

Lecture

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NYU

1. Consumption Savings Problem
2. Computing the Distribution
3. Stationary Equilibrium + Transition Dynamics
4. Why do I care??

THEORY

- [Pham \(2009\)](#) Continuous-time Stochastic Control – maybe too finance
- Lecture notes
 - ↪ [Caldentey \(????\)](#) Stochastic processes and optimal control — nice lecture notes Enio uses them
 - ↪ [Ross \(????\)](#) *Stochastic Control in Continuous Time* — alternative to math books [Fleming and Soner \(2006\)](#), [Øksendal \(2003\)](#), [Øksendal and Sulem \(2007\)](#)

THEORY (MACRO)

- Moll's website
- [Bayer and Wälde \(2015\)](#) – recent discovery, discuss the kind of SDE driven by a Markov chain
 - ↪ [Sennewald \(2007\)](#) (theory paper), [Walde \(2008\)](#) (book on intertemporal optimization),
- [Stokey \(2009\)](#) book – *Impulse control Problem*

NUMERIC

- [Achdou, Han, Lasry, Lions, and Moll \(2016\)](#) (mainly the [numerical appendix](#)), Moll's website (tons of examples and materials)
- [Forsyth and Vetzal \(2012\)](#) (Also has some slides) – good introduction to “viscosity solutions”
- Interested? Check applications ...
 - ↪ HANK by [Kaplan, Moll, and Violante \(2016\)](#), **PHACT**
 - ↪ [Nuño and Moll \(2017\)](#) (improved notation)
 - ↪ [Thomas and Nuño \(2016\)](#) (impulse control)

Consumption Savings Problem

Problem of Household

$$\max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \{ra_t + z_t - c_t\}dt$$

z_t : is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w - b)dq - (w - b)dQ, \quad q \sim \text{Poisson}(\lambda_1), \quad Q \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' consumption and saving decision is summarized by [HJB equation](#)

$$\rho v(a, z_k) = \max_c \left\{ u(c) + v_a(a, z)[ra + z_k - c] \right\} + \lambda_k \left[v(a, z_{-k}) - v(a, z_k) \right] \quad (1)$$

Where this came from? Check [Lagos lecture notes](#) for an heuristic argument.

Theoretical results analogous to discrete time:

- Value function satisfy the HJB equation
- Verification theorems: solution of HJB + ... \rightarrow value function
- Alternatively, one can show HJB has a unique “nice” solution which is the value function ([viscosity solution](#))

Households

Keynes-Ramsey rules

Before solving the HJB FE let's see what we can do. Analytical results from [Bayer and Wälde \(2015\)](#)

Envelope condition:

$$\rho V_a(a, b) = rV_a(a, b) + V_{aa}(a, b)\{ra + b - c(a, b)\} + \lambda_1 [V_a(a, w) - V_a(a, b)]$$

Differential of $V_a(a, z)$ — CVF, “Itô formula”

$$da_t = \{ra_t + z_t - c_t\}dt$$

$$dz_t = (w - b)dq - (w - b)dQ, \quad q \sim \text{Poisson}(\lambda_1), \quad Q \sim \text{Poisson}(\lambda_2)$$

$$dV_a(a, b) = \underbrace{V_{aa}\{ra + b - c(a, b)\}}_{\text{normal term}} dt + \underbrace{[V_a(a, w) - V_a(a, b)]}_{\text{jump terms}} dq_t$$

From optimization $V_a(a, z) = u'(c(a, z))$. Combining both equations to get rid of V_{aa} we have

$$\begin{aligned} du'(c(a, b)) = & \left\{ (\rho - r)u'(c(a, b)) - \lambda_1 u'(c(a, b)) \left[\frac{u'(c(a, w))}{u'(c(a, b))} - 1 \right] \right\} dt + \\ & + [u'(c(a, w)) - u'(c(a, b))] dq_t \end{aligned}$$

Households

Keynes-Ramsey rules

Applying “Itô lemma” to get consumption over time

$$dc(a, b) = \frac{u'(c(a, b))}{-u''(c(a, b))} \left\{ r - \rho - \lambda_1 \left[1 - \frac{u'(c(a, w))}{u'(c(a, b))} \right] \right\} dt + [c(a, w) - c(a, b)] dq_t \quad (2)$$

$$dc(a, w) = \frac{u'(c(a, w))}{-u''(c(a, w))} \left\{ r - \rho + \underbrace{\lambda_2 \left[\frac{u'(c(a, b))}{u'(c(a, w))} - 1 \right]}_{\text{prec. savings}} \right\} dt + \underbrace{[c(a, b) - c(a, w)]}_{\text{jumps}} dQ_t \quad (3)$$

$$\text{neoclassical growth model } \dot{c}(t) = \frac{u'(c)}{-u''(c)} (r - \rho)$$

Looking at period between jumps. What the signs tell us?

