American Economic Review 2014, 104(4): 1446–1460
http://dx.doi.org/10.1257/aer.104.4.1446

Heterogeneity and Aggregation: Implications for Labor-Market Fluctuations: Comment[†]

By Shuhei Takahashi*

Chang and Kim (2007) develop an incomplete asset markets model incorporating discrete labor supply and idiosyncratic labor productivity. Their results resolve long-standing puzzles for business cycle models. Specifically, they produce a low correlation between aggregate hours worked and labor productivity (0.23) and a labor wedge with 76 percent the volatility of output. I show that these results arise from errors in their computational method. I resolve their model using a corrected method and find a strong, positive correlation between hours and productivity (0.80). Fluctuations in the labor wedge decrease to 24 percent of those in output. (*JEL* D31, E32, J22, J24, J31)

How does inequality in wealth affect the behavior of employment and labor productivity in equilibrium business cycle models? This is a difficult question to answer because it needs a model that replicates the wealth distribution in the data reasonably well. In this respect, Chang and Kim (2007) is the seminal contribution. They develop an incomplete asset markets model incorporating discrete labor supply choice and idiosyncratic labor productivity. Calibrating the stochastic process for idiosyncratic productivity to micro-level wage data, they generate a distribution of wealth that closely matches that found in the US data.

Chang and Kim (2007) find that, when driven by exogenous shocks to aggregate total factor productivity, their heterogeneous-agent model reproduces the low correlation between total hours worked and labor productivity found in the US data (model: 0.23; versus data: 0.08).[1] This resolves the so-called hours-productivity puzzle present in typical equilibrium business cycle models, i.e., the result that hours and productivity comove strongly when driven by exogenous technology shocks.

In most equilibrium business cycle models (e.g., Kydland and Prescott 1982; Hansen 1985; and Prescott 1986), aggregate productivity shocks primarily shift labor demand. This leads to a strong, positive correlation between hours worked and labor productivity. To resolve this counterfactual prediction, several papers

* Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan (e-mail: takahashi@kier.kyoto-u.ac.jp). This paper is based on the second chapter of my dissertation submitted to the Ohio State University. I am grateful to Aubhik Khan, Bill Dupor, and Julia Thomas for their valuable comments and suggestions. I also would like to thank a referee for his/her insightful comments. I declare that I have no relevant or material financial interests that relate to the research described in this paper. Any remaining errors are my own.

† Go to http://dx.doi.org/10.1257/aer.104.4.1446 to visit the article page for additional materials and author disclosure statement(s).

[1] Labor productivity is defined as output per labor hour.

add shocks that move labor supply. For example, Benhabib, Rogerson, and Wright (1991) assume shocks to the home production technology, whereas Christiano and Eichenbaum (1992) consider government spending shocks. The finding of Chang and Kim (2007) suggests that these additional shocks are not needed to explain the key regularity of the labor market, once heterogeneity across households is taken into account.

In the economy of Chang and Kim (2007), households that differ in their labor productivity decide whether to work in any given period. Because asset markets are incomplete, households' labor supply decisions also depend on their wealth. As a result, the joint distribution of assets and productivity across households influences the composition of workers with different levels of productivity. This allows the possibility that the comovement of total hours worked with aggregate labor productivity may differ from that in representative-agent business cycle models.

This paper finds that, in spite of its rich heterogeneity across households, the model of Chang and Kim (2007) actually generates a strong comovement of hours worked with labor productivity, as in representative-agent models. I show here that their finding of a low correlation arises from errors in their computational method. First, throughout the simulation step of their Krusell and Smith (1998)-based solution algorithm, they mishandle the discontinuous within-the-period value function associated with not working. As a result, they incorrectly introduce noises in the employment choices of households that are near their borrowing limit and would otherwise always choose to work. Second, their method fails to solve for the equilibrium wage throughout their simulation step.

I resolve the Chang and Kim model correcting both of the problems in their solution method mentioned above. First, when determining households' employment in each period of my simulation, I use the beginning-of-the-period value function, avoiding the use of the discontinuous within-the-period value function of not working. This eliminates the noises in the employment decisions of households near their borrowing limit, and I correctly find that they always work. Second, in keeping with the algorithm originally devised by Krusell and Smith (1997), I compute the equilibrium wage and associated total employment in each period of the simulation step. With these corrections to the solution method, I find that the Chang and Kim model generates a strong, positive correlation between hours and productivity (0.80).

Specifically, my corrected method determines households' employment choice in each period by first solving the consumption-saving problem conditional on working or not working and then comparing these values. In this calculation, I use the conventional, beginning-of-the-period value function to compute future discounted life-time utility. As the value function is solved at a set of grid points on the aggregate and individual state, interpolation is used to compute the value not on the grid points. As the beginning-of-the-period value function is continuous, the computation provides an accurate approximation of discounted life-time utility. As a result, my method accurately computes the values of working and not working. Further, if households cannot maintain positive consumption without working, they must choose to work. Hence, I correctly find that households with high levels of debt and near their borrowing limit always work.

In contrast, Chang and Kim (2007) obtain the values of working and not working by directly interpolating the within-the-period value functions of working and not working, respectively, and compare these values to determine households' employment choice. This approach saves computation time, since it avoids solving the consumption-saving problem in each period of the simulation. However, the shortcut is not innocuous. Crucially, for households with a large amount of debt who are near their borrowing limit, the within-the-period value function associated with not working has a discontinuity in the aggregate state. This discontinuity exists because in the absence of labor income, such households cannot sustain positive consumption when the aggregate state generates a high real interest rate. In such cases, the value associated with not working approaches negative infinity.[2] By interpolating the discontinuous within-the-period value function, Chang and Kim (2007) generate non-negligible errors in the value of not working for households with high levels of debt. These, in turn, cause noises in the employment decisions of such households even though the true within-the-period value functions indicate that they would always choose to work. When they do not work, total labor hours fall. Further, as most of households in this problem region have low current labor productivity, the employment share of low-productivity households declines, raising aggregate labor productivity. This creates a downward bias in the correlation between hours and productivity.

