Topics in Computational Economics

Lecture 11

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Today's Lecture

Topics:

- Measure theory
- Integration
- The L_p spaces

References:

- Cheney, Analysis for Applied Mathematics, Chapter 8
- Pollard, A Users Guide to Measure Theoretic Probability
- Cinlar, Probability and Stochastics

Motivation

Why measure theory?

Example. Consider the symbol $\mathbb{E}$ here:

$$\max \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to

$$a_{t+1} = R(a_t - c_t) + W_{t+1}, \quad a_0 \text{ given}$$

To define $\mathbb{E}$ we need measure theory

Example. Consider the space $cb\mathbb{R}^S$ with $S = [a, b]$ and metric

$$d(f, g) = \int_a^b |f(x) - g(x)|dx$$

This is indeed a metric space

But it is **not** complete

Moreover, the ordinary Riemann integral is not very general

- What if we want to change S to some abstract set?
- How can we deal with more general functions?

To generalize we use measure theory

Prequel: Definition of a σ-algebra

Let S be any set

A nonempty family of subsets $\mathscr{S}$ of S is called a σ-**algebra** of S if

1. $A \in \mathscr{S} \implies A^c \in \mathscr{S}$
2. $A_1, A_2, \ldots \subset \mathscr{S} \implies \cup_n A_n \in \mathscr{S}$

Implication: $\emptyset$ and S are both in $\mathscr{S}$ (why?)

Are the following families σ-algebras on $\mathbb{R}^n$?

1. $\mathscr{S} := \{\emptyset, \mathbb{R}^n\}$
2. $\mathscr{S} :=$ set of all subsets of $\mathbb{R}^n$
3. $\mathscr{S} := \{\emptyset, B, B^c, \mathbb{R}^n\}$ for some $B \subset \mathbb{R}^n$
4. $\mathscr{S} :=$ all rectangles in $\mathbb{R}^n$
5. $\mathscr{S} :=$ all open subsets of $\mathbb{R}^n$
6. $\mathscr{S} :=$ all compact subsets of $\mathbb{R}^n$

Generated σ-algebras

Let S be any set and let $\mathscr{A}$ be a set of subsets

The **σ-algebra generated by $\mathscr{A}$** is the smallest σ-algebra containing $\mathscr{A}$

- The intersection of all σ-algebras on S containing $\mathscr{A}$

Well defined, typically written as $\sigma(\mathscr{A})$

Example. If $\mathscr{A} := \{B, S\}$, then $\sigma(\mathscr{A}) = \{\emptyset, B, B^c, S\}$

Facts

- If $\mathscr{A} \subset \mathscr{B}$, then $\sigma(\mathscr{A}) \subset \sigma(\mathscr{B})$
- If $\mathscr{A}$ is a σ-algebra, then $\sigma(\mathscr{A}) = \mathscr{A}$

The Borel Sets

Let S be a metric space

The **Borel** subsets of S are defined as $\sigma(\mathscr{O})$, the σ-algebra generated by the open subsets of S

- write $\mathscr{B}$ or $\mathscr{B}(S)$

When $S = \mathbb{R}^n$, the set $\mathscr{B}$ is very large, containing

- all open sets
- all closed sets
- all rectangles
- all compact sets

Ex. Show that every countable subset of a metric space S is Borel measurable

Summary: Borel sets contain almost every set you'll every use in analysis

Ex. Show that $\mathscr{B}(S)$ can also be defined as $\sigma(\mathscr{C})$ where $\mathscr{C}$ is

- the closed subsets of S
- the compact subsets of S, when S is σ-finite
- then open intervals, when $S = \mathbb{R}$

Lebesgue Measure

How to assign a "natural" measure $\lambda(B)$ of "size" to $B \subset \mathbb{R}^n$

- length in $\mathbb{R}$, area in $\mathbb{R}^2$, volume in $\mathbb{R}^3$, etc.

Some sets are easy to measure

- For a rectangle, area is the product of the sides
- For a circle, area is πr^2

But how about arbitrary subsets?

Maybe we can bootstrap up from rectangles by approximation?

