User Guide 02
State-Dependent Pricing

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Abstract

These notes present a second example of Reiters Projection+Perturbation approach to solve heterogenous agents models in the presence of aggregate uncertainty. For this application, we choose a **menu-cost model** in the spirit of Golosov and Lucas (2007). The model itself has been well studied in the literature - see Midrigan (2011) and Vavra (2013) subsequent contributions - where it is usually solved using Krusell et al. (1998) method. To see an application of the same methods to a similar menu-cost model as the one discussed in these notes, check Costain and Nakov (2011).

1. Model

1.1. Household

There is a representative household with utility function

$$E_0 \sum_{t=1}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{N_t^{1+1/\varphi}}{1+1/\varphi} \right] \tag{1.1}$$

Consumption is a CES aggregate of differentiated goods $\{c_t(h) : h \in [0,1]\}$, with elasticity of substitution ϵ

$$C_t = \left(\int c_t(h)^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

Optimal allocation across the differentiated goods implies the following demand function

$$c_t(h) = \big(p_t(h)/P_t\big)^{-\epsilon} C_t \tag{1.2}$$

where P_t is the price index defined by

$$P_t \equiv \left(\int p_t(h)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \tag{1.3}$$

Household's nominal period budget constraint is

$$\underbrace{\int p_t(h) c_t(h) dh}_{P_t C_t} + R_t^{-1} B_t = w_t N_t + B_{t-1} + T_t$$

where B_t is the nominal bond holdings purchased at t with nominal return of R_t between $t \mapsto t+1$ and T_t are the nominal dividend payments from the firms. Hence, households choose $\{C_t, N_t, B_t\}$ to maximize (1.1) subject to household's nominal period budget constraint. Optimality requires the following conditions

$$N^{1/\varphi} = \frac{1}{\chi} C^{-\sigma} w_t \tag{1.4}$$

$$R_t^{-1} = \beta E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\} \tag{1.5}$$

$$D_{t,t+\tau} = \beta \left(\frac{C_{t+\tau}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+\tau}} \tag{1.6}$$

1.2. Firms

Each firm h produces output $y_t(h)$ with labor $\ell_t(h)$ as the only input and is subject to an aggregate and an idiosyncratic productivity $Z_t, a_t(h)$

$$y_t(h) = Z_t a_t(h) \ell_t(h) \tag{1.7}$$

Firms act as monopolistic competitors and choose their prices to maximize its market value

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} D_{t,t+\tau} \Pi_{t+\tau}(h) \tag{1.8}$$

where $D_{t,t+\tau}$ is the nominal stochastic discount factor of the household introduced below and $\Pi_{t+\tau}(h)$ are the nominal profits in period t. The firm can change its price only upon payment of fixed cost $\xi \in \left[0, \bar{\xi}\right]$ denoted in units of labor. Specifically, each period a firm draws a cost from the time-invariant distribution H and decides whether or not to change its price. Given these restrictions, firm's flow profits are given by

$$\Pi_t(h) = p_t(h) y_t(h) - W_t \ell_t(h) - \xi_t(h) W_t \mathbb{1}\left\{p_t(h) \neq p_{t-1}(h)\right\} \tag{1.9}$$

The firm understands that its sales $y_t(h)$ depend upon the price charged for the good according to (1.2).

Recuvisve Formulation. We start by expressing everything in real terms, since the nominal price level should be irrelevant for the equilibrium. Define the real profit function in terms of idiosyncratic productivity and relative prices by incorporating the demand function (1.2)

$$\Pi_t^R(a, \tilde{p}) = C_t \tilde{p}^{-\epsilon} \left(\tilde{p} - \frac{w_t}{Z_t a} \right) \tag{1.10}$$

We can write the problem recursively as

$$\begin{aligned} V_t \left(a_t(h), \frac{p_{t-1}(h)}{P_t}, \xi_t; \boldsymbol{s} \right) = \max_p \Bigg\{ & \Pi_t^R \left(a_t(h), \frac{p}{P_t} \right) - \mathbb{1}\left\{p \neq p_{t-1}(h)\right\} \xi_t w_t + \\ & + \mathbb{E}_t \left[D_{t,t+1}^R V_{t+1} \left(a_{t+1}(h), \frac{p}{P_{t+1}}, \xi_{t+1}, \boldsymbol{s} \right) \right] \Bigg\} \end{aligned} \tag{1.11}$$

where $\boldsymbol{s}$ is the aggregate state vector - see Subsection 1.4 for a discussion of what is a sufficient aggregate state for this economy. It is computationally convenient to rewrite the problem in terms of two value functions

$\{V^A, V^N\}$, which denote respectively the value of adjusting and not adjusting. Moreover, we can get rid of ξ as a state variable by defining beginning- of-period expected value of a firm *prior* to its fixed cost draw, but after the determination of idiosyncratic productivity and aggregate state

$$v(a, x; \boldsymbol{s}) := \int_{\xi} \max\Big\{V^A(a, x; \boldsymbol{s}) - \xi w(\boldsymbol{s}),\ V^N(a, x; \boldsymbol{s})\Big\} dH(\xi) \tag{1.12}$$

where

$$V^A(a; \boldsymbol{s}) = \max_{\tilde{p}} \left\{ \Pi^R\left(a, \tilde{p}, \boldsymbol{s}\right) + \mathbb{E}\left[D^R(\boldsymbol{s}, \boldsymbol{s}) v\left(a, \frac{\tilde{p}}{\pi(\boldsymbol{s}')};\ \boldsymbol{s}'\right)\right]\right\} \tag{1.13}$$

$$V^N(a, x; \boldsymbol{s}) = \Pi^R\left(a, x; \boldsymbol{s}\right) + \mathbb{E}\left[D^R(\boldsymbol{s}, \boldsymbol{s}) v\left(a', \frac{x}{\pi(\boldsymbol{s}')};\ \boldsymbol{s}'\right)\right] \tag{1.14}$$

