CompEcon Workshop

Continuous time methods

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References

Theory

- ? Continuous-time Stochastic Control – maybe too finance
- Lecture notes
  - ↪ ? Stochastic processes and optimal control — nice lecture notes Enio uses them
  - ↪ ? *Stochastic Control in Continuous Time* — alternative to math books ?, ?, ?

Theory (macro)

- Moll's website
- ? – recent discovery, dicuss the kind of SDE driven by a Markov chain
  - ↪ ? (theory paper), ? (book on intertemporal optimization),
- ? book – *Impulse control Problem*

Numeric

- ? (mainly the numerical appendix), Moll's website (tons of examples and materials)
- ? (Also has some slides) – good introduction to "viscosity solutions"
- Interested? Check applications ...
  - ↪ HANK by ?, PHACT
  - ↪ ? (improved notation)
  - ↪ ? (impulse control)

Consumption Savings Problem

Problem of Household

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \{ra_t + z_t - c_t\} dt$$

z_t: is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w-b)dq - (w-b)dQ, \quad q \sim \text{Poisson}(\lambda_1), \; Q \sim \text{Poisson}(\lambda_2)$$

$$0.2a_t \geq \underline{a}$$

Individuals' consumption and saving decision is summarized by HJB equation

$$\rho v(a, z_k) = \max_c \left\{ u(c) + v_a(a, z)[ra + z_k - c] \right\} + \lambda_k \left[v(a, z_{-k}) - v(a, z_k) \right] \tag{1}$$

Where this came from? Check *Lagos* lecture notes for an heuristic argument.

Theoretical results analogous to discrete time:

- Value function satisfy the HJB equation
- Verification theorems: solution of HJB + ...$\longrightarrow$ value function
- Alternatively, one can show HJB has a unique "nice" solution which is the value function (viscosity solution)

Before solving the HJB FE let's see what we can do. Analytical results from ?

Envelope condition:

$$\rho V_a(a,b) = rV_a(a,b) + V_{aa}(a,b)\{ra + b - c(a,b)\} + \lambda_1 \Big[V_a(a,w) - V_a(a,b)\Big]$$

Differential of $V_a(a,z)$ — CVF, "Itô formula"

$$\begin{aligned} da_t &= \{ra_t + z_t - c_t\}dt \\ dz_t &= (w-b)dq - (w-b)dQ, \quad q \sim \text{Poisson}(\lambda_1),\ Q \sim \text{Poisson}(\lambda_2) \end{aligned}$$

$$\mathrm{d}V_a(a,b) = \underbrace{V_{aa}\{ra + b - c(a,b)\}}_{\text{normal term}}\mathrm{d}t + \underbrace{\Big[V_a(a,w) - V_a(a,b)\Big]}_{\text{jump terms}}\mathrm{d}q_t$$

From optimization $V_a(a,z) = u'\big(c(a,z)\big)$. Combining both equations to get rid of V_{aa} we have

$$\begin{aligned} \mathrm{d}u'\big(c(a,b)\big) = &\left\{(\rho - r)u'\big(c(a,b)\big) - \lambda_1 u'\big(c(a,b)\big)\left[\frac{u'\big(c(a,w)\big)}{u'\big(c(a,b)\big)} - 1\right]\right\}\mathrm{d}t + \\ &+ \Big[u'\big(c(a,w)\big) - u'\big(c(a,b)\big)\Big]\,\mathrm{d}q_t \end{aligned}$$

Applying "Itô lemma" to get consumption over time

$$\mathrm{d}c(a,b) = \frac{u'(c(a,b))}{-u''(c(a,b))}\left\{r-\rho-\lambda_1\left[1-\frac{u'(c(a,w))}{u'(c(a,b))}\right]\right\}\mathrm{d}t + \Big[c(a,w)-c(a,b)\Big]\,\mathrm{d}q_t \tag{2}$$

$$\mathrm{d}c(a,w) = \frac{u'(c(a,w))}{-u''(c(a,w))}\left\{r-\rho+\underbrace{\lambda_2\left[\frac{u'(c(a,b))}{u'(c(a,w))}-1\right]}_{\text{prec. savings}}\right\}\mathrm{d}t + \underbrace{\Big[c(a,b)-c(a,w)\Big]\,\mathrm{d}Q_t}_{\text{jumps}} \tag{3}$$

neoclassical growth model $\dot{c}(t) = \frac{u'(c)}{-u''(c)}\left(r-\rho\right)$

Looking at period between jumps. What the signs tell us?

