Popularity Signals in Trial-Offer Markets

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Abstract

This paper considers trial-offer and freemium markets where consumer preferences are modeled by a multinomial logic model with social influence and position bias. The social signal for a product i is of the form ϕ_i^r, i.e., its market share raised to power r. The paper shows that, when $0 < r < 1$ and a static position assignment (e.g., a quality ranking) is used, the market converges to a unique equilibrium where the market shares depend only on product quality, not their initial appeals or the early dynamics. When $r > 1$, the market become unpredictable and goes most likely to a monopoly for some product. Which product becomes a monopoly depends on the initial conditions of the market. These theoretical results are complemented by an agent-based simulation which indicates that convergence is fast when $0 < r < 1$ and that the quality ranking dominates the well-known popularity ranking in terms of market efficiency.

These results shed a new light on the role of social influence which has been believed to produce unpredictability, inequalities, and inefficiencies in markets. In contrast, this paper shows that, with a proper social signal and a proper position assignment for the products, the market becomes predictable and inequalities and inefficiencies can be controlled.

1 Introduction

The impact of social influence and product visibilities on consumer behavior in trial-offer and fremium markets has been explored in a variety of settings (e.g., [17, 18, 20]). Social influence can be dispensed through different types of social signals: A market place may report the number of past purchases of a product, its consumer rating, and/or its consumer recommendations, depending on the market and/or the marketing platform. Recent studies [8, 20] however came to the conclusion that the popularity signal (i.e., the number of past purchases or the market share) has a much stronger impact on consumer behavior than the average consumer rating signal.[1] These two experimental studies were conducted in very different settings, using the Android application platform in one case and hotel selection in the other. Consumer preferences are also influenced by product visibilities, a phenomenon that has been widely observed in internet advertisement (e.g., [6]), in online stores such as Expedia, Amazon, and iTunes, as well as physical retail stores (see, e.g., [14]).

As a result, when running a trial-offer market, a decision maker needs to address at least two key issues:

1. Whether to use a social signal to influence customers and which signal to use?
2. How to rank the products to exploit position bias, social influence, product appeals, and product qualities? For instance, should the products be ranked by quality or by popularity?

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[1]The music market iTunes shows the normalized market share of each song of an album.

This paper considers these two issues and, in particular, the role of social influence. Indeed, despite its widespread use, there is considerable debate in the scientific community about the benefits of social influence. In a recent paper, Hu et al. [10] studied a newsvendor problem with two substitutable products with the same quality in which consumer preferences are affected by past purchases. The authors showed that the market is unpredictable but can become less so if one of products has an initial advantage. Such unpredictabilty was in fact the main conclusion of the MusicLab experiment performed by Salganik et al. [17]. In the MusicLab, participants were presented a list of unknown songs from unknown bands, each song being described by its name and band. The participants were partitioned into two groups exposed to two different experimental conditions: the *independent* condition and the *social influence* condition. In the independent group, participants were shown the songs in a random order and they were allowed to listen to a song and then download it if they wish. In the second group (social influence condition), participants were shown the songs in popularity order, i.e., when assigning the most popular songs to the most visible positions. Moreover, these participants were also shown a social signal, i.e., the number of times each song was downloaded too. In order to investigate the impact of social influence, participants in the second group were distributed in eight "worlds" evolving completely independently. In particular, participants in one world had no visibility about the downloads and the rankings in the other worlds. The MusicLab is an ideal experimental example of a trial-offer market where each song represents a product, and listening and downloading a song represent trying and purchasing a product respectively. The results by Salganik et al. [17] show that the different worlds evolve in differently from one another, and significantly so, providing evidence that social influence may introduce unpredictability, inequalities, and inefficiency in the market.

This debate about the role of social influence led Kleinberg [11] to state that *"developing an expressive computational model for this phenomenon is an interesting open question".* The main contribution of this paper is to provide such a computational model for trial-offer markets with social influence and a comprehensive analyis of various social signals with techniques from stochastic approximation [13]. More precisely, this paper studies the dynamics of trial-offer and freemium markets that are modeled with a generalized multinomial logit featuring a social signal of the form ϕ_i^r ($r \geq 0$) for product i, where ϕ_i is the market share of i.

The paper builds on the work by Krumme et al. [12] who proposed a framework in which consumer choices are captured by a multinomial logit model whose product utilities depend on the song appeal, position bias, and a social influence representing past purchases. Abeliuk et al. [2] provided a theoretical and experimental analysis of such trial-offer markets using different ranking policies. They proved that social influence is always beneficial in expectation when the objective is to maximize the expected number of purchases with a greedy heuristic known as *performance ranking*. Van Hentenryck et al. [19] studied the performance of the *quality ranking* which ranks products by their intrinsic quality[2]: They show that the quality ranking is in fact asymptotically optimal and leads to a monopoly for the product of highest quality.

The main theoretical contributions of this paper can be summarized as follows:

1. When $0 < r < 1$ and a static ranking is used, the market converges to a unique equilibrium where the market shares of the products depend only on the product quality. Moreover, products with higher quality receive larger market shares than products of lower quality, introducing a notion of fairness in the market and reducing inequalities introduced when $r = 1$.

2. When $r > 1$ and a static ranking is used, the market converges almost surely to a monopoly for some product: Which product obtains the entire market share depends on the initial

[2]The quality of a product is here defined as the probability that a consumer would purchase/download the product once she has tried the product out

condition and the early dynamics. This generalizes a result by Ceyhan et al. [5] who studied a similar model without position bias and with products of the same quality.

These theoretical results indicate that, when the social influence signal is a sublinear function of the market share and a static ranking of the products (e.g., the quality ranking) is used, the market is entirely predictable, depends only on the product quality, and does not lead to a monopoly. This contrast with the case of $r = 1$ where the market is entirely predictable but goes to a monopoly for the product of highest quality (assuming the quality ranking) [19] and the case of $r > 1$ where the market becomes unpredictable (even with a static ranking).

The theoretical contributions are complemented by computational results from an agent-based simulation that uses the experimental setting by Krumme et al. [12]. These simulation results show that the market converges quickly towards an equilibrium when using sublinear social signals and the quality ranking. They also show that the quality ranking outperforms the popularity ranking in maximizing the efficiency of the market. The popularity ranking is also shown to have some significant drawbacks in some settings.

These results thus provide a comprehensive characterization of the role of social influence and should help decision makers in designing markets to meet their objectives, balancing market efficiency, predictability, and inequalities. In particular, our results show that social signals do not necessarily make markets unpredictable and inefficient or necessarily introduce a Matthew effect where the winner takes all. In fact, natural rankings, e.g., the quality ranking, make markets completely predictable if the social signal is not too strong. Moreover, sublinear social signals enable to balance the market shares appropriately as a function of the product quality. It is only with a strong social signal $r > 1$ or specific ranking policies such as the popularity ranking with $r \geq 1$ that make the market unpredictable and inefficient and create inequalities.

The remaining of this paper is organized as follows. Section 2 introduces trial-offer markets and the generalized multinomial logit model for consumer preferences considered here. Section 3 reviews some necessary mathematical preliminaries. Section 4 proves that trial-offer markets can be modeled as Robbins-Monro algorithms. Section 5 derives the equilibria for the market as a function of the social signal. Section 6 presents the convergence results. Section 7 reports the results from the agent-based simulation. Section 8 discusses the results and concludes the paper.

2 The Trial-Offer Model

This section formalizes trial-offer markets by discrete dynamic processes. A marketplace consists of a set N of n items. Each item $i \in N$ is characterized by two values:

1. its *appeal* $a_i > 0$ which represents the inherent preference of trying item i;

2. its *quality* $q_i > 0$ which represents the conditional probability of purchasing item i given that it was tried.

This paper assumes that the appeals and the qualities are known. Abeliuk et al. [2] has shown that these values can be recovered accurately and quickly, either before or during the market execution.

At the beginning of the market, the firm decides upon a ranking for the item, i.e., an assignment of items to positions in the marketplace. Each position j has a visibility v_j which represents the inherent probability of trying an item in position i. A ranking σ is a permutation of the items and $\sigma(i) = j$ means that item i is placed in position j ($j \in N$).

When customer t enters the market, she observes all the items and their market shares $\phi^t = (\phi_1^t, \ldots, \phi_n^t)$ where

$$\phi^t \in \Delta^{n-1} = \{x = (x_1, \ldots, x_n) \in \Re^n \ \mid \ 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^{n} x_i = 1\}.$$

The consumer selects an item to try. The probability that the customer tries item i is given by $P_i(\sigma, \phi^t)$ where

$$P_i(\sigma, \phi) = \frac{v_{\sigma(i)} f(\phi_i)}{\sum_{j=1}^{n} v_{\sigma(j)} f(\phi_j)} \tag{1}$$

and f is a continuous, positive, and nondecreasing function. This probability generalizes traditional multinomial logic model. Finally, the customer decides whether to buy the sampled item i and the probabilty that she purchases item i (after having tried it) is given by q_i.

