Matching and Saving in Continuous Time

Bayer and Walde

Sargent Reading Group July 19, 2017

MAIN POINT: INTERESTING:

> Our paper additionally differs from Bayer and Waelde's in the following three respects. First, they focus on stationary problems only whereas we also analyze transition dynamics. Second, they study a version of the model with the "natural borrowing constraint" implying that households never actually hit that constraint. In contrast, we also study tighter constraints and obtain two theoretical results concerning the speed at which this constraint is hit and how it affects the equilibrium interest rate. Third and on a more technical level, they characterize households' saving behavior in terms of a differential equation for its consumption policy function. In contrast, we do this in terms of the HJB equation for the value function, the standard formulation in optimal control theory, allowing us to take advantage of advances made in this field. Of course the two approaches are related and their equation for consumption can be derived from the envelope condition of the HJB equation.

1. MODEL

Definition. A stochastic process $q(t)$ is a Poisson process with arrival rate λ if

(i) *q(0) = 0*

(ii) *the process has independent increments*

(iii) *in any interval of length $\tau - t$ the increment $q(\tau) - q(t)$ is Poisson distributed with mean $\lambda(\tau - t)$*

The most common way to present the Poisson process is by looking at the distribution of the increment $q(t+h) - q(t)$ over a very small interval h. In case the process $q(t)$ is Poisson, its increment $dq(t)$ is characterized by

$$dq(t) = \begin{cases} 0 & \text{with prob } 1 - \lambda dt \\ 1 & \text{with prob } \lambda dt \end{cases} \tag{1.1}$$

Let wealth of the household evolve according to

$$da(t) = \{ra(t) + z(t) - c\}dt \tag{1.2}$$

Labor income $z \in \{b, w\}$ follows a stochastic differential equation

$$dz(t) = \Delta dq_\mu - \Delta dq_s, \quad \Delta \equiv w - b \tag{1.3}$$

Hamilton-Jacobi-Bellman

$$\rho V(a_t, z_t) = \max_{c_t} \left\{ u(c_t) + \frac{1}{\mathrm{d}t} \mathbb{E}_t \, \mathrm{d}V(a_t, z_t) \right\} \tag{1.4}$$

Computing the differential $\mathrm{d}V(a_t, z_t)$ and forming expectations gives

$$\rho V(a,z) = \max_c Big\{u(c) + V_a(a,z)\{ra + z - c\}Big\} + \mu(z)\Big[V(a,w) - V(a,z)\Big] + s(z)\Big[V(a,b) - V(a,z)\Big] \tag{1.5}$$

$$\underline{\text{foc}}\text{: } V_a(a,z) = u'(c(a,z)) \tag{1.6}$$

Next we compute $\mathrm{d}V_a(a,z)$

$$dV_a(a,z) = V_{aa}(a,z)\{ra + z - c\}dt + \Big[V_a(a,w) - V_a(a,z)\Big]dq_\mu + \Big[V_a(a,b) - V_a(a,z)\Big]dq_s \tag{1.7}$$

Differentiating the HJB with respect to a yields

$$(\rho - r)V_a(a,z) = s(z)\Big[V_a(a,b) - V_a(a,z)\Big] + \mu(z)\Big[V_a(a,w) - V_a(a,z)\Big] + \{ra + z - c\}dtV_{aa}(a,z) \tag{1.8}$$

Combining (1.7) with (1.8) we have

$$dV_a(a,z) = \Big[(\rho - r)V_a(a,z) - s(z)\Big[V_a(a,b) - V_a(a,z)\Big] - \mu(z)\Big[V_a(a,w) - V_a(a,z)\Big]\Big] + \\ + \Big[V_a(a,w) - V_a(a,z)\Big]dq_\mu + \Big[V_a(a,b) - V_a(a,z)\Big]dq_s$$

