Solving Linear Rational Expectations Models

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1 General form of the models

The models we are interested in can be cast in the form

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + \Psi z(t) + \Pi\eta(t) \tag{1}$$

$t = 1, \ldots, T$, where $z(t)$ is an exogenous i.i.d. random disturbance, and $\eta(t)$ is an expectational error, satisfying $E_t\eta(t+1) = 0$, all t. The $\eta(t)$ terms are not given exogenously, but instead are treated as determined as part of the model solution. Models with more lags, or with lagged expectations, or with expectations of more distant future values, can be accommodated in this framework by expanding the y vector.

This paper uses a notation in which time arguments or subscripts relate consistently to the information structure: variables dated t are always known at t.

2 Using diagonalization

2.1 Discrete time

In this section we consider the special case of

$$y(t) = \Gamma_1 y(t-1) + \Psi z(t) + \Pi\eta(t)\ , \tag{2}$$

We assume that the system matrix Γ_1 can be diagonalized to

$$\Gamma_1 = P\Lambda P^{-1}\ , \tag{3}$$

where P is the matrix of right-eigenvectors of Γ_1, and Λ is the diagonal matrix of eigenvalues of Γ_1. Multiplying the system on the left by P^{-1}, and defining $w = P^{-1}y$, we arrive at

$$w(t) = \Lambda w(t-1) + P^{-1}\cdot\left(\Psi z(t) + \Pi\eta(t)\right)\ . \tag{4}$$

The matrix is diagonal, so that the system breaks into unrelated components:

$$w_j\left(t\right)=\lambda_j w_j\left(t-1\right)+P_{j,:}^{-1}\cdot\left(\Psi z(t)+\Pi\eta(t)\right)\;,\tag{5}$$

If the disturbance term (including the combined effects of z and η on the equation) is zero and $\lambda_j \neq 1$, the model has the deterministic steady-state solution

$$w_j(t)=0\;,\tag{6}$$

If $|\lambda_j|>1$, then $E_s\left[w_j\left(t+s\right)\right]$ grows in absolute value at the rate of $|\lambda_j|^t$ as $t\to\infty$, for any solution other than that given in (6). Now consider the restriction on growth,

$$\lim_{t\to\infty} E_s w_j(t)\bar{\xi}^{-t}=0\tag{7}$$

In order for this condition to hold, every one of the variables w_j corresponding to a $|\lambda_j|>\bar{\xi}$ and to a $P_{j,:}^{-1}\neq 0$ must satisfy (6).

Of course a problem must have special structure in order for it to turn out that there is a j such that $P_{j,:}^{-1}=0$. This is the justification for the (potentially misleading) common practice of assuming that if any linear combination of y's is constrained to grow slower than $\bar{\xi}^t$, then all roots exceeding $\bar{\xi}$ in absolute value must be suppressed in the solution. If (6) does hold for all t then we can see from (5) that this entails

$$P_{j,:}^{-1}\left(\Psi z+\Pi\eta\right)=0\;.\tag{8}$$

Collecting all the rows of P^{-1} corresponding to j's for which (8) holds into a single matrix $P^{U\cdot}$ (where the U stands for "unstable"), we can write

$$P^{U\cdot}\left(\Psi z+\Pi\eta\right)=0\;.\tag{9}$$

Existence problems arise if the endogenous shocks η cannot adjust to offset the exogenous shocks z in (9). We might expect this to happen if $P^{U\cdot}$ has more rows than has columns. This accounts for the usual notion that there are existence problems if the number of unstable roots exceeds the number of "jump variables". However, the precise condition is that columns of $P^{U\cdot}\Pi$ span the space spanned by the columns of $P^{U\cdot}\Psi$, i.e.

$$\mathrm{span}\left(P^{U\cdot}\Psi\right)\subset\mathrm{span}\left(P^{U\cdot}\Pi\right)\;.\tag{10}$$

In order for the solution to be unique, it must be that (9) pins down not only the value of $P^{U\cdot}\Pi\eta$, but also all the other error terms in the system that are influenced by η. That is, from knowledge of $P^{U\cdot}\Pi\eta$ we must be able to determine $P^{S\cdot}\Pi\eta$, where $P^{S\cdot}$ is made up of all the rows of P^{-1} not included in $P^{U\cdot}$. Formally, the solution is unique if and only if

$$\mathrm{span}\left(\Pi'\left(P^{S\cdot}\right)'\right)\subset\mathrm{span}\left(\Pi'\left(P^{U\cdot}\right)'\right)\;.\tag{11}$$

In this case we will have

$$P^{S\cdot}\Pi\eta = \Phi P^{U\cdot}\Pi\eta \tag{12}$$

for some matrix Φ.

