Lecture

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References

Theory

- Pham (2009) Continuous-time Stochastic Control – maybe too finance
- Lecture notes
  ↪ Caldentey (????) Stochastic processes and optimal control — nice lecture notes Enio uses them
  ↪ Ross (????) *Stochastic Control in Continuous Time* — alternative to math books Fleming and Soner (2006), Øksendal (2003), Øksendal and Sulem (2007)

Theory (macro)

- Moll's website
- Bayer and Wälde (2015) – recent discovery, dicuss the kind of SDE driven by a Markov chain
  ↪ Sennewald (2007) (theory paper), Walde (2008) (book on intertemporal optimization),
- Stokey (2009) book – *Impulse control Problem*

Numeric

- Achdou, Han, Lasry, Lions, and Moll (2016) (mainly the numerical appendix), Moll's website (tons of examples and materials)
- Forsyth and Vetzal (2012) (Also has some slides) – good introduction to "viscosity solutions"
- Interested? Check applications . . .
  ↪ HANK by Kaplan, Moll, and Violante (2016), **PHACT**
  ↪ Nuño and Moll (2017) (improved notation)
  ↪ Thomas and Nuño (2016) (impulse control)

Consumption Savings Problem

Problem of Household

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \left\{ r a_t + z_t - c_t \right\} dt$$

z_t: is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w - b)dq - (w - b)dQ, \quad q \sim \text{Poisson}(\lambda_1), \ Q \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' consumption and saving decision is summarized by HJB equation

$$\rho v(a, z_k) = \max_c \left\{ u(c) + v_a(a, z)[ra + z_k - c] \right\} + \lambda_k \left[v(a, z_{-k}) - v(a, z_k) \right] \qquad (1)$$

Where this came from? Check *Lagos* lecture notes for an heuristic argument.

Theoretical results analogous to discrete time:

- Value function satisfy the HJB equation
- Verification theorems: solution of HJB $+ \ldots \rightarrow$ value function
- Alternatively, one can show HJB has a unique "nice" solution which is the value function (viscosity solution)

Households
Keynes-Ramsey rules

Before solving the HJB FE let's see what we can do. Analytical results from Bayer and Wälde (2015)

Envelope condition:

$$\rho V_a(a,b) = rV_a(a,b) + V_{aa}(a,b)\{ra + b - c(a,b)\} + \lambda_1 \left[V_a(a,w) - V_a(a,b)\right]$$

Differential of $V_a(a,z)$ — CVF, "Itô formula"

$$\begin{aligned} da_t &= \{ra_t + z_t - c_t\}dt \\ dz_t &= (w-b)dq - (w-b)dQ, \quad q \sim \text{Poisson}(\lambda_1),\ Q \sim \text{Poisson}(\lambda_2) \end{aligned}$$

$$\mathrm{d}V_a(a,b) = \underbrace{V_{aa}\{ra + b - c(a,b)\}}_{\text{normal term}}\mathrm{d}t + \underbrace{\left[V_a(a,w) - V_a(a,b)\right]}_{\text{jump terms}}\mathrm{d}q_t$$

From optimization $V_a(a,z) = u'\left(c(a,z)\right)$. Combining both equations to get rid of V_{aa} we have

$$\begin{aligned} \mathrm{d}u'\left(c(a,b)\right) = &\left\{(\rho - r)u'\left(c(a,b)\right) - \lambda_1 u'\left(c(a,b)\right)\left[\frac{u'\left(c(a,w)\right)}{u'\left(c(a,b)\right)} - 1\right]\right\}\mathrm{d}t + \\ &+ \left[u'\left(c(a,w)\right) - u'\left(c(a,b)\right)\right]\mathrm{d}q_t \end{aligned}$$