Replacing the shadow price for the foc and using the CVF one more time we get that consumption for the employed individual follows

$$dc(a_w, w) = -\frac{u'\big(c(a_w,w)\big)}{u''\big(c(a_w,w)\big)}\left\{r - \rho + s\left[\frac{u'\big(c(a_w,b)\big)}{u'\big(c(a_w,w)\big)} - 1\right]\right\}dt + \Big[c(a_w,b) - c(a_w,w)\Big]dq_s \tag{1.9}$$

while for the unemployed

$$dc(a_b, b) = -\frac{u'\big(c(a_b,b)\big)}{u''\big(c(a_b,b)\big)}\left\{r - \rho + \mu\left[\frac{u'\big(c(a_b,w)\big)}{u'\big(c(a_b,b)\big)} - 1\right]\right\}dt + \Big[c(a_b,w) - c(a_b,b)\Big]dq_\mu \tag{1.10}$$

COMMENTS:

- additional s term — consumption growth is faster under the risk of a job loss
- additional term contains ratio of marginal utilities — suggests that it relates to *precautionary savings* (Why?). If relative consumption shrinks as wealth rises $\frac{d}{da}\frac{c(a,w)}{c(a,b)} < 0$, reducing this gap and smoothing consumption is best achieved by fast capital accumulation, which goes hand in hand in equilibrium with fast consumption growth.

Discrete time analogue

Consider the discrete time analogue

$$1 = \frac{1 + r_t}{1 + \rho}\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\right] \tag{1.11}$$

Assume that log per capita consumption follows a random walk with drift

$$c_t = \mu + c_{t-1} + \sigma_c\varepsilon_t, \quad \varepsilon \sim \mathcal{N} \tag{1.12}$$

Under these assumptions, it is not hard to show that

$$\mathbb{E}_t\Delta c_{t+1} = \frac{1}{\sigma}\Big[\log(1 + r_t) - \log(1 + \rho)\Big] + \frac{1}{2}\sigma\sigma_c^2 \tag{1.13}$$

where we see that future consumption risk, captured in the last term of (1.13) tilts the consumption profile upward in expectation, relative to the consumption growth rule derived under no uncertainty. Consumption growth is higher in the presence of risk since households find it optimal to postpone consumption for precautionary motives.[a]

[a]This was taken from Krueger (2016).

1.1. PHASE DIAGRAM

Note by looking at (1.9) that in periods between jumps

$$\frac{dc(a_w,w)}{dt} \geq 0 \Leftrightarrow \frac{u'\big(c(a_w,b)\big)}{u'\big(c(a_w,w)\big)} = 1 - \frac{r - \rho}{s} \tag{1.14}$$

so consumption if the employed workes rises iff current consumption relative to consumption in the other state is sufficiently high.

Assumption: Relative consumption falls in wealth $\frac{\mathrm{d}}{\mathrm{d}a}\frac{c(a,b)}{c(a,w)} < 0$.

Proposition *Consider the case $0 < r \leq \rho$. Define the threshold level a_w^* by*

$$\frac{u'\left(c(a_w^*, b)\right)}{u'\left(c(a_w^*, w)\right)} = 1 - \frac{r - \rho}{s} \tag{1.15}$$

Then (i) Consumption of employed workers is increasing on $[\underline{a}, a_w^*]$ and decreasing $a > a_w^*$; (ii) consumption of unemployed workers always decrease

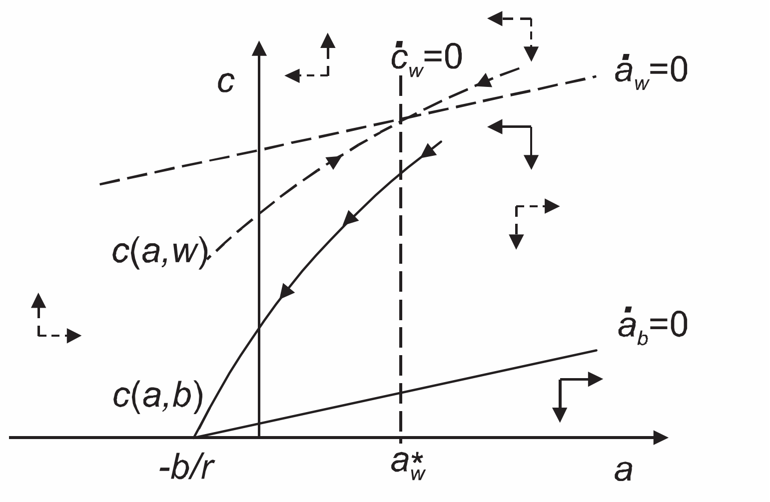
The dynamics of consumption and wealth can be illustrated in the wealth-consumption space. This illustration results from focusing on the evolution between jumps and by eliminating time as exogenous variable. How we do that? Compute the derivatives of consumption with respect to wealth and consider wealth as the exogenous variable to obtain a two-dimensional system of non-autonomous ODEs. The dynamics between jumps therefore follows

