Calibration d'un triangle à taux de change

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Plan

Context

We consider the calibration problem between two exchange rates

- $S^1 = \text{EUR/USD}$, $S^2 = \text{GBP/USD}$
- Under the condition $S^{12} = \text{EUR / GBP} = S^1/S^2$
- Three local volatility surfaces : $\sigma_1(t, S^1), \sigma_2(t, S^2)$ and $\sigma_{12}(t, S^{12})$

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Calibration des taux

Garman Kohlhagen model

- FX rate model for $S^1 = \mathrm{EUR/USD}$

$$dS_t^1/S_t^1 = (r_t^f - r_t^d)\,dt + \sigma_1(t, S_t^1)\,dW_t^1$$

- where r_t^d is the domestic interest rate (USD) and r_t^f is the foreign interest rate (EUR)
- W_t^1 is a Brownian motion under $\mathbb{Q}^f$ the risk neutral measure of the USD dolar.
- Calibration : adjust σ_1 with Dupire's formula.

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Dupire's formula

If we consider just the calibration problem of one exchange rate

- $S^1 = \mathrm{EUR/USD}$
- $dS_t = (r-d)S_t dt + \sigma_t S_t dW_t$.
- The Dupire's equation can be used to deduce the volatility function $\sigma(.,.)$ from option prices :

$$\sigma(T,K) = \sqrt{2\frac{\frac{\partial C}{\partial T} + (r-d)K\frac{\partial C}{\partial K}}{K^2\frac{\partial^2 C}{\partial K^2}}}$$

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The FX triangle calibration problem

- Local correlation model :

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\begin{aligned}
dS_t^1/S_t^1 &= (r_t^d - r_t^1)\,dt + \sigma_1(t, S_t^1)\,dW_t^1 \\
dS_t^2/S_t^2 &= (r_t^d - r_t^2)\,dt + \sigma_2(t, S_t^2)\,dW_t^2 \\
d\left\langle W^1, W^2 \right\rangle_t &= \rho(t, S_t^1, S_t^2)\,dt
\end{aligned}
$$

- Calibration condition

$$
\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2(t,S_t^1) + \sigma_2^2(t,S_t^2) - 2\rho(t,S_t^1,S_t^2)\sigma_1^2(t,S_t^1)\sigma_2^2(t,S_t^2) | S_t^{12}\right] = \sigma_{12}^2(t, S_t^{12})
$$

- Where $\mathbb{Q}^f$ is the risk neutral measure associated to S^2.

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\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{S_T^2}{S_0^2}\exp\left(\int_0^T (r_t^2 - r_t^d)\,dt\right)
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- Local correlation model :

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$$\frac{d\mathbb{Q}^f}{d\mathbb{Q}} = \frac{S_T^2}{S_0^2}\exp\left(\int_0^T (r_t^2 - r_t^d)\,dt\right)$$

Proof

- Applying Ito's formula :

$$\frac{dS_t^{12}}{S_t^{12}} = (r_t^2 - r_t^1)\,dt + \sigma_1\,dW_t^1 - \sigma_2\,dW_t^2 + \sigma_2^2\,dt - \sigma_1\sigma_2\,d\left\langle W^1, W^2\right\rangle_t$$
$$= (r_t^2 - r_t^1)\,dt + \sigma_1\left(t, S_t^1\right)\,dW_t^{1,f} - \sigma_2(t, S_t^2)\,dW_t^{2,f}$$

- where :

$$W_t^{1,f} = W_t^1 - \int_0^t \rho(s, S_s^1, S_s^2)\sigma_2(s, S_s^2)\,ds$$
$$W_t^{2,f} = W_t^2 - \int_0^t \sigma_2(s, S_s^2)\,ds$$

are (by Girsanov Theorem and the Novikov's condition) Brownian motions under the risk neutral measure $\mathbb{Q}^f$

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&\quad + \sigma_2^2\,dt - \sigma_1\sigma_2\,d\left\langle W^1, W^2\right\rangle_t \\
&= (r_t^2 - r_t^1)\,dt + \sigma_1\left(t, S_t^1\right)\,dW_t^{1,f} - \sigma_2(t, S_t^2)\,dW_t^{2,f}
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are (by Girsanov Theorem and the Novikov's condition) Brownian motions under the risk neutral measure $\mathbb{Q}^f$