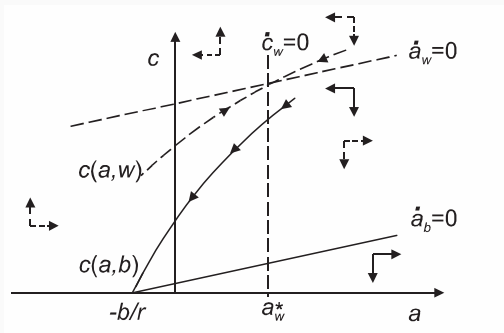
Proposition. Consider the case $0 < r \leq \rho$. Define the threshold level a_w^* by

$$\frac{u'(c(a_w^*, b))}{u'(c(a_w^*, w))} = 1 + \frac{\rho - r}{\lambda_2} \quad (4)$$

Then (i) Consumption of employed workers is increasing on $[\underline{a}, a_w^*]$ and decreasing $a > a_w^*$; (ii) consumption of unemployed workers always decrease

Properties of this system can be illustrated in the usual phase diagram

POLICIES



Note:

- Results help build some intuition on the problem. Look at [Bayer and Wälde \(2015\)](#) for much more. . .
- Now we change the approach.
Instead of looking at households' saving behavior in terms of a differential equation for its consumption policy function, we will focus on the HJB equation for the value function and how to solve it numerically.
- draw heavily on Moll's notes

Problem of Household

$$\max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \{ra_t + z_t - c_t\} dt$$

z_t is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w - b)dq_\mu - (w - b)dq_s, \quad q_\mu \sim \text{Poisson}(\lambda_1), \quad q_s \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' value function must satisfy [HJB equation](#)¹

$$\rho v_k(a) = \max_c \left\{ u(c) + v'_k(a)[ra + z_k - c] \right\} + \lambda_k [v_{-k}(a) - v_k(a)] \quad (5)$$

Borrowing constraint shows only as [state constraint boundary condition](#)

$$u'(c_i(\underline{a})) = v'_i(\underline{a}) \geq u'(r\underline{a} + z_i) \quad (6)$$

which ensures $s_i(\underline{a}) = r\underline{a} + z_i - c_i(\underline{a}) \geq 0$ so that the borrowing constraint is never violated.

¹change notation

Consider the **first-order condition** for consumption

$$u'(c) = \partial_a v(a, z) \quad (7)$$

$$u'(c) \geq \beta \int \partial_a v(a', z') dF(z'|z), \quad a' = z + (1 + r)a - c \quad (8)$$

Continuous time advantages:

1. “today” = “tomorrow” — *foc* is static
2. HJB is not stochastic — evolution of stochastic process is captured by additive terms
3. *foc* always holds with equality

Borrowing constraint shows only as **state constraint boundary condition**

$$u'(c_i(\underline{a}, t)) = \partial_a v_i(\underline{a}, t) \geq u'(r_t \underline{a} + z_i) \quad (9)$$

which ensures $c_i(\underline{a}, t) \leq r_t \underline{a} + z_i$ so that the borrowing constraint is never violated.

Numeric solution HJB

Finite difference methods: replace derivatives by differences. Simple right? Well developed theory... some slides on it

Recall our [HJB equation](#)

$$\rho v_k(a) - \sup_{c \in \Gamma_k(a)} \left\{ u(c) + \mathcal{D}^c v_k(a) \right\} = 0 \quad (10)$$

where

$$\mathcal{D}^c \phi_k(a) = \phi'_k(a)[ra + z_k - c] + \lambda_k [\phi_{-k}(a) - \phi_k(a)]$$

Define a grid $\{a_1, a_2, \dots, a_i, \dots\}$ and let $v_k = (v_k(a_1), \dots, v_k(a_i), \dots)'$. Discretizing this equation requires deciding upon