It is clear that a firm will choose to change its price only if the net value of doing so is at least as great as the continuation under the relative price x. Therefore, firms will follow a **threshold rule**. In particular, let $\widetilde{\xi}(a, x)$ describe the fixed cost that leaves a type (a, x) firm indifferent between adjusting/not adjusting

$$V^a(a; \boldsymbol{s}) - w(\boldsymbol{s})\widetilde{\xi}(a, x; \boldsymbol{s}) = V^n(a, x; \boldsymbol{s}) \tag{1.15}$$

Thus, within each group of firms with idiosyncratic state (a, x), a fraction $H\big(\widetilde{\xi}(a, x; \boldsymbol{s})\big)$ chooses to incur on the menu-cost and change their prices while a fraction $H\big(\widetilde{\xi}(a, x; \boldsymbol{s})\big)$ find it optimal not to adjust their prices. Given this observation, the firm's value function v in (1.12) can be expressed as

$$v(a, x;\ \cdot\ ) = \int_0^{\tilde{\xi}(a,x)} \Big[V^A(a, x;\ \cdot\ ) - \xi w(\cdot)\Big] d\xi + \Big[1 - H\Big(\tilde{\xi}(a, x;)\Big)\Big] V^N(a, x;\ \cdot\ ) \tag{1.16}$$

1.3. Monetary Policy

The monetary policy follows a Taylor interest rate rule

$$\frac{R_t}{R^*} = \exp\left(\epsilon_t\right) \left(\frac{P_t/P_{t-1}}{\pi^*}\right)^{\phi_\pi} \left(\frac{C_t}{C^*}\right)^{\phi_y} \tag{1.17}$$

where π^* is the inflation target at steady-state. For all the exercises, we will take it to be equal to 1.

What about inflation equation?

We are still missing an equilibrium equation for inflation. Consider the simple case where firm's do not differ on their idiosyncratic productivity. For a given ϵ, the firms that would adjust price given inflation $\pi(\boldsymbol{s})$ is given by

$$\mathcal{C}(\xi;\boldsymbol{s}) := \left\{ \tilde{p}_- : \xi > \tilde{\xi}\left(\frac{\tilde{p}_-}{\pi(\boldsymbol{s})}\right) \right\}$$

From the definition of the price index in (1.3) we have

$$\begin{aligned}
\pi(\boldsymbol{s})^{1-\epsilon} &= \int_0^{\bar{\xi}} \int \left[\pi(\boldsymbol{s}) \cdot \tilde{p}\left(\frac{\tilde{p}_-}{\pi(\boldsymbol{s})}, \xi; \boldsymbol{s}\right)\right]^{1-\epsilon} d\tilde{\Psi}_- dH(\xi) \\
&= \left(p^* \pi(\boldsymbol{s})\right)^{1-\epsilon} \underbrace{\left(\int_0^{\bar{\xi}} \int_{\tilde{p}_- \notin \mathcal{C}(\xi;\boldsymbol{s})} d\widetilde{\Psi}_- dH(\xi)\right)}_{\Omega(\boldsymbol{s})} + \int_0^{\bar{\xi}} \int_{\tilde{p}_- \in \mathcal{C}(\xi;\boldsymbol{s})} \tilde{p}_-^{1-\epsilon} d\widetilde{\Psi}_- dH(\xi)
\end{aligned}$$

Rearranging

$$\pi(\boldsymbol{s}) = \left(\frac{1-\Omega(\boldsymbol{s})}{1-\Omega(\boldsymbol{s})\left(p^*\right)^{1-\epsilon}}\right)^{\frac{1}{1-\epsilon}} \left(\frac{1}{1-\Omega(\boldsymbol{s})} \int_0^{\bar{\xi}} \int_{\tilde{p}_- \in \mathcal{C}(\xi;\boldsymbol{s})} \tilde{p}_-^{1-\epsilon} d\widetilde{\Psi}_- dH(\xi)\right)^{\frac{1}{1-\epsilon}}$$

in the simple Calvo model with probability of changing prices of θ we would get

$$\pi(\boldsymbol{s}) = \left(\frac{1-\theta}{1-\theta\left(p^*\right)^{1-\epsilon}}\right)^{\frac{1}{1-\epsilon}}$$

In the computation, instead of working with this condition, we will be working on the equivalent restriction that real price of consumption aggregate is equal to 1

$$1 = \int \tilde{p}^{1-\epsilon} d\tilde{\Psi}(\boldsymbol{s})$$

The apparently simpler relation uses the updated distribution $\widetilde{\Psi}$, which already incorporates the decision of firms to adjust or not their prices facing a inflation $\pi(\boldsymbol{s})$

1.4. Equilibrium

Aggregate state. In the recursive competitive equilibrium, what described the aggregate state $\boldsymbol{s}$ of the economy is the realization of aggregate shocks $\psi = \{Z, \epsilon\}$ and the distribution of last period production relative prices $\widetilde{\Psi}_{-1}$. To see why this state is sufficient, imagine there exists a map from it to inflation $\pi(\boldsymbol{s})$. Given this map, we can compute beginning of period - prior to adjustment decision - relative prices Ψ. Policy functions of the firm, which also depend on maps $C(\boldsymbol{s}), w(\boldsymbol{s})$ that come from household optimization and market clearing, imply then a mapping $\Psi \to \widetilde{\Psi}$. Note that (1.3) constrains the distribution $\widetilde{\Psi}$, which is what helps to pin down inflation $\pi(\boldsymbol{s})$ we started with[1]. It is my hope that the implementation will make clearer.

