Math Preliminaries

1. Intoduction

Fleming and Soner (2006), Caldentey, Pham (2009)

Definition. *A stochastic process X is a Markov process if*

- *For any sequence of times $t_1 < t_2 < \ldots < t_m < t$ and $B \in \mathcal{B}$*

$$Pr\Big(X(t) \in B | X(t_1), X(t_2), \ldots, X(t_m)\Big) = Pr\Big(X(t) \in B | X(t_m)\Big)$$

- *The* transition probability *of X defined by*

$$\hat{P}(s, y, t, B) := Pr\Big(X(t) \in B | X(s) = y\Big)$$

is borel measurable for fixed s, t, B and a probability measure on $\mathcal{B}$ for fixed s, y, t.

- *The Chapman-Kolmogorov equation*

$$\hat{P}(s, y, t, B) = \int \hat{P}(r, x, t, B)\hat{P}(s, y, r, dx)$$

holds for $s < r < t$.

Definition. *A stochastic process $q(t)$ is a Poisson process with arrival rate λ if*

(i) *q(0) = 0*

(ii) *the process has independent increments*

(iii) *in any interval of length $\tau - t$ the increment $q(\tau) - q(t)$ is Poisson distributed with mean $\lambda(\tau - t)$*

The most common way to present the Poisson process is by looking at the distribution of the increment $q(t+h) - q(t)$ over a very small interval h. In case the process $q(t)$ is Poisson, its increment $dq(t)$ is characterized by

$$dq(t) = \begin{cases} 0 & \text{with prob } 1 - \lambda dt \\ 1 & \text{with prob } \lambda dt \end{cases} \tag{1.1}$$

NOTE: Insert some material on continuous time Markov Chain?

Steven Lalley Material

$$abaa \tag{1.2}$$

Definition. *A continuous-time Markov chain with finite or countable state space $\mathcal{Z}$ is a family $\{z_t\}_{t\geq 0}$ of $\mathcal{Z}$-valued random variables such that*

1. *the sample paths $t \mapsto z_t$ are right-continuous $\mathcal{Z}$-valued step functions*

2. *the process z_t satisfies the* Markov property, *that is, for any set of times $t_0 < t_1 < \ldots < t_i < t_{i+1}$ and states $z_0, z_1, \ldots, z_{i+1}$*

$$\begin{aligned} Pr\Big(z(t_{i+1}) = x_{i+1} | z(t_k) = x_k \ \forall k \leq i\Big) &= Pr\Big(z(t_{i+1}) = x_{i+1} | z(t_i) = x_i\Big) \\ &= p_{t_{i+1}-t_i}(x_i, x_{i+1}) \end{aligned} \tag{1.3}$$

where

$$p_t(x,y) = Pr(z(t) = y | z(0) = x)$$

The probabilities $p_t(x,y)$ are called transition probabilities for the Markov chain. It is often advantageous to view them as being arranged in matrices

$$P_t = [p_t(x,y)]_{x,y\in\mathcal{Z}} \tag{1.4}$$

Note that the family $\{P_t\}_{t\geq 0}$ obey Chapman-Kolmogorov equations

$$P_{t+s} = P_t P_s$$

The paths of a continuous-time Markov chain are step functions, the jumps occur at discrete set of time points $0 < T = T_1 < T_2 < \ldots$. The following theorem characterizes properties of the first jump time and post-jump process.

Theorem. *For every state x, there is a positive parameter $\lambda_x > 0$ such that under P^x the distribution of the first jump time is exponential with parameter λ_x.*

Theorem. *Let X_t be a continuous-time Markov chain with first jump time T. Then there is a stochastic matrix $A = (a_{x,y})_{x,y\in\mathcal{Z}}$ such that for every pair $x.y$ and all $t > 0$*

$$Pr\Big(X(T) = y \mid T = t\Big) = a_{x,y} \tag{1.5}$$

so that $X(T)$ is independent of the time T at which the jump occurs.

Definition. *The infinitesimal generator of a continuous-time Markov chain is the matrix $Q = (q_{x,y})_{x,y\in\mathcal{Z}}$ with entries*

$$q_{x,y} = \begin{cases} \lambda_x a_{x,y} & \text{if } y \neq x \\ -\lambda_x & o.w. \end{cases} \tag{1.6}$$

where λ_x is the parameter of the holding distribution for state x and $A = (a_{x,y})_{x,y\in\mathcal{Z}}$ is the transition probability matrix of the embedded jump chain.

