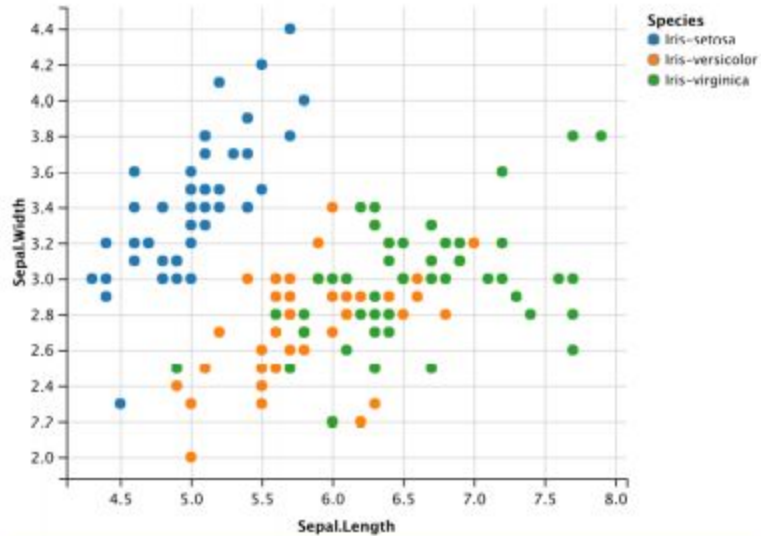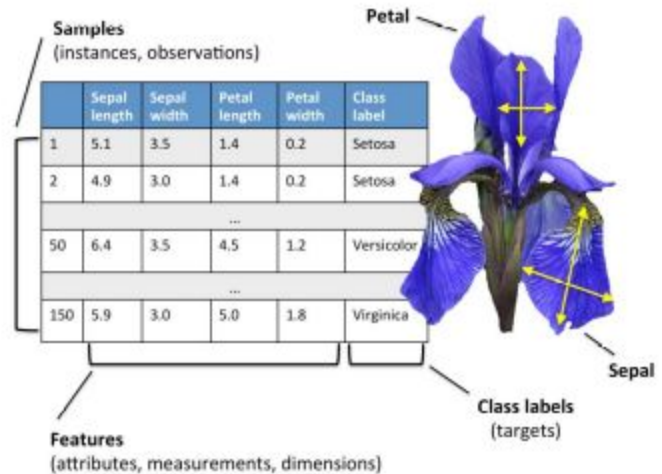# What happen for **all dimensions**

An example of a dataset (a point can be considered a vector through the origin).

Iris flower data set

Any vector can be expressed in terms of:
1. Projection directions unit vectors (v_1, v_2, …).
2. The lengths of projections onto them (s_{a1}, s_{a2}, …).

Extend this conclusion for handling a bunch of vectors

Extend this conclusion for handling a matrix (list of vectors)

Singular value decomposition
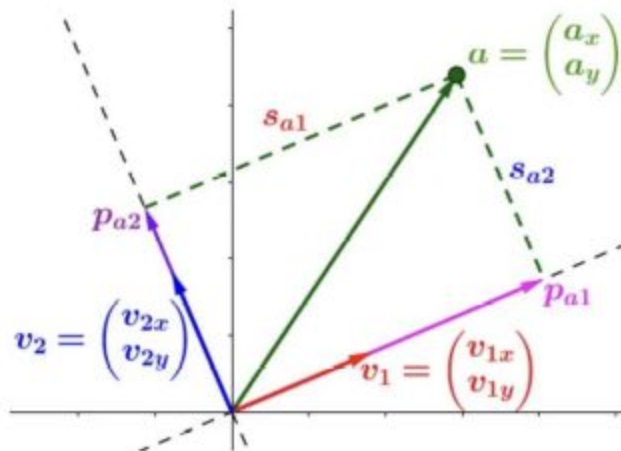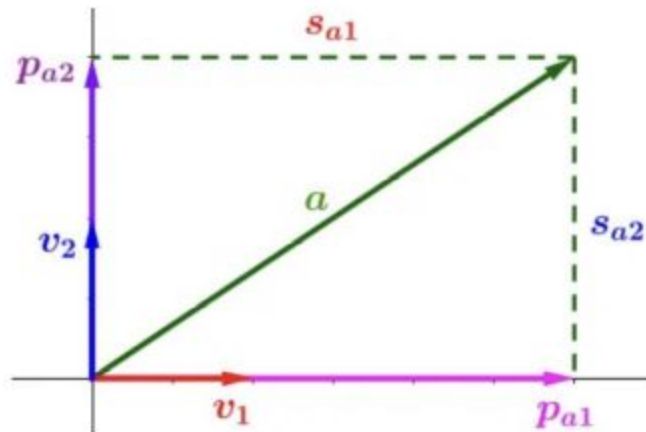
$$A = U D V^T$$

