CompEcon Workshop

Continuous time methods

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References

Theory: Stochastic Calulus and Stochastic Control

- Pham (2009) Continuous-time Stochastic Control – maybe too finance
- Lecture notes
  - ↪ Caldentey (????) Stochastic processes and optimal control — nice lecture notes Enio uses them
  - ↪ Ross (????) *Stochastic Control in Continuous Time* — alternative to math books Fleming and Soner (2006), Øksendal (2003), Øksendal and Sulem (2007)

Macro

- Moll's website, Nuno syllabus
- Stokey (2009) book – *Impulse control Problem*
- Bayer and Wälde (2015) – recent discovery, dicuss the kind of SDE driven by a Markov chain
  - ↪ Sennewald (2007) (theory paper), Walde (2008) (book on intertemporal optimization),

Numeric

- Achdou, Han, Lasry, Lions, and Moll (2016) (mainly the numerical appendix), Moll's website (tons of examples and materials)
- Forsyth and Vetzal (2012) (Also has some slides) – good introduction to "viscosity solutions"
- Interested? Check applications ...
  - ↪ **HANK** by Kaplan, Moll, and Violante (2016) (Transition Dynamics)
  - ↪ **PHACT** (Reiter + HACT)
  - ↪ Nuño and Moll (2017) (social optimum in models with heterogeneous agents)
  - ↪ Thomas and Nuño (2016) (impulse control)

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Consumption Savings Problem

Problem of Household

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \left\{ ra_t + z_t - c_t \right\} dt$$

z_t: is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w-b)dq - (w-b)dQ, \quad q \sim \text{Poisson}(\lambda_1),\ Q \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' consumption and saving decision is summarized by HJB equation

$$\rho v(a, z_k) = \max_c \left\{ u(c) + v_a(a, z)[ra + z_k - c] \right\} + \lambda_k \left[v(a, z_{-k}) - v(a, z_k) \right] \qquad (1)$$

Where this came from? Check *Lagos* lecture notes for an heuristic argument.

Theoretical results analogous to discrete time:

- Value function satisfy the HJB equation
- Verification theorems: solution of HJB + ... $\rightarrow$ value function
- Alternatively, one can show HJB has a unique "nice" solution which is the value function (viscosity solution)

Households

Keynes-Ramsey rules

Before solving the HJB FE let's see what we can do. Analytical results from Bayer and Wälde (2015)

Envelope condition:

$$\rho V_a(a,b) = rV_a(a,b) + V_{aa}(a,b)\Big\{ra + b - c(a,b)\Big\} + \lambda_1\Big[V_a(a,w) - V_a(a,b)\Big]$$

Differential of $V_a(a,z)$ — CVF, "Itô formula"

$$\begin{aligned} da_t &= \Big\{ra_t + z_t - c_t\Big\}dt \\ dz_t &= (w-b)dq - (w-b)dQ, \quad q \sim \text{Poisson}(\lambda_1),\ Q \sim \text{Poisson}(\lambda_2) \end{aligned}$$

$$\mathrm{d}V_a(a,b) = \underbrace{V_{aa}\Big\{ra + b - c(a,b)\Big\}}_{\text{normal term}}\mathrm{d}t + \underbrace{\Big[V_a(a,w) - V_a(a,b)\Big]}_{\text{jump terms}}\mathrm{d}q_t$$

From optimization $V_a(a,z) = u'\left(c(a,z)\right)$. Combining both equations to get rid of V_{aa} we have

$$\begin{aligned} \mathrm{d}u'\left(c(a,b)\right) = &\left\{(\rho - r)u'\left(c(a,b)\right) - \lambda_1 u'\left(c(a,b)\right)\left[\frac{u'\left(c(a,w)\right)}{u'\left(c(a,b)\right)} - 1\right]\right\}\mathrm{d}t + \\ &+ \Big[u'\left(c(a,w)\right) - u'\left(c(a,b)\right)\Big]\mathrm{d}q_t \end{aligned}$$