Theorem. *The transition probabilities $p_{x,y}$ of a finite continuous time Markov chain satisfy the following diffrential equations, called the Kolmogorov equations*

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y) = \sum_{z\in\mathcal{Z}} q(x,z) p_t(z,y) \tag{kb}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y) = \sum_{z\in\mathcal{Z}} p_t(x,z) q(z,y) \tag{kf}$$

Proof. — The Chapman-Kolmogorov equation imply that for any $t, \epsilon > 0$

$$\begin{aligned} \frac{1}{\epsilon}\Big[p_{t+\epsilon}(x,y) - p_t(x,y)\Big] &= \frac{1}{\epsilon}\sum_{z\in\mathcal{Z}} \Big[p_\epsilon(x,z) - \delta_x(z)\Big] p_t(z,y) \\ &= \frac{1}{\epsilon}\sum_{z\in\mathcal{Z}} p_t(x,z)\Big[p_\epsilon(z,y) - \delta_y(z)\Big] \end{aligned}$$

Let's prove the backward equation. Consider how the Markov chain might find its way from state x at time 0 to state $z \neq x$ at time ϵ. Either there is just one jump, from x to z, or there are two or more jumps before ϵ. For an small ϵ, the probability of two or more transitions is $o(\epsilon)$, so using the independence and transition probabilities of the jump chain we have

$$p_\epsilon(x,z) \approx (\lambda_x \epsilon) a_{x,z} \quad \text{for } z \neq x, \qquad p_\epsilon(x,x) \approx 1 - \lambda_x \epsilon$$

Since $q_{x,z} = \lambda_x a_{x,z}$ for $z \neq x$ and $-\lambda_x$ otherwise, substituting the approximations on the first Chapman-Kolmogorov equation and taking the limit as $\epsilon \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y) = \sum_{z \in \mathcal{Z}} q_{x,z} p_t(z,y) = \left(QP_t\right)_{x,y} \tag{1.7}$$

which agree with the backward equation (kb).

Going to the forward equation, observe that

$$p_\epsilon(z,y) = \lambda_z a_{z,y} \text{ for } z \neq y, \qquad p_\epsilon(y,y) \approx 1 - \lambda_y \epsilon$$

substituting the approximations on the first Chapman-Kolmogorov equation and taking the limit as $\epsilon \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y) = \sum_{z \in \mathcal{Z}} p_t(x,z) q_{z,y} = \left(P_t Q\right)_{x,y} \tag{1.8}$$

■

Definition. *A probability distribution π on the state space $\mathcal{Z}$ is called a stationary distribution for the Markov chain if*

$$\pi^T P_t = \pi^T, \qquad \forall t \geq 0 \tag{1.9}$$

In practice, it is often difficult to calculate stationary distributions by directly solving the equations (1.9), in part because it isn't always possible to solve the Kolmogorov equations in closed form. Nevertheless, the Kolmogorov equations lead to another characterization of stationary distributions that often leads more useful.

Lemma. *A probability distribution π is stationary if and only if*

$$\pi^T Q = 0 \tag{1.10}$$

Proof. — Suppose that π is stationary distribution. Taking the time derivative on (1.9) at $t = 0$ we obtain (1.10). Conversely, suppose that π satisfies (1.10). Multiply both sides to the right by P_t. From the Kolmogorov equations, this implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \pi^T P_t = 0, \qquad \forall t$$

which mean $\pi^T P_t$ is constant. Since $\pi^T P_0 = \pi_T$, we conclude that π is a stationary distribution. ■

1.1. Diffusion Processes and PDE's

The objective here is to establish the connection between diffusion processes and PDE's. Material is taken form Caldentey and Fleming and Soner (2006)

Let us define a family of linear operators $S_{s,t}$ associated with a markov process. For all bounded, real valued, measurable function ϕ and $s < t$ let

$$\left(S_{s,t}\phi\right)(y) := \int f(x)\hat{P}(s,y,t,dx) \equiv \mathbb{E}_{s,y}\left[\phi(X(t))\right]$$

For a given Markov process X, we also define its *generator*.

Definition. *The* infinitesimal generator $\mathcal{L}_t$ *of X_t is defined by*

$$\left(\mathcal{L}_t\phi\right)(x) = \lim_{h \to 0} \frac{1}{h}\left[S_{t,t+h}\phi(x) - \phi(x)\right] \tag{1.11}$$

for all bounded, real valued, measurable functions f such that the limit exists.