The second crucial difference in my numerical approach involves the determination of equilibrium wages. Following the method originally developed by Krusell and Smith (1997) for the bond market in their model, in each period of the model simulation, I repeatedly solve for households' employment decisions over a sequence of candidate wage rates until I find the market-clearing wage. In contrast, Chang and Kim (2007) determine aggregate labor as the sum of labor supplied at households' perceived wage, which is embedded in the computation of the value functions as a function of the aggregate state.[3] The wage in the period is then recorded as that at which firms demand the level of labor supplied. Crucially, the perceived and equilibrium wages are different. Although the difference is relatively small, it produces a downward bias in the correlation between the wage and aggregate employment. For example, when the perceived wage is higher than the equilibrium wage, the recorded employment is higher than its true equilibrium level. In contrast, the recorded wage is lower than the true equilibrium wage. Hence, the comovement between hours worked and labor productivity is understated relative to its true value.

Importantly, while Chang and Kim (2007) argue that their model can reproduce the large fluctuations in the labor wedge found in the US data, this result also arises from the errors in their solution method.[4] Specifically, they report that the labor wedge is 76 percent as volatile as output in their model, which is fairly close to that found in the data (92 percent). However, I find that, when the model is solved with my corrected method, the variability of the labor wedge falls to only 24 percent that of output.

[2] Chang and Kim (2007) assign a large negative value to those cases.

[3] Conventionally, the use of such perceived prices is restricted to the step that solves agents' value functions in applications of the Krusell and Smith (1998) algorithm and is not directly used in the following simulation step. See, for example, Krusell and Smith (1997) and Khan and Thomas (2003).

[4] The labor wedge is defined as the ratio of the marginal rate of substitution of leisure for consumption relative to labor productivity.

The rest of this paper proceeds as follows. Section I lays out the model. Section II explains my method and Chang and Kim's (2007) original method. Section III compares the model's business cycle statistics when solved with these two methods. Section IV concludes.

I. Model

Chang and Kim (2007) describe their model in detail. Hence, I review it only briefly here.

A. *Households*

A continuum of households of measure one have the identical utility function $u(c, h)$, where c is consumption and h is hours worked. Labor is indivisible: $h \in \{\overline{h}, 0\}$. Households earn labor income wxh, where w is the equilibrium wage rate per efficiency unit of labor and x is person-specific labor productivity. Asset markets are incomplete. Households hold a single asset, physical capital a, and earn (or pay) the equilibrium rental rate r. They also face a borrowing constraint, $a \geq \overline{a}$.

Let λ be aggregate total factor productivity and μ represent the joint distribution of a and x across households. This is the aggregate state. The household distribution evolves according to the equilibrium mapping, $\mu' = T(\lambda, \mu)$.[5]

Define $V(a, x; \lambda, \mu)$ as the beginning-of-the-period value for a household characterized by (a, x). This beginning-of-the-period value reflects the household's current labor supply choice:

$$V(a, x; \lambda, \mu) = \max\{V^E(a, x; \lambda, \mu), V^N(a, x; \lambda, \mu)\}. \tag{1}$$

The household's within-the-period value conditional on working is $V^E(a, x; \lambda, \mu)$, which satisfies

$$V^E(a, x; \lambda, \mu) = \max_{a'}\{u(c, \overline{h}) + \beta E[V(a', x'; \lambda', \mu') \,|\, x, \lambda, \mu]\}, \tag{2}$$

$$\text{subject to } c = w(\lambda, \mu)x\overline{h} + [1 + r(\lambda, \mu)]a - a' \text{ and } a' \geq \overline{a}.$$

Similarly, $V^N(a, x; \lambda, \mu)$ is the household's within-the-period value conditional on not working, and it is defined as

$$V^N(a, x; \lambda, \mu) = \max_{a'}\{u(c, 0) + \beta E[V(a', x'; \lambda', \mu') \,|\, x, \lambda, \mu]\}, \tag{3}$$

$$\text{subject to } c = [1 + r(\lambda, \mu)]a - a' \text{ and } a' \geq \overline{a}.$$

[5] Variables with a prime are next period values.

B. *Firms*

A representative firm produces the final good Y using capital K and labor L as inputs. The production function is $Y = \lambda K^{1-\alpha} L^{\alpha}$. The firm maximizes static profits by choosing $K(\lambda, \mu)$ and $L(\lambda, \mu)$. The first-order conditions are

$$r(\lambda, \mu) = (1 - \alpha)\lambda K(\lambda, \mu)^{-\alpha} L(\lambda, \mu)^{\alpha} - \delta \tag{4}$$

and

$$w(\lambda, \mu) = \alpha\lambda K(\lambda, \mu)^{1-\alpha} L(\lambda, \mu)^{\alpha-1}, \tag{5}$$

where δ is the capital depreciation rate.

C. *General Equilibrium*

A recursive competitive equilibrium consists of a set of functions,

$$(w, r, V, V^E, V^N, c, a', h, K, L, T),$$

that satisfy the following conditions.

(i) Households' Optimization:
$V(a, x; \lambda, \mu)$, $V^E(a, x; \lambda, \mu)$, and $V^N(a, x; \lambda, \mu)$ satisfy (1), (2), and (3), while $c(a, x; \lambda, \mu)$, $a'(a, x; \lambda, \mu)$, and $h(a, x; \lambda, \mu)$ are the associated policy functions.

(ii) Firm's Optimization:
The representative firm chooses $K(\lambda, \mu)$ and $L(\lambda, \mu)$ to satisfy (4) and (5).