SVD finds a way to express the operation of vector decomposition using matrices.
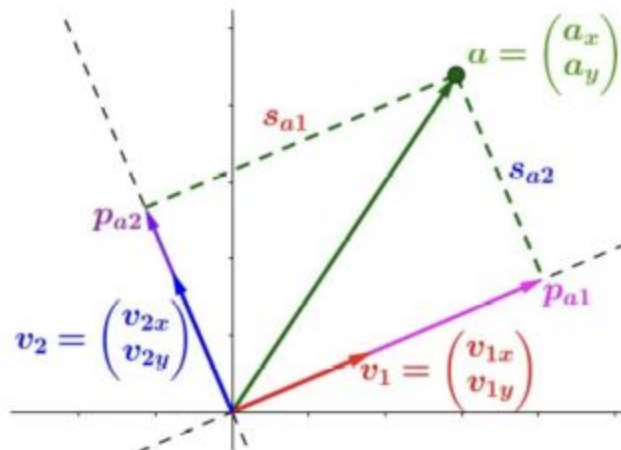
singular vectors

Singular values

singular vectors

Recall Vector Projection

Decompose (project) the vector $\boldsymbol{a}$ along unit vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

Recall Vector Projection

s_{a_1} and s_{a_2}: the lengths of projection
(Vector form)

$$a^T \cdot v_1 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = s_{a1}$$

$$a^T \cdot v_2 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = s_{a2}$$

Recall Vector Projection
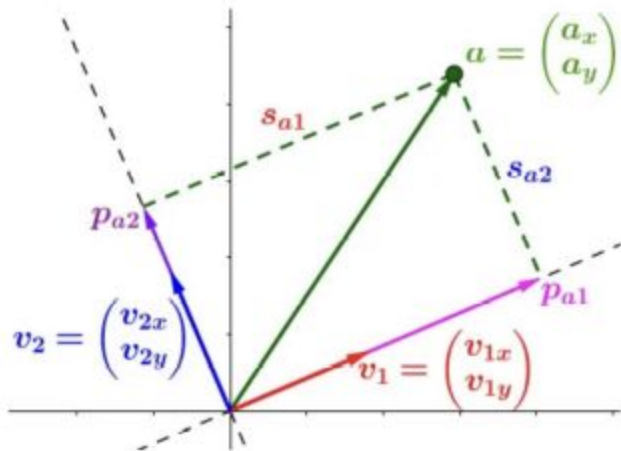
s_{a_1} and s_{a_2}: the lengths of projection
(Matrix form)

$$a^T \cdot v_1 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = s_{a1}$$

$$a^T \cdot v_2 = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = s_{a2}$$

$$a^T \cdot V = \begin{pmatrix} a_x & a_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} \end{pmatrix}$$
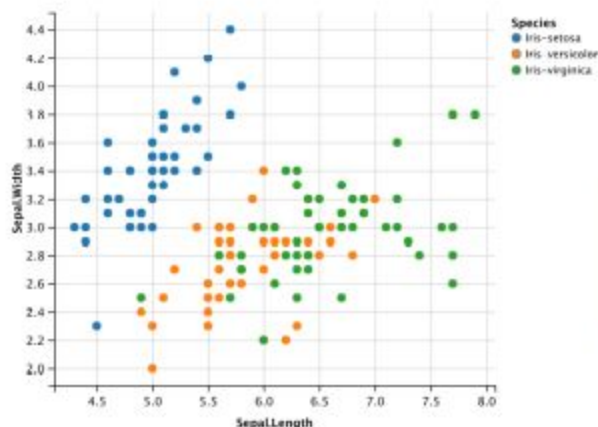
Recall Vector Projection

