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Computing sunspot equilibria in linear rational expectations models

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Abstract

We provide a computationally simple method of analyzing the effects of fundamental and sunspot shocks in linear rational expectations models when the equilibrium is indeterminate. Under indeterminacy sunspots can affect model dynamics through endogenous forecast errors. Moreover, the effect of fundamental shocks on forecast errors is not uniquely determined. The solution method is illustrated with a New Keynesian dynamic stochastic equilibrium model that can be solved analytically. Under a passive interest-rate rule, the response of inflation to an unanticipated interest rate cut is ambiguous: there are some equilibria in which inflation increases and others in which prices fall.

JEL classification: C62; C63; E00

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1. Introduction

It is well known that linear rational expectations (LRE) models can have multiple equilibria. If the equilibrium is not unique it is possible to construct sunspot equilibria in which non-fundamental stochastic disturbances influence model dynamics. Under indeterminacy the realization of a sunspot variable can affect economic agents' beliefs. In response to these shocks, agents may adjust their behavior, which induces fluctuations that would not be present in a unique RE equilibrium. Moreover, the dynamic effects

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of fundamental shocks to agents' preferences and technologies are restricted, but not uniquely determined without further assumptions.

In this paper, we characterize the complete set of solutions to LRE models with indeterminacies and provide a method to compute them. The paper builds upon the approach advocated by Sims (2000) which in turn generalizes the methods by Blanchard and Kahn (1980), King and Watson (1998), and Klein (2000). Sims' approach is particularly convenient for our purposes since it exploits the notion of rational expectations forecast errors as a solution device. Roughly speaking, given a sequence of fundamental shocks to agents' preferences and technologies, a solution to the LRE model is a sequence of rational expectations forecast errors under which the endogenous variables do not explode. Under determinacy, the forecast errors are uniquely determined by the fundamental shocks. Under indeterminacy the forecast errors can be decomposed in two components, one is due to the fundamental shocks, and the other one is caused by sunspot shocks. Although all of the cited papers derive conditions for equilibrium non-uniqueness, none explicitly shows how to introduce sunspot shocks into a LRE model and compute the resulting equilibria.

Our paper may also be of interest to the growing literature on quantitative implications of sunspots on business cycle dynamics.[1] The paper provides a formal justification for the common practice to model uncertainty due to sunspots similar to fundamental shocks by including them in the set of exogenous driving processes. We argue that sunspots can be interpreted as triggers for belief shocks that lead agents to revise their forecasts of the endogenous variables.

The paper is organized as follows. While our analysis applies to any LRE model, we begin by introducing a simple log-linearized dynamic stochastic general equilibrium (DSGE) model in Section 2. This model is used to illustrate the general analysis that follows. Section 3 presents a canonical form for LRE models and defines the notation that is used throughout the paper. The full set of solutions of the canonical LRE model under indeterminacy is derived in Section 4. In Section 5, the solution method is applied to the New Keynesian monetary business cycle model introduced in Section 2. Due to its simplicity this model can be solved analytically. We show that in this model under a passive interest-rate rule the response of output and inflation to an unanticipated interest rate cut is ambiguous: while output rises, there are some equilibria in which inflation increases and others in which prices fall. Section 6 concludes.

2. A simple example

To illustrate the analysis of LRE models under indeterminacy we consider a simple New Keynesian monetary business cycle model of an economy with monopolistically competitive firms that face adjustment costs when changing prices.[2] Goods are produced with variable labor input only. Agents can smooth their consumption stream by

[1] Examples include Farmer and Guo (1994), Perli (1998), Schmitt-Grohé (1997, 2000), and Weder (2000). For a comprehensive survey of the indeterminacy literature see Benhabib and Farmer (1999).

[2] Further details on the model specification and derivation of the equilibrium relationships can be found in Lubik and Marzo (2001).

purchasing government bonds that pay a nominal rate of interest. The central bank affects the economy's equilibrium by means of an interest rate policy.

The linearized reduced form of this model is described by the following equations:

$$\mathbb{E}_t[\tilde{y}_{t+1}] + \sigma \mathbb{E}_t[\tilde{\pi}_{t+1}] = \tilde{y}_t + \sigma \tilde{R}_t, \tag{1}$$

$$\beta \mathbb{E}_t[\tilde{\pi}_{t+1}] = \tilde{\pi}_t - \kappa \tilde{y}_t, \tag{2}$$

$$\tilde{R}_t = \psi \tilde{\pi}_t + \varepsilon_t. \tag{3}$$

All variables are in log-deviations from a unique steady state, where $\tilde{y}$ is output, $\tilde{\pi}$ is inflation, and $\tilde{R}$ is the nominal interest rate. $\sigma > 0$, $\kappa > 0$, and $0 < \beta < 1$ are parameters. Eq. (1) is an intertemporal Euler equation, while (2) governs inflation dynamics and is derived from firms' optimal price-setting problem. The monetary authority uses rule (3) to adjust the nominal interest rate in response to changes in its inflation target. $\psi \geqslant 0$ measures the elasticity of the interest rate response. The economy is perturbed by a single fundamental shock ε_t, which satisfies $\mathbb{E}_{t-1}[\varepsilon_t] = 0$. Substituting (3) into (1) results in a two-equation system in $\tilde{y}$ and $\tilde{\pi}$ only.