Proposition. *Consider the case* $0 < r \leq \rho$. *Define the threshold level* a_w^* *by*

$$\frac{u'(c(a_w^*,b))}{u'(c(a_w^*,w))} = 1 + \frac{\rho - r}{\lambda_2} \tag{4}$$

Then (i) Consumption of employed workers is increasing on $[\underline{a}, a_w^*]$ *and decreasing* $a > a_w^*$; *(ii) consumption of unemployed workers always decrease*

$[-b/r, a_w^*]$ and one can easily imagine a distribution of wealth over the range [

Properties of this system can be illustrated in the usual phase diagram

4 Stability of the wealth-employment process

POLICIES

We would now like to formally understand the stability properties of the model
As the fundamental state variables are wealth (2) and the employment status (
ual, the process we are interested in is the wealth-employment process $X_\tau \equiv ($
other variables (like control variables or e.g. factor rewards in a general equil
are known deterministic functions of the state variables. Hence, if we underst
governing the state variables, we also understand the properties of all other
model. The state-space of this process X_τ is $\mathbf{X} \equiv [-b/r,\ a_w^*] \times \{w, b\}$ and has a
required for the state space in the general ergodicity theory for Markov proc
review in section 4.1 below. Moreover, for the sake of simplicity, we now set
$t = 0$ – following the usual practice in the mathematical literature.

The goal of this section is a proof of *stability* of the Markov process X_τ in th
want to show that the distribution of X_τ converges for $\tau \to \infty$ to a unique limit
(no matter what the initial value X_0). (See def. 4.10 for the precise meaning of t

The general structure of the stability or ergodicity proof is quite usual:

Note:

Change

- Results help build some intuition on the problem. Look at ? for much more...
- Now we change the approach.
  Instead of looking at households' saving behavior in terms of a differential equation for its consumption policy function, we will focus on the HJB equation for the value function and how to solve it numerically.
- draw heavily on Moll's notes

Problem of Household

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \{ra_t + z_t - c_t\} dt$$

z_t is a ct markov chain on $\{b, w\}$ with intensities λ_1, λ_2

$$dz_t = (w-b)dq_\mu - (w-b)dq_s, \quad q_\mu \sim \text{Poisson}(\lambda_1), \; q_s \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' value function must satisfy HJB equation[1]

$$\rho v_k(a) = \max_c \left\{ u(c) + v_k'(a)[ra + z_k - c] \right\} + \lambda_k \left[v_{-k}(a) - v_k(a) \right] \tag{5}$$

Borrowing constraint shows only as *state constraint boundary condition*

$$u'\left(c_i(\underline{a})\right) = v_i'(\underline{a}) \geq u'\left(r\underline{a} + z_i\right) \tag{6}$$

which ensures $s_i(\underline{a}) = r\underline{a} + z_i - c_i(\underline{a}) \geq 0$ so that the borrowing constraint is never violated.

[1] change notation

Consider the first-order condition for consumption

$$\text{cont time:} \qquad u'(c) = \partial_a v(a,z) \tag{7}$$

$$\text{disc time:} \qquad u'(c) \geq \beta \int \partial_a v(a', z')\, \mathrm{d}F(z'|z), \quad a' = z + (1+r)a - c \tag{8}$$

Continuous time **advantages**:

1. "today" = "tomorrow" — *foc* is static
2. HJB is not stochastic — evolution of stochastic process is captured by additive terms
3. **Borrowing constraint** shows only as *state constraint boundary condition*

Numeric solution HJB

Finite difference methods: replace derivatives by differences. Simple right? Well developed theory... some slides on it

Recall our HJB equation

$$\rho v_k(a) - \sup_{c \in \Gamma_k(a)} \left\{ u(c) + \mathcal{D}^c v_k(a) \right\} = 0 \tag{9}$$

where

$$\mathcal{D}^c \phi_k(a) = \phi_k'(a)[ra + z_k - c] + \lambda_k \left[\phi_{\text{-}k}(a) - \phi_k(a) \right]$$