Usually we aim at writing the system in a form that can be simulated from arbitrary initial conditions, delivering a solution path that does not violate the stability conditions. We can construct such a system by assembling the equations of the form delivered by the stability conditions (6), together with the lines of (5) that determine w_S, the components of w not determined by the stability conditions, and use (9) to eliminate dependence on η. Specifically, we can use the system

$$\begin{bmatrix} w_S\,(t) \\ w_U\,(t) \end{bmatrix} = \begin{bmatrix} \Lambda_S \\ 0 \end{bmatrix} w_S(t-1) + \begin{bmatrix} I & -\Phi \\ 0 & 0 \end{bmatrix} P^{-1}\Psi z\,. \tag{13}$$

To arrive at an equation in y, we use $y = Pw$ to transform (13) into

$$y(t) = P_{\cdot S}\Lambda_S P^{S\cdot} y(t-1) + \left(P_{\cdot S}P^{S\cdot} - P_{\cdot S}\Phi P^{U\cdot}\right)\Psi z\,. \tag{14}$$

Labeling the three matrix coefficients in (14), we can give it the form

$$y(t) = \Theta_1 y(t-1) + \Theta_z z(t)\,, \tag{15}$$

which can in turn be used to characterize the impulse responses of y, according to

$$y(t+s) - E_t y(t+s) = \sum_{v=0}^{s-1} \Theta_1^v \Theta_z z(t+s-v)\,. \tag{16}$$

Of course to completely characterize the mapping from initial conditions and z realizations to y, we need in addition to (16) a formula for the predictable part of y, i.e.

$$E_t y(t+s) = \Theta_1^s y(t)\,. \tag{17}$$

However all the information needed to compute both (16) and (17) is contained in a report of the coefficient matrices for (14). Note that (14), while it insures that the second row of (13),

$$w_U(t) = P^{U\cdot} y(t) = 0\,, \tag{18}$$

holds for all t after the initial date $t = 0$, it does not in itself impose (18) at $t = 0$, which in fact is required by the solution.

3 Discrete time, solving forward

In this section we consider the generic canonical form (1), allowing for possibly singular Γ_0. We assume there is the bound $\bar{\xi}$ on the maximal growth rate of any component of y. We find conditions that prevent such explosive growth as follows. First we compute a QZ decomposition

$$\begin{aligned} Q'\Lambda Z' &= \Gamma_0 \\ Q'\Omega Z' &= \Gamma_1 \ . \end{aligned} \tag{19}$$

In this decomposition, $Q'Q = Z'Z = I$, where Q and Z are both possibly complex and the $'$ symbol indicates transposition and complex conjugation. Also Ω and Λ are possibly complex and are upper triangular. The QZ decomposition always exists. Letting $w(t) = Z'y(t)$, we can multiply (1) by Q to obtain

$$\Lambda w(t) = \Omega w(t-1) + Q\Pi\eta(t) + Q\Psi z(t) \ . \tag{20}$$

Though the QZ decomposition is not unique, the collection of values for the ratios of diagonal elements of Ω and Λ, $\{\omega_{ii}/\lambda_{ii}\}$ (called the set of generalized eigenvalues), is usually unique (if we include ∞ as a possible value). The generalized eigenvalues are indeterminate only when Γ_0 and Γ_1 have zero eigenvalues corresponding to the same eigenvector.[1] We can always arrange to have the largest of the generalized eigenvalues in absolute value appear at the lower right. In particular, let us suppose that we have partitioned (4) so that $|\omega_{ii}/\lambda_{ii}| \geqslant \bar{\xi}$ for all $i > k$ and $|\omega_{ii}/\lambda_{ii}| < \bar{\xi}$ for all $i \leq k$. Then (4) can be expanded as

$$\begin{aligned} &\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \cdot \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \cdot \begin{bmatrix} w_1(t-1) \\ w_2(t-1) \end{bmatrix} \\ &+ \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix} (\Psi z(t) + \Pi\eta(t)) \end{aligned} \tag{21}$$

Note that some diagonal elements of Λ_{22}, but not of Λ_{11}, may be zero. Also note that zeros at the same position i on the diagonals of both Λ and Ω cannot occur unless the equation system is incomplete, meaning that some equation is exactly a linear combination of the others.