The vector ϕ^t of market shares at time t is computed in terms of the vector $d^t = (d_1^t, \ldots, d_n^t)$ of purchases at time t, i.e.,

$$\phi_i^t = \frac{d_i^t}{\sum_{j=1}^{n} d_j^t}.$$

If item i is selected at time t, then the purchase vector becomes

$$d_j^{t+1} = \begin{cases} d_j^t + 1 & \text{if } j = i; \\ d_j^t & \text{otherwise.} \end{cases}$$

Initially, for simplicity, the vector d^0 is initialized with the product appeals, i.e., $d_i^0 = a_i$.[3] To analyze this process, we divide time into discrete periods such that each new period begins when a new consumer arrives. Hence, the length of each time period is not constant.

The objective of the firm running this market is to maximize the total expected number of purchases. To achieve this, the key managerial decision of the firm is what is known as the ranking policy [2], which consists in deciding the permutation σ used to display the items.

Given a static ranking policy σ, $\{\phi^t\}_{t>0}$ represents a discrete stochastic dynamic process and we are interested in studying its asymptotic behavior. Abeliuk et al. [2] considered the quality ranking which ranks the products in decreasing order of quality q_i and assigns to the positions in decreasing order of visibility v_j. When f is the identity function and the quality ranking is used, the market converges to a monopoly for the product of highest quality [2].

The main goal of this paper is to study how $\{\phi^t\}_{t>0}$ evolves over time for various functions f for a specific static ranking σ (e.g., the quality ranking). We are particularly interested in asymptotic behavior and in the case where $f(x) = x^r$ with $r > 0$ and its subcases $0 < r < 1$, $r = 1$, and $r > 1$, which we refer to as the sub-linear, linear and super-linear case respectively. As mentioned, the case $r = 1$ was already settled by Abeliuk et al. [2].

For notational simplicity, we assume that the ranking is fixed and is the identity function $\sigma(i) = i$ and omit it from the formulas. If the qualities and visibilities also satisfies $q_1 \geq \ldots \geq q_n$ and $v_1 \geq \ldots \geq v_n$, we obtain the quality ranking but the results in this paper hold for any static ranking.

The following lemma relates the two phases of the trial-offer market and characterizes the probability that the next purchase is item i. It generalizes a similar result in [19].

Lemma 2.1. *The probability $p_i(\phi)$ that the next purchase is the product i given the market share vector ϕ is given by*

$$p_i(\phi) = \frac{v_i q_i f(\phi_i)}{\sum_{j=1}^{n} v_j q_j f(\phi_j)}. \tag{2}$$

[3] The initialization can be justified by viewing the discrete dynamic process as an urn and balls process, where the appeals are the initial sets of balls.

Proof. The probability that item i is purchased in the first step is given by

$$p_i^{1st}(\phi) = \frac{v_i f(\phi_i)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} q_i.$$

The probability that item i is purchased in the second step and no item was purchased in the first step is given by

$$p_i^{2nd}(\phi) = \left(\frac{\sum\limits_{j=1}^{n} v_j f(\phi_j)(1-q_j)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} \right) \frac{v_i f(\phi_i)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} q_i.$$

More generally, the probability that item i is purchased in step m while no item was purchased in earlier steps is given by

$$p_i^{mth}(\phi) = \left(\frac{\sum\limits_{j=1}^{n} v_j f(\phi_j)(1-q_j)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} \right)^{m-1} \frac{v_i f(\phi_i)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} q_i. \tag{3}$$

Let $a = (\sum\limits_{j=1}^{n} v_j f(\phi_j) q_j)/(\sum\limits_{j=1}^{n} v_j f(\phi_j))$, Equation (3) becomes

$$p_i^{mth}(\phi) = \left(1-a\right)^{m-1} \frac{v_i f(\phi_i)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} q_i.$$

Hence the probability that the next purchase is item i is given by

$$p_i(\phi) = \sum_{m=0}^{\infty} \left(1-a\right)^{m} \frac{v_i f(\phi_i)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} q_i.$$

Since

$$\sum_{m=0}^{\infty} \left(1-a\right)^{m} = \frac{1}{a},$$

the probability that the next purchase is item i is given by

$$p_i(\phi) = \frac{v_i q_i f(\phi_i)}{\sum\limits_{j=1}^{n} v_j q_j f(\phi_j)}.$$

□

3 Mathematical Preliminaries

This section reviews some basic definitions and concepts that are useful in the rest of the paper.

Initial Value Problems An *Initial Value Problem* (IVP) is given by a first-order system of differential equations and a (vectorial) initial condition:

$$\frac{dy}{dt} = F(y,t), \quad y(0) = x \tag{4}$$

where F is some vector field. This paper only considers *autonomous* systems where there is no explicit dependance of the time t, i.e., $F(y,t) = F(y)$.

Definition 3.1. *A vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is said locally Lipschitz for the norm $\| \cdot \|$ if, for each pair $x, y \in \mathbb{R}^n$, there exists a constant $l > 0$ such that*

$$\|F(x) - F(y)\| \leq l\|x - y\|.$$

Whenever the vector field F is bounded and locally Lipschitz, the IVP problem (4) has a richer structure, characterized by the Local Existence and Uniqueness Theorem (e.g., [9, p. 385]).

Theorem 3.2. *If F is a bounded locally Lipschitz vector field on $\mathbb{R}^n$, then the autonomous IVP (4) admits a local unique solution $y_x : \mathbb{R} \to \mathbb{R}^n$.*

For convenience, we denote $y_x(t)$ by $\psi^t(x)$ and call $\psi = (\psi^t)_{t\in\mathbb{R}}$ the *(solution) flow* induced by the vector field F. This paper models dynamical systems whose behaviors over time are modeled by a differential equation

$$\frac{dy}{dt} = F(y). \tag{5}$$

The concept of equilibrium is central to such studies.

Definition 3.3. *A vector $y^* \in \mathbb{R}^n$ is an equilibrium for differential equation (5) if $F(y^*) = 0$.*

We are interested in equilibria that satisfy (at least) certain stability criterion.

Definition 3.4. *An equilibrium y^* is said to be stable for Equation (5) if, given $\epsilon > 0$, there exists $\delta > 0$ such that $\|\psi^t(y) - y^*\| < \epsilon$ for all $t > 0$ and for all y such that $\|y - y^*\| < \delta$.*

Definition 3.5. *An equilibrium y^* is said to be asymptotically stable for Equation (5) if y^* is stable and*

$$\lim_{t\to\infty} y(t) = y^*.$$

4 Trial-Offer Markets as Robbins-Monro Algorithms

This section shows that the discrete stochastic process ϕ^t $(t \geq 0)$ can be modeled as a Robbins-Monro Algorithm (RMA) [7, 13]).

Definition 4.1 (Robbins-Monro Algorithm)**.** A *Robbins-Monro Algorithm* (RMA) is a discrete time stochastic processes $\{x^t\}_{t\geq 0}$ whose general structure is specified by

$$x^{k+1} - x^k = \gamma^{k+1}[F(x^k) + U^{k+1}], \tag{6}$$

where

- x^k takes its values in some Euclidean space (e.g., $\mathbb{R}^n$);
- γ^k is deterministic and satisfies $\gamma^k > 0$, $\sum_{t\geq 1} \gamma^t = \infty$, and $\lim_{t\to\infty} \gamma^t = 0$;
- $F : \mathbb{R}^n \to \mathbb{R}^n$ is a deterministic continuous vector field;

- $\mathbb{E}[U^{k+1}|\mathcal{F}^k] = 0$, where $\mathcal{F}^k$ is the natural filtration of the entire process.[4]

Recall that the probability that the next purchase is item i at time k is given by $p_i(\phi^k)$ and denote by e^k the random unit vector whose j^{th} entry is 1 if item j is the next purchase and 0 otherwise. The market share at time $k+1$ is given by

$$\phi^{k+1} = \frac{D^k \phi^k}{D^k + 1} + \frac{e^k}{D^k + 1}$$

where $D^k = \sum_{t=0}^{k} \sum_{i=1}^{n} d_i^t$. It follows that

$$\begin{aligned}\phi^{k+1} &= \frac{(D^k+1)\phi^k}{D^k+1} - \frac{\phi^k}{D^k+1} + \frac{e^k}{D^k+1} \\ &= \phi^k + \frac{1}{D^k+1}(e^k - \phi^k) \\ &= \phi^k + \frac{1}{D^k+1}(\mathbb{E}[e^k|\phi^k] - \phi^k + e^k - \mathbb{E}[e^k|\phi^k]) \\ &= \phi^k + \frac{1}{D^k+1}(p(\phi^k) - \phi^k + e^k - \mathbb{E}[e^k|\phi^k])\end{aligned}$$

This last equality can be reformulated as

$$\phi^{k+1} = \phi^k + \gamma^{k+1}(F(\phi^k) + U^{k+1}) \tag{7}$$

where $\gamma^{k+1} = \frac{1}{D^k+1}$, $F(\phi) = p(\phi) - \phi$, and $U^{k+1} = e_k - \mathbb{E}[e^k|\phi^k]$. Note that the function F captures the difference between the probabilities of purchasing the items (given the market share) and the market shares at each time step. We can now prove that the discrete dynamic process $\{\phi^t\}_{t\geq 0}$ can be modeled as a Robbins-Monro algorithm.

