

# Lecture

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NYU

1. Consumption Savings Problem
2. Distribution
3. Transition Dynamics

## THEORY

- [Pham \(2009\)](#) Continuous-time Stochastic Control (finance)
- Lecture notes
  - ↪ [Caldentey \(????\)](#) Stochastic processes and optimal control — nice lecture notes Enio uses them
  - ↪ [Ross \(????\)](#) *Stochastic Control in Continuous Time* — alternative to math books [Fleming and Soner \(2006\)](#), [Øksendal \(2003\)](#), [Øksendal and Sulem \(2007\)](#)

## THEORY (MACRO)

- Moll's website
- [Bayer and Wälde \(2015\)](#)
  - ↪ [Sennewald \(2007\)](#) (theory paper), [Walde \(2008\)](#) (heuristic)
- [Stokey \(2009\)](#) book – *Impulse control Problem*

## NUMERIC

- [Achdou, Han, Lasry, Lions, and Moll \(2016\)](#) (mainly the numerical appendix), Moll's website (tons of examples and materials)
- [Forsyth and Vetzal \(2012\)](#) (Also has some slides) – good introduction to “viscosity solutions”
- Interested? Check ...
  - ↪ [Thomas and Nuño \(2016\)](#), [Nuño and Moll \(2017\)](#) (improve notation), [Kaplan, Moll, and Violante \(2016\)](#), PHACT

## Consumption Savings Problem

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## Problem of Household

$$\max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \{r_t a_t + z_t - c_t\} dt$$

$z_t$  is a ct markov chain on  $\{b, w\}$  with intensities  $\lambda_1, \lambda_2$

$$dz_t = (w - b)dq_{\mu} - (w - b)dq_s, \quad q_{\mu} \sim \text{Poisson}(\lambda_1), \quad q_s \sim \text{Poisson}(\lambda_2)$$

Individuals' consumption and saving decision is summarized by HJB equation

- time dependent

$$\rho v_k(a, t) = \max_c \left\{ u(c) + \partial_a v_k(a, t)[r_t a + z_k - c] \right\} + \lambda_k \left[ v_{-k}(a, t) - v_k(a, t) \right] + \partial_t v_k(a, t) \quad (1)$$

- stationary

$$\rho v_k(a) = \max_c \left\{ u(c) + v'_k(a)[ra + z_k - c] \right\} + \lambda_k \left[ v_{-k}(a) - v_k(a) \right] \quad (2)$$

## Problem of Household

$$\max_{\{c_t\}_{t \geq 0}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

$$\text{s.t. } da_t = \{r_t a_t + z_t - c_t\} dt$$

$z_t$  is a ct markov chain on  $\{b, w\}$  with intensities  $\lambda_1, \lambda_2$

$$dz_t = (w - b)dq_\mu - (w - b)dq_s, \quad q_\mu \sim \text{Poisson}(\lambda_1), \quad q_s \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' consumption and saving decision is summarized by HJB equation

- time dependent

$$\rho v_k(a, t) = \max_c \left\{ u(c) + \partial_a v_k(a, t)[r_t a + z_k - c] \right\} + \lambda_k \left[ v_{-k}(a, t) - v_k(a, t) \right] + \partial_t v_k(a, t) \quad (3)$$

- stationary

$$\rho v_k(a) = \max_c \left\{ u(c) + v'_k(a)[ra + z_k - c] \right\} + \lambda_k \left[ v_{-k}(a) - v_k(a) \right] \quad (4)$$

Consider the first-order condition for consumption

$$u'(c) = \partial_a v(a, z) \quad (5)$$

$$u'(c) \geq \beta \int \partial_a v(a', z') dF(z'|z), \quad a' = z + (1+r)a - c \quad (6)$$

Continuous time advantages:

1. “today” = “tomorrow” — *foc* is static
2. HJB is not stochastic — evolution of stochastic process is captured by additive terms
3. *foc* always holds with equality — Borrowing constraint shows only as *state constraint boundary condition*

$$u'(c_i(\underline{a}, t)) = \partial_a v_i(\underline{a}, t) \geq u'(r_t \underline{a} + z_i) \quad (7)$$

which ensures  $c_i(\underline{a}, t) \leq r_t \underline{a} + z_i$  so that the borrowing constraint is never violated.

We can write our HJB in a PDE notation

$$0 = F(\mathbf{x}, V, DV, D^2V) \quad (8)$$

where  $\mathbf{x} := (x, \tau)$ . How do we proceed to solve it??

