Example State-Dependent Pricing

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1 Intoduction

- Some comment

2 Model

- Production function

$$y_t(h) = Z_t a_t(h) \ell_t(h) \tag{2.1}$$

- Each firm h chooses prices $\{p_t\}_t$ in order to maximize its market value

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} D_{t,t+\tau} \Pi_{t+\tau}(h) \tag{2.2}$$

where $D_{t,t+\tau}$ is the nominal stochastic discount factor of the agent and $\Pi_{t+\tau}(h)$ are the nominal profits in period t given by

$$\Pi_t(h) = p_t(h) y_t(h) - W_t \ell_t(h) - \xi_t(h) W_t \mathbb{1}\{p_t(h) \neq p_{t-1}(h)\} \tag{2.3}$$

- Value function for the firm

$$\begin{aligned} V_t\left(a_t(h), \frac{p_{t-1}(h)}{P_t}, \xi_t;\ \cdot\ \right) = \max_p \Bigg\{ \Pi^R\left(a_t(h), \frac{p}{P_t}, \cdot\right) - \mathbb{1}\{p \neq p_{t-1}(h)\}\xi w + \\ + \mathbb{E}_t\left[D^R_{t,t+1} V_{t+1}\left(a_{t+1}(h), \frac{p}{P_{t+1}}, \xi_{t+1}, \cdot\right)\right]\Bigg\} \end{aligned} \tag{2.4}$$

- Rewriting the problem

$$v(a, x;\ \cdot\ ) = \int_\xi \max\left\{V^A(a, x;\ \cdot\ ) - \xi w(\cdot),\ V^N(a, x;\ \cdot\ )\right\} dH(\xi)$$

where

$$\begin{aligned} V^A(a, x;\ \cdot\ ) &= \max_{\tilde{p}} \left\{\Pi^R(a, \tilde{p}, \cdot) + \mathbb{E}\left[D^R(\cdot,\cdot) v\left(a, \tilde{p}\pi_{t+1}^{-1};\ \cdot\ \right)\right]\right\} \\ V^N(a, x;\ \cdot\ ) &= \Pi^R(a, x, \cdot) + \mathbb{E}\left[D^R(\cdot,\cdot) v\left(a', x\pi_{t+1}^{-1};\ \cdot\ \right)\right] \end{aligned} \tag{2.5}$$

The firm will choose to pay the fixed cost iff $V^A - \xi \geq V^N$. Hence, for each individual state a, x there is a unique threshold which makes the firm indifferent between these two options

$$\tilde{\xi}(a, \tilde{p};) = \frac{V^A(a, \tilde{p}) - V^N(a, \tilde{p})}{w}$$

- The firm value function V is therefore given by

$$v(a, x; \ \cdot\ ) = \int_0^{\tilde{\xi}(a,x)} \Big[V^A(a, x; \ \cdot\ ) - \xi w(\cdot)\Big] d\xi + \Big[1 - H\Big(\tilde{\xi}(a, x;)\Big)\Big] V^N(a, x; \ \cdot\ ) \tag{2.6}$$

2.1 Household

There is a representative household with preferences represented by the utility function

$$E_0 \sum_{t=1}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{N_t^{1+1/\varphi}}{1+1/\varphi}\right] \tag{2.7}$$

The total time endowment per period is normalized to 1, so . The household owns the firms and markets are complete.

2.2 Equilibrium

Equilibrium. *A recursive competitive equilibrium is a set of value functions* $\left\{v, V^A, V^N\right\}$*, policies* $\{\tilde{p}, \xi\}$ *for the firm and household* $\left\{C(), N()\right\}$*, and wage* $w(\cdot)$ *such that*

1. *Firm optimization*
   Taking $w(), Y()$ *as given the value function solves the Bellman equation in XX and the* $\{\tilde{p}, \xi\}$ *are the associated policies*

2. *Household optimization*

$$R_t \mathbb{E}\left\{\beta \frac{u_c(C_{t+1})}{u_c(C_t)} \frac{P_t}{P_{t+1}}\right\} = 1, \qquad N^{1/\varphi} = \frac{1}{\chi} C^{-\sigma} w(\cdot)$$

