Multivariate Calculus

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Surface Area of a Graph

Let $z = f(x, y)$ be a smooth function and let (x, y) vary in some region D of the xy-plane. In the vector form, we may write

$$\overrightarrow{r} = \langle x, y, f \rangle.$$

This is a parameterization of the graph of f: as the point (x, y) varies in D, the dependent vector $\overrightarrow{r}$ describes the surface of the graph.

At each point P of the graph we have two velocity vectors tangent to the graph:

$$\overrightarrow{r}_x = \langle 1, 0, f_x \rangle \quad \text{and} \quad \overrightarrow{r}_y = \langle 0, 1, f_y \rangle.$$

These vectors are neither zero nor parallel and determine an infinitesimal parallelogram with sidelengths

$$\|\overrightarrow{r}_x\| dx = \sqrt{1 + f_x^2}\ dx \quad \text{and} \quad \|\overrightarrow{r}_y\| dy = \sqrt{1 + f_y^2}\ dy$$

lying in the plane tangent to the graph at P. The flat parallelograms approximate small portions of the curved surface. The area of the parallelogram is

$$\|\overrightarrow{r}_x \times \overrightarrow{r}_y\| dxdy = \|\langle -f_x, -f_y, 1 \rangle\| dxdy = \sqrt{1 + f_x^2 + f_y^2}\ dxdy.$$

Note that the factor $\|\overrightarrow{r}_x \times \overrightarrow{r}_y\|$ simply tells us how the area of the infinitesimal rectangle $dx \times dy$ scales under f. Adding up the little pieces (integrating over D), one can argue that the total surface area of the graph is

$$A = \iint_D \|\overrightarrow{r}_x \times \overrightarrow{r}_y\| dxdy = \iint_D \sqrt{1 + f_x^2 + f_y^2}\ dxdy.$$

Example: area of the unit sphere

The surface area of the unit sphere $x^2 + y^2 + z^2 = 1$ is twice the area of the graph of $z = \sqrt{1 - x^2 - y^2}$ defined for $x^2 + y^2 < 1$:

$$A = 2 \iint_{x^2+y^2<1} \sqrt{1 + z_x^2 + z_y^2}\ dxdy.$$

We have $z_x = -\frac{x}{z}$, $z_y = -\frac{y}{z}$, and so

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} = \frac{1}{z}\ .$$

This gives

$$A = 2\iint_{x^2+y^2<1} \frac{1}{\sqrt{1-x^2-y^2}}\, dxdy$$
$$= 2\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-y^2-x^2}}\, dx\ dy$$

Observe that the inner integral is the same for every y:

$$\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-y^2-x^2}}\, dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-\frac{x^2}{1-y^2}}}\ \frac{dx}{\sqrt{1-y^2}}$$
$$= \int_{-1}^{1} \frac{1}{\sqrt{1-u^2}}\, du \qquad \left(u = \frac{x}{\sqrt{1-y^2}}\right)$$
$$= \arcsin u\Big|_{-1}^{1}$$
$$= \pi$$

Consequently, the area of the unit sphere is

$$A = 2\int_{-1}^{1} \pi dy = 4\pi.$$

Area of a Parameterized Surface

Assume now that we are able to conveniently describe a given surface S (which is not necessarily a graph of a function) in terms of variables (u, v) changing in some region R of the uv-plane. Let

$$\overrightarrow{r} = \langle x(u,v), y(u,v), z(u,v)\rangle$$

be a vector-function parameterizing S.

At each point of S we have two velocity vectors tangent to S:

$$\overrightarrow{r}_u = \langle x_u, y_u, z_u\rangle \quad \text{and} \quad \overrightarrow{r}_v = \langle x_v, y_v, z_v\rangle.$$

Suppose that these vectors are neither zero nor parallel at each typical point of S. Arguing as before, we may cover S by infinitesimal tangent parallelograms with sidelengths

$$\|\overrightarrow{r}_u\|du \quad \text{and} \quad \|\overrightarrow{r}_v\|dv.$$

Each flat parallelogram approximates a small portion of the curved surface. Its area is

$$\|\overrightarrow{r}_u \times \overrightarrow{r}_u\|dudv = \|\langle y_u z_v - y_v z_u, -x_u z_v + x_v z_u, x_u y_v - x_v y_u\rangle\|dudv.$$

The factor $\|\overrightarrow{r}_x \times \overrightarrow{r}_y\|$ (the Jacobian) tells us how the area of the infinitesimal rectangle $du\times dv$ scales as R is transformed into S. Adding up the little pieces (integrating over R), one can argue that the total area of the surface is

$$A(S) = \iint_R \|\overrightarrow{r}_u \times \overrightarrow{r}_v\| du dv.$$

Example: unit sphere, spherical coordinates

The unit sphere $x^2+y^2+z^2 = 1$ has $\rho = 1$ and so can be parameterized by θ and φ. We have

$$x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \cos\varphi,$$

where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. Thus R is simply the rectangle $[0, 2\pi] \times [0, \pi]$ in the $\theta\varphi$-plane. Write

$$\overrightarrow{r} = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle,$$

then

$$\begin{aligned}
\overrightarrow{r}_\theta &= \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0\rangle \\
\overrightarrow{r}_\varphi &= \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi\rangle \\
\overrightarrow{r}_\theta \times \overrightarrow{r}_\varphi &= \langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, -\sin\varphi\cos\varphi\rangle \\
&= -\sin\varphi\langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle \\
&= -\sin\varphi\ \langle x, y, z\rangle.
\end{aligned}$$

This gives

$$\begin{aligned}
\|\overrightarrow{r}_\theta \times \overrightarrow{r}_\varphi\| &= \sin\varphi\sqrt{x^2+y^2+z^2} \\
&= \sin\varphi.
\end{aligned}$$

Hence the area of the unit sphere is

$$\begin{aligned}
A &= \iint_R \|\overrightarrow{r}_\theta \times \overrightarrow{r}_\varphi\| d\theta d\varphi \\
&= \int_0^{2\pi}\int_0^{\pi} \sin\varphi d\varphi\ d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi d\varphi \\
&= 2\pi \cdot 2 = 4\pi.
\end{aligned}$$

Note that the actual integration is quite easy: spherical coordinates are tailored for spheres.

Example: unit sphere, cylindrical coordinates

This time let's employ cylindrical coordinates (r, θ, z). The unit sphere has $r^2 + z^2 = 1$, and so the vector function

$$\overrightarrow{v}(\theta, z) = \langle \sqrt{1 - z^2} \cos\theta, \sqrt{1 - z^2} \sin\theta, z \rangle, \quad 0 \le \theta \le 2\pi,\ |z| \le 1,$$

gives a parameterization. We have

$$\begin{aligned} \overrightarrow{v}_\theta &= \langle -r \sin\theta, r\cos\theta, 0 \rangle \\ \overrightarrow{v}_z &= \langle -\tfrac{z}{r} \cos\theta, -\tfrac{z}{r} \sin\theta, 1 \rangle \\ \overrightarrow{v}_\theta \times \overrightarrow{v}_z &= \langle r\cos\theta, r\sin\theta, z \rangle \\ \|\overrightarrow{v}_\theta \times \overrightarrow{v}_z\| &= 1. \end{aligned}$$

Consequently, the area of the unit sphere is

$$A = \int_0^{2\pi} \int_{-1}^{1} \|\overrightarrow{v}_\theta \times \overrightarrow{v}_z\| dz d\theta = \int_0^{2\pi} \int_{-1}^{1} dz d\theta = 4\pi.$$

This is even easier than the preceding example.