**Finite difference methods:** replace derivatives by differences. Simple right? Does it work?

Suppose we define a grid  $\{x_0, x_1, \dots, x_i, \dots\}$  and a set of timesteps  $\{i\Delta : i \in \mathbb{N}\}$ . Let  $V_i^n \approx V(x_i, \tau_n)$  be the approximate value of the solution at node  $x_i$  time  $\tau^n := T - t$ . Then we can write a general **discretization** of the HJB equation at node  $(x_i, \tau^{n+1})$

$$0 = S_i^{n+1} \left( (\Delta, \Delta x), V_i^{n+1}, \{V_j^m\}_{m \neq n+1, j \neq i} \right) \quad (9)$$



*Condition (Monotonicity)* . — The numerical scheme (9) is monotone if

$$S_i^{n+1}(\cdot, V_i^{n+1}, \{Y_j^m\}) \leq S_i^{n+1}(\cdot, V_i^{n+1}, \{Z_j^m\})$$

for all  $Y \geq Z$ .

*Condition (Stability)* . — The numerical scheme (9) is stable if for every  $\tilde{\Delta} > 0$  it has a solution which is uniformly bounded independently of  $\tilde{\Delta}$ .

*Condition (Consistency)* . — The numerical scheme (9) is consistent if for every smooth function  $\phi$  with bounded derivatives we have

$$S_i^{n+1}(\tilde{\Delta}, \phi(\mathbf{x}_i^{n+1}), \{\phi(\mathbf{x}_j^m)\}) \rightarrow F(\mathbf{x}, \phi, D\phi, D^2\phi)$$

as  $\tilde{\Delta} \rightarrow 0$  and  $\mathbf{x}_i^{n+1} \rightarrow \mathbf{x}$ .

**Theorem** *Barles and Souganidis (1990)* . *If the numerical scheme  $S$  (9) satisfies monotonicity, stability and consistency conditions, then its solution converges locally uniformly to the unique viscosity solution of (8).*

- Convergence here is about  $\tilde{\Delta} \rightarrow 0$
- For given  $\tilde{\Delta}$ , we have a system of  $I$  non-linear equations for each timestep that we must solve somehow. Theorem guarantees that the solution  $\{V_i^T\}$  of this system converges to the “viscosity solution” of the original PDE as  $\tilde{\Delta} \rightarrow 0$
- “viscosity solution” of the HJB is the the **value function**

# Discretization

Back to our example

Recall our *time-dependent* HJB equation as

$$\partial_\tau v_k(a, \tau) + \rho v_k(a, \tau) - \sup_{c \in \Gamma_k(a)} \left\{ u(c) + \mathcal{D}_{\tau-\tau}^c v_k(a, \tau) \right\} = 0 \quad (10)$$

where

$$\mathcal{D}_t^c \phi_k(a) = \partial_a \phi_k(a) [r_t a + z_k - c] + \lambda_k [\phi_{-k}(a) - \phi_k(a)]$$

Define a grid  $\{a_1, a_2, \dots, a_i, \dots\}$  and let  $v_k^n = (v_k(a_1, \tau^n), \dots, v_k(a_i, \tau^n), \dots)'$ . Discretizing this equation requires deciding upon

- which fd approximation to use: forward/backward differencing

$$\partial_a v_k(a) \approx \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}, \quad \partial_a v_k(a) \approx \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}},$$

- Implicit/explicit timestepping

# Discretization

Back to our example

Let  $\mathcal{D}^c$  be the discrete form of the differential operator  $\mathcal{D}^c$ , so that

$$(\mathcal{D}^c v)_{k,i} = \alpha_{k,i}(c)v_{k,i-1} + \beta_{k,i}(c)v_{k,i+1} - (\alpha_{k,i}(c) + \beta_{k,i}(c) + \lambda_i)v_{k,i} + \lambda_i v_{-k,i}$$

and the discretization

$$\frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta} - \sup_{c \in \Gamma_{k,i}} \left\{ u(c) + (\mathcal{D}^c v^{n(+1)})_{k,i} \right\} = 0 \quad (11)$$

where discretization can use forward, backward or central discretization. If

$$\alpha_{k,i}^{n+1} \geq 0, \quad \beta_{k,i}^{n+1} \geq 0$$

we say that (11) is *positive coefficient discretization*.

