1 derivation

- HMJ is
$$0 = \rho\theta V_t(\frac{C_t^{(1-\gamma)\theta}}{((1-\gamma)V_t)^{\frac{1}{\theta}}} - 1) + \frac{E[dV_t]}{dt}$$
We know V_t is homogeneous in C_t
$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma}G_t$$
Substituting in HMJ, we obtain
$$0 = \frac{\rho\theta}{1-\gamma}C_t^{1-\gamma}G_t(G_t^{-\frac{1}{\theta}} - 1) + \frac{C_t^{1-\gamma}}{1-\gamma}E\frac{dG_t}{dt} + \frac{G_t}{1-\gamma}E\frac{dC_t^{1-\gamma}}{dt}$$
Using Ito,
$$E[\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}] = \left[(1-\gamma)\mu_{Ct} - \frac{1}{2}(1-\gamma)\gamma\sigma_{Ct}^2\right]dt$$
We obtain
$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + E\frac{dG_t}{dt} + G_t((1-\gamma)\mu_{Ct} - \frac{1}{2}(1-\gamma)\gamma\sigma_{Ct}^2)$$

2 Bansal Yaron

$$\begin{aligned}
\frac{dC_t}{C_t} &= \mu_t dt + \nu_D\sqrt{\sigma_t}dZ_t \\
d\mu_t &= \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t}dZ_t^\mu \\
d\sigma_t &= \kappa_\sigma(1 - \sigma_t)dt + \nu_\sigma\sqrt{\sigma_t}dZ_t^\sigma
\end{aligned}$$

Name	BY04	BY04	This paper	This paper	Link
mean growth rate	μ	0.0015	$\bar{\mu}$	0.0015	$\mu = \bar{\mu}$
mean volatility	σ^2	0.00006084	ν_D	0.0078	$\sqrt{\sigma^2} = \nu_D$
growth persistence	ρ	0.979	κ_μ	0.0212	- $\log(\rho) = \kappa_\mu$
volatility persistence	ν_1	0.987	κ_σ	0.0131	$-\log(\nu_1) = \kappa_\sigma$
growth rate volatility	φ_e	0.044	ν_μ	0.0003432	$\varphi_e \times \sqrt{\sigma^2} = \nu_\mu$
volatility volatility	σ_w	0.0000023	ν_σ	0.0378	$\sigma_w/\sigma^2 = \nu_\sigma$
time discount	δ	0.998	ρ	0.002	$-\log(\delta) = \rho$
RRA	$1-\gamma$(RRA)	7.5 or 10	$1-\gamma$	-6.5 or -9	$1 -$ RRA $= 1-\gamma$
IES	ψ	1.5	ψ	1.5	$\psi = \psi$

This also means $\theta = (1-\gamma)/(1-1/\psi) =$ -19.50 or -27.

We write $G_t = G(\mu, \sigma)$ and we get

$$
\begin{aligned}
0 &= \rho\theta[G^{1-\frac{1}{\theta}} - G] + G((1-\gamma)\mu - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma) \\
&+ \kappa_\mu(\bar{\mu} - \mu)\frac{\partial G}{\partial \mu} + \kappa_\sigma(1-\sigma)\frac{\partial G}{\partial \sigma} \\
&+ \frac{1}{2}\nu_\mu^2\sigma\frac{\partial^2 G}{\partial \mu^2} + \frac{1}{2}\nu_\sigma^2\sigma\frac{\partial^2 G}{\partial \sigma^2}
\end{aligned}
$$

Boundary coundition = reflectiving barrier

$$
\begin{aligned}
\partial_\mu G(\underline{u}, \sigma) = 0 \\
\partial_\mu G(\overline{u}, \sigma) = 0 \\
\partial_\sigma G(u, \underline{\sigma}) = 0 \\
\partial_\sigma G(u, \overline{\sigma}) = 0
\end{aligned}
$$

Finite difference

$$
\begin{aligned}
\frac{G_{i,j}^{n+1} - G_{i,j}^n}{\Delta} &= \rho\theta[(G_{ij}^n)^{1-\frac{1}{\theta}} - G_{ij}^{n+1}] + G_{ij}^{n+1}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
&+ \kappa_{\mu_i}(\bar{\mu} - \mu_i)^+(G_{i+1,j}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\mu_i}(\bar{\mu} - \mu_i)^-(G_{i,j}^{n+1} - G_{i-1,j}^{n+1}) \\
&+ \kappa_{\sigma_j}(1 - \sigma_j)^+(G_{i,j+1}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\sigma_j}(1 - \sigma_j)^-(G_{i,j}^{n+1} - G_{i,j-1}^{n+1}) \\
&+ \frac{1}{2}\nu_{\mu_i}^2\sigma_j(G_{i+1,j}^{n+1} - 2G_{i,j}^{n+1} + G_{i-1,j}^{n+1}) + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j(G_{i,j+1}^{n+1} - 2G_{i,j}^{n+1} + G_{i,j-1}^{n+1})
\end{aligned}
$$

