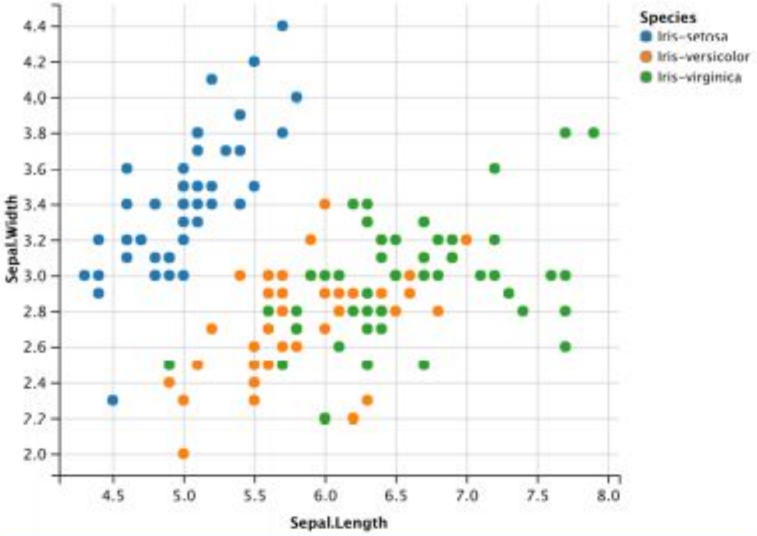
What happen for all dimensions



An example of a dataset (a point can be considered a vector through the origin).

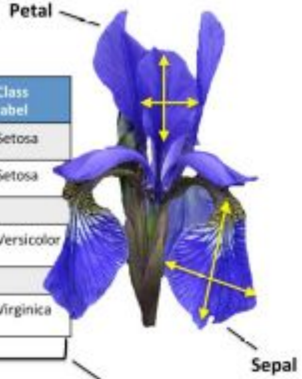
Iris flower data set

Samples
(instances, observations)

	Sepal length	Sepal width	Petal length	Petal width	Class label
1	5.1	3.5	1.4	0.2	Setosa
2	4.9	3.0	1.4	0.2	Setosa
			...		
50	6.4	3.5	4.5	1.2	Versicolor
			...		
150	5.9	3.0	5.0	1.8	Virginica

Features
(attributes, measurements, dimensions)

Class labels
(targets)



Any vector can be expressed in terms of:
1. Projection directions unit vectors (v_1, v_2, …).
2. The lengths of projections onto them (s_{a1}, s_{a2}, …).

Extend this conclusion for handling a bunch of vectors

Extend this conclusion for handling a matrix (list of vectors)

Singular value decomposition

$$A = U D V^{T}$$

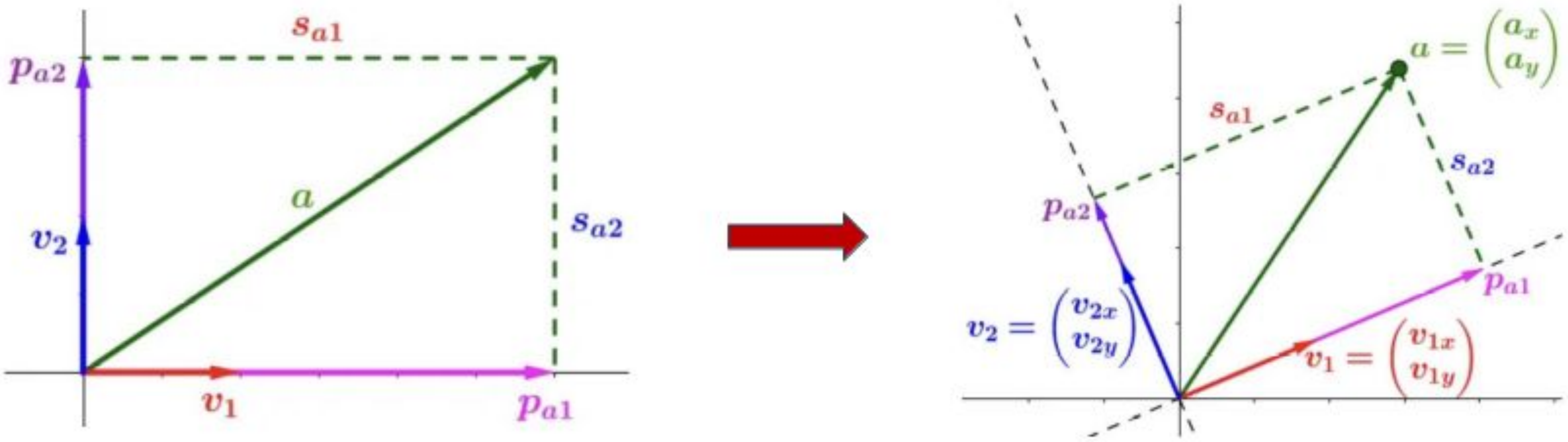
SVD finds a way to express the operation of vector decomposition using matrices.

