Simplified version of model in "The role of automatic stabilizers in the US business cycle."

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1 Overview

This document describes a simplified version of the model in McKay and Reis (2014). The main simplifications are that it omits most of the taxes and transfers and employment risk. The next section describes the model and derives the model equations used to solve it. Section 3 describes the method to solve the model.
The codes to solve the model are available at: `https://bitbucket.org/amckay/simplemckayreis`

2 Decision problems and model equations

The model described here is the model from McKay and Reis (2014) without taxes and a few other simplifications.

We assume that the economy is populated by two groups of households. The first group is relatively more patient and has access to a complete set of insurance markets in which they can insure all idiosyncratic risks. We can then talk of a representative patient household.

2.1 Patient household's problem

The representative patient household chooses $\{c_t, n_t\}$ to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \psi_1 \frac{n_t^{1+\psi_2}}{1+\psi_2}, \right] \tag{1}$$

subject to

$$p_t\left[c_t+k_{t+1}\right]+b_{t+1}-b_t=p_t\left[(i_{t-1}/p_t)b_t+d_t+(1-\tau_t)w_t\bar{s}n_t+(1+r_t)k_t-\frac{\zeta}{2}\left(\frac{\Delta k_{t+1}}{k_t}\right)^2k_t\right], \tag{2}$$

where k is capital, b is nominal bond holdings, d is a dividend from owning the intermediate goods firms, w is the aggregate wage, $\bar{s}$ is the skill of the patient household, and ζ is a parameter that controls the strength of a quadratic capital adjustment cost. τ_t is a proportional labor income tax.

Define x_{t+1} as real end-of-period savings in period t and $\pi_t = p_t/p_{t-1}$. Then we can rewrite the constraints as:

$$c_t+k_{t+1}+x_{t+1}=\frac{1+i_{t-1}}{\pi_t}x_t+d_t+(1-\tau_t)w_t\bar{s}n_t+(1+r_t)k_t-\frac{\zeta}{2}\left(\frac{\Delta k_{t+1}}{k_t}\right)^2k_t \tag{3}$$

Setting up the Lagrangian, with m_t as the Lagrange multiplier on constraint (3) the optimality conditions are:

$$\begin{aligned}
\beta^t c_t^{-1} &= m_t\\
m_t &= E_t\left[m_{t+1}\frac{1+i_t}{\pi_{t+1}}\right]\\
\beta^t\psi_1 n_t^{\psi_2} &= m_t(1-\tau_t)w_t\bar{s}\\
m_t\left(1+\zeta\frac{\Delta k_{t+1}}{k_t}\right) &= E_t\left[\left(1+r_{t+1}+\zeta\left(\frac{\Delta k_{t+2}}{k_{t+1}}\right)\frac{k_{t+2}}{k_{t+1}}-\frac{\zeta}{2}\left(\frac{\Delta k_{t+2}}{k_{t+1}}\right)^2\right)m_{t+1}\right].
\end{aligned}$$

These can be rearranged to give:

$$\psi_1 n_t^{\psi_2} = \left(\frac{1}{c_t}\right)(1-\tau_t)w_t\bar{s}, \tag{4}$$

$$\frac{1}{c_t} = \beta E_t\left\{\frac{1+i_t}{c_{t+1}\pi_{t+1}}\right\}, \tag{5}$$

$$\frac{1}{c_t}\left(1+\zeta\frac{\Delta k_{t+1}}{k_t}\right) = \beta E_t\left[\left(1+r_{t+1}+\zeta\left(\frac{\Delta k_{t+2}}{k_{t+1}}\right)\frac{k_{t+2}}{k_{t+1}}-\frac{\zeta}{2}\left(\frac{\Delta k_{t+2}}{k_{t+1}}\right)^2\right)\frac{1}{c_{t+1}}\right]. \tag{6}$$

Finally, notice that the patient household's stochastic discount factor is:

$$\lambda_{t,s} = \frac{\beta^s c_{t+s}^{-1}}{c_t^{-1}}. \tag{7}$$

2.2 Impatient households' problem

There is a measure ν of impatient households indexed by $h \in [0, \nu]$, so that an individual variable, say consumption, will be denoted by $c_t(h)$. They have the same period felicity function as patient households, but they are more impatient: $\hat{\beta} \leq \beta$. We assume that the impatient households do not own shares in the firms or own the capital stock. However, their savings can be used to finance capital accumulation by lending to the patient households through the bond market.

