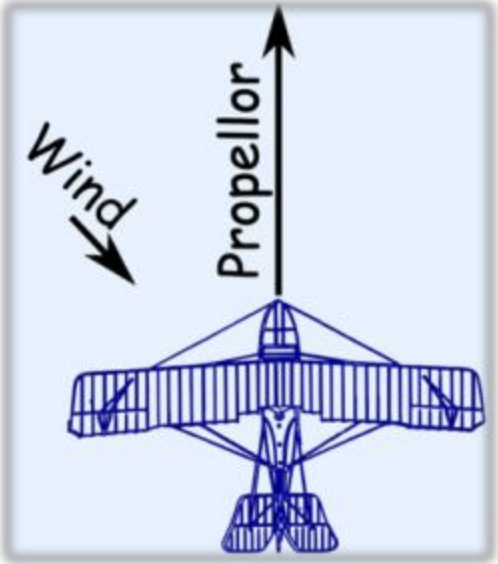
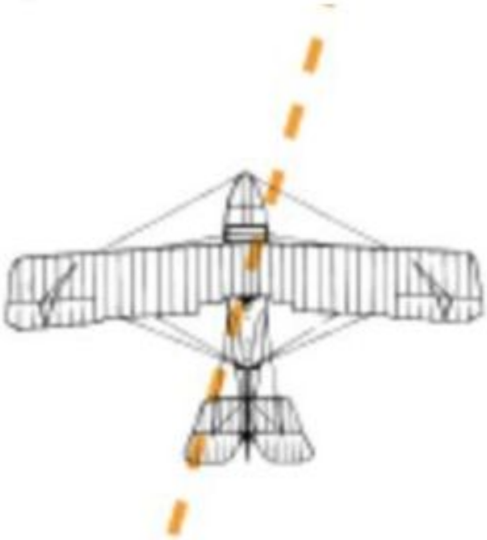
Why vector?

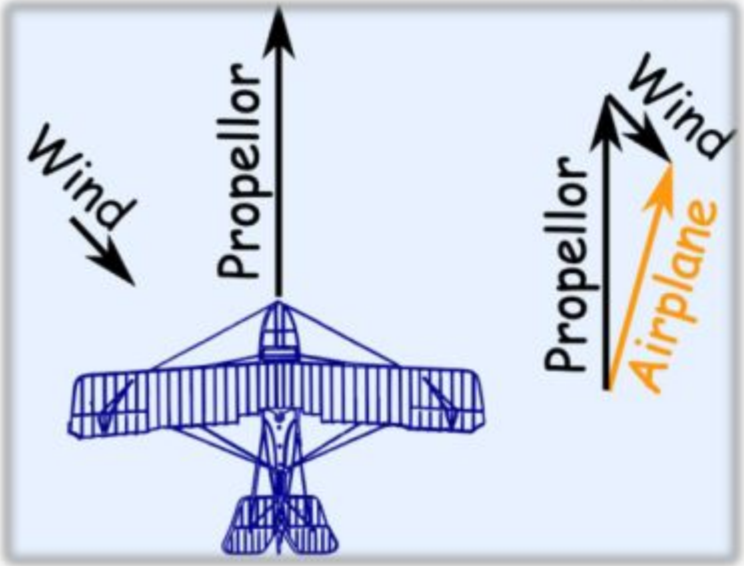




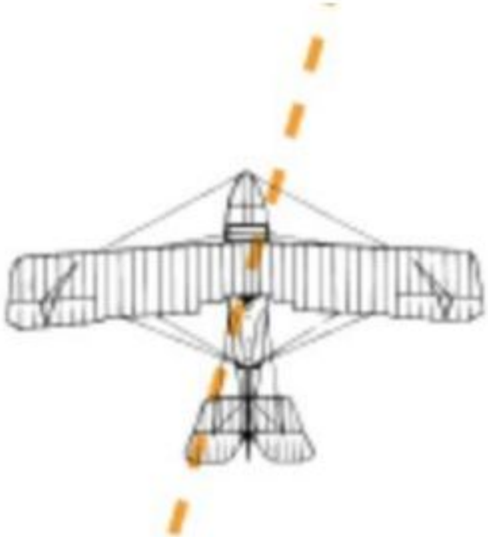
Example: A plane is flying along, pointing North, but there is a wind coming from the North-West.

If you watched the plane from the ground it would seem to be slipping sideways a little.

Why vector?



Example: A plane is flying along, pointing North, but there is a wind coming from the North-West.



If you watched the plane from the ground it would seem to be slipping sideways a little.

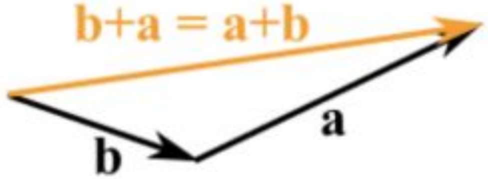
What vector?

This is a vector

A vector has **magnitude** (size) and **direction**

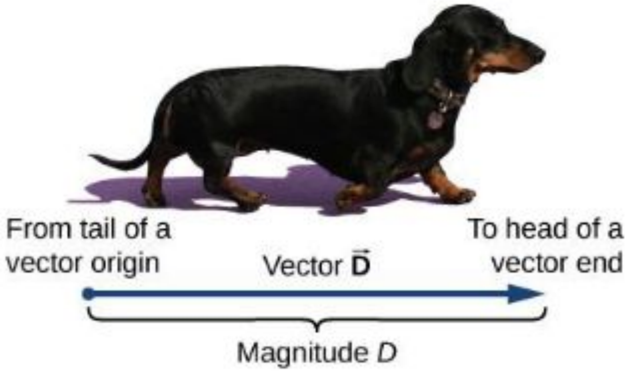
The length of the line shows its magnitude and the arrowhead points in the direction.

