FINITE DIFFERENCES SCHEMES FOR PDE

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1 Write Monotonous Scheme

- Take the following PDE
$$0 = u_t + F(t, x, v, \partial v, \partial^2 v)$$
with F elliptic, i.e.if $X - Y$ positive semi definite $S(x, v, \partial v, X) \geq S(x, v, \partial v, Y)$
- Denote a scheme a function
$$0 = S(\Delta a, v_j^{n+1}; ...)$$
Barles Souganadis: the scheme must be decreasing in all variables in ..., i.e. typically $v_{i+1}^{n+1}, v_{i-1}^{n+1}, v_{i+1}^n, v_i^n, v_{i+1}^n$ (i.e. current value at current position)
- Mote carlo chain
$$\rho V = \max_u f(x, u) + \alpha(x, u) DV(x) + \sigma^2(x, u) D^2 V(x)$$
Fully explicit finite difference schemes can be rewritten
$$V = \max_u V + \Delta f(x, u) + \sum p(x_k, y|u) V^h(y)$$
ie monotiniciy of explicit scheme in v_{i-1}^{n+1} and v_{i+1}^{n+1} iff can be interpreted as probability Monte carlo chain is a generalization of the finite difference scheme and is the right framework to understand cross derivative. Any monte carlo chain works if one had the following (denonting markov chain ξ_n^h)
$$E \Delta \xi = b(x) \Delta T$$
$$Var \Delta \xi = \sigma^2(x) \Delta T$$
- Another interpretation: if savings are positive, what matters is how the value function changes when wealth increases by a small amount; and vice versa when savings are negative. The right thing to do is therefore to approximate the derivative in the direction of the movement of the state.

1.1 Linear PDE

$$\rho V = f(V, x) + m(x) \partial V + a(x) \partial^2 V$$

HJB is particular kind of pde where $a(x)$ takes the marticular form

$$\rho V = \max_u f(x, u) + m(x, u) \partial V(x) + tr(\sigma \sigma') \partial^2 V(x)$$

How to satisfy monotonicity ?

- Implicit vs explicit One solution may be to usese Δt low enough so that decreasing in v_i^n. this forces the time step to be so small that the rounding error dominates the total computational error. A better solution is to use explicit with upwind scheme : ocnverged afor any Δt

- First derivative by upwind

$$S = \rho v_j - u(c_j) + \frac{v_{j+1} - v_j}{\Delta a} s_j^+ - \frac{v_j - v_{j-1}}{\Delta a} s_j^-$$

$$S = \rho v_j - u(c_j) + \frac{v_j - v_{j-1}}{\Delta a} c_{j,B} - \frac{v_{j+1} - v_j}{\Delta a} (w + r a_j)$$

  It satisfies monotonicity property

  - By construction, decreasing in v_{i-1}^{n+1} v_{i+1}^{n+1}
  - It's also decreasing in v_{i-1}^{n} and v_{i+1}^{n} due to the envelop theorem (c^n appears twice, once in $u(n)$, the other in the drift of asset, and derivative are equal)

- Second derivative

  - Naive approximation is

$$\partial_{ij} v = \frac{v_{i+1,j+1} + v_{i-1,j-1} - v_{i+1,j-1} - v_{i-1,j+1}}{4\Delta x^2}$$

    This is never monotonous because the term $v_{i+1,j-1}$ enters negatively and does not appear anywhere else

  - Better scheme

    * If $a_{ij}(x) \geq 0$

$$\begin{aligned}\partial_{ij} v = \frac{1}{2}(&\frac{v_{i+1,j+1} + v_{i-1,j-1} - 2v_{i,j}}{\Delta x_i \Delta x_j}\\ &- \frac{v_{i+1,j} + v_{i-1,j} - 2v_{i,j}}{(\Delta x_i)^2}\\ &- \frac{v_{i,j+1} + v_{i,j-1} - 2v_{i,j}}{(\Delta x_j)^2})\end{aligned}$$