singular vectors

Singular values

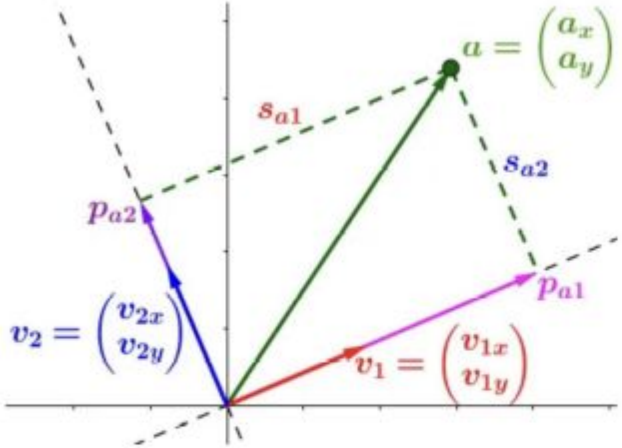
singular vectors

Recall Vector Projection



Decompose (project) the vector $\boldsymbol{a}$ along unit vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

Recall Vector Projection

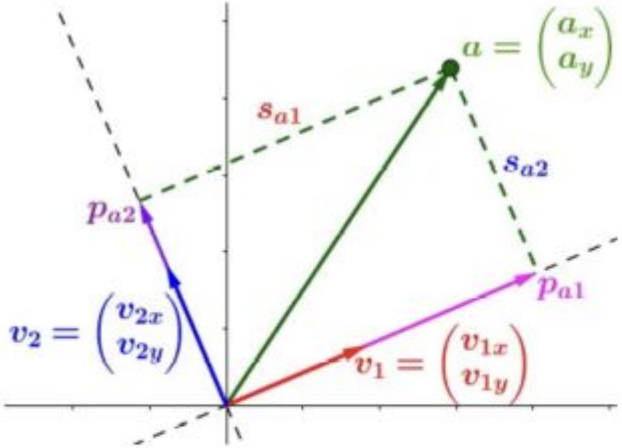


s_{a_1} and s_{a_2}: the lengths of projection (Vector form)

$$a^T \cdot v_1 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = s_{a1}$$

$$a^T \cdot v_2 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = s_{a2}$$

Recall Vector Projection



s_{a_1} and s_{a_2}: the lengths of projection (Matrix form)

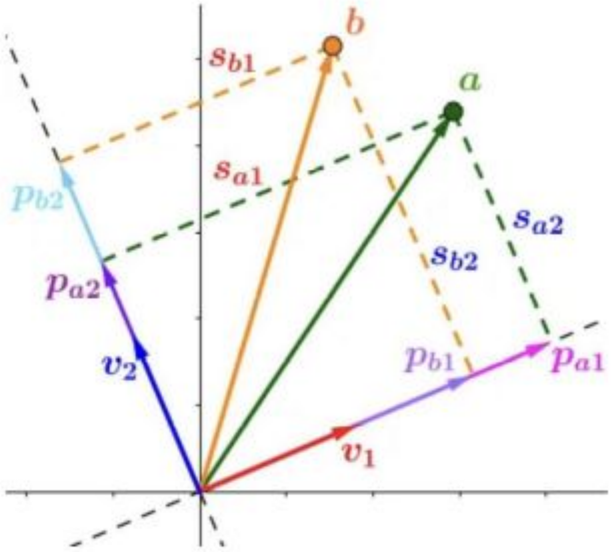
$$a^T \cdot v_1 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = s_{a1}$$

$$a^T \cdot v_2 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = s_{a2}$$

$$\Downarrow$$

$$a^T \cdot V = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} \end{pmatrix}$$