The endogenous variables appearing in expectations are handled in the following way. We introduce endogenous forecast errors η_t^y and η_t^π by defining $\eta_t^y = \tilde{y}_t - \mathbb{E}_{t-1}[\tilde{y}_t]$ and $\eta_t^\pi = \tilde{\pi}_t - \mathbb{E}_{t-1}[\tilde{\pi}_t]$, respectively. This convention of introducing LRE forecast errors will facilitate the analysis of the model. Solutions are subsequently constructed by expressing the forecast errors as functions of fundamental shocks, such as the monetary policy shock ε_t, appearing in Eq. (3), and sunspot shocks, which do not affect agents' tastes and technologies, but may influence equilibrium allocations. Let $\xi_t^y = \mathbb{E}_t[\tilde{y}_{t+1}]$ and $\xi_t^\pi = \mathbb{E}_t[\tilde{\pi}_{t+1}]$. The New Keynesian model can be represented as four-dimensional LRE system that includes the conditional expectations ξ_t^y and ξ_t^π as endogenous variables:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \sigma \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \tilde{\pi}_t \\ \xi_t^y \\ \xi_t^\pi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \sigma\psi \\ 0 & 0 & -\kappa & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_{t-1} \\ \tilde{\pi}_{t-1} \\ \xi_{t-1}^y \\ \xi_{t-1}^\pi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sigma \\ 0 \end{bmatrix} \varepsilon_t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \sigma\psi \\ -\kappa & 1 \end{bmatrix} \begin{bmatrix} \eta_t^y \\ \eta_t^\pi \end{bmatrix}. \tag{4}$$

3. General form of the LRE model

Representation (4) of the monetary business cycle model in the previous section conforms with the canonical form of LRE models considered in this paper:

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \tag{5}$$

where y_t is an $n\times 1$ vector of endogenous variables, ε_t is a $l\times 1$ vector of exogenous, serially uncorrelated random disturbances,[3] and η_t is a $k\times 1$ vector of expectation errors, satisfying $\mathbb{E}_{t-1}[\eta_t]=0$ for all t. Γ_0 and Γ_1 are $n\times n$ coefficient matrices, while Ψ is $n\times l$ and Π is $n\times k$. Sims (2000) develops a solution algorithm for (5) subject to restrictions on the rate of growth of y_t.[4] Moreover, he derives conditions that ensure the existence and uniqueness of a stable solution. Roughly, a stable solution exists if one can choose the expectation errors η_t as a function of the exogenous shocks ε_t to eliminate explosive components of y_t. The solution is unique if the mapping from ε_t to η_t is one to one. In the case of non-uniqueness, his algorithm provides one particular solution for the LRE system. Sims (2000, p. 13) mentions that "[i]f one is interested in generating the full set of non-unique solutions, one has to add back in, as additional "disturbances", the components of $[\eta_t]$ left undetermined". In the following section we show how to derive the full set of solutions of the LRE model in the presence of indeterminacy.

4. Sunspot solutions

The canonical system can be transformed through a generalized complex Schur decomposition (QZ) of Γ_0 and Γ_1. There exist matrices Q, Z, Λ, and Ω, such that $Q'\Lambda Z'=\Gamma_0$, $Q'\Omega Z'=\Gamma_1$, $QQ'=ZZ'=I_{n\times n}$, and Λ and Ω are upper-triangular. The QZ-decomposition always exists. Let $w_t=Z'y_t$ and pre-multiply (5) by Q to obtain

$$\begin{bmatrix}\Lambda_{11} & \Lambda_{12}\\ 0 & \Lambda_{22}\end{bmatrix}\begin{bmatrix}w_{1,t}\\ w_{2,t}\end{bmatrix}=\begin{bmatrix}\Omega_{11} & \Omega_{12}\\ 0 & \Omega_{22}\end{bmatrix}\begin{bmatrix}w_{1,t-1}\\ w_{2,t-1}\end{bmatrix}+\begin{bmatrix}Q_{1.}\\ Q_{2.}\end{bmatrix}(\Psi\varepsilon_t+\Pi\eta_t). \tag{6}$$

The second set of equations can be rewritten as

$$w_{2,t}=\Lambda_{22}^{-1}\Omega_{22}w_{2,t-1}+\Lambda_{22}^{-1}Q_{2.}(\Psi\varepsilon_t+\Pi\eta_t). \tag{7}$$

Without loss of generality, we assume that the system is ordered and partitioned such that the $m\times 1$ vector $w_{2,t}$ is purely explosive, where $0\leqslant m\leqslant n$.