Households
Keynes-Ramsey rules

Applying "Itô lemma" to get consumption over time

$$\mathrm{d}c(a,b) = \frac{u'\left(c(a,b)\right)}{-u''\left(c(a,b)\right)}\left\{r-\rho-\lambda_1\left[1-\frac{u'\left(c(a,w)\right)}{u'\left(c(a,b)\right)}\right]\right\}\mathrm{d}t + \left[c(a,w)-c(a,b)\right]\mathrm{d}q_t \qquad (2)$$

$$\mathrm{d}c(a,w) = \frac{u'\left(c(a,w)\right)}{-u''\left(c(a,w)\right)}\left\{r-\rho+\underbrace{\lambda_2\left[\frac{u'\left(c(a,b)\right)}{u'\left(c(a,w)\right)}-1\right]}_{\text{prec. savings}}\right\}\mathrm{d}t + \underbrace{\left[c(a,b)-c(a,w)\right]\mathrm{d}Q_t}_{\text{jumps}} \qquad (3)$$

neoclassical growth model $\dot{c}(t) = \frac{u'(c)}{-u''(c)}\left(r-\rho\right)$

Looking at period between jumps. What the signs tell us?

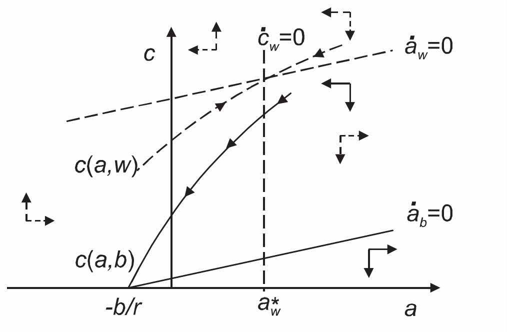
Proposition. *Consider the case $0 < r \leq \rho$. Define the threshold level a_w^* by*

$$\frac{u'\left(c(a_w^*,b)\right)}{u'\left(c(a_w^*,w)\right)} = 1 + \frac{\rho - r}{\lambda_2} \qquad (4)$$

Then (i) Consumption of employed workers is increasing on $[\underline{a}, a_w^*]$ and decreasing $a > a_w^*$; (ii) consumption of unemployed workers always decrease

Properties of this system can be illustrated in the usual phase diagram

Policies



Note:

Change

- Results help build some intuition on the problem. Look at Bayer and Wälde (2015) for much more. . .

- Now we change the approach.
  Instead of looking at households' saving behavior in terms of a differential equation for its consumption policy function, we will focus on the HJB equation for the value function and how to solve it numerically.

- draw heavily on Moll's notes

Problem of Household

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \left\{ ra_t + z_t - c_t \right\} dt$$

z_t is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w-b)dq_\mu - (w-b)dq_s, \quad q_\mu \sim \text{Poisson}(\lambda_1), \; q_s \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' value function must satisfy HJB equation[1]

$$\rho v_k(a) = \max_c \left\{ u(c) + v_k'(a)[ra + z_k - c] \right\} + \lambda_k \left[v_{\text{-}k}(a) - v_k(a) \right] \tag{5}$$

Borrowing constraint shows only as *state constraint boundary condition*

$$u'\left(c_i(\underline{a})\right) = v_i'(\underline{a}) \geq u'\left(r\underline{a} + z_i\right) \tag{6}$$

which ensures $s_i(\underline{a}) = r\underline{a} + z_i - c_i(\underline{a}) \geq 0$ so that the borrowing constraint is never violated.

[1] change notation

Continuous $\times$ Discrete time

Consider the first-order condition for consumption

$$\text{cont time:} \qquad u'(c) = \partial_a v(a, z) \tag{7}$$

$$\text{disc time:} \qquad u'(c) \geq \beta \int \partial_a v(a', z')\, \mathrm{d}F(z'|z), \quad a' = z + (1+r)a - c \tag{8}$$

Continuous time **advantages**:

1. "today" = "tomorrow" — *foc* is static
2. HJB is not stochastic — evolution of stochastic process is captured by additive terms
3. **Borrowing constraint** shows only as *state constraint boundary condition*

Numeric solution HJB

Finite Difference

Finite difference methods: replace derivatives by differences. Simple right? Well developed theory... some slides on it

Recall our HJB equation

$$\rho v_k(a) - \sup_{c \in \Gamma_k(a)} \left\{ u(c) + \mathcal{D}^c v_k(a) \right\} = 0 \tag{9}$$

where

$$\mathcal{D}^c \phi_k(a) = \phi_k'(a)[ra + z_k - c] + \lambda_k \left[\phi_{\text{-}k}(a) - \phi_k(a) \right]$$