Recall Vector Projection

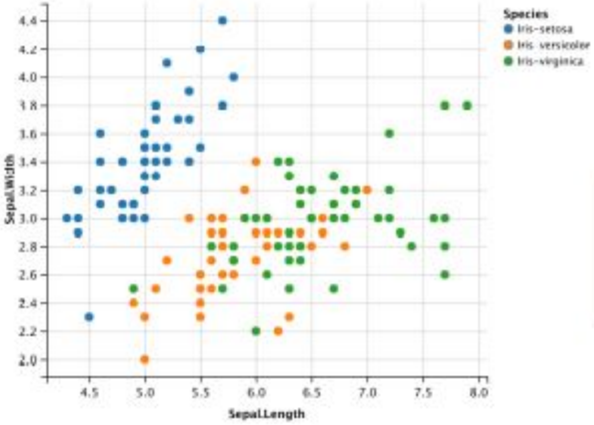


s_{a_1}, s_{a_2}: the lengths of projection of vector a

s_{b1}, s_{b2} : the lengths of projection of vector b

$$A \cdot V = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix} = S$$

Recall Vector Projection



n = no. of points, d = no. of dimensions, A = matrix containing points, V = matrix containing the decomposition axes, S = matrix containing lengths of projection.

$$A \cdot V = \underset{n \times d}{\begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} \cdot \underset{d \times d}{\begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} = \underset{n \times d}{\begin{pmatrix} s_{a1} & s_{a2} & \cdots \\ s_{b1} & s_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} = S$$

Generalize to any number of points and dimensions

Recall Vector Projection

$$A \cdot V = \underset{n \times d}{\begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} \cdot \underset{d \times d}{\begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} = \underset{n \times d}{\begin{pmatrix} s_{a1} & s_{a2} & \cdots \\ s_{b1} & s_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} = S$$

$$A \cdot V = S$$

Matrix of points

The dot product performs the projection

Matrix of decomposition axes

Matrix of the lengths of projections

V is orthogonal matrices

$$A = S\,V^{-1} = S\,V^{T}$$

Recall Vector Projection

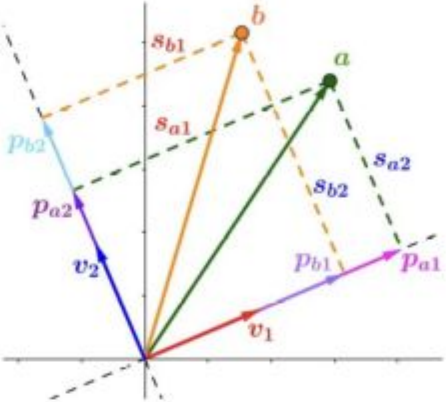
$$A = S\,V^{-1} = S\,V^{T}$$

Any set of vectors (A) can be expressed in terms of their lengths of projections (S) on some set of orthogonal axes (V).

Orthogonal Vectors

$$V = \underbrace{\begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}}_{d \times d}$$

V is orthogonal matrices



We are here!

Convention SVD

$$A = S\,V^{-1} = S\,V^T \neq A = U\,\Sigma\,V^T$$

$$\boxed{S = U\,\Sigma}$$

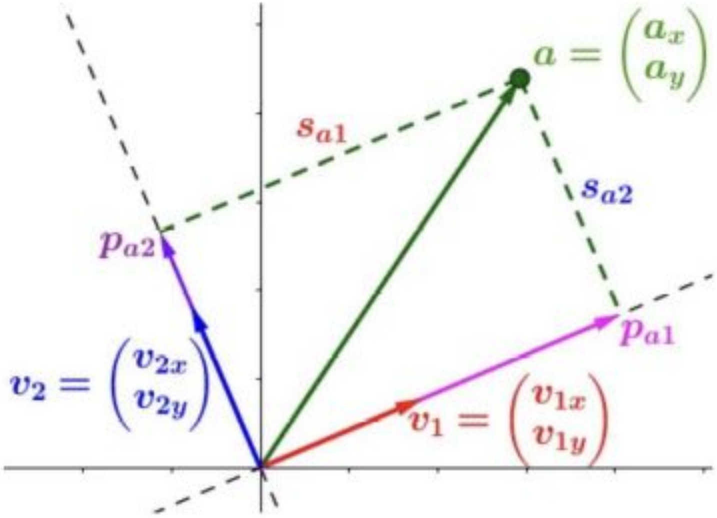
$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

