DATA 61

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Panel:

- Pascal Van Hentenryck (Supervisor)
- Lexing Xie (Chair)
- Patrik Haslum
- Gerardo Berbeglia

Motivation

Plenty of markets where social influence is relevant in the customer‘s choices.

- 1
- 2
- 3
- 4

MusicLab Experiment. Salganik et al. (2006)

Inequality and unpredictability in a cultural market. Negative effects of social influence.

MODEL & ASSUMPTIONS

Model and assumptions

Marketplace

A marketplace consists of a set N of n items. Each item $i \in N$ is characterized by two values:

1. its *appeal* $a_i > 0$,
2. its *quality* $q_i > 0$.

Recovering qualities

Bernoulli sampling: $\hat{q}_{i,t} = \frac{d_{i,t}}{s_{i,t}}$

Rankings

Each position j of the ranking has a visibility v_j which represents the inherent probability of trying an item in position j.

Market share and social signals

Definition

The market share of the product i at stage $t \geq 0$ is given by:

$$\phi_{i,t} = \frac{d_{i,t}}{\sum_j d_{j,t}}, \text{ if } t > 0; \quad \phi_{i,0} = \frac{a_i}{\sum_j a_j}$$

Its evolution over time will depend (among others) of some social signal:

Assumptions

The probability that the customer tries item i is given by $P_i(\phi^t)$ where

$$P_i(\phi) = \frac{v_i f(\phi_i)}{\sum_{j=1}^{n} v_j f(\phi_j)} \tag{1}$$

and f is a continuous, positive, and nondecreasing function. And q_i is the conditional probability of downloading song if it was tried.

Lemma 1

The probability $p_i(\phi)$ that the next purchase is the product i given the market share vector ϕ is given by

$$p_i(\phi) \ = \frac{\overline{q}_i f(\phi_i)}{\sum_{j=1}^{n} \overline{q}_j f(\phi_j)}, \tag{2}$$

where $\overline{q}_i = v_i q_i$

THEORETICAL BACKGROUND

Stochastic approximation algorithms

Definition: Robbins-Monro Algorithm

A *Robbins-Monro Algorithm* (RMA) is a discrete time stochastic processes $\{x_k\}_{k\geq 0}$ whose general structure is specified by

$$x_{k+1} - x_k = \gamma_{k+1}[F(x_k) + U_{k+1}], \tag{3}$$

where

- x_k takes its values in some Euclidean space (e.g., $\mathbb{R}^n$);
- γ_k is deterministic and satisfies $\gamma_k > 0$, $\sum_{k\geq 1} \gamma_k = \infty$, and $\lim_{k\to\infty} \gamma_k = 0$;
- $F : \mathbb{R}^n \to \mathbb{R}^n$ is a deterministic continuous vector field;
- $\mathbb{E}[U_{k+1}|\mathcal{F}^k] = 0$, where $\mathcal{F}^k$ is the natural filtration of the entire process.

The ODE Method

Under certain conditions on x_k, γ_k, and U_k, the asymptotic behavior of x_k in Equation (**??**), is closely related to the asymptotic behavior of the following continuous dynamic process:

$$\frac{dx_t}{dt} = F(x_t), \quad x_t \in \mathbb{R}^n. \tag{4}$$

Definition: Internally Chain Transitivity (ICT)

A closed set A is an ICT if for every $\epsilon > 0$ and every $x, y \in A$, there is a set $\{x_i\}_{i=0}^{n} \subset A$ such that $x_0 = x, x_n = y$, and $\|F(x_i) - x_{i+1}\| < \epsilon$, $\forall i \in \{0, ..., n-1\}$.

Theorem 1.

Let $\{x_n\}_{n\geq 0}$ be a RMA (**??**), where F is of class $\mathcal{C}^2$. Then, with probability 1, the limit set $L\{x_n\}_{n\geq 0}$ is internally chain transitive for Equation (**??**).

Differential equations and dynamical systems.