Theorem 4.2. *The discrete stochastic dynamic process $\{\phi^t\}_{t\geq 0}$ can be modeled as the Robbins-Monro algorithm.*

Proof. The above derivation showed that $\{\phi^t\}_{t\geq 0}$ can be expressed through Equation (7), i.e.,

$$\phi^{k+1} = \phi^k + \gamma^{k+1}(F(\phi^k) + U^{k+1})$$

where $\gamma^{k+1} = \frac{1}{D^k+1}$, $F(\phi) = p(\phi) - \phi$, and $U^{k+1} = e_k - \mathbb{E}[e^k|\phi^k]$. It is easy to see that $\gamma^k > 0$, $\sum_{t\geq 1} \gamma^t = \infty$, $\lim_{t\to\infty} \gamma^t = 0$, and that "noise" $\mathbb{E}[U^{k+1}|\mathcal{F}^k]$ is equal to zero. □

Robbins-Monro algorithms are particularly interesting because, under certain conditions on x^t, γ^t, and U^t, their asymptotic behavior, i.e., the values of x^t when $t \to \infty$ is closely related to the asymptotic behavior of the following continuous dynamic process:

$$\frac{dx^t}{dt} = F(x^t), \quad x^t \in \mathbb{R}^n. \tag{8}$$

This idea, called *the ODE Method*, was introduced by Ljung [15] and has been extensively studied (e.g., [4, 7, 13]). Consider again the RMA $\{x^t\}_{t\geq 0}$ defined in (6) and the following hypotheses:

H1: $\sup_t \mathbb{E}[\|U^{t+1}\|^2] < \infty$;

H2: $\sum_t (\gamma^t)^2 < \infty$;

[4] Because of this last condition on U^k, a Robbins-Monro Algorithm is also known as a *Martingale Difference Stochastic Approximation.*

H3: $\sup_t \|x^t\| < \infty$.

We will now present a theorem establishing the connection between the discrete stochastic process (6), and the continuous process defined (8). We first need a few concepts.

Definition 4.3. *A set $L \subset \mathbb{R}^K$ is an invariant for Ψ if $\Psi_t(L) = L$ for all $t \in \mathbb{R}$. If L is invariant for Ψ, we denote $\Psi_t^L(x)$ the restriction of $\Psi_t(x)$ for $x \in L$.*

Definition 4.4. *An attractor for Ψ^L is a nonempty compact invariant set $A \subset L$ such that*

$$\sup_{x \in B(A)} \lim_{t \to \infty} dist(\Psi_t^L(x), A) = 0.$$

where

$$dist(y, Z) := \inf_{z \in Z} \|y - z\|$$

is the distance of a point y to a set Z and $B(A) \subset L$ is an open set called the basin of attraction of A. If $A \neq L$, then A is called a proper attractor.

Definition 4.5. *A compact invariant set L is said to be internally chain transitive (ICT) if Ψ^L has no proper attractor.*

We are now ready to state the relationship between the limit set $L\{x_t\}_{t \geq 0}$ of a sample path $\{x_t\}_{t \geq 0}$ for Equation (6) and the limit sets of the solution to Equation (8). The result is due to Benaïm [3].

Theorem 4.6. *Let $\{x_t\}_{t \geq 0}$ be a Robbins-Monro algorithm (6) satisfying hypotheses $H1 - H3$ where F is locally Lipschitz. Then, with probability 1, the limit set $L\{x_t\}_{t \geq 0}$ is internally chain transitive for Equation (8).*

Since hypotheses H1, H2, and H3 are all satisfied by the discrete stochastic dynamic process $\{\phi^t\}_{t \geq 0}$, it remains to study the structure of the ICT set of Equation (8).

5 Equilibria of Trial-Offer Markets

This section characterizes the equilibria of the continuous dynamics

$$\frac{d\phi^t}{dt} = p(\phi^t) - \phi^t \quad (\phi^t \in \Delta^{n-1}) \tag{9}$$

associated with the RMA (7). For simplicity, we remove the visibilities by stating $\overline{q}_j = v_j q_j$. We are mainly interested in the case where $f(x) = x^r$ with $(0 < r < 1)$, since the case $r = 1$ has been settled in earlier work.

Theorem 5.1. *Let $f(x) = x^r, 0 < r < 1$. Then, there is a unique equilibrium to Equation (9) in $int(\Delta^{n-1})$, the interior of the simplex, specified by*

$$\phi^* = \frac{1}{\sum_j \overline{q}_j^{\frac{1}{1-r}}} [\overline{q}_1^{\frac{1}{1-r}}, ..., \overline{q}_n^{\frac{1}{1-r}}]$$

The remaining equilibria are on the boundary of the simplex.

Proof. An equilibrium to (9) must satisfy $p_i(\phi) = \phi_i$, i.e.,

$$\frac{\overline{q}_i(\phi_i)^r}{\sum_{j=0}^n \overline{q}_j(\phi_j)^r} = \phi_i$$

Assume first that $\phi_i > 0$ for all i. We have

$$\overline{q}_i(\phi_i)^{r-1} = \sum_{j=0}^{n} \overline{q}_j(\phi_j)^r.$$

Since this equality is valid for all i, we have

$$\overline{q}_1(\phi_1)^{r-1} = \ldots = \overline{q}_i(\phi_i)^{r-1} = \ldots = \overline{q}_n(\phi_n)^{r-1}$$

and

$$\overline{q}_i(\phi_i)^{r-1} = \overline{q}_j(\phi_j)^{r-1} \Leftrightarrow \phi_i = \left(\frac{\overline{q}_j}{\overline{q}_i}\right)^{\frac{1}{r-1}} \phi_j. \tag{10}$$

By summing for $i = 1$ to n, we obtain

$$1 = \sum_{i=1}^{n} \phi_i = \frac{\phi_j}{\overline{q}_j^{1/(1-r)}} \sum_{i=1}^{n} \overline{q}_i^{1/(1-r)}$$

and hence

$$\phi_j = \frac{\overline{q}_j^{1/(1-r)}}{\sum_{i=1}^n \overline{q}_i^{1/(1-r)}}$$

Finally, assume that there exists a set $Q \subset \{1,..,n\}$ ($|Q| < n$) of indexes such that $\phi_i = 0$ if $i \in Q$. We can remove these elements and proceed in a similar fashion as above: If $j \notin Q$, then we have

$$\phi_j = \frac{\overline{q}_j^{1/(1-r)}}{\sum_{i \notin Q} \overline{q}_i^{1/(1-r)}}.$$

This last type of equilibrium lies on the boundary of the simplex and, if $|Q| = n - 1$, then the equilibrium is a vertex of the simplex. □

Observe that the equilibrium $\phi^* \in int(\Delta^K)$ for the case $0 < r < 1$ has some very interesting properties: It is unique, which means that the market is completely predictable. Moreover, if $\overline{q}_i \geq \overline{q}_j$, then $\phi_i^* \geq \phi_j^*$, which endows the market with a basic notion of *fairness*. Finally, the market does not go to a monopoly contrary to the case $r \geq 1$.

6 Convergence of the Continuous Dynamics

Our next result characterizes the ICT of the continuous dynamics. We start with a useful lemma which indicates that submarkets can also be modeled as RMAs.

Lemma 6.1. *Consider a trial-offer market defined by n items and the submarket obtained by considering only $n - 1$ items. This submarket can also be modeled by an RMA.*

Proof. Let $\Phi^t = [\phi_1^t, \phi_2^t, ..., \phi_n^t]$ be the market share for the n-item trial-offer market at stage t. Consider a new process $\{\Psi^t\}_{t \geq 0}$ consisting of $n - 1$ products only. We show that this process can also be modeled as a RMA. The key is to prove that the probability of purchasing product j in stage t follows Equation (2).

Consider any item $i \in \{1, .., n\}$ such that $\phi_i^t \neq 1$. Without loss of generality, assume that $i = n$, define

$$\psi_i^t = \frac{\phi_i^t}{1 - \phi_n^t}, \quad (i < n),$$

and consider the following events:

- $A = \{\text{product } n \text{ is not purchased at stage } t\}$
- $B = \{\text{product } j \neq n \text{ is purchased at stage } t\}$.

Since $B \subseteq A$, $Pr[B \cap A] = Pr[B] = \dfrac{\overline{q}_j (\phi_j^t)^r}{\sum_{i=1}^n \overline{q}_i (\phi_i^t)^r}$. On the other hand

$$Pr[A] = 1 - \frac{\overline{q}_n (\phi_n^t)^r}{\sum_{i=1}^n \overline{q}_i (\phi_i^t)^r} = \frac{\sum_{j=1}^{n-1} \overline{q}_j (\phi_j^t)^r}{\sum_{i=1}^n \overline{q}_i (\phi_i^t)^r},$$

and therefore

$$Pr[B|A] = \frac{\overline{q}_j (\phi_j^t)^r}{\sum_{i=1}^{n-1} \overline{q}_i (\phi_i^t)^r} \cdot \frac{(1 - \phi_n^t)^r}{(1 - \phi_n^t)^r} = \frac{\overline{q}_j (\psi_j^t)^r}{\sum_{i=1}^{n-1} \overline{q}_i (\psi_i^t)^r}.$$

Since $\psi_i^t \geq 0$ and $\sum_{i=1}^{n-1} \psi_i^t = \sum_{i=1}^{n-1} \frac{\phi_i^t}{1-\phi_n^t} = \frac{1}{1-\phi_n^t} \sum_{i=1}^{n-1} \phi_i^t = 1$, the ψ_i^t are well-defined market shares. Since all the remaining properties from the original model Φ^t still holds, $\{\Psi^t\}_{t \geq 0}$ can be modeled as a $n - 1$ dimensional RMA. $\square$

We are now in position to prove one of the main results of this paper.

