# Lecture

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July 20, 2017

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# Table of contents

# References

## Theory

- Pham (finance)
- ?, ?

## Numeric

- ? (mainly the numerical appendix), Moll's website (more examples)
- ? (good slides)

# Consumption Savings Problem

## Problem of Household

$$\max_{\{c_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

$$\text{S.t } da_t = \left\{ r_t a_t + z_t - c_t \right\} dt$$

$z_t$ is a ct markov chain on $\{b, w\}$ with intensities $\lambda_1, \lambda_2$

$$dz_t = (w - b) dq_\mu - (w - b) dq_s, \quad q_\mu \sim \text{Poisson}(\lambda_1),\ q_s \sim \text{Poisson}(\lambda_2)$$

$$a_t \geq \underline{a}$$

Individuals' consumption and saving decision is summarized by HJB equation

- time dependent

$$\rho v_k(a,t) = \max_c \left\{ u(c) + \partial_a v_k(a,t)[r_t a + z_k - c] \right\} + \lambda_k \left[ v_{\text{-}k}(a,t) - v_k(a,t) \right] + \partial_t v_k(a,t) \quad (1)$$

- stationary

$$\rho v_k(a) = \max_c \left\{ u(c) + v_k'(a)[ra + z_k - c] \right\} + \lambda_k \left[ v_{\text{-}k}(a) - v_k(a) \right] \quad (2)$$

# Where did borrowing constraint go?

- In continuous time, an undistorted foc holds everywhere in the state space
- Borrowing constraint shows only as *state constraint boundary condition*

$$u'\left(c_i(\underline{a}, t)\right) = \partial_a v_i(\underline{a}, t) \geq u'\left(r_t \underline{a} + z_i\right) \tag{3}$$

which ensures $c_i(\underline{a}, t) \leq r_t \underline{a} + z_i$ so that the borrowing constraint is never violated.

# Continuous $\times$ Discrete time

Consider the first-order condition for consumption

$$u'(c) = \partial_a v(a, z) \tag{4}$$

$$u'(c) \geq \beta \int \partial_a v(a', z') \, \mathrm{d}F(z'|z), \quad a' = z + (1 + r)a - c \tag{5}$$

1. "today" = "tomorrow" — *foc* is static
2. HJB is not stochastic — evolution of stochastic process is captured by additive terms
3. *foc* always holds with equality — borrowing constraint only show up in boundary conditions

# Definition

In general form, the HJB equation is an equation of the form

$$0 = F(\boldsymbol{x}, V, DV, D^2 V) \tag{6}$$

where $\boldsymbol{x} := (x, \tau)$.

**Finite difference methods**: replace derivatives by differences.
Suppose we define a grid $\{x_0, x_1, \ldots, x_i, \ldots\}$ and a set of timesteps $\{i\Delta : i \in \mathbb{N}\}$ Let $V_i^n \approx V(x_i, \tau_n)$ be the approximate value of the solution at node $x_i$ time $\tau^n := T - t$. Then we can write a general discretization of the HJB equation at node $(sx_i, \tau^{n+1})$

$$0 = S_i^{n+1}\Big((\Delta, \Delta x), V_i^{n+1}, \{V_j^m\}\Big) \tag{7}$$

# Sufficient Conditions Convergence

*Condition (Monotonicity)* .— The numerical scheme (7) is monotone if

$$S_i^{n+1}(\cdot, V_i^{n+1}, \{Y_j^m\}) \leq S_i^{n+1}(\cdot, V_i^{n+1}, \{Z_j^m\})$$

for all $Y \geq Z$.

*Condition (Stability)* .— The numerical scheme (7) is stable if for every $\tilde{\Delta} > 0$ it has a solution which is uniformly bounded independently of $\tilde{\Delta}$.

*Condition (Consistency)* .— The numerical scheme (7) is consistent if for every smooth function $\phi$ with bounded derivatives we have

$$S_i^{n+1}(\tilde{\Delta}, \phi(\boldsymbol{x}_i^{n+1}), \{\phi(\boldsymbol{x}_j^m)\}) \to F(\boldsymbol{x}, \phi, D\phi, D^2\phi)$$

as $\tilde{\Delta} \to 0$ and $\boldsymbol{x}_i^{n+1} \to \boldsymbol{x}$.