[heuristic]
Consider a function $\psi(t,x)$. Then, under suitable restrictions on ψ we can write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S_{s,t}\psi(t,y) &= \lim_{h\to 0}\frac{1}{h}\Big[\mathbb{E}_{s,y}[\psi(t+h,X(t+h))-\psi(t,X(t))]\Big] \qquad (\pm\psi(t,X(t+h)))\\ &= \mathbb{E}_{s,y}[\psi_t(t,X(t))] + \mathbb{E}_{s,y}\left[\lim_{h\to 0}\frac{1}{h}\mathbb{E}_{t,X(t)}\left[\psi(t,X(t+h))-\psi(t,X(t))\right]\right]\\ &= \mathbb{E}_{s,y}\Big[\psi_t(t,X(t)) + \mathcal{L}_t\psi(t,X(t))\Big]\end{aligned}$$

This equation says the how much expectations in s about $\psi(t,X(t))$ changes when t moves further into the future. The result is that this change is given by the expected change of $\psi(t,X(t))$ itself, where the change is $\psi_t(t,X(t)) + \mathcal{L}_t(t,X(t))\psi$.

Integrating this last expression over t we get (*Dynkin's formula*)

$$\mathbb{E}_{s,y}[\psi(t,X(t))] = \psi(s,y) + \mathbb{E}_{s,y}\left[\int_s^t \psi_t(u,X(u)) + \mathcal{L}_u\psi(u,X(u))du\right] \tag{1.12}$$

Alternatively, consider the Cauchy linear parabolic partial differential equation.

$$rv - \frac{\partial v}{\partial t} - \mathcal{L}_t v = f, \qquad \text{on } [0,T)\times\mathbb{R}^n \tag{1.13}$$

$$v(T,\cdot) = g, \qquad \text{on } \mathbb{R}^n \tag{1.14}$$

where $\mathcal{L}_t$ is a differential operator of second order

$$(\mathcal{L}_t\varphi)(x) = b(t,x)\frac{\partial}{\partial x}\varphi(x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}\varphi(x) \tag{1.15}$$

with deterministic coefficients $b(t,x)$ and $\sigma(t,x)$.

Theorem. *Let v be a $C^{1,2}$ function and a solution to the Cauchy problem (1.13)-(1.14). Moreover, $X^{t,x}$ denotes the solution to a diffusion with drift $b(t,x)$ and volatility $\sigma(t,x)$ starting from x at time t. Then v admits a representation*

$$v(t,x) = \mathbb{E}_{t,x}\left[\int_t^T e^{-rs}f(s,X_s^{t,x})ds + e^{-r(T-t)}g(X_T^{t,x})\right] \tag{1.16}$$

This is a probabilistic formula for the solution of a PDE.

1.2. State Evolution

(a) Difusion Process

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbf{p})$ be a filtered probability space. Let W_t be a Brownian motion and $X_t\in\mathbb{R}$ denote the state of an agent at time t. The state evolves according to the diffusion process

$$dX_t = \mu(X_t,\alpha_t)dt + \sigma(X_t,\alpha_t)dW_t \tag{1.17}$$

where the drifts $b(\cdot)$ and $\sigma(\cdot)$ are deterministic measurable function on $\mathbb{R}\times A$ and α is a stochastic process which denotes the control.

For policy $\alpha_t = a$, the infinitesimal generator of the diffusion process (1.2) is a second-order elliptic partial differential operator

$$\mathcal{L}^a\phi(x) = \mu(x,\alpha)\frac{\partial\phi(x)}{\partial x} + \frac{1}{2}\sigma^2(x,\alpha)\frac{\partial^2\phi(x)}{\partial x^2} \tag{1.18}$$

For this case, (1.12) can be "formally" justified by Itô lemma. Forget about the control α for a second and consider the simple diffusion

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

If X is a solution to the SDE above, ψ a function of class $C^{1,2}$ Itô formula implies

$$\psi(t, X(t)) = \psi(s, X(s)) + \int_s^t \left(\frac{\partial\psi}{\partial t} + \mathcal{L}\psi\right)(s, X_s) + \int_s^t \psi_x(s, X_s)\sigma(s, X_s)dW_s$$

Taking $\mathbb{E}_{s,y}$ we get (1.12).