*Why do we care?*

We care because a *positive coefficient discretization* is also *monotone*. To see it check that

$$S_{k,i}^{n+1}(\tilde{\Delta}, v_{k,i}^{n+1}, v_{k,i+1}^{n(+1)}, v_{k,i-1}^{n(+1)}, v_{k,i}^n, v_{-k,i}^{n(+1)})$$

is a nonincreasing function of the neighbor nodes  $\{v_{\ell,j}^m\}$ . Check a example!

## Upwind scheme

In order to ensure a *positive coefficient discretization* our choice of central/forward/backward differencing will depend, in general, on the control  $c$ . A useful rule for this problem is to use the so-called *upwind scheme*.

IDEA: *Use forward difference whenever drift is positive, and use backward whenever it is negative.*

Suppose that we have the value of consumption  $c_{k,i}$  at a particular node. Let  $s_{k,i} = ra_i + z_k - c_{k,i}$ . In this case, the derivatives are approximated

$$\dots \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} s_{k,i}^+ + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} s_{k,i}^- + \dots$$

which in terms of our  $\alpha, \beta$

$$\alpha_{k,i}^{up} = -\frac{s_{k,i}^-}{a_i - a_{i-1}} \geq 0, \quad \beta_{k,i}^{up} = \frac{s_{k,i}^+}{a_{i+1} - a_i} \geq 0$$

But we don't know  $c_{k,i}$ !!! HJB equation is highly nonlinear, so we need an iterative method to solve it.

# Implicit timestepping

Start with a vector  $v^n$  and update  $v^{n+1}$  according to

$$\frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^n) + \frac{v_{k,i+1}^{n+1} - v_{k,i}^{n+1}}{a_{i+1} - a_i} [s_{k,i}^{F,n}]^+ + \frac{v_{k,i}^{n+1} - v_{k,i-1}^{n+1}}{a_i - a_{i-1}} [s_{k,i}^{B,n}]^- + \lambda_k [v_{-k,i}^{n+1} - v_{k,i}^{n+1}] \quad (12)$$

- Compute the policy from the foc  $\left( u'(c_{k,i}^n) = \partial_a v_{k,i}^n \right)$  for the backward AND forward derivative of the value function.
- Define  $s_{k,i}^{B,n} = ra_i + z_k - c_{k,i}^{B,n}$ ,  $s_{k,i}^{F,n} = ra_i + z_k - c_{k,i}^{F,n}$ . Set

$$c_{k,i}^n = \mathbb{1} \left\{ s_{k,i}^{B,n} \leq 0 \right\} \times c_{k,i}^{B,n} + \mathbb{1} \left\{ s_{k,i}^{F,n} \geq 0 \right\} \times c_{k,i}^{F,n} + \mathbb{1} \left\{ s_{k,i}^{F,n} \leq 0 \leq s_{k,i}^{B,n} \right\} \times (ra_i + z_k)$$

- Collecting terms with the same subscripts on the right-hand side

$$\frac{v_{k,i}^{n+1} - v_{k,i}^n}{\Delta} + \rho v_{k,i}^{n+1} = u(c_{k,i}^n) + \alpha_{k,i} v_{k,i-1}^{n+1} + \beta_{k,i} v_{k,i+1}^{n+1} - (\alpha_{k,i} + \beta_{k,i} + \lambda_k) v_{k,i}^{n+1} + \lambda_i v_{-k,i}^{n+1} \quad (13)$$

where

$$\alpha_{k,i}^{up} = -\frac{[s_{k,i}^{B,n}]^-}{a_i - a_{i-1}} \geq 0, \quad \beta_{k,i}^{up} = \frac{[s_{k,i}^{F,n}]^+}{a_{i+1} - a_i} \geq 0$$

- Equation (13) is just a system of linear equations!!

# Implicit Timestepping

Equation (13) can be written in matrix notation as

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u(c^n) + \mathbf{A}^n v^{n+1}$$

where the sparse matrix  $\mathbf{A}$  looks like

$$\mathbf{A}^n = \begin{bmatrix} \gamma_{1,1} & \beta_{1,1} & 0 & 0 & \dots & \lambda_1 & 0 & \dots & & 0 \\ \alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & 0 & \dots & 0 & \lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \alpha_{1,l} & \gamma_{1,l} & 0 & 0 & \dots & \dots & 0 & \lambda_1 \\ \lambda_2 & 0 & \dots & \dots & \dots & \gamma_{2,1} & \beta_{2,1} & 0 & \dots & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & \dots & \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \lambda_2 & 0 & 0 & \dots & \dots & \alpha_{2,l} & \gamma_{2,l} \end{bmatrix}_{l+2 \times l+2}$$