(iii) Labor Market Clearing: $L(\lambda, \mu) = \int xh(a, x; \lambda, \mu)\mu([da \times dx])$.

(iv) Capital Market Clearing: $K(\lambda, \mu) = \int a\mu([da \times dx])$.

(v) Final Good Market Clearing:
$\int\{a'(a, x; \lambda, \mu) + c(a, x; \lambda, \mu)\}\mu([da \times dx]) = \lambda K(\lambda, \mu)^{1-\alpha}L(\lambda, \mu)^{\alpha} + (1 - \delta) \int a\mu([da \times dx])$.

(vi) Evolution of Household Distribution:
$T(\lambda, \mu)$ is consistent with household decisions. Specifically, for all $B \subseteq \mathcal{K}$, $\mu'(B, x') = \int_{(a,x)\,|\,a'(a,x;\lambda,\mu)\in B} \pi_x(x'|x)\mu(da|x)$,

where $\pi_x(x'|x)$ is the transition probability from x to x'.

II. Solution Methods

Two difficulties arise when solving the model described above. First, the distribution of wealth and productivity across households, μ, is an infinite-dimensional object and cannot be dealt with directly. Thus, it is approximated with the first moment of the asset distribution, the aggregate capital stock, K. Second, to solve the household optimization problem, it is necessary to specify the relationship between the approximate aggregate state (λ and K) and next period's aggregate capital K', the current equilibrium wage rate w, and the current equilibrium rental rate r. Both methods below assume that K', w, and r are forecasted using simple functions of λ and K. Specifically, the following rules are used: $\ln \hat{K}' = a_0 + a_1 \ln K + a_2 \ln \lambda$, $\ln \hat{w} = b_0 + b_1 \ln K + b_2 \ln \lambda$, and $\ln(\hat{r} + \delta) = d_0 + d_1 \ln K + d_2 \ln \lambda$.

The model is solved in the following three steps.[6] First, given the current forecasting rules for K', w, and r, the value functions of households are computed using value function iteration. The associated policy functions are also obtained during this step. Second, using these value and policy functions, the model's time-series data are generated through simulation. Third, the forecasting rules are updated using this simulated data. These three steps are repeated until the forecasting rules converge.

In the following subsections, I first explain my corrected method and then discuss differences between my method and the method implemented by Chang and Kim (2007).

A. *My Method*

The first step is to solve for $V(a, x; \lambda, K)$ given the forecasting rules for K', w, and r. To do so, I set grid points for a and K and discretize the stochastic processes for x and λ using Tauchen's (1986) method. Next, starting from an initial guess, I update $V(a, x; \lambda, K)$ as $V(a, x; \lambda, K) = \max\{V^E(a, x; \lambda, K), V^N(a, x; \lambda, K)\}$ by solving the following two problems:

$$(6) \quad V^E(a, x; \lambda, K) = \max_{a'} \Big\{ u(c, \overline{h}) + \beta \sum_{x'} \sum_{\lambda'} \pi_x(x'|x) \pi_\lambda(\lambda'|\lambda) V(a', x'; \lambda', \hat{K}'(\lambda, K)) \Big\},$$

$$\text{subject to } c = \hat{w}(\lambda, K) x \overline{h} + [1 + \hat{r}(\lambda, K)]a - a' \text{ and } a' \geq \overline{a},$$

and

$$(7) \quad V^N(a, x; \lambda, K) = \max_{a'} \Big\{ u(c, 0) + \beta \sum_{x'} \sum_{\lambda'} \pi_x(x'|x) \pi_\lambda(\lambda'|\lambda) V(a', x'; \lambda', \hat{K}'(\lambda, K)) \Big\}$$

$$\text{subject to } c = [1 + \hat{r}(\lambda, K)]a - a' \text{ and } a' \geq \overline{a}.$$

[6]The procedure was originally developed by Krusell and Smith (1997, 1998) for their model with heterogeneous households and has been used extensively in the literature. Some examples are Chang and Kim (2006, 2007) and Pijoan-Mas (2007). Khan and Thomas (2003, 2007, 2008) modify the method for their models that incorporate heterogeneity across firms.

As households' choices of a' and perceived next period capital stock $\hat{K}'$ may not be on their grid points, I use bivariate cubic interpolation in these variables to approximate the conditional expectations in (6) and (7). Once $V(a, x; \lambda, K)$ converges, this step is done.

The second step is to simulate the model economy using $V(a, x; \lambda, K)$ obtained above. Crucially, labor market clearing is imposed by finding the equilibrium wage and associated total employment in each period, as Krusell and Smith (1997) did for the bond market in their model. Specifically, I first guess the current equilibrium wage rate $\tilde{w}$. This gives a guess for the equilibrium rental rate of capital as $\tilde{r} = (1-\alpha)\lambda(\tilde{w}/\alpha\lambda)^{-\alpha/(1-\alpha)} - \delta$. Households' forecast of the future aggregate state is given by $\ln \hat{K}' = a_0 + a_1 \ln K + a_2 \ln \lambda$. To determine households' behavior, I solve their consumption-saving problem given a current labor supply choice under $\hat{K}', \tilde{w}$, and $\tilde{r}$, and then allow households to choose whether or not to work. Specifically, for a household characterized by (a, x), I compute the within-the-period value of working $v^E(a, x)$ and that of not working $v^N(a, x)$ by solving the following problems:

$$(8) \qquad v^E(a, x) = \max_{a'} \{u(c, \overline{h}) + \beta \sum_{x'} \sum_{\lambda'} \pi_x(x'|x)\pi_\lambda(\lambda'|\lambda)V(a', x'; \lambda', \hat{K}')\},$$

$$\text{subject to } c = \tilde{w}x\overline{h} + (1+\tilde{r})a - a' \text{ and } a' \geq \overline{a},$$

and

$$(9) \qquad v^N(a, x) = \max_{a'} \{u(c, 0) + \beta \sum_{x'} \sum_{\lambda'} \pi_x(x'|x)\pi_\lambda(\lambda'|\lambda)V(a', x'; \lambda', \hat{K}')\},$$

$$\text{subject to } c = (1+\tilde{r})a - a' \text{ and } a' \geq \overline{a}.$$