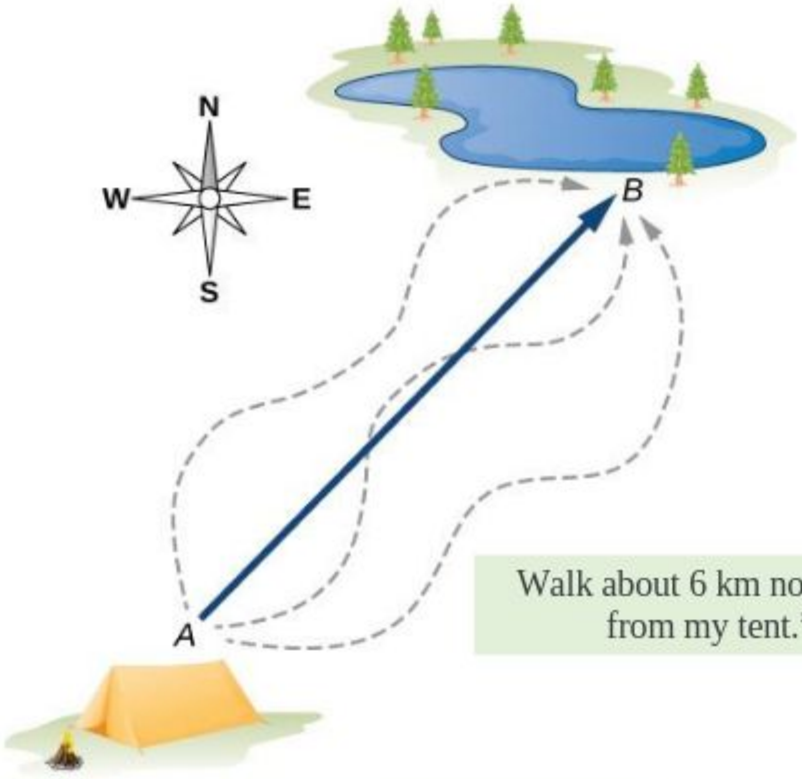
We can add two vectors by joining them head-to-tail



it doesn't matter which order we add them, we get the same result

Vector vs Scalar

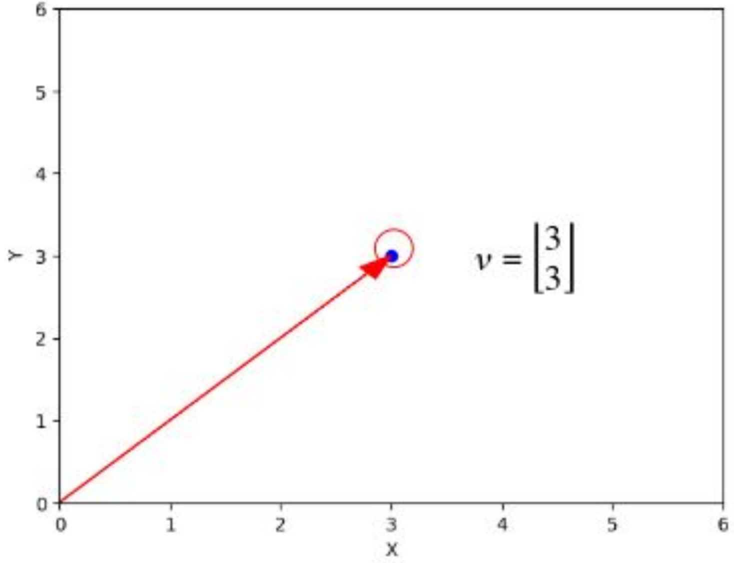




Walk about 6 km northeast from my tent."

You have discovered a terrific fishing hole 6 km from your tent

Vector

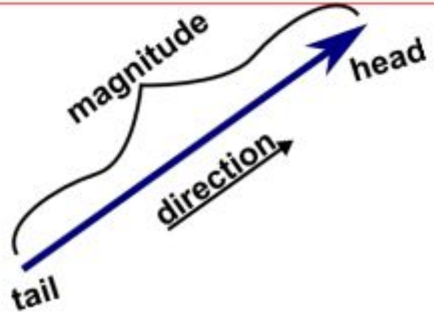


$$v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

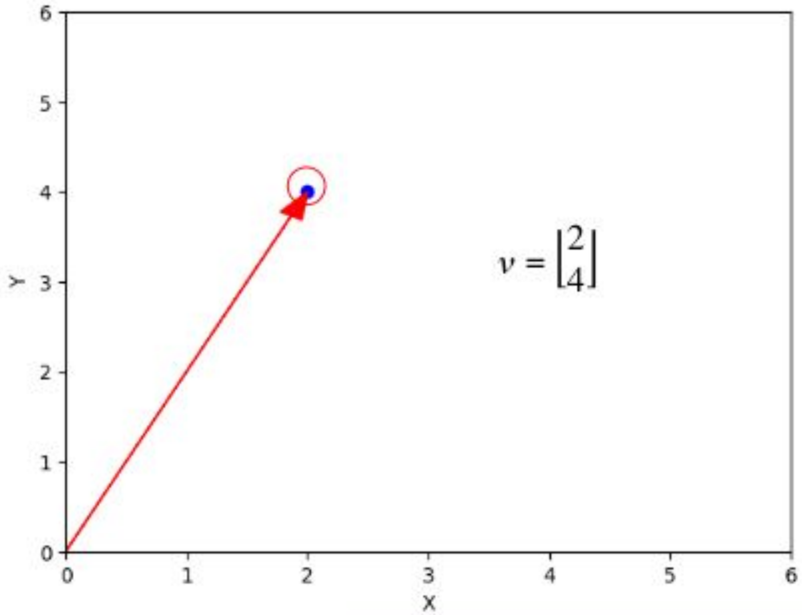
```python
import matplotlib.pyplot as plt
import numpy as np

def plot_vector2d(vector2d, origin=[0, 0], **options):
    return plt.arrow(origin[0], origin[1], vector2d[0], vector2d[1],
                        head_width=0.2,
            head_length=0.3, length_includes_head=True, **options)

u = np.array([3, 3])
plt.scatter(u[0], u[1], color=["b"])
plot_vector2d(u, color="r")
plt.xlabel("X")
plt.ylabel("Y")

plt.axis([0, 6, 0, 6])
plt.show()
```

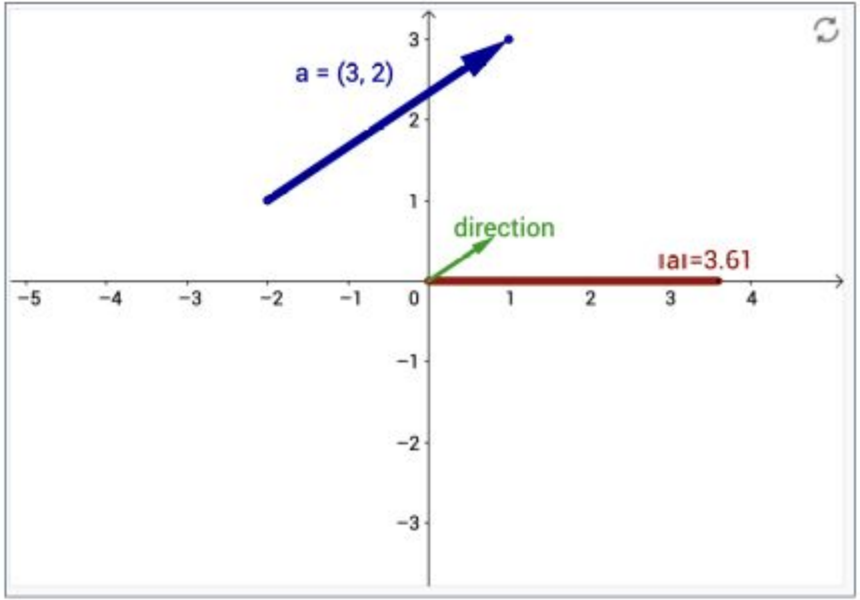
A vector is an object that has both a magnitude and a direction