$$\frac{dc(a,w)}{da}\{ra + w - c(a,w)\} = -\frac{u'\left(c(a,w)\right)}{u''\left(c(a,w)\right)}\left\{r - \rho + s\left[\frac{u'\left(c(a,b)\right)}{u'\left(c(a,w)\right)} - 1\right]\right\} \tag{1.16}$$

$$\frac{dc(a,b)}{da}\{ra + w - c(a,b)\} = -\frac{u'\left(c(a,b)\right)}{u''\left(c(a,b)\right)}\left\{r - \rho + \mu\left[\frac{u'\left(c(a,w)\right)}{u'\left(c(a,b)\right)} - 1\right]\right\} \tag{1.17}$$

Properties of this system can be illustrated in the usual phase diagram. The zero line on wealth follows from the budget constraint. The zero line from consumption follows from the proposition. The intersection between the zero motion lines fro $c(a_w, w)$ and a_w is the *temporary steady state*. The arrow-pairs for the employed worker shows that one can draw a saddle-path through the TSS. To the left, consumption and wealth rise, while to the right they fall.

POLICIES



Note:

Analytical solution to FPE. Achdou et al. (2016) An analytic solution to the Kolmogorov forward equation characterizing the stationary distribution with two income types for any given individual saving policy function. Summing the Kolmogorov Forward equation for the two income types we have

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\left[s(a, z_1)g(a, z_1) + s(a, z_2)g(a, z_2)\right] \tag{1.18}$$

for all a which implies that $s(a, z_1)g(a, z_1) + s(a, z_2)g(a, z_2)$ is constant. Since it has to be zero as $a \to \infty$, we have

$s(a,z_1)g(a,z_1)+s(a,z_2)g(a,z_2)=0$ for all a. Substituting this relationship on XX we have

$$0=-\frac{\mathrm{d}}{\mathrm{d}a}\left[s(a,z_i)g(a,z_i)\right]-\lambda_i g(a,z_i)+\lambda_j\frac{s(a,z_i)g(a,z_i)}{s(a,z_j)}\Leftrightarrow$$

$$g'(a,z_i)=-\left(\frac{s'(a,z_i)}{s(a,z_i)}+\frac{\lambda_i}{s(a,z_i)}+\frac{\lambda_j}{s(a,z_j)}\right)g(a,z_i) \tag{1.19}$$

Note that (1.19) are two independent ODEs for $g(\cdot,z_i)$ and they can be solved separately

$$g(a,z_i)=\frac{s(a_0,z_i)g(a_0,z_i)}{s(a,z_i)}\exp\left(-\int_{a_0}^{a}\left(\frac{\lambda_i}{s(u,z_i)}+\frac{\lambda_j}{s(u,z_j)}\right)du\right) \tag{1.20}$$

The relevant constant of integration is pinned down by the requirement that the "densities" integrate to one

$$\int_{\underline{a}}^{a_{max}} dG(a,z_1)=\frac{\lambda_2}{\lambda_1+\lambda_2}\qquad \int_{\underline{a}}^{a_{max}} dG(a,z_2)=\frac{\lambda_1}{\lambda_1+\lambda_2} \tag{1.21}$$

> NOTE: As shown in Achdou et al. (2014), the wealth distribution of the low-income type $g(a,z_1)$ features a Dirac mass at the borrowing constraint a (the left boundary of the state space). When discretizing the distribution using a finite difference method, there is technically a Dirac mass at every point in the state space.

References

Yves Achdou, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. Heterogeneous agent models in continuous time, 2016.

Dirk Krueger. *An Introduction to Macroeconomics with Household Heterogeneity (Lecture Notes)*. Publisher, 2016.