INTRODUCTION TO THE FINITE ELEMENT METHOD: 1D-2D CASES

3rd Workshop on Advances in CFD and LB Modelling of Interface Dynamics in Capillary Two-Phase Flows

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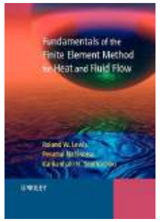
OUTLINE

- Intro to Finite Element Method
- Variational method: the weak form;
- Function approximations: The Galerkin method;
- 1D example;
- *Tasks*: 1D and 2D examples;
- Stream-Vorticity formulation and examples;
- *Hands-on*: Python scripts for 2D and Axisymmetric problems

BIBLIOGRAPHY

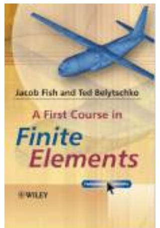
Basic

Fundamentals of the Finite Element Method for Heat and Fluid Flow
Authors: Roland W. Lewis, Perumal Nithiarasu, Kankanhally and N. Seetharamu



Basic

A First Course in Finite Elements
Authors:Jacob Fish and Sand Belytschko



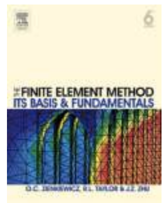
BIBLIOGRAPHY

Basic-advanced

The Finite Element Method - Linear Static and Dynamic Finite Element Analysis
Author: Thomas J.R. Hughues

Advanced

The Finite Element Method - Its Basis & Fundamentals
Authors: O.C. Zienkiewicz, R.L. Taylor & J.Z. Zhu



BRIEF HISTORY OF FEM

- Has been used since 1950's in solid mechanics
- in the 1970's FEM began to be used in CFD
- nowadays FEM is applicably to many engineering problems

Heat transfer, fluid flow, electromagnetic fields, solid mechanics, acoustics, biomechanics etc.

Finite Element Method - FEM	Finite Volume Method - FVM	Finite Difference Method - FDM
strong math	flux formulation	easy math
complex geometry	complex geometry	simple geometries
element geometry	conservative	grid systems
master element	low memory	low memory
high memory		not flexible
flexible		

FINITE ELEMENT METHOD

PDE → **weak form** → **ODE** → **linear system** → **approximated solution**

- PDE: Governing equations (heat equation, Maxwell equation, Navier-Stokes equation)
- weak form: Variational form
- Approximation functions: The Galerkin Method
- ODE: Set of Ordinary Differential Equations
- linear system: Ax=b
- approximated solution: solution for x, $x=A^{-1}b$, x=[u,v,w,p,T,c]

The approximation functions are combined with the weak form to obtain the discrete finite element equations.

CHOICE OF FUNCTIONS

Finite element function properties:

- the shape functions are 1 at the node and zero elsewhere;
- the sum of all shape function at the element is 1 everywhere, including boundary;
- the weight function is zero at boundary for Dirichlet b.c.

Chart:

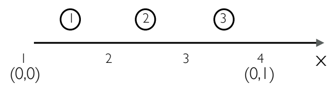
function	node, i	node, j	x
Ni	1	0	between 0 e 1
Nj	0	1	between 0 e 1
Ni+Nj	1	1	1

1D PROBLEM - STRONG FORM

Find u in $\Omega = [0,1]$ such that:

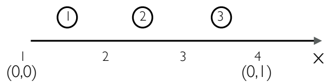
$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \left| \begin{array}{l} u(0) = 0 \\ \frac{du}{dx}(1) = 1 \end{array} \right. \longleftarrow \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$



MESH GENERATION

mesh:



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector

node	X
1	0
2	1/3
3	2/3
4	1

boundary vector

node	b.c. value
1	0

1D PROBLEM - WEAK FORM

Find u in H^1 with b.c. such that:

$$\int_\Omega w\left(\frac{d^2u}{dx^2}+u+1\right)d\Omega = 0$$

weight function

mathematical procedure (integration by parts)

$$\int_0^1 w\frac{d^2u}{dx^2}dx+\int_0^1 wudx+\int_0^1 dx=0$$

$$w\frac{du}{dx}\Big|_0^1-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wudx+\int_0^1 wdx=0$$

$$w\frac{du}{dx}\Big|_1-w\frac{du}{dx}\Big|_0-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wudx+\int_0^1 wdx=0$$

1D PROBLEM - WEAK FORM

$$w\frac{du}{dx}\Big|_1-w\frac{du}{dx}\Big|_0-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wudx+\int_0^1 wdx=0$$

apply boundary conditions to the equation
in the continuous form:

$$w(1)\cancelto{1}{\frac{du}{dx}(1)}-\cancelto{0}{w(0)}\frac{du}{dx}(0)-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wudx+\int_0^1 wdx=0$$

$$w(1)-\int_0^1\frac{du}{dx}\frac{dw}{dx}dx+\int_0^1 wudx+\int_0^1 wdx=0$$

GALERKIN METHOD

Approximate functions: $\hat{u}=\sum_{i=1}^4 N_i(x)u_i \quad \hat{w}=\sum_{j=1}^4 N_j(x)w_j$

$$w(1)\frac{du}{dx}(1)-w(0)\frac{du}{dx}(0)-\sum_{i=j=1}^4\int_0^1\frac{dN_i}{dx}u_i\frac{dN_j}{dx}w_jdx+$$
$$+\sum_{i=j=1}^4\int_0^1 N_iu_iN_jw_jdx+\sum_{j=1}^4\int_0^1 N_jw_jdx=0$$

$$\sum_{i=1}^4\sum_{j=1}^4\left(\int_0^1\boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}dx-\int_0^1\boxed{N_iN_jdx}\right)u_i=\sum_{j=1}^4\int_0^1\boxed{N_j}dx+\boxed{w(1)\frac{du}{dx}(1)-w(0)\frac{du}{dx}(0)}$$

stiffness matrix K_{ij} mass matrix M_{ij} right hand side. b_i boundary condition

GALERKIN METHOD

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}dx - \int_0^1 \boxed{N_iN_j dx}\right) {\color{red}u_i} = \sum_{j=1}^{4}\int_0^1 \boxed{N_j}dx + \boxed{w(1)\cancelto{1}{\frac{du}{dx}}(1) - \cancelto{0}{w}(0)\frac{du}{dx}(0)}$$

Replacing the shape function to the B.C.
Note that w_j is present at all members and can be removed!