3. *Market clearing*

   - *Bonds market*

     Simply requires that $B_t = 0$ *Include this?*

- *Labor market*

$$
\begin{aligned}
N &= \int \int_0^{\bar{\xi}} \left[\ell\Big(a, \tilde{p}(a,x,\xi;\boldsymbol{s}); \boldsymbol{s}\Big) + \mathbb{1}\Big\{\tilde{p}(a,x,\xi;\boldsymbol{s}) \neq x\Big\}\xi \right] dH(\xi)\ d\widetilde{\Psi} \\
&= \int \Bigg[H\Big(\tilde{\xi}(a,x;\boldsymbol{s})\Big)\ell\Big(a, \tilde{p}^a(a;\boldsymbol{s}); \boldsymbol{s}\Big) + \\
&\qquad \Big(1 - H\Big(\tilde{\xi}(a,x;\boldsymbol{s})\Big)\Big)\ \ell\Big(a, x; \boldsymbol{s}\Big) + \int_0^{\tilde{\xi}(a,x;\boldsymbol{s})} \xi dH(\xi) \Bigg] d\widetilde{\Psi}
\end{aligned} \tag{2.8}
$$

where $\ell(a, \tilde{p}; \boldsymbol{s}) = \frac{\tilde{p}^{-\epsilon} Y}{Za}$

- *Goods market*

$$
\begin{aligned}
C_t = Y_t &= \left(\int_0^1 y(h)^{\frac{\epsilon-1}{\epsilon}} dh \right)^{\frac{\epsilon}{\epsilon-1}} \\
1 &= \int \int_0^{\bar{\xi}} \tilde{p}(a,x,\xi;\boldsymbol{s})^{1-\epsilon} dH(\xi) d\widetilde{\Psi}
\end{aligned} \tag{2.9}
$$

where $y(a,x) = \Big(\tilde{p}(a,x)\Big)^{-\epsilon} Y$

4. *Law of motion Distribution*

3 Method

3.1 Finite Dimensional Approximation

Firm's Value Function

- Value functions are differently approximated in steady state and on the perturbation step

$$
v\big(a, x; \boldsymbol{s}\big) \approx \sum_{j=1}^{n_a} \sum_{i=1}^{n_x} \theta_{j,i}(\boldsymbol{s}) \psi_{j,i}(a,x) \tag{3.1}
$$

- With a particular approximation of the value function, we solve for the coefficients at steady state using collocation which forces the equation to hold exactly on a set of grid points $\big\{a_j, x_i\big\}_{j=1,n_A\ \ i=1,n}$

$$
\begin{aligned}
v\big(a_j, x_i; \boldsymbol{\theta}^*\big) = H\Big(\tilde{\xi}(a_j,x_i)\Big) &\left\{ \Pi^R\Big(a_j, \tilde{p}^a(a_j,x_i);\ \cdot\ \Big) + \beta \sum_{a_{j'} \in \mathcal{A}} \Pi(a_j, a_{j'}) v\left(a_{j'}, \frac{\tilde{p}^a(a_j,x_i)}{\pi^*}; \boldsymbol{\theta}^*\right) \right\} + \\
&\int_0^{\tilde{\xi}(a_j,x_i)} \xi dH(\xi) \quad + \\
&\Big(1 - H\Big(\tilde{\xi}(a_j,x_i)\Big)\Big) \left\{ \Pi^R\Big(a, x_i;\ \cdot\ \Big) + \beta \sum_{a' \in \mathcal{A}} \Pi(a,a') v\Big(a', \frac{x_i}{\pi^*}; \boldsymbol{\theta}^*\Big) \right\}
\end{aligned}
$$

where the decision rules are computed from

$$\tilde{\xi}(a_j, x_i) = \frac{V^A(a_j, x_i; \boldsymbol{\theta}^*) - V^N(a_j, x_i; \boldsymbol{\theta}^*)}{w^*} \tag{3.2}$$

$$0 = Y^* \tilde{p}^a(a_j) - \epsilon Y^* \left(\tilde{p}^a(a_j) - \frac{w^*}{a_j} \right) + \beta \Big(\tilde{p}^a(a_j) \Big)^{\epsilon} \left[\sum_{a \in \mathcal{A}} \Pi(a, a') \frac{\partial v\big(a', \tilde{p}^a(a_j)/\pi^*; \boldsymbol{\theta}^*\big)}{\partial x'} \right] \tag{3.3}$$