[1] Check the above box for a tentative explanation regarding this

Equilibrium *A recursive competitive equilibrium is a set of value functions* $\left\{v, V^A, V^N\right\}$, *policies* $\{\tilde{p}, \tilde{\xi}\}$ *for the firm, policies* $\left\{C(\boldsymbol{s}), N(\boldsymbol{s})\right\}$ *for the household, wage* $w(\boldsymbol{s})$, *interest rate* $R(\boldsymbol{s})$ *and inflation* $\pi(\boldsymbol{s})$, *and a law of motion* Γ *for the aggregate state* $\boldsymbol{s} \mapsto$

1. (Firm optimization)
   Taking $\{w(\boldsymbol{s}), C(\boldsymbol{s}), \pi(\boldsymbol{s})\}$ *as given the value function solves the Bellman equation in* (1.12)-(1.14) *and the* $\{\tilde{p}, \xi\}$ *are the associated policies*
2. (Household optimization)
$$R(\boldsymbol{s})\mathbb{E}\left\{\beta\left(\frac{C(\boldsymbol{s}')}{C(\boldsymbol{s}}\right)^{-\sigma}\frac{1}{\pi(\boldsymbol{s}')}\right\} = 1, \qquad N^{1/\varphi} = \frac{1}{\chi}C^{-\sigma}w(\cdot)$$
3. (Market clearing)
   - *Bonds market:* $B_t = 0$
   - *Labor market:*
$$\begin{aligned} N &= \int\int_0^{\bar{\xi}}\left[\ell\Big(a, \tilde{p}(a,x,\xi;\boldsymbol{s});\boldsymbol{s}\Big) + \mathbb{1}\Big\{\tilde{p}(a,x,\xi;\boldsymbol{s}) \neq x\Big\}\xi\right]dH(\xi)\ d\Psi \\ &= \int \ell\big(a,\tilde{p};\boldsymbol{s}\big)\widetilde{\Psi}\big(d[a\times\tilde{p}]\big) + \int_0^{\bar{\xi}}\xi\Bigg[\underbrace{\int \mathbb{1}\{\xi \le \tilde{\xi}(a,x;\boldsymbol{s})\}\Psi\big(d[a\times x]\big)}_{\Omega(\xi;\boldsymbol{s})}\Bigg]dH(\xi) \end{aligned} \tag{1.18}$$
     where $\ell(a,\tilde{p};\boldsymbol{s}) = \frac{\tilde{p}^{-\epsilon}Y}{Za}$.
   - *Goods market:*
$$C_t = Y_t = \left(\int_0^1 y(h)^{\frac{\epsilon-1}{\epsilon}}dh\right)^{\frac{\epsilon}{\epsilon-1}} \tag{1.19}$$
   - *The distribution over real prices should satisfy the condition*
$$1 = \left(\int \tilde{p}^{1-\epsilon}d\widetilde{\Psi}\right) \tag{1.20}$$
     which reflects the fact the real price level is one by definition.
4. (Law of motion Distribution)
5. (Law of motion for the aggregate shocks)

Using the functional form on the demand for labor note we can write the labor market clearing

$$\begin{aligned} N(\boldsymbol{s}) &= C(\boldsymbol{s})\int\frac{\tilde{p}^{-\epsilon}}{Za}\widetilde{\Psi}\big(d[a\times\tilde{p}]\big) + \int_0^{\bar{\xi}}\xi\cdot\Omega(\xi;\boldsymbol{s})dH(\xi) \\ &= C(\boldsymbol{s})\int\frac{\tilde{p}^{-\epsilon}}{Za}\widetilde{\Psi}\big(d[a\times\tilde{p}]\big) + \Omega(\boldsymbol{s}) \end{aligned} \tag{1.21}$$

2. Method

Check the notes on the Krusell-Smith model.

2.1. Finite Dimensional Approximation

Firm's Value Function

- Value functions are differently approximated in steady state and on the perturbation step. Here we discuss only the first, which relies on approximating the dependence on idiosyncratic states only[2]. Specifically, I approximate the value functions by

$$v(a,x;\theta) \approx \sum_{j=1}^{n_a}\sum_{i=1}^{n_x} \theta_{j,i} \cdot \psi_{j,i}(a,x) \tag{2.1}$$

where n_a, n_x denote the order of approximation for each dimension, $\{\psi_{i,j}\}$ the family of splines chosen and $\theta_{j,i}$ are the coefficients on these polynomials.

- With this particular approximation of the value function, we solve for the steady state coefficients using *collocation* which requires the functional equation for the value function to hold exactly on a set of grid points $\{a_j, x_i\}_{j=1,n_a\ i=1,n_x}$

$$\begin{aligned} v(a_j,x_i;\theta^*) = {} & H\Big(\tilde{\xi}(a_j,x_i)\Big)\left\{\Pi^R\Big(a_j,\tilde{p}^a(a_j)\Big) + \beta \sum_{a_{j'}\in\mathcal{A}} \Pi(a_j,a_{j'}) v\left(a_{j'}, \frac{\tilde{p}^a(a_j,x_i)}{\pi^*};\ \theta^*\right)\right\} + \\ & w^* \int_0^{\tilde{\xi}(a_j,x_i)} \xi dH(\xi) \quad + \\ & \Big(1 - H\Big(\tilde{\xi}(a_j,x_i)\Big)\Big)\left\{\Pi^R\Big(a_j,x_i\Big) + \beta \sum_{a_{j'}\in\mathcal{A}} \Pi(a_j,a_{j'}) v\Big(a_{j'}, \frac{x_i}{\pi^*};\ \theta^*\Big)\right\} \end{aligned}$$

where the decision rules must also satisfy the following at the grid points

$$\tilde{\xi}(a_j,x_i) = \frac{V^A(a_j;\theta^*) - V^N(a_j,x_i;\theta^*)}{w^*} \tag{2.2}$$

$$0 = Y^*\Big(\tilde{p}^a(a_j)\Big)^{-\epsilon} - \epsilon Y^*\Big(\tilde{p}^a(a_j)\Big)^{-\epsilon-1}\left(\tilde{p}^a(a_j) - \frac{w^*}{a_j}\right) + \beta\left[\sum_{a_{j'}\in\mathcal{A}} \Pi(a_j,a_{j'}) \frac{\partial v\big(a',\tilde{p}^a(a_j)/\pi^*;\theta^*\big)}{\partial x'}\frac{1}{\pi^*}\right] \tag{2.3}$$