Applying "Itô lemma" to get consumption over time

$$\mathrm{d}c(a,b) = \frac{u'\left(c(a,b)\right)}{-u''\left(c(a,b)\right)}\left\{r-\rho-\lambda_1\left[1-\frac{u'\left(c(a,w)\right)}{u'\left(c(a,b)\right)}\right]\right\}\mathrm{d}t + \left[c(a,w)-c(a,b)\right]\mathrm{d}q_t \qquad (2)$$

$$\mathrm{d}c(a,w) = \frac{u'\left(c(a,w)\right)}{-u''\left(c(a,w)\right)}\left\{r-\rho+\underbrace{\lambda_2\left[\frac{u'\left(c(a,b)\right)}{u'\left(c(a,w)\right)}-1\right]}_{\text{prec. savings}}\right\}\mathrm{d}t + \underbrace{\left[c(a,b)-c(a,w)\right]\mathrm{d}Q_t}_{\text{jumps}} \qquad (3)$$

neoclassical growth model $\dot{c}(t) = \frac{u'(c)}{-u''(c)}\left(r-\rho\right)$

Looking at period between jumps. What the signs tell us?

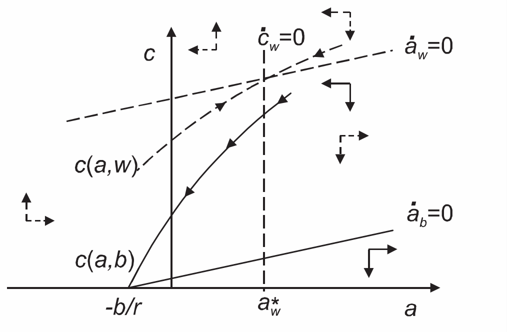
Proposition. *Consider the case* $0 < r \leq \rho$. *Define the threshold level* a_w^* *by*

$$\frac{u'\left(c(a_w^*,b)\right)}{u'\left(c(a_w^*,w)\right)} = 1 + \frac{\rho-r}{\lambda_2} \qquad (4)$$

Then (i) Consumption of employed workers is increasing on $[\underline{a}, a_w^*]$ *and decreasing* $a > a_w^*$; *(ii) consumption of unemployed workers always decrease*

Properties of this system can be illustrated in the usual phase diagram

POLICIES



Note:

Change

- Results help build some intuition on the problem. Look at Bayer and Wälde (2015) for much more. . .
- Now we change the approach.
  Instead of looking at households' saving behavior in terms of a differential equation for its consumption policy function, we will focus on the HJB equation for the value function and how to solve it numerically.
- draw heavily on Moll's notes

Problem of Household

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \left\{ r a_t + z_t - c_t \right\} dt$$

z_t is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w-b)dq_\mu - (w-b)dq_s, \quad q_\mu \sim \text{Poisson}(\lambda_1),\ q_s \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' value function must satisfy HJB equation[1]

$$\rho v_k(a) = \max_c \left\{ u(c) + v_k'(a)[ra + z_k - c] \right\} + \lambda_k \left[v_{\text{-}k}(a) - v_k(a) \right] \tag{5}$$

Borrowing constraint shows only as *state constraint boundary condition*

$$u'\left(c_i(\underline{a})\right) = v_i'(\underline{a}) \geq u'\left(r\underline{a} + z_i\right) \tag{6}$$

which ensures $s_i(\underline{a}) = r\underline{a} + z_i - c_i(\underline{a}) \geq 0$ so that the borrowing constraint is never violated.

[1] change notation

Continuous × Discrete time

Consider the first-order condition for consumption

$$u'\left(c\right) = \partial_a v(a, z) \tag{7}$$

$$u'(c) \geq \beta \int \partial_a v(a', z')\, \mathrm{d}F(z'|z), \quad a' = z + (1 + r)a - c \tag{8}$$

Continuous time advantages:

1. "today" = "tomorrow" — *foc* is static
2. HJB is not stochastic — evolution of stochastic process is captured by additive terms
3. *foc* always holds with equality
   Borrowing constraint shows only as *state constraint boundary condition*

$$u'\left(c_i(\underline{a}, t)\right) = \partial_a v_i(\underline{a}, t) \geq u'\left(r_t \underline{a} + z_i\right) \tag{9}$$

which ensures $c_i(\underline{a}, t) \leq r_t \underline{a} + z_i$ so that the borrowing constraint is never violated.