Theorem 6.2. *Under the social signal $f(x) = x^r, 0 < r < 1$ with $\phi^0 \in int(\Delta^n)$, the RMA $\{\phi^t\}_{t>0}$ converges to ϕ^* almost surely.*

Proof. The proof studies the asymptotic behaviour of the solutions of the following ODE:

$$\frac{d\phi^t}{dt} = p(\phi^t) - \phi^t. \tag{11}$$

Equation (11) is equivalent to

$$\frac{d\phi_i^t}{dt} = \frac{\overline{q}_i (\phi_i^t)^r}{\sum_j \overline{q}_j (\phi_j^t)^r} - \phi_i^t, \quad i \in \{1, ..., n\}.$$

Hence, we have

$$\begin{aligned}
&\frac{\overline{q}_i (\phi_i^t)^r}{\sum_j \overline{q}_j (\phi_j^t)^r} = \frac{d\phi_i^t}{dt} + \phi_i^t, \\
&\frac{1}{\sum_j \overline{q}_j (\phi_j^t)^r} = \frac{1}{\overline{q}_i (\phi_i^t)^r} [\frac{d\phi_i^t}{dt} + \phi_i^t] \quad \text{ if } \phi_i^t \neq 0, \\
&\frac{1}{\overline{q}_i (\phi_i^t)^r} [\frac{d\phi_i^t}{dt} + \phi_i^t] = \frac{1}{\overline{q}_j (\phi_j^t)^r} [\frac{d\phi_j^t}{dt} + \phi_j^t] \quad \forall i, j \in \{1, ..., n\}, \\
&\overline{q}_i^{-1} (\phi_i^t)^{-r} [\frac{d\phi_i^t}{dt} + \phi_i^t] = \overline{q}_j^{-1} (\phi_j^t)^{-r} [\frac{d\phi_j^t}{dt} + \phi_j^t], \\
&\overline{q}_i^{-1} [(\phi_i^t)^{-r} \frac{d\phi_i^t}{dt} + (\phi_i^t)^{1-r}] = \overline{q}_j^{-1} [(\phi_j^t)^{-r} \frac{d\phi_j^t}{dt} + (\phi_j^t)^{1-r}], \\
&e^{(1-r)t} (1-r) \overline{q}_i^{-1} [(\phi_i^t)^{-r} \frac{d\phi_i^t}{dt} + (\phi_i^t)^{1-r}] = e^{(1-r)t} (1-r) \overline{q}_j^{-1} [(\phi_j^t)^{-r} \frac{d\phi_j^t}{dt} + (\phi_j^t)^{1-r}], \\
&\frac{d}{dt} \left[e^{(1-r)t} \overline{q}_i^{-1} (\phi_i^t)^{1-r} \right] = \frac{d}{dt} \left[e^{(1-r)t} \overline{q}_j^{-1} (\phi_j^t)^{1-r} \right]
\end{aligned}$$

where the fourth equivalence is obtained by multipling both sides with $\mu(t) = (1-r)e^{(1-r)t}$. Taking the integral $\int_0^t dt$ of the last expression gives

$$e^{(1-r)t}\overline{q}_i^{-1}(\phi_i^t)^{1-r} - \overline{q}_i^{-1}(\phi_i^0)^{1-r} = e^{(1-r)t}\overline{q}_j^{-1}(\phi_j^t)^{1-r} - \overline{q}_j^{-1}(\phi_j^0)^{1-r} \tag{12}$$

and hence

$$\frac{(\phi_i^t)^{1-r}}{\overline{q}_i} - \frac{(\phi_j^t)^{1-r}}{\overline{q}_j} = e^{(r-1)t}\left[\frac{(\phi_i^0)^{1-r}}{\overline{q}_i} - \frac{(\phi_j^0)^{1-r}}{\overline{q}_j}\right]. \tag{13}$$

Consider Equation (13):

- if, for some $i \neq j$, $\dfrac{(\phi_i^0)^{1-r}}{\overline{q}_i} = \dfrac{(\phi_j^0)^{1-r}}{\overline{q}_j}$, then $\dfrac{(\phi_i^t)^{1-r}}{\overline{q}_i} = \dfrac{(\phi_j^t)^{1-r}}{\overline{q}_j}$, for all t;
- if $\dfrac{(\phi_i^0)^{1-r}}{\overline{q}_i} \neq \dfrac{(\phi_j^0)^{1-r}}{\overline{q}_j}$, the right-hand side of Equation (13) goes to zeroas $t \to \infty$ (because $r < 1$) and hence

$$\lim_{t\to\infty} \frac{(\phi_i^t)^{1-r}}{\overline{q}_i} - \frac{(\phi_j^t)^{1-r}}{\overline{q}_j} = 0. \tag{14}$$

We now prove by induction that the limits for the market shares exist. Consider first the case of 2 products. Since $\phi_2^t = (1-\phi_1^t)$, the market can be modeled as a one-dimensional RMA. By Theorem 1 in [16], the RMA converges under mild conditions trivially satisfied here. Assume now that a RMA with $k-1$ products converges and consider a market with k products. By Lemma 6.1, given a k-dimensional RMA $\Phi^t = [\phi_1^t, \phi_2^t, ..., \phi_k^t]$, we can create a $k-1$ dimensional RMA $\{\Psi^t\}_{t\geq 0}$ given by $\psi_i^t = \frac{\phi_i^t}{1-\phi_k^t}, \quad i < k,$. By induction, $\psi_i = \lim_{t\to\infty} \psi_i^t$ exists for all $i < k$ and therefore Equation (13) is equivalent to

$$\frac{(\phi_k^t)^{1-r}}{\overline{q}_k(1-\phi_k^t)^{1-r}} - \frac{(\psi_i^t)^{1-r}}{\overline{q}_i} = \frac{e^{(r-1)t}}{(1-\phi_k^t)^{1-r}}\left[\frac{(\phi_k^0)^{1-r}}{\overline{q}_k} - \frac{(\phi_i^0)^{1-r}}{\overline{q}_i}\right] \tag{15}$$

The right hand side of (15) goes to 0 when $t \to \infty$ and $\lim_{t\to\infty} \dfrac{(\psi_i^t)^{1-r}}{\overline{q}_i}$ exists. Hence $\lim_{t\to\infty} \dfrac{(\phi_n^t)^{1-r}}{\overline{q}_n(1-\phi_n^t)^{1-r}}$ also exists.

Now denote by ϕ_j the limit of ϕ_j^t for all $j \in \{1, ..., n\}$. Using Equation (14), the following equation holds for all $i, j \in \{1, ..., n\}$:

$$\frac{\phi_i^{1-r}}{\overline{q}_i} = \frac{\phi_j^{1-r}}{\overline{q}_j}. \tag{16}$$

Observe that, if there exists $l \in \{1, ..., n\}$ such that $\phi_l = 0$, Equation (16) implies that $\phi_i = 0$ for all i which is impossible since they sum to 1. Hence the limit process has strictly positive components and Equation (16) is equivalent to

$$\phi_i = \frac{\phi_j}{\overline{q}_j^{1/(1-r)}}\overline{q}_i^{1/(1-r)} \tag{17}$$

which is the equation that defines ϕ^* in Theorem 5.1 (see Equation (10)). As a result, when $\Phi_0 \in int(\Delta^{n-1})$, the only ICT set for the ODE (11) is the equilibrium ϕ^* and the RMA converges to ϕ^*. □

Consider now the case $r > 1$. We can apply the same reasoning as in Theorem 5.1 to characterize the equilibria. However, as we will show the dynamic behaviour is completely different due to the strength of the social signal. In particular, the inner equilibrium Φ^* is now unstable. The proof uses the following characterisation of unstability for one-dimensional RMA due to Renlund.