- Note that the conditional expectation of the future value function has been broken into two components: the expectation with respect to idiosyncratic shocks is taken explicitly by integration while the expectation with respect to the aggregate shocks is denoted by the expectation operator.

  θ'

Distribution

Two relevant distributions

- $\widetilde{\Psi}(a, x)$: distribution over beginning of period real prices
- $\Psi(a, \tilde{p})$: distribution over effective real prices (production relevant) and idio shocks $\mathbf{t}$

The $\widetilde{\Psi}$ distribution dynamics involves 3 different steps

1. decision of price adjustment adjustment
2. exogenous transition
3. deflation by inflation between $t \mapsto t+1$

The first option comes from Reiter (2009) and involves approximating both distributions by a finite number of mass points on a predefined grid on $\mathcal{X} \coloneqq \{x_i\}_{i=1}^{N_X}$ and productivity $\mathcal{A}$. Let $\widetilde{\Psi}(a_j, \tilde{p}_{-1})$ denote the fraction of firms at the beginning of the period with productivity level a_j and last period relative price of $\tilde{p}_{-1}$. The evolution of this distribution involves the three steps discussed above. For any pairs $\Big\{(a_j, \tilde{p}^i_{\text{-}1}), (a_j, x_{i'})\Big\} \subset \mathcal{A} \times \mathcal{X}$, the probability of moving from the first to the second is

$$prob(\quad ; \Pi) = \begin{cases} \frac{\Pi^{-1}\tilde{p}^i_{\text{-}1} - x_{i'-1}}{x_{i'} - x_{i'-1}} & \text{if } x_{i'} = \min\left\{x \in \mathcal{X} : x \geq \Pi^{-1}\tilde{p}^i_{\text{-}1}\right\} \\ \frac{x_{i'+1} - \Pi^{-1}\tilde{p}^i_{\text{-}1}}{x_{i'+1} - x_{i'}} & \text{if } x_{i'} = \max\left\{x \in \mathcal{X} : x < \Pi^{-1}\tilde{p}^i_{\text{-}1}\right\} \\ 0 & \text{ow} \end{cases} \tag{3.4}$$

Then, for any pair $(a_j, \tilde{p}^i) \in \mathcal{A} \times \mathcal{X}$, next period Ψ'

$$\Psi'\Big(a_j, \tilde{p}^{i'}\Big) = \omega_{j,i'} \sum_{i=1}^{N_X} \xi_{j,i} \times \widetilde{\Psi}(a_j, x_i) + (1 - \xi_{j,i'}) \times \widetilde{\Psi}(a_j, x_{i'}) \tag{3.5}$$

where $\omega_{j,i'}$

$$\omega_{j,i'} = \begin{cases} \frac{\tilde{p}_j - \tilde{p}^{i'-1}}{\tilde{p}^{i'} - \tilde{p}^{i'-1}} & \text{if } \tilde{p}_j \in \left[\tilde{p}^{i'-1}, \tilde{p}^{i'}\right] \\ \\ \frac{\tilde{p}^{i'+1} - \tilde{p}_j}{\tilde{p}^{i'+1} - \tilde{p}^{i'}} & \text{if } \tilde{p}_j \in \left[\tilde{p}^{i'}, \tilde{p}^{i'+1}\right] \\ \\ 0 & \text{o.w.} \end{cases}$$