Let $\mathscr{J}$ be the set of rectangles in $\mathbb{R}^n$ of the form

$$I = \times_{i=1}^{n}(a_i, b_i] \quad \text{for some} \quad a_i, b_i \in \mathbb{R}$$

Let λ be the map from $\mathscr{J}$ into $\mathbb{R}$ defined by

$$\lambda\left(\times_{i=1}^{n}(a_i, b_i]\right) = \prod_{i=1}^{n}(b_i - a_i)$$

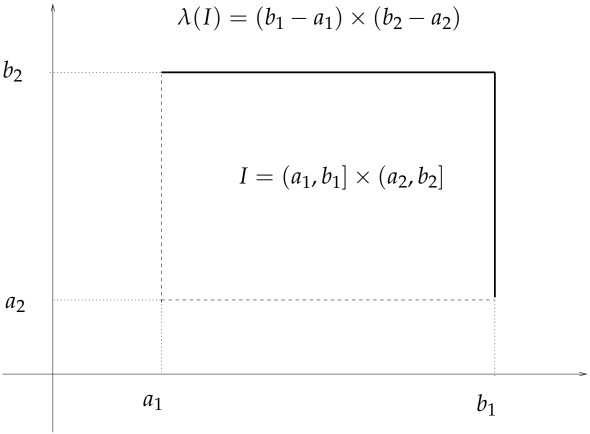


Figure: The measure of a rectangle $I \subset \mathbb{R}^2$

We want to extend the domain of λ to arbitrary subsets of $\mathbb{R}^n$

So now let's taken **any** $A \subset \mathbb{R}^n$

Natural idea:

$$\lambda(A) := \inf \sum_{k=1}^{\infty} \lambda(I_k) \tag{1}$$

subject to

- $I_k \in \mathscr{J}$ for all $k \in \mathbb{N}$
- $\cup_{k=1}^{\infty} I_k \supset A$

Called **Lebesgue outer measure**

A set function from all subsets of $\mathbb{R}^n$ to $[0, \infty]$

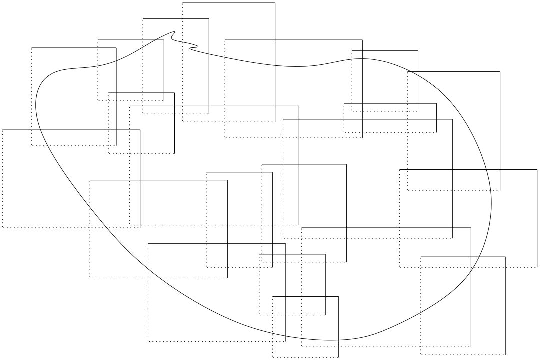


Figure: A covering of A by elements of $\mathscr{J}$

Example. If $C \subset \mathbb{R}^n$ is countable, then $\lambda(C) = 0$

Proof:

Let $C = \{x_k\}_{k\in\mathbb{N}}$, with $x_k \in \mathbb{R}^n$ for all k

Fix $\epsilon > 0$

Let I_k be a rectangle containing x_k with area $\leqslant \epsilon 2^{-k}$

Then $\sum_{k=1}^{\infty} \lambda(I_k) \leqslant \epsilon$

$$\therefore \quad 0 \leqslant \lambda(\{x_k\}_{k\in\mathbb{N}}) \leqslant \epsilon \qquad \forall \epsilon > 0$$

$$\therefore \quad \lambda(\{x_k\}_{n\in\mathbb{N}}) = 0$$

Good news about λ:

- Nonnegative, with $\lambda(\emptyset) = 0$
- Agrees with the usual notion of "size" for rectangles, triangles, circles, spheres, etc.
- Translation invariant:

$$\lambda(A + x) = \lambda(A) \quad \forall\, x \in \mathbb{R}^n, A \subset \mathbb{R}^n$$

- Monotone ($A \subset B$ implies $\lambda(A) \leqslant \lambda(B)$)
- $\lambda(A) = 0$ implies $\lambda(A \cup B) = \lambda(B)$

Bad news about λ:

- Additivity fails!

Theorem. (Vitali, 1905) There exists sets $A, B \subset \mathbb{R}$ such that

$$A \cap B = \emptyset \quad \text{and} \quad \lambda(A) + \lambda(B) > \lambda(A \cup B)$$

But surely the whole should equal sum of its parts

So must λ be abandoned?

