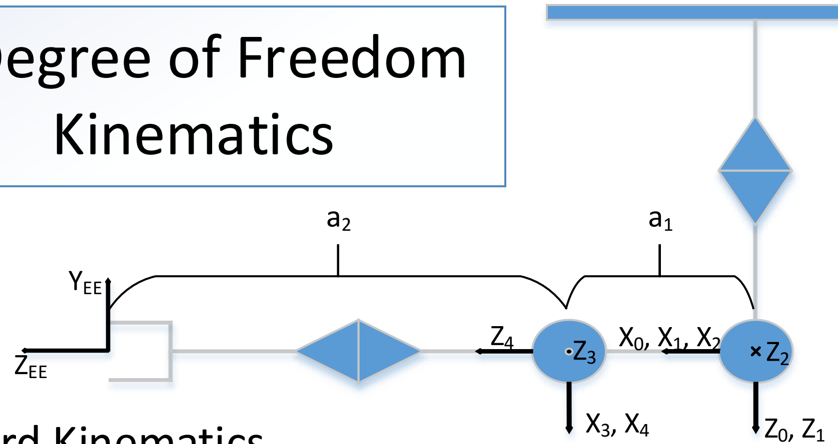
4 Degree of Freedom Kinematics

By: Austin Owens



DH Table

i	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	θ_1	0
2	-90°	0	θ_2	0
3	180°	a_1	θ_3+90°	0
4	90°	0	θ_4	0
EE	0	0	90°	a_2

Forward Kinematics

1

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-90°) & -\sin(-90°) & 0 \\ 0 & \sin(-90°) & \cos(-90°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(180°) & -\sin(180°) & 0 \\ 0 & \sin(180°) & \cos(180°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos(\theta_3+90°) & -\sin(\theta_3+90°) & 0 & 0 \\ \sin(\theta_3+90°) & \cos(\theta_3+90°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -s_3 & -c_3 & 0 & a_1 \\ -c_3 & s_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(90°) & -\sin(90°) & 0 \\ 0 & \sin(90°) & \cos(90°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^4_{EE}T = \begin{bmatrix} \cos(90°) & -\sin(90°) & 0 & 0 \\ \sin(90°) & \cos(90°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_{EE}T = {}^0_1T * {}^1_2T * {}^2_3T * {}^3_4T * {}^4_{EE}T = \begin{bmatrix} -c_1s_{2-3}s_4 + s_1c_4 & -c_1c_4s_{2-3} - s_1s_4 & c_1c_{2-3} & a_1c_1c_2 + a_2c_1c_{2-3} \\ -s_1s_{2-3}s_4 - c_1c_4 & c_1s_4 - s_1s_{2-3}c_4 & s_1c_{2-3} & a_1s_1c_2 + a_2s_1c_{2-3} \\ -c_{2-3}s_4 & -c_{2-3}c_4 & -s_{2-3} & -a_1s_2 - a_2s_{2-3} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Inverse Kinematics

2

$$ \boxed{X = a_1c_1c_2 + a_2c_1c_{2-3} \qquad Y = a_1s_1c_2 + a_2s_1c_{2-3} \qquad Z = -a_1s_2 - a_2s_{2-3}} $$

$$ X = c_1\overbrace{(a_1c_2 + a_2c_{2-3})}^{A} $$
$$ Y = s_1(a_1c_2 + a_2c_{2-3}) \longrightarrow \begin{matrix} X = c_1A \\ Y = s_1A \end{matrix} \longrightarrow \boxed{\theta_1 = \arctan 2(Y, X)} $$

3

$$ \boxed{X = a_1c_1c_2 + a_2c_1c_{2-3} \qquad Y = a_1s_1c_2 + a_2s_1c_{2-3} \qquad Z = -a_1s_2 - a_2s_{2-3}} $$

$$ \begin{matrix} \dfrac{X}{c_1} = a_1c_2 + a_2c_{2-3} \\ -Z = a_1s_2 + a_2s_{2-3} \end{matrix} \xrightarrow{\boxed{\bar{X} = \frac{X}{c_1}} \ \boxed{\bar{Z} = -Z}} \begin{matrix} a_2c_{2-3} = \bar{X} - a_1c_2 \\ a_2s_{2-3} = \bar{Z} - a_1s_2 \end{matrix} $$