A column vector containing the lengths of projections of each point on the 1st axis *v1*

A column vector containing the lengths of projections of each point on the 2nd axis *v2*

normalize these column vectors to make them of **unit length**

dividing each column vector by its magnitude, but in matrix form

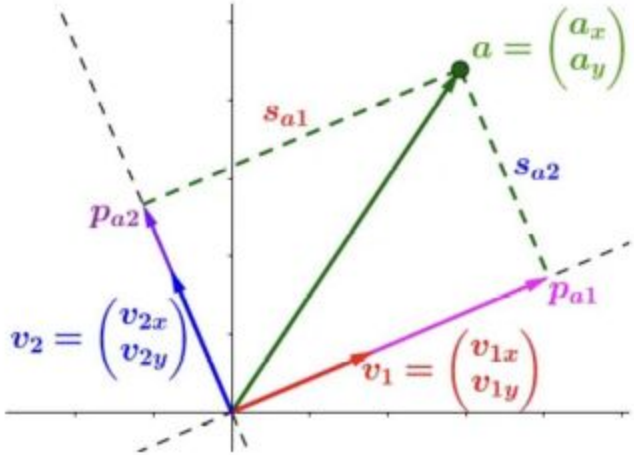


$$A \cdot V = \underset{n \times d}{\begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} \cdot \underset{d \times d}{\begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} = \underset{n \times d}{\begin{pmatrix} s_{a1} & s_{a2} & \cdots \\ s_{b1} & s_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}} = S$$

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

A column vector containing the lengths of projections of each point on the 1st axis *v1*

A column vector containing the lengths of projections of each point on the 2nd axis *v2*



$$M = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \longrightarrow M = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

$$\downarrow$$

$$M = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

$$\downarrow$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

Normalize the columns of $\boldsymbol{S}$

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

$$\text{Magnitude of 1st column} = \sigma_1 = \sqrt{(s_{a1})^2 + (s_{b1})^2}$$

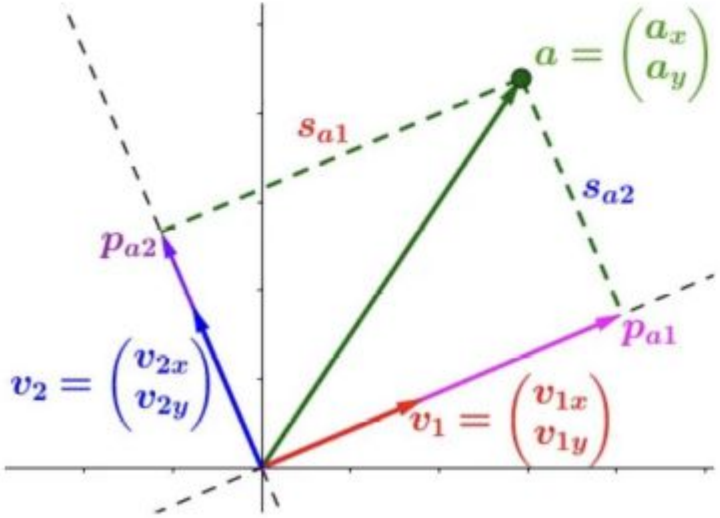
$$\text{Magnitude of 2nd column} = \sigma_2 = \sqrt{(s_{a2})^2 + (s_{b2})^2}$$

(σ_i) is the **square root of the sum of squared projection lengths,** of all points, onto the *i*th unit vector $\boldsymbol{v_i}$

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

A column vector containing the lengths of projections of each point on the 1st axis *v1*

A column vector containing the lengths of projections of each point on the 2nd axis *v2*



Normalize the columns of *S*

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

$$\text{Magnitude of 1st column} = \sigma_1 = \sqrt{(s_{a1})^2 + (s_{b1})^2}$$

$$\text{Magnitude of 2nd column} = \sigma_2 = \sqrt{(s_{a2})^2 + (s_{b2})^2}$$

$$S = \begin{pmatrix} \frac{s_{a1}}{\sigma_1} & \frac{s_{a2}}{\sigma_2} \\ \frac{s_{b1}}{\sigma_1} & \frac{s_{b2}}{\sigma_2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \underbrace{\begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}}_{\Sigma}$$

S - Explanation

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix} = \begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = U\,\Sigma$$

Lengths of projections on $v1$

Lengths of projections on v2

Lengths of projections on $v1$, but divided by σ1 to become a unit vector

Lengths of projections on $v2$, but divided by σ2 to become a unit vector

What about the sigmas? Why did we need to normalie ***S*** to find them?

