H'(x) = \sin(e^{\ln(x)})/x = \sin(x)/x$

4 Exemplo

$$\boldsymbol{F(x) = \int_x^{1} e^t\,\mathrm{d}t}$$

$F'(x) = -e^x$

5 Exemplo

$$\boldsymbol{H(x) = \int_x^{x^2} f(t)\,\mathrm{d}t}$$

$H'(x) = f(x^2)\,2\,x - f(x)$

6 Exemplo

$$\boldsymbol{H(x) = \int_x^{\tan(x)} f(t)\,\mathrm{d}t}$$

$H'(x) = f(\tan(x))/(1 + \tan^2(x)) - f(x)$

XXXIII | 01/06 - Teoremas do calculo integral

1 Teorema do Valor Médio

$$g \begin{cases} g : [a,b] \to \mathbb{R} \\ \text{continua em } [a,b] \end{cases}$$

$$\exists\, g(c) \in [a,b] : c * (b-a) = \\ = \int_a^b g(x)\,\mathrm{d}x$$

$$\min(g([a,b]))(b-a) \le \int_a^b g(x)\,\mathrm{d}x \le \\ \le \max(g([a,b]))(b-a)$$

$$\therefore\ \exists\, c \in [a,b] : g(c)(b-a) = \int_a^b g(x)\,\mathrm{d}x$$

2

$$\lim_{h\to 0} \int_x^{x+h} f(x)\,\mathrm{d}x = \lim_{h\to 0} h\, f(c)/h = \\ = f(c) : c \in [x, x+h] = f(x)$$

Integração por substituição

3 Exemplo

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

$$= \int_{\arcsin(-1)}^{\arcsin(1)} \sqrt{1-\sin^2(t)}\,\frac{\mathrm{d}\sin(t)}{\mathrm{d}t}\,\mathrm{d}t = \\ = \int_{-\pi/2}^{\pi/2} \cos^2(t)\,\mathrm{d}t = (1/2)\int_{-\pi/2}^{\pi/2} (\cos(2\,t)+1)\,\mathrm{d}t = \\ = 0.5\,\Delta\,(\sin(2\,t)/2 + t)\,\Big|_{-\pi/2}^{\pi/2} = \pi/2$$

4 Integração por Partes

$$\int_a^b f'(x)\,g(x)\,\mathrm{d}x = \\ = \Delta(f(x)\,g(x))\Big|_a^b - \int_a^b f(x)\,g'(x)\,\mathrm{d}x$$

4.1 Exercicio

$$\int_1^e \ln(x)\,\mathrm{d}x$$

$$= \Delta(x\,\ln(x))\Big|_1^e \int_1^e x\,x^{-1}\,\mathrm{d}x = \Delta(x\,\ln(x) - x)\Big|_1^e = \\ = e\,(1-e) - 1(0-1) = e - e^2 + 1$$

4.2 Aplicação: Area de uma elipse

$$\boldsymbol{\int_{-a}^{a} b\sqrt{a^2 - x^2}\,\mathrm{d}x/a}$$

$$
\begin{aligned}
&= \int_{-a/a}^{a/a} b\sqrt{a^2 - (a\,t)^2}\,\frac{\mathrm{d}a\,t}{\mathrm{d}t}\,\mathrm{d}t/a = \\
&= \int_{-1}^{1} b\sqrt{1 - \ t^2}\,a\,\mathrm{d}t = \\
&= a\,b\int_{\arcsin(-1)}^{\arcsin(1)} \sqrt{1 - \sin^2(t_2)}\,\frac{\mathrm{d}\sin(t_2)}{\mathrm{d}t_2}\,\mathrm{d}t_2 = \\
&= a\,b\int_{-\pi/2}^{\pi/2} \cos^2(t_2)\,\mathrm{d}t_2 = \\
&= \frac{a\,b}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos(2\,t_2))\,\mathrm{d}t_2 = \\
&= \frac{a\,b}{2}\Delta\,(t_2 + \sin(2\,t_2)/2)\,\Big|_{-\pi/2}^{\pi/2} = \frac{a\,b}{2}\pi/2
\end{aligned}
$$

XXXIV | 15/06 - Exercícios para o Teste

Q0 - a)

$$\boldsymbol{\int_{\varepsilon^3}^{8\pi^3} \frac{\cos(\sqrt[3]{x})}{3\sqrt[3]{x}}\mathrm{d}x = \int_{\varepsilon}^{2\pi} t\,\cos(t)\,\mathrm{d}t}$$

$$\boldsymbol{(x = t^3)}$$

$$\int_{\varepsilon^3}^{8\pi^3} \frac{\cos(\sqrt[3]{x})}{3\sqrt[3]{x}}\mathrm{d}x = \int_{\sqrt[3]{\varepsilon^3}}^{\sqrt[3]{8\pi^3}} \frac{\cos\left(\sqrt[3]{t^3}\right)}{3\sqrt[3]{t^3}}\mathrm{d}\left(t^3\right) =$$

$$= \int_{\varepsilon}^{2\pi} \frac{\cos(t)}{3t} 3t^2 \mathrm{d}t = \int_{\varepsilon}^{2\pi} t\,\cos(t)\mathrm{d}t$$