$$ \xrightarrow{\text{After squaring and adding}} a_2^{\,2}(\underbrace{\cancel{c_{2-3}^{\,2} + s_{2-3}^{\,2}}}_{1}) = \bar{X}^2 + \bar{Z}^2 - 2a_1(s_2\bar{Z} + c_2\bar{X}) + a_1(\underbrace{\cancel{c_2^{\,2} + s_2^{\,2}}}_{1}) $$

$$ \xrightarrow{\text{After rearranging}} s_2\bar{Z} + c_2\bar{X} = \frac{a_2^{\,2} - \bar{X}^2 - \bar{Z}^2 - a_1^{\,2}}{-2a_1} $$

$$ \longrightarrow s_2\bar{Z} + c_2\bar{X} = \frac{\bar{X}^2 + \bar{Z}^2 + a_1^{\,2} - a_2^{\,2}}{2a_1} \xrightarrow{\boxed{b_1 = \frac{\bar{X}^2 + \bar{Z}^2 + a_1^{\,2} - a_2^{\,2}}{2a_1}}} s_2\bar{Z} + c_2\bar{X} = b_1 $$

$$ \xrightarrow{\text{Rotation Matrix}} \begin{matrix} c_2\bar{Z} - s_2\bar{X} = \sigma_1 \\ s_2\bar{Z} + c_2\bar{X} = b_1 \end{matrix} \longrightarrow \begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix} * \begin{bmatrix} \bar{Z} \\ \bar{X} \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ b_1 \end{bmatrix} $$

$$ \xrightarrow{\text{Vector } \begin{bmatrix} \bar{Z} \\ \bar{X} \end{bmatrix} \,\&\, \begin{bmatrix} \sigma_1 \\ b_1 \end{bmatrix} \text{ are the same magnitude}} \bar{Z}^2 + \bar{X}^2 = \sigma_1^{\,2} + b_1^{\,2} \longrightarrow \boxed{\sigma_1 = \pm\sqrt{\bar{Z}^2 + \bar{X}^2 - b_1^{\,2}}} $$

$$ \xrightarrow{\text{Angle between } \begin{bmatrix} \bar{Z} \\ \bar{X} \end{bmatrix} \,\&\, \begin{bmatrix} \sigma_1 \\ b_1 \end{bmatrix}} \theta_2^{(1)} = \arctan 2(b_1, |\sigma_1|) - \arctan 2(\bar{X}, \bar{Z}) \longrightarrow \boxed{\theta_2^{(1)} = \arctan 2(\bar{Z}, \bar{X}) - \arctan 2(\sigma_1, b_1)} $$

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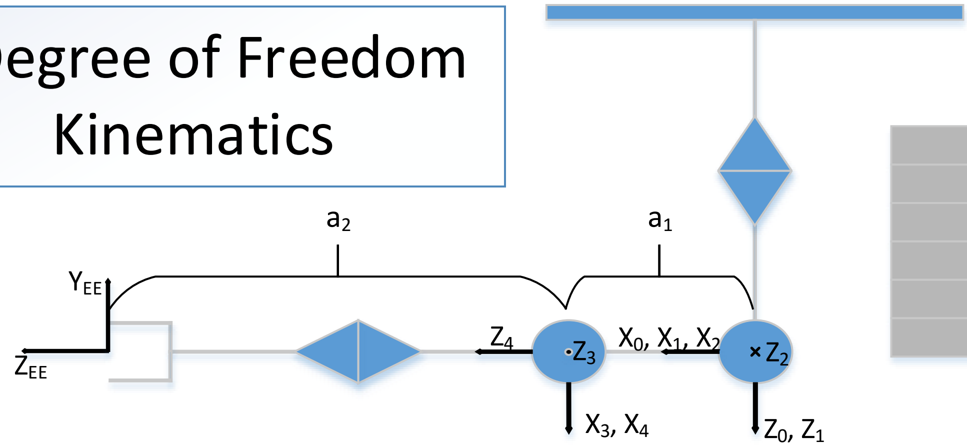

DH Table

i	α_{i-1}	a_{i-1}	θ_i	d_i
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3	180°	a_1	θ_3+90°	0
4	90°	0	θ_4	0
EE	0	0	90°	a_2