σ - Explanation

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix} = \begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = U\,\Sigma$$

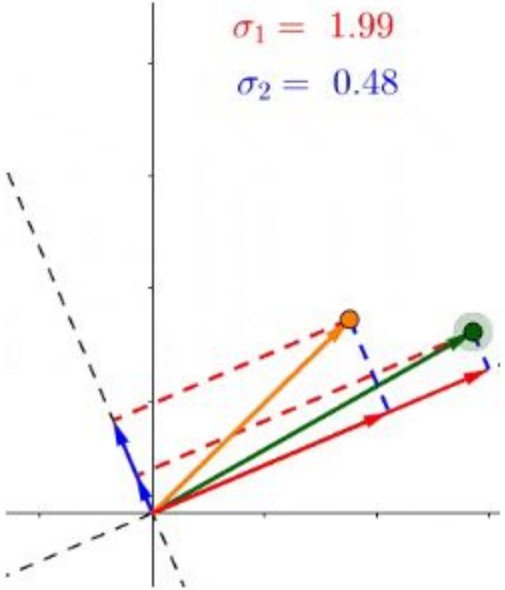
Lengths of projections on v1

Lengths of projections on v2

Lengths of projections on v1, but divided by σ1 to become a unit vector

Lengths of projections on v2, but divided by σ2 to become a unit vector

(σ_i) is the **square root of the sum of squared projection lengths,** of all points, onto the *i*th unit vector ***v*$_i$**



if $\sigma_1 > \sigma_2$, then most points are closer to $\boldsymbol{v}_1$ than $\boldsymbol{v}_2$, and vice versa.

Singular Value Decomposition

❖ **Formula**

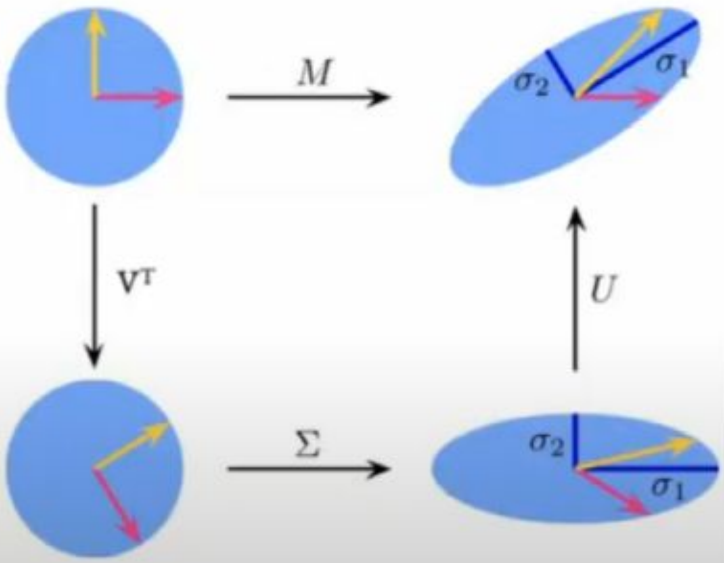
$$\mathbf{M} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^{\mathbf{T}}$$

Orthogonal Matrix | Diagonal Matrix | Orthogonal Matrix

U columns contain eigenvectors of matrix $\mathbf{MM}^t$

Σ is a diagonal matrix containing singular eigenvalues

V columns contain eigenvectors of matrix $\mathbf{M}^t\mathbf{M}$



❖ **Let's observe!**

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$\lambda_i^{(1)} = \lambda_i^{(2)}$$