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}dx - \int_0^1 \boxed{N_iN_j dx}\right) {\color{red}u_i} = \sum_{j=1}^{4}\int_0^1 \boxed{N_j}dx + \boxed{N_j(1)}$$

boundary condition (evaluated only at x=1)

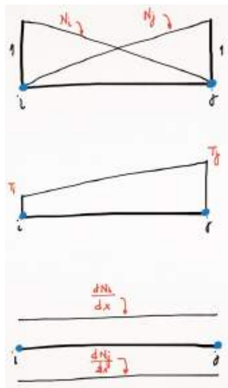
stiffness matrix K_{ij} mass matrix M_{ij} right hand side b_i

$$(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$$

FEM SHAPE FUNCTIONS

1D Problem - linear: $T(x) = \alpha_1 + \alpha_2 x$

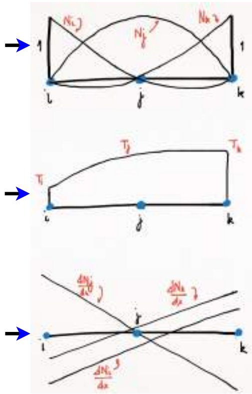
1D problem - quadratic: $T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$



shape function

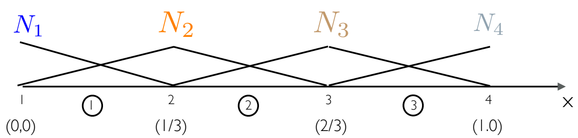
function

shape function derivative



1D PROBLEM - LINEAR

domain and shape functions:



element 1 $\Omega_1^e = [0, 1/3]$
$N_1 = -3x + 1$
$N_2 = 3x$

element 2 $\Omega_2^e = [1/3, 2/3]$
$N_2 = -3x + 2$
$N_3 = 3x - 1$

element 3 $\Omega_3^e = [2/3, 1]$
$N_3 = -3x + 3$
$N_4 = 3x - 2$

MATRIX FORM

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_1N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_1N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_2N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_2N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element 1
$\Omega_1^e = [0, 1/3]$
$N_1 = -3x + 1$
$N_2 = 3x$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

MATRIX FORM

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element 2
$\Omega_2^e = [1/3, 2/3]$
$N_2 = -3x + 2$
$N_3 = 3x - 1$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vector

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

MATRIX FORM

$$K_{33} - M_{33} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_4 N_3 dx$$

$$K_{44} - M_{44} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_4 N_4 dx$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

element 3
$\Omega_3^e = [2/3, 1]$
$N_3 = -3x + 3$
$N_4 = 3x - 2$

$$b_3 = \int_{2/3}^{1} N_3 dx$$

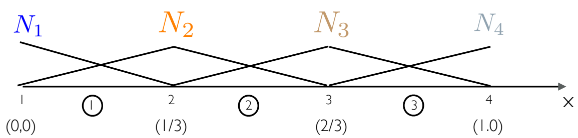
$$b_4 = \int_{2/3}^{1} N_4 dx$$

vector

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 1
$\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

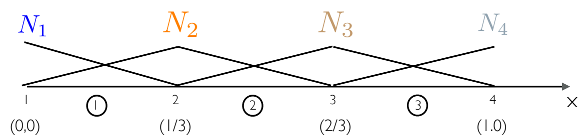
note that:

b.c. at $x = 0$ does not exist! $w(0) = 0$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 2
$\Omega_2^e = [1/3, 2/3]$

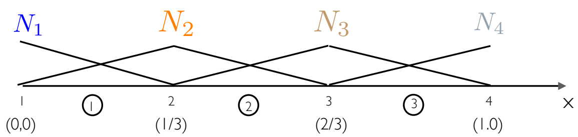
$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 3

$\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:

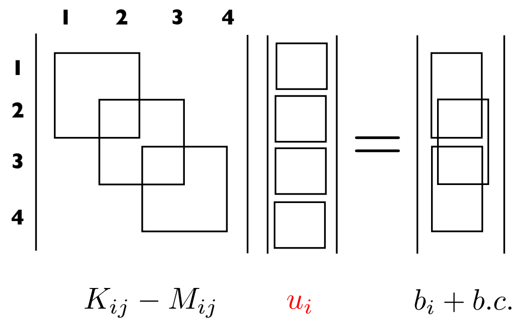
b.c. at $x = 1$

$$w(1) = N_4(1) = 3.1 - 2 = 1$$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 + \text{b.c. at } x = 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 + 1 \end{bmatrix}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$



ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```
for elem ← 1, NE do                                    ⟶ NE = Total number of elements
    for i_local ← 1, 2 do                              ⟶ i_local = [1, 2]
        i_global ← IEN[elem, i_local]                  ⟶ i_global = [v1, v2]
        for j_local ← 1, 2 do                          ⟶ j_local = [1, 2]
            j_global ← IEN[elem, j_local]              ⟶ j_global = [v1, v2]
            K[i_global, j_global] ← K[i_global, j_global] + k_elem[i_local, j_local]
        end for
    end for
end for
```

- loop on elements (1)
- loop on neighbors (i=1,2) and loop (j=1,2)
- conversion between local to global node
- assembling of stiffness matrix using SUM

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$\begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ 1 & KM_{11}^1 & KM_{12}^1 & & \\ 2 & KM_{21}^1 & KM_{22}^1 & & \\ 3 & & & & \\ 4 & & & & \end{array} \begin{Vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Vmatrix} = \begin{vmatrix} b_1^1 + \text{b.c.} \\ b_2^1 \\ \\ \\ \end{vmatrix}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} &
\left|\begin{matrix}
KM^1_{11} & KM^1_{12} & & \\
KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\
 & KM^2_{32} & KM^2_{33} & \\
 & & &
\end{matrix}\right\|
\left|\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}\right|
=
\left|\begin{matrix} b^1_1 + \text{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 \\ \ \end{matrix}\right|
\end{array}
$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} &
\left|\begin{matrix}
KM^1_{11} & KM^1_{12} & & \\
KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\
 & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} \\
 & & KM^3_{34} & KM^3_{44}
\end{matrix}\right\|
\left|\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}\right|
=
\left|\begin{matrix} b^1_1 + \text{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 + \text{b.c.} \end{matrix}\right|
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$u_1(0) = 0$

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} &
\left|\begin{matrix}
KM^1_{11} & KM^1_{12} & & \\
KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\
 & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} \\
 & & KM^3_{34} & KM^3_{44}
\end{matrix}\right\|
\left|\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{matrix}\right|
=
\left|\begin{matrix} b^1_1 + \text{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 + 1 \end{matrix}\right|
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

how to remove this line?

$$
\begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} &
\left|\begin{matrix}
KM^1_{11} & KM^1_{12} & & \\
KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\
 & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} \\
 & & KM^3_{34} & KM^3_{44}
\end{matrix}\right\|
\left|\begin{matrix} 0 \\ u_2 \\ u_3 \\ u_4 \end{matrix}\right|
=
\left|\begin{matrix} b^1_1 + \text{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 + 1 \end{matrix}\right|
\end{array}
$$

SETTING B.C.

writing down the equation of line 2

$$KM^1_{21} * {\color{red}u_1} + (KM^1_{22} + KM^2_{22}) * u_2 + KM^2_{23} = b^1_2 + b^2_2$$

subtracting $KM^1_{21} * {\color{red}u_1}$ from both sides:

$$(KM^1_{22} + KM^2_{22}) * u_2 + KM^2_{23} = b^1_2 + b^2_2 - KM^1_{21} * {\color{red}u_1}$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \left|\begin{array}{cccc} KM^1_{11} & KM^1_{12} & & \\ KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\ & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} \\ & & KM^3_{34} & KM^3_{44} \end{array}\right| \left|\begin{array}{c} 0 \\ u_2 \\ u_3 \\ u_4 \end{array}\right| = \left|\begin{array}{c} b^1_1 + \text{b.c.} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 + 1 \end{array}\right| \end{array}$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$\begin{array}{c c} & \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} & \left|\begin{array}{cccc} & & & \\ & KM^1_{22} + KM^2_{22} & KM^2_{23} & \\ & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} \\ & & KM^3_{34} & KM^3_{44} \end{array}\right| \left|\begin{array}{c} \\ u_2 \\ u_3 \\ u_4 \end{array}\right| = \left|\begin{array}{c} \\ b^1_2 + b^2_2 \\ b^2_3 + b^3_3 \\ b^3_3 + 1 \end{array}\right| \end{array}$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left|\begin{array}{ccc} KM^1_{22} + KM^2_{22} & KM^2_{23} & \\ KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} \\ & KM^3_{34} & KM^3_{44} \end{array}\right| \left|\begin{array}{c} u_2 \\ u_3 \\ u_4 \end{array}\right| = \left|\begin{array}{c} b^1_2 + b^2_2 - KM^1_{21} * {\color{red}u_1} \\ b^2_3 + b^3_3 \\ b^3_3 + 1 \end{array}\right| \end{array}$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