- Careful: Note that the foc in the code is different since the value function there is defined in terms of log price

Distribution Different from usual applications, we have two relevant distributions here

- $\Psi(a,x)$: distribution over beginning-of-period (prior to the adjustment decision) relative prices
- $\widetilde{\Psi}(a,\tilde{p})$: distribution over effective real prices (production relevant) and idio shocks

[2] Check the code for details on the approximation at the perturbation step. In short, we don't perturb the coefficients θ^* at steady state but instead the steady state value $v(a_j,x_i;\theta)$ at every node.

The Ψ distribution transition dynamics involves 3 different steps

1. decision of price adjustment adjustment
2. exogenous transition
3. deflation by inflation between $t \mapsto t+1$

Details on the discretization of each distribution can be checked on Krusell-Smith model notes. Check the code to see how I do the actual implementation of the three different steps.

Finite Dimensional System

$$
\left\{
\begin{array}{c}
\beta \left(\frac{Y'}{Y}\right)^{-\sigma} \frac{1}{\pi'} R - 1 \\
N^{1/\varphi} - \frac{1}{\chi} Y^{-\sigma} w \\
\\
N - \int \left[H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right) \frac{\left(\tilde{p}^a(a)\right)^{-\epsilon} Y}{Za} + \left(1 - H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\right) \frac{x^{-\epsilon} Y}{Za} + \int_0^{\tilde{\boldsymbol{\xi}}_{j,i}} \xi dH(\xi) \right] d\Psi(\pi) \\
N - Y \int \frac{\tilde{p}^{-\epsilon}}{Za} d\tilde{\Psi}'(\pi) + \int \left[\int_0^{\tilde{\boldsymbol{\xi}}_{j,i}} \xi dH(\xi) \right] d\Psi(\pi) \\
1 - \int \tilde{p}^{1-\epsilon} d\tilde{\Psi}'(\pi) \\
\\
\text{Distribution Dynamics} \\
\\
z' - \rho_z z - \sigma_z \omega_z' \\
\epsilon' - \rho_\epsilon \epsilon - \sigma_\epsilon \omega_\epsilon' \\
\\
v(a_j, x_i; \boldsymbol{\theta}) - H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right) \left\{ \Pi^R\left(a_j, \boldsymbol{p}_j^a; \cdot\right) + \beta \left(\frac{Y'}{Y}\right)^{-\sigma} \sum_{j'} \Pi[a_j, a_{j'}] \left(\boldsymbol{V}', v^*\right) \left(a_{j'}, \frac{\boldsymbol{p}_j^a}{\pi'}; \boldsymbol{\theta}'\right) \right\} \\
+ w \int_0^{\tilde{\boldsymbol{\xi}}_{j,i}} \xi dH(\xi) - \left(1 - H\left(\tilde{\boldsymbol{\xi}}_{j,i}\right)\right) \left\{ \Pi^R\left(a, x_i; \cdot\right) + \beta \sum_{j'} \Pi(a_j, a_{j'}) v\left(a_{j'}, \frac{x_i}{\pi'}; \boldsymbol{\theta}'\right) \right\} \\
\frac{\partial \Pi^R}{\partial \tilde{p}} \left(a_j, \boldsymbol{p}_j^a; \cdot\right) + \beta \left(\frac{Y'}{Y}\right)^{-\sigma} \sum_{j'} \Pi[a_j, a_{j'}] \frac{\partial v}{\partial x'} \left(a_{j'}, \frac{\boldsymbol{p}_j^a}{\pi'}; \boldsymbol{\theta}'\right) \frac{1}{\pi'}
\end{array}
\right.
$$

With all these approximations, the recursive equilibrium becomes computable. It resumes to a system of nonlinear equations f that satisfies

$$
\mathbb{E}\left[f\left(\mathbf{y}', \mathbf{y}, \mathbf{x}', \mathbf{x}\right) \right] = 0 \tag{2.4}
$$

where $\mathbf{y} = \left(Y, N, R, \boldsymbol{\pi}, w, \boldsymbol{\theta}, \boldsymbol{p}^a\right)$ are the *control* variables[3], $\mathbf{x} = \left(\widetilde{\Psi}_-, \psi\right)$ are the (endogenous and exogenous) *state* variables. This puts the model in the canonical form presented in.

[3]Although we have introduced $\boldsymbol{\theta}$ as the variable on the system f, we actually treat the value function differently on the steady-state and linearization steps. To compute the stationary equilibrium, the description above is precise. Now, let θ^* be the coefficients on the steady-state value function of the firm. At the linearization step, instead of perturbing the coefficients θ we use as variable

2.2. Stationary Equilibrium

In the stationary equilibrium, aggregate shocks are zero and distribution of effective prices converges to an ergodic distribution $\widetilde{\Psi}$ so aggregate state is constant. In terms of the finite dimensional system presented above, the stationary equilibrium are values $\mathbf{x}^*, \mathbf{y}^*$ such that $f\left(\mathbf{y}^*, \mathbf{y}^*, \mathbf{x}^*, \mathbf{x}^*\right) = 0$

The following algorithm searchs for an stationary equilibrium by nesting the firm's problem inside an outer loop over steady-state real-wage w and production Y^*