Define a grid $\{a_1, a_2, \ldots, a_i, \ldots\}$ and let $v_k = \left(v_k(a_1), \ldots, v_k(a_i), \ldots \right)'$. Discretizing this equation requires deciding upon

- which fd approximation to use: forward/backward differencing

$$v_k'(a) \approx \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}, \quad v_k'(a) \approx \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}},$$

Let $\mathscr{D}^c$ be the discrete form of the differential operator $\mathcal{D}^c$, so that

$$\left(\mathscr{D}^c v\right)_{k,i} = \alpha_{k,i}(c) v_{k,i-1} + \beta_{k,i}(c) v_{k,i+1} - \Big(\alpha_{k,i}(c) + \beta_{k,i}(c) + \lambda_k\Big) v_{k,i} + \lambda_k v_{\text{-}k,i}$$

and the discretization

$$\rho v_{k,i} - \sup_{c \in \Gamma_{k,i}} \left\{ u(c) + \left(\mathscr{D}^c v^{n(+1)}\right)_{k,i} \right\} = 0 \tag{10}$$

where discretization can use forward, backward or central discretization. If

$$\alpha_{k,i} \geq 0, \;\; \beta_{k,i} \geq 0$$

we say that (10) is *positive coefficient discretization*. We will search for a discretization that satisfies this condition — more on the reason later.

In order to ensure a *positive coefficient discretization* our choice of central/forward/backward differencing will depend, in general, on the control c. A useful rule for this problem is to use the so-called *upwind scheme*.

IDEA: *Use forward difference whenever drift is positive, and use backward whenever it is negative.*

Suppose that we have the value of consumption $c_{k,i}$ at a particular node. Let $s_{k,i} = ra_i + z_k - c_{k,i}$. In this case, the derivatives are approximated

$$\dots \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} s^+_{k,i} + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} s^-_{k,i} + \dots$$

which in terms of our α, β

$$\alpha^{up}_{k,i} = -\frac{s^-_{k,i}}{a_i - a_{i-1}} \geq 0, \quad \beta^{up}_{k,i} = \frac{s^+_{k,i}}{a_{i+1} - a_i} \geq 0$$

Discretized HJB equation is

$$\rho v_{k,i} = u(c_{k,i}) + \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}\Big[s_{k,i}(c)\Big]^{+} + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}}\Big[s_{k,i}(c)\Big]^{-} + \lambda_k\Big[v_{\text{-}k,i} - v_{k,i}\Big] \quad (11)$$

which can be written in matrix notation

$$\rho v = \mathbf{u} + \mathbf{A} v$$

But we don't know $c_{k,i}$! Rememeber that c satisfy the foc everywhere on the grid

$$u'\big(c_{k,i}\big) = v_k'(a_i)$$

so $\mathbf{c}(v), \mathbf{A}(v)$. HJB equation is highly nonlinear, so we need an iterative method to solve it.

Start with a vector v^n, solve for foc and update v^{n+1} according to

$$\frac{v_{k,i}^{n+1} - v_{k,i}^{n}}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^{n}) + \frac{v_{k,i+1}^{n+1} - v_{k,i}^{n+1}}{a_{i+1} - a_i}\left[s_{k,i}^{F,n}\right]^{+} + \frac{v_{k,i}^{n+1} - v_{k,i-1}^{n+1}}{a_i - a_{i-1}}\left[s_{k,i}^{B,n}\right]^{-} + \lambda_k\left[v_{\text{-}k,i}^{n+1} - v_{k,i}^{n+1}\right] \quad (12)$$

- Compute the policy from the foc $\left(u'\left(c_{k,i}^{n}\right) = \partial_a v_{k,i}^{n}\right)$ for the backward AND forward derivative of the value function.
- Define $s_{k,i}^{B,n} = ra_i + z_k - c_{k,i}^{B,n}$, $s_{k,i}^{F,n} = ra_i + z_k - c_{k,i}^{F,n}$. Set

$$c_{k,i}^{n} = \mathbb{1}\left\{s_{k,i}^{B,n} \le 0\right\} \times c_{k,i}^{B,n} + \mathbb{1}\left\{s_{k,i}^{F,n} \ge 0\right\} \times c_{k,i}^{F,n} + \mathbb{1}\left\{s_{k,i}^{F,n} \le 0 \le s_{k,i}^{B,n}\right\} \times (ra_i + z_k)$$