Proposition 6.3. *([16]) Consider a one-dimentional RMA with $F(x) = p(x) - x$. A point x^* is unstable if there exists a neighbourhood N_{x^*} around x^* such that $F(x)[x - x^*] \geq 0$ for all $x \in N_{x^*}$.*

Theorem 6.4. *For a social signal $f(x) = x^r$ with $r > 1$, the inner equilibrium is unstable.*

Proof. Let $x = [x_1, x_2]$ be the market share for the case of 2 products (at any time t). Since $x_2 = 1 - x_1$, $x_i^* = \dfrac{q_i^{1/(1-r)}}{q_1^{1/(1-r)} + q_2^{1/(1-r)}}$. Let $c_i = q_i^{1/(1-r)}$ ($\Rightarrow q_i = c_i^{1-r}$). We have

$$\begin{aligned}
[p(x_1) - x_1][x_1 - x_1^*] &= \left[\frac{c_1^{1-r} x_1^r}{c_1^{1-r} x_1^r + c_2^{1-r}(1-x_1)^r} - x_1\right]\left[x_1 - \frac{c_1}{c_1 + c_2}\right] \\
&= \frac{\left[c_1^{1-r} x_1^r (1-x_1) - c_2^{1-r}(1-x_1)^r x_1\right][x_1(c_1+c_2) - c_1]}{(c1 + c2)(c_1^{1-r} x_1^r + c_2^{1-r}(1-x_1)^r)} \\
&= \frac{[x_1(1-x_1)][c_1^{1-r} x_1^{r-1} - c_2^{1-r}(1-x_1)^{r-1}][x_1(c_1+c_2) - c_1]}{(c_1+c_2)(c_1^{1-r} x_1^r + c_2^{1-r}(1-x_1)^r)}.
\end{aligned}$$

Since the denominator is always positive and $x_1(1 - x_1) \geq 0$, we only need to study the sign of $[c_1^{1-r} x_1^{r-1} - c_2^{1-r}(1-x_1)^{r-1}][x_1(c_1+c_2) - c_1]$. Then,

$$\begin{aligned}
x_1(c_1+c_2) - c_1 > 0 &\Leftrightarrow x_1 > \frac{c_1}{c_1+c_2} \\
&\Leftrightarrow 1 - x_1 < 1 - \frac{c_1}{c_1+c_2} = \frac{c_2}{c_1+c_2} \\
&\Leftrightarrow [c_1^{1-r} x_1^{r-1} - c_2^{1-r}(1-x_1)^r x - 1] > [c_1^{1-r}\left(\frac{c_1}{c_1+c_2}\right)^{r-1} - c_2^{1-r}\left(\frac{c_2}{c_1+c_2}\right)^{r-1}]
\end{aligned}$$

Since $c_1^{1-r}\left(\frac{c_1}{c_1+c_2}\right)^{r-1} - c_2^{1-r}\left(\frac{c_2}{c_1+c_2}\right)^{r-1} = \frac{1}{(c_1+c_2)^{r-1}} - \frac{1}{(c_1+c_2)^{r-1}} = 0$, $[p(x_1) - x_1][x_1 - x_1^*] \geq 0$ for all $x_1 \in (0, 1)$. And hence x^* is an unstable equilibrium. $\square$

Theorem 6.5. *Consider the social signal $f(x) = x^r$ with $r > 1$. The RMA $\{\phi^t\}_{t \geq 0}$ converges almost surely to one of the equilibria $\phi \in Z_F := \{x \in \Delta^{n-1} : p(x) - x = 0\}$.*

Proof. The analysis of the ODE is the same as in Theorem 6.2 since the only restriction in the proof is $r \neq 1$. However, the interpretation of Equation (13) changes when $r > 1$.

We let $H_{i,0} := \dfrac{(\phi_i^0)^{1-r}}{\overline{q}_i}$ and order the products in decreasing order of $H_{i,0}$. Let $h : \{1, .., n\} \to \{1, .., n\}$ be the permutation that defines this order and denotes by h^{-1} its inverse permutation, i.e., $h^{-1}(i) = j$ means that product j is in the i-th position in permutation h. We have that $H_{h^{-1}(1),0} \geq \cdots \geq H_{h^{-1}(n),0}$. Define the following sets:

- $Q_0 = \{i \in \{1, .., n-1\} : H_{h^{-1}(i),0} = H_{h^{-1}(i+1),0}\}$,
- $Q_1 = \{i \in \{1, .., n-1\} : H_{h^{-1}(i),0} > H_{h^{-1}(i+1),0}\}$,

and consider the following case analysis:

i) If $|Q_0| = n-1$, then $H_{h^{-1}(i),0} = H_{h^{-1}(i+1),0}$ for all $1 < i < n-1$. By Equation (13), $\frac{(\phi^t_{h^{-1}(i)})^{1-r}}{\overline{q}_{h^{-1}(i)}} = \frac{(\phi^t_{h^{-1}(i+1)})^{1-r}}{\overline{q}_{h^{-1}(i+1)}}$, for all $t > 0$ and for all $1 < i < n-1$, which is the inner equilibrium.

ii) If $0 < |Q_0| < n-1$, select $i \notin Q_0$. Equation (13) implies that

$$\lim_{t\to\infty} \frac{(\phi^t_{h^{-1}(i)})^{1-r}}{\overline{q}_{h^{-1}(i)}} - \frac{(\phi^t_{h^{-1}(i+1)})^{1-r}}{\overline{q}_{h^{-1}(i+1)}} = \infty,$$

because $r > 1$ and hence $e^{(r-1)t} \to \infty$ when $t \to \infty$. It follows that $\lim_{t\to\infty} \phi^t_{h^{-1}(i)} = 0$ and the RMA necessarily converges to one of the equilibria that live in the boundary of the simplex (see Theorem 5.1).

iii) If $|Q_0| = 0$ then $|Q_1| = n-1$, Using a similar reasoning as in case (ii), it follows that $\lim_{t\to\infty} \phi^t_{h^{-1}(i)} = 0$ for all $1 < i < n-1$ and, since $\phi^t \in \Delta^{n-1}$ for all t, $\lim_{t\to\infty} \phi^t_{h^{-1}(n)} = 1$.

As a result, the only ICT for the differential equation (11) are equilibria and the RMA $\{\phi^t\}_{t\geq 0}$ converges almost surely to one of them. □

It is important to observe that, in the case $r > 1$, the initial condition affects the entire dynamics. This is in contrast with the case $r < 1$ for which the long-term behaviour is independent of the initial condition. This has obviously fundamental consequences for the predictability of the market. Note also that, by Theorem 6.4, ϕ^* is an unstable equilibrium when $r > 1$ and hence the RMA $\{\phi^t\}_{t\geq 0}$ most likely converges to one of the monopolies $\phi \in Z_F \setminus \{\phi^*\}$.

7 Agent-Based Simulation Results

To highlight and complement the theoretical results, We now report results from a agent-based simulation to highlight and complement the theoretical analysis. The agent-based simulation uses settings that model the MusicLab experiments discussed in [2, 12, 17]. As mentioned in the introduction, the MusicLab is a trial-offer market where participants can try a song and then decide to download it. The generative model of the MusicLab [12] uses the consumer choice preferences described in Section 2.

The Simulation Setting The agent-based simulation aims at emulating the MusicLab: Each simulation consists of N iterations and, at each iteration t,

1. the simulator randomly selects a song i according to the probabilities $p_i(\sigma, d)$, where σ is the ranking proposed by the policy under evaluation and d is the social influence signal;

2. the simulator randomly determines, with probability q_i, whether selected song i is downloaded. In the case of a download, the simulator increases the number of downloads of song i, i.e., $d_i^{t+1} = d_i^t + 1$, changing the market shares. Otherwise, $d_i^{t+1} = d_i^t$.

Every r iterations, a new list σ is computed using one of the ranking policies. In this paper, the simulation setting focuses mostly on two policies for ranking the songs:

- The *quality ranking* (Q-rank) that assigns the songs in decreasing order of quality to the positions in decreasing order of visibility (i.e., the highest quality song is assigned to the position with the highest visibility and so on);

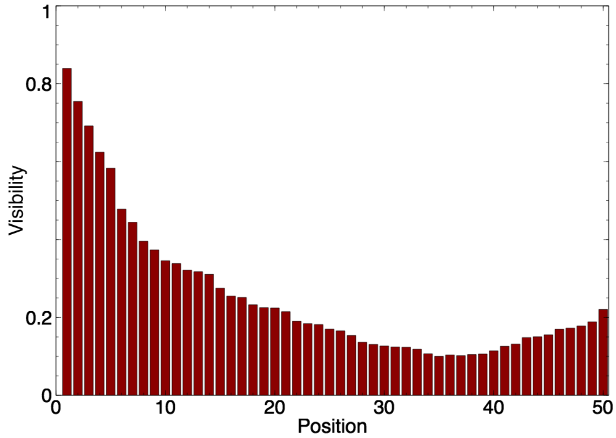

Figure 1: The visibility v_p (y-axis) of position p in the song list (x-axis) where $p = 1$ is the top position and $p = 50$ is the bottom position of the list which is displayed in a single column.

- The *popularity ranking* (D-rank) that assigns the songs in decreasing order of popularity (i.e., d_i^t) to the positions in decreasing order of visibility (i.e., the most popular song is assigned to the position with the highest visibility and so on);

Note that the popularity ranking was used in the original MusicLab, while the quality ranking is a static policy: the ranking remains the same for the entire simulation. The simulation setting, which aims at being close to the MusicLab experiments, considers 50 songs and simulations with 20,000 steps. The songs are displayed in a single column. The analysis in [12] indicated that participants are more likely to try songs higher in the list. More precisely, the visibility decreases with the list position, except for a slight increase at the bottom positions. Figure 1 depicts the visibility profile based on these guidelines, which is used in all computational experiments.

The paper also uses two settings for the quality and appeal of each song, which are depicted in Figure 2. In the first setting, the quality and the appeal were chosen independently according to a Gaussian distribution normalized to fit between 0 and 1. The second setting explores an extreme case where the appeal is negatively correlated with quality. The quality of each product is the same as in the first setting but the appeal is chosen such that the sum of appeal and quality is 1. The results were obtained by averaging the results of $W = 400$ simulations.