s_{a_1}, s_{a_2}: the lengths of projection of vector a

s_{b1}, s_{b2} : the lengths of projection of vector b
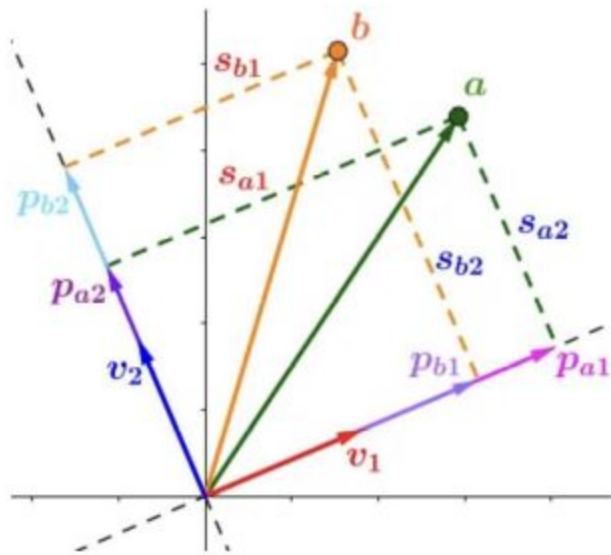
$$A \cdot V = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix} = S$$

Recall Vector Projection

n = no. of points, d = no. of dimensions, A = matrix containing points, V = matrix containing the decomposition axes, S = matrix containing lengths of projection.

$$A \cdot V = \begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} & \cdots \\ s_{b1} & s_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = S$$

$$n \times d \qquad\qquad d \times d \qquad\qquad n \times d$$

Generalize to any number of points and dimensions

Recall Vector Projection

$$A \cdot V = \begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} & \cdots \\ s_{b1} & s_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = S$$

$$n \times d \qquad\qquad d \times d \qquad\qquad n \times d$$

$$A \quad \cdot \quad V \quad = \quad S$$

Matrix of points

The dot product performs the projection

Matrix of decomposition axes

Matrix of the lengths of projections

V is orthogonal matrices

$$A = S\, V^{-1} = S\, V^T$$

Recall Vector Projection

$$A = S\,V^{-1} = S\,V^{T}$$

Any set of vectors (A) can be expressed in terms of their lengths of projections (S) on some set of orthogonal axes (V).
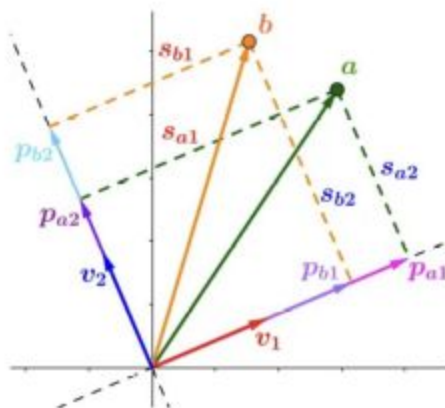
Orthogonal Vectors

$$V = \begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$d \times d$$

V is orthogonal matrices

$$A = S\,V^{-1} = S\,V^T \qquad \neq \qquad A = U\,\Sigma\,V^T$$
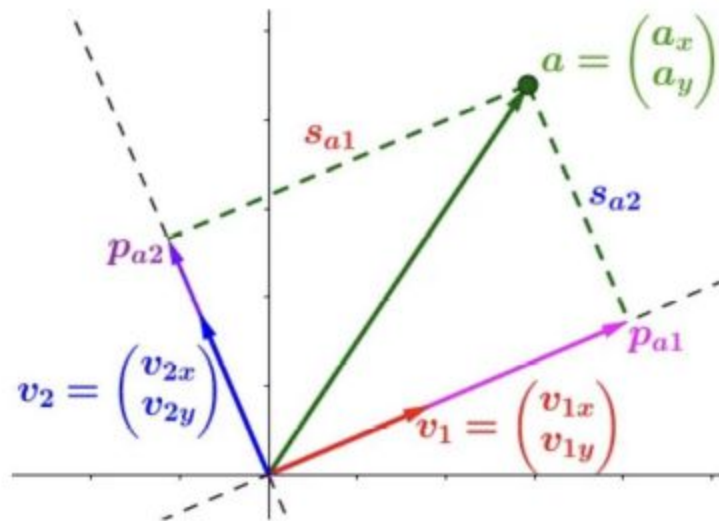
$$S = U\,\Sigma$$