An *Initial Value Problem* (IVP) is given by a first-order autonomous system of differential equations and a (vectorial) initial condition:

$$\frac{dy}{dt} = F(y), \quad y(0) = x. \tag{5}$$

Definition: Equilibria

A vector $y^* \in \mathbb{R}^n$ is an equilibrium for differential equation (**??**) if $F(y^*) = 0$.

Definition: Stability

An equilibrium y^* is said to be *stable* for Equation (**??**) if, given $\epsilon > 0$, there exists $\delta > 0$ such that $\|y(t) - y^*\| < \epsilon$ for all $t > 0$ and for all y such that $\|y - y^*\| < \delta$. We say that y^* is *asymptotic stable* if also satisfies

$$\lim_{t\to\infty} y(t) = y^*.$$

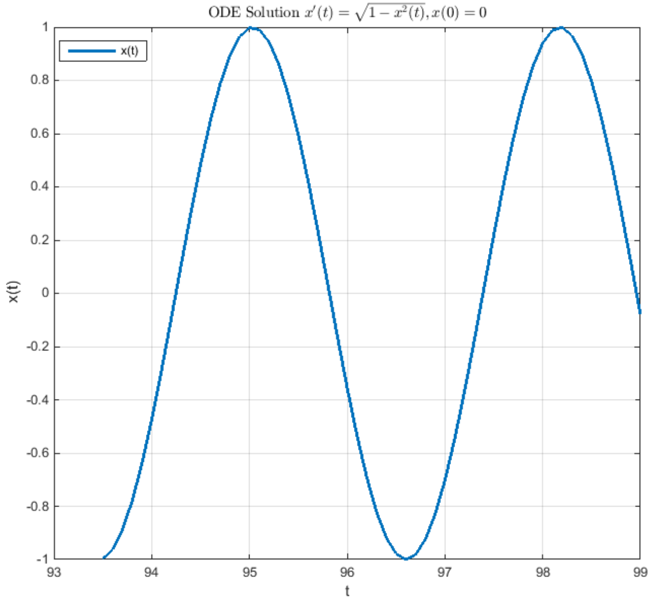
Stochastic approximation algorithms: Example

Consider $(x_n)_{n\in\mathbb{N}}$ a 1-dimensional RMA (**??**), with vector field $F(x) = 2\sqrt{1-x^2}$ and $\gamma_{n+1} = \frac{1}{n+1}$. Then, the associated continuous dynamic (**??**) with initial condition $x(0) = x_0$, has a solution $x(t) = \sin[2t + \arcsin(x_0)]$.

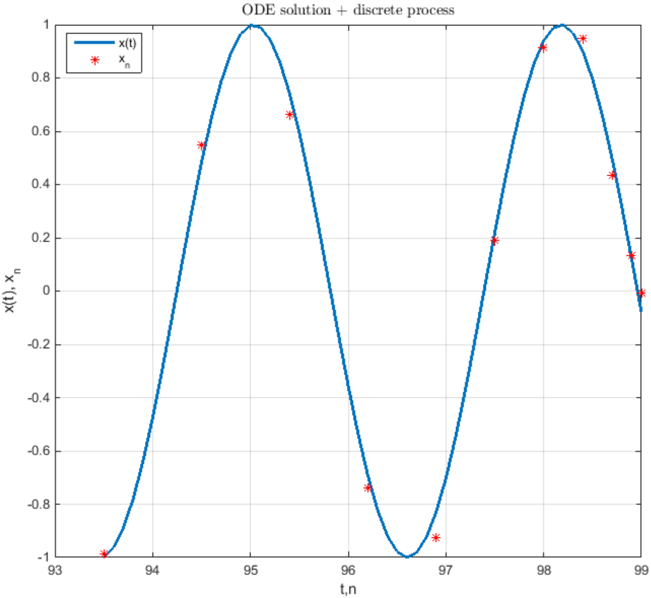
We set $\tau_n = \sum_{i=1}^{n} \gamma_i, \tau_0 = 0$, and with this we define an affine interpolated process $Z(t)$ given by:

$$Z(t) = x_n + [t - \tau_n]\frac{x_{n+1} - x_n}{\gamma_{n+1}}, \quad \tau_n \geq t \geq \tau_{n+1}$$