7.1 Convergence

We first illustrate the convergence of the market for various popularity signals ($r < 1$) using the quality ranking. In order to visualize the results, we focus on only 5 songs, where the qualities, appeals, and visibilities are given by

q = [0.8, 0.72, 0.68, 0.65, 0.60]
a = [0.38, 0.35, 0.46, 0.27, 0.62]
v = [0.8, 0.75, 0.69, 0.62, 0.58]

The simulation is run for 10^5 time steps for the social signals $f(x) = x^r (r \in \{0.1, 0.25, 0.5, 0.75\})$ and Figure 3 depicts the simulation results. Observe that the equilibrium ϕ^* (dashed lines) changes because it depends of the value of r. Interestingly, for social signals with $r \leq 0.5$, the convergence of the process seems to occur around 10^4 time steps even when they start with a strong initial distortion due to the appeals of the songs. The simulations show clear differences in behavior depending on r and, when r moves closer to 1, the market tends to exhibit a monopolistic behaviour for the song with the best quality (confirming the results obtained in [19]).

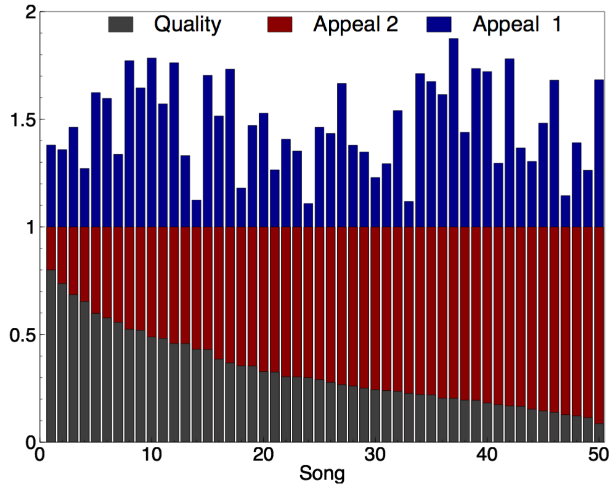

Figure 2: The Quality q_i (gray) and Appeal a_i (red and blue) of song i in the two settings. The settings only differ in the appeal of songs, and not in the quality of songs. In the first setting, the quality and the appeal for the songs were chosen independently according to a Gaussian distribution normalized to fit between 0 and 1. The second setting explores an extreme case where the appeal is negatively correlated with the quality used in setting 1. In this second setting, the appeal and quality of each song sum to 1.

Figure 4 shows how the market is distributed in the equilibrium for 6 other songs taken from the MusicLab. The qualities, appeals, and visibilities are given by

q = [0.8,0.72,0.65,0.57,0.52,0.4887]
A = [0.38,0.36,0.27,0.60,0.77,0.78]
v = [0.8,0.75,0.62,0.48,0.40,0.35]

In the figure, the songs are ordered from left to right by increasing quality and the social signals are of the form $f(x) = x^r, \quad r \in \{0.1, 0.25, 0.5, 0.75\}$. Songs with better qualities (i.e., the top 2 songs on the right in this case) have larger market shares and their market shares increases with r. In contrast, the market shares of the lower-quality songs (i.e., the bottom four songs on the left in this case) decreases when r increases. These results indicate that social influence has a beneficial effect on the market: It drives customers towards the better products, while not going to a monopology as long as $r < 1$.

7.2 Market Predictability

This section depicts the predictability of the market for various values of r and the number of downloads per song as a function of its quality. Figures 5 and 6 depict the results for the two quality/appeal settings discussing previously. The figures display the results of 200 experiments for each setting. Each simulation contributes 50 data points per simulation, i.e., the number of downloads for each song and all the data points for the 200 simulations are displayed in the figures. In the plots, the x-axis represents the song qualities and the y-axis the number of downloads. A dot at location (q, d) indicates that the song with quality q had d downloads in a simulation. Obviously, there can be several dots at the same location. For $r \in \{0.5, 0.75\}$, the market is highly predictable and there is little variation in the song downloads. For $r = 1$, the market converges to a monopoly for the song of highest quality, confirming the results from [2, 19]. Finally, for $r = 2$, the market exhibits significant unpredictability, as suggested by the

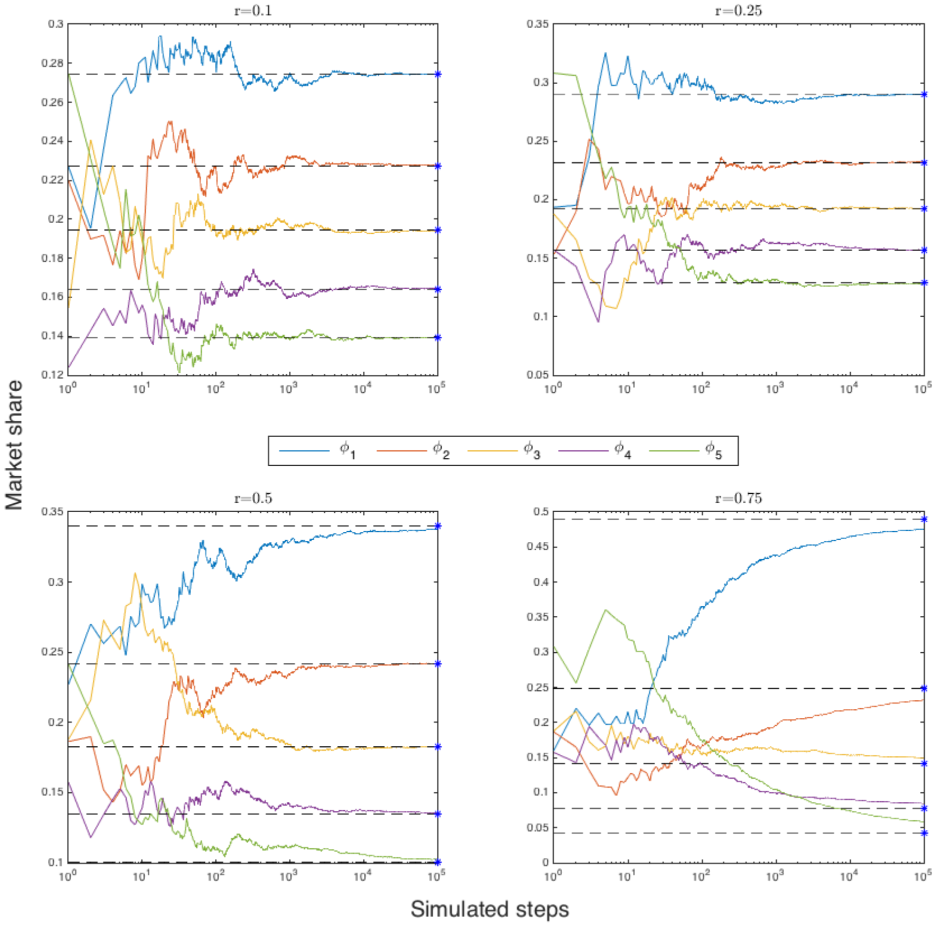

Figure 3: Evolution of market shares of 5 songs using a social signal $f(x) = x^r$, $r \in \{0.1, 0.25, 0.5, 0.75\}$. Dashed lines are the values of the equilibrium for each song.

theoretical results. In this case, the equilibria are monopolies for various songs but it is hard to predict which song will dominate the market. Note also that the unpredictability of the market increases significantly for $r = 2$ when the appeal and quality of the songs are negatively correlated. This is not the case for $r \in \{0.5, 0.75\}$ and less so for $r = 1$.

7.3 Performance of the Market

Figures 7 and 8 report results about the performance of the markets as a function of the social influence signals. The figures report the average number of downloads over time for the quality and popularity rankings as a function of the social signals for the signals. There are a few observations that deserve mention.

1. For the quality ranking, the expected number of downloads increase with the strength of the social signal as r converges to 1. The equilibrium when $r = 1$ is optimal asymptotically and assigns the entire market share to the song of highest quality. When $r = 2$, the situation is more complicated. In the second setting, when the simulation is run for more time steps, the market efficiency decreases slightly compared to $r = 1$, which is consistent with the

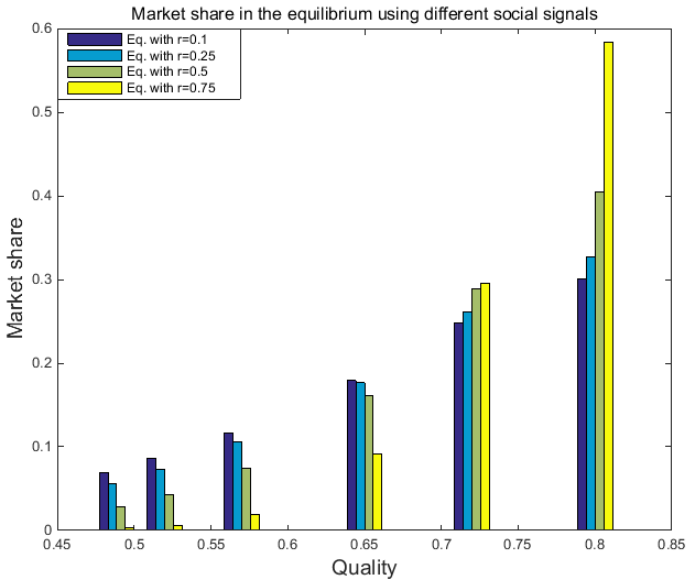

Figure 4: Market share of 6 songs respect their qualities, using a social signal $f(x) = x^r, \quad r \in \{0.1, 0.25, 0.5, 0.75\}$.

theory since there is no guarantee that the monopoly for $r > 1$ is is for the song of highest quality.