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

A column vector containing the lengths of projections of each point on the 1st axis *v1*

A column vector containing the lengths of projections of each point on the 2nd axis *v2*

normalize these column vectors to make them of **unit length**

dividing each column vector by its magnitude, but in matrix form

$$A \cdot V = \begin{pmatrix} a_x & a_y & \cdots \\ b_x & b_y & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} v_{1x} & v_{2x} & \cdots \\ v_{1y} & v_{2y} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} s_{a1} & s_{a2} & \cdots \\ s_{b1} & s_{b2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = S$$

$$n \times d \qquad\qquad d \times d \qquad\qquad n \times d$$

$$S = \begin{pmatrix} \boxed{s_{a1}} & \boxed{s_{a2}} \\ \boxed{s_{b1}} & \boxed{s_{b2}} \end{pmatrix}$$

A column vector containing the lengths of projections of each point on the 1st axis *v1*

A column vector containing the lengths of projections of each point on the 2nd axis *v2*
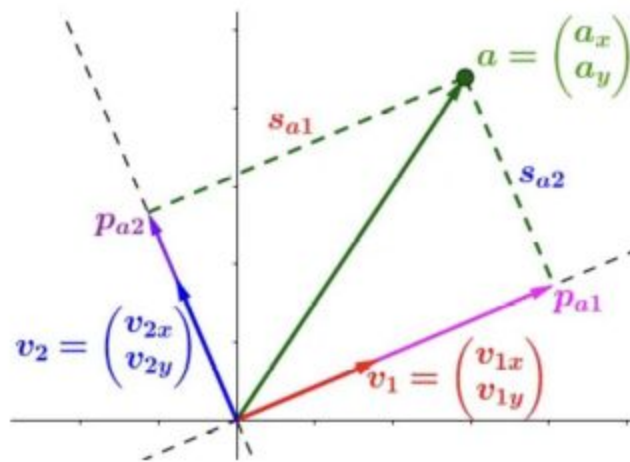
$$M = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \longrightarrow M = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$$

Normalize the columns of S

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

Magnitude of 1st column $= \sigma_1 = \sqrt{(s_{a1})^2 + (s_{b1})^2}$

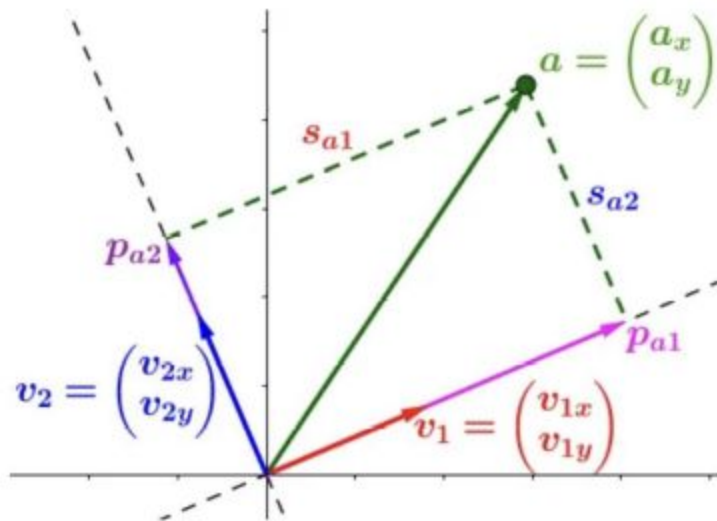Magnitude of 2nd column $= \sigma_2 = \sqrt{(s_{a2})^2 + (s_{b2})^2}$

(σ_i) is the **square root of the sum of squared projection lengths,** of all points, onto the *i*th unit vector v_i

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

A column vector containing the lengths of projections of each point on the 1st axis *v1*