Numeric solution HJB

Definition

We can write our HJB

$$\rho v_k(a) = \max_c \left\{ u(c) + v_k'(a)[ra + z_k - c] \right\} + \lambda_k \left[v_{\text{-}k}(a) - v_k(a) \right]$$

in a PDE notation

$$0 = F(\boldsymbol{x}, V, DV, D^2 V) \tag{10}$$

where $\boldsymbol{x} := (x, \tau)$. How do we proceed to solve it??

$\hookrightarrow$ **Finite difference methods**: replace derivatives by differences. Simple right?

Suppose we define a grid $\{x_0, x_1, \dots, x_i, \dots\}$ and a set of timesteps $\{i\Delta : i \in \mathbb{N}\}$ Let $V_i^n \approx V(x_i, \tau_n)$ be the approximate value of the solution at node x_i time $\tau^n := T - t$. Then we can write a general **discretization** of the HJB equation at node (x_i, τ^{n+1})

$$0 = S_i^{n+1}\Big((\Delta, \Delta x), V_i^{n+1}, \{V_j^m\}_{m \neq n+1, j \neq i}\Big) \tag{11}$$

Sufficient Conditions Convergence

Condition (Monotonicity) .— The numerical scheme (11) is monotone if

$$S_i^{n+1}(\cdot, V_i^{n+1}, \{Y_j^m\}) \leq S_i^{n+1}(\cdot, V_i^{n+1}, \{Z_j^m\})$$

for all $Y \geq Z$.

Condition (Stability) .— The numerical scheme (11) is stable if for every $\tilde{\Delta} > 0$ it has a solution which is uniformly bounded independently of $\tilde{\Delta}$.

Condition (Consistency) .— The numerical scheme (11) is consistent if for every smooth function ϕ with bounded derivatives we have

$$S_i^{n+1}(\tilde{\Delta}, \phi(\boldsymbol{x}_i^{n+1}), \{\phi(\boldsymbol{x}_j^m)\}) \to F(\boldsymbol{x}, \phi, D\phi, D^2\phi)$$

as $\tilde{\Delta} \to 0$ and $\boldsymbol{x}_i^{n+1} \to \boldsymbol{x}$.

Sufficient Conditions Convergence

Theorem *Barles and Souganidis (1990)* . *If the numerical scheme S (11) satisfies monotonicity, stability and consistency conditions, then its solution converges locally uniformly to the unique viscosity solution of (10).*

- Convergence here is about $\tilde{\Delta} \to 0$
- For given $\tilde{\Delta}$, we have a system of I non-linear equations for each timestep that we must solve somehow. Theorem guarantees that the solution $\{V_i^\tau\}$ of this system converges to the "viscosity solution" of the original PDE as $\tilde{\Delta} \to 0$
- "viscosity solution" of the HJB is the the **value function**

Recall our *time-dependent* HJB equation as

$$\partial_\tau v_k(a,\tau) + \rho v_k(a,\tau) - \sup_{c \in \Gamma_k(a)} \left\{ u(c) + \mathcal{D}^c_{T-\tau} v_k(a,\tau) \right\} = 0 \qquad (12)$$

where

$$\mathcal{D}^c_t \phi_k(a) = \partial_a \phi_k(a)[r_t a + z_k - c] + \lambda_k \left[\phi_{\text{-}k}(a) - \phi_k(a) \right]$$

Define a grid $\{a_1, a_2, \ldots, a_i, \ldots\}$ and let $v^n_k = \left(v_k(a_1, \tau^n), \ldots, v_k(a_i, \tau^n), \ldots \right)'$. Discretizing this equation requires deciding upon

- which fd approximation to use: forward/backward differencing

$$\partial_a v_k(a) \approx \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}, \quad \partial_a v_k(a) \approx \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}},$$

- Implicit/explicit timestepping

Discretization

Back to our example

Let $\mathscr{D}^c$ be the discrete form of the differential operator $\mathcal{D}^c$, so that

$$\left(\mathscr{D}^c v\right)_{k,i} = \alpha_{k,i}(c) v_{k,i-1} + \beta_{k,i}(c) v_{k,i+1} - \Big(\alpha_{k,i}(c) + \beta_{k,i}(c) + \lambda_i\Big) v_{k,i} + \lambda_i v_{\text{-}k,i}$$

and the discretization

$$\frac{v_{k,i}^{n+1} - v_{k,i}^{n}}{\Delta} - \sup_{c \in \Gamma_{k,i}} \left\{ u(c) + \left(\mathscr{D}^c v^{n(+1)}\right)_{k,i} \right\} = 0 \tag{13}$$

where discretization can use forward, backward or central discretization. If

$$\alpha_{k,i}^{n+1} \geq 0, \;\; \beta_{k,i}^{n+1} \geq 0$$

we say that (13) is *positive coefficient discretization*.