Following Chang and Kim (2007), I assume that λ changes continuously in this simulation step. Hence, in order to approximate the conditional expectations in (8) and (9), I also use univariate cubic spline interpolation in λ in addition to bivariate cubic spline interpolation in a' and $\hat{K}'$. If $v^E(a, x) \geq v^N(a, x)$, then the household with a and x chooses to work, or $h(a, x) = \overline{h}$. The next period asset $a'(a, x)$ is obtained from (8). If $v^E(a, x) < v^N(a, x)$, then the household does not work, or $h(a, x) = 0$. In this case, (9) gives $a'(a, x)$. Critically, if households have high levels of debt and can have positive consumption only under employment, they must work. The amount of labor supplied in the aggregate is $L^S = \int xh(a, x)\mu([da \times dx])$. I repeatedly solve the households' problem searching for $\tilde{w}$ such that L^S equals the amount of labor demanded: $L^D = (\alpha\lambda/\tilde{w})^{1/(1-\alpha)}K$. Once such $\tilde{w}$ is found, aggregate variables are calculated: $L = \int xh(a, x)\mu([da \times dx])$, $K' = \int a'(a, x)\mu([da \times dx])$, $H = \int h(a, x)\mu([da \times dx])$, $Y = \lambda K^{1-\alpha}L^\alpha$, $I = K' - (1-\delta)K$, $C = Y - I$, and $r = (1-\alpha)\lambda K^{-\alpha}L^\alpha - \delta$. The distribution across households μ is updated as consistent with households' saving choice $a'(a, x)$ and the stochastic process for x. This simulation is done for 3,500 periods.

Lastly, I apply ordinary least squares to this simulated data and update the forecasting rules for K', w, and r. These three steps are repeated until the forecasting rules converge.

B. *Departures from the Chang and Kim (2007) Method*

The main difference between my method and Chang and Kim's (2007) method lies in the simulation step. As shown above, I only use the beginning-of-the-period value function $V(a, x; \lambda, K)$ in the simulation and do not use the within-the-period value functions, $V^E(a, x; \lambda, K)$ and $V^N(a, x; \lambda, K)$. In contrast, Chang and Kim (2007) use $V^E(a, x; \lambda, K)$ and $V^N(a, x; \lambda, K)$, and the associated policy functions $a'^E(a, x; \lambda, K)$ and $a'^N(a, x; \lambda, K)$ in their simulation step.[7] Further, they do not impose labor market clearing in each period of their simulation.

Specifically, for a household with a and x, Chang and Kim (2007) compute the value of working $\tilde{v}^E(a, x)$ and that of not working $\tilde{v}^N(a, x)$ by directly evaluating $V^E(a, x; \lambda, K)$ and $V^N(a, x; \lambda, K)$, respectively, at the current values of λ and K. In this computation, they use bivariate polynomial interpolation in λ and K, and then linear interpolation in a. When $\tilde{v}^E(a, x) \geq \tilde{v}^N(a,x)$, the household with a and x chooses to work, or $h(a, x) = \overline{h}$, and next period's asset holding $a'(a, x)$ is computed by directly interpolating $a'^E(a, x; \lambda, K)$ in λ, K, and a. If $\tilde{v}^E(a, x) < \tilde{v}^N(a, x)$, then the household with a and x does not work, or $h(a, x) = 0$. In this case, $a'^N(a, x; \lambda, K)$ is used to determine $a'(a, x)$. Crucially, Chang and Kim (2007) compare $\tilde{v}^E(a, x)$ and $\tilde{v}^N(a, x)$ without checking whether each household can maintain positive consumption even under unemployment. Once households' decisions are obtained, aggregate labor is calculated using $L = \int xh(a, x)\mu([da \times dx])$. The wage rate is then determined from labor demand: $w = \alpha\lambda K^{1-\alpha} L^{\alpha-1}$. Other aggregate variables are calculated in the same way as in my method.

C. *Discussion*

There are two problems in the solution method implemented by Chang and Kim (2007). First, when determining the within-the-period value to households of choosing not to work $\tilde{v}^N(a, x)$, they interpolate the within-the-period value function associated with not working $V^N(a, x; \lambda, K)$ for λ and K. Critically, for households with high levels of debt near their borrowing limit, $V^N(a, x; \lambda, K)$ has a discontinuity in λ and K. This discontinuity arises because without working, such households cannot sustain positive consumption under a high λ and low K, which produce a high real interest rate, and their value approaches negative infinity.

Chang and Kim (2007) derive $V^N(a, x; \lambda, K)$ by setting $c = 10^{-20}$ and $a' = \overline{a}$ for those cases. Figure 1 plots $V^N(a, x; \lambda, K)$ over λ and K at $a = -1.98$ ($\overline{a} = -2.0$) and $x = 1.0$ (mean productivity), as recorded using their solution method.[8] The

[7] Chang and Kim (2007) define $a'^E(a, x; \lambda, K)$ as the saving function when households choose to work and $a'^N(a, x; \lambda, K)$ as the saving function when households choose not to work.

[8] The figure looks the same for any x.