entries of row  $i$

$$\left[ \underbrace{\alpha_{k,i}}_{\text{inflow } i-1} \quad \underbrace{-(\alpha_{k,i} + \beta_{k,i} + \lambda_k)}_{\text{outflow}} \quad \underbrace{\beta_{k,i}}_{\text{inflow } i+1} \right] \begin{bmatrix} v_{k,i-1} \\ v_{k,i} \\ v_{k,i+1} \end{bmatrix}$$

## Distribution

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- We now know how to solve the Household problem. But interesting questions require dealing with distributions.
- Denote by  $g_i(a, t)$   $i = 1, 2$  the joint density of income  $z_i$  and wealth  $a$ .
- The evolution of the density given a fixed initial distribution  $g_i(a, 0)$  is described by the *Kolmogorov forward equation*
  - time dependent

$$\frac{\partial}{\partial t} g(a, t) = -\frac{\partial}{\partial a} [s_k(a, t) g_k(a, t)] - \lambda_k g_k(a, t) + \lambda_{-k} g_{-k}(a, t) \quad (14)$$

- stationary

$$0 = -\frac{d}{da} [s_k(a) g_k(a)] - \lambda_k g_k(a) + \lambda_{-k} g_{-k}(a) \quad (15)$$

# Kolmogorov Forward equation

Stationary

Consider the stationary KFE

$$0 = -\frac{d}{da} \left[ s(a, z_k) g(a, z_k) \right] - \lambda_k g(a, z_k) + \lambda_{-k} g(a, z_k)$$

with the following discretization

$$0 = -\frac{(s_{k,i}^F)^+ g_{k,i} - (s_{k,i-1}^F)^+ g_{k,i-1}}{\Delta a} - \frac{(s_{k,i+1}^B)^- g_{k,i+1} - (s_{k,i}^B)^- g_{k,i}}{\Delta a} - \lambda_k g_{k,i} + \lambda_{-k} g_{-k,i} \quad (16)$$

Collecting terms with the same subscripts on the right-hand side

$$0 = \underbrace{\frac{(s_{k,i-1}^F)}{\Delta a}}_{\beta_{k,i-1}} g_{k,i-1} + \underbrace{\left( \frac{(s_{k,i}^B)}{\Delta a} - \frac{(s_{k,i}^F)}{\Delta a} - \lambda_k \right)}_{\gamma_{k,i}} g_{k,i} + \underbrace{\left( -\frac{(s_{k,i+1}^B)}{\Delta a} \right)}_{\alpha_{k,i+1}} g_{k,i+1} + \lambda_{-k} g_{-k,i}$$

which in matrix notation reads  $\mathbf{A}^T \mathbf{g} = \mathbf{0}$ .

This makes sense: the operation is exactly the same as that used for finding the stationary distribution of a discrete Poisson process (continuous-time Markov chain). The matrix  $\mathbf{A}$  captures the evolution of the stochastic process over a very short interval — it is our discretized *infinitesimal generator* of our state — and to find the stationary distribution, one solves the eigenvalue problem  $\mathbf{A}^T \mathbf{g} = \mathbf{0}$ .

# Kolmogorov Forward equation

Time dependent

Consider the time dependent

$$\frac{\partial}{\partial t} g(a, t) = - \frac{\partial}{\partial a} \left[ s_k(a, t) g_k(a, t) \right] - \lambda_k g_k(a, t) + \lambda_{-k} g_{-k}(a, t)$$

Given an initial condition, the KFE can be easily solve through a implicit method

$$\frac{g^{n+1} - g^n}{\Delta} = \left( \mathbf{A}^{n(+1)} \right)^T g^{n+1} \quad (17)$$

# Kolmogorov Forward equation

How to think on mass point boundary?