Vector is a matrix with single row or single column



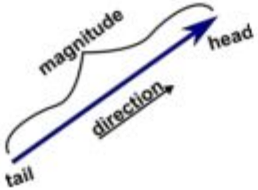
Vector is a matrix with single row or single column



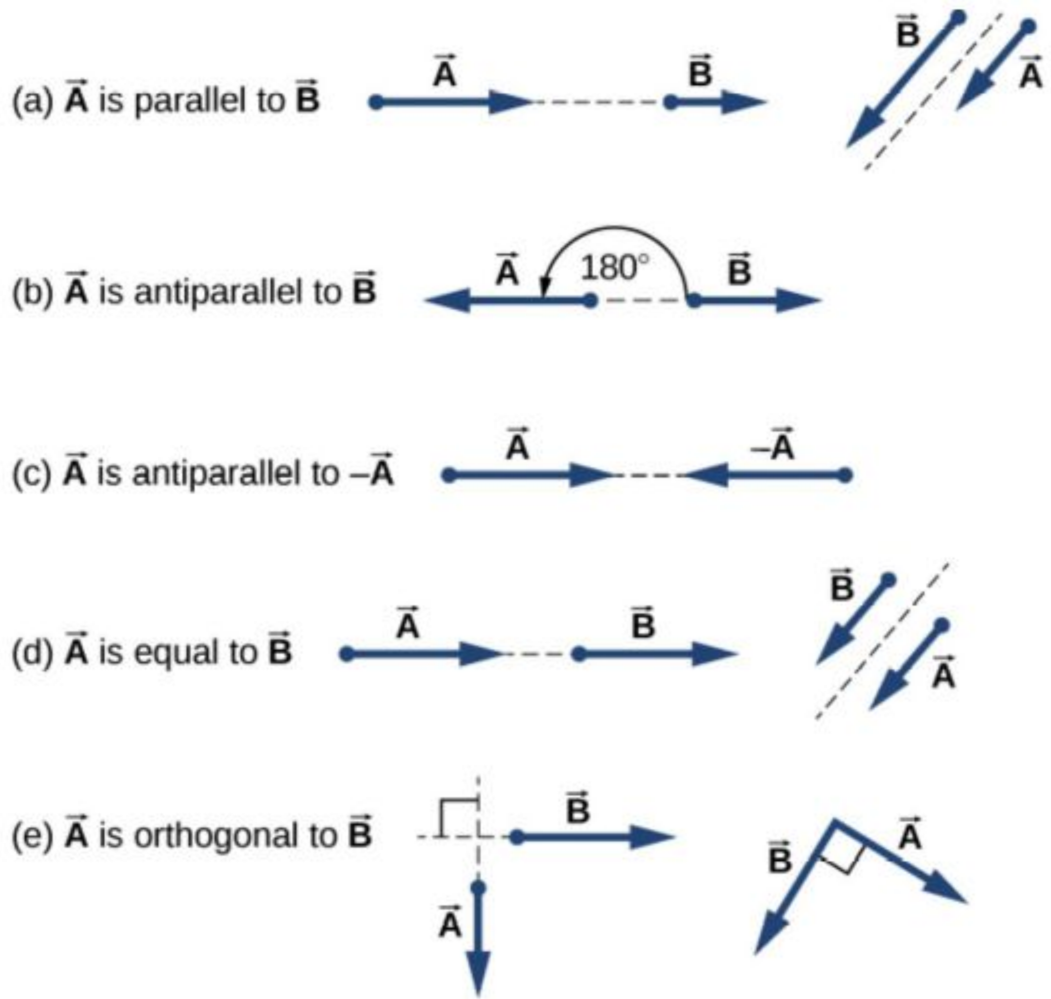


A vector is an object that has both a magnitude and a direction

Vector is a matrix with single row or single column



Relations between vectors



CHECK YOUR UNDERSTANDING

1knot = 1.852 km/h

Two motorboats named *Alice* and *Bob* are moving on a lake. Given the information about their velocity vectors in each of the following situations, indicate whether their velocity vectors are equal or otherwise.

(a) *Alice* moves north at 6 knots and *Bob* moves west at 6 knots.
(b) *Alice* moves west at 6 knots and *Bob* moves west at 3 knots.
(c) *Alice* moves northeast at 6 knots and *Bob* moves south at 3 knots.
(d) *Alice* moves northeast at 6 knots and *Bob* moves southwest at 6 knots.
(e) *Alice* moves northeast at 2 knots and *Bob* moves closer to the shore northeast at 2 knots.

Eigenvector

A eigenvector v, is a non-zero vector that satisfies the following equation:

$$Av = \boxed{\lambda v}$$

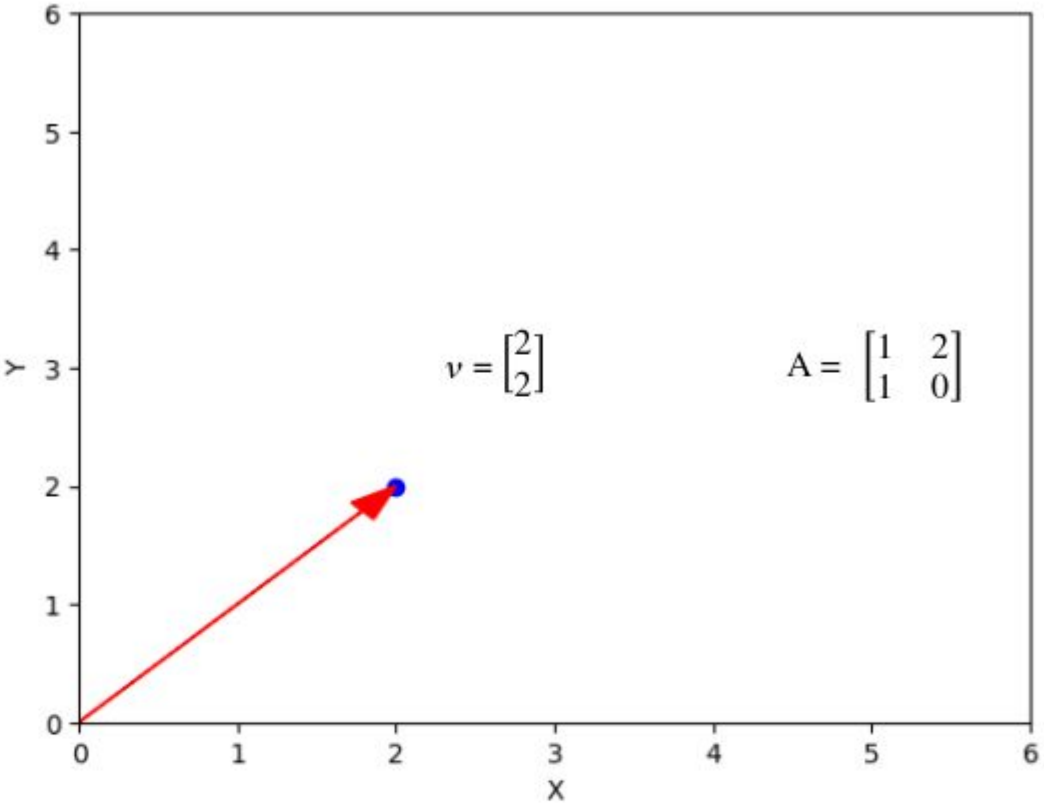
Square matrix **A**

Old column vector **v**

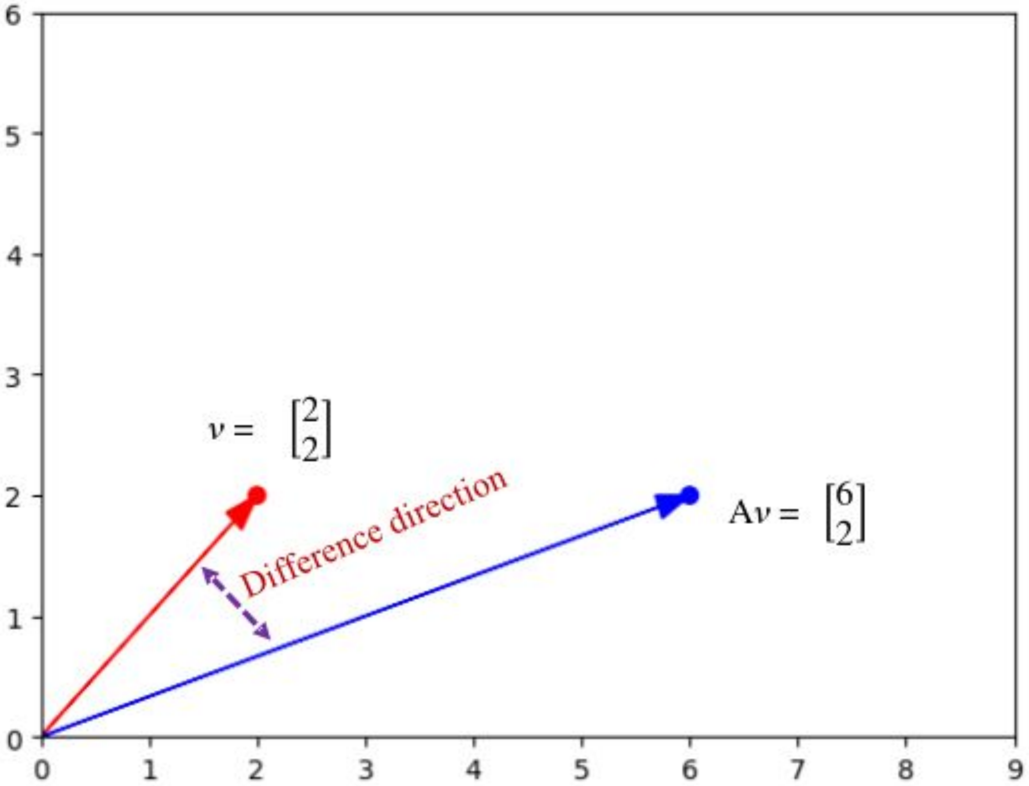
New column vector λv

Vector v is call eigenvector of matrix A

If we multiply Matrix A by vector v, the new vector λv does not change direction after the transformation



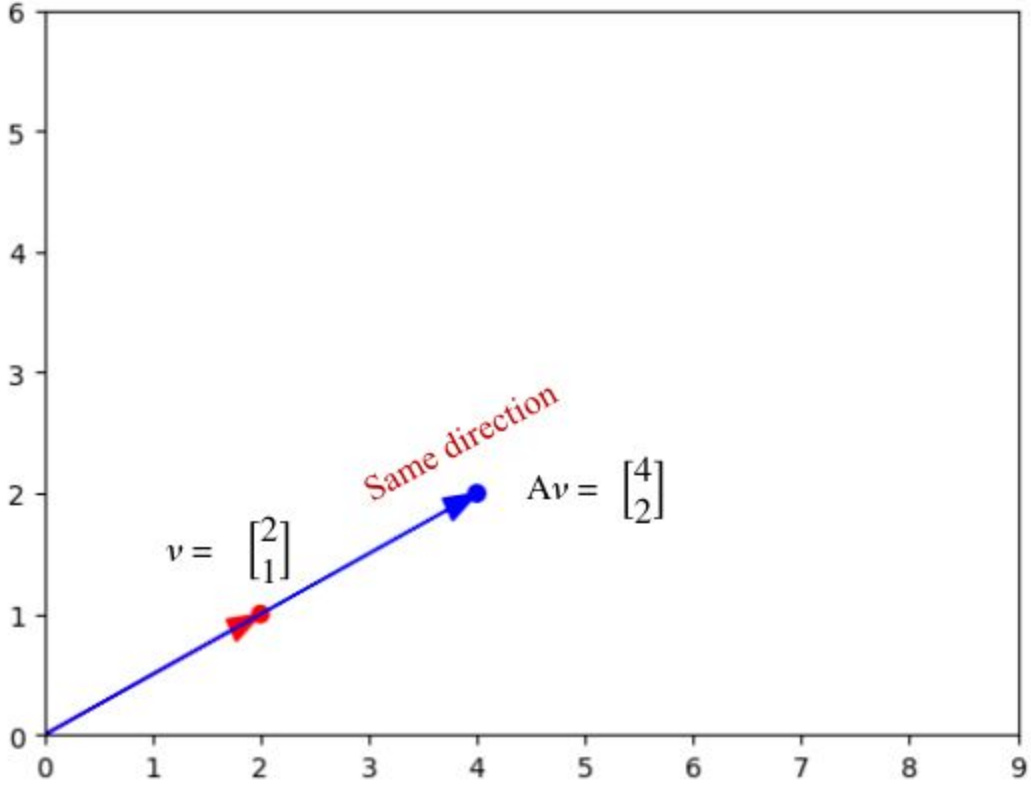
$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \lambda v$$