Individual impatient households choose consumption, hours of work, and bond holdings $\{c_t(h), n_t(h), b_{t+1}(h)\}$ to maximize:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \hat{\beta}^t \left[\log c_t(h) - \psi_1 \frac{n_t(h)^{1+\psi_2}}{1+\psi_2} \right]. \tag{8}$$

Also like patient households, households can save using risk-free nominal bonds so their budget constraint is:

$$\hat{p}_t c_t(h) + b_{t+1}(h) = (1 + i_{t-1}) b_t(h) + p_t \left[(1 - \tau_t) w_t s_t(h) n_t(h) \right], \tag{9}$$

together with a borrowing constraint, $b_{t+1}(h) \geq 0$. Unlike patient households, impatient households face uninsurable idiosyncratic risk to their skill, $s_t(h)$. We assume that s follows a Markov chain where $\omega_{ss'}$ is the probability of moving from state s to s'.

The idiosyncratic state of a household is its real bond holdings $\tilde{b}$ and its skill level s. Let $\mathbf{S}$ be the set of all values of $s(h)$ and let $\mathcal{S}$ be the collection of aggregate state variables. Then the problem of a household with real assets $\tilde{b}$ and skill s can be written as

$$V(\tilde{b}, s, \mathcal{S}) = \max_{c,n} \left\{ \log(c) - \psi_1 \frac{n^{1+\psi_2}}{1+\psi_2} + \hat{\beta} \mathbb{E}_t \sum_{s' \in \mathbf{S}} \omega_{ss'} V(x/\pi(\mathcal{S}'), s', \mathcal{S}') \right\}$$

subject to

$$c + x = (1 + i(\mathcal{S}_{-1})) \, \tilde{b} + w(\mathcal{S}) sn$$

where x is end-of-period savings, $i(\mathcal{S}_{-1})$ refers to the interest rate determined in the previous period. Here the expectation operator is over aggregate states.

From this problem, one can derive an Euler equation and a labor supply condition that are analogous to those for the patient household's problem. One difference, however, is that in these analogous expressions the expectation operator reflects an expectation over idiosyncratic uncertainty as well as over aggregate uncertainty.

2.3 Final goods' producers

A competitive sector for final goods combines intermediate goods according to the production function:

$$y_t = \left(\int_0^1 y_t(j)^{1/\mu_t} dj \right)^{\mu_t}, \tag{10}$$

where $y_t(j)$ is the input of the j^{th} intermediate input. There are shocks to the elasticity of substitution across intermediates that generate exogenous movements in desired markups, $\mu_t > 1$.

The representative firm in this sector takes as given the final-goods price p_t, and pays $p_t(j)$ for each of its inputs. Cost minimization together with zero profits imply that:

$$y_t(j) = \left(\frac{p_t(j)}{p_t} \right)^{\mu_t/(1-\mu_t)} y_t, \tag{11}$$

$$p_t = \left(\int_0^1 p_t(j)^{1/(1-\mu_t)} dj \right)^{1-\mu_t}. \tag{12}$$

2.4 Intermediate goods and corporate income taxes

There is a unit continuum of intermediate-goods monopolistic firms, each producing variety j using a production function:

$$y_t(j) = a_t k_t(j)^{\alpha} \ell_t(j)^{1-\alpha}, \tag{13}$$

where a_t is productivity, $k_t(j)$ is capital used, and $\ell_t(j)$ is effective labor.

The labor market clearing condition is

$$\int_0^1 \ell_t(j) dj = \int_0^{\nu} s_t(h) n_t(h) dh + \bar{s} n_t. \tag{14}$$

The demand for labor on the left-hand side comes from the intermediate firms. The supply on the right-hand side comes from employed households, adjusted for their productivity.

The firm maximizes

$$d_t(j) \equiv \frac{p_t(j)}{p_t} y_t(j) - w_t \ell_t(j) - (r_t + \delta) k_t(j) - \xi, \tag{15}$$

taking into account the demand function in equation (11). The firm's costs are the wage bill to workers, the rental of capital at rate r_t plus depreciation of a share δ of the capital used, and a fixed cost ξ.