Finite-Dimensional system

$$
\begin{cases}
\beta\left(\frac{Y'}{Y}\right)^{-\sigma}\frac{1}{\Pi'}R-1 \\
N^{1/\varphi}-\frac{1}{\chi}Y^{-\sigma}w \\
\\
N-\int\left[H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\frac{\left(\tilde{p}^{a}(a)\right)^{-\epsilon}Y}{Za}+\left(1-H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\right)\frac{x^{-\epsilon}Y}{Za}+\int_{0}^{\tilde{\boldsymbol{\xi}}_{j,i}}\xi dH(\xi)\right]d\widetilde{\Psi}(\Pi) \\
1-\int\left[H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\left(\tilde{p}^{a}(a)\right)^{1-\epsilon}+\left(1-H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\right)x^{1-\epsilon}\right]d\widetilde{\Psi}(\Pi) \\
1-\int\tilde{p}^{1-\epsilon}d\Psi'(\Pi) \\
\\
\text{Distribution Dynamics} \\
\\
z'-\rho_{z}z-\sigma_{z}\omega_{z}' \\
\\
\boldsymbol{V}_{j,i}-H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\left\{\Pi^{R}\left(a_{j},\boldsymbol{p}_{j}^{a};\,\cdot\,\right)+\beta\left(\frac{Y'}{Y}\right)^{-\sigma}\sum_{j'}\Pi[a_{j},a_{j'}]\left(\boldsymbol{V}',v^{*}\right)\left(a_{j'},\frac{\boldsymbol{p}_{j}^{a}}{\Pi'}\right)\right\} \\
+\int_{0}^{\tilde{\boldsymbol{\xi}}_{j,i}}\xi dH(\xi)-\left(1-H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\right)\left\{\Pi^{R}\left(a,x_{i};\,\cdot\,\right)+\beta\sum_{j'}\Pi(a_{j},a_{j'})\left(\boldsymbol{V}',v^{*}\right)\left(a_{j'},\frac{x_{i}}{\Pi'}\right)\right\} \\
\frac{\partial\Pi^{R}}{\partial\tilde{p}}\left(a_{j},\boldsymbol{p}_{j}^{a};\,\cdot\,\right)+\beta\left(\frac{Y'}{Y}\right)^{-\sigma}\sum_{j'}\Pi[a_{j},a_{j'}]\frac{\partial\left(\boldsymbol{V}',v^{*}\right)}{\partial x'}\left(a_{j'},\frac{\boldsymbol{p}_{j}^{a}}{\Pi'}\right)\frac{1}{\Pi'}
\end{cases}
$$

With all these approximations, the recursive equilibrium becomes computable.

function f that satisfies

$$\mathbb{E}\Big[f\big(\mathbf{y}',\mathbf{y},\mathbf{x}',\mathbf{x}\big)\Big]=0 \tag{3.6}$$

where $\mathbf{y}=\big(Y,N,R,\Pi,w,\boldsymbol{V},\boldsymbol{p}^{a},\boldsymbol{\xi}\big)$ are the control variables, $\mathbf{x}=\big(\Psi,Z\big)$ are the (endogenous and exogenous) state variables.

This puts the model in the canonical form presented in Schmitt-Grohe and Uribe (2004).

3.2 Steady State

OPTION 02

1. Guess a value a pair w^{*},Y^{*}
2. Given $\left(w^{*},Y^{*}\right)$, compute the firm's value function.
3. Using the firm's decision rules, compute the invariant distribution.

4. Check the market-clearing conditions

$$1 = \int \tilde{p}^{1-\epsilon} \Psi\big(da, d\tilde{p}\big)$$

$$N^* = \int \ell\big(a, \tilde{p}\big) \Psi\big(da, d\tilde{p}\big) + \int \left(\int^{\tilde{\xi}(a,x)} \zeta dH(\zeta) \right) \widetilde{\Psi}\big(da, dx\big)$$

4 Linear Rational Expectational Difference Equations

4.1 Preliminaries

Definition. Let $P \in \mathbb{C} \to \mathbb{C}^{n\times n}$ be a matrix-valued function of a complex variable (a matrix pencil). Then the set of its generalized eigenvalues $\lambda(P)$ is defined as

$$P(z) := \left\{ z \in \mathbb{C} : |P(z)| = \mathbf{0} \right\}$$

When $P(z)$ writes as AzB, we denote this set as $\lambda(A, B)$. Then there exists a vector V such that $BV = \lambda AV$.