1 & 26/9 & -55/18 & 0 & 0 \\
2 & -55/18 & 52/9 & -55/18 & 0 \\
3 & 0 & -55/18 & 52/9 & -55/18 \\
4 & 0 & 0 & -55/18 & 26/9
\end{array}
\left|\begin{array}{c} \\ 0 \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} \\ 1/6 \\ 1/3 \\ 1/3 \\ {\color{red}1/6+1} \end{array}\right|$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$\begin{array}{c|ccc|}
 & 1 & 2 & 3 \\
1 & 52/9 & -55/18 & 0 \\
2 & -55/18 & 52/9 & -55/18 \\
3 & 0 & -55/18 & 26/9
\end{array}
\left|\begin{array}{c} \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} \\ 1/3 - KM_{21}^{1} * \cancelto{0}{{\color{red}u_1}} \\ 1/3 \\ {\color{red}7/6} \end{array}\right|$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

$$\begin{array}{c|ccc|}
 & 1 & 2 & 3 \\
1 & 52/9 & -55/18 & 0 \\
2 & -55/18 & 52/9 & -55/18 \\
3 & 0 & -55/18 & 26/9
\end{array}
\left|\begin{array}{c} \\ u_2 \\ u_3 \\ u_4 \end{array}\right|
=
\left|\begin{array}{c} \\ 1/3 \\ 1/3 \\ {\color{red}7/6} \end{array}\right|$$

SOLVING LINEAR SYSTEM

linear system of equations: $(K_{ij} - M_{ij}){\color{red}u_i} = b_i + b.c.$

solving for ${\color{red}u_i}$:

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij}){\color{red}u_i} = (K_{ij} - M_{ij})^{-1}b_i + \text{b.c.}$$

$${\color{red}u_i} = (K_{ij} - M_{ij})^{-1}b_i + \text{b.c.}$$

How to compute $(K_{ij} - M_{ij})^{-1}$?

How to solve the linear system?

direct methods: **not recommended!**

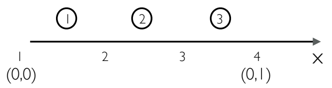
iterative methods: **recommended!**

SOLUTION

Find u in $\Omega = [0, 1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \left| \begin{array}{l} u(0) = 0 \\ \dfrac{du}{dx}(1) = 1 \end{array} \right. \longleftarrow \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$



$$u_2 = 1.049$$

$$u_3 = 1.874$$

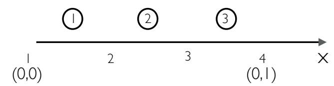
$$u_4 = 2.386$$

1D PROBLEM

Find u in $\Omega = [0, 1]$ such that:

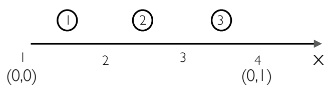
$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \left| \begin{array}{l} u(0) = 0 \\ \dfrac{du}{dx}(1) = -u \end{array} \right. \longleftarrow \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$



MESH GENERATION

mesh:



IEN matrix

element	node 1	node 2
1	1	2
2	2	3
3	3	4

ID vector

node	not b.c.
1	0
2	1
3	2
4	3

Coordinate vector

node	X
1	0
2	1/3
3	2/3
4	1

boundary vector

node	b.c. value
1	0

1D PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_\Omega w\left(\frac{d^2u}{dx^2} + u + 1\right) d\Omega = 0$$

weight function

mathematical procedure (integration by parts)

$$\int_0^1 w\frac{d^2u}{dx^2}dx + \int_0^1 wudx + \int_0^1 dx = 0$$

$$w\frac{du}{dx}\bigg|_0^1 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wudx + \int_0^1 wdx = 0$$

$$w\frac{du}{dx}\bigg|_1 - w\frac{du}{dx}\bigg|_0 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wudx + \int_0^1 wdx = 0$$

1D PROBLEM - WEAK FORM

$$w\frac{du}{dx}\Big|_1 - w\frac{du}{dx}\Big|_0 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wu\,dx + \int_0^1 w\,dx = 0$$

apply boundary conditions to the equation in the continuous form:

$$w(1)\cancelto{-u}{\frac{du}{dx}}(1) - \cancelto{0}{w(0)}\frac{du}{dx}(0) - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wu\,dx + \int_0^1 w\,dx = 0$$

$$-w(1)u(1) - \int_0^1 \frac{du}{dx}\frac{dw}{dx}dx + \int_0^1 wu\,dx + \int_0^1 w\,dx = 0$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{4} N_i(x)u_i \quad \hat{w} = \sum_{j=1}^{4} N_j(x)w_j$

$$w(1)\frac{du}{dx}(1) - w(0)\frac{du}{dx}(0) - \sum_{i=j=1}^{4}\int_0^1 \frac{dN_i}{dx}u_i\frac{dN_j}{dx}w_j dx +$$