Why do we care?

We care because a *positive coefficient discretization* is also *monotone*. To see it check that

$$S_{k,i}^{n+1}\left(\tilde{\Delta}, v_{k,i}^{n+1}, v_{k,i+1}^{n(+1)}, v_{k,i-1}^{n(+1)}, v_{k,i}^{n}, v_{\text{-}k,i}^{n(+1)}\right)$$

is a nonincreasing function of the neighbor nodes $\{v_{\ell,j}^m\}$. Check a example!

Upwind scheme

In order to ensure a *positive coefficient discretization* our choice of central/forward/backward differencing will depend, in general, on the control c. A useful rule for this problem is to use the so-called *upwind scheme*.

IDEA: *Use forward difference whenever drift is positive, and use backward whenever it is negative.*

Suppose that we have the value of consumption $c_{k,i}$ at a particular node. Let $s_{k,i} = ra_i + z_k - c_{k,i}$. In this case, the derivatives are approximated

$$\dots \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} s_{k,i}^{+} + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} s_{k,i}^{-} + \dots$$

which in terms of our α, β

$$\alpha_{k,i}^{up} = -\frac{s_{k,i}^{-}}{a_i - a_{i-1}} \geq 0, \quad \beta_{k,i}^{up} = \frac{s_{k,i}^{+}}{a_{i+1} - a_i} \geq 0$$

But we don't know $c_{k,i}$!!! HJB equation is highly nonlinear, so we need an iterative method to solve it.

Implicit timestepping

Start with a vector v^n and update v^{n+1} according to

$$\frac{v_{k,i}^{n+1} - v_{k,i}^{n}}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^{n}) + \frac{v_{k,i+1}^{n+1} - v_{k,i}^{n+1}}{a_{i+1} - a_i}\left[s_{k,i}^{F,n}\right]^{+} + \frac{v_{k,i}^{n+1} - v_{k,i-1}^{n+1}}{a_i - a_{i-1}}\left[s_{k,i}^{B,n}\right]^{-} + \lambda_k\left[v_{\text{-}k,i}^{n+1} - v_{k,i}^{n+1}\right] \tag{14}$$

- Compute the policy from the foc $\left(u'\left(c_{k,i}^{n}\right) = \partial_a v_{k,i}^{n}\right)$ for the backward AND forward derivative of the value function.
- Define $s_{k,i}^{B,n} = ra_i + z_k - c_{k,i}^{B,n}$, $s_{k,i}^{F,n} = ra_i + z_k - c_{k,i}^{F,n}$. Set

$$c_{k,i}^{n} = \mathbb{1}\left\{s_{k,i}^{B,n} \leq 0\right\} \times c_{k,i}^{B,n} + \mathbb{1}\left\{s_{k,i}^{F,n} \geq 0\right\} \times c_{k,i}^{F,n} + \mathbb{1}\left\{s_{k,i}^{F,n} \leq 0 \leq s_{k,i}^{B,n}\right\} \times (ra_i + z_k)$$

- Collecting terms with the same subscripts on the right-hand side

$$\frac{v_{k,i}^{n+1} - v_{k,i}^{n}}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^{n}) + \alpha_{k,i} v_{k,i-1}^{n+1} + \beta_{k,i} v_{k,i+1}^{n+1} - \left(\alpha_{k,i} + \beta_{k,i} + \lambda_k\right) v_{k,i}^{n+1} + \lambda_i v_{\text{-}k,i}^{n+1} \tag{15}$$

where

$$\alpha_{k,i}^{up} = -\frac{\left[s_{k,i}^{B,n}\right]^{-}}{a_i - a_{i-1}} \geq 0, \quad \beta_{k,i}^{up} = \frac{\left[s_{k,i}^{F,n}\right]^{+}}{a_{i+1} - a_i} \geq 0$$

- Equation (15) is just a system of linear equations!!