(b) Random evolution with Markov chain parameters[1]

Let z_t be a finite state Markov chain with state space Z. On any time interval where $z(t) = z_i$ is constant, the state X satisfies the ordinary differential equation

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = f(X_t, z_i) \tag{1.19}$$

Let $t \in [0, t_1]$ and $\tau_1 < \tau_2 < \ldots < \tau_m$ denote the sucessive jump times of process z_t. The process $X(t)$ is therefore defined by

$$\frac{\mathrm{d}X}{\mathrm{d}t} = f(X_t, z(\tau_i^+)),\ \tau_i \le t < \tau_{i+1} \tag{1.20}$$

For each bounded, real valued, measurable function $\phi(x,z)$ such that $\phi(\cdot, z) \in C^1$ we have that the infinitesimal generator is given by

$$\mathcal{D}\phi(x, z_i) = f(x, z_i)\frac{\partial\phi(x, z_i)}{\partial x} + \sum_{j\neq i} q_{z_i,z_j}\left[\phi(x, z_j) - \phi(x, z_i)\right] \tag{1.21}$$

A formal statement of this CVF can be found on Sennewald (2007)

[heuristic]

This can be heuristically derived using a discrete time approximation. Suppose there are $|Z| = 2$ and consider periods of length h. The value of z_t is z_i and that value is kept unchanged during h with a probability $p_i(h) = e^{-\lambda_i h} \approx 1 - \lambda_i h$ and switchs to state z_j with probability $1 - p_i(h)$. The value of X evolves according to $X_{t+h} = X_t + hf(X_t, z_t)$ In that case,

$$E_{t,x,z_i}\left[\psi(X_{t+h}, z_{t+h})\right] = (1 - \lambda_i h)\psi(X_{t+h}, z_i) + \lambda_i h\psi(X_{t+h}, z_j)$$

Subtracting $\psi(x, z_i)$ and dividing by h

$$\frac{1}{h}E_{t,x,z_i}\left[\psi(X_{t+h}, z_{t+h}) - \psi(x, z_i)\right] = \frac{\psi(x + hf(x, z_i), z_i) - \psi(x, z_i)}{hf(x, z_i)}f(x, z_i) + \lambda_i\left[\psi(x + hf(x, z_i), z_j) - \psi(x + hf(x, z_i), z_i)\right]$$

Taking the limit $h \to 0$

$$\lim_{h\to 0}\frac{1}{h}E_{t,x,z_i}\left[\psi(X_{t+h}, z_{t+h}) - \psi(x, z_i)\right] = \frac{\partial\phi(x, z_i)}{\partial x} + \lambda_i\left[\psi(x, z_j) - \psi(x, z_i)\right]$$

which agrees with (1.21).

[1]Section taken from III.4 of Fleming and Soner (2006) and Sennewald (2007).

1.3. Kolmogorov Equations

The discussion is based on Karlin and Taylor (1981).

Let $\{X(t)\}$ be a regular diffusion with infinitesimal parameters $\mu(x), \sigma(x)$. Denote $P(t,x,y) = \Pr\left[X(t) \leq y | X(0) = x\right]$ the transition distribution function of $X(t)$ with density $p(t,x,y)$.

Kolmogorov Backward equation

Fix a bounded, piecewise continuous function g and define

$$u(t,x) := \mathbb{E}\left[g(X(t))|X(0) = 0\right] \tag{1.22}$$

By law of iterated expectations, for any $h > 0$

$$\begin{aligned} u(t+h,x) &= \mathbb{E}\left[g(X(t+h))|X(0) = x\right] \\ &= \mathbb{E}\Big\{\left[g(X(t+h))|X(h)\right]|X(0) = x\Big\} \\ &= \mathbb{E}\left[u(t,X(h))|X(0) = x\right] \end{aligned}$$

It follows that for any $h > 0$

$$\frac{u(t+h,x) - u(t,x)}{h} = \frac{1}{h}\mathbb{E}\left[u(t,X(h)) - u(t,x)|X(0) = x\right]$$

Taking the limit as $h \to 0$ and using Ito's lemma on the RHS

$$\frac{\partial u(t,x)}{\partial t} = \mu(x)\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u(t,x)}{\partial x^2} \tag{1.23}$$

The appropriate initial condtion is $u(0,x) = g(x)$. For the indicator function, $g = \mathbb{1}_{(,y]}$ we have $u(t,x) = P(t,x,y)$ which by (1.3)

$$\frac{\partial P(t,x,y)}{\partial t} = \mu(x)\frac{\partial P(t,x,y)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 P(t,x,y)}{\partial x^2}$$

which is the Kolmogorov backward equation. The boundary condition in this case

$$P(0,x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$$

We can also write it terms of density $p(t,x,y)$

$$\frac{\partial p(t,x,y)}{\partial t} = \mu(x)\frac{\partial p(t,x,y)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 p(t,x,y)}{\partial x^2}$$

Note that in this case, the boundary condition is different, as $t \to 0$ the density function $p(t,x,y)$ collapses to a mass point at $x = y$.