H. Lebesgue: Keeps λ but restrict its domain to well behaved sets

One such class of sets is the Borel sets...

Theorem. For $A, B \in \mathscr{B}(\mathbb{R}^n)$, we have

$$A \cap B = \emptyset \implies \lambda(A \cup B) = \lambda(A) + \lambda(B)$$

In fact, on $\mathscr{B}$, Lebesgue outer measure is not just additive but **countably additive**:

$$\{A_n\} \subset \mathscr{B}, \text{ disjoint} \implies \lambda(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \lambda(A_n)$$

When restricted to $\mathscr{B}$, Lebesgue outer measure λ is simply called **Lebesgue measure**

The Lebesgue Integral

The Riemann integral lets us integrate continuous functions on $\mathbb{R}^n$

We want an integral that

1. agrees with the Riemann integral on continuous functions
2. can integrate lots of other functions too
3. can extend to more abstract settings

Borel Measurable Functions

We call $f\colon \mathbb{R}^n \to \mathbb{R}$ **Borel measurable** and write $f \in m\mathscr{B}$ if

$$f^{-1}(B) \in \mathscr{B}, \quad \forall\, B \in \mathscr{B}$$

A necessary and sufficient condition is

$$f^{-1}(G) \in \mathscr{B}, \quad \forall \text{ open } G \subset \mathbb{R}$$

Example.

- Every continuous function is Borel measurable
- But so are lots of other functions, like $\mathbb{1}_{\mathbb{Q}}$

Proof that

$$f^{-1}(G) \in \mathscr{B},\ \forall G \in \mathscr{O} \quad \Longrightarrow \quad f^{-1}(B) \in \mathscr{B},\ \forall B \in \mathscr{B}$$

Let $\mathscr{M} := \{A \in \mathbb{R} : f^{-1}(A) \in \mathscr{B}\}$

The set $\mathscr{M}$ is a σ-algebra. For example, $A \in \mathscr{M}$ implies $A^c \in \mathscr{M}$ because

$$A \in \mathscr{M} \implies f^{-1}(A) \in \mathscr{B} \implies (f^{-1}(A))^c = f^{-1}(A^c) \in \mathscr{B}$$

Moreover, by assumption $\mathscr{O} \subset \mathscr{M}$

$$\therefore \quad \mathscr{B} := \sigma(\mathscr{O}) \subset \sigma(\mathscr{M}) = \mathscr{M}$$

This is exactly what we wish to show

Like continuity, measurability is preserved under simple algebraic operations

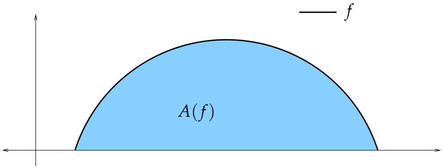
- $f \in \mathscr{B}$ and $\alpha \in \mathbb{R} \implies \alpha f \in \mathscr{B}$
- $f, g \in \mathscr{B} \implies f + g \in \mathscr{B}$
- $f, g \in \mathscr{B} \implies f * g \in \mathscr{B}$
- $f \in \mathscr{B}$ and $f > 0 \implies 1/f \in \mathscr{B}$

Unlike continuity, measurability is preserved under pointwise limits

- $f_n \in \mathscr{B}$ for all n and $f_n \to f$ pointwise on $S \implies f \in \mathscr{B}$

Given nonnegative $f \in m\mathscr{B}$, let

$$A(f) := \left\{ (x, y) \in \mathbb{R}^{n+1} \mid 0 < y < f(x) \right\}$$



Fact. f Borel measurable $\implies A(f)$ is a Borel set in $\mathbb{R}^{n+1}$

The **Lebesgue integral** of f is

$$\int f \,\mathrm{d}\lambda := \lambda(A(f))$$

Notes

- There are other (equivalent) ways to define $\int f \,\mathrm{d}\lambda$
- $\int f \,\mathrm{d}\lambda$ sometimes written $\lambda(f)$