$$+ \sum_{i=j=1}^{4}\int_0^1 N_i u_i N_j w_j dx + \sum_{j=1}^{4}\int_0^1 N_j w_j dx = 0$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \underbrace{\boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}}_{\text{stiffness matrix } K_{ij}}dx - \int_0^1 \underbrace{\boxed{N_iN_j}}_{\text{mass matrix } M_{ij}}dx\right)u_i = \sum_{j=1}^{4}\int_0^1 \underbrace{\boxed{N_j}}_{\text{right hand side. } b_i}dx + \underbrace{\boxed{w(1)\cancelto{-u}{\frac{du}{dx}}(1) - \cancelto{0}{w(0)}\frac{du}{dx}(0)}}_{\text{boundary condition}}$$

$$(K_{ij} - M_{ij})u_i = b_i + b.c.$$

GALERKIN METHOD

note that the b.c. has $-u(1)$:

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \underbrace{\boxed{\frac{dN_i}{dx}\frac{dN_j}{dx}}}_{\text{stiffness matrix } K_{ij}}dx - \int_0^1 \underbrace{\boxed{N_iN_j}}_{\text{mass matrix } M_{ij}}dx\right)u_i = \sum_{j=1}^{4}\int_0^1 \underbrace{\boxed{N_j}}_{\text{right hand side. } b_i}dx + \underbrace{\boxed{w(1)\cancelto{-u}{\frac{du}{dx}}(1) - \cancelto{0}{w(0)}\frac{du}{dx}(0)}}_{\text{boundary condition}}$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left(\int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx - \int_0^1 N_iN_j dx\right)u_i + w(1)u(1) = \sum_{j=1}^{4}\int_0^1 N_j dx - \cancelto{0}{w(0)}\frac{du}{dx}(0)$$

Replacing the shape function to the B.C.
Note that w_j is present at all members and can be removed!

$$\sum_{i=1}^{4}\sum_{j=1}^{4}\left[\int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx - \int_0^1 N_iN_j dx + N_i(1)N_j(1)\right]u_i = \sum_{j=1}^{4}\int_0^1 N_j dx$$

ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

for $elem \leftarrow 1, NE$ **do** $\longrightarrow$ NE = Total number of elements
  for $i_{\text{local}} \leftarrow 1,2$ **do** $\longrightarrow i_{\text{local}} = [1,2]$
    $i_{\text{global}} \leftarrow IEN[elem, i_{\text{local}}]$ $\longrightarrow i_{\text{global}} = [v_1, v_2]$
    for $j_{\text{local}} \leftarrow 1,2$ **do** $\longrightarrow j_{\text{local}} = [1,2]$
      $j_{\text{global}} \leftarrow IEN[elem, j_{\text{local}}]$ $\longrightarrow j_{\text{global}} = [v_1, v_2]$
      $K[i_{\text{global}}, j_{\text{global}}] \leftarrow K[i_{\text{global}}, j_{\text{global}}] + k_{\text{elem}}[i_{\text{local}}, j_{\text{local}}]$
    end for
  end for
end for

- loop on elements (1)
- loop on neighbors (i=1,2) and loop (j=1,2)
- conversion between local to global node
- assembling of stiffness matrix using SUM

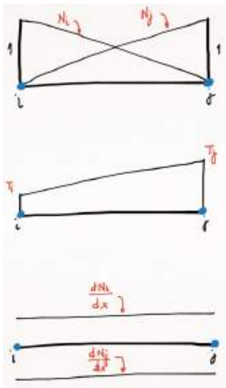
FEM SHAPE FUNCTIONS

1D Problem - linear:

$T(x) = \alpha_1 + \alpha_2 x$

1D problem - quadratic:

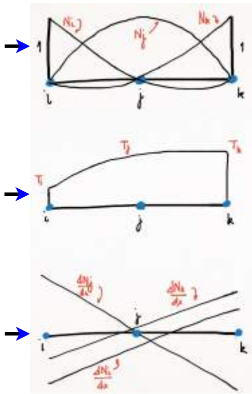
$T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$



shape function

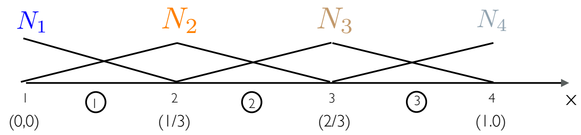
function

shape function derivative



1D PROBLEM - LINEAR

domain and shape functions:



element 1 $\Omega_1^e = [0, 1/3]$

$N_1 = -3x + 1$

$N_2 = 3x$

element 2 $\Omega_2^e = [1/3, 2/3]$

$N_2 = -3x + 2$

$N_3 = 3x - 1$

element 3 $\Omega_3^e = [2/3, 1]$

$N_3 = -3x + 3$

$N_4 = 3x - 2$

MATRIX FORM

$$K_{11} - M_{11} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_1 N_1 dx$$

$$K_{12} - M_{12} = \int_0^{1/3} \frac{dN_1}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_1 N_2 dx$$

$$K_{21} - M_{21} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_1}{dx}dx - \int_0^{1/3} N_2 N_1 dx$$

$$K_{22} - M_{22} = \int_0^{1/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_0^{1/3} N_2 N_2 dx$$

matrix

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

element 1
$\Omega_1^e = [0, 1/3]$
$N_1 = -3x + 1$
$N_2 = 3x$

$$b_1 = \int_0^{1/3} N_1 dx$$

$$b_2 = \int_0^{1/3} N_2 dx$$

vector

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

MATRIX FORM

$$K_{22} - M_{22} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_2 N_2 dx$$

$$K_{23} - M_{23} = \int_{1/3}^{2/3} \frac{dN_2}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_2 N_3 dx$$

$$K_{32} - M_{32} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_2}{dx}dx - \int_{1/3}^{2/3} N_3 N_2 dx$$

$$K_{33} - M_{33} = \int_{1/3}^{2/3} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{1/3}^{2/3} N_3 N_3 dx$$

matrix

$$\begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