- Collecting terms with the same subscripts on the right-hand side

$$\frac{v_{k,i}^{n+1} - v_{k,i}^{n}}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^{n}) + \alpha_{k,i} v_{k,i-1}^{n+1} + \beta_{k,i} v_{k,i+1}^{n+1} - \left(\alpha_{k,i} + \beta_{k,i} + \lambda_k\right) v_{k,i}^{n+1} + \lambda_i v_{\text{-}k,i}^{n+1} \quad (13)$$

where

$$\alpha_{k,i}^{up} = -\frac{\left[s_{k,i}^{B,n}\right]^{-}}{a_i - a_{i-1}} \ge 0, \quad \beta_{k,i}^{up} = \frac{\left[s_{k,i}^{F,n}\right]^{+}}{a_{i+1} - a_i} \ge 0$$

- Equation (13) is just a system of linear equations on v^{n+1}!!

Finite Differences

Implicit Timestepping

Equation (13) can be written in matrix notation as

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = \mathbf{u}(c^n) + \mathbf{A}^n v^{n+1}$$

where the sparse matrix **A** looks like

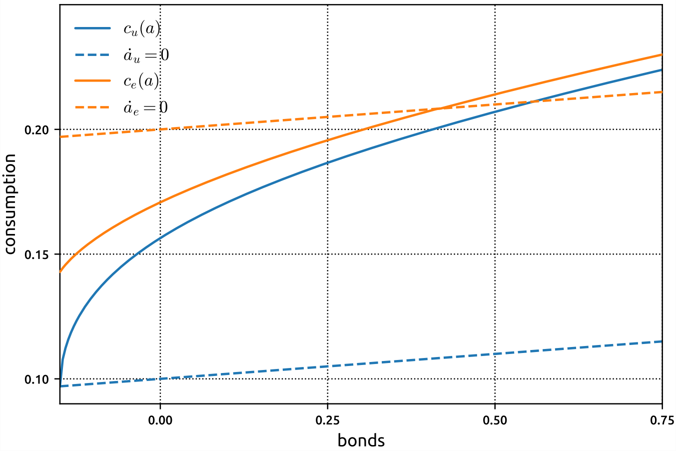
$$\mathbf{A}^n = \begin{bmatrix}
\gamma_{1,1} & \beta_{1,1} & 0 & 0 & \dots & \lambda_1 & 0 & \dots & & & 0 \\
\alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & 0 & \dots & 0 & \lambda_1 & 0 & \dots & & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & \alpha_{1,I} & \gamma_{1,I} & 0 & 0 & \dots & \dots & 0 & \lambda_1 \\
\lambda_2 & 0 & \dots & \dots & \dots & \gamma_{2,1} & \beta_{2,1} & 0 & \dots & \dots & 0 \\
0 & \lambda_2 & 0 & \dots & \dots & \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & 0 & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \dots & \dots & \lambda_2 & 0 & 0 & \dots & \dots & \alpha_{2,I} & \gamma_{2,I}
\end{bmatrix}_{I+2 \times I+2}$$

entries of row i

$$\begin{bmatrix} \underbrace{\alpha_{k,i}}_{\text{inflow } i-1} & \underbrace{-\left(\alpha_{k,i} + \beta_{k,i} + \lambda_k\right)}_{\text{outflow}} & \underbrace{\beta_{k,i}}_{\text{inflow } i+1} \end{bmatrix} \begin{bmatrix} v_{k,i-1} \\ v_{k,i} \\ v_{k,i+1} \end{bmatrix}$$

