References

Numeric

- Achdou, Han, Lasry, Lions, and Moll (2016) (mainly the numerical appendix), Moll's website (more examples)
- Forsyth and Vetzal (2012) (good slides)

Definition

In general form, the HJB equation is an equation of the form

$$0 = F(\mathbf{x}, V, DV, D^2V) \tag{1}$$

where $\mathbf{x} \coloneqq (x, \tau)$.

Finite difference methods: replace derivatives by differences.
Suppose we define a grid $\{x_0, x_1, \ldots, x_i, \ldots\}$ and a set of timesteps $\{i\Delta : i \in \mathbb{N}\}$ Let $V_i^n \approx V(x_i, \tau_n)$ be the approximate value of the solution at node x_i time $\tau^n \eqqcolon\coloneqq T - t$. Then we can write a general discretization of the HJB equation at node (x_i, τ^{n+1})

$$0 = S_i^{n+1}\Big((\Delta, \Delta x), V_i^{n+1}, \{V_j^m\}\Big) \tag{2}$$

Sufficient Conditions Convergence

Condition (Monotonicity) .— The numerical scheme (2) is monotone if

$$S_i^{n+1}(\cdot, V_i^{n+1}, \{Y_j^m\}) \leq S_i^{n+1}(\cdot, V_i^{n+1}, \{Z_j^m\})$$

for all $Y \geq Z$.

Condition (Stability) .— The numerical scheme (2) is stable if for every $\tilde{\mathbf{\Delta}} > 0$ it has a solution which is uniformly bounded independently of $\tilde{\mathbf{\Delta}}$.

Condition (Consistency) .— The numerical scheme (2) is consistent if for every smooth function ϕ with bounded derivatives we have

$$S_i^{n+1}(\tilde{\mathbf{\Delta}}, \phi(\mathbf{x}_i^{n+1}), \{\phi(\mathbf{x}_j^m)\}) \to F(\mathbf{x}, \phi, D\phi, D^2\phi)$$

as $\tilde{\mathbf{\Delta}} \to 0$ and $\mathbf{x}_i^{n+1} \to \mathbf{x}$.

Theorem Barles and Souganidis (1990) . *If the numerical scheme (2) satisfies monotonicity, stability and consistency conditions, then its solution converges locally uniformly to the unique viscosity solution of (1).*

Recall that we can write our HJB equation as

$$\partial_\tau v_k(a,\tau) + \rho v_k(a,\tau) - \sup_{c\in\Gamma_k(a)} \left\{ u(c) + \mathcal{D}^c_{T-\tau} v_k(a,\tau) \right\} = 0 \tag{3}$$

where

$$\mathcal{D}^c_t \phi_k(a) = \partial_a \phi_k(a)[r_t a + z_k - c] + \lambda_k \Big[\phi_{\text{-}k}(a) - \phi_k(a) \Big]$$

Define a grid $\{a_1, a_2, \ldots, a_i, \ldots\}$ and let $v_k^n = \Big(v_k(a_1,\tau^n), \ldots, v_k(a_i,\tau^n), \ldots \Big)'$. Discretizing this equation requires deciding upon

- forward/backward differencing

$$\partial_a v_k(a) \approx \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}, \quad \partial_a v_k(a) \approx \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}},$$

- Implicit/explicit timestepping

Let $\mathscr{D}^c$ be the discrete form of the differential operator $\mathcal{D}^c$, so that

$$\left(\mathscr{D}^c v\right)_{k,i} = \alpha_{k,i}(c) v_{k,i-1} + \beta_{k,i}(c) v_{k,i+1} - \Big(\alpha_{k,i}(c) + \beta_{k,i}(c) + \lambda_i\Big) v_{k,i} + \lambda_i v_{\text{-}k,i}$$

and the discretization

$$\frac{v^{n+1}_{k,i} - v^n_{k,i}}{\Delta} - \sup_{c \in \Gamma_{k,i}} \left\{ u(c) + \left(\mathscr{D}^c v^{n(+1)}\right)_{k,i} \right\} = 0 \tag{4}$$

where discretization can use forward, backward or central discretization. If $\alpha^{n+1}_{k,i} \geq 0$, $\beta^{n+1}_{k,i} \geq 0$ we say that (4) is *positive coefficient discretization*. *Why do we care?* We care because a *positive coefficient discretization* is also *monotone*. To see it check that

$$S^{n+1}_{k,i}\Big(\tilde{\Delta}, v^{n+1}_{k,i}, v^{n(+1)}_{k,i+1}, v^{n(+1)}_{k,i-1}, v^n_{k,i}, v^{n(+1)}_{\text{-}k,i}\Big)$$

is a nonincreasing function of the neighbor nodes $\{v^m_{\ell,j}\}$. Check a example!