Implicit Timestepping

Equation (15) can be written in matrix notation as

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u(c^n) + \boldsymbol{A}^n v^{n+1}$$

where the sparse matrix $\boldsymbol{A}$ looks like

$$\boldsymbol{A}^n = \begin{bmatrix}
\gamma_{1,1} & \beta_{1,1} & 0 & 0 & \cdots & \lambda_1 & 0 & \cdots & & & 0 \\
\alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \alpha_{1,I} & \gamma_{1,I} & 0 & 0 & \cdots & \cdots & 0 & \lambda_1 \\
\lambda_2 & 0 & \cdots & \cdots & \cdots & \gamma_{2,1} & \beta_{2,1} & 0 & \cdots & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & \cdots & \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \lambda_2 & 0 & 0 & \cdots & \cdots & \alpha_{2,I} & \gamma_{2,I}
\end{bmatrix}_{I+2 \times I+2}$$

entries of row i

$$\begin{bmatrix} \underbrace{\alpha_{k,i}}_{\text{inflow } i-1} & \underbrace{-(\alpha_{k,i} + \beta_{k,i} + \lambda_k)}_{\text{outflow}} & \underbrace{\beta_{k,i}}_{\text{inflow } i+1} \end{bmatrix} \begin{bmatrix} v_{k,i-1} \\ v_{k,i} \\ v_{k,i+1} \end{bmatrix}$$

Computing the Distribution

Distributions

- We now know how to solve the Household consumption/savings problem
- But interesting questions require dealing with distributions
- Denote by $g_i(a,t)$ $i = 1,2$ the joint density of income z_i and welath a.
- The evolution of the density given a fixed initial distribution $g_i(a,0)$ is described by the *Kolmogorov forward equation*
  - time dependent

$$\frac{\partial}{\partial t} g(a,t) = -\frac{\partial}{\partial a}\Big[s_k(a,t)g_k(a,t)\Big] - \lambda_k g_k(a,t) + \lambda_{\text{-}k} g_{\text{-}k}(a,t) \tag{16}$$

  - stationary

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s_k(a)g_k(a)\Big] - \lambda_k g_k(a) + \lambda_{\text{-}k} g_{\text{-}k}(a) \tag{17}$$

Kolmogorov Forward equation

Stationary

Consider the stationary KFE

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s(a, z_k)g(a, z_k)\Big] - \lambda_k g(a, z_k) + \lambda_{\text{-}k} g(a, z_{\text{-}k})$$

with the following discretization

$$0 = -\frac{\left(s^F_{k,i}\right)^+ g_{k,i} - \left(s^F_{k,i-1}\right)^+ g_{k,i-1}}{\Delta a} - \frac{\left(s^B_{k,i+1}\right)^- g_{k,i+1} - \left(s^B_{k,i}\right)^- g_{k,i}}{\Delta a} - \lambda_k g_{k,i} + \lambda_{-k} g_{\text{-}k,i} \quad (18)$$

Collecting terms with the same subscripts on the right-hand side

$$0 = \underbrace{\frac{\left(s^F_{k,i-1}\right)}{\Delta a}}_{\beta_{k,i-1}} g_{k,i-1} + \underbrace{\left(\frac{\left(s^B_{k,i}\right)}{\Delta a} - \frac{\left(s^F_{k,i}\right)}{\Delta a} - \lambda_k\right)}_{\gamma_{k,i}} g_{k,i} + \underbrace{\left(-\frac{\left(s^B_{k,i+1}\right)}{\Delta a}\right)}_{\alpha_{k,i+1}} g_{k,i+1} + \lambda_{-k} g_{\text{-}k,i}$$

which in matrix notation reads $\boldsymbol{A}^T g = \mathbf{0}$.