Kolmogorov Forward equation

Let $\eta(t,y)$ be an arbitrary smooth function satisfying the following recursion

$$\eta(t+s,y) = \int \eta(r,\xi)p(s,\xi,y) \tag{1.24}$$

Differentiate both sides with respect to with respect to s and use the backward equation satisfied by $p(s,\xi,y)$ to get

$$\frac{\partial \eta(t+s,y)}{\partial t} = \int \eta(t,\xi)\left[\mu(\xi)\frac{\partial p(t,\xi,y)}{\partial \xi} + \frac{1}{2}\sigma^2(\xi)\frac{\partial^2 p(t,\xi,y)}{\partial \xi^2}\right]d\xi$$

Next, assuming that the contribution from the boundaries vanish, integration by parts gives

$$\frac{\partial \eta(t+s,y)}{\partial t} = \int p(s,\xi,y)\left\{-\frac{\partial}{\partial \xi}\left[\mu(\xi)\eta(t,\xi)\right] + \frac{1}{2}\frac{\partial^2}{\partial \xi^2}\left[\sigma^2(\xi)\eta(t,\xi)\right]\right\} d\xi$$

As $s \to 0$, $p(s,\xi,y)$ approaches the delta measure at y so we get

$$\frac{\partial \eta(t,y)}{\partial t} = -\frac{\partial}{\partial y}\left[\mu(y)\eta(t,y)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sigma^2(y)\eta(t,y)\right] \tag{1.25}$$

In particular, the choice $\eta(t,y) = p(t,x,y)$ obeys (1.24) and therefore satisfies (1.25). The Kolmogorov equation is adjoint to KBE.

Kolmogorov Forward Equation Moll

Consider the discrete time analogue to the continuous time economy described in the notes. Individuals wealth evolves as $a_{t+\Delta} = a_t + \Delta s(a_t, z_t)$. After saving decision is made, next period income $z_{t+\Delta}$ is realized: it changes from z_i to z_j with probability $\Delta\lambda_i$. Define the CDF

$$G(a,z_i,t) = \Pr\big(a_t \le a, z_t = z_i\big)$$

which evolves between t and $t+\Delta$ according to

$$\begin{aligned} G(a,z_i,t+\Delta) &= (1-\Delta\lambda_i)\Pr\big(a_t \le a - \Delta s(a,z_i), z_t = z_i\big) + \Delta\lambda_j \Pr\big(a_t \le a - \Delta s(a,z_j), z_t = z_i\big) \\ &= (1-\Delta\lambda_i)G(a-\Delta s(a,z_i),z_i,t) + \Delta\lambda_j G(a-\Delta s(a,z_j),z_j,t) \end{aligned}$$

Subtracting $G(a,z_i,t)$ from both sides and dividing by Δ

$$\frac{G(a,z_i,t+\Delta)-G(a,z_i,t)}{\Delta} = \frac{G(a-\Delta s(a,z_i),z_i,t)-G(a,z_i,t)}{\Delta s(a,z_i)} s(a,z_i) - \lambda_i G(a-\Delta s(a,z_i),z_i,t) + \lambda_j G(a-\Delta s(a,z_j),z_j,t)$$

Taking the limit as $\Delta \to 0$

$$\partial_t G(a,z_i,t) = -s(a,z_i)\partial_a G(a,z_i,t) - \lambda_i G(a,z_i,t) + \lambda_j G(a,z_j,t)$$

Differentiating wrt a and using we have

$$\partial_t g(a,z_i,t) = -\partial_a\Big[s(a,z_i)g(a,z_i,t)\Big] - \lambda_i g(a,z_i,t) + \lambda_j g(a,z_j,t) \tag{1.26}$$

Kolmogorov Forward Equation Bayer and Wälde (2015)[2]

Assume there is a function f having as arguments the state variables a and z. *Heuristically*, the differential of this function[3]

$$\begin{aligned} df\big(a(t),z(t)\big) = f_a(ra(t)+z(t)-c)dt + \Big[f\big(a(t),z(t)+\Delta\big) - f\big(a(t),z(t)\big)\Big]dq_\mu + \\ + \Big[f\big(a(t),z(t)-\Delta\big) - f\big(a(t),z(t)\big)\Big]dq_s \end{aligned} \tag{1.27}$$

Most times we will be interested in expected changes. Applying the condition expectation operator and dividing by dt yields[4]

[3]For formal presentation of this change-of-variables formula (CVF) see .