Intermediate firms set prices subject to nominal rigidities a la Calvo (1983) with probability of price revision θ. Since they are owned by the patient households, they use their stochastic discount factor, $\lambda_{t,t+s}$, to choose price $p_t(j)^*$ at a revision date with the aim of maximizing expected future profits:

$$\mathbb{E}_t \left[\theta \sum_{s=0}^{\infty} (1-\theta)^s \lambda_{t,t+s} d_{t+s}(j) \right] \quad \text{subject to:} \quad p_{t+s}(j) = p_t(j)^*. \tag{16}$$

A firm that sets its price at date t chooses $p_t^*, \{y_s(j), k_s(j), l_s(j)\}_{s=t}^{\infty}$ to solve

$$\max E_t \sum_{s=t}^{\infty} \lambda_{t,s} (1-\theta)^{s-t} \left\{ \frac{p_t^*}{p_s} y_s(j) - w_s l_s(j) - (r_s + \delta) k_s(j) - \xi \right\},$$

subject to

$$\begin{aligned} y_s(j) &= \left(\frac{p_t^*}{p_s} \right)^{\mu/(1-\mu)} y_s \\ y_s(j) &= a_s k_s(j)^{\alpha} l(j)^{1-\alpha}. \end{aligned}$$

where the first constraint is the demand for the firm's good and the second its production function. Substituting in the demand curve gives the modified problem:

$$\max_{p_t^*, \{k_s(j), l_s(j)\}_{s=t}^{\infty}} E_t \sum_{s=t}^{\infty} \left[\left(\frac{p_t^*}{p_s} \right)^{1/(1-\mu)} y_s - w_s l_s(j) - (r_s + \delta) k_s(j) - \xi \right] \lambda_{t,s} (1-\theta)^{s-t}$$

subject to

$$\left(\frac{p_t^*}{p_s} \right)^{\mu/(1-\mu)} y_s = a_s k_s(j)^{\alpha} l(j)^{1-\alpha}.$$

The first order conditions with respect to $k_s(j)$ and $l_s(j)$ are:

$$(r_s + \delta) = M_s \alpha a_s k_s(j)^{\alpha-1} l_s(j)^{1-\alpha}. \tag{17}$$

$$w_s = M_s (1-\alpha) a_s k_s(j)^{\alpha} l_s(j)^{-\alpha}, \tag{18}$$

where M_s is the Lagrange multiplier on the production function constraint at date s, which is real marginal cost at date s.

We can derive several useful features of the solution from these two optimality conditions. First, taking their ratio:

$$\frac{w_s}{r_s + \delta} = \frac{1-\alpha}{\alpha} \frac{k_s(j)}{l_s(j)},$$

so that all firms have the same capital-labor ratio and, by market clearing, $k_s(j)/l_s(j) = k_s/l_s$ for all firms.

Second, these optimality conditions allow us already to derive the expression for dividends as a function of factor prices. Total factor payments are

$$(r_s + \delta) k_s = M_s \alpha a_s k_s^{\alpha} l_s^{1-\alpha}, \tag{19}$$

$$w_s l_s = M_s (1-\alpha) a_s k_s^{\alpha} l_s^{1-\alpha}. \tag{20}$$

The aggregate dividend of the intermediate goods firms is then

$$d_t \equiv= \int_0^1 d_t^i(j) dj = \int_0^1 \left[\frac{p_t(j)}{p_t} y_t(j) - w_t(j) l_t(j) - (r_t + \delta) k_t(j) - \xi \right] dj$$

and by market clearing this becomes

$$d_t = y_t - M_t a_t k_t^{\alpha} l_t^{1-\alpha} - \xi. \tag{21}$$

Finally, we turn to the optimality condition with respect to p_t^*:

$$\mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s} (1-\theta)^{s-t} \left[\frac{1}{1-\mu} \left(\frac{p_t^*}{p_s} \right)^{1/(1-\mu)-1} \frac{y_s}{p_s} - M_s \frac{\mu}{1-\mu} \left(\frac{p_t^*}{p_s} \right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s} \right] = 0,$$

which we can rewrite as

$$\mathbb{E}_t \sum_{s=t}^{\infty} \frac{1}{1-\mu} \left(\frac{p_t^*}{p_s} \right)^{1/(1-\mu)-1} \frac{y_s}{p_s} \lambda_{t,s} (1-\theta)^{s-t} = \mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s} (1-\theta)^{s-t} M_s \frac{\mu}{1-\mu} \left(\frac{p_t^*}{p_s} \right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s}$$