A non-explosive solution of the LRE model (5) for y_t exists if $w_{2,0}=0$,[5] and for every vector ε_t, one can find a vector η_t that offsets the impact of ε_t on $w_{2,t}$:

$$\underbrace{Q_{2.}\Psi}_{m\times l}\;\underbrace{\varepsilon_t}_{l\times 1}+\underbrace{Q_{2.}\Pi}_{m\times k}\;\underbrace{\eta_t}_{k\times 1}=\underbrace{0}_{m\times 1}. \tag{8}$$

[3] This assumption is not very restrictive. The laws of motion for serially correlated disturbances can always be appended to the system and the vector y_t expanded.

[4] Sims' solution method generalizes the method proposed by Blanchard and Kahn (1980) in two main directions. First, it allows treatment of singular Γ_0 cases. Second, it does not require the researcher to separate the list of endogenous variables y_t into 'jump' and 'predetermined' variables. It recognizes that it is the structure of the coefficient matrices that implicitly pins down the solution. Instead of imposing ex ante which *individual* variables are 'predetermined' Sims' algorithm determines endogenously the *linear combination* of variables that has to be 'predetermined' for a solution to exist.

[5] Decompose Z (obtained from the QZ decomposition of Γ_0 and Γ_1) into $[Z_{.1},Z_{.2}]$ such that $w_{2,t}=Z_{.2}'y_t$. The condition $w_{2,0}=0$ requires the linear combinations of y_0 associated with the unstable roots of the LRE system to be zero.

The vector η_t, however, need not be unique. For instance, if the number of expectation errors k exceeds the number of explosive components m, Eq. (8) does not provide enough restrictions to uniquely determine the elements of η_t. Hence, it is possible to introduce expectation errors that are unrelated to the fundamental uncertainty ε_t without destabilizing the system.

Sims (2000) showed that a necessary and sufficient condition for a stable solution to exist is that the column space of $Q_{2.}\Psi$ be contained in that of $Q_{2.}\Pi$. This condition generalizes the usual procedure of 'counting roots' as to which a solution exists if $k \geqslant m$. The generalization allows for the possibility of linearly dependent rows in $Q_{2.}\Pi$, that is we may have existence even if $m > k$. This insight is summarized in Lemma 1.

Lemma 1. *Statements* (i) *and* (ii) *are equivalent.*

(i) *For every* $\varepsilon_t \in \mathbb{R}^l$ *there exists an* $\eta_t \in \mathbb{R}^k$ *such that Eq.* (8) *is satisfied.*
(ii) *There exists a* (*real*) $k \times l$ *matrix* λ *such that* $Q_{2.}\Psi = Q_{2.}\Pi\lambda$.

Proof. Statement (ii) implies (i), because one can choose $\eta_t = -\lambda\varepsilon_t$. To show the converse, let the jth column of λ be equal to an η_t that solves Eq. (8) for $\varepsilon_t = I_{.j}$, where $I_{.j}$ is the jth column of the $l \times l$ identity matrix, $j = 1, \ldots, l$. $\square$

Subsequently, we will examine solutions to Eq. (8) in terms of the forecast errors η_t. Since the rows of the matrix $Q_{2.}\Pi$ are potentially linearly dependent, it is convenient to work with its singular value decomposition:[6]

$$Q_{2.}\Pi = \begin{bmatrix} U_{.1} & U_{.2} \end{bmatrix} \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{.1}' \\ V_{.2}' \end{bmatrix} = \underbrace{U}_{m\times m} \underbrace{D}_{m\times k} \underbrace{V'}_{k\times k} = \underbrace{U_{.1}}_{m\times r} \underbrace{D_{11}}_{r\times r} \underbrace{V_{.1}'}_{r\times k}, \tag{9}$$

where D_{11} is a diagonal matrix and U and V are orthonormal matrices. Thus, the m explosive components of y_t generate only $r \leqslant m$ restrictions for the expectation errors η_t. Under the assumption that there exists at least one stable solution, we can deduce from Lemma 1 that there exists a $k \times l$ matrix λ such that the stability condition can be rewritten as

$$\underbrace{U_{.1}D_{11}}_{m\times r} \left(\underbrace{V_{.1}'\lambda\varepsilon_t + V_{.1}'\eta_t}_{r\times 1} \right) = \underbrace{0}_{m\times 1}. \tag{10}$$

The relationship provides r equations to determine the k-dimensional vector of forecast errors η_t.