In order to ensure a *positive coefficient discretization* our choice of central/forward/backward differencing will depend, in general, on the control c. A useful rule for this problem is to use the so-called *upwind scheme*.

Idea: Use forward difference whenever drift is positive, and use backward whenever it is negative.

Suppose that we have the value of consumption $c_{k,i}$ at a particular node. Let $s_{k,i} = ra_i + z_k - c_{k,i}$. In this case, the derivatives are approximated

$$\dots \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} s^+_{k,i} + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} s^-_{k,i} + \dots$$

which in terms of our α, β

$$\alpha^{up}_{k,i} = -\frac{s^-_{k,i}}{a_i - a_{i-1}} \geq 0, \quad \beta^{up}_{k,i} = \frac{s^+_{k,i}}{a_{i+1} - a_i} \geq 0$$

But we don't know $c_{k,i}$! Need an iterative method due to max operator.

Implicit timestepping

Start with a vector v^n and update v^{n+1} according to

$$\frac{v_{k,i}^{n+1}-v_{k,i}^{n}}{\Delta}+\rho v_{k,i}^{n+1}=u(c_{k,i}^{n})+\left(v_{k,i}^{n+1}\right)^{F}\left[s_{k,i}^{F,n}\right]^{+}+\left(v_{k,i}^{n+1}\right)^{B}\left[s_{k,i}^{B,n}\right]^{-}$$
$$+\lambda_k\left[v_{-k,i}^{n+1}-v_{k,i}^{n+1}\right] \tag{5}$$

- Compute the policy from the foc $\left(u'\left(c_{k,i}^{n}\right)=\partial_a v_{k,i}^{n}\right)$ for the backward AND forward derivative of the value function.
- Define $s_{k,i}^{B,n}=ra_i+z_k-c_{k,i}^{B,n}$, $s_{k,i}^{F,n}=ra_i+z_k-c_{k,i}^{F,n}$. Set

$$c_{k,i}^{n}=\mathbb{1}_{\left\{s_{k,i}^{B,n}\leq 0\right\}}c_{k,i}^{B,n}+\mathbb{1}_{\left\{s_{k,i}^{F,n}\geq 0\right\}}c_{k,i}^{F,n}+\mathbb{1}_{\left\{s_{k,i}^{F,n}\leq 0\leq s_{k,i}^{B,n}\right\}}(ra_i+z_k)$$

- Collecting terms with the same subscripts on the right-hand slide a

$$\frac{v_{k,i}^{n+1}-v_{k,i}^{n}}{\Delta}+\rho v_{k,i}^{n+1}=u(c_{k,i}^{n})+\alpha_{k,i}v_{k,i-1}^{n+1}+\beta_{k,i}v_{k,i+1}^{n+1}-\left(\alpha_{k,i}+\beta_{k,i}+\lambda_k\right)v_{k,i}^{n+1}+\lambda_i v_{-k,i}^{n+1} \tag{6}$$

where

$$\alpha_{k,i}^{up}=-\frac{\left[s_{k,i}^{B,n}\right]^{-}}{a_i-a_{i-1}}\geq 0,\quad \beta_{k,i}^{up}=\frac{\left[s_{k,i}^{F,n}\right]^{+}}{a_{i+1}-a_i}\geq 0$$

- Equation (6) is just a system of linear equations!!

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$$f(x) = ax^2 + bx + c$$

Some more content

References

ACHDOU, Y., J. HAN, J.-M. LASRY, P.-L. LIONS, AND B. MOLL (2016): “Heterogeneous Agent Models in Continuous Time,” .

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