## II. Systems with control delays

For the system:

$$\dot{x}(t) = Ax(t) + B_0 u(t) + B_1 u(t-h), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \qquad (2.1)$$

the following feedback law

$$u(t) = Fx(t) + F\int_{-h}^{0} e^{-(h+\theta)A} B_1 u(t+\theta)\, d\theta, \qquad (2.2)$$

with $F_{m\times n}$, yields a finite spectrum of the closed-loop system under some controllability conditions.

Let's try to understand what the authors have done.

The same result is true for the general system:

$$\dot{x}(t) = Ax(t) + \int_{-h}^{0} d\beta(\theta)\, u(t+\theta) \qquad (2.3)$$

where $\beta(\cdot)_{n\times m}$ is a matrix function of bounded variation which is a sum of an absolutely continuous function and a finite number of jump discontinuities, or:

$$\dot{x}(t) = Ax(t) + \sum_{i=0}^{N} B_i u(t-h_i) + \int_{-h}^{0} \tilde{B}(\theta)\, u(t+\theta)\, d\theta, \qquad (2.4)$$

where $0 = h_0 \leq h_1 \ldots \leq h_N = h$, and $\theta \to \tilde{B}(\theta)$ is in $L^{\infty}((-h,0),\mathbb{R}^{mn})$.

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A matrix function of bounded variation is a function whose value is a matrix and whose total variance (in some proper sense) is finite.

Definition: Let $f(x)$ be a real-valued function defined on a closed interval $[a,b]$; $f(x)$ is called a matrix function of bounded variation if there exists a constant $K$ such that

$$\sum_{\nu=1}^{p} \| f(A_\nu) - f(A_{\nu-1}) \| \le K$$

for all partitions $aI \le A_0 \le A_1 \le \cdots \le A_p \le bI$.

⇝ An absolutely continuous function is a function whose total variation over any interval can be made arbitrarily small, provided the length of the interval is sufficiently small. Formally, a function $g(t)$ is absolutely continuous on an interval $[a,b]$ if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for any finite collection of disjoint intervals $\{(t_i, t_i')\}$ within $[a,b]$,

$$\sum |t_i' - t_i| < \delta \Rightarrow \sum |g(t_i') - g(t_i)| < \epsilon$$

For a matrix function, this means that each entry of the matrix has this continuously varying property.

⇝ A function with a finite number of jump discontinuities is one that, in addition to being absolutely continuous at almost all points, has a finite number of points where it undergoes "jumps". At such points, the function changes values instantaneously, as opposed to varying continuously.

## Putting it all together

$$F(t) = F_{ac}(t) + \sum_{i=1}^{n} \Delta F_i H(t-t_i)$$

where

$F_{ac}(t)$ is an absolutely continuous function.

$\Delta F_i(t)$ are the matrices representing the jumps in the function.

$H(t-t_i)$ is the heaviside function (step function) which is 0 for $t < t_i$ and 1 for $t \geq t_i$.

In the context of control, this means that the system's response can be decomposed into a smoothly varying part (absolutely continuous) and a part that undergoes sudden changes at specific points (finite jumps). This is important because controller analysis and design can be different for continuous and discontinuous parts of the function.

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The feedback for (2.3) has the form

$$u(t) = Fx(t) + F \int_{-h}^{0} \int_{\tau}^{0} e^{(\tau-\theta)A} d\beta(\tau) u(t+\theta)\, d\theta \qquad (2.5)$$

where $\tau$ is the integration variable of the first integral.

Consider solutions to (2.3), (2.5) corresponding to some initial conditions $x(0) \in \mathbb{R}^n$, $u(\theta) = \phi(\theta)$ $\theta \in (-h, 0)$, $\phi(\cdot) \in L^1((-h,0), \mathbb{R}^m)$. The function $t \to x(t)$ is absolutely continuous. Suppose that for $t \geq 0$, $t \to u(t)$ is only $L^1_{loc}$. Rewritten (2.5):

$t + \theta = \xi \to \theta = \xi - t$ $\qquad u(t) = Fx(t) + F \int_{-h}^{0} \int_{\tau}^{0} e^{(\tau - \xi + t)A} d\beta(\tau) u(\xi)\, d\xi$

$$u(t) = Fx(t) + F \int_{-h}^{0} e^{(t+\tau)A} \int_{t+\tau}^{t} e^{-\xi A} d\beta(\tau) u(\xi)\, d\xi$$