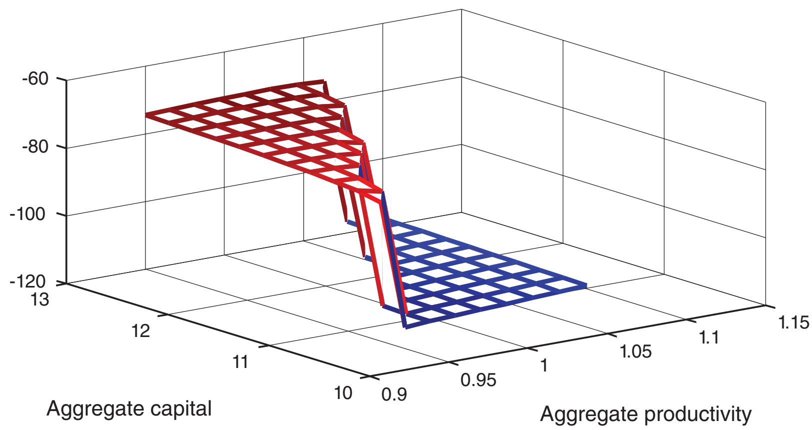

Figure 1. The Within-the-Period Value Function of not Working Recorded by the Chang and Kim (2007) Method

Note: This figure plots the within-the-period value function when households do not work $V^N(a, x; \lambda, K)$ over λ and K, for households with high levels of debt and close to their borrowing limit ($a = -1.98$ and $x = 1.0$), as recorded using the Chang and Kim (2007) method.

function falls sharply in the high λ and low K directions. Because of this discontinuity, when interpolating $V^N(a, x; \lambda, K)$ with respect to λ and K, non-negligible approximation errors arise in the within-the-period value of not working $\tilde{v}^N(a, x)$ as shown in Figure 2, and the employment decisions of households near their borrowing limit are distorted. Specifically, as Figure B1 in the online Appendix shows, the method of Chang and Kim (2007) finds that such households choose not to work under some λ and K. In contrast, the true within-the-period value functions imply that they always work because $V^E(a, x; \lambda, K) \geq V^N(a, x; \lambda, K)$ for all λ and K at $a = -1.98$ and $x = 1.0$. Thus, the Chang and Kim (2007) method understates the level of labor hours and overstates fluctuations in hours.

Furthermore, as idiosyncratic productivity is persistent, many of the households close to their borrowing limit have low current labor productivity. When they choose not to work, total labor hours fall. At the same time, labor productivity measured at the aggregate level rises because the employment share of high-productivity workers increases. Accordingly, the noises in the employment decisions introduced by the approximation errors generate *a downward bias in the correlation between hours and productivity at the aggregate level*. By contrast, my method uses the beginning-of-the-period value function $V(a, x; \lambda, K) = \max\{V^E(a, x; \lambda, K), V^N(a, x; \lambda, K)\}$, which is continuous. This allows me to obtain accurate approximations of the values of working and not working, and hence I correctly determine employment decisions of households, including those with large amounts of debt near their borrowing limit.

Second, the method of Chang and Kim (2007) skips a step that ensures labor market clearing in each period of the simulation. It essentially determines the amount of labor in each period as the sum of labor supplied at households' forecasted wage, which is embedded in the computation of $V^E(a, x; \lambda, K)$ and $V^N(a, x; \lambda, K)$. The wage in the period is then obtained by substituting the aggregate labor supplied into the firm's labor demand function. Figure 3 illustrates this point using a supply-demand graph of the labor market. Although the difference is small, the

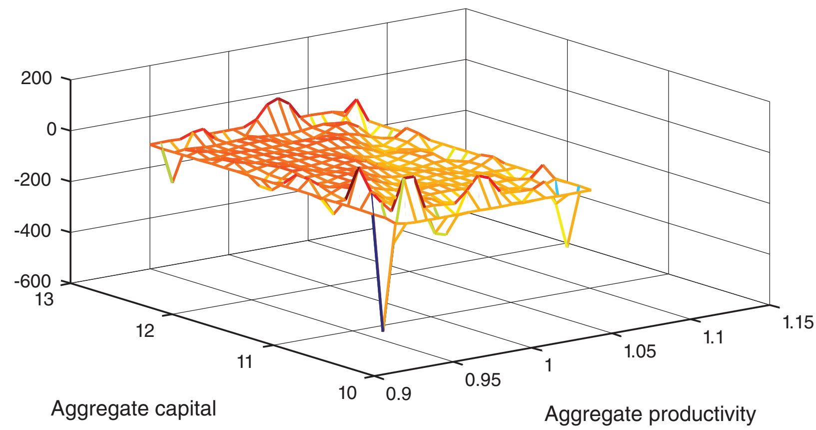

Figure 2. The Within-the-Period Value of not Working for Households Computed by the Chang and Kim (2007) Method

Note: This figure plots the within-the-period value of not working $\tilde{v}^N(a, x)$ for households with high levels of debt and close to their borrowing limit ($a = -1.98$ and $x = 1.0$), as obtained using the Chang and Kim (2007) method.

forecasted and equilibrium wages do not coincide. As shown in panel A of Figure 3, when the forecasted wage $\hat{w}$ is higher than the equilibrium wage w^*, the recorded labor L exceeds the equilibrium labor, or $L > L^*$. By contrast, the recorded wage w is lower than the equilibrium wage, or $w < w^*$. Panel B shows the opposite case, where $L < L^*$ and $w > w^*$. Hence, both errors understate the comovement of the wage with aggregate labor relative to the equilibrium counterpart. Since aggregate labor and hours worked are highly correlated in the model, as are the wage and labor productivity, the correlation between hours and productivity is also understated. My corrected implementation of the Krusell and Smith (1997) solution method eliminates this bias by finding the equilibrium wage and associated labor in each period of the simulation.