[6] Singular value decomposition procedures implemented in matrix-oriented software packages such as GAUSS and MATLAB usually return the matrices U, D, and V.

4.1. The full set of stable solutions

We assume that the forecast errors are a linear function of the fundamental shocks ε_t and a $p \times 1$ vector of sunspot shocks ζ_t with the property $\mathbb{E}_{t-1}[\zeta_t] = 0$. In the context of the model presented in Section 2 the sunspot shocks do not affect agents' preferences or production technologies. Nevertheless, they potentially affect equilibrium dynamics through the forecast errors η_t. Let

$$\eta_t = A_1\varepsilon_t + A_2\zeta_t, \tag{11}$$

where A_1 is a $k \times l$ matrix and A_2 is a $k \times p$ matrix. Substituting (11) into (10) yields the following stability condition:

$$U_{.1}D_{11}(V_{.1}'\lambda + V_{.1}'A_1)\varepsilon_t + U_{.1}D_{11}V_{.1}'A_2\zeta_t = 0 \tag{12}$$

for all values of ε_t and ζ_t.

In order to satisfy (12) for all ζ_t it is necessary that A_2 is orthogonal to $V_{.1}$. The orthogonal space of $V_{.1}$ is spanned by the columns of the $k \times (k-r)$ matrix $V_{.2}$ which is obtained as a byproduct of the singular value decomposition of $Q_{2.}\Pi$. Hence, $A_2 = V_{.2}M_2$, where M_2 is a $(k-r) \times p$ matrix that does not depend on the coefficients of the LRE system. If $k = r$ then the null space of $V_{.1}$ is empty and $A_2 = 0$.

Since V is orthonormal $V_{.1}V_{.1}' + V_{.2}V_{.2}' = I_{k\times k}$. Without loss of generality the columns of A_1 can be expressed as linear combinations of the columns of $V_{.1}$ and $V_{.2}$:

$$A_1 = V_{.1}\tilde{A}_1 + V_{.2}M_1, \tag{13}$$

where the $r \times l$ matrix $\tilde{A}_1 = V_{.1}'A_1$ and the $(k-r) \times l$ matrix $M_1 = V_{.2}'A_1$. The stability condition (12) requires that

$$0 = U_{.1}D_{11}(V_{.1}'\lambda + V_{.1}'A_1) = U_{.1}D_{11}(V_{.1}'\lambda + \tilde{A}_1). \tag{14}$$

The second equality holds because $V_{.1}'V_{.1} = I_{r\times r}$ and $V_{.1}'V_{.2} = 0$. We deduce that

$$\tilde{A}_1 = -V_{.1}'\lambda. \tag{15}$$

Since U is orthonormal and $U_{.1}'U_{.1} = I_{r\times r}$, it is straightforward to verify that $V_{.1}'\lambda$ is uniquely determined by Condition (ii) of Lemma 1:

$$V_{.1}'\lambda = D_{11}^{-1}U_{.1}'Q_{2.}\Psi. \tag{16}$$

Thus, we proved the following proposition.

Proposition 1. *Let ζ_t be a $p \times 1$ vector of sunspot shocks with the property $\mathbb{E}_{t-1}[\zeta_t] = 0$. Suppose that condition (i) of Lemma 1 is satisfied. The full set of solutions for the forecast errors in the LRE model (5) is*

$$\eta_t = (-V_{.1}D_{11}^{-1}U_{.1}'Q_{2.}\Psi + V_{.2}M_1)\varepsilon_t + V_{.2}M_2\zeta_t, \tag{17}$$

where M_1 is a $(k-r) \times l$ matrix and M_2 is a $(k-r) \times p$ matrix. If $k = r$ the second and third term drop out and the solution is unique.

Indeterminacy arises if the dimension k of the vector of forecast errors exceeds the number of stability restrictions r. According to Proposition 1 indeterminacy has two consequences. First, sunspot fluctuations ζ_t can influence equilibrium dynamics if $M_2 \neq 0$. Proposition 1 does not restrict the elements of the matrix M_2. While the effect of the sunspot shock is not uniquely determined, it does perturb the forecast errors in the directions of the columns of $V_{.2}$. The $(k-r) \times 1$ vector $\zeta_t^* = M_2\zeta_t$ can be regarded as a reduced form sunspot shock. The dimension of ζ_t^* automatically matches the degree of indeterminacy in the model. The solution for the expectation errors η_t can be substituted into the LRE model (5). The contemporaneous impact of the reduced form sunspot shocks onto the endogenous variables is given by

$$\frac{\partial \Gamma_0 y_t}{\partial \zeta_t^{*\prime}} = \underbrace{\Pi V_{.2}}_{n\times(k-r)}. \tag{18}$$

The columns of $\Pi V_{.2}$ contain the directions in which sunspot shocks can perturb the endogenous variables.