1. Guess a pair w^*, Y^*
2. Given $\left(w^*, Y^*\right)$, compute the firm's value function.
3. Using the firm's decision rules, compute the invariant distribution.
4. Check the market-clearing and consistency conditions

$$
\begin{aligned}
1 &= \int \tilde{p}^{1-\epsilon} \Psi\left(da, d\tilde{p}\right) \\
N^* &= \int \ell\left(a, \tilde{p}\right) \Psi\left(d[a \times \tilde{p}]\right) + \int \left(\int^{\tilde{\xi}(a,x)} \zeta dH(\zeta) \right) \widetilde{\Psi}\left(d[a \times x]\right)
\end{aligned}
$$

$\{\boldsymbol{V}_{i,j}\}$ with steady-state values $\boldsymbol{V}^*_{i,j} = v(a_j, x_i; \theta^*)$. To evaluate points outside the grid, we use

$$
\left(\boldsymbol{V}, v^*\right)(a, x) = v^*(a, x; \theta^*) + \frac{x - x_i}{x_{i+1} - x_i} d\boldsymbol{V}_{i+1,j} + \frac{x_{i+1} - x}{x_{i+1} - x_i} d\boldsymbol{V}_{i,j}
$$

Check the codes for more details.

3. Linear Rational Expectational Difference Equations

Linearizing (2.4) around the steady-state yields a first-order linear expectational difference equation system of the form

$$f_{\mathbf{y}'} E_t\Big[\underbrace{(\mathbf{y}_{t+1} - \bar{\mathbf{y}})}_{\tilde{\mathbf{y}}_{t+1}}\Big] + f_{\mathbf{y}}\tilde{\mathbf{y}}_t + f_{\mathbf{x}'} E_t\Big[\tilde{\mathbf{x}}_{t+1}\Big] + f_{\mathbf{x}}\tilde{\mathbf{x}}_t = 0 \tag{3.1}$$

I am going to present two different methods to solve the linear rational expectation model (3.1). Section 3.2 shows the Klein's method which follows closely the Blanchard and Kahn approach, while section 3.3 discusses Sim's method.

The differences between the two methods lies in the how they pin down the one-step-ahead prediction error of the endogenous variables. As is made clear in Klein's paper

> In Sims (1996), the approach is as follows. First, the equation is written without expectational terms but with an endogenous prediction error process. The system is then transformed into a triangular one using the generalized Schur form, and the unstable block of equations is isolated. This block is then solved forward, and the endogenous prediction error process is solved for by imposing the informational restriction that the solution must be adapted to the given filtration. At this stage, no extraneous assumptions (e.g. about what variables are predetermined) are invoked; all information about the solution is given in the coefficient matrices of the difference equation itself.
>
> By contrast, the approach in this paper follows very closely the one used in Blanchard and Kahn (1980). No endogenous prediction error is introduced, and the unstable block of the triangular system is solved forward without having to solve for the prediction error separately. Instead, the endogenous prediction error process is solved for when solving the stable block of equations and use is then made of extraneous assumptions which generalize Blanchard and Kahn's assumption of certain variables being predetermined.

Before discussing the methods, let me just show how (3.1) can be cast into the canonical forms of both models

Klein's form. Let's start by putting (3.1) on Klein's notation. Let $A := \begin{bmatrix} f_{\mathbf{x}'} & f_{\mathbf{y}'} \end{bmatrix}$ and $B := -\begin{bmatrix} f_{\mathbf{x}} & f_{\mathbf{y}} \end{bmatrix}$, from what we get

$$A E_t\left\{\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{y}_{t+1} \end{bmatrix}\right\} = B \begin{bmatrix} \mathbf{x}_t \\ \mathbf{y}_t \end{bmatrix} \tag{3.2}$$

Sims's form. Let $X_t := (\tilde{\mathbf{x}}_t;\ \tilde{\mathbf{y}}_t)$. To put the model into Sims' format, we need to get rid of the expectational terms in

$$\Gamma_0 \mathbb{E}_t\Big[X_{t+1}\Big] = \Gamma_1 X_t$$

where Γ_0, Γ_1 are defined as before. To do so we substitute $E_t\big\{X_{t+1}\big\}$ by the combination of its *ex post* realizations plus the appropriate forecast errors.[4] Once we do that, we get to Sims' canonical form.

[4]For each $y_{i,t+1}$, whenever we have $E_t\{y_{i,t+1}\}$ we substitute to $y_{i,t+1} - \eta_{i,t+1}$, where $\eta_{i,t+1}$ are **endogenous** expectational errors determined as part of the solution. Endogenous states are all pre-determined, so $E_t\{x_{1,t+1}\} = x_{t+1}$, while exogenous states have exogenous forecast error ω_{t+1}, so we can write $E_t\{\mathbf{x}_{2,t+1}\} = \mathbf{x}_{2,t+1} - \omega_{t+1}$

Definition *Let $P \in \mathbb{C} \to \mathbb{C}^{n\times n}$ be a matrix-valued function of a complex variable (a matrix pencil). Then the set of its generalized eigenvalues $\lambda(P)$ is defined as*

$$P(z) := \left\{ z \in \mathbb{C} : |P(z)| = \mathbf{0} \right\}$$

When $P(z)$ writes as $Az - B$, we denote this set as $\lambda(A, B)$. Then there exists a vector V such that $BV = \lambda AV$.

Theorem (*The complex generalized Schur form*). Let A and B belong to $\mathbb{C}^{n\times n}$ and be such that $P(z) = Az-B$ is a regular matrix pencil. Then there exist **unitary** (orthogonal) $n \times n$ matrices of complex numbers Q and Z such that

1. $S = QAZ$ is upper-triangular
2. $T = QBZ$ is upper-triangular
3. For each i, s_{ii}, t_{ii} are not both zero
4. $\lambda(A, B) = \left\{ \frac{t_{ii}}{s_i i} : \ s_{ii} \neq 0 \right\}$
5. The pairs (s_{ii}, t_{ii}) can be arranged in any order

Note that the set $\lambda(A, B)$ may have fewer than n elements, since if A is singular, we may have $s_{ii} = 0$ for some i. The missing generalized eigenvalues will be called infinite. These, together with the finite generalized eigenvalues with $\{\lambda_i\} > 1$ will be denominated *unstable*.