Theorem (The complex generalized Schur form)**.**
Let A and B belong to $\mathbb{C}^{n\times n}$ and be such that $P(z) = Az - B$ is a regular matrix pencil. Then there exist unitary $n \times n$ matrices of complex numbers Q and Z such that

1. *$S = QAZ$ is upper-triangular*
2. *$T = QBZ$ is upper-triangular*
3. *For each i, s_{ii}, t_{ii} are not both zero*
4. *$\lambda(A, B) = \left\{ \frac{t_{ii}}{s_i i} : \; s_{ii} \neq 0 \right\}$*
5. *The pairs (s_{ii}, t_{ii}) can be arranged in any order*

Theorem (*Singular Value Decomposition*)**.**
Every $m \times n$ complex matrix A can be factored into a product of three matrices

$$A = USV' \tag{4.1}$$

called a singular value decomposition (SVD) *where*

- *$U_{m\times m}, V_{n\times n}$ are orthogonal matrices*
- *S is a diagonal matrix with entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min\{n,m\}} \geq 0$.*

*The values σ_i are called the **singular values** of A, while the column vectors u_j, v_i are the left/right **singular vectors**.*

4.2 Klein

Linearizing (3.6) around the steady-state yields a first-order linear expectational difference equation system of the form

$$f_{\mathbf{y}'} E_t \Big[\underbrace{(\mathbf{y}_{t+1} - \bar{\mathbf{y}})}_{\tilde{\mathbf{y}}_{t+1}} \Big] + f_{\mathbf{y}} \tilde{\mathbf{y}}_t + f_{\mathbf{x}'} E_t \Big[\tilde{\mathbf{x}}_{t+1} \Big] + f_{\mathbf{x}} \tilde{\mathbf{x}}_t = 0 \tag{4.2}$$

which we can put on Klein's form letting $A = -\begin{bmatrix} f_{\mathbf{x}'} & f_{\mathbf{y}'} \end{bmatrix}$ and $B = \begin{bmatrix} f_{\mathbf{x}} & f_{\mathbf{y}} \end{bmatrix}$. Then the system can be written as

$$AE_t \left\{ \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} \right\} = B \begin{bmatrix} x_t \\ y_t \end{bmatrix} \tag{4.3}$$

Consider the generalized Schur decomposition of A, B

$$QAZ = S \qquad QBZ = T \tag{4.4}$$

where A, B are upper triangular and Q, Z are orthonormal matrices. Let S and T be arranged in such a way that the n_s stable eigenvalue come first. Partition the rows of Z conformably as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

DEfine the auxiliary variables W_t as

$$w_t := Z^H \left[x_t' \; y_t' \right]' = \begin{bmatrix} s_t \\ u_t \end{bmatrix} \tag{4.5}$$

where the transformed variable w_t is divided into $n_s \times 1$ stable and $n_u \times 1$ unstable components. Premultiply the system by Q to get

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \left\{ \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} \right\} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} \tag{4.6}$$

Since the generalized eigenvalues of the matrix pencil $S_{22}z - T_{22}$ are all unstable, the unique stable solution for u_t is found by solving forward

$$u_t = T_{22}^{-1} S_{22} E_t\{u_{t+1}\} = 0 \tag{4.7}$$

The first block then implies

$$E_t\left\{s_{t+1}\right\} = S_{11}^{-1} T_{11} s_t \tag{4.8}$$

Recalling the definition of w_t

$$x_{t+1} = \begin{bmatrix} Z_{11} & Z_{12} \end{bmatrix} \begin{bmatrix} s_{t+1} \\ u_t \end{bmatrix} \tag{4.9}$$

we have

$$z_{11}\Big(s_{t+1} - E_t\{s_{t+1}\}\Big) = \eta \epsilon_{t+1} \tag{4.10}$$

Taken together, these define the unique solution for s_t given s_0 and the exogenous process ϵ

$$s_{t+1} = S_{11}^{-1} T_{11} s_t + Z_{11}^{-1} \eta \epsilon_{t+1} \tag{4.11}$$