Define a grid $\{a_1, a_2, \ldots, a_i, \ldots\}$ and let $v_k = \Big(v_k(a_1), \ldots, v_k(a_i), \ldots \Big)'$. Discretizing this equation requires deciding upon

- which fd approximation to use: forward/backward differencing

$$v_k'(a) \approx \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}, \quad v_k'(a) \approx \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}},$$

Let $\mathscr{D}^c$ be the discrete form of the differential operator $\mathcal{D}^c$, so that

$$\left(\mathscr{D}^c v\right)_{k,i} = \alpha_{k,i}(c)v_{k,i-1} + \beta_{k,i}(c)v_{k,i+1} - \Big(\alpha_{k,i}(c) + \beta_{k,i}(c) + \lambda_k\Big)v_{k,i} + \lambda_k v_{\text{-}k,i}$$

and the discretization

$$\rho v_{k,i} - \sup_{c\in\Gamma_{k,i}} \left\{u(c) + \left(\mathscr{D}^c v^{n(+1)}\right)_{k,i}\right\} = 0 \tag{10}$$

where discretization can use forward, backward or central discretization. If

$$\alpha_{k,i} \geq 0,\ \beta_{k,i} \geq 0$$

we say that (10) is *positive coefficient discretization*. We will search for a discretization that satisfies this condition — more on the reason later.

In order to ensure a *positive coefficient discretization* our choice of central/forward/backward differencing will depend, in general, on the control c. A useful rule for this problem is to use the so-called *upwind scheme*.

IDEA: *Use forward difference whenever drift is positive, and use backward whenever it is negative.*

Suppose that we have the value of consumption $c_{k,i}$ at a particular node. Let $s_{k,i} = ra_i + z_k - c_{k,i}$. In this case, the derivatives are approximated

$$\dots \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} s^+_{k,i} + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} s^-_{k,i} + \dots$$

which in terms of our α, β

$$\alpha^{up}_{k,i} = -\frac{s^-_{k,i}}{a_i - a_{i-1}} \geq 0, \quad \beta^{up}_{k,i} = \frac{s^+_{k,i}}{a_{i+1} - a_i} \geq 0$$

Discretized HJB equation is

$$\rho v_{k,i} = u(c_{k,i}) + \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}\Big[s_{k,i}(c)\Big]^+ + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}}\Big[s_{k,i}(c)\Big]^- + \lambda_k\Big[v_{-k,i} - v_{k,i}\Big] \tag{11}$$

which can be written in matrix notation

$$\rho v = \mathbf{u} + \mathbf{A}v$$

But we don't know $c_{k,i}$! Rememeber that c satisfy the foc everywhere on the grid

$$u'\left(c_{k,i}\right) = v_k'(a_i)$$

so $\mathbf{c}(v), \mathbf{A}(v)$. HJB equation is highly nonlinear, so we need an iterative method to solve it.

Start with a vector v^n, solve for foc and update v^{n+1} according to

$$\frac{v_{k,i}^{n+1}-v_{k,i}^{n}}{\Delta}+\rho v_{k,i}^{n+1}=u(c_{k,i}^{n})+\frac{v_{k,i+1}^{n+1}-v_{k,i}^{n+1}}{a_{i+1}-a_i}\left[s_{k,i}^{F,n}\right]^{+}+\frac{v_{k,i}^{n+1}-v_{k,i-1}^{n+1}}{a_i-a_{i-1}}\left[s_{k,i}^{B,n}\right]^{-} \\ +\lambda_k\left[v_{\text{-}k,i}^{n+1}-v_{k,i}^{n+1}\right] \tag{12}$$

- Compute the policy from the foc $\left(u'\left(c_{k,i}^{n}\right)=\partial_a v_{k,i}^{n}\right)$ for the backward AND forward derivative of the value function.
- Define $s_{k,i}^{B,n}=ra_i+z_k-c_{k,i}^{B,n}$, $s_{k,i}^{F,n}=ra_i+z_k-c_{k,i}^{F,n}$. Set

$$c_{k,i}^{n}=\mathbb{1}\left\{s_{k,i}^{B,n}\leq 0\right\}\times c_{k,i}^{B,n}+\mathbb{1}\left\{s_{k,i}^{F,n}\geq 0\right\}\times c_{k,i}^{F,n}+\mathbb{1}\left\{s_{k,i}^{F,n}\leq 0\leq s_{k,i}^{B,n}\right\}\times(ra_i+z_k)$$