A column vector containing the lengths of projections of each point on the 2nd axis *v2*

$$S = \begin{pmatrix} \frac{s_{a1}}{\sigma_1} & \frac{s_{a2}}{\sigma_2} \\ \frac{s_{b1}}{\sigma_1} & \frac{s_{b2}}{\sigma_2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$U \qquad \Sigma$$

Normalize the columns of S

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix}$$

Magnitude of 1st column $= \sigma_1 = \sqrt{(s_{a1})^2 + (s_{b1})^2}$

Magnitude of 2nd column $= \sigma_2 = \sqrt{(s_{a2})^2 + (s_{b2})^2}$

S - Explanation

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix} = \begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = U\,\Sigma$$

Lengths of projections on $v1$

Lengths of projections on $v2$

Lengths of projections on $v1$, but divided by $\sigma1$ to become a unit vector

Lengths of projections on $v2$, but divided by $\sigma2$ to become a unit vector

WHAT IS IT?

What about the sigmas? Why did we need to normalie S to find them?

σ - Explanation

$$S = \begin{pmatrix} s_{a1} & s_{a2} \\ s_{b1} & s_{b2} \end{pmatrix} = \begin{pmatrix} u_{a1} & u_{a2} \\ u_{b1} & u_{b2} \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = U \Sigma$$

Lengths of projections on v1

Lengths of projections on v2

Lengths of projections on v1, but divided by σ1 to become a unit vector

Lengths of projections on v2, but divided by σ2 to become a unit vector

WHAT IS IT?

(σ_i) is the **square root of the sum of squared projection lengths,** of all points, onto the *i*th unit vector v_i

$\sigma_1 = 1.99$

$\sigma_2 = 0.48$

if $\sigma_1 > \sigma_2$, then most points are closer to v_1 than v_2, and vice versa.
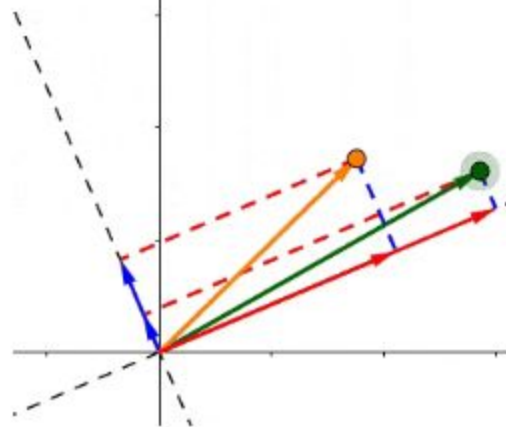
Singular Value Decomposition

❖ Formula

$$M = U\,\Sigma\,V^T$$

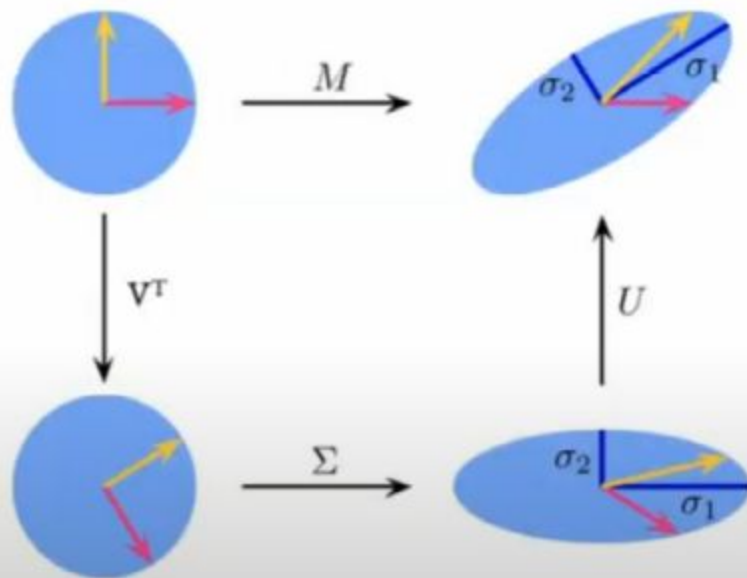Orthogonal Matrix — Diagonal Matrix — Orthogonal Matrix

U columns contain eigenvectors of matrix MM^t

Σ is a diagonal matrix containing singular eigenvalues

V columns contain eigenvectors of matrix M^tM

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Singular Value Decomposition