Because of the way we have grouped the generalized eigenvalues, the lower block of equations in (20) is purely explosive. It has a solution that does not explode any faster than the disturbances z so long as we solve it "forward" to make w_2 a function of future

[1] This would imply that a linear combination of the equations contains no y's, i.e. that there is effectively one equation fewer for determining the y's than would appear from the order of the system.

z's. That is, if we label the last additive term in (21) $x(t)$ and set $M = \Omega_{22}^{-1} \cdot \Lambda_{22}$,

$$\begin{aligned} Z'_{\cdot 2} y(t) = w_2(t) &= M w_2(t+1) - \Omega_{22}^{-1} x_2(t+1) \\ &= M^2 \cdot w_2(t+2) - M \cdot \Omega_{22}^{-1} \cdot x_2(t+2) - \Omega_{22}^{-1} \cdot x_2(t+1) \\ &= -\sum_{s=1}^{\infty} M^{s-1} \cdot \Omega_{22}^{-1} \cdot x_2(t+s)\,. \end{aligned} \tag{22}$$

The last equality in (22) follows on the assumption that $M^t w_2(t) \to 0$ as $t \to \infty$. Note that in the special case of $\lambda_{ii} = 0$ there are equations in (21) containing no current values of w. While these cases do not imply explosive growth, the corresponding components of (22) are still valid. For example, if the lower right element of Λ is zero, the last equation of (21) has the form

$$0 \cdot w_n(t) = \omega_{nn} \cdot w_n(t-1) + x_n(t). \tag{23}$$

Solving for $w_n(t-1)$ produces the corresponding component of (22). Since $\lambda_{ii} = 0$ corresponds to a singularity in Γ_0, the method we are describing handles such singularities transparently.

Note that (22) asserts the equality of something on the left that is known at time t to something on the right that is a combination of variables dated $t+1$ and later. Since taking expectations as of date t leaves the left-hand side unchanged, we can write

$$Z'_{\cdot 2} y(t) = w_2(t) = -E_t \left[\sum_{s=1}^{\infty} M^{s-1} \cdot \Omega_{22}^{-1} \cdot x_2(t+s) \right] \tag{24}$$

$$= -\sum_{s=1}^{\infty} M^{s-1} \cdot \Omega_{22}^{-1} \cdot x_2(t+s)\,. \tag{25}$$

$$= -\sum_{s=1}^{\infty} M^{s-1} \Omega_{22}^{-1} Q_{2\cdot} \big(\Psi z(t+s) + \Pi \eta(t+s) \big) = 0 \tag{26}$$

Taking E_{t+1} on both sides of (26), we see that (26) is satisfied if and only if

$$Q_{2\cdot} \Pi \eta(t+1) = -Q_{2\cdot} \Psi z(t+1), \qquad \forall t \tag{27}$$

So a necessary and sufficient condition for the existence of a solution[2] satisfying (27) is that the column space of $Q_{2\cdot}\Psi$ be contained in that of $Q_{2\cdot}\Pi$, i.e.

$$\operatorname{span}(Q_{2\cdot}\Psi) \subset \operatorname{span}(Q_{2\cdot}\Pi)\,. \tag{28}$$

[2]Note that here it is important to our analysis that there are no hidden restrictions on variation in z that cannot be deduced from the structure of the equation system. In an equation system in which there are two exogenous variables with $z_1(t) = 2z_2(t-2)$, for example, our analysis requires that this restriction connecting the two exogenous variables be included as an equation in the system and the number of exogenous variables be reduced to one.