- Collecting terms with the same subscripts on the right-hand side

$$\frac{v_{k,i}^{n+1}-v_{k,i}^{n}}{\Delta}+\rho v_{k,i}^{n+1}=u(c_{k,i}^{n})+\alpha_{k,i}v_{k,i-1}^{n+1}+\beta_{k,i}v_{k,i+1}^{n+1}-\left(\alpha_{k,i}+\beta_{k,i}+\lambda_k\right)v_{k,i}^{n+1}+\lambda_i v_{\text{-}k,i}^{n+1} \tag{13}$$

where

$$\alpha_{k,i}^{up}=-\frac{\left[s_{k,i}^{B,n}\right]^{-}}{a_i-a_{i-1}}\geq 0,\quad \beta_{k,i}^{up}=\frac{\left[s_{k,i}^{F,n}\right]^{+}}{a_{i+1}-a_i}\geq 0$$

- Equation (13) is just a system of linear equations on v^{n+1}!!

Equation (13) can be written in matrix notation as

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = \mathbf{u}(c^n) + \mathsf{A}^n v^{n+1}$$

where the sparse matrix A looks like

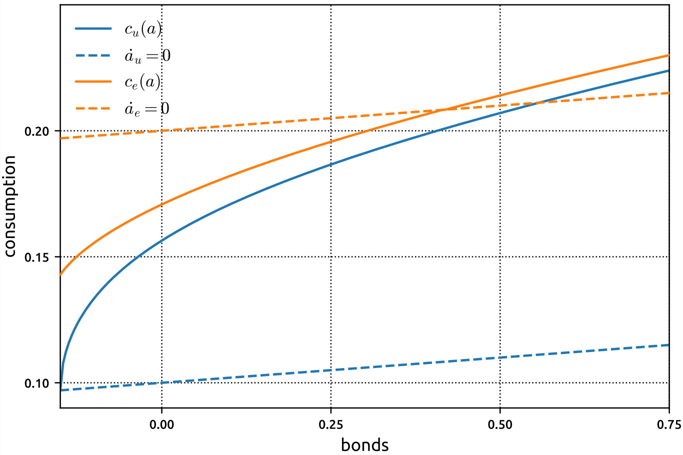
$$\mathsf{A}^n = \begin{bmatrix}
\gamma_{1,1} & \beta_{1,1} & 0 & 0 & \cdots & \lambda_1 & 0 & \cdots & & & 0 \\
\alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \alpha_{1,I} & \gamma_{1,I} & 0 & 0 & \cdots & \cdots & 0 & \lambda_1 \\
\lambda_2 & 0 & \cdots & \cdots & \cdots & \gamma_{2,1} & \beta_{2,1} & 0 & \cdots & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & \cdots & \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \lambda_2 & 0 & 0 & \cdots & \cdots & \alpha_{2,I} & \gamma_{2,I}
\end{bmatrix}_{I+2\times I+2}$$

entries of row i

$$\begin{bmatrix} \underbrace{\alpha_{k,i}}_{\text{inflow } i-1} & \underbrace{-(\alpha_{k,i} + \beta_{k,i} + \lambda_k)}_{\text{outflow}} & \underbrace{\beta_{k,i}}_{\text{inflow } i+1} \end{bmatrix} \begin{bmatrix} v_{k,i-1} \\ v_{k,i} \\ v_{k,i+1} \end{bmatrix}$$