Properties

1. $A \in \mathscr{B} \implies \int \mathbb{1}_A \,\mathrm{d}\lambda = \lambda(A)$
2. $f \leqslant g \implies \int f \,\mathrm{d}\lambda \leqslant \int g \,\mathrm{d}\lambda$
3. $\int (\alpha f + \beta g) \,\mathrm{d}\lambda = \alpha \int f \,\mathrm{d}\lambda + \beta \int g \,\mathrm{d}\lambda$
4. $f_n \uparrow f$ pointwise $\implies \int f_n \,\mathrm{d}\lambda \uparrow \int f \,\mathrm{d}\lambda$

So far we've only discussed how to integrate nonnegative $f \in m\mathscr{B}$

So now take any $f \in m\mathscr{B}$

Decompose as $f = f^+ - f^-$

Fact. $f \in m\mathscr{B} \implies f^+ \in m\mathscr{B}$ and $f^- \in m\mathscr{B}$

Define

$$\int f \, \mathrm{d}\lambda := \int f^+ \, \mathrm{d}\lambda - \int f^- \, \mathrm{d}\lambda$$

This is the general Lebesgue integral on $\mathbb{R}^n$

Integration over a subset $B \in \mathscr{B}(\mathbb{R}^n)$ is defined by

$$\int_B f \, \mathrm{d}\lambda := \int \mathbb{1}_B f \, \mathrm{d}\lambda$$

Theorem. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $a \leqslant b$, then

$$\int \mathbb{1}_{[a,b]} f \, \mathrm{d}\lambda = \int_a^b f(x) dx$$

where the right hand side is the regular Riemann integral

Abstract Measure

In probability theory, $\mathbb{P}(A)$ means "the probability of A"

Viewed as a function, $\mathbb{P}$ resembles Lebegue measure:

- a set function
- nonnegative
- $\mathbb{P}(\emptyset) = 0$
- A and B disjoint implies $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

So let's define **abstract measure** with λ, $\mathbb{P}$ as special cases

Let S be a set and let $\mathscr{S}$ be a σ-algebra on S

A **measure** μ on $(S, \mathscr{S})$ is a function from $\mathscr{S}$ to $[0, \infty]$ with the properties

(a) $\mu(\emptyset) = 0$

(b) μ is countably additive

The triple $(S, \mathscr{S}, \mu)$ is called a **measure space**

Ex.

- Show that if $\mu(S) < \infty$, then (b) $\implies$ (a)
- Show that if $E, F \in \mathscr{S}$ and $E \subset F$, then $\mu(E) \leqslant \mu(F)$

A measure P on $(S, \mathscr{S})$ is called a

- **probability measure** if $P(S) = 1$
- **Borel probability measure** if $P(S) = 1$ and $\mathscr{S} = \mathscr{B}(S)$

Theorem There is a one-to-one correspondence between

1. the collection of all CDFs on $\mathbb{R}$
2. the set of all Borel probability measures on $\mathbb{R}$

If P is a Borel probability measure, then the corresponding CDF is

$$F(x) = P((-\infty, x]) \qquad (x \in \mathbb{R})$$

Abstract Integral

Let

- S be a set
- $\mathscr{S}$ be a σ-algebra on S

A function $f: S \to \mathbb{R}$ called $\mathscr{S}$-**measurable** if

$$f^{-1}(G) \in \mathscr{S}, \quad \forall \text{ open } G \subset \mathbb{R}$$

Notation:

- $m\mathscr{S} :=$ all such functions
- $m\mathscr{S}^+ :=$ the nonnegative functions in $m\mathscr{S}$

Theorem For each measure μ on $(S, \mathscr{S})$ there exist a unique functional $L\colon m\mathscr{S}^+ \to [0, \infty]$ such that

1. $L\mathbb{1}_A = \mu(A)$ for all $A \in \mathscr{S}$
2. $f \leqslant g \implies Lf \leqslant Lg$
3. $L(\alpha f + \beta g) = \alpha Lf + \beta Lg$
4. $f_n \uparrow f$ pointwise on $S \implies Lf_n \uparrow Lf$ pointwise on S

The value Lf is also written as

$$\mu(f) \quad \text{or} \quad \int f \, \mathrm{d}\mu \quad \text{or} \quad \int f(x)\mu(\mathrm{d}x)$$

and called the **integral** of f with respect to μ

The above theorem takes care of all $f \in m\mathscr{S}^+$

For $f \in m\mathscr{S}$ we can decomposed f as

$$f = f^+ - f^-$$

Now let

$$\int f \, \mathrm{d}\mu := \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu$$

