1 Derivation

HJB is

$$0 = \rho\theta V_t\left(\frac{C_t^{\frac{1-\gamma}{\theta}}}{((1-\gamma)V_t)^{\frac{1}{\theta}}} - 1\right) + \frac{E[dV_t]}{dt}$$

Let's define G_t such that

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma} G_t$$

By Ito,

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t E\frac{\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt} + E\frac{dG_t}{dt} + E\frac{dG_t\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt}$$

Denoting μ_C and σ_C the geometric drift of C_t, we have

$$\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}} = ((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C^2)dt + (1-\gamma)\sigma_C dW_t$$

Injecting this expression into HJB and denoting μ_G, σ_G the arithmetic drift and volatility of C_t

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C'\sigma_C) + \mu_G + \sigma_G'(1-\gamma)\sigma_C$$

2 Long run risk model

2.1 Derivation

We now assume that the evolution of consumption is driven by two state variables μ_t and σ_t:

$$\begin{aligned}
\frac{dC_t}{C_t} &= \mu_t dt + \nu_D\sqrt{\sigma_t}dZ_t \\
d\mu_t &= \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t}dZ_t^\mu \\
d\sigma_t &= \kappa_\sigma(1 - \sigma_t)dt + \nu_\sigma\sqrt{\sigma_t}dZ_t^\sigma
\end{aligned}$$

We write $G_t = G(\mu, \sigma)$ and we get the PDE

$$\begin{aligned}
0 = \rho\theta[G^{1-\frac{1}{\theta}} - G] &+ G((1-\gamma)\mu - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma) \\
&+ \kappa_\mu(\bar{\mu} - \mu)\frac{\partial G}{\partial \mu} + \kappa_\sigma(1-\sigma)\frac{\partial G}{\partial \sigma} \\
&+ \frac{1}{2}\nu_\mu^2\sigma\frac{\partial^2 G}{\partial \mu^2} + \frac{1}{2}\nu_\sigma^2\sigma\frac{\partial^2 G}{\partial \sigma^2}
\end{aligned}$$

Boundary coundition = reflectiving barrier

$$
\begin{aligned}
\partial_\mu G(\underline{u}, \sigma) = 0 \\
\partial_\mu G(\overline{u}, \sigma) = 0 \\
\partial_\sigma G(u, \underline{\sigma}) = 0 \\
\partial_\sigma G(u, \overline{\sigma}) = 0
\end{aligned}
$$

We can solve this PDE through the following finite difference scheme:

$$
\begin{aligned}
0 &= \rho\theta[(G_{ij}^{n+1})^{1-\frac{1}{\theta}} - G_{ij}^{n+1}] + G_{ij}^{n+1}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
&+ \kappa_{\mu_i}(\bar{\mu} - \mu_i)^+(G_{i+1,j}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\mu_i}(\bar{\mu} - \mu_i)^-(G_{i,j}^{n+1} - G_{i-1,j}^{n+1}) \\
&+ \kappa_{\sigma_j}(1 - \sigma_j)^+(G_{i,j+1}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\sigma_j}(1 - \sigma_j)^-(G_{i,j}^{n+1} - G_{i,j-1}^{n+1}) \\
&+ \frac{1}{2}\nu_{\mu_i}^2\sigma_j(G_{i+1,j}^{n+1} - 2G_{i,j}^{n+1} + G_{i-1,j}^{n+1}) + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j(G_{i,j+1}^{n+1} - 2G_{i,j}^{n+1} + G_{i,j-1}^{n+1})
\end{aligned}
$$

with usual ghost nodes to satisfy boundary conditions. The scheme satisfies the monotonicity condition of the Barles-Souganadis Theorem.

- monotonicity in ${G_{i+1,j}}_{i-1}^{n+1}, {G_{i-1,j}}_{i-1}^{n+1}, {G_{i,j-1}}_{i-1}^{n+1}, {G_{i,j-1}}_{i+1}^{n+1}$ by upwinding
- monoticity in $G_{i,j}^n$ because ...there is no term in $G_{i,j}^n$. We do need a fully explicit scheme : if $(G_{ij}^{n+1})^{1-\frac{1}{\theta}}$ was replaced by $(G_{ij}^n)^{1-\frac{1}{\theta}}$, the scheme would not be decreasing in G_{ij}^n for $\theta < 0$.