❖ **Let's observe!**

(1)

$$AA^T = \begin{pmatrix} 1 & 2 \\ 2 & 13 \end{pmatrix}$$

$$\lambda_1 = 13.32 \qquad \lambda_2 = 0.675$$

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} -0.16 \\ -0.98 \end{pmatrix} \qquad u_2 = \begin{pmatrix} -0.98 \\ 0.16 \end{pmatrix}$$

(2)

$$A^T A = \begin{pmatrix} 5 & 6 \\ 6 & 9 \end{pmatrix}$$

$$\lambda_1 = 13.32 \qquad \lambda_2 = 0.675$$

$$v_1 = \begin{pmatrix} -0.58 \\ -0.81 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -0.81 \\ 0.58 \end{pmatrix}$$

$$\lambda_i^{(1)} = \lambda_i^{(2)}$$

$$AA^T u_i = \lambda_i u_i \qquad\qquad A^T A v_i = \lambda_i v_i$$

singular value $\sigma = \sqrt{\lambda}$

$$\sigma_1 = \sqrt{\lambda_1} = 3.65$$

$$\sigma_2 = \sqrt{\lambda_2} = 0.82$$

$$A v_i = \sigma_i u_i$$

$$A = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -0.16 & -0.98 \\ -0.98 & 0.16 \end{pmatrix} \begin{pmatrix} 3.65 & 0 \\ 0 & 0.82 \end{pmatrix} \begin{pmatrix} -0.58 & -0.81 \\ -0.81 & 0.58 \end{pmatrix}$$

$$A = U \Sigma V$$

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Singular Value Decomposition

Revisit

$$A = U \Sigma V^T$$

$$\underbrace{A}_{m \times n} = \underbrace{U}_{m \times m} \; \underbrace{\Sigma}_{m \times n} \; \underbrace{V^T}_{n \times n}$$
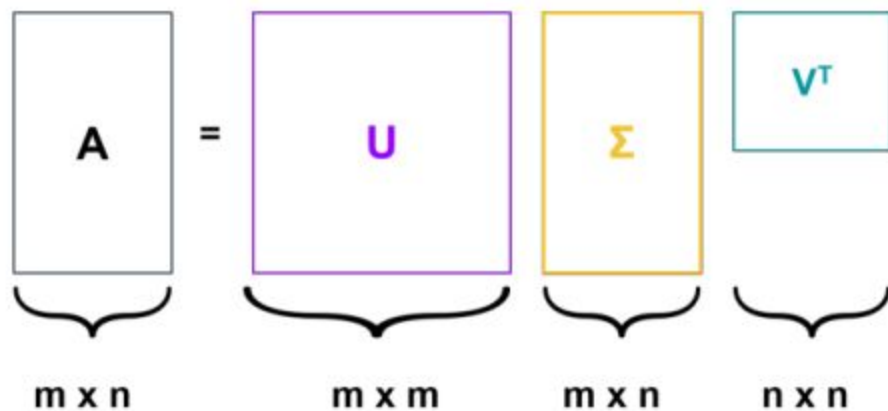
Singular Value Decomposition

Ü Revisit

A_r = U' Σ' V^T

$m \times n$ $\quad$ $m \times$ **k** $\quad$ **k** $\times n$ $\quad$ $n \times n$

Singular Value Decomposition

Example

$$A = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 3 & 1 \\ 4 & 6 & 1 \\ 5 & 7 & 7 \end{bmatrix}$$

$$A = U \quad \Sigma \quad V^T$$

m x n   m x m   m x n   n x n

$$U = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix}$$
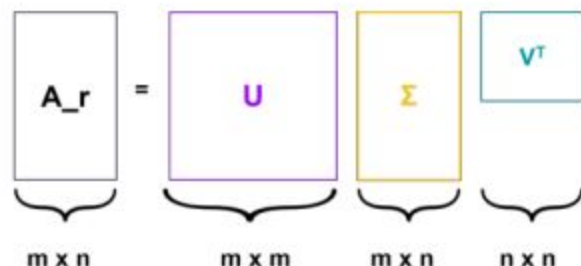
$$\Sigma = \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

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Singular Value Decomposition