The function f is called **integrable** if both terms on RHS are finite

If μ is a probablity measure, the integral is usually written as

$$\mathbb{E} f := \int f \, \mathrm{d}\mu$$

and called the **expectation**

Example. Let X be a finite random variable on $(\Omega, \mathscr{F}, \mathbb{P})$:

$$X(\omega) = \sum_{k=1}^{K} x_k \mathbb{1}_{A_k}(\omega)$$

where

1. $A_k \in \mathscr{F}$ for each k
2. the set $\{A_k\}$ are disjoint

Then, by the properties of the integral,

$$\mathbb{E}\left(X\right) = \mathbb{E}\left[\sum_{k=1}^{K} x_k \mathbb{1}_{A_k}\right] = \sum_{k=1}^{K} x_k \mathbb{E}\left[\mathbb{1}_{A_k}\right] = \sum_{k=1}^{K} x_k \mathbb{P}(A_k)$$

This aligns with our intuition

Two Famous Convergence Theorems

Monotone Convergence Theorem. If $\{f_n\}$ is a sequence of measurable functions with $f_n \uparrow f$ pointwise and $\int f_1 \, \mathrm{d}\mu > -\infty$, then

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$$

Dominated Convergence Theorem. Let $\int |g| \, \mathrm{d}\mu < \infty$ and let $\{f_n\}$ be a measurable sequence with $|f_n| \leqslant g$ for all n. If $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$$

The Banach space L_p

Let $(S, \mathscr{S}, \mu)$ be any measure space

Theorem. $m\mathscr{S}$ is a vector space and a linear subspace of $\mathbb{R}^S$

Proof: It suffices to show that $m\mathscr{S}$ is a linear subspace of $\mathbb{R}^S$

Measurability is preserved under simple algebraic operations

In particular,

$$f, g \in m\mathscr{S} \quad \text{and } \alpha, \beta \in \mathbb{R} \quad \Longrightarrow \quad \alpha f + \beta g \in m\mathscr{S}$$

That's all we need to show

Adding a Norm

For $f \in m\mathscr{S}$ and $p \geqslant 1$, let

$$\|f\|_p := \left[\int |f|^p \, \mathrm{d}\mu\right]^{1/p}$$

$$L_p(\mu) := \text{all } f \in m\mathscr{S} \text{ with } \|f\|_p < \infty$$

Fact. If $f \in L_p(\mu)$ and $\alpha \in \mathbb{R}$, then $\|\alpha f\|_p = |\alpha| \|f\|_p$

Fact. (Minkowski) If $f, g \in L_p(\mu)$, then $f + g \in L_p(\mu)$ with

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$$

In particular, $L_p(\mu)$ is a vector space

Given $p \geqslant 1$, is

$$\|f\|_p := \left[\int |f|^p \, \mathrm{d}\mu\right]^{1/p}$$

a norm on $L_p(\mu)$?

In fact only a **semi-norm** on $L_p(\mu)$

- all properties of a norm except $\|f\|_p = 0$ does not imply $f = 0$

Example.

$$\|\mathbb{1}_{\mathbb{Q}}\|_1 = \int \mathbb{1}_{\mathbb{Q}} \, \mathrm{d}\lambda = \lambda(\mathbb{Q}) = 0$$

However, $\|f\|_p = 0$ implies $\mu\{x \in S : f(x) \neq 0\} = 0$

Similarly, $\|f - g\|_p = 0$ implies

$$\mu\{x \in S : f(x) \neq g(x)\} = 0$$

We say that $f = g$ **μ-almost everywhere**

Let's agree to **identify** elements of $L_p(\mu)$ that are equal μ-a.e.

Then $\|\cdot\|_p$ is a norm on $L_p(\mu)$

Completeness of the L_p spaces

As above let

- $(S, \mathscr{S}, \mu)$ be any measure space
- $p \geqslant 1$
- $L_p(\mu) :=$ all $f \in m\mathscr{S}$ with $\|f\|_p < \infty$

Theorem $(L_p(\mu), \|\cdot\|_p)$ is a Banach space