(1)

$$AA^T = \begin{pmatrix} 1 & 2 \\ 2 & 13 \end{pmatrix}$$

$\lambda_1 = 13.32$ $\lambda_2 = 0.675$

$u_1 = \begin{pmatrix} -0.16 \\ -0.98 \end{pmatrix}$ $u_2 = \begin{pmatrix} -0.98 \\ 0.16 \end{pmatrix}$

(2)

$$A^T A = \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix}$$

$\lambda_1 = 13.32$ $\lambda_2 = 0.675$

$v_1 = \begin{pmatrix} -0.58 \\ -0.81 \end{pmatrix}$ $v_2 = \begin{pmatrix} -0.81 \\ 0.58 \end{pmatrix}$

$$AA^T u_i = \lambda_i u_i \qquad A^T A v_i = \lambda_i v_i$$

singular value $\sigma = \sqrt{\lambda}$ $\quad \sigma_1 = \sqrt{\lambda_1} = 3.65$, $\quad \sigma_2 = \sqrt{\lambda_2} = 0.82$

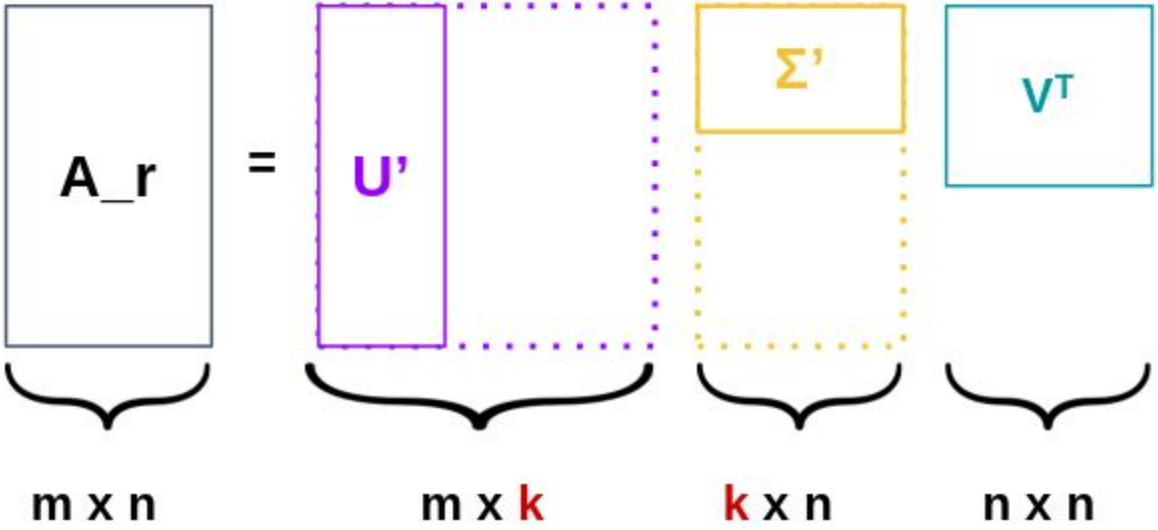
$$A v_i = \sigma_i u_i$$

$$A = (u_1 \ \ u_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -0.16 & -0.98 \\ -0.98 & 0.16 \end{pmatrix} \begin{pmatrix} 3.65 & 0 \\ 0 & 0.82 \end{pmatrix} \begin{pmatrix} -0.58 & -0.81 \\ -0.81 & 0.58 \end{pmatrix}$$