This makes sense: the operation is exactly the same as that used for finding the stationary distribution of a discrete Poisson process (continuous-time Markov chain). The matrix A captures the evolution of the stochastic process over a very short interval — it is our discretized *infinitesimal generator* of our state — and to find the stationary distribution, one solves the eigenvalue problem $\boldsymbol{A}^T g = \mathbf{0}$.

[Pass slide to Transition dynamics]

Consider the time dependent

$$\frac{\partial}{\partial t} g(a,t) = -\frac{\partial}{\partial a}\Big[s_k(a,t) g_k(a,t)\Big] - \lambda_k g_k(a,t) + \lambda_{\text{-}k} g_{\text{-}k}(a,t)$$

Given an initial condition, the KFE can be easily solve through a implicit method

$$\frac{g^{n+1} - g^n}{\Delta} = \left(\boldsymbol{A}^{n(+1)}\right)^T g^{n+1} \tag{19}$$

In the case wiht a dirac mass on the boundary constraint

$$M_1(t) + \int_0^a g_1(x,t)dx = G_1(a,t), \qquad \int_0^a g_2(x,t)dx = G_2(a,t) \tag{20}$$

and we may write

$$\frac{\partial}{\partial t} g_1(a,t) = -\frac{\partial}{\partial a}\Big[s_1(a,t)g_1(a,t)\Big] - \lambda_1 g_1(a,t) + \lambda_2 g_2(a,t) \tag{21}$$

$$\frac{\partial}{\partial t} g_2(a,t) = -\frac{\partial}{\partial a}\Big[s_2(a,t)g_2(a,t)\Big] - \lambda_2 g_2(a,t) + \lambda_1 g_1(a,t) + \underbrace{\lambda_1 M_1 \mathbb{1}_{\{a=0\}}}_{\text{flow? mass?}} + \underbrace{\lambda_1 M_1 \delta_0(a)}_{\text{flow? mass?}} \tag{22}$$

$$\frac{\partial}{\partial t} M_1(t) = -\lim_{\epsilon \to 0} s_1(\underline{a}+\epsilon, t) g_1(\underline{a}+\epsilon, t) - \lambda_1 M_1 \tag{23}$$

Integrating the KFE between $\underline{a} + \epsilon$ and ∞

$$\frac{\partial}{\partial t}\int_{\underline{a}+\epsilon} g_1(a,t)\,\mathrm{d}a = s_1(\underline{a}+\epsilon,t)g_1(\underline{a}+\epsilon,t) - \lambda_1 \int_{\underline{a}+\epsilon} g_1(a,t)\,\mathrm{d}a + \lambda_2 \int_{\underline{a}+\epsilon} g_2(a,t)\,\mathrm{d}a$$

Using the definition on distributions and taking the limit as $\epsilon \to 0$

$$-\frac{\partial}{\partial t}\Big(M_1(t) - G_1(t)\Big) = \lim_{\epsilon\to 0} s_1(\underline{a}+\epsilon,t)g_1(\underline{a}+\epsilon,t) + \lambda_1\Big(M_1(t) - G_1(t)\Big) + \lambda_2 G_2(t)$$

but note that $\frac{\partial}{\partial t}G_1(t) = -\lambda_1 G_1(t) + \lambda_2 G_2(t)$. That leave us with (23).

Stationary Equilibrium + Transion Dynamics

Experiment

•

Equilibrium (stationary)

$$\rho v_k(a) = \max_c \Big\{ u(c) + \partial_a v_k(a) \Big\} + \lambda_k \Big[v_\ell(a) - v_k(a) \Big] \qquad \Big[\text{HJB}\Big]$$

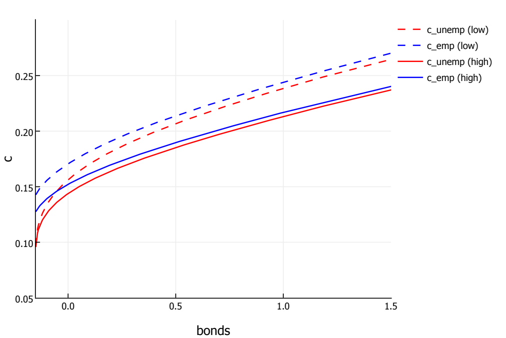
$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}a}\big[s_k(a) g_k(a)\big] - \lambda_k g_k(a) + \lambda_\ell g_\ell(a) \\ 1 &= \int_{\underline{a}}^{\infty} \Big(g_1(a) + g_2(a) \Big) da \end{aligned} \qquad \Big[\text{KFE}\Big]$$

$$0 = \int_{\underline{a}}^{\infty} a \Big(g_1(a) + g_2(a) \Big) da \qquad \Big[\text{Equil}\Big]$$

