

# Ficha 1: Noções básicas de topologia na recta real

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## Questão 1

$$\{A, B, C, D\} \in \mathbb{R}$$

$$\mathbf{1-a)} \quad A = ]-1, \sqrt{2}]$$

$$\mathbf{1-b)} \quad B = \{e^{-n} : n \in \mathbb{N}\} \cup \{(1 + 3/n)^n : n \in \mathbb{N}\}$$

$$\text{int}(A) = (-1, \sqrt{2});$$

$$\text{Ext}(A) = \mathbb{R} \setminus [1, \sqrt{2}];$$

$$\text{Font}(A) = \{-1, \sqrt{2}\}.$$

$$\text{Int}(B) = \emptyset;$$

$$\text{Ext}(B) = \mathbb{R} \setminus B - \{e^3, 0\};$$

$$\text{Font}(B) = \{0, e^3\}$$

**Nota:**

$$\lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha$$

$$\mathbf{1-c)} \quad C = (0, 1) \cap \mathbb{Q}$$

$$\mathbf{1-d)} \quad D = \{1/(n + \sqrt{n}) : n \in \mathbb{N}\} \cap \mathbb{Q}$$

$$\text{Int}(C) = \emptyset$$

$$\text{Ext}(C) = \mathbb{R} \setminus [0, 1]$$

$$\text{Font}(C) = [0, 1]$$

$$\frac{1}{n + \sqrt{n}} = \frac{1}{m^2 + m} \quad \forall \{\{m, n\} \in \mathbb{N} : n = m^2\} \implies$$

$$\text{Int}(D) = \emptyset$$

$$\text{Ext}(D) = \mathbb{R} \setminus D - \{0\}$$

$$\text{Font}(D) = D + \{0\}$$

## Questão 2

$$\mathbf{2-a)} \quad f(x) = \ln(-x^2 + 2x)$$

$$\begin{aligned} D &= \{x \in \mathbb{R} : -x^2 + 2x > 0\}; \quad -x^2 + 2x = x(-x + 2) > 0 \implies \\ \implies D &= \{x \in \mathbb{R} : 0 < x < 2\} \implies \\ \implies \text{Sup}(D) &= 2; \quad \text{Inf}(D) = 0; \quad \text{Max}(D) = \text{Min}(D) = \emptyset \end{aligned}$$

**2-b)**  $g(x) = \sqrt[6]{\pi^2 - x^2} \tan(x)$

$$\begin{aligned} D &= \{x \in \mathbb{R} : \pi^2 - x^2 \geq 0\}; \pi^2 - x^2 \geq 0 \implies x^2 \leq \pi^2 \implies \\ \implies D &= \{x \in \mathbb{R} : |x| \leq \pi\} \implies \\ \implies \text{Sup}(D) &= \text{Max}(D) = \pi; \text{Inf}(D) = \text{Min}(D) = -\pi \end{aligned}$$

### Questão 3

$$A = \{(1/n) : n \in \mathbb{N}\}$$

**3-a)**

$$1 \in A \iff \exists n \in \mathbb{N} : 1/n = 1 \iff n = 1$$

$$\begin{aligned} 1 \notin \text{Acum}(A) &\iff \nexists \epsilon \in \mathbb{R} : V_\epsilon(1) \cap A - \{1\} \neq \emptyset \iff \\ &\iff V_\epsilon(1) \cap A - \{1\} = \emptyset \forall \epsilon \in \mathbb{R} : \epsilon < 0.5 \end{aligned}$$

$$\begin{aligned} 0 \in \text{Acum}(A) &\iff \nexists \epsilon \in \mathbb{N} : V_\epsilon(0) \cap A - \{0\} = \emptyset \iff \\ &\iff V_\epsilon(0) \cap A - \{0\} = [0, \epsilon] \cap \{1/n : n \in \mathbb{N}\} \neq \emptyset \forall \epsilon \in \mathbb{R}, \forall n \in \mathbb{N} \iff \\ &\iff \nexists \epsilon \in \mathbb{R} : \epsilon > 0 \wedge \epsilon < 1/n \forall n \in \mathbb{N} \end{aligned}$$