- which fd approximation to use: forward/backward differencing

$$v'_k(a) \approx \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}, \quad v'_k(a) \approx \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}},$$

Finite Differences

Discretization

Let \mathcal{D}^c be the discrete form of the differential operator \mathcal{D}^c , so that

$$(\mathcal{D}^c v)_{k,i} = \alpha_{k,i}(c) v_{k,i-1} + \beta_{k,i}(c) v_{k,i+1} - (\alpha_{k,i}(c) + \beta_{k,i}(c) + \lambda_k) v_{k,i} + \lambda_k v_{-k,i}$$

and the discretization

$$\rho v_{k,i} - \sup_{c \in \Gamma_{k,i}} \left\{ u(c) + (\mathcal{D}^c v^{n(+1)})_{k,i} \right\} = 0 \quad (11)$$

where discretization can use forward, backward or central discretization. If

$$\alpha_{k,i} \geq 0, \quad \beta_{k,i} \geq 0$$

we say that (11) is *positive coefficient discretization*. We will search for a discretization that satisfies this condition — more on the reason later.

Finite Differences

Upwind scheme

In order to ensure a *positive coefficient discretization* our choice of central/forward/backward differencing will depend, in general, on the control c . A useful rule for this problem is to use the so-called *upwind scheme*.

IDEA: Use forward difference whenever drift is positive, and use backward whenever it is negative.

Suppose that we have the value of consumption $c_{k,i}$ at a particular node. Let $s_{k,i} = ra_i + z_k - c_{k,i}$. In this case, the derivatives are approximated

$$\dots \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} s_{k,i}^+ + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} s_{k,i}^- + \dots$$

which in terms of our α, β

$$\alpha_{k,i}^{up} = -\frac{s_{k,i}^-}{a_i - a_{i-1}} \geq 0, \quad \beta_{k,i}^{up} = \frac{s_{k,i}^+}{a_{i+1} - a_i} \geq 0$$

Discretized HJB equation is

$$\rho v_{k,i} = u(c_{k,i}) + \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} [s_{k,i}(c)]^+ + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} [s_{k,i}(c)]^- + \lambda_k [v_{-k,i} - v_{k,i}] \quad (12)$$

which can be written in matrix notation

$$\rho v = u(c) + A v$$

But we don't know $c_{k,i}$! Remember that c satisfy the foc everywhere on the grid

$$u'(c_{k,i}) = v'_k(a_i)$$

so $c(v)$, $A(v)$. HJB equation is highly nonlinear, so we need an iterative method to solve it.

Finite Differences

Implicit Timestepping

Start with a vector v^n , solve for foc and update v^{n+1} according to

$$\frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^n) + \frac{v_{k,i+1}^{n+1} - v_{k,i}^{n+1}}{a_{i+1} - a_i} [s_{k,i}^{F,n}]^+ + \frac{v_{k,i}^{n+1} - v_{k,i-1}^{n+1}}{a_i - a_{i-1}} [s_{k,i}^{B,n}]^- + \lambda_k [v_{-k,i}^{n+1} - v_{k,i}^{n+1}] \quad (13)$$

- Compute the policy from the foc $\left(u'(c_{k,i}^n) = \partial_a v_{k,i}^n\right)$ for the backward AND forward derivative of the value function.
- Define $s_{k,i}^{B,n} = ra_i + z_k - c_{k,i}^{B,n}$, $s_{k,i}^{F,n} = ra_i + z_k - c_{k,i}^{F,n}$. Set

$$c_{k,i}^n = \mathbb{1} \left\{ s_{k,i}^{B,n} \leq 0 \right\} \times c_{k,i}^{B,n} + \mathbb{1} \left\{ s_{k,i}^{F,n} \geq 0 \right\} \times c_{k,i}^{F,n} + \mathbb{1} \left\{ s_{k,i}^{F,n} \leq 0 \leq s_{k,i}^{B,n} \right\} \times (ra_i + z_k)$$

- Collecting terms with the same subscripts on the right-hand side

$$\frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^n) + \alpha_{k,i} v_{k,i-1}^{n+1} + \beta_{k,i} v_{k,i+1}^{n+1} - \left(\alpha_{k,i} + \beta_{k,i} + \lambda_k \right) v_{k,i}^{n+1} + \lambda_i v_{-k,i}^{n+1} \quad (14)$$

where

$$\alpha_{k,i}^{up} = -\frac{[s_{k,i}^{B,n}]^-}{a_i - a_{i-1}} \geq 0, \quad \beta_{k,i}^{up} = \frac{[s_{k,i}^{F,n}]^+}{a_{i+1} - a_i} \geq 0$$

- Equation (14) is just a system of linear equations on v^{n+1} !!