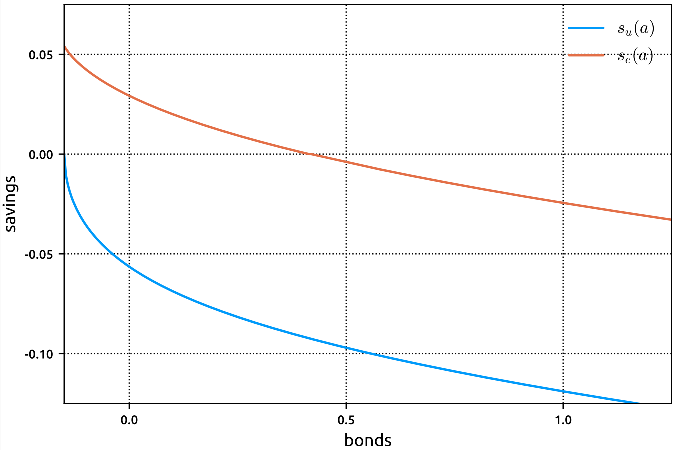
Iterate until $v^{n+1} \approx v^n$

Solution



$c_u(a)$
$\dot{a}_u = 0$
$c_e(a)$
$\dot{a}_e = 0$
0.20
0.15
0.10
consumption
0.00
0.25
0.50
0.75
bonds

Solution



Backgroud Finite Difference

Why does it work?

Definition

Time dependent version of HJB

$$\rho v_k(a) = \max_c \left\{ u(c) + v_k'(a)[ra + z_k - c] \right\} + \lambda_k \left[v_{\text{-}k}(a) - v_k(a) \right]$$

in a PDE notation

$$0 = F(\boldsymbol{x}, v, Dv, D^2 v) \tag{14}$$

where $\boldsymbol{x} := (a, z)$. Suppose we define a grid $\{a_0, a_1, \dots, a_i, \dots\}$. Let $v_{k,i} \approx v_k(a_i)$ be the approximate value of the solution. Then we can write a general **discretization** of the HJB equation at node (a_i, z_k)

$$0 = S_{k,i}\left(\tilde{\Delta}, v_{k,i}, \{v_{m,j}\}_{m \neq k, j \neq i}\right) \tag{15}$$

Sufficient Conditions Convergence

Condition (Monotonicity) .— The numerical scheme (15) is monotone if

$$S_{k,i}(\cdot, v_{k,i}, \{y_{m,j}\}) \leq S_{k,i}(\cdot, v_{k,i}, \{z_{m,j}\})$$

for all $y \geq z$.

Condition (Stability) .— The numerical scheme (15) is stable if for every $\tilde{\Delta} > 0$ it has a solution which is uniformly bounded independently of $\tilde{\Delta}$.

Condition (Consistency) .— The numerical scheme (15) is consistent if for every smooth function ϕ with bounded derivatives we have

$$S_{k,i}(\tilde{\Delta}, \phi(\boldsymbol{x}_{k,i}), \{\phi(\boldsymbol{x}_{m,j})\}) \to F(\boldsymbol{x}, \phi, D\phi, D^2\phi)$$

as $\tilde{\Delta} \to 0$ and $\boldsymbol{x}_{k,i} \to \boldsymbol{x}$.

Sufficient Conditions Convergence

Theorem Barles and Souganidis (1990) . *If the numerical scheme S (15) satisfies monotonicity, stability and consistency conditions, then its solution converges locally uniformly to the unique viscosity solution of (14).*

- Convergence here is about $\tilde{\Delta} \to 0$
- For given $\tilde{\Delta}$, we have a system of I non-linear equations that we must solve somehow (Implicit scheeme). Theorem guarantees that the solution $\{v_{k,i}\}$ of this system converges to the "viscosity solution" of the original PDE as $\tilde{\Delta} \to 0$
- "viscosity solution" of the HJB is the the **value function**
- A *positive coefficient discretization* is also *Monotone*. To see it check that

$$S_{k,i}\left(\tilde{\Delta}, v_{k,i}, v_{k,i+1}, v_{k,i-1}, v_{k,i}, v_{\text{-}k,i}\right)$$

  is a nonincreasing function of the neighbor nodes $\{v_{m,j}\}$. Check a example!