In the case with a dirac mass on the boundary constraint

$$M_1(t) + \int_0^a g_1(a, t) = G_1(a, t), \quad \int_0^a g_2(a, t) = G_2(a, t) \quad (18)$$

and we may write

$$\frac{\partial}{\partial t} g_1(a, t) = -\frac{\partial}{\partial a} [s_1(a, t) g_1(a, t)] - \lambda_1 g_1(a, t) + \lambda_2 g_2(a, t) \quad (19)$$

$$\frac{\partial}{\partial t} g_2(a, t) = -\frac{\partial}{\partial a} [s_2(a, t) g_2(a, t)] - \lambda_2 g_2(a, t) + \lambda_1 g_1(a, t) + \lambda_1 M_1 \delta_0(a) \quad (20)$$

$$\frac{\partial}{\partial t} M_1(t) = -\lim_{\epsilon \rightarrow 0} s_1(\underline{a} + \epsilon, t) g_1(\underline{a} + \epsilon, t) - \lambda_1 M_1 \quad (21)$$

# Kolmogorov Forward equation

How to think on mass point boundary?

Integrating the KFE between  $\underline{a} + \epsilon$  and  $\infty$

$$\frac{\partial}{\partial t} \int_{\underline{a}+\epsilon} g_1(a, t) = s_1(\underline{a} + \epsilon, t)g_1(\underline{a} + \epsilon, t) - \lambda_1 \int_{\underline{a}+\epsilon} g_1(a, t) + \lambda_2 \int_{\underline{a}+\epsilon} g_2(a, t)$$

Using the definition on distributions and taking the limit as  $\epsilon \rightarrow 0$

$$-\frac{\partial}{\partial t} (M_1(t) - G_1(t)) = \lim_{\epsilon \rightarrow 0} s_1(\underline{a} + \epsilon, t)g_1(\underline{a} + \epsilon, t) + \lambda_1 (M_1(t) - G_1(t)) + \lambda_2 G_2(t)$$

but note that  $\frac{\partial}{\partial t} G_1(t) = -\lambda_1 G_1(t) + \lambda_2 G_2(t)$ . That leave us with (21).

## Transion Dynamics

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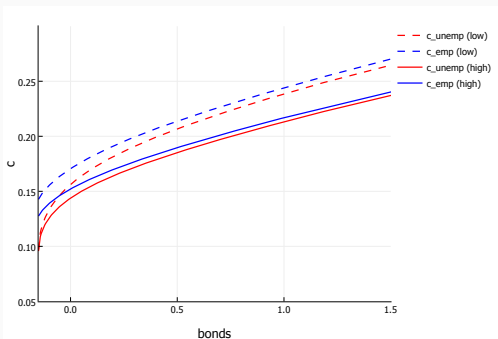
$$\rho v_k(a) = \max_c \left\{ u(c) + \partial_a v_k(a) \right\} + \lambda_k \left[ v_\ell(a) - v_k(a) \right] \quad [\text{HJB}]$$

$$0 = \frac{d}{da} [s_k(a)g_k(a)] - \lambda_k g_k(a) + \lambda_\ell g_\ell(a)$$
$$1 = \int_{\underline{a}}^{\infty} (g_1(a) + g_2(a)) da \quad [\text{KFE}]$$

$$0 = \int_{\underline{a}}^{\infty} a (g_1(a) + g_2(a)) da \quad [\text{Equil}]$$

Suppose an increase in the unemployment risk  $\lambda_2$ . What would you expect?

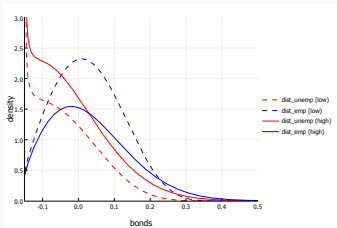
## CONSUMPTION POLICY



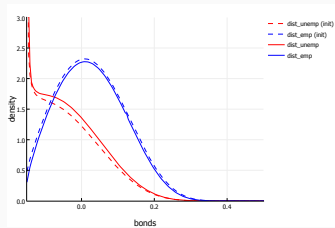
*Note:*



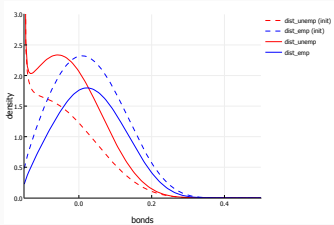
# TRANSITION DYNAMICS



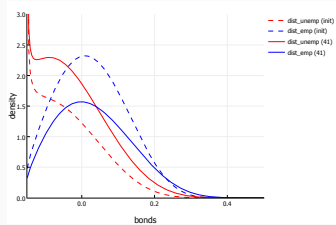
init  $\times$  end



$t = 0.125$



$t = 2.0$



$t = 5.0$

Note:

Here is some content on the slide

$$f(x) = ax^2 + bx + c$$

Some more content

## References

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