**3-b)**

$$A' = [-1, \sqrt{2}] \quad B' = \{0, \epsilon^3\} \quad C' = [0, 1] \quad D' = \{0\}$$

### Questão 4

$$X \subset \mathbb{R}$$

4-a)

$$\begin{aligned}
x \in \text{Fr}(X) &\iff \{V_\epsilon(x) \cap X \neq \emptyset \wedge V_\epsilon(x) \not\subset X\} \forall \epsilon \in \mathbb{R} \implies \\
&\implies \exists x \in \text{Fr}(X) : x = V_\epsilon(x) \cap X; \quad X' = \{x \in \mathbb{R} : V_\epsilon(x) \cap X - \{x\} \neq \emptyset\} \implies \\
&\implies \exists x \in \text{Fr}(X) : x \notin X'
\end{aligned}$$

4-b)

$$V_\epsilon(x) \cap X = \{x\} \implies V_\epsilon(x) \cap X \neq \emptyset \wedge V_\epsilon(x) \not\subset X \iff x \in \text{Fr}(X)$$

4-c)

$$\begin{aligned}
\text{Fr}(\text{Ext}(X)) &= \{x \in \mathbb{R} : V_\epsilon(x) \cap \text{Ext}(X) \neq \emptyset \wedge V_\epsilon(x) \not\subset \text{Ext}(X)\}; \\
\forall x \in \mathbb{R} : V_\epsilon(x) \cap X &= V_\epsilon(x) - \{x\} \implies x \in \text{Fr}(X) \wedge x \notin \text{Fr}(\text{Ext}(X)); \\
\forall x \in \mathbb{R} : V_\epsilon(x) \cap X &= \{x\} \implies x \in \text{Fr}(X) \wedge x \notin \text{Fr}(\text{Int}(X)) \wedge x \in \text{Fr}(\text{Ext}(X)); \\
\therefore \text{Fr}(\text{Ext}(X)) &\neq \text{Fr}(\text{Int}(X)) \neq \text{Fr}(X) \neq \text{Fr}(\text{Ext}(X))
\end{aligned}$$

4-d) **Duvida**

$$\begin{aligned}
X \text{ é um conjunto fechado} &\implies \text{Fr}(X) \subset X; \\
\mathbb{R} \text{ é um conjunto fechado} &\implies \text{Fr}(\mathbb{R}) = \{-\infty, \infty\} \not\subset \mathbb{R}
\end{aligned}$$

4-e)

$$\begin{aligned}
X' = \{x \in \mathbb{R} : V_\epsilon(x) \cap X - \{x\} \neq \emptyset\} &\implies (V_\epsilon(x) - \{x\}) \cap X = \emptyset \forall x \in \mathbb{R} \setminus X' \implies \\
\implies \mathbb{R} \setminus X' \text{ é um grupo aberto} &\implies X' \text{ é um grupo fechado}
\end{aligned}$$

## Questão 5

$$A = \left\{ x \in \mathbb{R} : \frac{\ln(x^2 + 1)}{x^2 - 16} \geq 0 \right\}; \quad B = \{x \in \mathbb{R} : |x^2 - 18| \leq 18\}$$

**5-a)**

$$\begin{aligned} A &= \left\{ x \in \mathbb{R} : \frac{\ln(x^2 + 1)}{x^2 - 16} \geq 0 \right\} = \\ &= \{x \in \mathbb{R} : x^2 + 1 > 0 \wedge x^2 - 16 \neq 0 \wedge x^2 - 16 \geq 0\} = \{x \in \mathbb{R} : |x| > 4\} = \\ &= \{x \in \mathbb{R} \cap ((-\infty, -4) \cup (4, \infty))\} \\ B &= \{x \in \mathbb{R} : |x^2 - 18| \leq 18\} = \{x \in \mathbb{R} : x^2 \leq 36 \wedge x^2 \geq 0\} = \{x \in \mathbb{R} : |x| \leq 6\} \\ &= \{x \in \mathbb{R} \cap [-6, 6]\} \end{aligned}$$