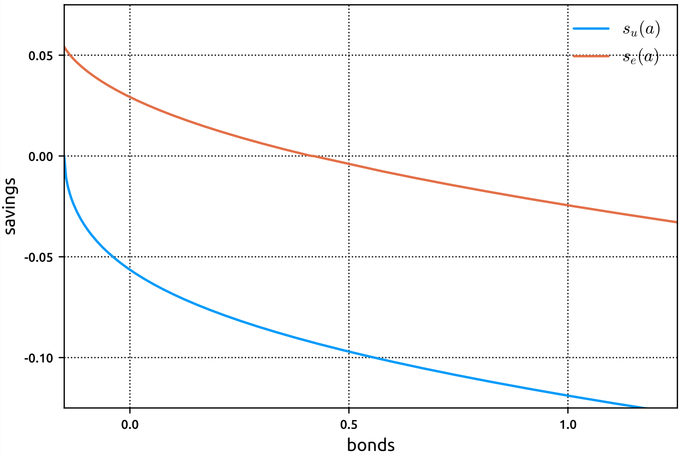
Iterate until $v^{n+1} \approx v^n$

Solution



$c_u(a)$
$\dot{a}_u = 0$
$c_e(a)$
$\dot{a}_e = 0$
0.20
0.15
0.10
consumption
0.00
0.25
0.50
0.75
bonds

Solution



Backgroud Finite Difference

Why does it work?

Time dependent version of HJB

$$\rho v_k(a) = \max_c \left\{ u(c) + v_k'(a)[ra + z_k - c] \right\} + \lambda_k \left[v_{\text{-}k}(a) - v_k(a) \right]$$

in a PDE notation

$$0 = F(\mathbf{x}, v, Dv, D^2 v) \tag{14}$$

where $\mathbf{x} := (a, z)$. Suppose we define a grid $\{a_0, a_1, \ldots, a_i, \ldots\}$. Let $v_{k,i} \approx v_k(a_i)$ be the approximate value of the solution. Then we can write a general **discretization** of the HJB equation at node (a_i, z_k)

$$0 = S_{k,i}\left(\tilde{\boldsymbol{\Delta}}, v_{k,i}, \{v_{m,j}\}_{m \neq k, j \neq i}\right) \tag{15}$$

Condition (Monotonicity) .— The numerical scheme (15) is monotone if

$$S_{k,i}(\cdot, v_{k,i}, \{y_{m,j}\}) \leq S_{k,i}(\cdot, v_{k,i}, \{z_{m,j}\})$$

for all $y \geq z$.

Condition (Stability) .— The numerical scheme (15) is stable if for every $\tilde{\boldsymbol{\Delta}} > 0$ it has a solution which is uniformly bounded independently of $\tilde{\boldsymbol{\Delta}}$.

Condition (Consistency) .— The numerical scheme (15) is consistent if for every smooth function ϕ with bounded derivatives we have

$$S_{k,i}(\tilde{\boldsymbol{\Delta}}, \phi(\mathbf{x}_{k,i}), \{\phi(\mathbf{x}_{m,j})\}) \to F(\mathbf{x}, \phi, D\phi, D^2\phi)$$

as $\tilde{\boldsymbol{\Delta}} \to 0$ and $\mathbf{x}_{k,i} \to \mathbf{x}$.

Theorem ? . *If the numerical scheme S* (15) *satisfies monotonicity, stability and consistency conditions, then its solution converges locally uniformly to the unique viscosity solution of* (14).

- Convergence here is about $\tilde{\Delta} \to 0$
- For given $\tilde{\Delta}$, we have a system of I non-linear equations that we must solve somehow (Implicit scheeme). Theorem guarantees that the solution $\{v_{k,i}\}$ of this system converges to the "viscosity solution" of the original PDE as $\tilde{\Delta} \to 0$
- "viscosity solution" of the HJB is the the **value function**
- A *positive coefficient discretization* is also *Monotone*. To see it check that

$$S_{k,i}\left(\tilde{\Delta}, v_{k,i}, v_{k,i+1}, v_{k,i-1}, v_{k,i}, v_{\text{-}k,i}\right)$$

is a nonincreasing function of the neighbor nodes $\{v_{m,j}\}$. Check a example!