III. Results

Tables 1 and 2 compare the business cycle properties of the US economy with those of the Chang and Kim (2007) model economy.[9] The column labeled "Data" presents the business cycle statistics of the United States taken from Chang and Kim (2007). The column labeled "CK" contains the model moments reported by Chang and Kim (2007). Subsequent columns report results for the same model solved in three ways.[10] "Adjusted replication" is the result when I solve the model with Chang

[9] All parameter values are taken from Chang and Kim (2007) (Table A1). The utility function is $u(c, h) = \ln c - Bh^{1+1/\gamma}/(1 + 1/\gamma)$. Both aggregate and idiosyncratic productivity follow AR(1) processes. Specifically, aggregate TFP follows $\ln \lambda' = \rho_\lambda \ln \lambda + \varepsilon'_\lambda$, where $\varepsilon'_\lambda \sim N(0, \sigma^2_\lambda)$. Idiosyncratic productivity follows $\ln x' = \rho_x \ln x + \varepsilon'_x$, where $\varepsilon'_x \sim N(0, \sigma^2_x)$.

[10] I use the same aggregate productivity history for all these simulations.

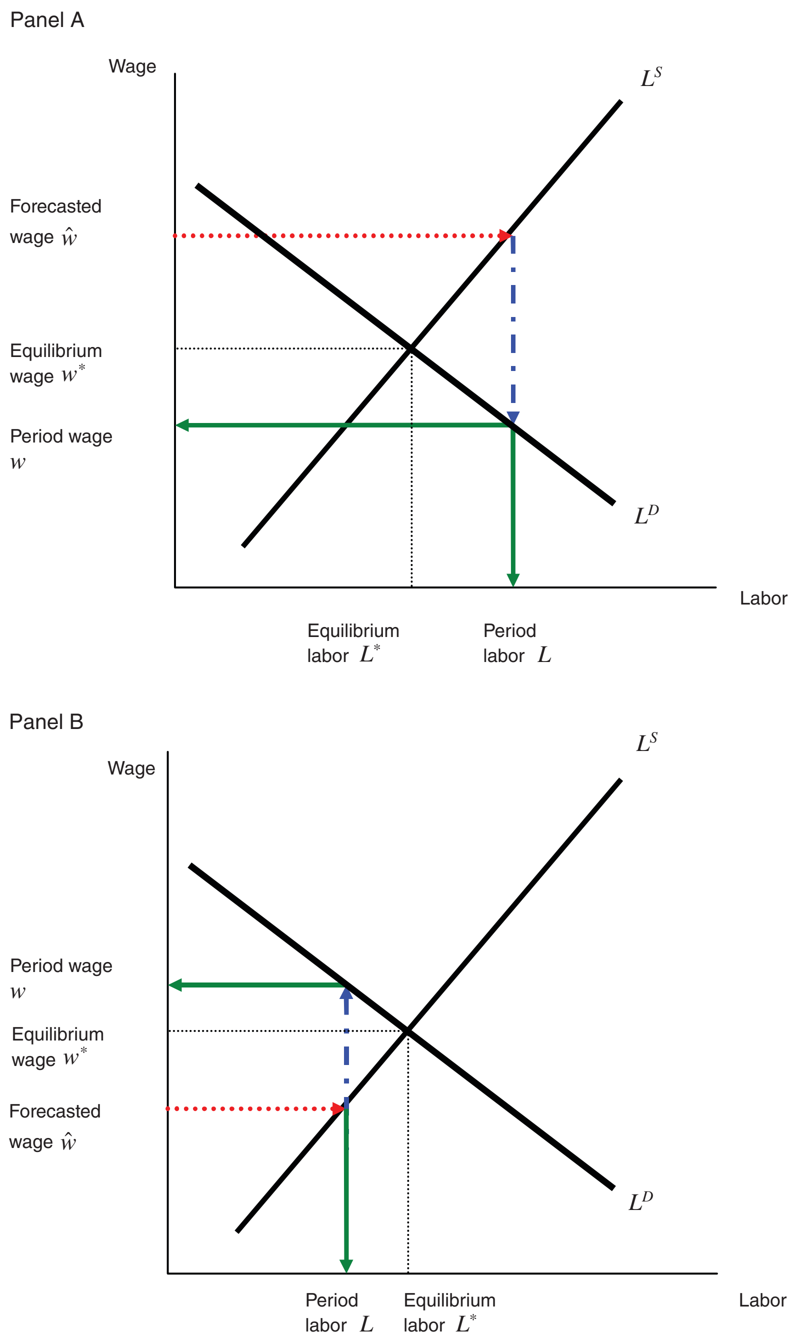

Figure 3. Determination of Wage and Aggregate Labor in the Chang and Kim (2007) Method

Notes: This figure illustrates how the Chang and Kim (2007) method determines the wage rate and aggregate labor in each period of their simulation. Panel A illustrates the situation in which the forecasted wage is higher than the equilibrium wage, whereas panel B illustrates the opposite case.

Table 1—Volatilities of Aggregate Variables

	Data	CK	Adjusted replication	With one correction	Corrected method
σ_H	0.82	0.76	0.71	0.60	0.57
σ_{wedge}	0.92	0.76	0.67	0.31	0.24
σ_Y	2.06	1.28	1.33	1.31	1.30
σ_C	0.45	0.39	0.39	0.33	0.33
σ_I	2.41	3.06	3.07	3.10	3.08
σ_L	—	0.50	0.47	0.44	0.41
$\sigma_{Y/H}$	0.50	0.50	0.51	0.49	0.49

Notes: This table shows the volatilities of aggregate variables. Variables are logged and then detrended using the Hodrick-Prescott (HP) filter with a smoothing parameter of 1,600. The volatility of output σ_Y is defined as the standard deviation of output (multiplied by 100). Other volatilities are expressed relative to σ_Y. The column labeled "Data" shows the US data taken by Chang and Kim (2007), whereas the column labeled "CK" presents the model result reported by Chang and Kim (2007). Subsequent three columns report results of my simulation. "Adjusted replication" is the result when solving the model with the Chang and Kim (2007) method but keeping the productivity distribution constant, as described in footnote 11. "With one correction" is the result when I correct the problem of the discontinuous value function in the Chang and Kim (2007) method, leaving their failure to compute equilibrium wages at each date. "Corrected method" is the result when I correct both problems.