Finite Differences

Implicit Timestepping

Equation (14) can be written in matrix notation as

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u(c^n) + \mathbf{A}^n v^{n+1}$$

where the sparse matrix \mathbf{A} looks like

$$\mathbf{A}^n = \begin{bmatrix} \gamma_{1,1} & \beta_{1,1} & 0 & 0 & \dots & \lambda_1 & 0 & \dots & 0 \\ \alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & 0 & \dots & 0 & \lambda_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \alpha_{1,I} & \gamma_{1,I} & 0 & 0 & \dots & \dots & 0 & \lambda_1 \\ \lambda_2 & 0 & \dots & \dots & \dots & \gamma_{2,1} & \beta_{2,1} & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & \dots & \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \lambda_2 & 0 & 0 & \dots & \dots & \alpha_{2,I} & \gamma_{2,I} \end{bmatrix}_{I+2 \times I+2}$$

entries of row i

$$\left[\underbrace{\alpha_{k,i}}_{\text{inflow } i-1} \quad \underbrace{-(\alpha_{k,i} + \beta_{k,i} + \lambda_k)}_{\text{outflow}} \quad \underbrace{\beta_{k,i}}_{\text{inflow } i+1} \right] \begin{bmatrix} v_{k,i-1} \\ v_{k,i} \\ v_{k,i+1} \end{bmatrix}$$

Iterate until $v^{n+1} \approx v^n$

Background Finite Difference

Time dependent version of HJB

$$\rho v_k(a) = \max_c \left\{ u(c) + v'_k(a)[ra + z_k - c] \right\} + \lambda_k [v_{-k}(a) - v_k(a)]$$

in a PDE notation

$$0 = F(\mathbf{x}, v, Dv, D^2v) \quad (15)$$

where $\mathbf{x} := (a, z)$. Suppose we define a grid $\{a_0, a_1, \dots, a_i, \dots\}$. Let $v_{k,i} \approx v_k(a_i)$ be the approximate value of the solution. Then we can write a general **discretization** of the HJB equation at node (a_i, z_k)

$$0 = S_{k,i} \left(\tilde{\Delta}, v_{k,i}, \{v_{m,j}\}_{m \neq k, j \neq i} \right) \quad (16)$$

Condition (Monotonicity) . — The numerical scheme (16) is monotone if

$$S_{k,i}(\cdot, v_{k,i}, \{y_{m,j}\}) \leq S_{k,i}(\cdot, v_{k,i}, \{z_{m,j}\})$$

for all $y \geq z$.

Condition (Stability) . — The numerical scheme (16) is stable if for every $\tilde{\Delta} > 0$ it has a solution which is uniformly bounded independently of $\tilde{\Delta}$.

Condition (Consistency) . — The numerical scheme (16) is consistent if for every smooth function ϕ with bounded derivatives we have

$$S_{k,i}(\tilde{\Delta}, \phi(\mathbf{x}_{k,i}), \{\phi(\mathbf{x}_{m,j})\}) \rightarrow F(\mathbf{x}, \phi, D\phi, D^2\phi)$$

as $\tilde{\Delta} \rightarrow 0$ and $\mathbf{x}_{k,i} \rightarrow \mathbf{x}$.

Theorem *Barles and Souganidis (1990)* . *If the numerical scheme S (16) satisfies monotonicity, stability and consistency conditions, then its solution converges locally uniformly to the unique viscosity solution of (15).*

- Convergence here is about $\tilde{\Delta} \rightarrow 0$
- For given $\tilde{\Delta}$, we have a system of I non-linear equations that we must solve somehow (Implicit scheme). Theorem guarantees that the solution $\{v_{k,i}\}$ of this system converges to the “viscosity solution” of the original PDE as $\tilde{\Delta} \rightarrow 0$
- “viscosity solution” of the HJB is the the **value function**
- A *positive coefficient discretization* is also *Monotone*. To see it check that

$$S_{k,i}(\tilde{\Delta}, v_{k,i}, v_{k,i+1}, v_{k,i-1}, v_{k,i}, v_{-k,i})$$

is a nonincreasing function of the neighbor nodes $\{v_{m,j}\}$. Check a example!