We can now use the relation $\left[x_t' \; y_t'\right]' = Zw_t$ to find a recursive representation of the solution. Using the restriction on the control

$$
\begin{aligned}
Z_{21}^H x_t + Z_{22}^H y_t = 0 \Rightarrow y_t &= -\left(Z_{22}^H\right)^{-1} Z_{21}^H x_t \\
&= \left(Z_{22}^H\right)^{-1} Z_{22}^H Z_{21} Z_{11}^{-1} x_t \\
&= Z_{21} Z_{11} x_t
\end{aligned}
\tag{4.12}
$$

As for x_t, note that

$$
\begin{aligned}
s_t &= Z_{11}^H x_t + Z_{12}^H y_t \\
&= \left(Z_{11}^H + Z_{12}^H Z_{21} Z_{11}^{-1}\right) x_t \\
&= \left(Z_{11}^H + \left(Z_{11}^{-1} - Z_{11}^H\right)\right) x_t \\
&= Z_{11}^{-1} x_t
\end{aligned}
$$

Hence

$$
x_{t+1} = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} x_t + \eta \epsilon_{t+1} \tag{4.13}
$$

4.3 Sims

Consider the following model

$$
\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \epsilon_t + \Pi \eta_t \tag{4.14}
$$

where y_t is a n vector of endogenous variables, Γ_0, Γ_1 are $n \times n$ coefficients matrices , ϵ_t is a $\ell \times 1$ vector of exogenous random disturbances, η_t is an $k \times 1$ vector of expectational errors satisfying $E_t\left[\eta_{t+1}\right] = 0$. Note that η_t terms are not given exogenously, instead they are determined as part of the solution.

This method uses the notation that in which time arguments relate consistently to the information structure, meaning that variables date t are always know at date t.

4.3.1 Special Case

Consider the special case of (4.14)

$$
y_t = \Gamma y_{t-1} + \Psi \epsilon_t + \Pi \eta_t
$$

Assume that the matrix Γ can be diagonalized to

$$
\Gamma = P \Lambda P^{-1}
$$

where P is the matrix of right-eigenvectors of Γ, P^{-1} is the matrix of left-eigenvectors and Λ is the diagonal matrix of eigenvalues. Multiplying the system by P^{-1} and defining $w = P^{-1} y$ we arrive at

$$
w_t = \Lambda w_{t-1} + P^{-1}\left(\Psi \epsilon_t + \Pi \eta_t\right) \tag{4.15}
$$

Since Γ is diagonal the system breaks into unrelated components

$$w_{j,t} = \lambda_j w_{j,t-1} + \tilde{P}_{j\cdot}\Big(\Psi\epsilon_t + \Pi\eta_t\Big) \tag{4.16}$$

If the disturbance term zero and $\lambda_j \neq 1$, the model has a deterministic steady-state solution

$$w_{j,t} = 0 \tag{4.17}$$

If $|\lambda_j| > 1$, then $E_t\big[w_{j,t+\tau}\big]$ diverges as $\tau \to \infty$ for any solution other then $w_{j,t} = 0$. If we are looking for a *stationary equilibrium*, every one of the variables w_j corresponding to $|\lambda_j| > 1$ and to $P_{j,:} \neq 0$ must be set to its steady-state value. If (4.17) holds for all t then (4.16) implies

$$\tilde{P}_{j\cdot}\Big(\Psi\epsilon_t + \Pi\eta_t\Big) = 0 \tag{4.18}$$

Collecting all the rows of P^{-1} for which (4.18) holds into a single matrix $\tilde{P}^U$, we can write

$$\tilde{P}^U\Big(\Psi\epsilon + \Pi\eta\Big) = 0 \tag{4.19}$$

Existence problems arise if the endogenous shocks η cannot adjust to offset the exogenous disturbances ϵ. This accounts for the usual notion that there are existence problems if the number of *unstable roots* exceeds the number of *jump variables*. The precise condition here is that the columns of $\tilde{P}^U\Pi$ span the space spanned by the columns of $\tilde{P}^U\Psi$, i.e.