Computing the Distribution

Distributions

- We now know how to solve the Household consumption/savings problem
- But interesting questions require dealing with distributions
- Denote by $g_i(a,t)$ $i = 1, 2$ the joint density of income z_i and welath a.
- The evolution of the density given a fixed initial distribution $g_i(a, 0)$ is described by the *Kolmogorov forward equation*
  - time dependent

$$\frac{\partial}{\partial t} g(a,t) = -\frac{\partial}{\partial a}\Big[s_k(a,t)g_k(a,t)\Big] - \lambda_k g_k(a,t) + \lambda_{\text{-}k} g_{\text{-}k}(a,t) \tag{16}$$

  - stationary

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s_k(a)g_k(a)\Big] - \lambda_k g_k(a) + \lambda_{\text{-}k} g_{\text{-}k}(a) \tag{17}$$

Consider the stationary KFE

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s(a, z_k)g(a, z_k)\Big] - \lambda_k g(a, z_k) + \lambda_{\text{-}k} g(a, z_{\text{-}k})$$

with the following discretization

$$0 = -\frac{\left(s^F_{k,i}\right)^+ g_{k,i} - \left(s^F_{k,i-1}\right)^+ g_{k,i-1}}{\Delta a} - \frac{\left(s^B_{k,i+1}\right)^- g_{k,i+1} - \left(s^B_{k,i}\right)^- g_{k,i}}{\Delta a} - \lambda_k g_{k,i} + \lambda_{-k} g_{\text{-}k,i} \tag{18}$$

Collecting terms with the same subscripts on the right-hand side

$$0 = \underbrace{\frac{\left(s^F_{k,i-1}\right)}{\Delta a}}_{\beta_{k,i-1}} g_{k,i-1} + \underbrace{\left(\frac{\left(s^B_{k,i}\right)}{\Delta a} - \frac{\left(s^F_{k,i}\right)}{\Delta a} - \lambda_k\right)}_{\gamma_{k,i}} g_{k,i} + \underbrace{\left(-\frac{\left(s^B_{k,i+1}\right)}{\Delta a}\right)}_{\alpha_{k,i+1}} g_{k,i+1} + \lambda_{-k} g_{\text{-}k,i}$$

which in matrix notation reads $\mathbf{A}^T g = \mathbf{0}$. Numerically, this is very efficient bs we have already computed $\mathbf{A}$.

This makes sense: the operation is exactly the same as that used for finding the stationary distribution of a discrete Poisson process (continuous-time Markov chain). The matrix A captures the evolution of the stochastic process over a very short interval — it is our discretized *infinitesimal generator* of our state — and to find the stationary distribution, one solves the eigenvalue problem $\mathbf{A}^T g = \mathbf{0}$.

Stationary Equilibrium + Transion Dynamics

Definition. *A stationary recursive competitive equilibrium is*

$$\Big(v, c, s, r, g\Big)$$

such that ...

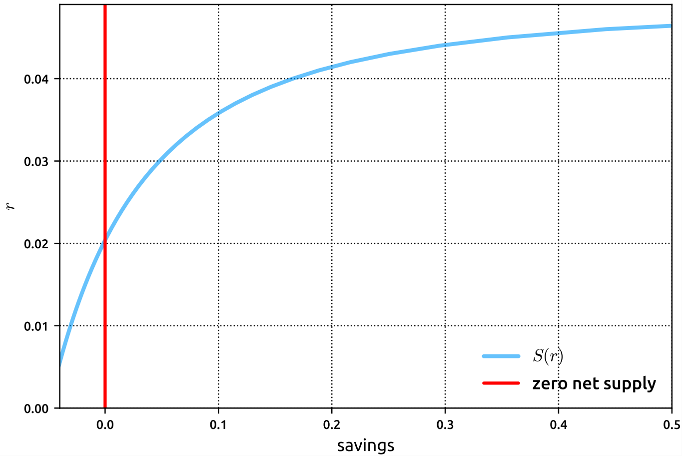
$$\rho v_k(a) = \max_c \Big\{ u(c) + v_k'(a)\big[ra + z_k - c\big] \Big\} + \lambda_k \Big[v_\ell(a) - v_k(a) \Big] \qquad \Big[\text{HJB}\Big]$$

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}a}\big[s_k(a)g_k(a)\big] - \lambda_k g_k(a) + \lambda_\ell g_\ell(a) \\ 1 &= \int_{\underline{a}}^{\infty} \Big(g_1(a) + g_2(a)\Big) da \end{aligned} \qquad \Big[\text{KFE}\Big]$$

$$0 = \int_{\underline{a}}^{\infty} a\Big(g_1(a) + g_2(a)\Big) da \qquad \Big[\text{Equil}\Big]$$

Stationary Equilibrium

Stationary Equilibrium



hypothetical thought experiments of the following form.