Computing the Distribution

- We now know how to solve the Household consumption/savings problem
- But interesting questions require dealing with distributions
- Denote by $g_i(a, t)$ $i = 1, 2$ the joint density of income z_i and wealth a .
- The evolution of the density given a fixed initial distribution $g_i(a, 0)$ is described by the *Kolmogorov forward equation*
 - time dependent

$$\frac{\partial}{\partial t} g(a, t) = -\frac{\partial}{\partial a} [s_k(a, t) g_k(a, t)] - \lambda_k g_k(a, t) + \lambda_{-k} g_{-k}(a, t) \quad (17)$$

- stationary

$$0 = -\frac{d}{da} [s_k(a) g_k(a)] - \lambda_k g_k(a) + \lambda_{-k} g_{-k}(a) \quad (18)$$

Kolmogorov Forward equation

Stationary

Consider the stationary KFE

$$0 = -\frac{d}{da} \left[s(a, z_k) g(a, z_k) \right] - \lambda_k g(a, z_k) + \lambda_{-k} g(a, z_{-k})$$

with the following discretization

$$0 = -\frac{(s_{k,i}^F)^+ g_{k,i} - (s_{k,i-1}^F)^+ g_{k,i-1}}{\Delta a} - \frac{(s_{k,i+1}^B)^- g_{k,i+1} - (s_{k,i}^B)^- g_{k,i}}{\Delta a} - \lambda_k g_{k,i} + \lambda_{-k} g_{-k,i} \quad (19)$$

Collecting terms with the same subscripts on the right-hand side

$$0 = \underbrace{\frac{(s_{k,i-1}^F)}{\Delta a}}_{\beta_{k,i-1}} g_{k,i-1} + \underbrace{\left(\frac{(s_{k,i}^B)}{\Delta a} - \frac{(s_{k,i}^F)}{\Delta a} - \lambda_k \right)}_{\gamma_{k,i}} g_{k,i} + \underbrace{\left(-\frac{(s_{k,i+1}^B)}{\Delta a} \right)}_{\alpha_{k,i+1}} g_{k,i+1} + \lambda_{-k} g_{-k,i}$$

which in matrix notation reads $\mathbf{A}^T \mathbf{g} = \mathbf{0}$. Numerically, this is very efficient bs we have already computed \mathbf{A} .

This makes sense: the operation is exactly the same as that used for finding the stationary distribution of a discrete Poisson process (continuous-time Markov chain). The matrix \mathbf{A} captures the evolution of the stochastic process over a very short interval — it is our discretized *infinitesimal generator* of our state — and to find the stationary distribution, one solves the eigenvalue problem $\mathbf{A}^T \mathbf{g} = \mathbf{0}$.

Kolmogorov Forward equation

Time dependent

[Pass slide to Transition dynamics]

Consider the time dependent

$$\frac{\partial}{\partial t} g(a, t) = -\frac{\partial}{\partial a} \left[s_k(a, t) g_k(a, t) \right] - \lambda_k g_k(a, t) + \lambda_{-k} g_{-k}(a, t)$$

Given an initial condition, the KFE can be easily solve through a implicit method

$$\frac{g^{n+1} - g^n}{\Delta} = \left(\mathbf{A}^{n(+1)} \right)^T g^{n+1} \quad (20)$$

Stationary Equilibrium + Transion Dynamics

-

$$\rho v_k(a) = \max_c \left\{ u(c) + \partial_a v_k(a) \right\} + \lambda_k \left[v_\ell(a) - v_k(a) \right] \quad [\text{HJB}]$$