Second, the effect of fundamental shocks ε_t on the endogenous variables

$$\frac{\partial \Gamma_0 y_t}{\partial \varepsilon_t'} = (I - \Pi V_{.1} D_{11}^{-1} U_{.1}' Q_{2.})\Psi + \Pi V_{.2} M_1 \tag{19}$$

is not uniquely determined, as the elements of the matrix M_1 are not pinned down by the structure of the LRE model. If M_1 is arbitrarily set equal to zero, then the component of the forecast error due to the fundamental shocks, $A_1\varepsilon_t$, is orthogonal to the contribution of the sunspot shocks, $A_2\zeta_t$. Any algorithm that delivers the matrices $-V_{.1}D_{11}^{-1}U_{.1}'Q_{2.}\Psi$ and a $k \times (k-r)$ matrix that spans the column space of $V_{.2}$ can be used to compute the full set of stable solutions of the LRE model.

4.2. Belief shocks

The derivation of the full set of stable solutions did not provide any economic intuition about how sunspot shocks might influence equilibrium dynamics under indeterminacy. We will subsequently formalize the following interpretation: sunspots trigger belief shocks ζ_t^b that lead to a revision of forecasts. Consider representation (4) of the New Keynesian model that includes the conditional expectations ξ_t^y and ξ_t^π in the vector of endogenous variables. Recall that, for instance,

$$\tilde{y}_t = \xi_{t-1}^y + \eta_t^y, \tag{20}$$

where η_t^y is the forecast error between period $t-1$ and t. Suppose that based on a sunspot the output expectation is revised between $t-1$ and t by ζ_t^y. Thus,

$$\tilde{y}_t = (\xi_{t-1}^y + \zeta_t^y) + \tilde{\eta}_t^y, \tag{21}$$

where $\xi_{t-1}^y + \zeta_t^y$ corresponds to the revised forecast and $\tilde{\eta}_t^y$ is the error of this revised forecast. Under indeterminacy the belief shock ζ_t^y may affect allocation and prices in the model economy.

More generally, let ζ_t^b be a k-dimensional vector of belief shocks that is of the same dimension as η_t and has the property $\mathbb{E}_{t-1}[\zeta_t^b] = 0$. The LRE system is now of the form

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi\varepsilon_t + \Pi(\tilde{\eta}_t + \zeta_t^b) = \Gamma_1 y_{t-1} + \begin{bmatrix} \Psi & \Pi \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \zeta_t^b \end{bmatrix} + \Pi\tilde{\eta}_t. \tag{22}$$

The belief shocks are regarded as exogenous and will be treated in the same way as the fundamental shocks ε_t. The stability condition (8) changes to

$$[\, Q_{2.}\Psi \quad Q_{2.}\Pi \,] \begin{bmatrix} \varepsilon_t \\ \zeta_t^b \end{bmatrix} + Q_{2.}\Pi\tilde{\eta}_t = 0. \tag{23}$$

Using Proposition 1 we now express the errors $\tilde{\eta}_t$ of the revised forecast as a function of the fundamental shocks and the belief shocks to ensure stability:

$$\tilde{\eta}_t = -V_{.1}D_{11}^{-1}U_{.1}'[\, Q_{2.}\Psi \quad Q_{2.}\Pi \,] \begin{bmatrix} \varepsilon_t \\ \zeta_t^b \end{bmatrix} + V_{.2} \begin{bmatrix} M_1^{\varepsilon}\varepsilon_t \\ M_1^{\zeta}\zeta_t^b \end{bmatrix}.$$

The overall forecast error is

$$\eta_t = \tilde{\eta}_t + \zeta_t = (-V_{.1}D_{11}^{-1}U_{.1}'Q_{2.}\Psi + V_{.2}M_1^{\varepsilon})\varepsilon_t + V_{.2}(V_{.2}' + M_1^{\zeta})\zeta_t^b, \tag{24}$$

since $Q_{2.}\Pi = U_{.1}D_{11}V_{.1}'$ and the orthogonality of V implies that $I - V_{.1}V_{.1}' = V_{.2}V_{.2}'$.