Table 2—Correlations of Aggregate Variables

	Data	CK	Adjusted replication	With one correction	Corrected method
Corr(H, Y/H)	0.08	0.23	0.33	0.70	0.80
Corr(H, $wedge$)	0.85	0.87	0.84	0.91	0.95
Corr(Y, C)	0.69	0.84	0.83	0.88	0.89
Corr(Y, I)	0.90	0.98	0.97	0.99	0.99
Corr(Y, H)	0.86	0.87	0.88	0.94	0.96
Corr(Y, L)	—	0.92	0.92	0.94	0.95
Corr(Y, Y/H)	0.57	0.68	0.75	0.90	0.94

Notes: This table shows the contemporaneous correlations of aggregate variables. Variables are logged and then detrended using the Hodrick-Prescott (HP) filter with a smoothing parameter of 1,600. The column labeled "Data" shows the US data taken by Chang and Kim (2007), whereas the column labeled "CK" presents the model result reported by Chang and Kim (2007). Subsequent three columns report results of my simulation. "Adjusted replication" is the result when solving the model with the Chang and Kim (2007) method but keeping the productivity distribution constant, as described in footnote 11. "With one correction" is the result when I correct the problem of the discontinuous value function in the Chang and Kim (2007) method, leaving their failure to compute equilibrium wages at each date. "Corrected method" is the result when I correct both problems.

and Kim's (2007) original method.[11] "Corrected method" is the result when I correct both problems in Chang and Kim's method noted in the previous section. Lastly, "With one correction" is an intermediate set of results obtained when I correct their problem about the discontinuous value function while leaving in place their failure to compute equilibrium wages at each date. The inclusion of this intermediate set of results isolates the impact of each of the two problems in the Chang and Kim (2007)

[11] Chang and Kim (2007) simulate the model economy including 200,000 households. Hence, the productivity distribution across households changes over time in their simulation. This affects the model's aggregate dynamics. My method eliminates this noise by approximating the household distribution at 2,000 (wealth a) $\times$ 17 (productivity x) points and thereby holding the productivity distribution constant throughout my simulation.

solution method: the inaccurate determination of employment of households close to their borrowing limit and the absence of labor market clearing.

Most importantly, after correcting both errors in their solution method, I find that the model of Chang and Kim (2007) generates a strong, positive correlation between hours worked and labor productivity (0.80) far from the nearly zero correlation found in the US data (0.08). Hence, in contrast to the authors' previous reporting, their model fails to solve the hours-productivity puzzle. More broadly, despite its rich heterogeneity in wealth and productivity across households, the model has business cycle properties similar to those of the baseline real business cycle model, which assumes a representative agent.

As explained in Section IIC, the solution method implemented by Chang and Kim (2007) understates the comovement of hours worked with labor productivity and overstates fluctuations in hours. Indeed, they report a substantially lower correlation between hours and productivity (0.23) and a higher volatility of hours (0.76) when compared to the values obtained under the corrected method (0.80 and 0.57, respectively). Further, their method often understates total hours worked, as discussed in Section IIC and shown in Figure B2 of the online Appendix.

When equilibrium wage determination is ignored in the simulation, the correlation between hours worked and labor productivity becomes lower (0.70) relative to its value when the labor market clears (0.80). This finding highlights the importance of ensuring labor market clearing, although the incorrect determination of the labor supply of households near their borrowing limit is largely responsible for the low correlation between hours and productivity reported in Chang and Kim (2007).

Lastly, although the correlation between the labor wedge and hours worked is little affected, the variability of the labor wedge in the Chang and Kim (2007) model is sensitive to how the model is solved. The labor wedge is defined as the ratio of the marginal rate of substitution of leisure for consumption relative to labor productivity and computed as $\ln wedge = \ln H^{1/\gamma} C - \ln Y/H$, assuming that $\gamma = 1.5$ at the aggregate level. Chang and Kim (2007) find a volatile labor wedge as consistent with the US data ($\sigma_{wedge} = 0.76$ in the model versus 0.92 in the data). In contrast, I obtain a substantially less volatile labor wedge (0.24) when I correct the errors in their solution method.[12]

IV. Conclusion

Chang and Kim (2007) develop a heterogeneous-agent model that reproduces the inequality of wealth found in the US data. They argue that their model explains the key regularities of the US labor market by reproducing the low correlation between total hours worked and labor productivity and the volatile labor wedge. I have shown here that these results arise from errors in the authors' computational method and that their model actually generates a strong, positive comovement of hours with productivity and small fluctuations in the labor wedge. My finding establishes that an incomplete asset markets model with discrete labor supply choice and idiosyncratic

[12] Table A2 reports the coefficients and accuracy measures for the forecasting rules. Two measures are presented: the coefficients of determination R^2 and the standard deviation of the forecasting error $\hat{\sigma}$. These statistics suggest that the forecast rules are quite accurate.

labor productivity cannot account for the above stylized facts about the US labor market dynamics if the only source of aggregate fluctuations is shocks to total factor productivity.[13]

Appendix

Table A1—Parameter Values

Parameter	Description	Value
β	Discount factor	0.98267
B	Disutility parameter	166.3
γ	Individual labor-supply elasticity	0.4
$\bar{h}$	Working hours	1/3
$\bar{a}$	Borrowing constraint	–2.0
α	Labor's share	0.64
δ	Capital depreciation rate	0.025
ρ_x	Persistence in idiosyncratic productivity	0.929
σ_x	Volatility of innovation to idiosyncratic productivity	0.227
ρ_λ	Persistence in aggregate productivity	0.95
σ_λ	Volatility of innovation to aggregate productivity	0.007

Notes: This table lists parameter values. All the values are taken from Chang and Kim (2007). One period in the model corresponds to one quarter.