$$\frac{p_t^*}{p_t} = \frac{p_t \mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s} \left(1-\theta\right)^{s-t} M_s \mu_t \left(\frac{p_t}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s}}{p_t \mathbb{E}_t \sum_{s=t}^{\infty} \left(\frac{p_t}{p_s}\right)^{\mu/(1-\mu)} \frac{y_s}{p_s} \lambda_{t,s} \left(1-\theta\right)^{s-t}} \equiv \frac{\bar{p}_t^A}{\bar{p}_t^B}. \tag{22}$$

This equation gives the solution for p_t^*. It is useful to write $\bar{p}_t^A$ and $\bar{p}_t^B$ recursively. To that end,

$$\begin{aligned}
\bar{p}_t^A &= p_t \mathbb{E}_t \sum_{s=t}^{\infty} \lambda_{t,s} \left(1-\theta\right)^{s-t} M_s \mu_t \left(\frac{p_t}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s} \\
&= M_t \mu_t y_t + \\
&\mathbb{E}_t p_{t+1} \pi_{t+1}^{-1} \mathbb{E}_{t+1} \lambda_{t,t+1} \left(1-\theta\right) \left(\frac{p_t}{p_{t+1}}\right)^{\mu/(1-\mu)-1} \\
&\times \sum_{s=t+1}^{\infty} \lambda_{t+1,s} \left(1-\theta\right)^{s-t-1} M_s \mu \left(\frac{p_{t+1}}{p_s}\right)^{\mu/(1-\mu)-1} \frac{y_s}{p_s} \\
&= M_t \mu_t y_t + \mathbb{E}_t \left[\lambda_{t,t+1} \left(1-\theta\right) \pi_{t+1}^{-\mu/(1-\mu)} \bar{p}_{t+1}^A\right],
\end{aligned} \tag{23}$$

where $\pi_{t+1} \equiv p_{t+1}/p_t$. Similar logic for $\bar{p}_t^B$ yields

$$\bar{p}_t^B = y_t + \mathbb{E}_t \left[\lambda_{t,t+1} \left(1-\theta\right) \pi_{t+1}^{-\mu/(1-\mu)-1} \bar{p}_{t+1}^B\right]. \tag{24}$$

Next, comes the relationship between p_t^* and inflation. The price index is

$$p_t = \left(\int_0^1 p_t(j)^{1/(1-\mu)} dj\right)^{1-\mu}$$

and with Calvo pricing we have

$$\begin{aligned}
p_t &= \left(\left(1-\theta\right) \int_0^1 \left(p_{t-1}(j)\right)^{1/(1-\mu)} dj + \theta \left(p_t^*\right)^{1/(1-\mu)}\right)^{1-\mu} \\
&= \left(\left(1-\theta\right) p_{t-1}^{1/(1-\mu)} + \theta \left(p_t^*\right)^{1/(1-\mu)}\right)^{1-\mu}.
\end{aligned}$$

Therefore

$$\pi_t = \left(\frac{1-\theta}{1-\theta\left(\frac{p_t^*}{p_t}\right)^{1/(1-\mu)}}\right)^{1-\mu}. \tag{25}$$

Finally, note that because the capital-labor ratio is constant across firms, the production of variety j follows:

$$y_t(j) = a_t \left(\frac{k_t}{l_t}\right)^{\alpha} l_t(j).$$

The demand for variety j can be written in terms of the relative price to arrive at

$$\left(\frac{p_t(j)}{p_t}\right)^{\mu/(1-\mu)} y_t = a_t \left(\frac{k_t}{\ell_t}\right)^{\alpha} \ell_t(j).$$

Integrating both sides yields

$$\int_0^1 \left(\frac{p_t(j)}{p_t}\right)^{\mu/(1-\mu)} dj y_t = a_t \left(\frac{k_t}{\ell_t}\right)^{\alpha} \int_0^1 \ell_t(j) dj.$$

By market clearing we have then that:

$$S_t y_t = a_t k_t^{\alpha} \ell_t^{1-\alpha}, \tag{26}$$

where

$$S_t = \int_0^1 \left(\frac{p_t(j)}{p_t}\right)^{\mu/(1-\mu)} dj.$$

S_t reflects the efficiency loss due to price dispersion and it evolves according to

$$S_t = (1-\theta) S_{t-1} \pi_t^{-\mu/(1-\mu)} + \theta \left(\frac{p_t^*}{p_t}\right)^{\mu/(1-\mu)}. \tag{27}$$

Throughout this subsection, we have dropped most of the t subscripts on μ_t. When the equations in this subsection are linearized around the zero-inflation steady state, the markup shock only enters equation (23).