# Sufficient Conditions Convergence

**Theorem ?** . *If the numerical scheme (7) satisfies monotonicity, stability and consistency conditions, then its solution converges locally uniformly to the unique viscosity solution of (6).*

# Discretization

## Back to our example

Recall our *time-dependent* HJB equation as

$$\partial_\tau v_k(a,\tau) + \rho v_k(a,\tau) - \sup_{c \in \Gamma_k(a)} \left\{ u(c) + \mathcal{D}^c_{T-\tau} v_k(a,\tau) \right\} = 0 \tag{8}$$

where

$$\mathcal{D}^c_t \phi_k(a) = \partial_a \phi_k(a)[r_t a + z_k - c] + \lambda_k \left[ \phi_{\text{-}k}(a) - \phi_k(a) \right]$$

Define a grid $\{a_1, a_2, \ldots, a_i, \ldots\}$ and let $v^n_k = \Big( v_k(a_1, \tau^n), \ldots, v_k(a_i, \tau^n), \ldots \Big)'$. Discretizing this equation requires deciding upon

- which fd approximation to use: forward/backward differencing

$$\partial_a v_k(a) \approx \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i}, \quad \partial_a v_k(a) \approx \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}},$$

- Implicit/explicit timestepping

Let $\mathscr{D}^c$ be the discrete form of the differential operator $\mathcal{D}^c$, so that

$$\left(\mathscr{D}^c v\right)_{k,i} = \alpha_{k,i}(c) v_{k,i-1} + \beta_{k,i}(c) v_{k,i+1} - \Big(\alpha_{k,i}(c) + \beta_{k,i}(c) + \lambda_i\Big) v_{k,i} + \lambda_i v_{\text{-}k,i}$$

and the discretization

$$\frac{v^{n+1}_{k,i} - v^n_{k,i}}{\Delta} - \sup_{c \in \Gamma_{k,i}} \left\{ u(c) + \left(\mathscr{D}^c v^{n(+1)}\right)_{k,i} \right\} = 0 \tag{9}$$

where discretization can use forward, backward or central discretization. If

$$\alpha^{n+1}_{k,i} \geq 0, \;\; \beta^{n+1}_{k,i} \geq 0$$

we say that (9) is *positive coefficient discretization*.

*Why do we care?*
We care because a *positive coefficient discretization* is also *monotone*. To see it check that

$$S^{n+1}_{k,i}\Big(\tilde{\Delta}, v^{n+1}_{k,i}, v^{n(+1)}_{k,i+1}, v^{n(+1)}_{k,i-1}, v^n_{k,i}, v^{n(+1)}_{\text{-}k,i}\Big)$$

is a nonincreasing function of the neighbor nodes $\{v^m_{\ell,j}\}$. Check a example!

# Upwind scheme

In order to ensure a *positive coefficient discretization* our choice of central/forward/backward differencing will depend, in general, on the control $c$. A useful rule for this problem is to use the so-called *upwind scheme*.

IDEA: *Use forward difference whenever drift is positive, and use backward whenever it is negative.*

Suppose that we have the value of consumption $c_{k,i}$ at a particular node. Let $s_{k,i} = ra_i + z_k - c_{k,i}$. In this case, the derivatives are approximated

$$\ldots \frac{v_{k,i+1} - v_{k,i}}{a_{i+1} - a_i} s_{k,i}^{+} + \frac{v_{k,i} - v_{k,i-1}}{a_i - a_{i-1}} s_{k,i}^{-} + \ldots$$

which in terms of our $\alpha, \beta$

$$\alpha_{k,i}^{up} = -\frac{s_{k,i}^{-}}{a_i - a_{i-1}} \geq 0, \quad \beta_{k,i}^{up} = \frac{s_{k,i}^{+}}{a_{i+1} - a_i} \geq 0$$

But we don't know $c_{k,i}$!!! HJB equation is highly nonlinear, so we need an iterative method to solve it.

# Implicit timestepping

Start with a vector $v^n$ and update $v^{n+1}$ according to

$$\frac{v_{k,i}^{n+1}-v_{k,i}^{n}}{\Delta}+\rho v_{k,i}^{n+1}=u(c_{k,i}^{n})+\frac{v_{k,i+1}^{n+1}-v_{k,i}^{n+1}}{a_{i+1}-a_i}\left[s_{k,i}^{F,n}\right]^{+}+\frac{v_{k,i}^{n+1}-v_{k,i-1}^{n+1}}{a_i-a_{i-1}}\left[s_{k,i}^{B,n}\right]^{-} \\ +\lambda_k\left[v_{\text{-}k,i}^{n+1}-v_{k,i}^{n+1}\right] \tag{10}$$