$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \lambda v$$

Conclusion: vector v is not an egenvector of matrix A



$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \lambda v$$

Conclusion: vector v is an egenvector of matrix A

Eigenvalue

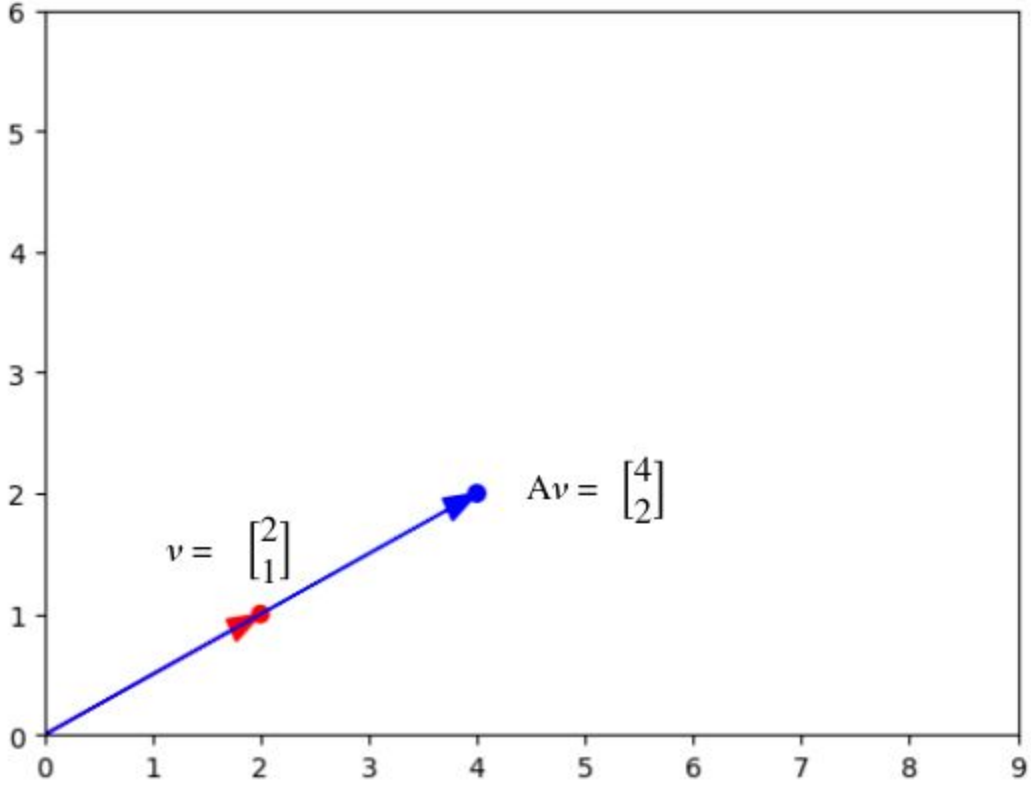
Eigenvalue tell us how much the eigenvector changes in size when multiplied with the matrix

$$Av = \lambda \boxed{v}$$

Square matrix **A**

Old column vector **v**

Eigenvalue



$$A = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}$$

$$Av = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix} = \lambda v$$

$$\lambda v = \lambda \begin{bmatrix}2\\1\end{bmatrix}$$

$$\lambda = 2$$

Conclusion: vector v is an egenvector of matrix A

Eigenvector

A eigenvector v, is a non-zero vector that satisfies the following equation:

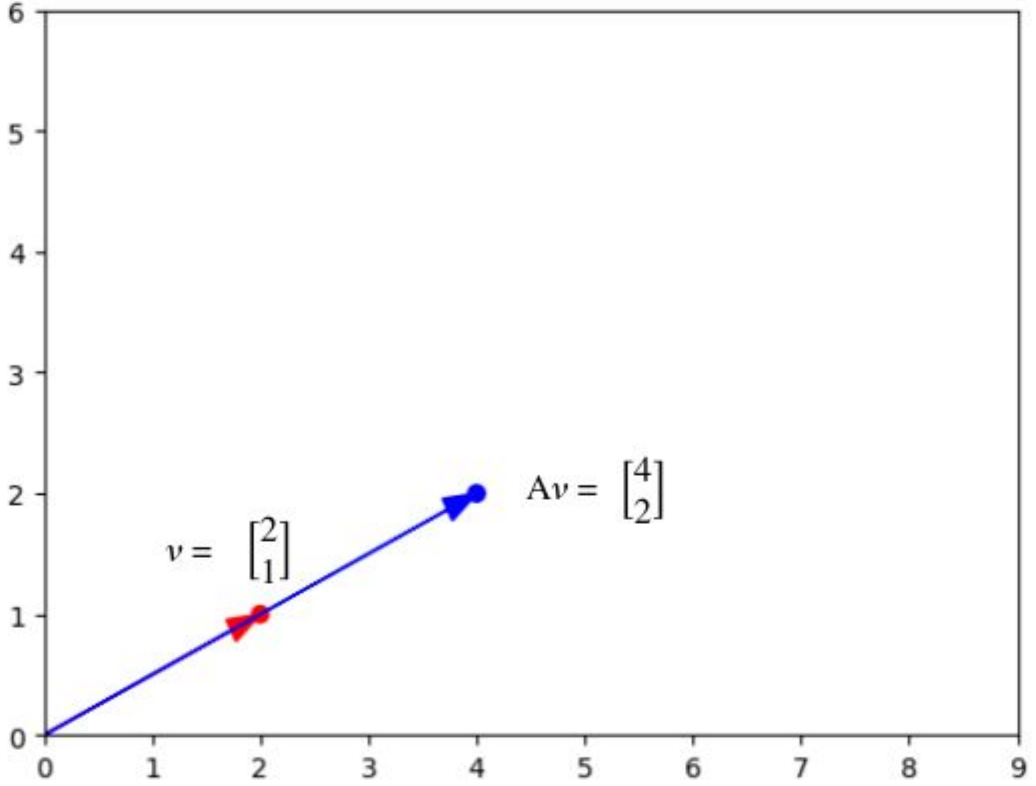
$$Av = \boxed{\lambda v}$$

Square matrix **A**

Egenvector

New column vector λv

New column vector λv has same direction eigenvector.
New column vector λv maybe eitheir longer or shorter than eigenvector because of eigenvalue λ.