Thus, the notion of a belief shock allows us to generate the full set of stable solutions to the LRE system. Under determinacy $V_{.2} = 0$ and the belief shock ζ_t^b does not affect the dynamics of the system. In the case of indeterminacy the effect of the belief shock is ambiguous due to the presence of the matrix M_1^{ζ}. One can only deduce that ζ_t^b perturbs the forecast errors η_t in the direction of the columns of $V_{.2}$. Since the dimension k of ζ_t is usually greater than the degree of indeterminacy $k - r$ different realizations of the belief shock can generate the same equilibrium dynamics. For instance, the New Keynesian model has a one-dimensional indeterminacy under a passive monetary policy. Ex post it is not possible to distinguish a sunspot shock that led to a revision of the inflation forecast from one that altered the output forecast.

Despite this identification problem, the notion of a belief shock that leads to a forecast revision can help us provide some economic intuition about equilibrium dynamics under indeterminacy. From a computational perspective the use of belief shocks is attractive because solution (24) with $M_1^{\varepsilon} = 0$ and $M_1^{\zeta} = 0$ can be directly obtained with the algorithm described in Sims (2000).

5. The simple example revisited

While in most cases the solution of LRE systems can only be obtained numerically, the New Keynesian model presented in Section 2 can be solved analytically. To derive the solution one can exploit the block-triangular structure of representation (4) and calculate the expectation errors based on the two-dimensional subsystem for the conditional expectations $\xi_t = [\xi_t^y, \xi_t^\pi]'$:

$$\xi_t = \underbrace{\begin{bmatrix} 1 + \dfrac{\kappa\sigma}{\beta} & \sigma(\psi - \dfrac{1}{\beta}) \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} \end{bmatrix}}_{\Gamma_1^*} \xi_{t-1} + \underbrace{\begin{bmatrix} \sigma \\ 0 \end{bmatrix}}_{\Psi^*} \varepsilon_t + \underbrace{\begin{bmatrix} 1 + \dfrac{\kappa\sigma}{\beta} & \sigma(\psi - \dfrac{1}{\beta}) \\ -\dfrac{\kappa}{\beta} & \dfrac{1}{\beta} \end{bmatrix}}_{\Pi^*} \eta_t. \tag{25}$$

For this simple system the Jordan decomposition suffices to find the unstable components. Replace Γ_1^* by its Jordan decomposition $J\Lambda J^{-1}$, define $w_t = J^{-1}\xi_t$, and rewrite the model as

$$w_t = \Lambda w_{t-1} + J^{-1}\Psi^*\varepsilon_t + J^{-1}\Pi^*\eta_t. \tag{26}$$

Straightforward algebra yields the eigenvalues that appear in the diagonal matrix Λ:

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(1 + \frac{\kappa\sigma + 1}{\beta}\right) \mp \frac{1}{2}\sqrt{\left(\frac{\kappa\sigma + 1}{\beta} - 1\right)^2 + \frac{4\kappa\sigma}{\beta}(1 - \psi)}. \tag{27}$$

It can be shown[7] that the determinacy properties of this model hinge solely on the policy parameter ψ. If $\psi > 1$ then both λ_1 and λ_2 are greater than one in absolute value and the stable solution is unique. If $0 \leqslant \psi_1 < 1$ the system has only one unstable root and there exist multiple stable solutions.

Subsequently, we will apply the solution method discussed in Section 4 to derive representations for the forecast errors η_t. Since $\tilde{y}_t = \xi_{t-1}^y + \eta_t^y$ and $\tilde{\pi}_t = \xi_{t-1}^\pi + \eta_t^\pi$ the matrix A_1 can be interpreted as the initial effect of an unanticipated change ε_t of the interest rate on output and inflation. Likewise A_2 is the initial impact of the sunspot shock ζ_t.

5.1. Determinacy

In a determinate equilibrium the only stable solution of (25) is $\xi_t = 0 \ \forall t$ which is obtained if $\xi_0 = 0$ and

$$\Psi^*\varepsilon_t + \Pi^*\eta_t = 0. \tag{28}$$

Note that this is the equivalent of Eq. (8) in the general case for $k = r = m = 2$. The expectation errors η_t are then uniquely determined as functions of the structural shock ε_t:

$$\eta_t = \begin{bmatrix} -\dfrac{\sigma}{1 + \kappa\sigma\psi} \\ -\dfrac{\kappa\sigma}{1 + \kappa\sigma\psi} \end{bmatrix} \varepsilon_t. \tag{29}$$

The equilibrium dynamics are not affected by sunspot shocks. Since $\xi_t = 0$ the evolution of output and inflation is given by

$$\begin{bmatrix} \tilde{y}_t \\ \tilde{\pi}_t \end{bmatrix} = -\frac{\sigma}{1 + \kappa\sigma\psi}\begin{bmatrix} 1 \\ \kappa \end{bmatrix} \varepsilon_t. \tag{30}$$

The model exhibits no dynamics because the solution suppresses the two roots of the autoregressive matrix Γ_1^*. An unanticipated monetary contraction leads to a one-period fall in output and inflation. The parameter κ determines the inflation–output trade-off. After the impact of the shock both variables return to their initial values.