element 2
$\Omega_2^e = [1/3, 2/3]$
$N_2 = -3x + 2$
$N_3 = 3x - 1$

$$b_2 = \int_{1/3}^{2/3} N_2 dx$$

$$b_3 = \int_{1/3}^{2/3} N_3 dx$$

vector

$$\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$$

MATRIX FORM

$$K_{33} - M_{33} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_3 N_3 dx$$

$$K_{34} - M_{34} = \int_{2/3}^{1} \frac{dN_3}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_3 N_4 dx$$

$$K_{43} - M_{43} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_3}{dx}dx - \int_{2/3}^{1} N_4 N_3 dx$$

matrix

$$\begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}$$

$$K_{44} - M_{44} = \int_{2/3}^{1} \frac{dN_4}{dx}\frac{dN_4}{dx}dx - \int_{2/3}^{1} N_4 N_4 dx + N_4(1)N_4(1)$$

element 3
$\Omega_3^e = [2/3, 1]$
$N_3 = -3x + 3$
$N_4 = 3x - 2$

$$b_3 = \int_{2/3}^{1} N_3 dx$$

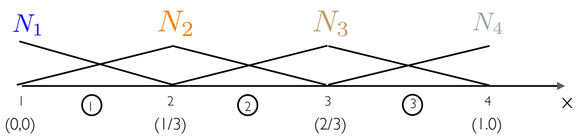
$$b_4 = \int_{2/3}^{1} N_4 dx$$

vector

$$\begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 1
$\Omega_1^e = [0, 1/3]$

$$K_1^e - M_1^e = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

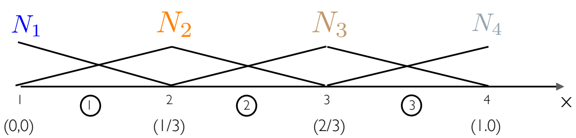
$$K_1^e - M_1^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

note that:
b.c. at $x = 0$
does not exist!
$w(0) = 0$

$$b_1^e = \begin{bmatrix} b_1 + \text{b.c. at } x = 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 2
$\Omega_2^e = [1/3, 2/3]$

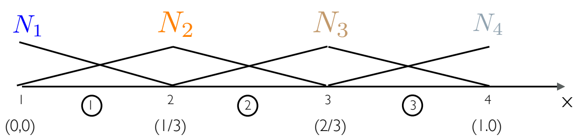
$$K_2^e - M_2^e = \begin{bmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{bmatrix} - \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix}$$

$$K_2^e - M_2^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{52}{18} \end{bmatrix}$$

$$b_2^e = \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

MATRIX FORM

domain and shape functions:



element 3
$\Omega_3^e = [2/3, 1]$

$$K_3^e - M_3^e = \begin{bmatrix} K_{33} & K_{34} \\ K_{43} & K_{44} \end{bmatrix} - \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K_3^e - M_3^e = \begin{bmatrix} 3 - \frac{2}{18} & -3 - \frac{1}{18} \\ -3 - \frac{1}{18} & 3 - \frac{2}{18} + 1 \end{bmatrix} = \begin{bmatrix} \frac{52}{18} & -\frac{55}{18} \\ -\frac{55}{18} & \frac{70}{18} \end{bmatrix}$$

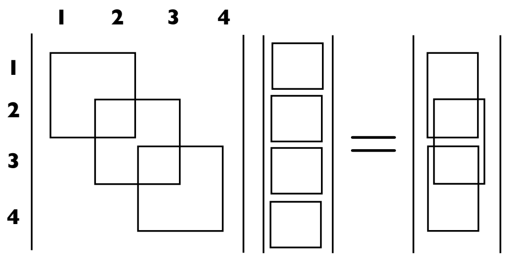
note that:
b.c. at $x = 1$

$$b_3^e = \begin{bmatrix} b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}$$

$$w(1)u(1) = N_4(1) * N_4(1)$$
$$= (3.1 - 2)(3.1 - 2)$$
$$= 1$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$



$K_{ij} - M_{ij}$ u_i $b_i + b.c.$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & u_1 & & b^1_1 + \text{b.c.} \\
2 & KM^1_{21} & KM^1_{22} & & & u_2 & = & b^1_2 \\
3 & & & & & u_3 & & \\
4 & & & & & u_4 & &
\end{array}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & u_1 & & b^1_1 + \text{b.c.} \\
2 & KM^1_{21} & \begin{array}{c}KM^1_{22}\\+\\KM^2_{22}\end{array} & KM^2_{23} & & u_2 & & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & KM^2_{33} & & u_3 & = & b^2_3 \\
4 & & & & & u_4 & &
\end{array}$$

ASSEMBLING

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & u_1 & & b^1_1 + \text{b.c.} \\
2 & KM^1_{21} & \begin{array}{c}KM^1_{22}\\+\\KM^2_{22}\end{array} & KM^2_{23} & & u_2 & & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & \begin{array}{c}KM^2_{33}\\+\\KM^3_{33}\end{array} & KM^3_{43} & u_3 & = & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & KM^3_{44} & u_4 & & b^3_3
\end{array}$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$u_1(0) = 0$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & u_1 & & b^1_1 + \text{b.c.} \\
2 & KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & & u_2 & = & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} & u_3 & & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & KM^3_{44} + 1 & u_4 & & b^3_3
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

replace this equation by b.c.

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & u_1 & & b^1_1 \\
2 & KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & & u_2 & = & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} & u_3 & & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & KM^3_{44} + 1 & u_4 & & b^3_3
\end{array}
$$

SETTING B.C.

writing down the equation of node 2