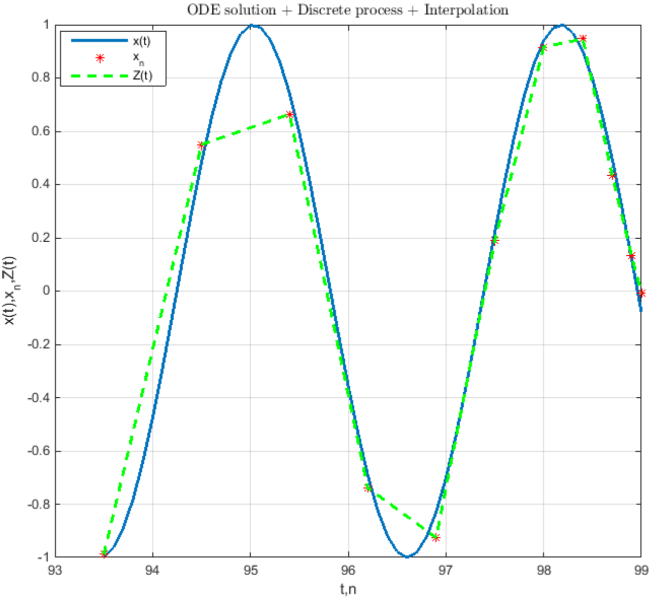
Stochastic approximation algorithms: Example



Stochastic approximation algorithms: Example



Stochastic approximation algorithms: Example



RESULTS

Market share as a Robbins Monro Algorithm

If $D^k = \sum_{t=0}^{k} \sum_{i=1}^{n} d_i^t$

$$\phi^k = \frac{D^k \phi^k}{D^k} \Rightarrow \phi^{k+1} = \frac{D^k \phi^k}{D^k + 1} + \frac{e^k}{D^k + 1}$$

It follows that

$$\begin{aligned}
\phi^{k+1} &= \frac{(D^k + 1)\phi^k}{D^k + 1} - \frac{\phi^k}{D^k + 1} + \frac{e^k}{D^k + 1} \\
&= \phi^k + \frac{1}{D^k + 1}(\mathbb{E}[e^k | \phi^k] - \phi^k + e^k - \mathbb{E}[e^k | \phi^k]) \\
&= \phi^k + \underbrace{\frac{1}{D^k + 1}}_{\gamma^{k+1}}(\underbrace{p(\phi^k) - \phi^k}_{F(\phi^k)} + \underbrace{e^k - \mathbb{E}[e^k | \phi^k]}_{U^{k+1}}).
\end{aligned}$$

Continuous dynamic given by

$$\frac{d\phi^t}{dt} = p(\phi^t) - \phi^t \quad (\phi^t \in \Delta^{n-1}) \tag{6}$$

Equilibria

Theorem 2.

Let $f(x) = x^r, 0 < r < 1$. Then, there is a unique equilibrium to Equation (**??**) in $int(\Delta^{n-1})$, the interior of the simplex, specified by

$$\phi^* = \frac{1}{\sum_j \overline{q}_j^{\frac{1}{1-r}}} [\overline{q}_1^{\frac{1}{1-r}}, ..., \overline{q}_n^{\frac{1}{1-r}}]$$

The remaining equilibria are on the boundary of the simplex.

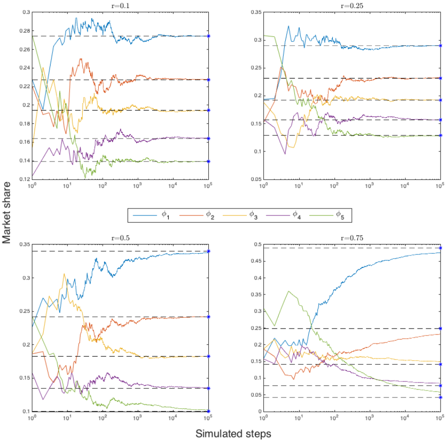
Remark: If assume that there exists a set $Q \subset \{1, .., n\}$ ($|Q| < n$) of indexes such that $\phi_i = 0$ if $i \in Q$. The remaining coordinates are given as follows: If $j \notin Q$, then we have

$$\phi_j = \frac{\overline{q}_j^{1/(1-r)}}{\sum_{i \notin Q} \overline{q}_i^{1/(1-r)}}.$$

Equilibria and stability

Main Theorem

Under the social signal $f(x) = x^r, 0 < r < 1$ with $\phi^0 \in int(\Delta^n)$, the RMA $\{\phi^t\}_{t>0}$ converges to ϕ^* almost surely.