- Compute the policy from the foc $\left(u'\left(c_{k,i}^{n}\right)=\partial_a v_{k,i}^{n}\right)$ for the backward AND forward derivative of the value function.
- Define $s_{k,i}^{B,n}=ra_i+z_k-c_{k,i}^{B,n}$, $s_{k,i}^{F,n}=ra_i+z_k-c_{k,i}^{F,n}$. Set

$$c_{k,i}^{n}=\mathbb{1}\left\{s_{k,i}^{B,n}\leq 0\right\}\times c_{k,i}^{B,n}+\mathbb{1}\left\{s_{k,i}^{F,n}\geq 0\right\}\times c_{k,i}^{F,n}+\mathbb{1}\left\{s_{k,i}^{F,n}\leq 0\leq s_{k,i}^{B,n}\right\}\times(ra_i+z_k)$$

- Collecting terms with the same subscripts on the right-hand side

$$\frac{v_{k,i}^{n+1}-v_{k,i}^{n}}{\Delta}+\rho v_{k,i}^{n+1}=u(c_{k,i}^{n})+\alpha_{k,i}v_{k,i-1}^{n+1}+\beta_{k,i}v_{k,i+1}^{n+1}-\Big(\alpha_{k,i}+\beta_{k,i}+\lambda_k\Big)v_{k,i}^{n+1}+\lambda_i v_{\text{-}k,i}^{n+1} \tag{11}$$

where

$$\alpha_{k,i}^{up}=-\frac{\left[s_{k,i}^{B,n}\right]^{-}}{a_i-a_{i-1}}\geq 0,\quad \beta_{k,i}^{up}=\frac{\left[s_{k,i}^{F,n}\right]^{+}}{a_{i+1}-a_i}\geq 0$$

- Equation (11) is just a system of linear equations!!

# Implicit Timestepping

Equation (11) can be written in matrix notation as

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u(c^n) + \boldsymbol{A}^n v^{n+1}$$

where the sparse matrix $\boldsymbol{A}$ looks like

$$\boldsymbol{A}^n = \begin{bmatrix}
\gamma_{1,1} & \beta_{1,1} & 0 & 0 & \cdots & \lambda_1 & 0 & \cdots & & & 0 \\
\alpha_{1,2} & \gamma_{1,2} & \beta_{1,2} & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \alpha_{1,I} & \gamma_{1,I} & 0 & 0 & \cdots & \cdots & 0 & \lambda_1 \\
\lambda_2 & 0 & \cdots & \cdots & \cdots & \gamma_{2,1} & \beta_{2,1} & 0 & \cdots & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & \cdots & \alpha_{2,2} & \gamma_{2,2} & \beta_{2,2} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \lambda_2 & 0 & 0 & \cdots & \cdots & \alpha_{2,I} & \gamma_{2,I}
\end{bmatrix}_{I+2 \times I+2}$$

entries of row $i$

$$\begin{bmatrix} \underbrace{\alpha_{k,i}}_{\text{inflow } i-1} & \underbrace{-(\alpha_{k,i} + \beta_{k,i} + \lambda_k)}_{\text{outflow}} & \underbrace{\beta_{k,i}}_{\text{inflow } i+1} \end{bmatrix} \begin{bmatrix} v_{k,i-1} \\ v_{k,i} \\ v_{k,i+1} \end{bmatrix}$$

# Distribution

# Distributions

- We now know how to solve the Household problem. But interesting questions require dealing with distributions.
- Denote by $g_i(a,t)$ $i = 1, 2$ the joint density of income $z_i$ and welath $a$.
- The evolution of the density given a fixed initial distribution $g_i(a,0)$ is described by the *Kolmogorov forward equation*
  - time dependent

$$\frac{\partial}{\partial t} g(a,t) = -\frac{\partial}{\partial a}\Big[s_k(a,t) g_k(a,t)\Big] - \lambda_k g_k(a,t) + \lambda_{\text{-}k} g_{\text{-}k}(a,t) \tag{12}$$

  - stationary

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s_k(a) g_k(a)\Big] - \lambda_k g_k(a) + \lambda_{\text{-}k} g_{\text{-}k}(a) \tag{13}$$

Consider the stationary KFE

$$0 = -\frac{\mathrm{d}}{\mathrm{d}a}\Big[s(a, z_k)g(a, z_k)\Big] - \lambda_k g(a, z_k) + \lambda_{\text{-}k} g(a, z_{\text{-}k})$$

with the following discretization

$$0 = -\frac{\left(s^F_{k,i}\right)^+ g_{k,i} - \left(s^F_{k,i-1}\right)^+ g_{k,i-1}}{\Delta a} - \frac{\left(s^B_{k,i+1}\right)^- g_{k,i+1} - \left(s^B_{k,i}\right)^- g_{k,i}}{\Delta a} - \lambda_k g_{k,i} + \lambda_{-k} g_{\text{-}k,i} \quad (14)$$