[4]From

If we are given a Poisson process q with arrival rate λ and a cádlág process X — continuous from the right with left limits — then the following relation holds true

$$\mathbb{E}_s\left[\int_s^t X_{\tau-}dq_\tau\right] = \lambda\mathbb{E}_s\left[\int_s^t X_\tau d\tau\right]$$

$$\frac{1}{dt}\mathbb{E}_t df\big(a(t),z(t)\big) = f_a(ra(t)+z(t)-c)dt + \mu(z(t))\Big[f\big(a(t),z(t)+\Delta\big) - f\big(a(t),z(t)\big)\Big] + \\ + s(z(t))\Big[f\big(a(t),z(t)-\Delta\big) - f\big(a(t),z(t)\big)\Big] \tag{1.28}$$

In what follows, we denote this expression by the operator $\mathcal{A}$, which is, in more precise terms, defined as the *infinitesimal operator* of the joint process

$$\mathcal{A}f(a,z) := \lim_{h\to 0}\frac{1}{h}\mathbb{E}_{a,z}\Big[f\big(a(h),z(h)\big) - f(a,z)\Big] \tag{1.29}$$

We should view $\mathcal{A}$ as an operator mapping function on $A\times Z$ to itself. From now onwards and consider the process $X(t) = (a(t),z(t))$. The expected value of $f(X(\tau))$ is given by *Dynkin's formula*

$$\mathbb{E}f(X(\tau)) = \mathbb{E}f(X(t)) + \mathbb{E}\left[\int_t^\tau \mathcal{A}f(X_s)ds\right] \tag{1.30}$$

Let us now differentiate (1.30) with respect to τ

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{E}f\big(X(\tau)\big) = \mathbb{E}\mathcal{A}f(X_\tau) \tag{1.31}$$

If we introduce the densities defined previously, we may express condition (1.31) as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{E}f\big(X(\tau)\big) = \int \mathcal{A}f(a,w)p(a,w,\tau)da + \int \mathcal{A}f(a,b)p(a,b,\tau)da \tag{1.32}$$

Substituting for $\mathcal{A}$ and using integration by parts to move the derivatives in $\mathcal{A}f$ into the density we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{E}f\big(X(\tau)\big) = \sum_{z\in\{w,z\}}\int f(a,z) - \frac{\partial}{\partial a}\Big[s(a,z)p(a,z,\tau)\Big]da + \\ \int\bigg(s\Big[f(a,b)-f(a,w)\Big]p(a,w,\tau) + \mu\Big[f(a,w)-f(a,b)\Big]p(a,b,\tau)\bigg)da \tag{1.33}$$

Alternatively,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbb{E}f\big(X(\tau)\big) = \sum_{z\in\{w,z\}}\int f(a,z)\frac{\partial p(a,z,\tau)}{\partial \tau}da \tag{1.34}$$

Equating (1.33) with (1.34) we get

$$\int f(a,w)\bigg[-\frac{\partial}{\partial a}\Big[s(a,w)p(a,w,\tau)\Big] - sp(a,w,\tau) + \mu p(a,b,\tau) - \frac{\partial p(a,w,\tau)}{\partial\tau}\bigg]da + \\ + \int f(a,b)\bigg[-\frac{\partial}{\partial a}\Big[s(a,b)p(a,b,\tau)\Big] + sp(a,w,\tau) - \mu p(a,b,\tau) - \frac{\partial p(a,b,\tau)}{\partial\tau}\bigg]da = 0 \tag{1.35}$$

which can only be true for an aribitrary f if the inside bracket expressions are zero. Note that these are the Fokker-Planck equations.

1.4. HJB with Poisson Process

Achdou et al. (2016) Consider the income fluctuation problem in discrete time. Periods are of length Δ, individuals discount future with discount factor $\beta(\Delta) = e^{-\rho\Delta}$. Individuals with income z_i keep their income with probability $p_i(\Delta) = e^{-\lambda_i\Delta}$ and switch to state z_j with probability $1 - p_i(\Delta)$. The Bellman equation for this problem

$$v_i(a_t) = \max_{c_t} u(c_t)\Delta + \beta(\Delta)\Big[p_i(\Delta)v_i(a_{t+\Delta}) + \big(1 - p_i(\Delta)\big)v_j(a_{t+\Delta})\Big] \tag{1.36}$$

$$a_{t+\Delta} = a_t + \Delta\big(r_t a_t + z_i - c_t\big) \tag{1.37}$$

For small Δ,

$$\beta(\Delta) \approx 1 - \rho\Delta, \qquad p_i(\Delta) \approx 1 - \lambda_i\Delta$$