Theorem (Singular Value Decomposition). Every $m \times n$ complex matrix A can be factored into a product of three matrices

$$A = USV' \tag{3.3}$$

called a *singular value decomposition (SVD)* where

- $U_{m\times m}, V_{n\times n}$ are orthogonal matrices - $UU' = VV' = I$
- S is a diagonal matrix having the first r diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ and all other entries zero.

The values σ_i are called the **singular values** of A, while the column vectors u_j, v_i are the left/right **singular vectors**. Note that the SVD allows us to write the matrix A as

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

where the vectors $u_1,\ i = 1, \ldots, r$ (resp. $v_1,\ i = 1, \ldots, r$) are mutually orthogonal.

It is convenient at this point to state some properties of the SVD that will be useful for the applications later on.

Claim.

1. The rank of A is the cardinality of the nonzero singular values
2. An orthonormal basis spanning $\mathcal{N}(A)$ is given by the last $n-r$ columns of V
3. An orthonormal basis spanning the range of A is given by the first r columns of U, i.e.,

$$\mathcal{R}(A) = \mathcal{R}\big(\{u_1, \ldots, u_r\}\big)$$

3.2. Klein

Consider the generalized Schur decomposition of A, B

$$QAZ = S \qquad QBZ = T \tag{3.4}$$

where A, B are upper triangular and Q, Z are orthonormal matrices. Let S and T be arranged in such a way that the n_s stable eigenvalue come first. Partition the rows of Z conformably with the classification of eigenvalues and the columns with the sizes of $\mathbf{x}, \mathbf{y}$

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

At this point, let's assume that

Assumption. — Z_{11} is square and invertible.

Note that the assumption requires $n_s - n_x$. This means that there should be as many state variables - variables with exogenously given initial values and prediction error - as there are stable eigenvalues. Now, in order to find a solution define the auxiliary variables w_t as

$$w_t := Z^H \big[\mathbf{x}_t' \;\; \mathbf{y}_t'\big]' = \begin{bmatrix} s_t \\ u_t \end{bmatrix} \tag{3.5}$$

where the transformed variable w_t is divided into $n_s \times 1$ stable and $n_u \times 1$ unstable components. Premultiply the system by Q to get

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \left\{ \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} \right\} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} \tag{3.6}$$

We start by solving for u_t. Since the generalized eigenvalues of $\big(S_{22}, T_{22}\big)$ are all unstable, the solution is found by solving forward in time

$$u_t = \lim_{s \to \infty} \big(T_{22}^{-1} S_{22}\big)^s E_t\{u_{t+1}\} = 0 \tag{3.7}$$

where last inequality comes from the fact we are interested in stable equilibrium paths u_t. Using this on the first block we get

$$E_t\big\{s_{t+1}\big\} = S_{11}^{-1} T_{11} s_t \tag{3.8}$$

Recalling the definition of s_t

$$\mathbf{x}_{t+1} = \begin{bmatrix} Z_{11} & Z_{12} \end{bmatrix} \begin{bmatrix} s_{t+1} \\ u_t \end{bmatrix}$$

and our definition of $\mathbf{x}_{t+1}$ as pre-determined in the sense of Klein

$$Z_{11}\Big(s_{t+1} - E_t\{s_{t+1}\}\Big) = \eta\epsilon_{t+1} \tag{3.9}$$

Taken together, (3.8) and (3.9) define the unique solution for s_t given the exogenous process ϵ

$$s_{t+1} = S_{11}^{-1}T_{11}s_t + Z_{11}^{-1}\eta\epsilon_{t+1} \tag{3.10}$$

with $s_0 = Z_{11}^{-1}\mathbf{k}_0$. We can now go back to the relation $Z^H\begin{bmatrix}\mathbf{x}_t' & \mathbf{y}_t'\end{bmatrix}' =: w_t$ to find a recursive solution only in terms of the variables of interest. Using the restriction on the control (3.7)

$$\begin{aligned} Z_{21}^H\mathbf{x}_t + Z_{22}^H\mathbf{y}_t = 0 \Rightarrow \mathbf{y}_t &= -\big(Z_{22}^H\big)^{-1}Z_{21}^H\mathbf{x}_t \\ &= \big(Z_{22}^H\big)^{-1}Z_{22}^H Z_{21}Z_{11}^{-1}\mathbf{x}_t \\ &= Z_{21}Z_{11}^{-1}\mathbf{x}_t \end{aligned}$$

As for state variables $\mathbf{x}_t$, note that

$$\begin{aligned} s_t &= Z_{11}^H\mathbf{x}_t + Z_{12}^H\mathbf{y}_t \\ &= \Big(Z_{11}^H + Z_{12}^H Z_{21}Z_{11}^{-1}\Big)\mathbf{x}_t \\ &= \Big(Z_{11}^H + \Big(Z_{11}^{-1} - Z_{11}^H\Big)\Big)\mathbf{x}_t \\ &= Z_{11}^{-1}\mathbf{x}_t \end{aligned}$$

which together with (3.10) implies

$$\mathbf{x}_{t+1} = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}\mathbf{x}_t + \eta\epsilon_{t+1}$$

Putting it together we have

$$\begin{cases} \mathbf{x}_{t+1} = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}\mathbf{x}_t + \eta\epsilon_{t+1}, & \mathbf{x}_0 \text{ given} \\ \mathbf{y}_t = Z_{21}Z_{11}^{-1}\mathbf{x}_t \end{cases} \tag{3.11}$$

3.3. Sims

We concentrate in solving model that can be cast into the following form

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi\epsilon_t + \Pi\eta_t \tag{3.12}$$

where y_t is a n vector of endogenous variables, Γ_0, Γ_1 are $n \times n$ coefficients matrices , ϵ_t is a $\ell \times 1$ vector of *exogenous* random disturbances, η_t is an $k \times 1$ vector of *endogenous* expectational errors satisfying $E_t\big[\eta_{t+1}\big] = 0$. Note that η_t terms are not given exogenously, instead they are determined as part of the solution. This method uses the notation that in which time arguments relate consistently to the information structure, meaning that variables date t are always know at date t.

