# Error models for the water surface elevation and discharge simulations

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# 1 Model and hypotheses

## 1.1 Measurements errors:

### 1.1.1 Gauged WSE errors:

We have $N$ measurements of the WSE, $(\tilde{H}_{x_i,t_i})_{i=1,...,N}$ at given positions $x_i$ and times $t_i$. The error model is presented below:

$$\tilde{H}_{x_i,t_i} = H_{x_i,t_i} + \delta_i^{(H)}, \quad \text{with} \quad \delta_i^{(H)} \sim \mathcal{N}\left(0, u_i^{(H)}\right) \tag{1}$$

With $H_{x_i,t_i}$: the true values of the WSE [m]; $\delta_i^{(H)}$: measurement error [m]; $u_i^{(H)}$: standard deviation of the error [m]

### 1.1.2 Gauged discharge errors:

We have $N$ measurements of discharge, $(\tilde{Q}_{x_i,t_i})_{i=1,...,N}$ at given positions $x_i$ and times $t_i$. The error model is presented below:

$$\tilde{Q}_{x_i,t_i} = Q_{x_i,t_i} + \delta_i^{(Q)}, \quad \text{with} \quad \delta_i^{(Q)} \sim \mathcal{N}\left(0, u_i^{(Q)}\right) \tag{2}$$

With $Q_{x_i,t_i}$: the true values of the discharge [m$^3$/s]; $\delta_i^{(Q)}$: measurement error [m$^3$/s]; $u_i^{(H)}$: standard deviation of the error [m$^3$/s].

## 1.2 WSE and discharge simulations:

WSE and discharge simulations are formalized as a function $\mathcal{M}$ representing the hydrodynamic model, for instance, MAGE code.

$$(\hat{H}_{x,t}, \hat{Q}_{x,t})(\boldsymbol{\theta}) = \mathcal{M}(K(x;\boldsymbol{\theta}), t), \quad \text{with} \quad K(x;\boldsymbol{\theta}) = \theta_0 + \sum_{d=1}^{D} \theta_d \, P_d(x) \tag{3}$$

With $\hat{Q}_{x,t}$: discharge [m$^3$/s] at the curvilinear coordinate $x$ along the river and at time $t$; $\hat{H}_{x,t}$: water surface elevation [m] at the same position and time; $K$: friction coefficient [m$^{1/3}$/s]; $P_d$: the Legendre polynomial of degree $d$; $\boldsymbol{\theta}$: coefficients related to Legendre polynomials.

Note that the covariate $x$ can be replaced by another covariate that depends on $x$, $C(x)$.

## 1.3 Structural error:

Error models are developed using the water surface elevations (WSE), but they are transferable to discharge by replacing the superscript $(H)$ with $(Q)$.

$$H_{x,t} = \hat{H}_{x,t} + \varepsilon_{x,t}^{(H)}, \quad \text{with} \quad \varepsilon_{x,t}^{(H)} \sim \mathcal{N}(0, \sigma_{x,t}^{(H)}) \tag{4}$$

$$\sigma_{x,t}^{(H)} = f^{(H)}(x, t, \boldsymbol{\gamma}^{(H)}) \tag{5}$$

Here some examples of the error model $f^{(H)}$:

### 1.3.1 Constant structural standard deviation in space and time:

$$f(x, t, \gamma) = \gamma \tag{6}$$

### 1.3.2 Constant structural standard deviation in time for a given position $x_0$:

$$f(x_0, t, \gamma_{x_0}) = \gamma_{x_0} \tag{7}$$

This model is valid only at position $x_0$. It is suitable for processing a time series at a given position $x_0$. We can repeat this error model at several positions $(x_1, \dots, x_M)$, but we cannot interpolate it outside these positions. For example, for two positions:

$$f(x, t, \boldsymbol{\gamma}) = \begin{cases} \gamma_1 & \text{if} \quad x = x_1 \\ \gamma_2 & \text{if} \quad x = x_2 \\ ?? & \text{if} \quad x \neq x_1 \ \& \ x \neq x_2 \end{cases} \tag{8}$$

### 1.3.3 Constant structural standard deviation in space for a given time $t_0$:

$$f(x, t_0, \gamma_{t_0}) = \gamma_{t_0} \tag{9}$$

This model is valid only at time $t_0$. It is suitable for processing a water surface elevation at a given time $t_0$. We can repeat this error model at several times $(t_1, \dots, t_M)$, but we cannot interpolate it outside these times.

### 1.3.4 Structural standard deviation varying in space and time:

Here a proposition:

$$f(x, t, \boldsymbol{\gamma}) = \gamma_1 + \gamma_2 \times \hat{H}_{x,t} \tag{10}$$

## 1.4 Total error:

We assume that the structural error is independent of the measurement error of the water surface elevation. By combining the previous equations, we arrive at the following total error model:

$$\tilde{H}_{x_i,t_i} = \hat{H}_{x_i,t_i}(\boldsymbol{\theta}) + \varepsilon^{(H)}_{x_i,t_i} + \delta^{(H)}_i \quad \text{with} \quad \varepsilon^{(H)}_{x_i,t_i} + \delta^{(H)}_i \sim \mathcal{N}\left(0, \sqrt{\sigma^{2(H)}_{x_i,t_i} + u^{2(H)}_i}\right) \tag{11}$$

# 2 Bayesian estimation

## 2.1 Information brought by the mesaurements: likelihood

According to the equation 11, the gauged WSE ($\tilde{H}_{x_i,t_i}$) is a realisation from a Gaussian distribution with mean $\hat{H}_{x_i,t_i}(\boldsymbol{\theta})$ (i.e. the simulated water surface elevation) and standard deviation $\sqrt{\sigma^{2(H)}_{x_i,t_i} + u^{2(H)}_i}$. Assuming that errors affecting all gauged WSE are mutually independent, the likelihood can be written as:

$$p(\tilde{\boldsymbol{H}}|\boldsymbol{\theta}, \gamma) = \prod_{i=1}^{N} d_{\mathcal{N}}\left(\tilde{H}_{x_i,t_i}; \hat{H}_{x_i,t_i}(\boldsymbol{\theta}), \sqrt{\sigma^{2(H)}_{x_i,t_i} + u^{2(H)}_i}\right) \tag{12}$$

Where $\tilde{\boldsymbol{H}} = (\tilde{H}_1, \dots, \tilde{H}_N)$ denote the $N$ gauged WSE and $d_{\mathcal{N}}(z; m, s)$, denote the probability density function (pdf) of a Gaussian distribution with mean $m$ and standard deviation $s$ evaluated at some value $z$.

TO DO: -Total likelihood for WSE and discharge. -Prior distribution -Posterior distribution