Equilibria

Proposition: (Renlund2010)

Consider a one-dimentional RMA with $F(x) = p(x) - x$. A point x^* is unstable if there exists a neighbourhood N_{x^*} around x^* such that $F(x)[x - x^*] \geq 0$ for all $x \in N_{x^*}$.

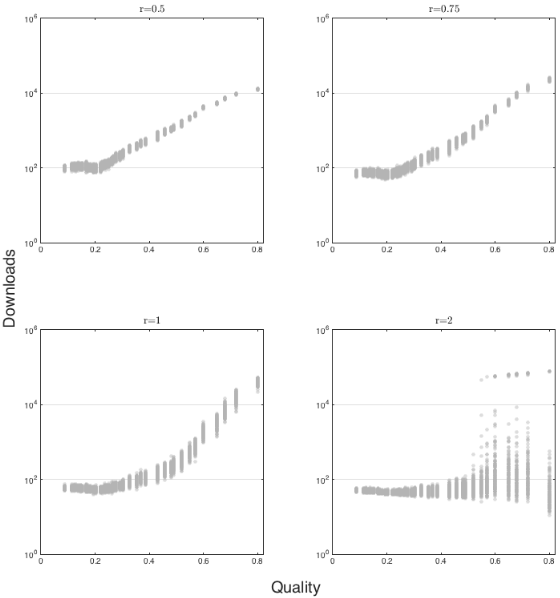
Theorem 4.

For a social signal $f(x) = x^r$ with $r > 1$, the inner equilibrium is unstable.

Theorem 5.

Consider the social signal $f(x) = x^r$ with $r > 1$. The RMA $\{\phi^t\}_{t\geq 0}$ converges almost surely to one of the equilibria $\phi \in Z_F := \{x \in \Delta^{n-1} : p(x) - x = 0\}$.

Predictability plots



PROGRESS AND FUTURE WORK

Submitted and accepted papers

Submitted

- Popularity Signals in Trial-Offer Markets. Operations Research.
- Asymptotic Optimality of Myopic Optimization in Trial-Offer Markets with Social Influence. 25th International Joint Conference on Artificial Intelligence (IJCAI16 New York City, USA. July 9-15).

Accepted

Aligning Popularity and Quality in Online Cultural Markets.The 10th International Conference of Web and Social Media (ICWSM 2016- Cologne, Germany. May 17-20).

Second Workshop on Algorithms and Dynamics for Games and Optimization. Santiago, Chile. Jan 25-29.

Current and future work

- Asymptotic study of popularity rankings (in progress, I will continue it during my visit to University of Michigan).
- *Side project:* Develop a formal generalisation of Polya Urn (currently well studied only for the case of 2 colors).
- Using the convergence results for sublinear functions using static rankings, create an alternative way to recover qualities*.

Scheme:

1. Define a time horizon (e.g 1 week), and divide it in 9 steps.
2. Order the songs according to some previous information (or random) and apply a social signal $f(x) = x^{0.1}$. Using the convergence theorem compute the values of $q_{i,1}$.
3. Order the songs using the values of $q_{i,1}$ and apply a social signal $f(x) = x^{0.2}$, obtaining $q_{i,2}$.
4. Repeat for the remaining values of $r \in \{0.3, ..., 0.9\}$.

DATA 61

THANKS!