Computing the Distribution

Distributions

- We now know how to solve the Household consumption/savings problem
- But interesting questions require dealing with distributions
- Denote by $g_i(a, t)$ $i = 1, 2$ the joint density of income z_i and welath a.
- The evolution of the density given a fixed initial distribution $g_i(a, 0)$ is described by the *Kolmogorov forward equation*
  - time dependent

$$\frac{\partial}{\partial t} g(a, t) = -\frac{\partial}{\partial a}\Big[s_k(a, t) g_k(a, t)\Big] - \lambda_k g_k(a, t) + \lambda_{\text{-}k} g_{\text{-}k}(a, t) \tag{16}$$

  - stationary

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s_k(a) g_k(a)\Big] - \lambda_k g_k(a) + \lambda_{\text{-}k} g_{\text{-}k}(a) \tag{17}$$

Consider the stationary KFE

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s(a, z_k)g(a, z_k)\Big] - \lambda_k g(a, z_k) + \lambda_{\text{-}k} g(a, z_{\text{-}k})$$

with the following discretization

$$0 = -\frac{\left(s^F_{k,i}\right)^+ g_{k,i} - \left(s^F_{k,i-1}\right)^+ g_{k,i-1}}{\Delta a} - \frac{\left(s^B_{k,i+1}\right)^- g_{k,i+1} - \left(s^B_{k,i}\right)^- g_{k,i}}{\Delta a} - \lambda_k g_{k,i} + \lambda_{-k} g_{\text{-}k,i} \quad (18)$$

Collecting terms with the same subscripts on the right-hand side

$$0 = \underbrace{\frac{\left(s^F_{k,i-1}\right)}{\Delta a}}_{\beta_{k,i-1}} g_{k,i-1} + \underbrace{\left(\frac{\left(s^B_{k,i}\right)}{\Delta a} - \frac{\left(s^F_{k,i}\right)}{\Delta a} - \lambda_k\right)}_{\gamma_{k,i}} g_{k,i} + \underbrace{\left(-\frac{\left(s^B_{k,i+1}\right)}{\Delta a}\right)}_{\alpha_{k,i+1}} g_{k,i+1} + \lambda_{-k} g_{\text{-}k,i}$$

which in matrix notation reads $\mathbf{A}^T g = \mathbf{0}$. Numerically, this is very efficient bs we have already computed $\mathbf{A}$.

This makes sense: the operation is exactly the same as that used for finding the stationary distribution of a discrete Poisson process (continuous-time Markov chain). The matrix A captures the evolution of the stochastic process over a very short interval — it is our discretized *infinitesimal generator* of our state — and to find the stationary distribution, one solves the eigenvalue problem $\mathbf{A}^T g = \mathbf{0}$.

Stationary Equilibrium + Transion Dynamics

Stationary Equilibrium

Definition. *A stationary recursive competitive equilibrium is*

$$\Big(v, c, s, r, g\Big)$$

such that ...

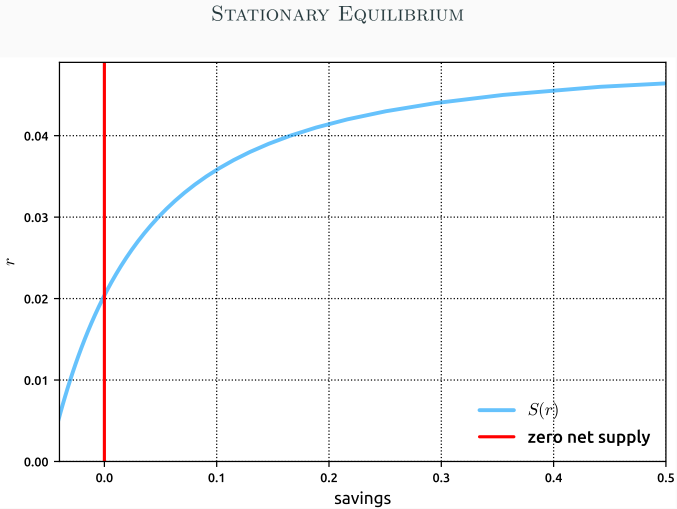
$$\rho v_k(a) = \max_c \Big\{ u(c) + v_k'(a)\big[ra + z_k - c\big]\Big\} + \lambda_k \Big[v_\ell(a) - v_k(a)\Big] \qquad \Big[\text{HJB}\Big]$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}a}\big[s_k(a)g_k(a)\big] - \lambda_k g_k(a) + \lambda_\ell g_\ell(a)$$

$$1 = \int_{\underline{a}}^{\infty} \Big(g_1(a) + g_2(a)\Big)\, da \qquad \Big[\text{KFE}\Big]$$

$$0 = \int_{\underline{a}}^{\infty} a\Big(g_1(a) + g_2(a)\Big)\, da \qquad \Big[\text{Equil}\Big]$$

Stationary Equilibrium



Transition Dynamics

hypothetical thought experiments of the following form.