Substituting these on (1.36) we have

$$v_i(a_t) = \max_{c_t} u(c_t)\Delta + (1 - \rho\Delta)\Big[v_i(a_{t+\Delta}) + \lambda_i\Delta\big(v_j(a_{t+\Delta}) - v_i(a_{t+\Delta})\big)\Big]$$

Rearranging and dividing by Δ

$$\rho v_i(a_t) = \max_{c_t} u(c_t) + (1 - \rho\Delta)\left[\frac{v_i(a_{t+\Delta}) - v_i(a_t)}{\Delta} + \lambda_i\big(v_j(a_{t+\Delta}) - v_i(a_{t+\Delta})\big)\right]$$

Taking $\Delta \to 0$

$$\begin{aligned}
\rho v_i(a_t) &= \max_{c_t}\; u(c_t) + \lim_{\Delta\to 0} \frac{v_i\Big(a_t + \Delta(r_t a_t + z_i - c_t)\Big) - v_i(a_t)}{\Delta(r_t a_t + z_i - c_t)}(r_t a_t + z_i - c_t) + \lambda_i\big[v_j(a_t) - v_i(a_t)\big] \\
&= \max_{c_t}\; u(c_t) + v_i'(a_t)\big(r_t a_t + z_i - c_t\big) + \lambda_i\big[v_j(a_t) - v_i(a_t)\big]
\end{aligned}$$

Formal derivation. Consider the problem

$$\begin{aligned}
&\sup_{c\in\mathcal{U}} \mathbb{E}_{a,z}\left[\int_0^\infty u(c_t)dt\right] \\
\text{S.t. } &da_t = \{ra_t + z_t - c_t\}dt \\
&dz_t = \Delta dq_\mu - \Delta dq_s
\end{aligned}$$

where $\mathcal{U}$ is a set of admissible controls. We will restrict attention to *Markov controls*, i.e. controls of the form $c_t = c(X_t)$. Define the value function $V(a,z)$ as the supremum of (1.4). Suppose that $c_t^* = \phi(a_t, z_t)$ is an optimal markov control. Let us define an suboptimal control $\tilde{c}$ as

$$\tilde{c}(a,z) = \begin{cases} c & \text{if } t \le \tau \\ \phi^*(a,z) & \text{if } t > \tau \end{cases}$$

In that case,

$$\begin{aligned}
V(a,z) &\ge \mathbb{E}_{a,z}\left[\int_0^\tau e^{-\rho t}u(c)dt + \int_\tau^\infty e^{-\rho t}u\Big((\phi^*(a_t, z_t)\Big)dt\right] \\
&= \mathbb{E}_{a,z}\left[\int_0^\tau e^{-\rho t}u(c)dt + e^{-\rho\tau}\int_0^\infty e^{-\rho t}u\Big((\phi^*(a_{t+\tau}, z_{t+\tau})\Big)dt\right] \\
&= \mathbb{E}_{a,z}\left[\int_0^\tau e^{-\rho t}u(c)dt + e^{-\rho\tau}V(a_\tau, z_\tau)\right]
\end{aligned} \tag{1.38}$$

Computing $\mathbb{E}_{a,z}\Big[e^{-\rho\tau}V(a_\tau, z_\tau)\Big]$ by *Dynkin's formula*

$$\mathbb{E}_{a,z}\Big[e^{-\rho\tau}V(a_\tau, z_\tau)\Big] = V(a,z) + \mathbb{E}_{a,z}\left[\int_0^\tau e^{-\rho t}\Big(-\rho V + \mathcal{D}^\phi V\Big)(a_t, z_t)dt\right] \tag{1.39}$$

Combining (1.38) and (1.39) we get

$$0 \geq \mathbb{E}_{a,z} \left\{ \int_0^\tau e^{-\rho t} \left[u(c) + \left(-\rho V + \mathcal{D}^\phi V \right)(a_t, z_t) \right] dt \right\}$$

Dividing by τ and taling the limit as $\tau \to 0$ we get

$$0 \geq u(c) + \mathcal{D}^\phi V(a,z) - \rho V(a,z)$$

Because at optimality the same reasoning goes through but with equalite, we conclude that

$$0 = \max \left\{ u(c) + \mathcal{D}^\phi V(a,z) \right\} - \rho V(a,z)$$

which is the *Haminlton-Jacobi-Bellman* equation for our problem.