4

$$\boxed{X = a_1c_1c_2 + a_2c_1c_{2-3} \qquad Y = a_1s_1c_2 + a_2s_1c_{2-3} \qquad Z = -a_1s_2 - a_2s_{2-3}}$$

$$\left.\begin{aligned} \frac{Y}{s_1} &= a_1c_2 + a_2c_{2-3} \\ Z &= -a_1s_2 - a_2s_{2-3} \end{aligned}\right. \xrightarrow{\boxed{\bar{Y} = \frac{Y}{s_1}}\ \boxed{\bar{Z} = -Z}} \begin{aligned} a_2c_{2-3} &= \bar{Y} - a_1c_2 \\ a_2s_{2-3} &= \bar{Z} - a_1s_2 \end{aligned}$$

Same procedures as first derivation except replace $\bar{X}$ with $\bar{Y}$

$$\boxed{b_2 = \frac{\bar{Y}^2 + \bar{Z}^2 + a_1^2 - a_2^2}{2a_1}} \qquad \boxed{\sigma_2 = \pm\sqrt{\bar{Z}^2 + \bar{Y}^2 - b_2^2}}$$

$$\boxed{\theta_2^{(2)} = \arctan 2(\bar{Z}, \bar{Y}) - \arctan 2(\sigma_2, b_2)}$$

5

$$\boxed{X = a_1c_1c_2 + a_2c_1c_{2-3} \qquad Y = a_1s_1c_2 + a_2s_1c_{2-3} \qquad Z = -a_1s_2 - a_2s_{2-3}}$$

$$\left.\begin{aligned} \frac{X}{c_1} &= a_1c_2 + a_2c_{2-3} \\ Z &= -a_1s_2 - a_2s_{2-3} \end{aligned}\right. \xrightarrow{\boxed{\bar{X} = \frac{X}{c_1}}\ \boxed{\bar{Z} = -Z}} \begin{aligned} a_1c_2 &= \bar{X} - a_2c_{2-3} \\ a_1s_2 &= \bar{Z} - a_2s_{2-3} \end{aligned}$$

Same procedures as first derivation except replace a_2 with a_1 and a_1 with a_2

$$\boxed{b_3 = \frac{\bar{X}^2 + \bar{Z}^2 + a_2^2 - a_1^2}{2a_2}} \qquad \boxed{\sigma_3 = \pm\sqrt{\bar{Z}^2 + \bar{X}^2 - b_3^2}}$$

$$\theta_{2-3} = \theta_2 - \theta_3 = \arctan 2(\bar{Z}, \bar{X}) - \arctan 2(\sigma_3, b_3) \longrightarrow \boxed{\theta_3^{(1)} = \arctan 2(\sigma_3, b_3) - \arctan 2(\sigma_1, b_1)}$$

6

$$\boxed{X = a_1c_1c_2 + a_2c_1c_{2-3} \qquad Y = a_1s_1c_2 + a_2s_1c_{2-3} \qquad Z = -a_1s_2 - a_2s_{2-3}}$$

$$\left.\begin{aligned} \frac{Y}{s_1} &= a_1c_2 + a_2c_{2-3} \\ Z &= -a_1s_2 - a_2s_{2-3} \end{aligned}\right. \xrightarrow{\boxed{\bar{Y} = \frac{Y}{s_1}}\ \boxed{\bar{Z} = -Z}} \begin{aligned} a_1c_2 &= \bar{Y} - a_2c_{2-3} \\ a_1s_2 &= \bar{Z} - a_2s_{2-3} \end{aligned}$$

Same procedures as first derivation except replace $\bar{X}$ with $\bar{Y}$, a_2 with a_1, and a_1 with a_2

$$\boxed{b_4 = \frac{\bar{Y}^2 + \bar{Z}^2 + a_2^2 - a_1^2}{2a_2}} \qquad \boxed{\sigma_4 = \pm\sqrt{\bar{Z}^2 + \bar{Y}^2 - b_4^2}}$$

$$\theta_{2-3} = \theta_2 - \theta_3 = \arctan 2(\bar{Z}, \bar{Y}) - \arctan 2(\sigma_4, b_4) \longrightarrow \boxed{\theta_3^{(2)} = \arctan 2(\sigma_4, b_4) - \arctan 2(\sigma_2, b_2)}$$