- Suppose the economy is in a stationary equilibrium, with a given government policy and all other exogenous elements that define preferences, endowments and technology
- **Unexpectedly**, either government policy or some exogenous elements of the economy (such as the labor productivity process) change
  This change was completely unexpected by all agents of the economy (a zero probability event), so that no anticipation actions were taken by any agent.
- We want to study the transition path induced by the exogenous change, from the old stationary equilibrium to a new stationary equilibrium (which may coincide with the old stationary equilibrium in case the exogenous change is of transitory nature, or may differ from it in case the exogenous change is permanent.

Transition Dynamics

The time-dependent analogue of the stationary system is

$$\rho v_k(a,t) = \max_c \left\{ u(c) + \partial_a v_k(a,t)\left[r(t)a + z_k - c\right] \right\} + \lambda_k \left[v_\ell(a) - v_k(a) \right] + \partial_t v_k(a,t) \qquad \left[\text{HJB}\right]$$

$$\partial_t g_k(a,t) = \partial_a \left[s_k(a,t) g_k(a,t) \right] - \lambda_k g_k(a,t) + \lambda_{\text{-}k} g_{\text{-}k}(a,t)$$
$$1 = \int_{\underline{a}}^{\infty} \Big(g_1(a,t) + g_2(a,t) \Big) da \qquad \left[\text{KFE}\right]$$

$$0 = \int_{\underline{a}}^{\infty} a \Big(g_1(a,t) + g_2(a,t) \Big) da \qquad \left[\text{Equil}\right]$$

where the density satisfies an initial condition and runs forwards

$$g_k(a,0) = g_k^0(a)$$

while the value function satisfies a terminal condition and runs backwards

$$v_k(a,T) = v_k^E(a)$$

We solve this system using the following algorithm. Guess a function $r^0(t)$ and then for $m = 1, 2, 3, \ldots$ follow

- Given $r^m(t)$, solve the HJB backwards in time to find $\{v_k^m(a,t), s_k^m(a,t)\}$
- Given $s_k^m(a,t)$ solve the KFE forward in time given initial condition to calculate the time path for $g_k(a,t)$
- Check market clearing for the whole path

$$S^m(t) = \int_{\underline{a}}^{\infty} a\Big(g_1^m(a,t) + g_2^m(a,t)\Big)da$$

- Update $r^{m+1}(t) = r^m(t) - \xi \frac{dS^m(t)}{dt}$

HJB:

Approximate the value function at I discrete points in the wealth dimension and N discrete points in the time dimension, and use the shorthand notation $v_{k,i}^n = v_k(a_i, t_n)$. The discrete approximation to the time-dependent HJB is

$$\rho v_{k,i}^n = u(c_{k,i}^{n+1}) + \left(v_{k,i}^n\right)' \left[r^n a + z_k - c_{k,i}^{n+1}\right] + \lambda_k \left[v_{-k,i}^n - v_{k,i}^n\right] + \frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta t} \tag{19}$$

which is exactly as we have before!!! Why?

KFE:

Consider the time dependent KFE

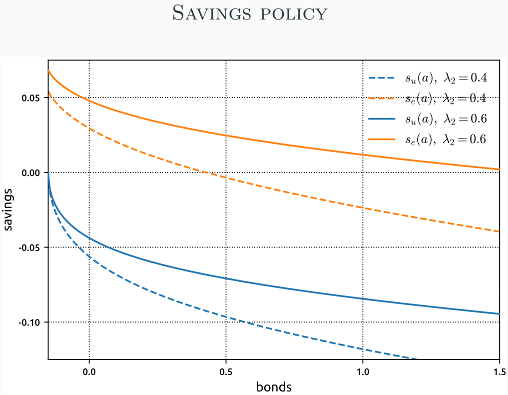
$$\frac{\partial}{\partial t} g(a,t) = -\frac{\partial}{\partial a}\left[s_k(a,t) g_k(a,t)\right] - \lambda_k g_k(a,t) + \lambda_{-k} g_{-k}(a,t)$$

Given an initial condition, the KFE can be easily solve through a implicit method

$$\frac{g^{n+1} - g^n}{\Delta} = \left(\mathbf{A}^{n(+1)}\right)^T g^{n+1} \tag{20}$$

Experiment

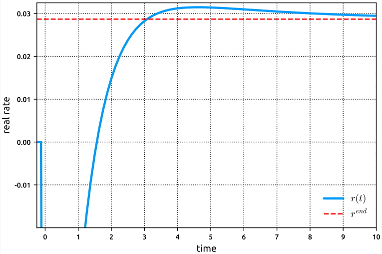
Suppose an increase in the unemployment risk λ_2. What would you expect has to happen to the interest rate?