Proof theorem 2.2:

$$\rightsquigarrow \begin{bmatrix} I\lambda - A & -N \\ -F & I - FM \end{bmatrix}\begin{bmatrix} I & -M \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix} =$$

$$\begin{bmatrix} I\lambda - A & -(I\lambda - A)M - N \\ -F & FM + I - FM \end{bmatrix}\begin{bmatrix} I & 0 \\ F & I \end{bmatrix} = \begin{bmatrix} I\lambda - A - (I\lambda - A)MF - NF & -(I\lambda - A)M - N \\ -F + FMF + F - FMF & FM + I - FM \end{bmatrix}$$

$$\det\left(\begin{bmatrix} I\lambda - A - (I\lambda - A)MF - NF & -(I\lambda - A)M - N \\ 0 & I \end{bmatrix}\right) = 0$$

$$\det\left[(I\lambda - A)(I - MF) - NF\right] = 0$$

$$\rightsquigarrow \int_\tau^0 (I\lambda - A) e^{(\lambda - A)\theta} d\theta = \frac{(I\lambda - A)}{\lambda - A} e^{(\lambda - A)\theta}\Big|_\tau^0 = I - e^{(\lambda - A)\tau}$$

$$(I\lambda - A)MF = (I\lambda - A)\int_{-h}^0 e^{\tau A}\int_\tau^0 e^{(\lambda - A)\theta} d\theta \, d\beta(\tau) F$$

$$= \int_{-h}^0 e^{\tau A} d\beta(\tau) F - \int_{-h}^0 e^{\tau A} e^{(\lambda - A)\tau} d\beta(\tau) F = B(A)F - NF$$

$$\det\left[(I\lambda - A)(I - MF) - NF\right] = \det\left[I\lambda - A - B(A)F\right] = 0$$

Example 2.1: $\dot{x}_1(t) = u(t-h)$

$\dot{x}_2(t) = x_1(t) + x_2(t)$

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1-4e^{h} & -4e^{h} \end{bmatrix}$$

$\rightsquigarrow$ $B(A) = \int_{-h}^{0} e^{\tau A}\, d\beta(\tau)$ I saw it in ref. [10]

$$= B_0 + e^{-Ah} B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ e^{-h}-1 & e^{-h} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-h}-1 \end{bmatrix}$$

$$u(t) = Fx(t) + F\int_{-h}^{0} e^{-(h+\theta)A} B_1 u(t+\theta)\, d\theta$$

i) $$e^{-(h+\theta)A} = \begin{bmatrix} 1 & 0 \\ e^{-(h+\theta)}-1 & e^{-(h+\theta)} \end{bmatrix} \therefore F e^{-(h+\theta)A} B_1 = \begin{bmatrix} 1-\cancel{4e^{h}}-4e^{-\theta}+\cancel{4e^{h}} & -4e^{-\theta} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

ii) $$u(t) = \begin{bmatrix} 1-4e^{h} & -4e^{h} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \int_{-h}^{0} (1-4e^{-\theta})\, u(t+\theta)\, d\theta$$

iii) $$w = \int_{-h}^{0} (1-4e^{-\theta})\, u(t+\theta)\, d\theta = \int_{-h}^{0} u(t+\theta)\, d\theta - 4\int_{-h}^{0} e^{-\theta} u(t+\theta)\, d\theta$$

$$\frac{W_1(s)}{u_1(s)} = \int_{-h}^{0} e^{s\theta}\, d\theta = \left.\frac{e^{s\theta}}{s}\right|_{-h}^{0} = \frac{1}{s} - \frac{e^{-hs}}{s}$$

$$\frac{W_2(s)}{u_2(s)} = 4\int_{-h}^{0} e^{-\theta} e^{s\theta}\, d\theta = 4\int_{-h}^{0} e^{(s-1)\theta}\, d\theta = 4\left.\frac{e^{(s-1)\theta}}{s-1}\right|_{-h}^{0} = \frac{4}{s-1} - \frac{4e^{h} e^{-hs}}{s-1}$$