Proof

- We consider now the Brownian Motion under $\mathbb{Q}^f$ defined as :

$$W_t^f = \int_0^t \frac{\sigma_1(s, S_s^1)\, dW_s^{1,f} - \sigma_2(s, S_s^2)\, dW_s^{2,f}}{a_s}\, ds$$

where
$a_t^2 = \sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - \rho(t, S_t^1, S_t^2)\sigma_1^2(t, S_t^1)\sigma_2^2(t, S_t^2)$

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- We have :

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- To end the proof, we use the Gyongy's theorem and thus we have :

$$\mathbb{E}(a_t^2 | S_t^{12}) = \sigma_{12}^2$$

which is equivalent to the calibration requirement

$$\mathbb{E}_\rho^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) - 2\rho(t, S_t^1, S_t^2)\sigma_1^2(t, S_t^1)\sigma_2^2(t, S_t^2) | S_t^{12}\right] = \sigma_{12}^2(t, S_t^{12})$$

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$$= \sigma_{12}^2(t, S_t^{12})$$

The FX triangle calibration problem

- Let us now pick two functions $a(t, S^1, S^2)$ and $b(t, S^1, S^2)$ such that $b(t, S^1, S^2)$ does not vanish and

$$a(t, S^1, S^2) + b(t, S^1, S^2)\rho(t, S^1, S^2) \equiv f\left(t, \frac{S^1}{S^2}\right)$$

- Then :

$$\sigma_{12}^2(t, \frac{S_t^1}{S_t^2}) = \mathbb{E}_{\rho}^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) + 2\frac{a(t, S_t^1, S_t^2)}{b(t, S_t^1, S_t^2)}\sigma_1(t, S_t^1)\sigma_2(t, S_t^2) | \frac{S_t^1}{S_t^2}\right]$$
$$- 2(a + b\rho)\left(t, \frac{S_t^1}{S_t^2}\right)\mathbb{E}_{\rho}^{\mathbb{Q}^f}\left[\frac{\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)}{b(t, S_t^1, S_t^2)} | \frac{S_t^1}{S_t^2}\right]$$

As consequence $\rho = \rho_{(a,b)}$ satisfies $\rho_{(a,b)} \in \mathcal{C}$ and

$$\rho_{(a,b)}(t, S_t^1, S_t^2) = -\frac{-a(t, S_t^1, S_t^2)}{b(t, S_t^1, S_t^2)} + \frac{1}{b(t, S_t^1, S_t^2)}$$
$$\left(\frac{\mathbb{E}_{\rho(a,b)}^{\mathbb{Q}^f}\left[\sigma_1^2(t, S_t^1) + \sigma_2^2(t, S_t^2) + 2\frac{a(t, S_t^1, S_t^2)}{b(t, S_t^1, S_t^2)}\sigma_1(t, S_t^1)\sigma_2(t, S_t^2) | \frac{S_t^1}{S_t^2} - \sigma_{12}^2(t, \frac{S_t^1}{S_t^2})\right]}{\mathbb{E}_{\rho}^{\mathbb{Q}^f}[\frac{\sigma_1(t, S_t^1)\sigma_2(t, S_t^2)}{b(t, S_t^1, S_t^2)} | \frac{S_t^1}{S_t^2}]}\right)$$

Numerical solution

- A numerical solution can be found using the particle method :
  1. Initialize k = 1 and set $\rho_{(a,b)}(t, S_t^1, S_t^2) = \frac{\sigma_1^2(0,S^1)+\sigma_1^2(0,S^1)-\sigma_{12}^2(0,\frac{S^1}{S^2})}{2\sigma_1^2(0,S^1)\sigma_2^2(0,S^2)}$ for all $t \in [t_0 = 0; t_i]$
  2. Simulate $(S_t^{1,i}, S_t^{2,i})_{1\leq i\leq N}$ from t_{k-1} to t_k using a discretization scheme
  3. For all S^{12} in a grid G_{t_k} of cross rate values, compute

$$f(t_k, S^{12}) = \frac{E_{t_k}^{num}(S^{12}) - \sigma^{12}(t_k, S^{12})}{2E_{t_k}^{den}(S^{12})}$$

     interpolate and extrapolate $f(t_k, .)$, for instance using cubic splines, and, for all $t \in [t_k, t_{k+1}]$, set

$$\rho_{(a,b)}(t, S^1, S^2) = \frac{1}{b(t, S^1, S^2)}\left(f\left(t_k, \frac{S^1}{S^2}\right) - a(t, S^1, S^2)\right)$$

  4. Set $k := k + 1$. Iterate steps 2 and 3 up the maturity date T.

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- Proof and understanding of the fundamental theorems and concepts
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End

Thanks