1 Derivation

HJB is

$$0 = \rho\theta V_t\left(\frac{C_t^{\frac{1-\gamma}{\theta}}}{((1-\gamma)V_t)^{\frac{1}{\theta}}} - 1\right) + \frac{E[dV_t]}{dt}$$

Let's define G_t such that

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma}G_t$$

By Ito,

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t E\frac{\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt} + E\frac{dG_t}{dt} + E\frac{dG_t\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt}$$

Denoting μ_C and σ_C the geometric drift of C_t, we have

$$\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}} = ((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C^2)dt + (1-\gamma)\sigma_C dW_t$$

Injecting this expression into HJB and denoting μ_G, σ_G the arithmetic drift and volatility of G_t

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C'\sigma_C) + \mu_G + \sigma_G'(1-\gamma)\sigma_C$$

2 Long run risk model

2.1 Derivation

We now assume that the evolution of consumption is driven by two state variables μ_t and σ_t:

$$\begin{aligned}
\frac{dC_t}{C_t} &= \mu_t dt + \nu_D\sqrt{\sigma_t}dZ_t \\
d\mu_t &= \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t}dZ_t^\mu \\
d\sigma_t &= \kappa_\sigma(1 - \sigma_t)dt + \nu_\sigma\sqrt{\sigma_t}dZ_t^\sigma
\end{aligned}$$

We write $G_t = G(\mu, \sigma)$ and we get the PDE

$$\begin{aligned}
0 = \rho\theta[G^{1-\frac{1}{\theta}} - G] + G((1-\gamma)\mu - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma) \\
+ \kappa_\mu(\bar{\mu} - \mu)\frac{\partial G}{\partial \mu} + \kappa_\sigma(1-\sigma)\frac{\partial G}{\partial \sigma} \\
+ \frac{1}{2}\nu_\mu^2\sigma\frac{\partial^2 G}{\partial \mu^2} + \frac{1}{2}\nu_\sigma^2\sigma\frac{\partial^2 G}{\partial \sigma^2}
\end{aligned}$$

2.1.1 Finite Difference Method

We can discretize this PDE on a grid using a Finite Difference Scheme.

$$
\begin{aligned}
0 &= \rho\theta[(G_{ij})^{1-\frac{1}{\theta}} - G_{ij}] + G_{ij}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
&+ \kappa_{\mu_i}(\bar{\mu} - \mu_i)\frac{G_{i+1,j} - G_{i+1,j}}{2\Delta\mu} \\
&+ \kappa_{\sigma_j}(1 - \sigma_j)\frac{G_{i,j+1} - G_{i,j-1}}{2\Delta\sigma} \\
&+ \frac{1}{2}\nu_{\mu_i}^2\sigma_j\frac{G_{i+1,j} - 2G_{i,j} + G_{i-1,j}}{(\Delta\mu)^2} + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j\frac{G_{i,j+1} - 2G_{i,j} + G_{i,j-1}}{(\Delta\sigma)^2}
\end{aligned}
$$

At the border of the grid, we remove the second derivative term and we use a second order approximation for the first derivative. For instance at $i = 1$, the scheme is

$$
\begin{aligned}
0 &= \rho\theta[(G_{1j})^{1-\frac{1}{\theta}} - G_{1j}] + G_{1j}((1-\gamma)\mu_1 - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
&+ \kappa_{\mu_1}(\bar{\mu} - \mu_1)\frac{-(3G_{0,j} - 4G_{1,j} + G_{2,j})}{2\Delta\mu} \\
&+ \kappa_{\sigma_j}(1 - \sigma_j)\frac{G_{1,j+1} - G_{1,j-1}}{2\Delta\sigma} \\
&+ \frac{1}{2}\nu_{\sigma_j}^2\sigma_j\frac{G_{1,j+1} - 2G_{1,j} + G_{1,j-1}}{(\Delta\sigma)^2}
\end{aligned}
$$

Denote Y the vector of $(G_{ij})_{1\leq i,j\leq n}$. The scheme defines a function F such that $F(Y) = 0$. We can solve for Y using one of these methods:

1. Use a non linear solver for the system $F(Y) = 0$
2. Use an ODE solver for the system $F(Y) = \dot{Y}$. The solution when $T \to +\infty$ is the solution of the PDE. This method is called "the method of lines".
3. A solution that would *not* work is to solve for G by iterating over time:

   $$\frac{G_{n+1} - G_n}{\Delta t} = F(G_n)$$

   This method can be seen as a special case of a non linear solver (in this context, it is called the Newton Method) or as a special case of an ODE solver (in this context, it is called the Euler method). The criterion for the convergence of such a method is that F is monotonous in G (Barles Souganadis theorem). This is not the case here, due to the non linear term $\rho\theta[(G_{ij})^{1-\frac{1}{\theta}} - G_{ij}]$

For the initial guess, we use the value function for the stationary problem $\sigma = 1$ and $\mu = \overline{\mu}$

2.1.2 Spectral Method (= collocation method)

Another way to solve the PDE is to look for a solution of the form

$$V(\mu, \sigma) = \sum_{kl} a_{kl} \phi_k(\mu) \psi_l(\sigma)$$

where ϕ, ψ denote any basis of functions (splines or chebyshev polynomials). The value function is characterized by its coordinates a_{kl} on this basis. Writing the PDE on a set of points gives a non linear system in term of the coordinates. Compared to the Finite Difference method, the non linear system solves for a_{kl} rather than V_{ij}. We use the same border condition and initial guess as the Finite Difference method.

2.2 Comparison

Name	BY04	BY04	This paper	This paper	Link
mean growth rate	μ	0.0015	$\bar{\mu}$	0.0015	$\mu = \bar{\mu}$
mean volatility	σ^2	0.00006084	ν_D	0.0078	$\sqrt{\sigma^2} = \nu_D$
growth persistence	ρ	0.979	κ_μ	0.0212	$-\log(\rho) = \kappa_\mu$
volatility persistence	ν_1	0.987	κ_σ	0.0131	$-\log(\nu_1) = \kappa_\sigma$
growth rate volatility	φ_e	0.044	ν_μ	0.0003432	$\varphi_e \times \sqrt{\sigma^2} = \nu_\mu$
volatility volatility	σ_w	0.0000023	ν_σ	0.0378	$\sigma_w/\sigma^2 = \nu_\sigma$
time discount	δ	0.998	ρ	0.002	$-\log(\delta) = \rho$
RRA	$1-\gamma$(RRA)	7.5 or 10	$1-\gamma$	-6.5 or -9	$1 - \text{RRA} = 1-\gamma$
IES	ψ	1.5	ψ	1.5	$\psi = \psi$

Also, $\theta = (1-\gamma)/(1-1/\psi) =$ -19.50 or -27. Let's express the wealth to consumption ratio K_t in term of state variables.

$$V = G_t K_t^{\gamma-1} \frac{W^{1-\gamma}}{(1-\gamma)}$$

FOC for consumption can be written

$$K_t^{-1} = \rho^\psi K_t^{\psi-1} G_t^{\frac{1-\psi}{1-\gamma}}$$

We conclude

$$K_t = \rho^{-1} G_t^{1/\theta}$$

Bansal Yaron find

$$\log K_t \propto A_1 \mu_t + A_2 \nu_D^2 \sigma_t$$

$$A1 = \frac{1-\frac{1}{\psi}}{1-0.997e^{-\kappa_\mu}}$$

$$A2 = 0.5\theta \frac{(1-\frac{1}{\psi})^2 + (A_1 0.997 \frac{\nu_\mu}{\nu_D})^2}{1-0.997e^{-\kappa_\sigma}}$$