$$KM^1_{21} * u_1 + (KM^1_{22} + KM^2_{22}) * u_2 + KM^2_{23} = b^1_2 + b^2_2$$

subtracting $KM^1_{21} * u_1$ from both sides:

$$(KM^1_{22} + KM^2_{22}) * u_2 + KM^2_{23} = b^1_2 + b^2_2 - KM^1_{21} * u_1$$

replacing equation of node 1 by the trivial equation:

$$1 * u_1 = 0$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & KM^1_{11} & KM^1_{12} & & & u_1 & & b^1_1 \\
2 & KM^1_{21} & KM^1_{22} + KM^2_{22} & KM^2_{23} & & u_2 & = & b^1_2 + b^2_2 \\
3 & & KM^2_{32} & KM^2_{33} + KM^3_{33} & KM^3_{43} & u_3 & & b^2_3 + b^3_3 \\
4 & & & KM^3_{34} & KM^3_{44} + 1 & u_4 & & b^3_3
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & 1 & 0 & & & u_1 & & 0 \\
2 & 0 & KM_{22}^1 + KM_{22}^2 & KM_{23}^2 & & u_2 & = & b_2^1 + b_2^2 - KM_{21}^1 * u_1 \\
3 & & KM_{32}^2 & KM_{33}^2 + KM_{33}^3 & KM_{43}^3 & u_3 & & b_3^2 + b_3^3 \\
4 & & & KM_{34}^3 & KM_{44}^3 + 1 & u_4 & & b_3^3
\end{array}
$$

SETTING B.C.

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & 1 & 0 & 0 & 0 & u_1 & & 0 \\
2 & 0 & 52/9 & -55/18 & 0 & u_2 & = & b_2^1 + b_2^2 - \cancelto{0}{KM_{21}^1 * u_1} \\
3 & 0 & -55/18 & 52/9 & -55/18 & u_3 & & b_3^2 + b_3^3 \\
4 & 0 & 0 & -55/18 & 26/9 + 1 & u_4 & & b_3^3
\end{array}
$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & 1 & 0 & 0 & 0 & u_1 & & 0 \\
2 & 0 & 52/9 & -55/18 & 0 & u_2 & = & 1/3 \\
3 & 0 & -55/18 & 52/9 & -55/18 & u_3 & & 1/3 \\
4 & 0 & 0 & -55/18 & 26/9 + 1 & u_4 & & 1/6
\end{array}
$$

SUBSTITUTING VALUES

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

$$
\begin{array}{c|cccc||c|c|c|}
 & 1 & 2 & 3 & 4 & & & \\
1 & 1 & 0 & 0 & 0 & u_1 & & 0 \\
2 & 0 & 52/9 & -55/18 & 0 & u_2 & = & 1/3 \\
3 & 0 & -55/18 & 52/9 & -55/18 & u_3 & & 1/3 \\
4 & 0 & 0 & -55/18 & 35/9 & u_4 & & 1/6
\end{array}
$$

SOLVING LINEAR SYSTEM

linear system of equations: $(K_{ij} - M_{ij})u_i = b_i + b.c.$

solving for u_i :

$$(K_{ij} - M_{ij})^{-1}(K_{ij} - M_{ij})u_i = (K_{ij} - M_{ij})^{-1}b_i + \text{b.c.}$$

$$u_i = (K_{ij} - M_{ij})^{-1}b_i + \text{b.c.}$$

How to compute $(K_{ij} - M_{ij})^{-1}$?

How to solve the linear system?

direct methods: **slow and high memory consumption!**

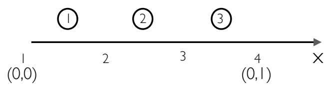
iterative methods: **faster!**

SOLUTION

Find u in $\Omega = [0,1]$ such that:

$$\frac{d^2u}{dx^2} + u + 1 = 0 \qquad \begin{array}{|l} u(0) = 0 \\ \dfrac{du}{dx}(1) = -u \end{array} \longleftarrow \text{boundary condition}$$

domain: $h_1 = h_2 = h_3 = 1/3$



$u_1 = 0.000$

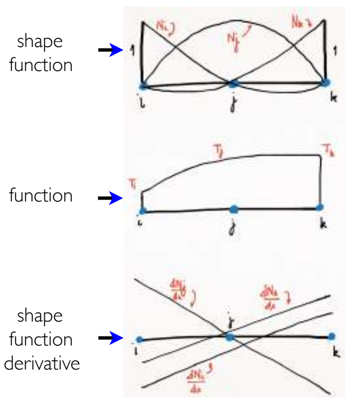
$u_2 = 0.251$

$u_3 = 0.363$

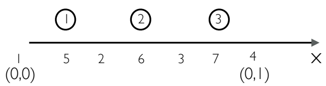
$u_4 = 0.328$

EXERCISE

Repeat exercise 1 and 2 for quadratic elements.

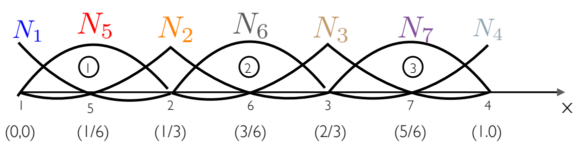


domain: $h_1 = h_2 = h_3 = 1/3$



EXERCISE

domain and shape functions:



element 1 $\quad \Omega_1^e = [0, 1/3]$

$N_1 = 18x^2 - 9x + 1$
$N_2 = 18x^2 - 3x$
$N_5 = -36x^2 + 12x$

element 2 $\quad \Omega_2^e = [1/3, 2/3]$

$N_2 = 18x^2 - 21x + 6$
$N_3 = 18x^2 - 15x + 3$
$N_6 = -36x^2 + 36x - 8$

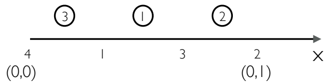
element 3 $\quad \Omega_3^e = [2/3, 1]$

$N_3 = 18x^2 - 33x + 15$
$N_4 = 18x^2 - 27x + 10$
$N_7 = -36x^2 + 60x - 24$

EXERCISE

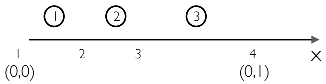
Repeat exercise 1 and 2 for different mesh numbering:

domain: $h_1 = h_2 = h_3 = 1/3$



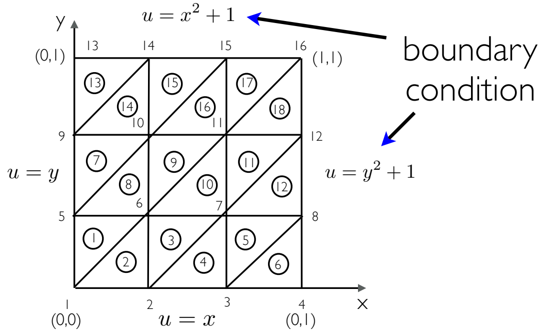
Repeat exercise 1 and 2 for different mesh spacing:

domain: $2h_1 = 2h_2 = h_3 = 1/2$



2D PROBLEM

Find u in $\Omega = [0,1] \times [0,1]$ such that:



Equation:

$$\nabla^2 u = 0$$

MESH GENERATION

element	node 1	node 2	node 3
1	1	6	5
2	1	2	6
3	2	7	6
4	2	3	7
5	3	8	7
6	3	4	8
7	5	10	9
8	5	6	10
9	6	11	10
10	6	7	11
11	7	12	11
12	7	8	12
13	9	14	13
14	9	10	14
15	10	15	14
16	10	11	15
17	11	16	15
18	11	12	16

node	not b.c.
1	0
2	0
3	0
4	0
5	0
6	1
7	2
8	0
9	0
10	3
11	4
12	0
13	0
14	0
15	0
16	0

node	X	Y
1	0	0
2	1/3	0
3	2/3	0
4	1	0
5	0	1/3
6	1/3	1/3
7	2/3	1/3
8	1	1/3
9	0	2/3
10	1/3	2/3
11	2/3	2/3
12	1	2/3
13	0	1
14	1/3	1
15	2/3	1
16	1	1

node	b.c. value
1	0
2	1/3
3	2/3
4	1
5	1/3
6	-
7	-
8	10/9
9	2/3
10	-
11	-
12	13/9
13	1
14	10/9
15	13/9
16	2

ASSEMBLING

Algorithm 1 Assembling algorithm for stiffness matrix K_{ij}