$$A = U\Sigma V$$

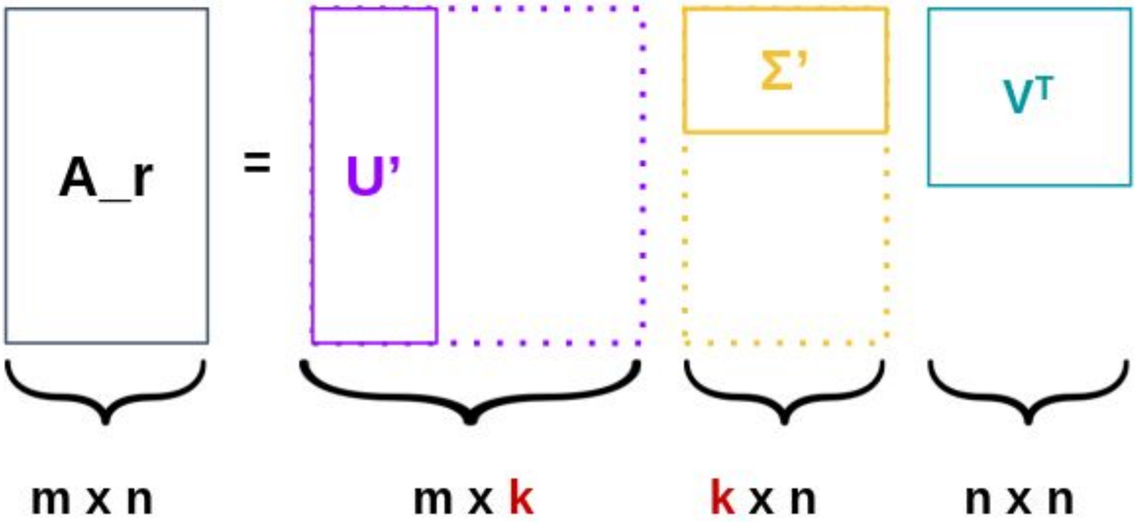
Singular Value Decomposition

Revisit



Singular Value Decomposition

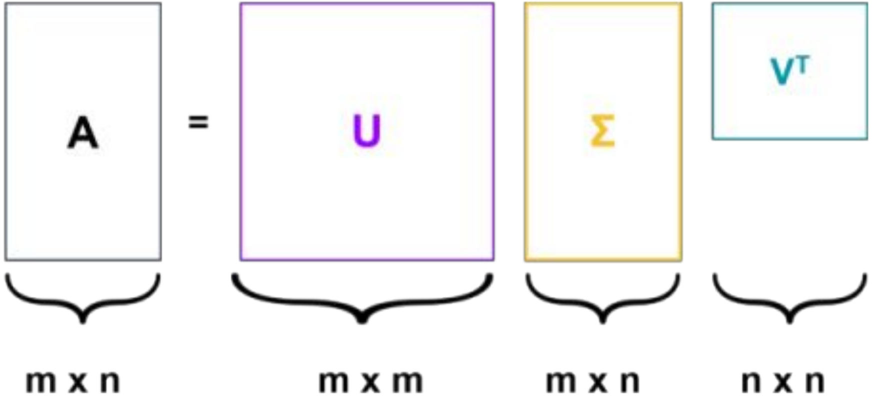
Revisit



Singular Value Decomposition

Example

$$A = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 3 & 1 \\ 4 & 6 & 1 \\ 5 & 7 & 7 \end{bmatrix}$$



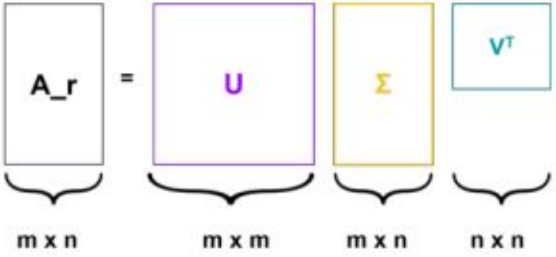
$$U = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

Singular Value Decomposition

Example: Reconstruction



$$A = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 3 & 1 \\ 4 & 6 & 1 \\ 5 & 7 & 7 \end{bmatrix} \qquad U = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix} \qquad V^T = \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

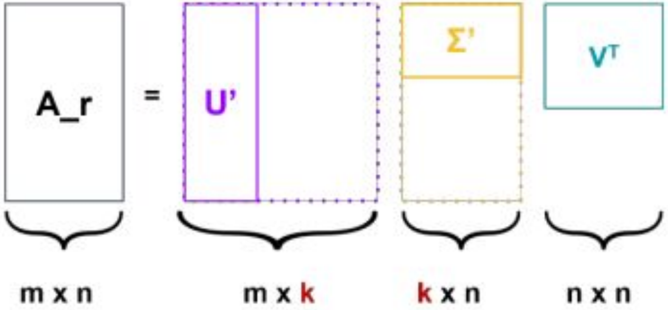
$$A\_r = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix} \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix} \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0e{+}00 & 6.0e{+}00 & 6.0e{+}00 \\ 1.0e{-}16 & 3.0e{+}00 & 1.0e{+}00 \\ 4.0e{+}00 & 6.0e{+}00 & 1.0e{+}00 \\ 5.0e{+}00 & 7.0e{+}00 & 7.0e{+}00 \end{bmatrix}$$