7

Solution #	σ_1	σ_3	$\theta_2^{(1)}$	$\theta_3^{(1)}$
~~1~~	~~+~~	~~+~~	~~$a(\bar{Z}, \bar{X})-a(\lvert\sigma_1\rvert, b_1)$~~	~~$a(\lvert\sigma_3\rvert, b_3)-a(\lvert\sigma_1\rvert, b_1)$~~
2	+	-	$a(\bar{Z}, \bar{X})-a(\lvert\sigma_1\rvert, b_1)$	$-a(\lvert\sigma_3\rvert, b_3)-a(\lvert\sigma_1\rvert, b_1)$
3	-	+	$a(\bar{Z}, \bar{X})+a(\lvert\sigma_1\rvert, b_1)$	$a(\lvert\sigma_3\rvert, b_3)+a(\lvert\sigma_1\rvert, b_1)$
~~4~~	~~-~~	~~-~~	~~$a(\bar{Z}, \bar{X})+a(\lvert\sigma_1\rvert, b_1)$~~	~~$-a(\lvert\sigma_3\rvert, b_3)+a(\lvert\sigma_1\rvert, b_1)$~~
Solution #	**σ_2**	**σ_4**	**$\theta_2^{(2)}$**	**$\theta_3^{(2)}$**
~~5~~	~~+~~	~~+~~	~~$a(\bar{Z}, \bar{Y})-a(\lvert\sigma_2\rvert, b_2)$~~	~~$a(\lvert\sigma_4\rvert, b_4)-a(\lvert\sigma_2\rvert, b_2)$~~
6	+	-	$a(\bar{Z}, \bar{Y})-a(\lvert\sigma_2\rvert, b_2)$	$-a(\lvert\sigma_4\rvert, b_4)-a(\lvert\sigma_2\rvert, b_2)$
7	-	+	$a(\bar{Z}, \bar{Y})+a(\lvert\sigma_2\rvert, b_2)$	$a(\lvert\sigma_4\rvert, b_4)+a(\lvert\sigma_2\rvert, b_2)$
~~8~~	~~-~~	~~-~~	~~$a(\bar{Z}, \bar{Y})+a(\lvert\sigma_2\rvert, b_2)$~~	~~$-a(\lvert\sigma_4\rvert, b_4)+a(\lvert\sigma_2\rvert, b_2)$~~

Suppose $a_1 = a_2$, then $b_1 = b_3$, $\sigma_1 = \sigma_3$, $b_2 = b_4$, and $\sigma_2 = \sigma_4$. Consequently solutions 1, 4, 5, and 8 would give $\theta_3 = 0$ for all $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ which is not possible. Therefore we keep only 4 solutions, 2, 3, 6, 7 which we rename as solutions 1, 2, 3, and 4.

Where:

$$a = \arctan 2() \quad \bar{X} = \frac{X}{c_1} \quad \bar{Y} = \frac{Y}{s_1} \quad \bar{Z} = -Z$$