    * if negative, a classical scheme is

$$\begin{aligned}\partial_{ij} v = \frac{1}{2}(&-\frac{v_{i+1,j-1} + v_{i-1,j+1} - 2v_{i,j}}{\Delta x_i \Delta x_j}\\ &+ \frac{v_{i+1,j} + v_{i-1,j} - 2v_{i,j}}{(\Delta x_i)^2}\\ &+ \frac{v_{i,j+1} + v_{i,j-1} - 2v_{i,j}}{(\Delta x_j)^2})\end{aligned}$$

    * This scheme is monotonous when the negative terms in $v_{i+1,j}$ and $v_{i,j+1}$ are compensated by the diagonal terms of the Hessian. More precisely, the scheme is monotonous iff

$$a_{ii}(x) \geq \sum |a_{ij}(x)|$$

- Rewrite in term of directional derivatives. Denote $a(x)$ such that the second order term is $\sum a_{i,j}(x)\delta_{ij}v$.
  * If $a = \sum \lambda_i \xi_i \xi_i$, the second order term can be rewritten only in term of directional derivatives:

    $$\sum \lambda_i D_\xi^2 v$$

    $D_\xi v$ is the directional derivative along the direction ξ

    $$D_\xi^2 v \approx \frac{v(t, x+\xi\Delta x) + v(t, x-\xi\Delta x) - 2v(x)}{|\xi|^2 \Delta x^2}$$

    The corresponding scheme is monotonous, similarly to the case with diagonal Hessian. How to find this decomposition?
  * Bonnans. Choose an order p and solve the problem

    $$\begin{aligned} &\min_{\lambda_i} ||a - \sum_{\xi \in \Xi_p} \lambda_i \xi_i \xi_i'||^2 \\ &\Xi_p = \{\xi \in Z^d \text{ s.t. } \max_i |\xi_i^d| \leq p\} \\ &\lambda_i \geq 0 \end{aligned}$$

    p needs to be equal to 5 so that mistake is not too large.
  * Me. For PDE obtained by HJB only. Denote $\sigma(s_t)$ the matrix such that $ds_t = \mu dt + \sigma dW_t$, the second order term has the form

    $$Tr(\sigma\sigma^T D^2 V)$$

    The matrix $\sigma\sigma^T$ is symetric therefore diagonalizable. There exists positive λ_i and directions ξ_i such that

    $$\sigma\sigma^T = \sum \lambda_i \xi_i \xi_i^T$$

    The issue is that $\xi \notin Z^d$. One can rewrite this as a scheme on the grid by using use a linear interpolation to approximate $v(t, x+\beta\Delta x)$ as a weighted average of points of the grid (with positive weights). One can always normalize v so that $v_1 = 1$. In particular, in two dimensions, one only needs to interpolate linearly along the second dimension, which simplifies the code (see image2015831195641.png)

1.2 Linear PDE with controls

$$\rho V = \sup_u f(x,u) + m(x,u)\partial V(x) + a(x,u)\partial^2 V(x)$$

Simplest solution is to substitute u by FOC

- The resulting scheme is monotonous Why? envelop theorem.

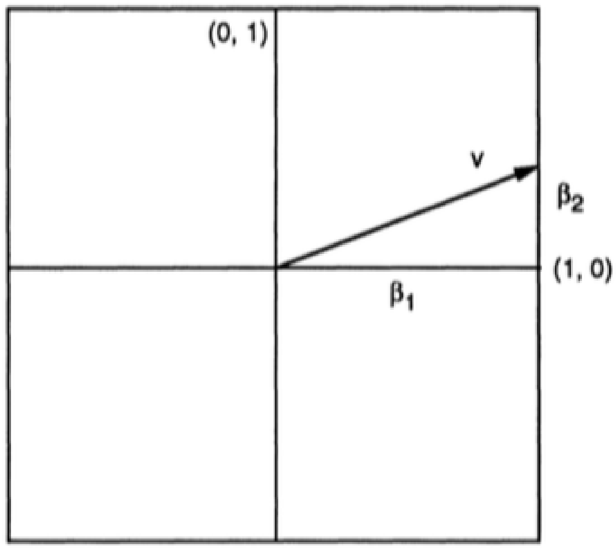

Figure 1: caption

- One caveat: you have to make sure you're using the same derivatives to compute the control and in the rest of the PDE so that envelop theorem applies. Since v is generally concave in a, $s_F \leq s_B$, where s_F is the drift computed using forward difference for consumption and s_B the backward. Therefore the following situation happens: positive difference for ∂V when computing c gives negative drift, while positive difference gives positive drift). For these grid points, set the derivative of the value function equal to the case where drift is zero (i.e stationary value)
- This circularity becomes too complicated with multiple controls . In this case, split the drift in two