- Suppose the economy is in a stationary equilibrium, with a given government policy and all other exogenous elements that define preferences, endowments and technology
- Unexpectedly, either government policy or some exogenous elements of the economy (such as the labor productivity process) change
  This change was completely unexpected by all agents of the economy (a zero probability event), so that no anticipation actions were taken by any agent.
- We want to study the transition path induced by the exogenous change, from the old stationary equilibrium to a new stationary equilibrium (which may coincide with the old stationary equilibrium in case the exogenous change is of transitory nature, or may differ from it in case the exogenous change is permanent.

The time-dependent analogue of the stationary system is

$$\rho v_k(a,t) = \max_c \left\{ u(c) + \partial_a v_k(a,t)\left[r(t)a + z_k - c\right] \right\} + \lambda_k \left[v_\ell(a) - v_k(a) \right] + \partial_t v_k(a,t) \qquad \text{[HJB]}$$

$$\partial_t g_k(a,t) = \partial_a \left[s_k(a,t) g_k(a,t) \right] - \lambda_k g_k(a,t) + \lambda_{\text{-}k} g_{\text{-}k}(a,t)$$
$$1 = \int_{\underline{a}}^{\infty} \Big(g_1(a,t) + g_2(a,t) \Big) da \qquad \text{[KFE]}$$

$$0 = \int_{\underline{a}}^{\infty} a\Big(g_1(a,t) + g_2(a,t) \Big) da \qquad \text{[Equil]}$$

where the density satisfies an initial condition and runs forwards

$$g_k(a,0) = g_k^0(a)$$

while the value function satisfies a terminal condition and runs backwards

$$v_k(a,T) = v_k^E(a)$$

We solve this system using the following algorithm. Guess a function $r^0(t)$ and then for $m = 1, 2, 3, \ldots$ follow

- Given $r^m(t)$, solve the HJB backwards in time to find $\{v_k^m(a,t), s_k^m(a,t)\}$
- Given $s_k^m(a,t)$ solve the KFE forward in time given initial condition to calculate the time path for $g_k(a,t)$
- Check market clearing for the whole path

$$S^m(t) = \int_{\underline{a}}^{\infty} a\Big(g_1^m(a,t) + g_2^m(a,t)\Big)da$$

- Update $r^{m+1}(t) = r^m(t) - \xi \frac{dS^m(t)}{dt}$

HJB:

Approximate the value function at I discrete points in the wealth dimension and N discrete points in the time dimension, and use the shorthand notation $v_{k,i}^n = v_k(a_i, t_n)$. The discrete approximation to the time-dependent HJB is

$$\rho v_{k,i}^n = u(c_{k,i}^{n+1}) + \left(v_{k,i}^n\right)' \left[r^n a + z_k - c_{k,i}^{n+1}\right] + \lambda_k \left[v_{-k,i}^n - v_{k,i}^n\right] + \frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta t} \tag{19}$$

which is exactly as we have before!!! Why?

KFE:

Consider the time dependent KFE

$$\frac{\partial}{\partial t} g(a,t) = -\frac{\partial}{\partial a}\left[s_k(a,t) g_k(a,t)\right] - \lambda_k g_k(a,t) + \lambda_{-k} g_{-k}(a,t)$$

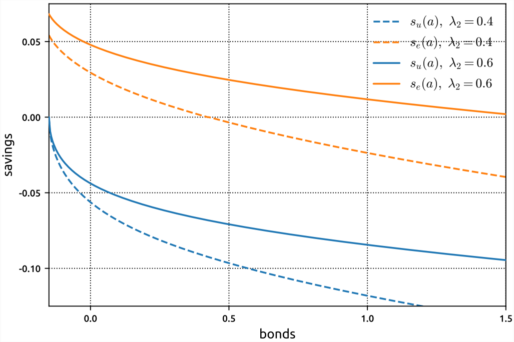
Given an initial condition, the KFE can be easily solve through a implicit method

$$\frac{g^{n+1} - g^n}{\Delta} = \left(\mathrm{A}^{n(+1)}\right)^T g^{n+1} \tag{20}$$

Experiment

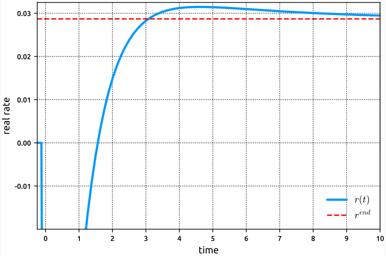
Suppose an increase in the unemployment risk λ_2. What would you expect has to happen to the interest rate?