$$\operatorname{span}\Big(\tilde{P}^U\Psi\Big) \subset \operatorname{span}\Big(\tilde{P}^U\Pi\Big) \tag{4.20}$$

Multiple solutions may exist when (4.20) puts too few restrictions. For the solution to be unique, it must be that (4.20) pins down not only the value of $\tilde{P}^U\Pi\eta$ but also$\tilde{P}^S\Pi\eta$ which resumes the impact of expectational shocks on the stable block of the system (4.15). Formally, we require the row space of $\tilde{P}^S\Pi$ to be included into the row space of $\tilde{P}^U\Pi$. In that case, there exists Φ such that

$$\tilde{P}^S\Pi = \Phi\tilde{P}^U\Pi \tag{4.21}$$

We can write the solution by assembling the equations representing the stability conditions (4.17) together with the lines of (4.15) that determine w_s and use (4.21) to eliminate the dependence over η.

$$\begin{bmatrix} w_t^S \\ w_t^U \end{bmatrix} = \begin{bmatrix} \Lambda_S \\ \mathbf{0} \end{bmatrix} w_{t-1}^S \begin{bmatrix} I & -\Phi \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^{-1}\Psi\epsilon_t \tag{4.22}$$

Multiply by P to arrive in y

$$y_t = \underbrace{P_{:,S}\Lambda_S\tilde{P}^S}_{\Theta_y} y_{t-1} + \underbrace{\Big(P_{:,S}\tilde{P}^S - P_{:,S}\Phi\tilde{P}^U\Big)\Psi}_{\Theta_\epsilon}\,\epsilon_t \tag{4.23}$$

4.3.2 General Case

Generalized Schur decomposition there exist matrices Q, Z, T and S such that

$$Q'SZ' = \Gamma_0, \qquad Q'TZ' = \Gamma_1 \tag{4.24}$$

$$QQ' = ZZ' = I_{n\times n} \tag{4.25}$$

Let us define $w_t = Z'y_t$ and pre-multiply the system by Q in order to get

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Big(\Psi\epsilon_t + \Pi\eta_t \Big) \tag{4.26}$$

Let's focus on the explosive part of the system

$$S_{22}w_{2,t} = T_{22}w_{2,t-1} + Q_2\Big(\Psi\epsilon_t + \Pi\eta_t\Big)$$

While the diagonal elements of S_{22} can be null, T_{22} is necessarily full rank. Therefore, we can solve forward for $w_{2,t-1}$. Start by leading the equation by one period and writing it in terms of $w_{2,t}$

$$w_{2,t} = Mz_{2,t+1} - T_{22}^{-1}Q_2\Big(\Psi\epsilon_{t+1} + \Pi\eta_{t+1}\Big)$$

where $M := T_{22}^{-1}S_{22}$. Recursive substitution of $w_{2,t+1}$ leads us to

$$w_{2,t} = -\sum_{i=1}^{\infty} M^{i-1}T_{22}^{-1}Q_2\Big(\Psi\epsilon_{t+i} + \Pi\eta_{t+i}\Big)$$

where we imposed $\lim M^t w_{2,t} = 0$ since we are searching for a non-explosive solution of the LRE model (4.14). Since y_t is known at time t, $w_{2,t} = E_t\{w_{2,t}\}$ which implies

$$w_{2,t} = -\sum_{i=1}^{\infty} M^{i-1}T_{22}^{-1}Q_2\Big(\Psi E_t\big\{\epsilon_{t+i}\big\} + \Pi E_t\big\{\eta_{t+i}\big\}\Big) = 0$$

which imposes a restriction on ϵ_t, η_t. If we go back to (4.26) and take into account $W_{2,t} = 0$ in the second block, this imposes

$$\underbrace{Q_2\Psi}_{n_u\times\ell}\ \underbrace{\epsilon_t}_{\ell\times 1} \quad + \quad \underbrace{Q_2\Pi}_{n_u\times k}\ \underbrace{\eta_t}_{k\times 1} = 0 \tag{4.27}$$

As before, existence problems arise if the endogenous shocks η cannot adjust to offset the exogenous shocks ϵ in (4.27). Note that the assertion in (4.27) is only possible because of the degree of freedom to choose η, otherwise it requires that exogenously evolving events always satisfy a deterministic equation.