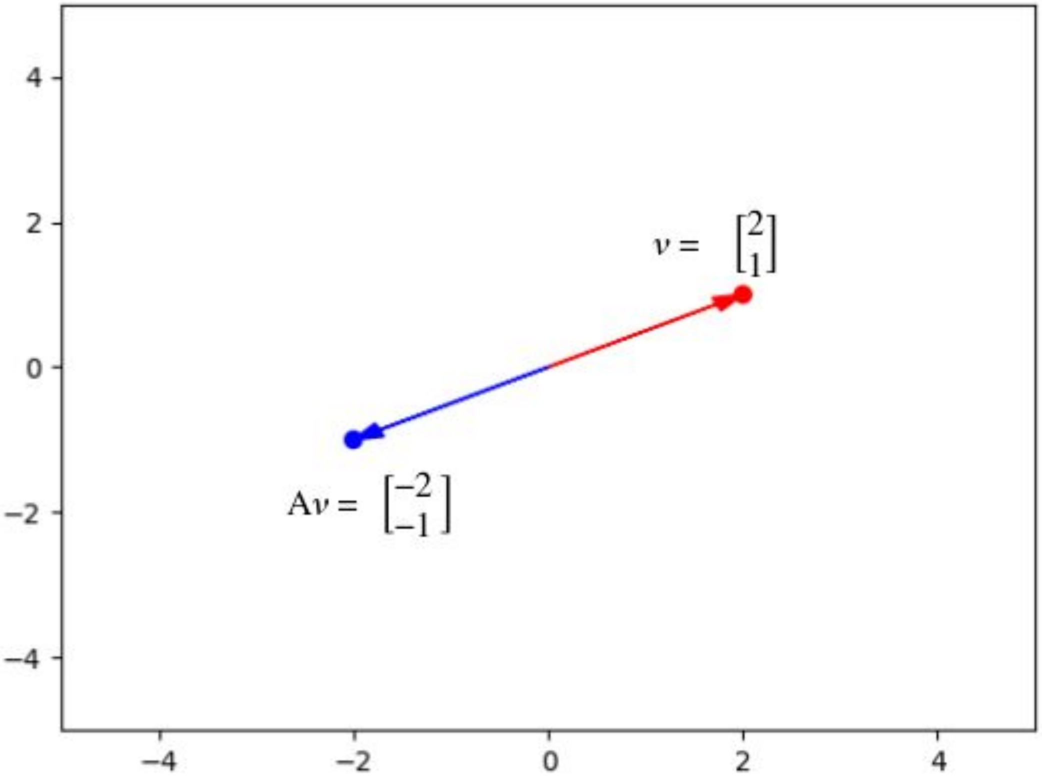
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \lambda v$$

$$\lambda v = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = 2$$

New column vector λv maybe eitheir longer or shorter than eigenvector .



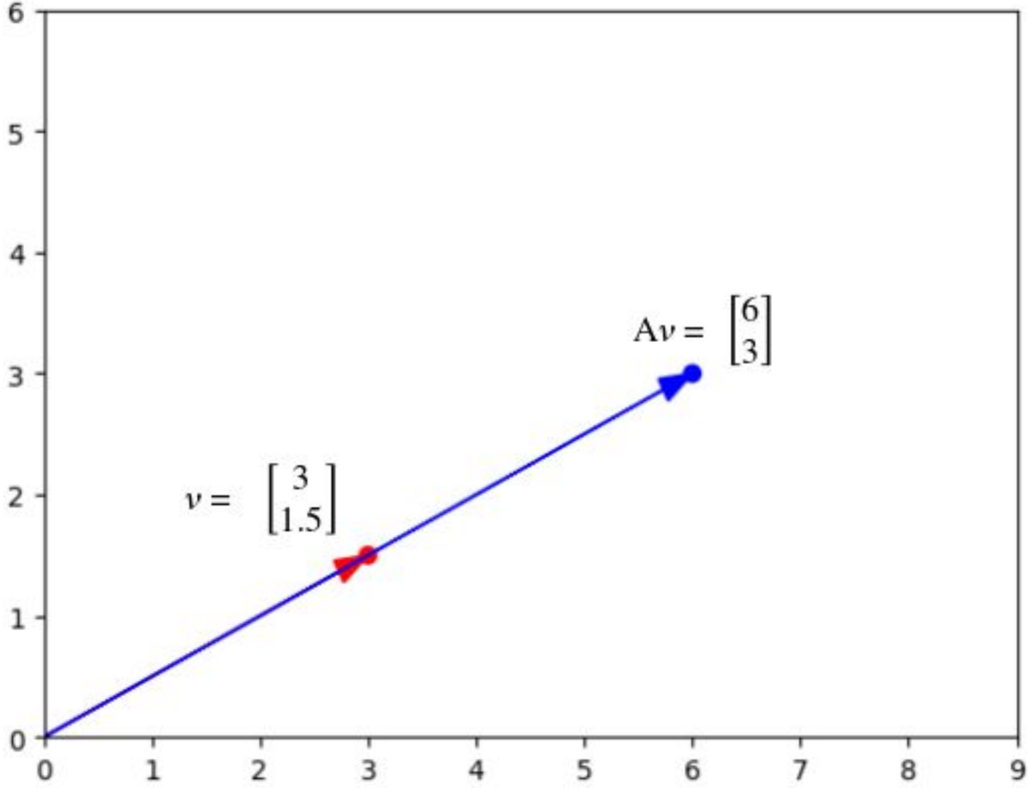
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \lambda v$$

$$\lambda v = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boxed{\lambda = 2}$$

Eigen value λ might be negative. The direction of new vector is reversed but still on the same line



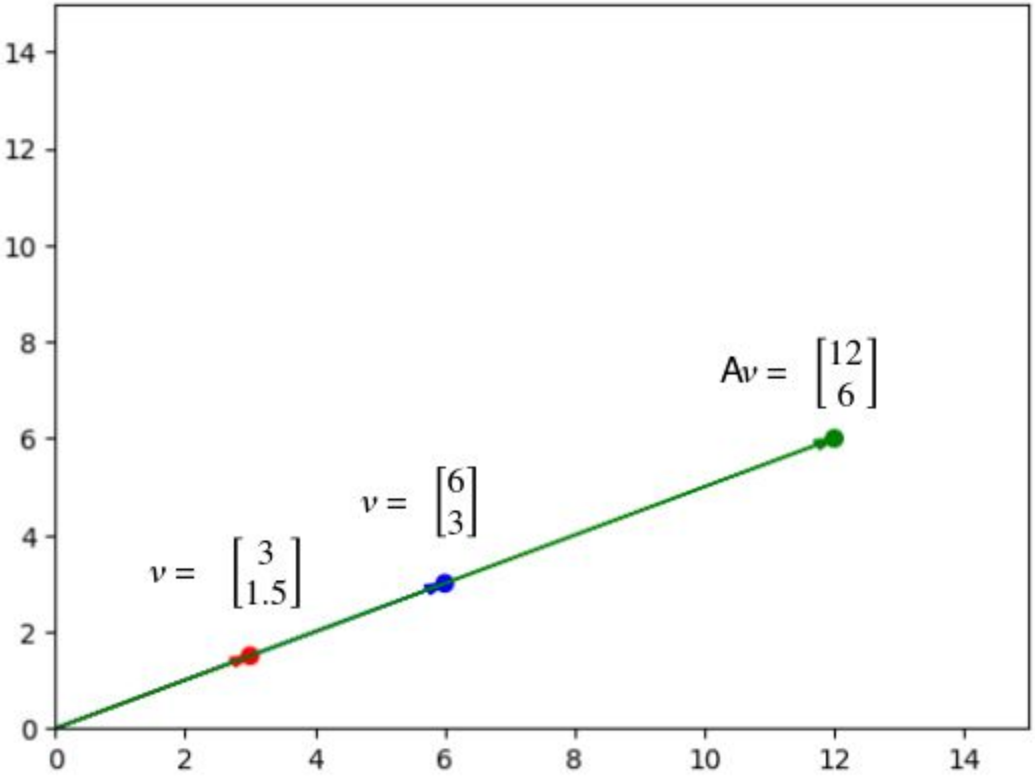
$$A = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}$$