Error = 1.0e-14

Singular Value Decomposition

Example: Reconstruction



$$A = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 3 & 1 \\ 4 & 6 & 1 \\ 5 & 7 & 7 \end{bmatrix} \qquad U = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix} \qquad V^T = \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

Compression
k=2

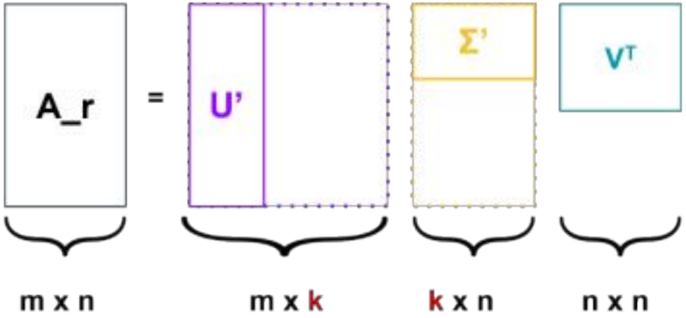
$$A\_r = \begin{bmatrix} -0.53, & -0.55 \\ -0.18, & 0.03 \\ -0.42, & 0.83 \\ -0.71, & -0.08 \end{bmatrix} \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \end{bmatrix} \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

$$= \begin{bmatrix} 1.78 & 5.33 & 6.34 \\ 1.1 & 2.05 & 1.49 \\ 4.37 & 5.68 & 1.16 \\ 3.93 & 7.92 & 6.53 \end{bmatrix}$$

Error = 2.45

Singular Value Decomposition

Example: Reconstruction



$$A = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 3 & 1 \\ 4 & 6 & 1 \\ 5 & 7 & 7 \end{bmatrix} \qquad U = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix} \qquad V^T = \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

Compression
k=1

$$A\_r = \begin{bmatrix} -0.53 \\ -0.18 \\ -0.42 \\ -0.71 \end{bmatrix} [15.39, \quad 0, \quad 0] \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

$$= \begin{bmatrix} 3.07 & 5.98 & 4.67 \\ 1.04 & 2.02 & 1.58 \\ 2.42 & 4.71 & 3.68 \\ 4.12 & 8.02 & 6.27 \end{bmatrix}$$

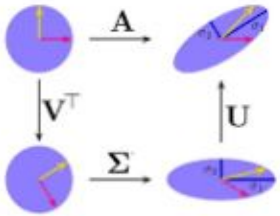
Error = 4.703

5 × 5 image

Grayscale Image

$$A = \begin{matrix} 255, & 255, & 2, & 2, & 2 \\ 255, & 255, & 3, & 5, & 10 \\ 4, & 4, & 255, & 253, & 255 \\ 3, & 2, & 255, & 255, & 255 \\ 1, & 3, & 255, & 255, & 254 \end{matrix}$$

Grayscale Image



$\mathbf{A} = \mathbf{U\Sigma V}^{\top}$

Singular Value Decomposition (SVD)

U - Matrix

-0.01,	**0.70**,	0.67,	-0.19,	0.02
-0.02,	**0.70**,	-0.67,	0.19,	-0.03
-0.57,	-0.01,	-0.16,	-0.48,	0.63
-0.57,	-0.02,	-0.04,	-0.29,	-0.75
-0.57,	-0.02,	0.21,	0.77,	0.12

```python
import numpy as np

image = [[255, 255,   2,   2,   2],
         [255, 255,   3,   5,  10],
         [4,    4, 255, 253, 255],
         [3,    2, 255, 255, 255],
         [1,    3, 255, 255, 254]]

U, S, V_T = np.linalg.svd(image)
print(U)
```

```
[[-0.01951968  0.70687216  0.67998675 -0.19152823  0.02976065]
 [-0.02857923  0.70651131 -0.67862357  0.19592488 -0.03329408]
 [-0.57631894 -0.01635574 -0.16367682 -0.48361327  0.63790052]
 [-0.57773825 -0.02059233 -0.04545047 -0.29639971 -0.7588649 ]
 [-0.57695512 -0.02195382  0.21961838  0.77665681  0.12334014]]
```