2. For the quality ranking, the improvements in market efficiency from social influence are quite substantial.

3. The popularity ranking is always dominated by the quality ranking and the benefits of the quality ranking increase as r increases.

4. The popularity ranking behaves in a catastrophic way when $r = 2$ in the second setting.

8 Discussion and Conclusion

This paper studied the role of social influence in trial-offer markets where customer preferences are modeled by a generalization of multinomial logic models. In this model, both position bias and social influence impact the products tried by consumers.

The main result of the paper is to show that trial-offer markets, when the ranking of the products is fixed, converge to a unique equilibrium for sublinear social signals of the form ϕ_i^r, where ϕ_i represents the market share of product i and $0 < r < 1$. Of particular interest is the fact that the equilibrium does not depend on the initial conditions, e.g., the product appeals: It only depends on the product qualities. Moreover, when the products are ranked by quality, i.e., the best products are assigned the highest visibilities, the equilibrium is such that the better products receive the largest market shares, which increase as r increases for the best products. The equilibrium for a sublinear social signal contrast with the case with $r = 1$, where the market goes to a monopoly for the highest quality product (under the quality ranking). In the sublinear case, the market shares reflect product quality but no product becomes a monopoly. The paper also shows that, when $r > 1$, the market becomes more unpredictable. In particular, the inner equilibrium, which assigns a market share to all products, is unstable and the market almost surely converges to a monopoly. However, this monopoly depends on the initial conditions.

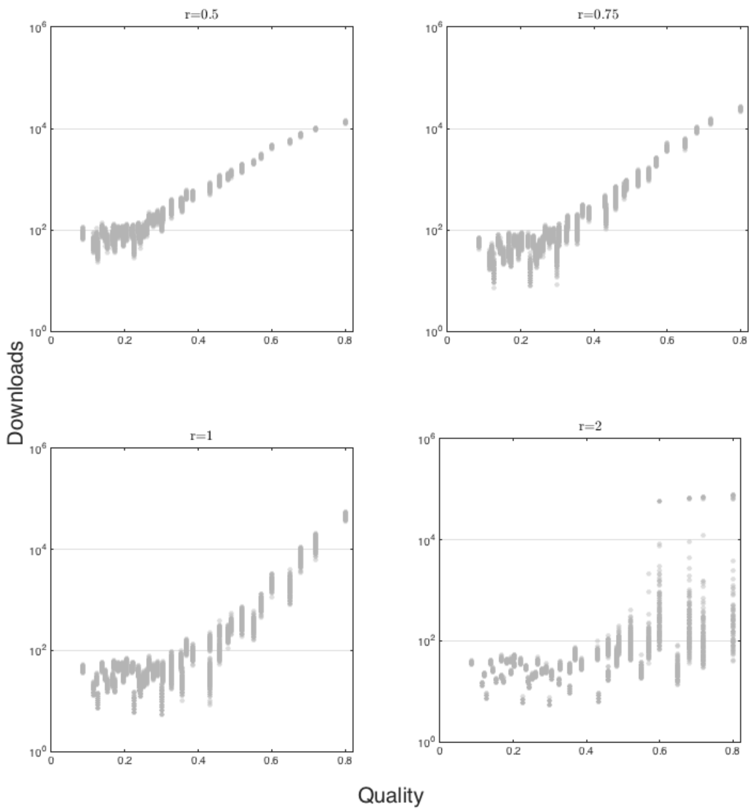

Figure 5: Distribution of Downloads Versus the Qualities, using Social Signals $f(x) = x^r, \quad r \in \{0.5, 0.75, 1, 2\}$. The results are for the first setting where the quality and appeal of each song are not correlated. The songs are ordered by increasing quality along the x-axis. The y-axis is the number of downloads.

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Simulation results on a setting close to the original MusicLab complemented the theoretical results. They show that the market converges quickly to the equilibrium for a sublinear social signal and that the convergence speed depends on the social signal. The simulation results also illustrate how the market shares of the highest (resp. lowest) quality products increase (resp. decrease) with r. As expected, when $r \leq 1$, the market is shown to be highly predictable, while it exhibits a lot of randomness when $r > 1$. The simulation results also show the benefits of social influence for market efficiency, i.e., the total number of purchases, and demonstrate that the quality ranking once again outperforms the popularity ranking.

Overall, these results shed a new light on the role of social influence in trial-offer markets and provide a comprehensive overview of the choices and tradeoffs available to firms interested in optimizing their markets with social influence. In particular, they show that social influence does not necessarily make markets unpredictable and is typically beneficial when the social signal is not too strong. Moreover, ranking the products by quality appears to be a much more effective policy than ranking products by popularity which may induce unpredictability and market inefficiency. The results also show that sublinear social signals give decision makers with the ability to trade market efficiency for more balanced market shares.

Perhaps the main message of this paper is that one has to be very careful in designing computational social science experiments. The findings by Salganik et al. [17] used the popularity

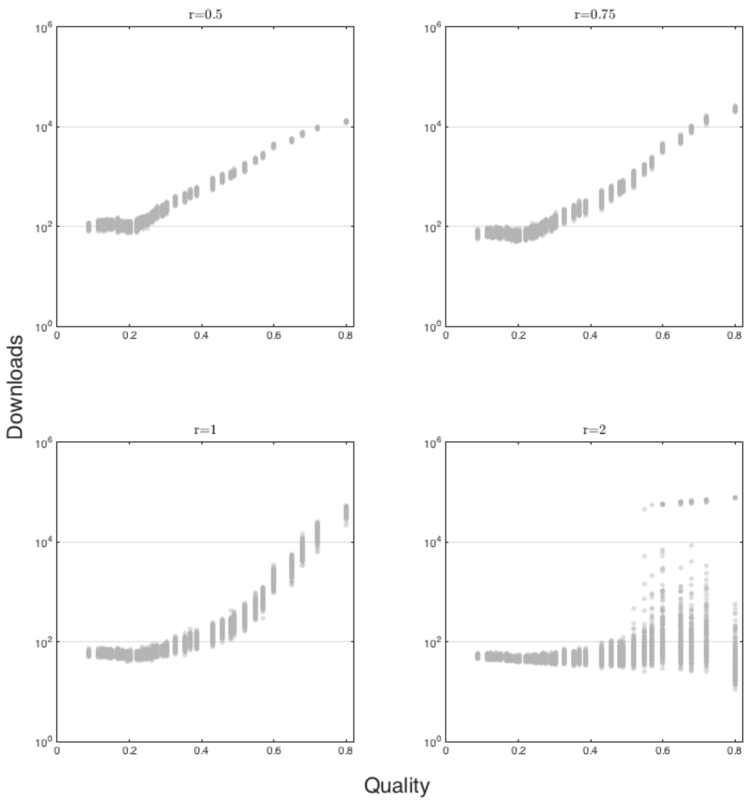

Figure 6: Distribution of Downloads Versus the Qualities, using Social Signals $f(x) = x^r, \quad r \in \{0.5, 0.75, 1, 2\}$. The results are for the first setting where the quality and appeal of each song are negatively correlated. The songs are ordered by increasing quality along the x-axis. The y-axis is the number of downloads.

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ranking, which significantly affected their conclusions about market unpredictability and efficiency. The theoretical and simulation results of this paper, together with those in [2, 19] for the case $r = 1$, show that the market is highly predictable when using any static ranking and $r \leq 1$. Moreover, the quality ranking is optimal asyptotically when $r = 1$ and dominates the popularity ranking in all our simulations which were modeled after the MusicLab. This does not diminish the value of the results by Salganik et al. [17] who isolated potential pathologies linked to social influence. But this paper shows that these pathologies are not inherent to the market but are a consequence of specific design choices in the experiment: The strength of the social signal and the ranking policy.

There are a few other observations that deserve to be mentioned. First, sublinear social influences are not always beneficial to the market contrary to the case $r = 1$ [2, 19]. Consider, once again, the quality ranking and assume that $q_1 \geq \ldots \geq q_n$. When there is no social signal, the probability of trying product i is given by

$$P_i^- = \frac{v_i a_i}{\sum_{j=1}^n v_j a_j}$$

if quality ranking is used and the expected number of purchases is

$$\sum_{i=1}^{n} P_i^- q_i.$$

With a sublinear social signal, the probability of trying product i at time t is

$$P_i(\phi^t) = \frac{v_i f(\phi_i^t)}{\sum_{j=1}^{n} v_j f(\phi_j^t)},$$

It is

$$P_i(\phi^*) = \frac{v_i f(\phi_i^*)}{\sum_{j=1}^{n} v_j f(\phi_j^*)},$$

at the equilibrium.

Example 8.1. *Consider a 2 dimensional trial-offer market with social signal $f(x) = x^{0.5}$, where the qualities, visibilities, and appeals are given by*

- $q_1 = 1,\ q_2 = 0.4,$
- $v_1 = 1,\ v_2 = 0.8.$
- $a_1 = 1,\ a_2 = 0.1,$

The expected number of purchases with the social signal at the equilibrium is about 0.88 with social influence and about 0.95 without the social signal.