Contrary to the schemes of Achdou et al., 2014, the scheme contains a non linear term in G_{ij}^{n+1}. We can solve this non-linear scheme by Newton method (first order taylor approximation of the non linear term), i.e. by iterating

$$
\begin{aligned}
0 &= \rho(G_{ij}^n)^{1-\frac{1}{\theta}} + \rho(\theta - 1)(G_{ij}^n)^{-\frac{1}{\theta}}G_{ij}^{n+1} \\
&- \rho\theta G_{ij}^{n+1} + G_{ij}^{n+1}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
&+ \kappa_{\mu_i}(\bar{\mu} - \mu_i)^+(G_{i+1,j}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\mu_i}(\bar{\mu} - \mu_i)^-(G_{i,j}^{n+1} - G_{i-1,j}^{n+1}) \\
&+ \kappa_{\sigma_j}(1 - \sigma_j)^+(G_{i,j+1}^{n+1} - G_{i,j}^{n+1}) + \kappa_{\sigma_j}(1 - \sigma_j)^-(G_{i,j}^{n+1} - G_{i,j-1}^{n+1}) \\
&+ \frac{1}{2}\nu_{\mu_i}^2\sigma_j(G_{i+1,j}^{n+1} - 2G_{i,j}^{n+1} + G_{i-1,j}^{n+1}) + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j(G_{i,j+1}^{n+1} - 2G_{i,j}^{n+1} + G_{i,j-1}^{n+1})
\end{aligned}
$$

This defines a linear semi explicit scheme [1]. Note that this scheme does not satisfy the monotonicity condition of the Barles-Souganadis Theorem since the derivative of the scheme wrt G_{ij}^n is

$$
\rho(1-\theta)(G_{ij}^n)^{-\frac{1}{\theta}}(1 - \frac{G_{ij}^{n+1}}{G_{ij}^n})
$$

[1] Actually, the schemes used in Achdou et al., 2014 can be obtained in this way.

Yet, this scheme converges because (i) the non linear fully implicit scheme satisfies Barles-Souganadis Theorem (ii) Newton method converges to the non linear scheme

Instead of the Newton method, we could use another off the shelf non linear solver. The NLsolve package in Julia uses the Powell Dog-leg method. In this case, the updating step is a mix of gradient and Newton steps.

2.2 Comparaison

Name	BY04	BY04	This paper	This paper	Link
mean growth rate	μ	0.0015	$\bar{\mu}$	0.0015	$\mu = \bar{\mu}$
mean volatility	σ^2	0.00006084	ν_D	0.0078	$\sqrt{\sigma^2} = \nu_D$
growth persistence	ρ	0.979	κ_μ	0.0212	- $\log(\rho) = \kappa_\mu$
volatility persistence	ν_1	0.987	κ_σ	0.0131	$-\log(\nu_1) = \kappa_\sigma$
growth rate volatility	φ_e	0.044	ν_μ	0.0003432	$\varphi_e \times \sqrt{\sigma^2} = \nu_\mu$
volatility volatility	σ_w	0.0000023	ν_σ	0.0378	$\sigma_w/\sigma^2 = \nu_\sigma$
time discount	δ	0.998	ρ	0.002	$-\log(\delta) = \rho$
RRA	$1-\gamma$(RRA)	7.5 or 10	$1-\gamma$	-6.5 or -9	$1 - \text{RRA} = 1-\gamma$
IES	ψ	1.5	ψ	1.5	$\psi = \psi$

Also, $\theta = (1-\gamma)/(1-1/\psi) =$ -19.50 or -27. Let's express the consumption to wealth ratio k_t in term of state variables.

$$V = G_t k_t^{1-\gamma} \frac{W^{1-\gamma}}{(1-\gamma)}$$

FOC for consumption can be written

$$k_t = \rho^\psi k_t^{1-\psi} G_t^{\frac{1-\psi}{1-\gamma}}$$

General equilibrium gives

$$k_t = \rho G_t^{-1/\theta}$$

Bansal Yaron find

$$\frac{1}{\theta}\log G_t - \log\rho \approx A_1\mu_t + A_2\nu_D^2\sigma_t$$

$$A1 = \frac{1-\frac{1}{\psi}}{1-0.997e^{-\kappa_\mu}}$$

$$A2 = 0.5\theta\frac{(1-\frac{1}{\psi})^2 + (A_1\kappa_1\frac{\nu_\mu}{\nu_D^2})^2}{1-0.997e^{-\kappa_\sigma}}$$