Stachuski
Problem Set 6

EXERCISE 1

Show that $\mathbf{X}'\mathbf{X}$ *is invertible.*

Answer:

Let $0 \neq \alpha \in \mathbb{R}^k$. Then

$$\begin{aligned}
\alpha'(X'X)\alpha &= (X\alpha)'(X\alpha) \\
&= \left(\sum_{j=1}^{k} \alpha_j x_j\right)' \underbrace{\left(\sum_{j=1}^{k} \alpha_j x_j\right)}_{z} \\
&= \sum_{i=1}^{n} z_i^2 \geq 0
\end{aligned}$$

where x_j is the j-th column of matrix X. But since $\{x_1, \ldots, x_k\}$ is LI, z as a linear combination of columns of X is nonzero and therefore we have inequality holding strictly. This implies that $X'X$ is *positive definite*.

Hence, all eigenvalues of $(X'X)$ are positive. To see this, let λ_j, z_j be an eigenvalue, eigenvector pair of $(X'X)$. Then

$$\lambda_j z_j = (X'X)z_j \Rightarrow \lambda_j z_j' z_j = z_j'(X'X)z_j > 0$$

but $\lambda_j \|z\|^2 > 0 \Leftrightarrow \lambda_j > 0$. Taking as given the result regarding the relaton of determinant of a matrix and its eigenvalues - $\det A = \Pi_{j=1}^{k} \lambda_j$ - we get that determinant of $X'X$ is positive, therefore invertible.

EXERCISE 2

Let $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$*. Show that if* $k = n$*, then* $\mathbf{P}$ *is the identity.*

Answer:

If $k = n$, then X is square and, because we assumed that it has LI columns, is invertible. Since X invertible implies X' invertible[1] we have

$$P = X(X'X)^{-1}X' = XX^{-1}(X')^{-1}X' = I$$

The result is also intuitive. From the orthogonal projection theorem we know that Py is the vector in span$X$ which best approximates y. Here, span$X = \mathbb{R}^n$, so $Py = y \ \forall y \in \mathbb{R}^n$, which implies $P = I$

EXERCISE 3

Show that the projection of $\mathbf{y} \in \mathbb{R}^n$ *onto span*$\{1\}$ *is the mean of elements of* y*.*

Answer:

Let $S =$ span $\{\mathbf{1}\}$. From the lecture notes, we have that proj S can be represented in a matrix form as $\mathbf{P} = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$. Therefore the projection of $y \in \mathbb{R}^n$ onto S is given by

$$\begin{aligned}
\mathbf{P}y &= \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'y \\
&= \mathbf{1}(n)^{-1}\sum_{i=1}^{n} y_i \\
&= \begin{bmatrix} n^1 \sum_{i=1}^n y_i \\ \vdots \\ n^1 \sum_{i=1}^n y_i \end{bmatrix}
\end{aligned}$$

EXERCISE 4

Let $y \in \mathbb{R}^n$ *and* $\mathbf{X} \in \mathcal{M}(n \times k)$ *have linearly independent columns. Let* $S =$ *span* $\{x_1, \dots, x_k\}$*,* $\mathbf{P} =$ *proj* S *and* $\mathbf{M} = \mathbf{I} - \mathbf{P}$*. Show that if* $x_1 = \mathbf{1}$*, then elements of* $\hat{u} = \mathbf{M}y$ *sum to zero.*

[1] The fact that X is investible mean that exists X^{-1} such that $XX^{-1} = X^{-1}X = I$ but in that case $(X^{-1})'X' = X'(X^{-1})' = I \Rightarrow (X^{-1})' = (X')^{-1}$

Answer:

By definition of $\mathbf{M}$, we can rewrite $\hat{u} = y - \mathbf{P}y$. From the **Orthogonal Projection Theorem** we have $y - \mathbf{P}y \perp S$. In particular, $\hat{u}$ must be orthogonal to $\mathbf{1}$, since $\mathbf{1} \in S$. Therefore, the result follows from the realization of $\sum_{i=1}^{n} \hat{u}_i = \langle \mathbf{1}, \hat{u} \rangle = 0$.

EXERCISE 5

Show that if S is a nonempty subset of $\mathbb{R}^n$, then $S \cap S^{\perp} = \{0\}$.

Answer:

The following is true

$$x \in S \cap S^{\perp} \Rightarrow \langle x, x \rangle = \|x\|^2 = 0 \Leftrightarrow x = 0$$

where the last inequality follows from the properties of the norm induced by $\langle \cdot, \cdot \rangle$.

However, from the asusmption on the statement only, we cannot be sure that $S \cap S^{\perp} \neq \emptyset$ - we would if S was assumed to be a linear subspace. So, we either have $S \cap S^{\perp} = \emptyset$ if $0 \notin S$ or the derised result if we add the assumption $0 \in S$.