The simple example is illuminating. It shows that the value of a sublinear social signal lies primarily in its ability to correct a misalignment of appeal and quality. If qualities and appeals are correlated, there is little use for a sublinear social influence signal. In contrast, when $r = 1$, social influence drives the market towards a monopoly, which leads to an asymptotically optimal market that assigns the entire market share to the highest quality product (which may be undesirable in practice). Note that, once the qualities and appeals have been recovered (using, say, Bernouilli sampling as suggested in [2]), it is easy to decide whether to use social influence when $r < 1$: Simply compare the expected number of purchases in both settings, using the equilibrium for the social influence case.

Observe also that the quality ranking is not necessarily optimal asymptotically when $r < 1$, in contrast to the case where $r = 1$. However, the best possible equilibrium (in terms of the expected number of purchases) can be computed in strongly polynomial time using the reduction to the linear fractional assignment problem proposed in [2].

Finally, note that this paper provides an alternative way to recover the quality of the products: Simply choose an arbitrary static rank and run the market with a sublinear signal until an equilibrium is reached. By Theorem 6.2, the market share of product i is proportional to $v_i q_i$, which allows us to recover the quality of the products. This observation is interesting for the following reason. When a sublinear social signal and a static ranking is used, the market share of a product is representative of its quality. This is not the case when $r > 1$ or when the popularity ranking is used.

Our current research is focusing on two main directions. First, we would like to validate the theory with a large-scale computational social science experiment to complement our simulation results. Second, we are generalizing these results to other settings, including assortment problems [1] and cascade models.

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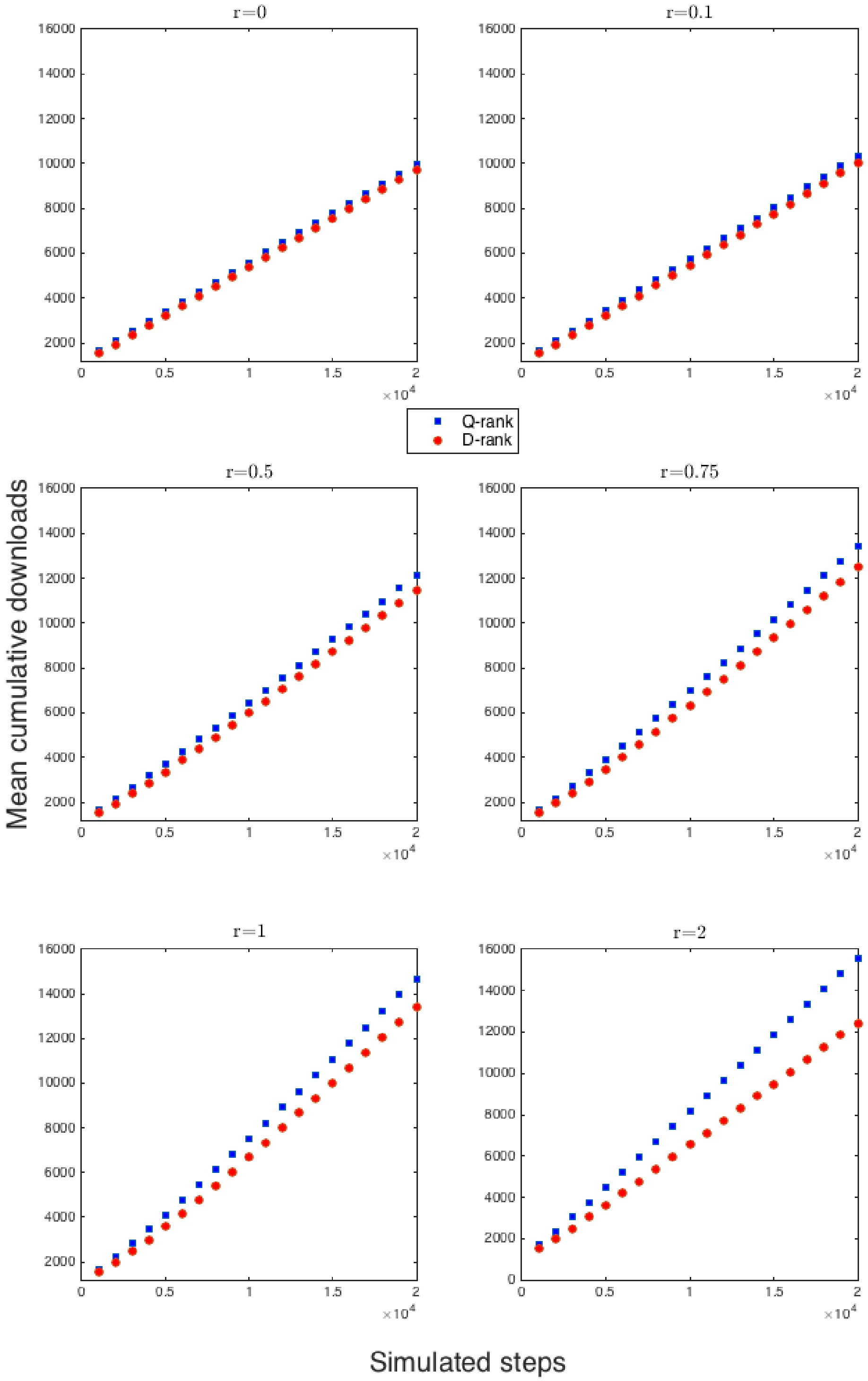

Figure 7: The Average Number of Downloads over Time for the Quality and Popularity Rankings for Various Social Signals in the First Setting for Song Appeal and Quality.

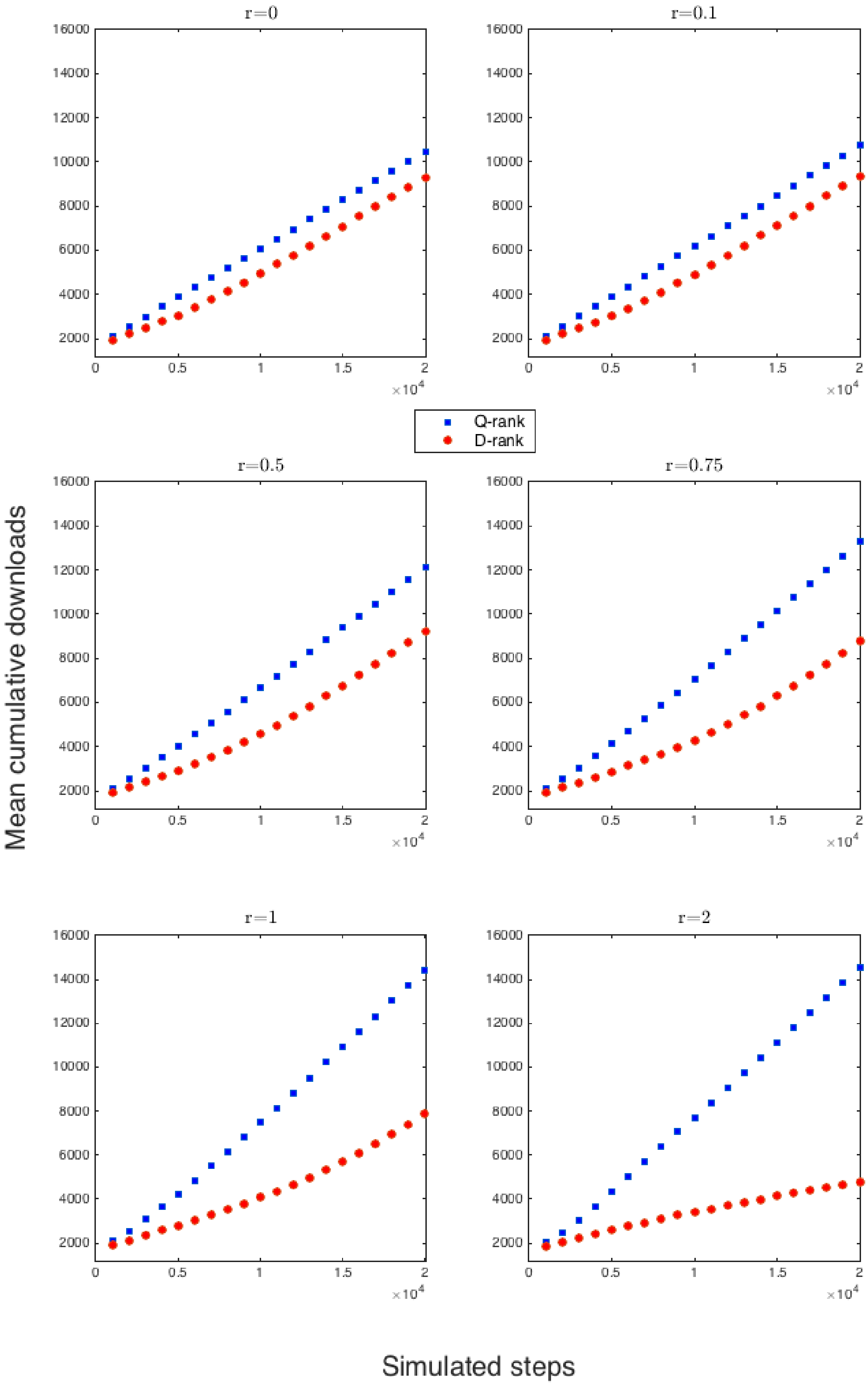

Figure 8: The Average Number of Downloads over Time for the Quality and Popularity Rankings for Various Social Signals in the Second Setting for Song Appeal and Quality.