2.5 Shocks, business cycles, and government

Monetary policy follows a simple Taylor rule:

$$i_t = \bar{i} + \phi \Delta \log(p_t) - \varepsilon_t, \tag{28}$$

with $\phi > 1$.

Each period the government issues a fixed real amount of debt, B. The market for debt clears if

$$B = \int_0^\nu x_t(h) dh + x_t. \tag{29}$$

The government levies a proportional labor-income tax to finance interest payments on this debt. The government budget constraint is

$$\frac{1 + i_{t-1}}{\pi_t} B = B + \tau_t w_t \left[\int_0^\nu s_t(h) n_t(h) dh + \bar{s} n_t \right]. \tag{30}$$

Three aggregate shocks hit the economy: technology, $\log(a_t)$, monetary policy, ε_t, and markups, $\log(\mu_t)$. Therefore, both aggregate-demand and aggregate-supply shocks may drive business cycles, and fluctuations may be efficient or inefficient. We assume that all shocks follow independent AR(1) processes for simplicity.

3 Numerical solution algorithm

The key steps involved in solving the model are: (i) to discretize the cross-sectional distributions and decision rules, (ii) to solve for the stationary equilibrium, (iii) to collect all of the many equations defining the approximate equilibrium, (iv) linearize the model equations, (v) solve the system with standard techniques for linear rational expectations models.

3.1 Discretizing the model

For each discrete type of household characterized by a skill level, we approximate the distribution of wealth by a histogram with 250 bins. We approximate the policy rules for savings and labor supply by two piece-wise linear splines with 100 knot points each. We deal with the borrowing constraint in the approximation of the policy functions by, following Reiter (2010), parameterizing the point at which the borrowing constraint is just binding, and then constructing a grid for higher levels of assets. As a result of these approximations, there are 450 variables for each type of worker.

3.2 Solving for the stationary equilibrium

We solve for a zero-inflation steady state. From the price-setting problem, Eq. (22), it follows that $M = 1/\mu$. The equilibrium interest rate is pinned down by the Euler equation of the patient household as in the standard growth model. This interest rate then determines the marginal product of capital and therefore the capital-labor ratio and wage. Using these prices we can solve for the impatient household's problem to find the steady state decision rules and simulating these yields the invariant distribution of bond holdings by impatient households.

3.3 System of equations

Keeping track of the wealth distribution We track real assets at the beginning of the period using Reiter's procedure to allocate households to the discrete grid in a way that preserves total assets. As we have nominal bonds in the model, we account for the effect of inflation in the evolution of the household's asset position. For each discrete type of household this provides 250 equations.

Solving for household decision rules We use the household's Euler equation and the household's labor supply condition to solve for their decision rules by imposing that these equations hold with equality at the spline knot points. This provides 200 for each type of household.

Aggregate equations In addition to those equations that relate to the solution of the household's problem and the distribution of wealth across households, we have equations that correspond to the patient household's savings and labor supply decisions, as well as those that correspond to the firms' problems. We use equations (3), (4), (5), (6), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), and exogenous AR(1) processes for ϵ_t, a_t, and μ_t. We introduce an auxiliary variable that carries an extra lag of capital, $k_t^{\text{lag}} = k_{t-1}$. We use these equations to solve for c_t, n_t, x_t, M_t, p_t^*/p_t, $\bar{p}_t^A$, $\bar{p}_t^B$, S_t, π_t, y_t, w_t, r_t, τ_t, k_t, k_t^{lag}, d_t, i_t, a_t, μ_t, and ϵ_t.

3.4 Linearization and solution

At this stage, we have a large system of non-linear equations that the discretized model must satisfy. We follow Reiter (2009, 2010) in linearizing this system around the stationary equilibrium using automatic differentiation and then solving the linearized system as a linear rational expectations model using the algorithm from Sims (2002).

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