$$Av = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}3\\1.5\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix} = \lambda v$$

$$\lambda v = \lambda \begin{bmatrix}3\\1.5\end{bmatrix}$$

$$\lambda = 2$$

We might find many eigenvectors for matrix A



$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \lambda v$$

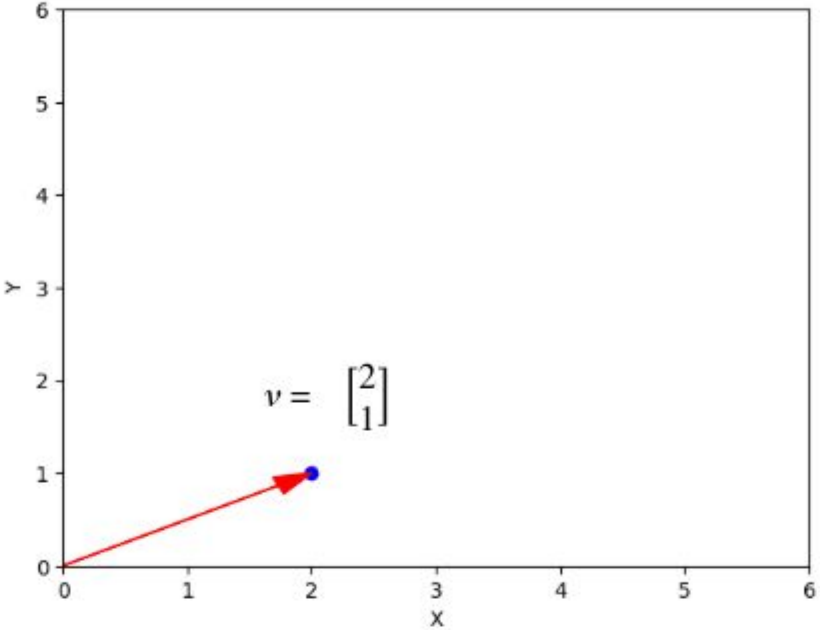
$$\lambda v = \lambda \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\lambda = 2$$

All vectors with the same direction are acctually eigenvector of Matrix A

Unit Length

Eigenvector length is normalized to 1



$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\|v\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\sqrt{2^2 + 1^2} = \sqrt{5}$$