Experiment

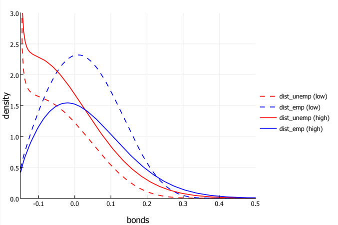
Suppose an increase in the unemployment risk λ_2. What would you expect has to happen to the interest rate? *Distributional effect of more unemployed in equilibrium makes me converge to an higher interest rate!*

CONSUMPTION POLICY

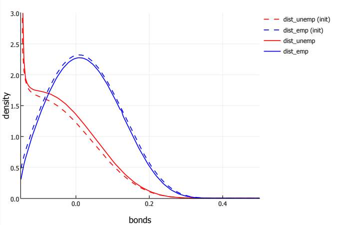


Note:

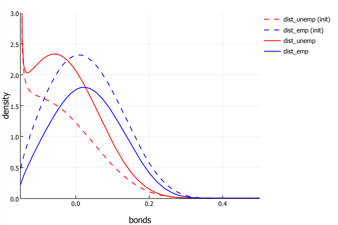
Transition Dynamics



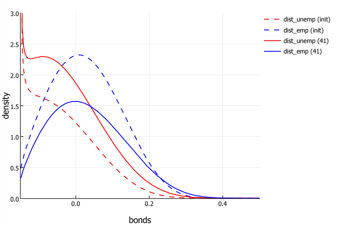
init $\times$ end



$t = 0.125$



$t = 2.0$



$t = 5.0$

Note:

Why do I care??

Example

- looks nice, but why should I pay a cost if I can do discrete time?
- It was my original tought, but recently...
- Consumption/savings problem + direct search labor market

$$V^u(a) = \max_{c,a'} u(c) + \beta \mathcal{R}^u(a') \qquad \mathcal{R}^u(\tilde{a}) = V^u(\tilde{a}) + \max_{\tilde{w}} p\big(\theta(\tilde{a},\tilde{w})\big)\Big[V^e(\tilde{a},\tilde{w}) - V^u(\tilde{a})\Big]$$

$$\text{S.t.} \quad c + \frac{a'}{1+r} = b + a, \quad a' \geq \underline{a}$$

euler equation for asset holdings

$$u'(c(\cdot)) \geq \beta(1+r)\Big\{(1-p(\theta(a',\tilde{w})))\,V^u_a(a') + p(\theta(a',\tilde{w}))V^e_a(a',\tilde{w}) + p'(\theta(a',\tilde{w}))\frac{\partial\theta(a',\tilde{w})}{\partial a}\Big[V^e(a',\tilde{w}) - V^u(a',\tilde{w})\Big]\Big\}$$

- EGM for consumption/savings, VFI for labor choice.
- Finding the equilibrium requires iterating over a lot of stuff

$$V(\cdot) \hookrightarrow \hat{\theta}(\cdot) \hookrightarrow \mathcal{J}(\cdot) \hookrightarrow \theta(\cdot)$$

- couldn't do it = (...

Example

- How does this look in continuous time, *today is tomorrow*

$$\rho V^u(a) = \max_c \Big\{ u(c) + V_a^u(b + ra - c) \Big\} + \underbrace{\lambda_u}_{\text{rate of search}} \max_{\tilde{w}} \Big\{ p(\theta(a, \tilde{w})) \Big[V^e(a, w) - V^u(a) \Big] \Big\} \tag{24}$$

- This is wayyyyy simpler and importantlty it doesn't seem I am throwing away anything of the economics

This is a title 10pt

Here is some content on the slide

$$f(x) = ax^2 + bx + c$$

Some more content

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