Table A2—Forecasting Rules

		CK	Adjusted replication	With one correction	Corrected method
$\hat{K}'$	a_0	0.1133	0.1132	0.1153	0.1154
	a_1	0.9537	0.9538	0.9529	0.9529
	a_2	0.0997	0.0993	0.1012	0.1012
	R^2	0.9999	1.0000	1.0000	1.0000
	$\hat{\sigma}$	0.0246	0.0220	0.0120	0.0077
$\hat{w}$	b_0	–0.2370	–0.2252	–0.2099	–0.2100
	b_1	0.4494	0.4445	0.4381	0.4381
	b_2	0.7997	0.8093	0.8203	0.8202
	R^2	0.9977	0.9989	0.9995	0.9998
	$\hat{\sigma}$	0.136	0.1005	0.0692	0.0414
$\hat{r} + \delta$	d_0	–1.3936	–1.4148	–1.4419	–1.4417
	d_1	–0.7989	–0.7902	–0.7788	–0.7788
	d_2	1.3559	1.3391	1.3195	1.3197
	R^2	0.9887	0.9949	0.9975	0.9991
	$\hat{\sigma}$	0.242	0.1789	0.1231	0.0737

Notes: This table shows the coefficients and accuracy of the forecasting rules. The rules are $\ln \hat{K}' = a_0 + a_1 \ln K + a_2 \ln \lambda$, $\ln \hat{w} = b_0 + b_1 \ln K + b_2 \ln \lambda$, and $\ln(\hat{r} + \delta) = d_0 + d_1 \ln K + d_2 \ln \lambda$. Two accuracy measures are the coefficients of determination R^2 and standard deviation of the forecasting error $\hat{\sigma}$ (percent). The column labeled "CK" presents the result reported by Chang and Kim (2007). Subsequent three columns report results of my simulation. "Adjusted replication" is the result when solving the model with the Chang and Kim (2007) method but keeping the productivity distribution constant, as described in footnote 11. "With one correction" is the result when I correct the problem of the discontinuous value function in the Chang and Kim (2007) method, leaving their failure to compute equilibrium wages at each date. "Corrected method" is the result when I correct both problems.

[13] Takahashi (2013) introduces uncertainty shocks affecting the dispersion of idiosyncratic labor productivity to the Chang and Kim (2007) model. The generalized model delivers a substantially lower correlation between hours worked and labor productivity and larger standard deviation of the labor wedge, when compared to the original model.

REFERENCES

Benhabib, Jess, Richard Rogerson, and Randall Wright. 1991. "Homework in Macroeconomics: Household Production and Aggregate Fluctuations." *Journal of Political Economy* 99 (6): 1166–87.

Chang, Yongsung, and Sun-Bin Kim. 2006. "From Individual to Aggregate Labor Supply: A Quantitative Analysis Based on a Heterogeneous Agent Macroeconomy." *International Economic Review* 47 (1): 1–27.

Chang, Yongsung, and Sun-Bin Kim. 2007. "Heterogeneity and Aggregation: Implications for Labor-Market Fluctuations." *American Economic Review* 97 (5): 1939–56.

Christiano, Lawrence J., and Martin Eichenbaum. 1992. "Current Real-Business-Cycle Theories and Aggregate Labor-Market Fluctuations." *American Economic Review* 82 (3): 430–50.

Hansen, Gary D. 1985. "Indivisible Labor and the Business Cycle." *Journal of Monetary Economics* 16 (3): 309–27.

Khan, Aubhik, and Julia K. Thomas. 2003. "Nonconvex Factor Adjustments in Equilibrium Business Cycle Models: Do Nonlinearities Matter?" *Journal of Monetary Economics* 50 (2): 331–60.

Khan, Aubhik, and Julia K. Thomas. 2007. "Inventories and the Business Cycle: An Equilibrium Analysis of (S,s) Policies." *American Economic Review* 97 (4): 1165–88.

Khan, Aubhik, and Julia K. Thomas. 2008. "Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics." *Econometrica* 76 (2): 395–436.

Krusell, Per, and Anthony A. Smith, Jr. 1997. "Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns." *Macroeconomic Dynamics* 1 (2): 387–422.

Krusell, Per, and Anthony A. Smith, Jr. 1998. "Income and Wealth Heterogeneity in the Macroeconomy." *Journal of Political Economy* 106 (5): 867–96.

Kydland, Finn E., and Edward C. Prescott. 1982. "Time to Build and Aggregate Fluctuations." *Econometrica* 50 (6): 1345–70.

Pijoan-Mas, Josep. 2007. "Pricing Risk in Economies with Heterogeneous Agents and Incomplete Markets." *Journal of the European Economic Association* 5 (5): 987–1015.

Prescott, Edward C. 1986. "Theory Ahead of Business Cycle Measurement." *Federal Reserve Bank of Minneapolis Quarterly Review* 10 (4): 9–22.

Takahashi, Shuhei. 2013. "Time-Varying Wage Risk, Incomplete Markets, and Business Cycles." Unpublished.

Tauchen, George. 1986. "Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions." *Economics Letters* 20 (2): 177–81.

This article has been cited by:

1. Yongsung Chang, Sun-Bin Kim. 2014. Heterogeneity and Aggregation: Implications for Labor-Market Fluctuations: Reply. *American Economic Review* **104**:4, 1461-1466. [Abstract] [View PDF article] [PDF with links]