1 Derivation

HJB is

$$0 = \rho\theta V_t\left(\frac{C_t^{\frac{1-\gamma}{\theta}}}{((1-\gamma)V_t)^{\frac{1}{\theta}}} - 1\right) + \frac{E[dV_t]}{dt}$$

Let's define G_t such that

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma} G_t$$

By Ito,

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t E\frac{\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt} + E\frac{dG_t}{dt} + E\frac{dG_t\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt}$$

Denoting μ_C and σ_C the geometric drift of C_t, we have

$$\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}} = ((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C^2)dt + (1-\gamma)\sigma_C dW_t$$

Injecting this expression into HJB and denoting μ_G, σ_G the arithmetic drift and volatility of G_t

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C'\sigma_C) + \mu_G + \sigma_G'(1-\gamma)\sigma_C$$

2 Long run risk model

2.1 Derivation

We now assume that the evolution of consumption is driven by two state variables μ_t and σ_t:

$$\begin{aligned}
\frac{dC_t}{C_t} &= \mu_t dt + \nu_D\sqrt{\sigma_t}dZ_t \\
d\mu_t &= \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t}dZ_t^\mu \\
d\sigma_t &= \kappa_\sigma(1-\sigma_t)dt + \nu_\sigma\sqrt{\sigma_t}dZ_t^\sigma
\end{aligned}$$

We write $G_t = G(\mu, \sigma)$ and we get the PDE

$$\begin{aligned}
0 = {} & \rho\theta[G^{1-\frac{1}{\theta}} - G] + G((1-\gamma)\mu - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma) \\
& + \kappa_\mu(\bar{\mu}-\mu)\frac{\partial G}{\partial \mu} + \kappa_\sigma(1-\sigma)\frac{\partial G}{\partial \sigma} \\
& + \frac{1}{2}\nu_\mu^2\sigma\frac{\partial^2 G}{\partial \mu^2} + \frac{1}{2}\nu_\sigma^2\sigma\frac{\partial^2 G}{\partial \sigma^2}
\end{aligned}$$

Boundary coundition = reflectiving barrier

$$\partial_\mu G(\underline{u}, \sigma) = 0$$
$$\partial_\mu G(\overline{u}, \sigma) = 0$$
$$\partial_\sigma G(u, \underline{\sigma}) = 0$$
$$\partial_\sigma G(u, \overline{\sigma}) = 0$$

I think the best way to incorporate these boundary conditions is to adopt an upwinding scheme. The upwinding scheme says to approximate the first derivative by the forward difference when the drift of a state variable is positive, and by the backward difference when the drift of a state variable is negative.

$$\begin{aligned}
0 = {} & \rho\theta[(G_{ij})^{1-\frac{1}{\theta}} - G_{ij}] + G_{ij}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
& + \kappa_{\mu_i}(\bar{\mu} - \mu_i)^+ \frac{G_{i+1,j} - G_{i,j}}{\Delta\mu} + \kappa_{\mu_i}(\bar{\mu} - \mu_i)^- \frac{G_{i,j} - G_{i-1,j}}{\Delta\mu} \\
& + \kappa_{\sigma_j}(1 - \sigma_j)^+ \frac{G_{i,j+1} - G_{i,j}}{\Delta\sigma} + \kappa_{\sigma_j}(1 - \sigma_j)^- \frac{G_{i,j} - G_{i,j-1}}{\Delta\sigma} \\
& + \frac{1}{2}\nu_{\mu_i}^2\sigma_j \frac{G_{i+1,j} - 2G_{i,j} + G_{i-1,j}}{(\Delta\mu)^2} + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j \frac{G_{i,j+1} - 2G_{i,j} + G_{i,j-1}}{(\Delta\sigma)^2}
\end{aligned}$$

At the frontier of the state space, the second order derivative uses the value of G at nodes not on the grid ("ghost nodes"): set the value of these nodes to the value of nodes at the frontier.

Denote y the vector of $(G_{ij})_{1 \le i,j \le n})$. The scheme can be solved in two ways

- Use a non linear solver to solve $F(y) = 0$, using for instance the Powell or Newton methods. Both methods require to specify the Jacobian of F.
- Use a ODE solver to solve $F(y) = \dot{y}$. The solution when $T \to +\infty$ is the solution of the PDE. Some MATLAB solvers like ode23s also accept the Jacobian of F as an imput. This makes the solution faster, but this requires to program a little bit more.

2.2 Comparaison

Name	BY04	BY04	This paper	This paper	Link
mean growth rate	μ	0.0015	$\bar{\mu}$	0.0015	$\mu = \bar{\mu}$
mean volatility	σ^2	0.00006084	ν_D	0.0078	$\sqrt{\sigma^2} = \nu_D$
growth persistence	ρ	0.979	κ_μ	0.0212	$-\log(\rho) = \kappa_\mu$
volatility persistence	ν_1	0.987	κ_σ	0.0131	$-\log(\nu_1) = \kappa_\sigma$
growth rate volatility	φ_e	0.044	ν_μ	0.0003432	$\varphi_e \times \sqrt{\sigma^2} = \nu_\mu$
volatility volatility	σ_w	0.0000023	ν_σ	0.0378	$\sigma_w/\sigma^2 = \nu_\sigma$
time discount	δ	0.998	ρ	0.002	$-\log(\delta) = \rho$
RRA	$1-\gamma$(RRA)	7.5 or 10	$1-\gamma$	-6.5 or -9	$1 - \text{RRA} = 1-\gamma$
IES	ψ	1.5	ψ	1.5	$\psi = \psi$

Also, $\theta = (1-\gamma)/(1-1/\psi)$ = -19.50 or -27. Let's express the consumption to wealth ratio k_t in term of state variables.

$$V = G_t k_t^{1-\gamma} \frac{W^{1-\gamma}}{(1-\gamma)}$$

FOC for consumption can be written

$$k_t = \rho^{\psi} k_t^{1-\psi} G_t^{\frac{1-\psi}{1-\gamma}}$$

General equilibrium gives

$$k_t = \rho G_t^{-1/\theta}$$

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$$\frac{1}{\theta} \log G_t - \log \rho \approx A_1 \mu_t + A_2 \nu_D^2 \sigma_t$$

$$A1 = \frac{1 - \frac{1}{\psi}}{1 - 0.997 e^{-\kappa_\mu}}$$

$$A2 = 0.5\theta \frac{(1-\frac{1}{\psi})^2 + (A_1 \kappa_1 \frac{\nu_\mu}{\nu_D^2})^2}{1 - 0.997 e^{-\kappa_\sigma}}$$