Collecting terms with the same subscripts on the right-hand side

$$0 = \underbrace{\frac{\left(s^F_{k,i-1}\right)}{\Delta a}}_{\beta_{k,i-1}} g_{k,i-1} + \underbrace{\left(\frac{\left(s^B_{k,i}\right)}{\Delta a} - \frac{\left(s^F_{k,i}\right)}{\Delta a} - \lambda_k\right)}_{\gamma_{k,i}} g_{k,i} + \underbrace{\left(-\frac{\left(s^B_{k,i+1}\right)}{\Delta a}\right)}_{\alpha_{k,i+1}} g_{k,i+1} + \lambda_{-k} g_{\text{-}k,i}$$

which in matrix notation reads $\boldsymbol{A}^T g = \mathbf{0}$.

This makes sense: the operation is exactly the same as that used for finding the stationary distribution of a discrete Poisson process (continuous-time Markov chain). The matrix A captures the evolution of the stochastic process over a very short interval — it is our discretized *infinitesimal generator* of our state — and to find the stationary distribution, one solves the eigenvalue problem $\boldsymbol{A}^T g = \mathbf{0}$.

Consider the time dependent

$$\frac{\partial}{\partial t} g(a,t) = -\frac{\partial}{\partial a}\Big[s_k(a,t) g_k(a,t)\Big] - \lambda_k g_k(a,t) + \lambda_{\text{-}k} g_{\text{-}k}(a,t)$$

Given an initial condition, the KFE can be easily solve through a implicit method

$$\frac{g^{n+1} - g^n}{\Delta} = \left(\boldsymbol{A}^{n(+1)}\right)^T g^{n+1} \tag{15}$$

In the case wiht a dirac mass on the boundary constraint

$$M_1(t) + \int_0^a g_1(a,t) = G_1(a,t), \qquad \int_0^a g_2(a,t) = G_2(a,t) \tag{16}$$

and we may write

$$\frac{\partial}{\partial t} g_1(a,t) = -\frac{\partial}{\partial a}\Big[s_1(a,t) g_1(a,t)\Big] - \lambda_1 g_1(a,t) + \lambda_2 g_2(a,t) \tag{17}$$

$$\frac{\partial}{\partial t} g_2(a,t) = -\frac{\partial}{\partial a}\Big[s_2(a,t) g_2(a,t)\Big] - \lambda_2 g_2(a,t) + \lambda_1 g_1(a,t) + \lambda_1 M_1 \delta_0(a) \tag{18}$$

$$\frac{\partial}{\partial t} M_1(t) = -\lim_{\epsilon \to 0} s_1(\underline{a} + \epsilon, t) g_1(\underline{a} + \epsilon, t) - \lambda_1 M_1 \tag{19}$$

Integrating the KFE between $\underline{a} + \epsilon$ and $\infty$

$$\frac{\partial}{\partial t}\int_{\underline{a}+\epsilon} g_1(a,t) = s_1(\underline{a}+\epsilon,t)g_1(\underline{a}+\epsilon,t) - \lambda_1\int_{\underline{a}+\epsilon} g_1(a,t) + \lambda_2\int_{\underline{a}+\epsilon} g_2(a,t)$$

Using the definition on distributions and taking the limit as $\epsilon \to 0$

$$-\frac{\partial}{\partial t}\Big(M_1(t) - G_1(t)\Big) = \lim_{\epsilon\to 0} s_1(\underline{a}+\epsilon,t)g_1(\underline{a}+\epsilon,t) + \lambda_1\Big(M_1(t) - G_1(t)\Big) + \lambda_2 G_2(t)$$

but note that $\frac{\partial}{\partial t}G_1(t) = -\lambda_1 G_1(t) + \lambda_2 G_2(t)$. That leave us with (19).

# Equilibrium (stationary)

$$\rho v_k(a) = \max_c \Big\{ u(c) + \partial_a v_k(a) \Big\} + \lambda_k \Big[ v_\ell(a) - v_k(a) \Big] \qquad \Big[\text{HJB}\Big]$$

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}a} \big[ s_k(a) g_k(a) \big] - \lambda_k g_k(a) + \lambda_\ell g_\ell(a) \\ 1 &= \int_{\underline{a}}^{\infty} \Big( g_1(a) + g_2(a) \Big) da \end{aligned} \qquad \Big[\text{KFE}\Big]$$

$$0 = \int_{\underline{a}}^{\infty} a \Big( g_1(a) + g_2(a) \Big) da \qquad \Big[\text{Equil}\Big]$$

# This is a title 10pt

Here is some content on the slide

$$f(x) = ax^2 + bx + c$$

Some more content

## References i