  $$S = \rho v_j - u(c_j) + \frac{v_j - v_{j-1}}{\Delta a} c_{j,B} - \frac{v_{j+1} - v_j}{\Delta a}(w + ra_j)$$

  It is done in the Moll notes with fixed cost

  $$\rho V = \max_c u(c) + V_b(r^b - \chi(d,a) - c) + V_a(r^a a + d) V_b \chi_d(d,a) = V_a$$

- Discrete maximiation is handled through splitting method (optimial time / fixed cost). Suppose there is fixed cost to convert illiquid asset a into liguid asset. Denote $v^*(w)$ the value function conditionally to paying the fixed cost. It is only a function of wealth.

$$v^*(a+b) \equiv \max_{a',b'} v(a',b')$$
$$a' + b'* = a + b - \kappa$$

Then the HJB equation is the same, for v but we have new constraint

$$\forall a, b v(a,b) \geq v^*(a+b)$$

THis problem is solved in two step

$$v^{n+1/2} \ldots \text{(usual HJB)}$$
$$v^{n+1}(a',b') = \max(\max_{a'+b'=a+b-\kappa} v^{n+1/2}(a',b'), v^{n+1/2}(a',b'))$$

1.3 Non linear PDE

$$\rho V = f(x,V) + m(x, \partial V(x))\partial V(x) + a(x, \partial V(x))\partial^2 V(x)$$

- Case of ditella μ_x depends on derivative ξ_ν etc. The same situation appears when substituting out optimal control by derivative, but in this case the envelop theorem does not apply
- Choose different direction in $\partial V(x)$ (even within a grid) for different terms, so that we make sure everything works

Is there existence theorem for the scheme?

2 Solve Finite Difference scheme

2.1 Non Linear Scheme

$$\rho V = f(x,V) + m(x)\partial V(x) + a(x)\partial^2 V(x)$$

- This typically happens with Epstein Zin utility (Bansal Yaron) or with optimal control (see below)
- The fully explicit scheme becomes a non linear problem. This can be reduced into a squanece of linear problem using Newton method (or any other method).
- The newton step looks like a semi explicit scheme with time iteration.

2.2 Borders

- Upwind scheme has an adantage when using boundary conditions
  - saving is negative at the top of state space therefore forward difference is not used and no boundary condition needs to be imposed
  - since upwind scheme selects a particular derivative when saving (ie drift) is negative, and since condition is about the drift, then we just replace the backward difference derivative by $u'(z + ra_1)$.
- First derivative $v_{I+1} - v_I$ is handled through upwiding
  - Exogeneous state constraint typically does not appear thanks to upwinding (i.e. when we're at the border of state, the drift is mean reverting)
  - True endogeneous state constraint (like $a \geq 0$) or artificial endogeneous state constraint (like $a \leq a_m ax$ - not really necessary but "sometimes helps numerical stability") Since FOC holds even at the constraint, a constraint of c is just a constraint on v'.

    $$v'(\underline{a}) \geq u'(z + r\underline{a})$$

    Note that this constraint binds iff the saving is negative. Thanks to upwind scheme, the boundary condition is therefore

    $$\partial v_1^b = u'(z + r\underline{a})$$
- Second derivative $v_{I+1} + v_{I-1} - 2v_I$
  - if exogeneous mean reverting state variable (like μ), use reflecting barrier $v'(x) = 0$ and therefore $v_{I+1} = v_I$.
  - if endogeneous state variable
    - policy funciton $c(a_{max})$ is approximately linear, and so from FOC this allows to express second derivative in term of first derivative

      $$v''(a_{max}) = -\gamma v'(a_{max})^{1+1/\gamma} \overline{c}$$