3.3.1. Special Case

Consider the special case of when the Γ_0 is non-singular. Hence, we can represent (3.12) as

$$y_t = \Gamma y_{t-1} + \Psi \epsilon_t + \Pi \eta_t$$

Assume that the matrix Γ can be diagonalized to

$$\Gamma = P \Lambda P^{-1}$$

where P is the matrix of right-eigenvectors of Γ, P^{-1} is the matrix of left-eigenvectors and Λ is the diagonal matrix of eigenvalues. Multiplying the system by P^{-1} and defining $w := P^{-1}y$ we arrive at

$$w_t = \Lambda w_{t-1} + P^{-1}\Big(\Psi \epsilon_t + \Pi \eta_t\Big) \tag{3.13}$$

Since Γ is diagonal the system breaks into unrelated components

$$w_{j,t} = \lambda_j w_{j,t-1} + \tilde{P}_{j\cdot}\Big(\Psi \epsilon_t + \Pi \eta_t\Big) \tag{3.14}$$

If the disturbance term, including the combined effects of ϵ and η, is zero and $\lambda_j \neq 1$, the model has a deterministic steady-state solution

$$w_{j,t} = 0 \tag{3.15}$$

Moreover, for any $|\lambda_j| > 1$, then $E_t\big[w_{j,t+\tau}\big]$ diverges as $\tau \to \infty$ for any solution other then $w_{j,t} = 0$. If we are looking for a *stationary equilibrium*, every one of the variables w_j corresponding to $|\lambda_j| > 1$ and to $P_{j,:} \neq 0$ must be set to its steady-state value. If we impose (3.15) for all t in (3.14) we get

$$\tilde{P}_{j\cdot}\Big(\Psi \epsilon_t + \Pi \eta_t\Big) = 0 \tag{3.16}$$

Collecting all the rows of P^{-1} for which (3.16) holds into a single matrix $\tilde{P}^U$, we can write

$$\tilde{P}^U\Big(\Psi \epsilon + \Pi \eta\Big) = 0 \tag{3.17}$$

Existence problems arise if the endogenous shocks η cannot adjust to offset the exogenous disturbances ϵ. This accounts for the usual notion that there are existence problems if the number of *unstable roots* exceeds the number of *jump variables*. The precise condition here is that the columns of $\tilde{P}^U\Pi$ span the space spanned by the columns of $\tilde{P}^U\Psi$, i.e.

$$\operatorname{span}\Big(\tilde{P}^U\Psi\Big) \subset \operatorname{span}\Big(\tilde{P}^U\Pi\Big) \Leftrightarrow \tilde{P}^U\Psi = \tilde{P}^U\Pi \underbrace{\lambda}_{k\times\ell} \tag{3.18}$$

From (3.17), we have an expression that will determine $\tilde{P}^U\Pi\eta$ from information of the stochastic process ϵ_t. However, multiple solutions may exist when (3.18) puts too few restrictions. For the solution to be unique, it must be that (3.17) pins down not only the value of $\tilde{P}^U \Pi \eta$ but also $\tilde{P}^S \Pi \eta$, which resumes the impact of expectational shocks on the stable block of the system (3.13). Formally, we require the row space of $\tilde{P}^S \Pi$ to be included into the row space of $\tilde{P}^U \Pi$. In that case, there exists Φ such that

$$\tilde{P}^S \Pi = \Phi \tilde{P}^U \Pi \tag{3.19}$$

If this is the case, we can write the solution by assembling the equations representing the stability conditions (3.15) together with the lines of (3.13) that determine w for the stable block and use (3.19) to eliminate the dependence over η

$$\begin{bmatrix} w_t^S \\ w_t^U \end{bmatrix} = \begin{bmatrix} \Lambda_S \\ \mathbf{0} \end{bmatrix} w_{t-1}^S \begin{bmatrix} I & -\Phi \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \tilde{P}^S \\ \tilde{P}^U \end{bmatrix} \Psi \epsilon_t \tag{3.20}$$

Multiply by P to go back to y

$$y_t = \underbrace{P_{:,S} \Lambda_S \tilde{P}^S}_{\Theta_y} y_{t-1} + \underbrace{\Big(P_{:,S} \tilde{P}^S - P_{:,S} \Phi \tilde{P}^U \Big) \Psi \, \epsilon_t}_{\Theta_\epsilon} \tag{3.21}$$

3.3.2. General Case

First, we compute the *Generalized Schur decomposition* to find matrices Q, Z, T and S such that

$$Q' S Z' = \Gamma_0, \qquad Q' T Z' = \Gamma_1 \tag{3.22}$$

$$Q Q' = Z Z' = I_{n \times n} \tag{3.23}$$

Let us define $w_t = Z' y_t$ and pre-multiply the system by Q in order to get

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Big(\Psi \epsilon_t + \Pi \eta_t \Big) \tag{3.24}$$

Let's focus on the explosive part of the system

$$S_{22} w_{2,t} = T_{22} w_{2,t-1} + Q_2 \Big(\Psi \epsilon_t + \Pi \eta_t \Big)$$