The monotinicity condition in $G_{i+1,j}^{n+1}, G_{i-1,j}^{n+1}, G_{i,j+1}^{n+1}, G_{i,j-1}^{n+1}$ is always satisfied. The monotonicity condition in $G_{i,j}^n$ is not necessary satisfied when $\theta < 1$ since then $\rho\theta(G_{ij}^n)^{1-1/\theta}$ is decreasing in $G_{i,j}^n \Rightarrow$. There are two solutions

- Substitute G_{ij}^{n+1} by G_{ij}^n in some terms increasing in G_{ij}^n. This does not work for all possible parameter values. Scheme implemented by the type BansalYaronL.

  $$
  \begin{aligned}
  \frac{G_{i,j}^{n+1} - G_{i,j}^n}{\Delta} &= \rho\theta(G_{ij}^n)^{1-\frac{1}{\theta}} \\
  &- (\rho\theta)^- G_{ij}^n - (\rho\theta)^+ G_{ij}^{n+1} \\
  &+ ((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2))G_{ij}^n \\
  &+ \kappa_{\mu_i}(\bar{\mu} - \mu_i)^+(G_{i+1,j}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\mu_i}(\bar{\mu} - \mu_i)^-(G_{i,j}^{n+1} - G_{i-1,j}^{n+1}) \\
  &+ \kappa_{\sigma_j}(1 - \sigma_j)^+(G_{i,j+1}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\sigma_j}(1 - \sigma_j)^-(G_{i,j}^{n+1} - G_{i,j-1}^{n+1}) \\
  &+ \frac{1}{2}\nu_{\mu_i}^2\sigma_j(G_{i+1,j}^{n+1} - 2G_{i,j}^{n+1} + G_{i-1,j}^{n+1}) + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j(G_{i,j+1}^{n+1} - 2G_{i,j}^{n+1} + G_{i,j-1}^{n+1})
  \end{aligned}
  $$

  Actually, one can always choose Δ low enough that the scheme is increasing in G_{ij}^n (since G is bounded on the state space). But then this introduces rounding errors.

- Substitute G_{ij}^n by G_{ij}^{n+1} in $\rho\theta(G_{ij}^n)^{1-1/\theta}$. Then one needs to solve a non-iterative non-linear equation instead of a iterative linear one. It's not really slower. Scheme implemented by the type BansalYaronNL.

$$\begin{aligned}
0 &= \rho\theta[(G_{ij}^{n+1})^{1-\frac{1}{\theta}} - G_{ij}^{n+1}] + G_{ij}^{n+1}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
&+ \kappa_{\mu_i}(\bar{\mu}-\mu_i)^+(G_{i+1,j}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\mu_i}(\bar{\mu}-\mu_i)^-(G_{i,j}^{n+1} - G_{i-1,j}^{n+1}) \\
&+ \kappa_{\sigma_j}(1-\sigma_j)^+(G_{i,j+1}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\sigma_j}(1-\sigma_j)^-(G_{i,j}^{n+1} - G_{i,j-1}^{n+1}) \\
&+ \frac{1}{2}\nu_{\mu_i}^2\sigma_j(G_{i+1,j}^{n+1} - 2G_{i,j}^{n+1} + G_{i-1,j}^{n+1}) + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j(G_{i,j+1}^{n+1} - 2G_{i,j}^{n+1} + G_{i,j-1}^{n+1})
\end{aligned}$$

3 Comparaison

Denote k_t the consumption to wealth ratio

$$V = G_t k_t^{1-\gamma}\frac{W^{1-\gamma}}{(1-\gamma)}$$

FOC for consumption

$$k_t = \rho^{\psi}k_t^{1-\psi}G_t^{\frac{1-\psi}{1-\gamma}}$$

therefore

$$k_t = \rho G_t^{-1/\theta}$$

Bansal Yaron find

$$\begin{aligned}
\frac{1}{\theta}\log G_t - \log\rho &\approx A_1\mu_t + A_2\nu_D^2\sigma_t \\
A1 &= \frac{1-\frac{1}{\psi}}{1-0.997e^{-\kappa_\mu}} \\
A2 &= 0.5\theta\frac{(1-\frac{1}{\psi})^2 + (A_1\kappa_1\frac{\nu_\mu}{\nu_D^2})^2}{1-0.997e^{-\kappa_\sigma}}
\end{aligned}$$