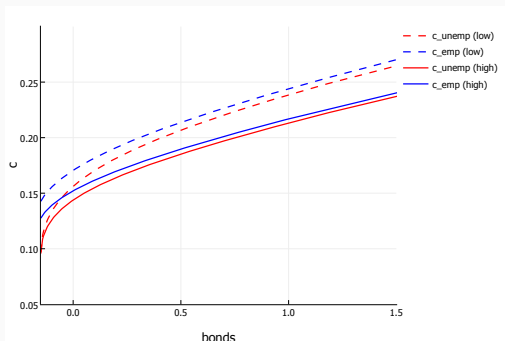
$$\begin{aligned} 0 &= \frac{d}{da} [s_k(a)g_k(a)] - \lambda_k g_k(a) + \lambda_\ell g_\ell(a) \\ 1 &= \int_{\underline{a}}^{\infty} (g_1(a) + g_2(a)) da \end{aligned} \quad [\text{KFE}]$$

$$0 = \int_{\underline{a}}^{\infty} a (g_1(a) + g_2(a)) da \quad [\text{Equil}]$$

Experiment

Suppose an increase in the unemployment risk λ_2 . What would you expect has to happen to the interest rate? *Distributional effect of more unemployed in equilibrium makes me converge to a higher interest rate!*

CONSUMPTION POLICY

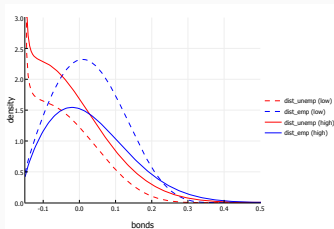


Note:

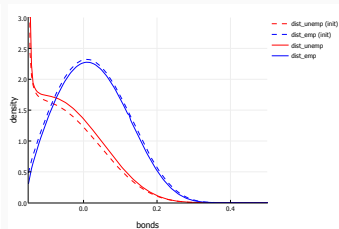
Definition and how to compute!!

-

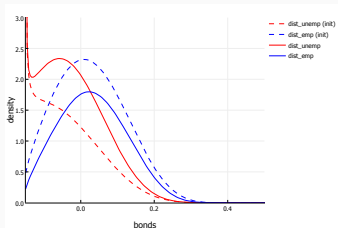
TRANSITION DYNAMICS



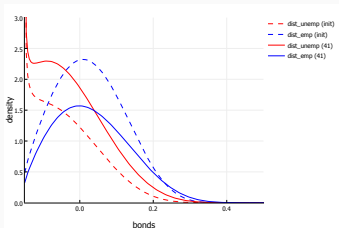
init \times end



$t = 0.125$



$t = 2.0$



$t = 5.0$

Note:

Why do I care??

Example

- looks nice, but why should I pay a cost if I can do discrete time?
- It was my original thought, but recently. . .
- Consumption/savings problem + direct search labor market

$$V^u(a) = \max_{c, a'} u(c) + \beta \mathcal{R}^u(a') \quad \mathcal{R}^u(\tilde{a}) = V^u(\tilde{a}) + \max_{\tilde{w}} p(\theta(\tilde{a}, \tilde{w})) \left[V^e(\tilde{a}, \tilde{w}) - V^u(\tilde{a}) \right]$$

$$\text{S.t. } c + \frac{a'}{1+r} = b + a, \quad a' \geq \underline{a}$$

euler equation for asset holdings

$$u'(c(\cdot)) \geq \beta(1+r) \left\{ (1-p(\theta(a', \tilde{w}))) V_a^u(a') + p(\theta(a', \tilde{w})) V_a^e(a', \tilde{w}) + p'(\theta(a', \tilde{w})) \frac{\partial \theta(a', \tilde{w})}{\partial a} \left[V^e(a', \tilde{w}) - V^u(a', \tilde{w}) \right] \right\}$$

- EGM for consumption/savings, VFI for labor choice.
- Finding the equilibrium requires iterating over a lot of stuff

$$V(\cdot) \hookrightarrow \hat{\theta}(\cdot) \hookrightarrow \mathcal{J}(\cdot) \hookrightarrow \theta(\cdot)$$

- couldn't do it = (...

Example

- How does this look in continuous time, *today is tomorrow*

$$\rho V^u(a) = \max_c \left\{ u(c) + V_a^u(b + ra - c) \right\} + \underbrace{\lambda_u}_{\text{rate of search}} \max_{\tilde{w}} \left\{ p(\theta(a, \tilde{w})) \left[V^e(a, w) - V^u(a) \right] \right\} \quad (21)$$

- This is wayyyyy simpler and importantly it doesn't seem I am throwing away anything of the economics

Here is some content on the slide

$$f(x) = ax^2 + bx + c$$

Some more content

References

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