```
for elem ← 1, NE do                                        ⟶ NE = Total number of elements
    for i_local ← 1, 3 do                                  ⟶ i_local = [1, 2, 3]
        i_global ← IEN[elem, i_local]                      ⟶ i_global = [v1, v2, v3]
        for j_local ← 1, 3 do                              ⟶ j_local = [1, 2, 3]
            j_global ← IEN[elem, j_local]                  ⟶ j_global = [v1, v2, v3]
            K[i_global, j_global] ← K[i_global, j_global] + k_elem[i_local, j_local]
        end for
    end for
end for
```

- loop on elements (1)
- loop on neighbors (i=1,2,3) and loop (j=1,2,3)
- conversion between local to global node
- assembling of stiffness matrix using SUM

2D PROBLEM - WEAK FORM

Find u in H^1 such that:

$$\int_\Omega w\left(\nabla^2 u\right) d\Omega = 0$$

weight function

mathematical procedure (Green theorem)

$$\int_\Omega w\nabla^2 u d\Omega = \int_\Omega w\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) d\Omega = 0$$

$$\oint_\Gamma w\,\nabla u\,d\Gamma - \int_\Omega \nabla w \cdot \nabla u\,d\Omega = 0$$

2D PROBLEM - WEAK FORM

$$\oint_\Gamma w\,\nabla u\,d\Gamma - \int_\Omega \nabla w \cdot \nabla u\,d\Omega = 0$$

apply boundary conditions to the equation
in the continuous form:

0 (since w=0 at Dirichlet b.c.)

$$\oint_\Gamma w\,\cancel{\nabla u}\,d\Gamma - \int_\Omega \nabla w \cdot \nabla u\,d\Omega = 0$$

$$\int_\Omega \nabla w \cdot \nabla u d\Omega = 0$$

GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{16} N_i(x,y)u_i \quad \hat{w} = \sum_{j=1}^{16} N_j(x,y)w_j$

$$\boxed{\mathbf{k}^e = \int_\Omega B^T D B d\Omega}$$ **formula**

where:

$$B = \begin{bmatrix} \frac{\partial N(x,y)}{\partial x} \\ \frac{\partial N(x,y)}{\partial y} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial x} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial y} \end{bmatrix} \quad D = k\mathbf{I}$$

coefficient

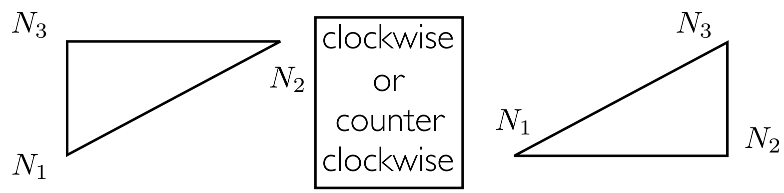
if k constant and isotropic: $\mathbf{k}^e = \frac{k}{4A} B^T B$

coefficient

area

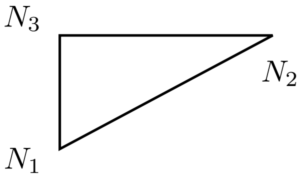
GALERKIN METHOD

Approximated functions: $\hat{u} = \sum_{i=1}^{16} N_i(x,y)u_i \quad \hat{w} = \sum_{j=1}^{16} N_j(x,y)w_j$

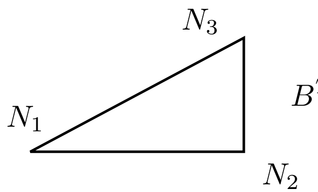


$$B = \frac{1}{2A}\begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

ELEMENT MATRIX



$$B^T B = \frac{1}{4A}\begin{bmatrix} 1/9 & 0 & -1/9 \\ 0 & 1/9 & -1/9 \\ -1/9 & -1/9 & 2/9 \end{bmatrix}$$



$$B^T B = \frac{1}{4A}\begin{bmatrix} 1/9 & -1/9 & 0 \\ -1/9 & 2/9 & -1/9 \\ 0 & -1/9 & 1/9 \end{bmatrix}$$

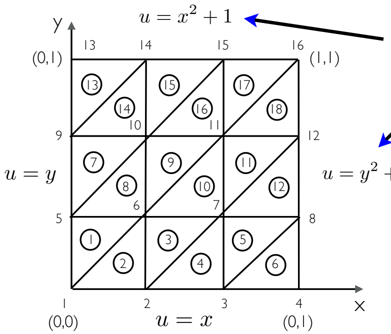
ASSEMBLING

linear system of equations: $K_{ij}u_i = b_i + \text{b.c.}$

$$\begin{array}{c|cccc|} & \mathbf{6} & \mathbf{7} & \mathbf{10} & \mathbf{11} \\ \hline \mathbf{6} & 4 & -1 & -1 & 0 \\ \mathbf{7} & -1 & 4 & 0 & -1 \\ \mathbf{10} & -1 & 0 & 4 & -1 \\ \mathbf{11} & 0 & -1 & -1 & 4 \end{array} \begin{vmatrix} u_6 \\ u_7 \\ u_{10} \\ u_{11} \end{vmatrix} = \begin{vmatrix} 2/3 \\ 16/9 \\ 16/9 \\ 26/9 \end{vmatrix}$$

SOLUTION

Find u in $\Omega = [0,1] \times [0,1]$ such that:

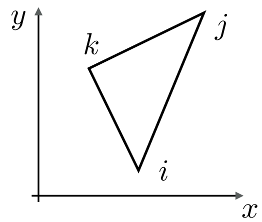


boundary condition

Equation:
$\nabla^2 u = 0$

$$u_6 = 0.611$$
$$u_7 = 0.889$$
$$u_{10} = 0.889$$
$$u_{11} = 1.167$$

TRIANGLE FORMULAE



$$\mathbf{N}(x,y) = [N_i \; N_j \; N_k]$$

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y)$$

$$a_i = x_j y_k - x_k y_j \qquad b_i = y_j - y_k \qquad c_i = x_k - x_j$$
$$a_j = x_k y_i - x_i y_k \qquad b_j = y_k - y_i \qquad c_k = x_i - x_k$$
$$a_k = x_i y_i - x_j y_i \qquad b_k = y_i - y_j \qquad c_k = x_j - x_i$$

TRIANGLE FORMULAE

stiffness matrix

$$\mathbf{k^e} = \frac{k}{4A}\begin{bmatrix} b_i^2 + c_i^2 & b_ib_j + c_ic_j & b_ib_k + c_ic_k \\ b_ib_j + c_ic_j & b_j^2 + c_j^2 & b_kb_j + c_kc_j \\ b_ib_k + c_ic_k & b_kb_j + c_kc_j & b_k^2 + c_k^2 \end{bmatrix}$$

mass matrix

$$\mathbf{m^e} = \frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

gradient matrices

$$\mathbf{g^e_x} = \frac{1}{6}\begin{bmatrix} b_i & b_j & b_k \\ b_i & b_j & b_k \\ b_i & b_j & b_k \end{bmatrix}$$

$$\mathbf{g^e_y} = \frac{1}{6}\begin{bmatrix} c_i & c_j & c_k \\ c_i & c_j & c_k \\ c_i & c_j & c_k \end{bmatrix}$$

GESAR

The Group for Environmental Studies in Reservoirs

FIRST FLUID SOLVER: 2D STREAM-VORTICITY FORMULATION

3rd Workshop on Advances in CFD and LB Modelling of Interface Dynamics in Capillary Two-Phase Flows

Gustavo R. ANJOS

http://www.uerj.br http://2phaseflow.org

http://www.gesar.uerj.br http://gustavorabello.github.io

Kobe - Japan

October 10th, 2018

GOVERNING EQUATIONS

vorticity transport:

$$\frac{\partial \omega_z}{\partial t} + \mathbf{v}\cdot\nabla\omega_z = \nu\nabla^2\omega_z$$

stream function:

$$\nabla^2\psi = -\omega_z$$

auxiliary:

$$\frac{\partial\psi}{\partial y} = v_x$$

$$\frac{\partial\psi}{\partial x} = -v_y$$

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

Required 2 boundary conditions for ψ and ω_z

MATRICIAL EQUATIONS

vorticity transport:

$$\left(\frac{\mathbf{M}}{\Delta t} + \nu\mathbf{K} + \mathbf{v}\cdot\mathbf{G}\right){\omega_z}^{n+1} = \frac{\mathbf{M}}{\Delta t}{\omega_z}^{n} + \mathbf{b.c.}$$

stream function:

$$\mathbf{K}\psi = \mathbf{M}\omega_z + \mathbf{b.c.}$$

boundary conditions:

$\psi \longrightarrow$ constant

$\omega_z \longrightarrow$ variable

auxiliary:

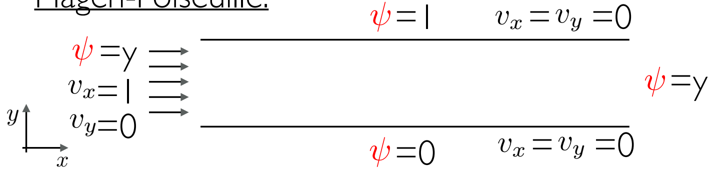
$$\mathbf{M}v_x = \mathbf{G_y}\psi$$

$$\mathbf{M}v_y = -\mathbf{G_x}\psi$$

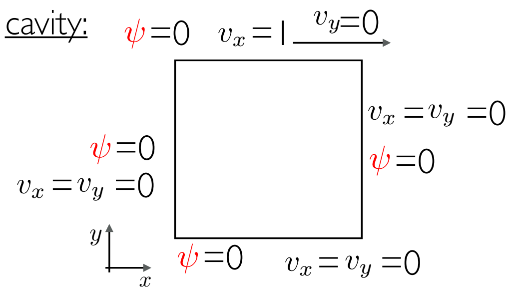
$$\mathbf{M}\omega_z = \mathbf{G_x}v_y - \mathbf{G_y}v_x$$

EXAMPLES

Hagen-Poiseuille:



cavity:



streamfunction b.c.

$$\psi = \int (v_x dy - v_y dx)$$

vorticity b.c.
solve ω_z every dt:

$$\mathbf{M}\omega_z = \mathbf{G_x} v_y - \mathbf{G_y} v_x$$