While the diagonal elements of S_{22} can be null, T_{22} is necessarily full rank. Therefore, we can solve forward for $w_{2,t-1}$. Start by leading the equation by one period and writing it in terms of $w_{2,t}$

$$w_{2,t} = M z_{2,t+1} - T_{22}^{-1} Q_2 \Big(\Psi \epsilon_{t+1} + \Pi \eta_{t+1} \Big)$$

where $M := T_{22}^{-1} S_{22}$. Recursive substitution of $w_{2,t+1}$ leads us to

$$w_{2,t} = - \sum_{i=1}^{\infty} M^{i-1} T_{22}^{-1} Q_2 \Big(\Psi \epsilon_{t+i} + \Pi \eta_{t+i} \Big)$$

where we imposed $\lim M^t w_{2,t} = 0$ since we are searching for a non-explosive solution of the LRE model (3.12). Since y_t is known at time t, $w_{2,t} = E_t\{w_{2,t}\}$ which implies

$$w_{2,t} = -\sum_{i=1}^{\infty} M^{i-1} T_{22}^{-1} Q_2 \Big(\Psi E_t\big\{\epsilon_{t+i}\big\} + \Pi E_t\big\{\eta_{t+i}\big\} \Big) = 0$$

which imposes a restriction on paths for ϵ_t and η_t. If we go back to (3.24) and take into account $W_{2,t} = 0$ in the second block, this imposes

$$\underbrace{Q_2\Psi}_{n_u \times \ell} \underbrace{\epsilon_t}_{\ell \times 1} \quad + \quad \underbrace{Q_2\Pi}_{n_u \times k} \underbrace{\eta_t}_{k \times 1} = 0 \tag{3.25}$$

Note that the assertion in (3.25) is only possible because we have the degree of freedom to choose η, otherwise it requires that exogenously evolving events always satisfy a deterministic equation. As before, existence problems arise if the endogenous shocks η cannot adjust to offset the exogenous shocks ϵ in (3.25). Sufficient conditions for the existence of a unique stable solution are given below

Assumption. —

1. The columns space of $Q_2\Psi$ is contained in that of $Q_2\Pi$.
2. There exists an $n_s \times n_u$ matrix Φ such that

$$Q_1\Pi = \Phi\big(Q_2\Pi\big)$$

How do we practically apply/check the conditions of the above assumption? Since the rows of the matrix $Q_2\Pi$ are potentially linearly dependent it is convenient to work with its Singular Value Decomposition discussed in the preliminaries

$$Q_2\Pi = \underbrace{U}_{n_u \times n_u} \underbrace{S}_{n_u \times k} \underbrace{V}_{k \times k} = \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V'_{1.} \\ V'_{2.} \end{bmatrix} = \underbrace{U_{.1}}_{n_u \times r} \cdot \underbrace{S_{11}}_{r \times r} \cdot \underbrace{V'_{1.}}_{r \times k}$$

where S_{11} is a diagonal and U, V are orthonormal matrices. From previous claim, we can check whether $Q_2\Pi$'s column space includes $Q_2\Psi$'s by checking $\big(I - U_1 U_1'\big)T = 0$ where T comes from the SVD decomposition of $Q_2\Psi = TRW'$. If this holds, then it is easy to check

$$Q_2\Psi = Q_2\Pi \underbrace{\Big(\big(V_1 S_{11}^{-1} U_1'\big) \cdot Q_2\Psi \Big)}_{\lambda}$$

is satisfied. Hence, we can rewrite (3.25)

$$U_1 S_{11} \underbrace{\big(V_1' \lambda \epsilon_t + V_1' \eta_t\big)}_{r \times 1} = \underbrace{0}_{n_u \times 1} \tag{3.26}$$

We therefore now have r restrictions to identify the k-dimensional vector of expectation errors.

References

James Costain and Anton Nakov. Distributional dynamics under smoothly state-dependent pricing. *Journal of Monetary Economics*, 58(6):646–665, 2011.

Mikhail Golosov and Robert Lucas. Menu costs and phillips curves. *Journal of Political Economy*, 115(2): 171–199, 2007.

Per Krusell, Anthony A. Smith, and Jr. Income and Wealth Heterogeneity in the Macroeconomy. *Journal of Political Economy*, 106(5):867–896, October 1998. URL https://ideas.repec.org/a/ucp/jpolec/v106y1998i5p867-896.html.

Virgiliu Midrigan. Menu costs, multiproduct firms, and aggregate fluctuations. *Econometrica*, 79(4):1139–1180, 2011. ISSN 1468-0262. doi: 10.3982/ECTA6735. URL http://dx.doi.org/10.3982/ECTA6735.

Joseph S. Vavra. Inflation Dynamics and Time-Varying Volatility: New Evidence and an Ss Interpretation. NBER Working Papers 19148, National Bureau of Economic Research, Inc, June 2013. URL https://ideas.repec.org/p/nbr/nberwo/19148.html.

A. Perturbing the value Function

function in Reiter: `BellmanIterGivenPolicy1dd`

We adopt the following approximation to perturb the value function

$$\operatorname{itp}\Big(\boldsymbol{V}, v^*\Big)(z_j, \tilde{x}) = v^*(z_j, \boldsymbol{x}) + \frac{\tilde{x} - \bar{x}_i}{\bar{x}_{i+1} - \bar{x}_i}\Big(\boldsymbol{V}_{j,i+1} - v^*(z_j, \bar{x}_{i+1})\Big) + \frac{\bar{x}_{i+1} - \tilde{x}}{\bar{x}_{i+1} - \bar{x}_i}\Big(\boldsymbol{V}_{j,i} - v^*(z_j, \bar{x}_i)\Big) \tag{A.1}$$

In steady state, $\boldsymbol{V}_t[j, i] = v^*(z_j, x_i)$ so that in steady state the whole expression reduces to simply $v^*(a, \tilde{x})$.