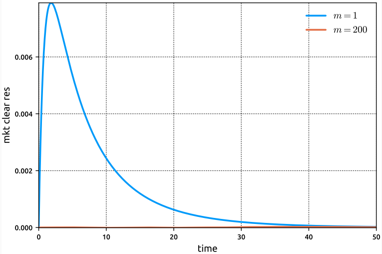
Distributional effect of more unemployed in equilibrium makes me converge to an higher interest rate!

Transition
Real rate path

TRANSITION

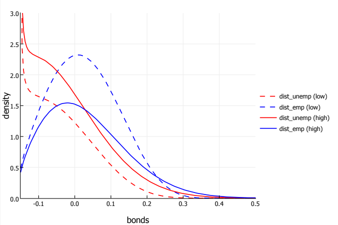


(a)

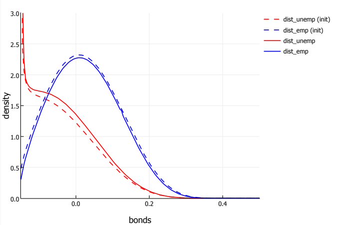


(b)

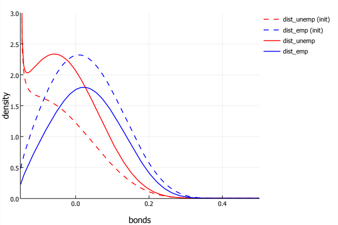
Transition Dynamics



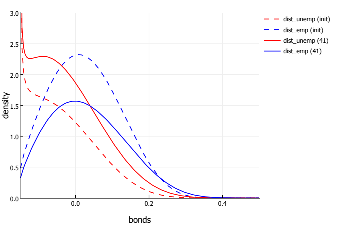
init $\times$ end



$t = 0.125$



$t = 2.0$



$t = 5.0$

Note:

Why do I care??

Example

- looks nice, but why should I pay a cost if I can do discrete time?
- It was my original tought, but recently...
- Consumption/savings problem + direct search labor market

$$V^u(a) = \max_{c,a'} u(c) + \beta \mathcal{R}^u(a') \qquad \mathcal{R}^u(\tilde{a}) = V^u(\tilde{a}) + \max_{\tilde{w}} p\big(\theta(\tilde{a},\tilde{w})\big)\Big[V^e(\tilde{a},\tilde{w}) - V^u(\tilde{a})\Big]$$

$$\text{S.t.} \quad c + \frac{a'}{1+r} = b + a, \quad a' \geq \underline{a}$$

euler equation for asset holdings

$$u'(c(\cdot)) \geq \beta(1+r)\Big\{\big(1-p(\theta(a',\tilde{w}))\big)V^u_a(a') + p(\theta(a',\tilde{w}))V^e_a(a',\tilde{w}) + p'(\theta(a',\tilde{w})\frac{\partial\theta(a',\tilde{w})}{\partial a}\Big[V^e(a',\tilde{w}) - V^u(a',\tilde{w})\Big]\Big\}$$

- EGM for consumption/savings, VFI for labor choice.
- Finding the equilibrium requires iterating over a lot of stuff

$$V(\cdot) \hookrightarrow \hat{\theta}(\cdot) \hookrightarrow \mathcal{J}(\cdot) \hookrightarrow \theta(\cdot)$$

- couldn't do it = (...

Example

- How does this look in continuous time, *today is tomorrow*

$$\rho V^u(a) = \max_c \Big\{ u(c) + V_a^u(b + ra - c) \Big\} + \underbrace{\lambda_u}_{\text{rate of search}} \max_{\tilde{w}} \Big\{ p(\theta(a, \tilde{w})) \Big[V^e(a, w) - V^u(a) \Big] \Big\} \tag{21}$$

- This is wayyyyy simpler and importantlty it doesn't seem I am throwing away anything of the economics

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