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Proof of Theorem 2

We know the equilibria are given by
$p(x) = x \Leftrightarrow p_i(x) = x_i, \forall i \in \{1, ..., K\}$, then
$p_i(\Phi) = \phi_i \Leftrightarrow \frac{\overline{q}_i(\phi_i)^r}{\sum_{j=0}^{K} \overline{q}_j(\phi_j)^r} = \phi_i$
if $\phi_i > 0$ for all i, and then we have the following:
$p_i(\Phi) = \phi_i \Leftrightarrow \overline{q}_i(\phi_i)^{r-1} = \sum_{j=0}^{K} \overline{q}_j(\phi_j)^r$,
then

$$\overline{q}_i(\phi_i)^{r-1} = \overline{q}_j(\phi_j)^{r-1} \Leftrightarrow \phi_i = \left(\frac{\overline{q}_i}{\overline{q}_j}\right)^{\frac{1}{1-r}} \phi_j$$

Adding from $i = 1$ to $i = K$ we have
$1 = \sum_{i=1}^{K} \phi_i = \frac{\phi_j}{\overline{q}_j^{1/(1-r)}} \sum_{i=1}^{K} \overline{q}_i^{1/(1-r)}$, and in consecuence
$\phi_j = \frac{\overline{q}_j^{1/(1-r)}}{\sum_{i=1}^{K} \overline{q}_i^{1/(1-r)}}, \quad j \in \{1, ..., K\}$ are the coordinates of the equilibrium.

Proof of Theorem 3

The proof studies the asymptotic behaviour of the solutions of the following ODE:

$$\frac{d\phi^t}{dt} = p(\phi^t) - \phi^t. \tag{7}$$

Hence, we have

$$\begin{aligned}
\frac{\overline{q}_i(\phi_i^t)^r}{\sum_j \overline{q}_j(\phi_j^t)^r} &= \frac{d\phi_i^t}{dt} + \phi_i^t, \\
\frac{1}{\sum_j \overline{q}_j(\phi_j^t)^r} &= \frac{1}{\overline{q}_i(\phi_i^t)^r}\Big[\frac{d\phi_i^t}{dt} + \phi_i^t\Big] \quad \text{if } \phi_i^t \neq 0, \\
\overline{q}_i^{-1}[(\phi_i^t)^{-r}\frac{d\phi_i^t}{dt} + (\phi_i^t)^{1-r}] &= \overline{q}_j^{-1}[(\phi_j^t)^{-r}\frac{d\phi_j^t}{dt} + (\phi_j^t)^{1-r}], \\
\frac{d}{dt}\left[e^{(1-r)t}\overline{q}_i^{-1}(\phi_i^t)^{1-r}\right] &= \frac{d}{dt}\left[e^{(1-r)t}\overline{q}_j^{-1}(\phi_j^t)^{1-r}\right]
\end{aligned}$$

Integrating and re-ordering terms we have:

$$\frac{(\phi_i^t)^{1-r}}{\overline{q}_i} - \frac{(\phi_j^t)^{1-r}}{\overline{q}_j} = e^{(r-1)t}\left[\frac{(\phi_i^0)^{1-r}}{\overline{q}_i} - \frac{(\phi_j^0)^{1-r}}{\overline{q}_j}\right]. \quad (8)$$

if $\frac{(\phi_i^0)^{1-r}}{\overline{q}_i} \neq \frac{(\phi_j^0)^{1-r}}{\overline{q}_j}$, the right-hand side of Equation (**??**) goes to zero as $t \to \infty$ (because $r < 1$) and hence

$$\lim_{t\to\infty} \frac{(\phi_i^t)^{1-r}}{\overline{q}_i} - \frac{(\phi_j^t)^{1-r}}{\overline{q}_j} = 0. \quad (9)$$

Now denote by ϕ_j the limit of ϕ_j^t for all $j \in \{1, ..., n\}$. Using Equation (**??**), the following equation holds for all $i, j \in \{1, ..., n\}$:

$$\frac{\phi_i^{1-r}}{\overline{q}_i} = \frac{\phi_j^{1-r}}{\overline{q}_j} \Leftrightarrow \phi_i = \frac{\phi_j}{\overline{q}_j^{1/(1-r)}}\overline{q}_i^{1/(1-r)} \quad (10)$$

which is the equation that defines ϕ^* in Theorem 2.