References

Michael Reiter. Solving heterogeneous-agent models by projection and perturbation. *Journal of Economic Dynamics and Control*, 33(3):649–665, March 2009.

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A Perturbing the value Function

function in Reiter: `BellmanIterGivenPolicy1dd`

We adopt the following approximation to perturb the value function

$$\text{itp}\Big(\boldsymbol{V}, v^*\Big)(z_j, \tilde{x}) = v^*(z_j, \boldsymbol{x}) + \frac{\tilde{x} - \bar{x}_i}{\bar{x}_{i+1} - \bar{x}_i}\Big(\boldsymbol{V}_{j,i+1} - v^*(z_j, \bar{x}_{i+1})\Big) + \frac{\bar{x}_{i+1} - \tilde{x}}{\bar{x}_{i+1} - \bar{x}_i}\Big(\boldsymbol{V}_{j,i} - v^*(z_j, \bar{x}_i)\Big) \tag{A.1}$$

In steady state, $\boldsymbol{V}_t[j,i] = v^*(z_j, x_i)$ so that in steady state the whole expression reduces to simply $v^*(a, \tilde{x})$.

B How to do the perturbation of policy

`policyPtyerturb1dd` for perturbation of policies

The policy outside steady state must satisfy

$$x_t'\Big(z, x\Big) = \underset{x'}{\text{argmax}}\left\{U(z, x, x'; X_t) + \beta\, \mathbb{E}_t\left[\sum_{z' \in \mathcal{Z}} \Pi(z, z')\text{itp}\Big[\boldsymbol{V}_{t+1}, v^*\Big](z', x')\right]\right\} \tag{B.1}$$

Since it is optimal, it must satisfy the *foc* for all realizations of aggregate state

$$\mathbb{E}_t\left[U_{x'}\Big(z, x, \boldsymbol{x}_t'(z, x); \boldsymbol{X}_t\Big) + \beta \sum_{z \in \mathcal{Z}} \Pi(z, z')\left(\frac{\partial v^*(z', \boldsymbol{x}_t'(z, x))}{\partial x'} + \frac{d\boldsymbol{V}_{t+1}[j, i+1] - d\boldsymbol{V}_{t+1}[j, i]}{\bar{x}_{i+1} - \bar{x}_i}\right)\right] = 0 \tag{dvdc}$$

We know that at steady-state the policy computed in Step XX satisfy

$$0 = U_{x'}\Big(z, x, x'(z, x; X^*); X^*\Big) + \beta \sum_{z \in \mathcal{Z}} \Pi(z, z')\left(\frac{\partial v^*\Big(z', x'(z, x; X^*)\Big)}{\partial x'} + \frac{d\boldsymbol{V}^*_{j,i+1} - d\boldsymbol{V}^*_{j,i}}{\bar{x}_{i+1} - \bar{x}_i}\right)$$

while the *soc* again at steady state values satisfy

$$U_{x'x'}\Big(z, x, x'(z, x; X^*); X^*\Big) + \beta \sum_{z \in \mathcal{Z}} \Pi(z, z')\frac{\partial^2 v^*\Big(z', x'(z, x; X^*)\Big)}{\partial x'^2} \tag{d2vdc2}$$

Hence, to compute how the policy changes given variations on other equilibrium quantities we can use the *Implicit function theorem* to get

$$\frac{\partial x'\Big(z, x; X^*\Big)}{\partial X} = -\frac{1}{\text{d2vdc2}}\Big(\text{d2vdc2.der}\Big) \tag{B.2}$$