The following Theorem taken from Sennewald (2007) offers a verification theorem for the sufficiency of HJB equation

Theorem. *Let a C^1 function J satify*

$$\rho \mathcal{J}(x) \geq u(c) + \mathcal{D}^\phi \mathcal{J}(x) \qquad \forall x \in X, \forall c \tag{1.40}$$

and suppose in addition that there exists an admissible policy ϕ^* such that

$$\rho \mathcal{J}(x) = u(\phi^*(x)) + \mathcal{D}^{\phi^*} \mathcal{J}(x) \qquad \forall x \in X, \forall c \tag{1.41}$$

If furthermore, the limiting condition

$$\lim_{t \to \infty} \mathbb{E} \left[e^{-\rho t} \mathcal{J}(X^{\phi,x}) \right] \geq 0$$

holds for all admissible policy, with equality for ϕ^*, then $\mathcal{J}$ is the value function and ϕ^* is optimal.

Proof. — Let ϕ be an arbitrary admissible policy. Then inequality (1.40) gives

$$-\rho \mathcal{J}(x) + \mathcal{D}^\phi \mathcal{J}(x) \leq -u(\phi(x)) \qquad \forall x \in X$$

Applying the CVF to the C^1-function $e^{-\rho t} \mathcal{J}(x)$ and taking expectations we get the following version of the Dynkin's formula

$$\mathbb{E}_x \left[e^{-\rho t} \mathcal{J}(X_t^\phi) \right] = \mathcal{J}(x) + \mathbb{E}_x \int_0^t e^{-\rho t} \left[-\rho \mathcal{J}(X_\tau^\phi) + \mathcal{D}^\phi \mathcal{J}(X_\tau^\phi) \right] d\tau$$

which by the previous inequality imply

$$\mathcal{J}(x) \geq \mathbb{E}_x \int_0^t e^{-\rho t} u(\phi(X_\tau^\phi)) d\tau + \mathbb{E}_x \left[e^{-\rho t} \mathcal{J}(X_t^\phi) \right]$$

Letting $t \to \infty$ and by assumption on the limiting term, we have $\mathcal{J}(x) \geq \mathbb{E}_x \int_0^\infty e^{-\rho t} u(\phi(X_\tau^\phi)) d\tau$. By repeating the same steps with ϕ^*, we may also conclude $\mathcal{J}(x) = \mathbb{E}_x \int_0^\infty e^{-\rho t} u(\phi^*(X_\tau^{\phi^*})) d\tau$, which completes the proof. ∎

> NOTE: Include the good reference on Ross?

1.5. Analytical solution for KFE

Remember that Kolmogorov equations for the two state Markov problem is given by

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a} \left[s_i(a) g_i(a) \right] - \lambda_i g_i(a) + \lambda_{-i} g_{-i}(a) \tag{1.42}$$

Summing up the two we conclude

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a} \left[s_1(a) g_1(a) + s_2(a) g_2(a) \right] \Rightarrow 0 = s_1(a) g_1(a) + s_2(a) g_2(a)$$

Going back to (**??**)

$$g_i'(a) = -\left(\frac{s_i'(a)}{s_i(a)} + \frac{\lambda_i}{s_i(a)} + \frac{\lambda_{\text{-}i}}{s_{\text{-}i}(a)}\right) g_i(a)$$

Importantly, we have two independent ODEs for g_1 and g_2 rather than the coupled system of two ODEs we started out with, and they can be solved separately. The general solution is

$$g_i(a) = \frac{s_i(a_0)g_i(a_0)}{s_i(a)} \exp\left(-\int_{a_0}^{a}\left(\frac{\lambda_1}{s_1(u)} + \frac{\lambda_2}{s_2(u)}\right) du\right) \tag{1.43}$$

for some fixed a_0. Using $\lim_{\epsilon\to 0} s_1(\underline{a}+\epsilon)g_1(\underline{a}+\epsilon) = -\lambda_{-1}M_1$. as a boundary condition for (1.43), we have

$$g_1(a) = -\frac{\lambda_1 M_1}{s_1(a)} \exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_1}{s_1(u)} + \frac{\lambda_2}{s_2(u)}\right) du\right) \tag{1.44}$$

$$g_1(a) = \frac{\lambda_1 M_1}{s_1(a)} \exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_1}{s_1(u)} + \frac{\lambda_2}{s_2(u)}\right) du\right) \tag{1.45}$$

$$equation \tag{1.46}$$

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