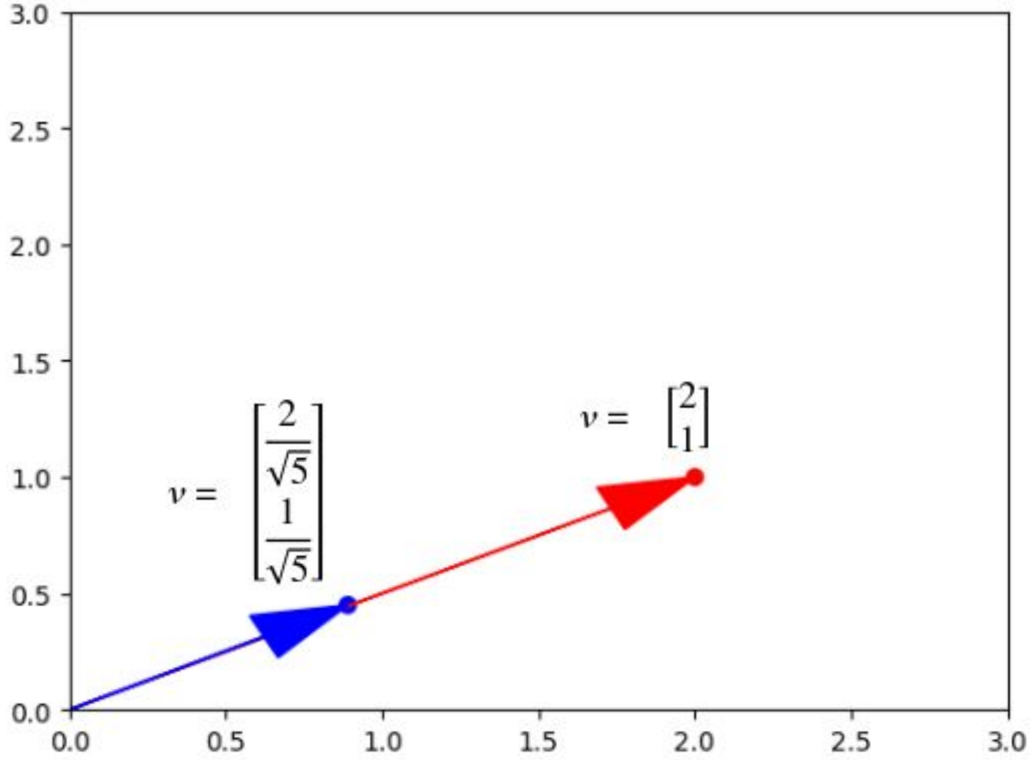
$$2^2 + 1^2 = 5$$

$$\frac{2^2}{5} + \frac{1^2}{5} = 1$$

$$\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 = 1$$

Unit Length

Eigenvector length is normalized to 1



$$v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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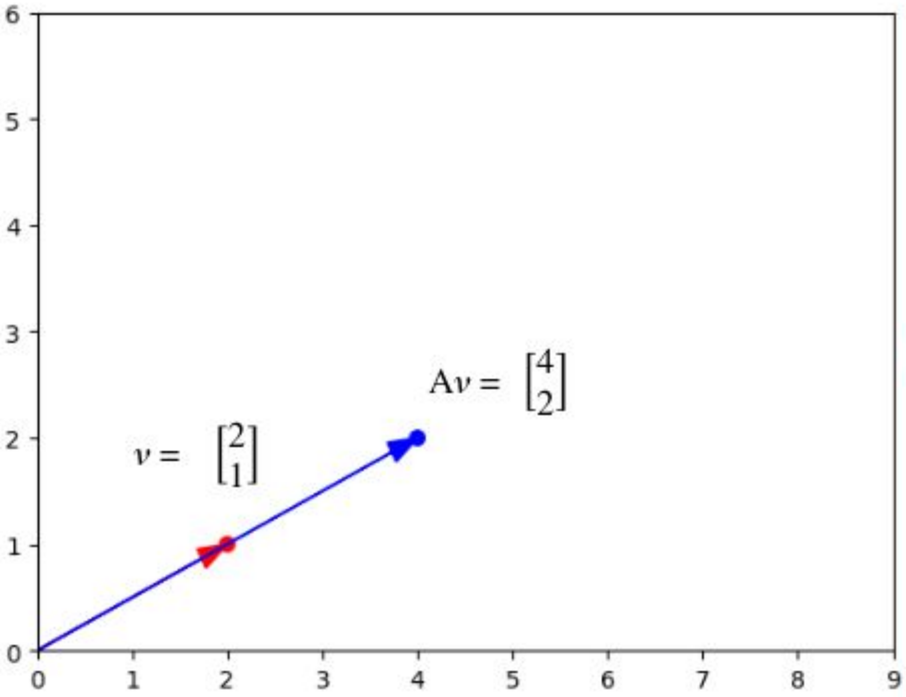
Eigenvector



$A = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} \quad v = \begin{bmatrix}2\\1\end{bmatrix}$

$Av = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix} = \lambda v$

$\lambda v = \lambda \begin{bmatrix}2\\1\end{bmatrix}$

$\lambda = 2$

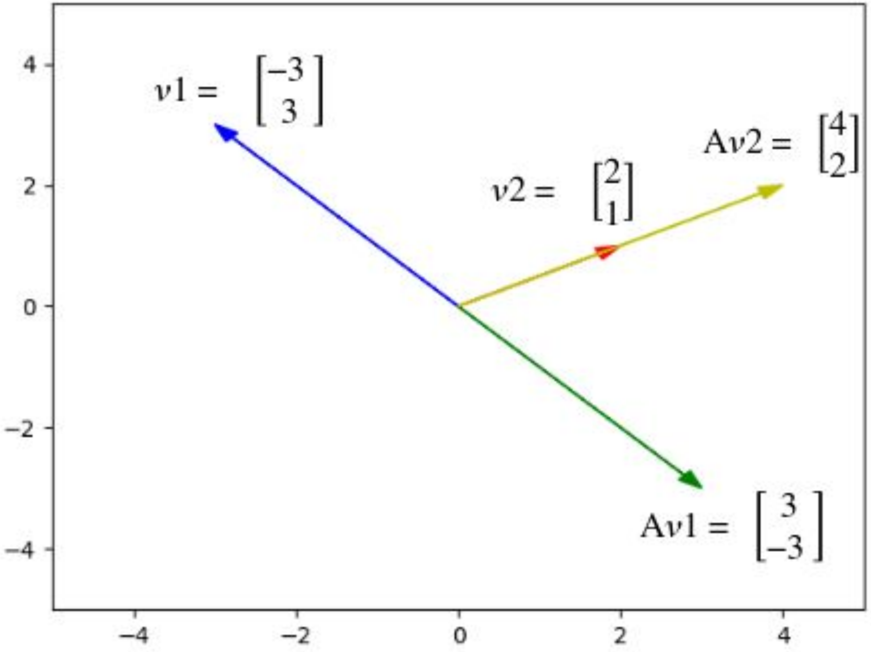
$A = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} \quad v = \begin{bmatrix}-3\\3\end{bmatrix}$

$Av = \begin{bmatrix}1 & 2\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}-3\\3\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix} = \lambda v$

$\lambda v = \lambda \begin{bmatrix}-3\\3\end{bmatrix}$

$\lambda = -1$

Eigenvector



$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \lambda v2$$

$\lambda = 2$

$$\lambda v2 = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad v1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$$Av = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \lambda v1$$

$\lambda = -1$

$$\lambda v1 = \lambda \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

For matrix A (2,2), we will have 2 eigenvalues.
If matrix A(3,3), we will have 3 eigenvalues, and so on

Eigenvalue and Eigenvector

Procedure and example

$$A = \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix}$$

$$Av = \lambda v$$

$$\Leftrightarrow Av - \lambda v = 0$$

$$\Leftrightarrow Av - \lambda I v = 0$$

$$\Leftrightarrow (A - \lambda I)v = 0$$

If v is non-zero then we can solve for λ using just the ***determinant***

$$|A - \lambda I| = 0$$

Start with $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

Which is

$$\begin{vmatrix} -6 - \lambda & 3 \\ 4 & 5 - \lambda \end{vmatrix} = 0$$

Calculating that determinant gets:

$$(-6 - \lambda)(5 - \lambda) - 3 \times 4 = 0$$

Which then gets us this

$$\lambda^2 + \lambda - 42 = 0$$

And solving it gets:

$$\lambda = -7 \quad \text{or } \lambda = 6$$

https://textbooks.math.gatech.edu/ila/index.html

Eigenvalue and Eigenvector

Procedure and example

Start with

$$Av = \lambda v$$

Put in the values we know

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix}$$

After multiplying we get these two equations:

$$\begin{aligned} -6x + 3y &= 6x \\ 4x + 5y &= 6y \end{aligned}$$

Bringing all to left hand side:

$$\begin{aligned} -12x + 3y &= 0 \\ 4x - 1y &= 0 \end{aligned}$$

Either equation reveals that $y = 4x$, so the ***eigenvector*** is any *nonzero* multiple of this:

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

And we get the solution shown at the top of the page:

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \times 1 + 3 \times 4 \\ 4 \times 1 + 5 \times 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \end{pmatrix}$$

... and also...

$$6 \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \end{pmatrix}$$

So $Av = \lambda v$

Eigenvalue and Eigenvector

Procedure and example

Start with

$$Av = \lambda v$$

Put in the values we know

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix}$$

After multiplying we get these two equations:

$$\begin{aligned} -6x + 3y &= 6x \\ 4x + 5y &= 6y \end{aligned}$$

Bringing all to left hand side:

$$\begin{aligned} -12x + 3y &= 0 \\ 4x - 1y &= 0 \end{aligned}$$

Either equation reveals that $y = 4x$, so the ***eigenvector*** is any *nonzero* multiple of this:

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \xrightarrow{\text{normalize}} \begin{pmatrix} 0.24 \\ 0.97 \end{pmatrix}$$

Approximation

Verify:

$$\begin{pmatrix} -6 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 0.24 \\ 0.97 \end{pmatrix} = \begin{pmatrix} 1.44 \\ 5.82 \end{pmatrix}$$

... and also...

$$6 \begin{pmatrix} 0.24 \\ 0.97 \end{pmatrix} = \begin{pmatrix} 1.44 \\ 5.82 \end{pmatrix}$$

So $Av = \lambda v$