$$b_1 = \frac{\bar{X}^2 + \bar{Z}^2 + a_1^2 - a_2^2}{2a_1}$$

$$\sigma_1 = \pm\sqrt{\bar{Z}^2 + \bar{X}^2 - b_1^2}$$

$$b_2 = \frac{\bar{Y}^2 + \bar{Z}^2 + a_1^2 - a_2^2}{2a_1}$$

$$\sigma_2 = \pm\sqrt{\bar{Z}^2 + \bar{Y}^2 - b_2^2}$$

$$b_3 = \frac{\bar{X}^2 + \bar{Z}^2 + a_2^2 - a_1^2}{2a_2}$$

$$\sigma_3 = \pm\sqrt{\bar{Z}^2 + \bar{X}^2 - b_3^2}$$

$$b_4 = \frac{\bar{Y}^2 + \bar{Z}^2 + a_2^2 - a_1^2}{2a_2}$$

$$\sigma_4 = \pm\sqrt{\bar{Z}^2 + \bar{Y}^2 - b_4^2}$$

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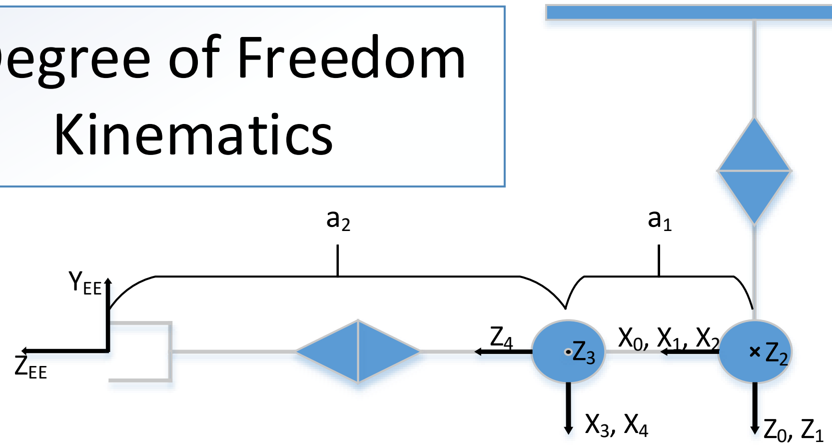

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4	90°	0	θ_4	0
EE	0	0	90°	a_2

Summary

$$\theta_1 = \arctan 2(Y, X)$$

In the cases where θ_1= 0° or θ_1=180° , $\bar{Y}$ is undefined making solution 3 and 4 incorrect. In the cases where θ_1= 90° or θ_1=270°, $\bar{X}$ is undefined making solution 1 and 2 incorrect. If θ_1 does not equal 0°, 90°, 180°, or 270°, solution 1 and 3 will be the same value and solution 2 and 4 will be the same value.

Solution 1

$$\theta_2^{(1)} = \arctan 2(\bar{Z}, \bar{X}) - \arctan 2(|\sigma_1|, b_1)$$
$$\theta_3^{(1)} = -\arctan 2(|\sigma_3|, b_3) - \arctan 2(|\sigma_1|, b_1)$$

Solution 2

$$\theta_2^{(1)} = \arctan 2(\bar{Z}, \bar{X}) + \arctan 2(|\sigma_1|, b_1)$$
$$\theta_3^{(1)} = \arctan 2(|\sigma_3|, b_3) + \arctan 2(|\sigma_1|, b_1)$$

$\theta_1 \neq 90°$
$\theta_1 \neq 270°$

Solution 3

$$\theta_2^{(2)} = \arctan 2(\bar{Z}, \bar{Y}) - \arctan 2(|\sigma_2|, b_2)$$
$$\theta_3^{(2)} = -\arctan 2(|\sigma_4|, b_4) - \arctan 2(|\sigma_2|, b_2)$$

Solution 4

$$\theta_2^{(2)} = \arctan 2(\bar{Z}, \bar{Y}) + \arctan 2(|\sigma_2|, b_2)$$
$$\theta_3^{(2)} = \arctan 2(|\sigma_4|, b_4) + \arctan 2(|\sigma_2|, b_2)$$

$\theta_1 \neq 0°$
$\theta_1 \neq 180°$

Where:

$$\bar{X} = \frac{X}{c_1} \qquad \bar{Y} = \frac{Y}{s_1} \qquad \bar{Z} = -Z$$

$$b_1 = \frac{\bar{X}^2 + \bar{Z}^2 + a_1^2 - a_2^2}{2a_1} \qquad \sigma_1 = \pm\sqrt{\bar{Z}^2 + \bar{X}^2 - b_1^2}$$

$$b_2 = \frac{\bar{Y}^2 + \bar{Z}^2 + a_1^2 - a_2^2}{2a_1} \qquad \sigma_2 = \pm\sqrt{\bar{Z}^2 + \bar{Y}^2 - b_2^2}$$

$$b_3 = \frac{\bar{X}^2 + \bar{Z}^2 + a_2^2 - a_1^2}{2a_2} \qquad \sigma_3 = \pm\sqrt{\bar{Z}^2 + \bar{X}^2 - b_3^2}$$

$$b_4 = \frac{\bar{Y}^2 + \bar{Z}^2 + a_2^2 - a_1^2}{2a_2} \qquad \sigma_4 = \pm\sqrt{\bar{Z}^2 + \bar{Y}^2 - b_4^2}$$

θ_4=All Real Values