**5-b)**

$$\begin{aligned} A \cap B &= [-6, -4) \cup (4, 6]; \\ \text{Inf}(A \cap B) &= \{4\}; \quad \text{Min}(A \cap B) = \emptyset; \quad \text{Sup}(A \cap B) = \text{Max}(A \cap B) = \{6\} \end{aligned}$$

## Questão 6

**6-a)**      $\text{Int}(X) = (0, 1) \wedge X' = [0, 1] \cup \{e\}$

$$X = [0, 1] \cup \{(1 + 1/x)^x : x \in \mathbb{N}\}$$

**6-b)**      $\text{Ext}(X) = (-\infty, 0) \wedge \text{Int}(X) = \emptyset$

$$X = \{x \in \mathbb{Q} : x < 0\}$$

**6-c)**      $X' = \mathbb{Z}$

$$X = \left\{ x + \frac{1}{y} : x \in \mathbb{Z} \wedge y \in \mathbb{N} \right\}$$

**6-d)**  $X' = (0, 1)$

$\nexists X : X'$  é um conjunto aberto

**6-e)**  $\text{Fr}(X) = [0, 1]$

$X = [0, 1] \cap \mathbb{Q}$

## Questão 7

$$f(x) = \frac{\sqrt{2-x^2} \ln(x+1)}{\sin(x)}$$

$$\begin{aligned} D &= \{x \in \mathbb{R} : 2 - x^2 \geq 0 \wedge x + 1 \geq 0 \wedge \sin(x) \neq 0\} = \\ &= \{x \in \mathbb{R} : |x| \leq \sqrt{2} \wedge x \geq -1 \wedge x \neq \pi n \ \forall n \in \mathbb{Z}\} = \\ &= \{x \in \mathbb{R} : -1 \leq x \leq \sqrt{2} \wedge x \neq \pi n \ \forall n \in \mathbb{Z}\} = \left\{x \in \mathbb{R} \cap [-1, 0) \cup (0, \sqrt{2}]\right\} \end{aligned}$$

$$\begin{aligned} (A \cup D)' &= [-1, \sqrt{2}] \cup \{x \in \mathbb{R} : V_\epsilon(x) \cap A - \{x\} \neq \emptyset\} = \\ &= [-1, \sqrt{2}] \cup \left[-\frac{4}{3}, \frac{4}{3}\right] = \left[-\frac{4}{3}, \sqrt{2}\right] \end{aligned}$$

## Questão 8

$$f(x) = \sin(x)/x; \quad f : (0, \infty) \rightarrow \mathbb{R}$$

$$\begin{aligned} f(x) = \{y \in \mathbb{R} : y = \sin(x)/x \ \forall x \in \mathbb{R} \cap (0, \infty)\}; \quad x > 0 \wedge -1 \leq \sin(x) \leq 1 &\implies \\ \implies -1 < f(x) < 1 \end{aligned}$$

$$\begin{aligned} \text{Fr}(f(x)) \not\subset f(x) &\iff \\ \iff \exists y \in \mathbb{R} : V_\epsilon(y) \cap f(x) - \{y\} \neq \emptyset \wedge V_\epsilon(y) \not\subset f(x) \wedge y \notin f(x) &\iff \\ \iff 1 \in \text{Fr}(f(x)) \end{aligned}$$

$$\begin{aligned} \text{Int}(f(x)) \neq f(x) &\iff \exists y \in f(x) : V_\epsilon(y) \not\subset f(x) \iff \\ \iff g(x) = \{y \in \mathbb{R} : y = \sin(x)/x \ \forall x \in [\pi, 2\pi]\} \subset f(x); g(x) = [m, 0] &\implies \\ \implies \exists m \in f(x) \cap (-1, 0] : f(x) = [m, 1) \end{aligned}$$