Assuming a solution exists, we can combine (27), or its equivalent in terms of w (22), with some linear combination of equations in (21) to obtain a new complete system in w that is stable. However, the resulting system will not be directly useful for generating simulations or distributions of y from specified processes for z unless we can free it from references to the endogenous error term η. From (27), we have an expression that will determine $Q_{2\cdot}\Pi\eta(t)$ from information available at t and a known stochastic process for z. However the system also involves a different linear transformation of η, $Q_{1\cdot}\Pi\eta(t)$. It is possible that knowing $Q_{2\cdot}\Pi\eta(t)$ is not enough to tell us the value of $Q_{1\cdot}\Pi\eta(t)$, in which case the solution to the model is not unique. In order that the solution be unique it is necessary and sufficient that the row space of $Q_{1\cdot}\Pi$ be contained in that of $Q_{2\cdot}\Pi$. In that case we can write

$$Q_{1\cdot}\Pi = \Phi Q_{2\cdot}\Pi \tag{29}$$

for some matrix Φ. Premultiplying (21) by $[I \quad -\Phi]$ gives us a new set of equations, free of references to η, that can be combined with (22) to give us

$$\begin{aligned} &\begin{bmatrix} \Lambda_{11} & \Lambda_{12}-\Phi\Lambda_{22} \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \\ &\qquad = \begin{bmatrix} \Omega_{11} & \Omega_{12}-\Phi\Omega_{22} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} w_1(t-1) \\ w_2(t-1) \end{bmatrix} \\ &\qquad + \begin{bmatrix} Q_{1\cdot}-\Phi Q_{2\cdot} \\ 0 \end{bmatrix} \Psi z(t) - E_t \begin{bmatrix} 0 \\ \sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1} Q_{2\cdot}\Psi z(t+s) \end{bmatrix} . \end{aligned} \tag{30}$$

This can be translated into a system in y of the form

$$y(t) = \Theta_1 y(t-1) + \Theta_0 z(t) + \Theta_y \sum_{s=1}^{\infty} \Theta_f^{s-1}\Theta_z E_t z(t+s) . \tag{31}$$

The details of the translation are

$$\begin{gathered} H = Z \begin{bmatrix} \Lambda_{11}^{-1} & -\Lambda_{11}^{-1}\left(\Lambda_{12}-\Phi\Lambda_{22}\right) \\ 0 & I \end{bmatrix} ; \quad \Theta_1 = Z_{\cdot 1}\Lambda_{11}^{-1}\begin{bmatrix} \Omega_{11} & (\Omega_{12}-\Phi\Omega_{22}) \end{bmatrix} Z ; \\ \Theta_0 = H \cdot \begin{bmatrix} Q_{1\cdot}-\Phi Q_{2\cdot} \\ 0 \end{bmatrix} \cdot \Psi ; \\ \Theta_y = -H_{\cdot 2} ; \quad \Theta_f = M ; \quad \Theta_z = \Omega_{22}^{-1} Q_{2\cdot}\Psi . \end{gathered} \tag{32}$$

The system defined by (31) and (32) can always be computed and is always a complete equation system for y satisfying the condition that its solution grow slower than $\bar{\xi}^t$, even if there is no solution for η in (27) or the solution for $Q_{1\cdot}\Pi$ in (29) is not unique. If there is no solution to (27), then (31)-(32) implicitly restricts the way z enters the system, achieving stability by contradicting the original specification. If the solution to (29) is not unique, then the absence of η from (31)-(32) contradicts the original specification. If the solution is not unique, but Φ is computed to satisfy (29), the (31)-(32) system generates one of the multiple solutions to (1) that grows slower that $\bar{\xi}^t$. If one is interested in generating the full set of non-unique solutions, one has to add back in, as additional "disturbances", the components of $Q_1.\Pi\eta$ left undetermined by (29).

To summarize, we have the following necessary and sufficient conditions for existence and uniqueness of solutions satisyfing the bounded rate of growth condition:

A. A necessary and sufficient condition that (1) has a solution meeting the growth condition for arbitrary assumptions on the covariance matrix of serially uncorrelated z's is (28), i.e. that the column space spanned by $Q_2.\Psi$ be contained in that of $Q_2.\Pi$.

B. A necessary and sufficient condition that any solution to (1) be unique is that the row space of $Q_1.\Pi$ be contained in that of $Q_2.\Pi$.

When condition A is met, a solution is defined by (31)-(32). In the special case of $E_t z\,(t+1) = 0$, the last term of (31) (involving Θ_y, Θ_f and Θ_z) drops out.