[7] See Bullard and Mitra (2002) or Lubik and Marzo (2001).

5.2. Indeterminacy

If the inflation elasticity of the interest rate rule is less than one, then the eigenvalue $\lambda_1 < 1$, and only the second element of the 2×1 vector w_t is explosive. Since $r = m = 1 < k$ there is one-dimensional indeterminacy. Let $[B]_{2.}$ denote the second row of a 2×2 matrix B. Then the stability condition for Eq. (26) is of the form

$$[J^{-1}\Psi^*]_{2.}\varepsilon_t + [J^{-1}\Pi^*]_{2.}\eta_t = 0, \tag{31}$$

which can be rewritten as

$$-\kappa\sigma\varepsilon_t - \kappa\lambda_2\eta_t^y + [\lambda_2 - 1 - \kappa\sigma\psi]\eta_t^\pi = 0. \tag{32}$$

The solution suppresses only one root of the matrix of autoregressive coefficients Γ_1^*. Hence, unlike in the determinacy case, the effect of shocks on output and inflation typically does not vanish after one period.

In Section 4 we proceeded by computing a singular value decomposition for the impact matrix of the forecast errors on the unstable component of the LRE system. In the context of this example it corresponds to decomposing

$$[J^{-1}\Pi^*]_{2.} = 1[d \quad 0]\begin{bmatrix} -\dfrac{\kappa\lambda_2}{d} & \dfrac{\lambda_2 - 1 - \kappa\sigma\psi}{d} \\ \dfrac{\lambda_2 - 1 - \kappa\sigma\psi}{d} & \dfrac{\kappa\lambda_2}{d} \end{bmatrix}, \tag{33}$$

where $d = \sqrt{(\kappa\lambda_2)^2 + (\lambda_2 - 1 - \kappa\sigma\psi)^2}$. According to Proposition 1 the full set of solutions for the forecast errors is of the form

$$\eta_t = -\frac{\kappa\sigma}{d^2}\begin{bmatrix} \kappa\lambda_2 \\ -(\lambda_2 - 1 - \kappa\sigma\psi) \end{bmatrix}\varepsilon_t + \frac{1}{d}\begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix}(M_1\varepsilon_t + \zeta_t^*), \tag{34}$$

where M_1 is 1×1 and ζ_t^* is a one-dimensional reduced form sunspot shock. It can be shown that both λ_2 and $\lambda_2 - 1 - \kappa\sigma\psi$ are positive in the indeterminacy region for all values $\kappa \geq 0$, $\sigma \geq 0$. Thus, a positive realization of the reduced form sunspot shock ζ_t^* increases both output and inflation.

The presence of M_1 reflects the fact that the effects of the fundamental shocks on output, inflation, and interest rates are ambiguous under indeterminacy. To illustrate the variety of dynamics that can arise if $\psi < 1$, we consider two particular choices of M_1. First, suppose that the contributions of ε_t and ζ_t^* to the forecast error η_t are orthogonal, that is, $M_1 = 0$. In this case, an unanticipated interest rate cut reduces inflation.

Second, notice that the solution to Eq. (28), obtained under determinacy, also provides a relationship between fundamental shocks and expectation errors that satisfies Eq. (31):

$$\eta_t = -\frac{\sigma}{1 + \kappa\sigma\psi}\begin{bmatrix} 1 \\ \kappa \end{bmatrix}\varepsilon_t + \frac{1}{d}\begin{bmatrix} \lambda_2 - 1 - \kappa\sigma\psi \\ \kappa\lambda_2 \end{bmatrix}\zeta_t^*. \tag{35}$$