Savings policy

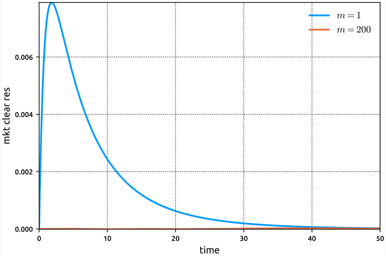


Distributional effect of more unemployed in equilibrium makes me converge to an higher interest rate!

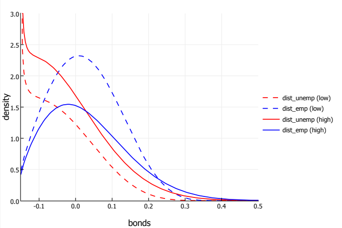
Transition



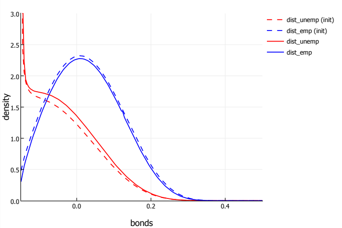
(a)



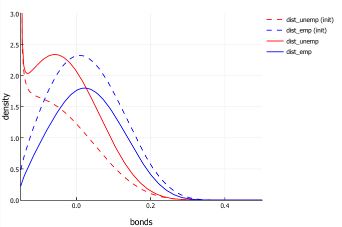
(b)



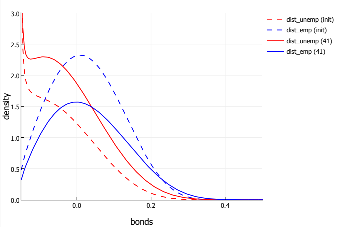
init × end



$t = 0.125$



$t = 2.0$



$t = 5.0$

Note:

Why do I care??

Example

- looks nice, but why should I pay a cost if I can do discrete time?
- It was my original tought, but recently...
- Consumption/savings problem + direct search labor market

$$V^u(a) = \max_{c,a'} u(c) + \beta \mathcal{R}^u(a') \qquad \mathcal{R}^u(\tilde{a}) = V^u(\tilde{a}) + \max_{\tilde{w}} p\big(\theta(\tilde{a},\tilde{w})\big)\Big[V^e(\tilde{a},\tilde{w}) - V^u(\tilde{a})\Big]$$

$$\text{S.t.} \quad c + \frac{a'}{1+r} = b + a, \quad a' \geq \underline{a}$$

euler equation for asset holdings

$$u'(c(\cdot)) \geq \beta(1+r)\Big\{(1-p(\theta(a',\tilde{w})))V^u_a(a') + p(\theta(a',\tilde{w}))V^e_a(a',\tilde{w}) + p'(\theta(a',\tilde{w})\frac{\partial\theta(a',\tilde{w})}{\partial a}\Big[V^e(a',\tilde{w}) - V^u(a',\tilde{w})\Big]\Big\}$$

- EGM for consumption/savings, VFI for labor choice.
- Finding the equilibrium requires iterating over a lot of stuff

$$V(\cdot) \hookrightarrow \hat{\theta}(\cdot) \hookrightarrow \mathcal{J}(\cdot) \hookrightarrow \theta(\cdot)$$

- couldn't do it = (...

- How does this look in continuous time, *today is tomorrow*

$$\rho V^u(a) = \max_c \left\{ u(c) + V^u_a(b + ra - c) \right\} + \underbrace{\lambda_u}_{\text{rate of search}} \max_{\tilde{w}} \left\{ p(\theta(a, \tilde{w})) \left[V^e(a, w) - V^u(a) \right] \right\} \tag{21}$$

- This is wayyyyy simpler and importantlty it doesn't seem I am throwing away anything of the economics