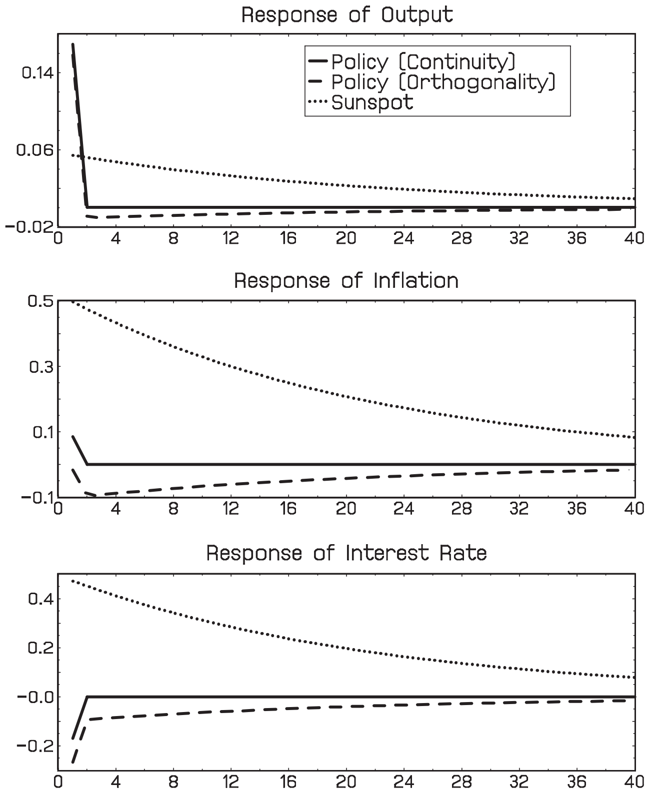

Fig. 1. Impulse responses of output, inflation, and interest rate to an unanticipated 25-basis-point interest rate cut and a reduced form sunspot shock (scaled by 0.5, dotted lines) under two assumptions on the rational expectations forecast errors: 'continuity' (—— lines) and 'orthogonality' (- - - lines). The parameters are $\beta = 0.99$, $\kappa = 0.5$, $\psi = 0.95$, and $\sigma = 1$.

This particular solution corresponds to

$$M_1 = \frac{\sigma}{d}\left(1 - \frac{\lambda_2(1+\kappa^2)}{1+\kappa\sigma\psi}\right)$$

and preserves some of the properties that the model exhibits under determinacy. Both output and inflation rise in response to an unanticipated interest rate reduction. The response of output and inflation to the fundamental shock has no persistence because by construction it does not affect the conditional expectations ξ_t^y and ξ_t^π. Moreover, the impulse response function of the endogenous variables with respect to ε_t are continuous in the parameter ψ at the boundary between the determinacy and the indeterminacy region.

We summarize these results by plotting impulse response functions for output, inflation, and interest rates in Fig. 1. We choose the parameter values $\beta = 0.99$, $\kappa = 0.5$, $\psi = 0.95$, and $\sigma = 1$. The magnitude of the unanticipated interest rate cut ε_t is 25 basis points. The following two solutions are considered: 'orthogonality' ($M_1 = 0$ in Eq. (34))

and 'continuity' (35). For both choices of M_1 output rises in response to the policy shock by about 0.15 percent. Under the solution that imposes continuity output returns to its steady state immediately. According to the orthogonality solution output becomes slightly negative after one period and returns slowly to its steady state afterwards.

Most striking is the difference among the inflation responses. Whereas under the continuity solution inflation rises in response to an interest rate cut, prices fall under the $M_1 = 0$ solution. In both equilibria that are illustrated in Fig. 1 the interest rate falls in response to the policy shock. Fig. 1 also depicts the response of output, inflation, and interest rates to a sunspot shock. All three variables move in the same direction. The effect on inflation and interest rates is about 10 times as strong as on output. Thus, sunspot shocks in this model mostly contribute to the variation of inflation and interest rates.

6. Conclusion

In this paper, we provide a method for quantitative theorists to study the implications of sunspot shocks in stochastic general equilibrium models when the rational expectations equilibrium is not unique. We show that sunspot equilibria can be computed by decomposing endogenous forecast errors into two parts. One part is due to fundamental shocks to agents' preferences and the available production technologies. The other component is caused by sunspot shocks.

We characterize the full set of sunspot equilibria in the presence of indeterminacies. Our analysis shows that indeterminacy has two consequences. First, sunspot shocks can influence equilibrium allocations and prices in the economy. Second, the dynamic effects of the fundamental shocks on the endogenous variables are not uniquely identified without further assumptions. In the context of the New Keynesian DSGE model we considered the assumption that endogenous expectation formation does not abruptly change as the economy moves across the boundary between the determinacy and the indeterminacy region. We believe that this last assumption is intuitively very appealing and could serve as a benchmark to which other equilibria can be compared.

This paper might also of interest to empirical researchers. When estimating dynamic stochastic general equilibrium models, it is common practice to restrict the parameter space to regions where indeterminacy does not occur. But since the possibility of indeterminacy is an integral feature of LRE models, this could lead to serious model misspecification. Our results enable researchers to properly account for indeterminacies in empirical models by computing the full set of solutions to the LRE system and specifying a joint probability distribution for the fundamental and sunspot shocks. Such an approach is pursued in Lubik and Schorfheide (2002).

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