U Example: Reconstruction

$$A = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 3 & 1 \\ 4 & 6 & 1 \\ 5 & 7 & 7 \end{bmatrix}$$

$$U = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$
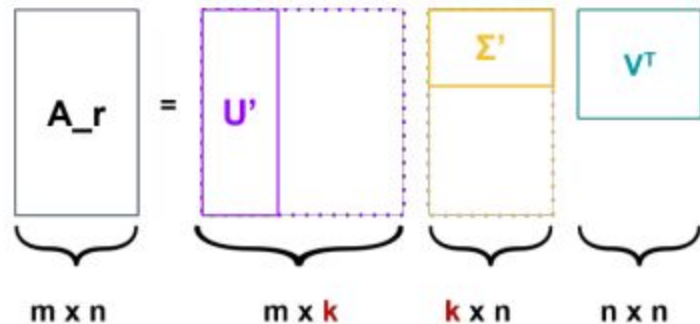
Compression
k=2

$$A\_r = \begin{bmatrix} -0.53, & -0.55 \\ -0.18, & 0.03 \\ -0.42, & 0.83 \\ -0.71, & -0.08 \end{bmatrix} \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \end{bmatrix} \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

$$= \begin{bmatrix} 1.78 & 5.33 & 6.34 \\ 1.1 & 2.05 & 1.49 \\ 4.37 & 5.68 & 1.16 \\ 3.93 & 7.92 & 6.53 \end{bmatrix}$$

Error = 2.45

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Singular Value Decomposition

U Example: Reconstruction

$$A_r = U' \quad \Sigma' \quad V^T$$

m x n   m x **k**   **k** x n   n x n

$$A = \begin{bmatrix} 1 & 6 & 6 \\ 0 & 3 & 1 \\ 4 & 6 & 1 \\ 5 & 7 & 7 \end{bmatrix}$$

$$U = \begin{bmatrix} -0.53, & -0.55, & -0.44, & -0.47 \\ -0.18, & 0.03, & -0.63, & 0.76 \\ -0.42, & 0.83, & -0.21, & -0.3 \\ -0.71, & -0.08, & 0.61, & 0.34 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 15.39, & 0, & 0 \\ 0, & 4.01, & 0 \\ 0, & 0, & 2.45 \\ 0, & 0, & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$
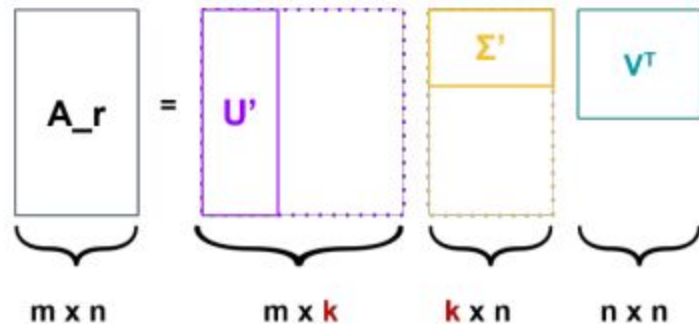
Compression k=1

$$A_r = \begin{bmatrix} -0.53 \\ -0.18 \\ -0.42 \\ -0.71 \end{bmatrix} [15.39, \quad 0, \quad 0] \begin{bmatrix} -0.38, & -0.73, & -0.57 \\ 0.58, & 0.29, & -0.76 \\ 0.72, & -0.62, & 0.32 \end{bmatrix}$$

$$= \begin{bmatrix} 3.07 & 5.98 & 4.67 \\ 1.04 & 2.02 & 1.58 \\ 2.42 & 4.71 & 3.68 \\ 4.12 & 8.02 & 6.27 \end{bmatrix}$$
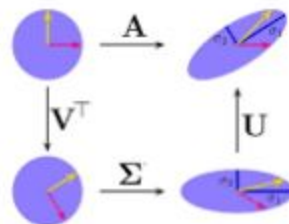
Error = 4.703

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A =
```
255, 255,   2,   2,   2
255, 255,   3,   5,  10
  4,   4, 255, 253, 255
  3,   2, 255, 255, 255
  1,   3, 255, 255, 254
```

Grayscale Image

5 × 5 image
Grayscale Image

A = UΣVᵀ

Singular Value Decomposition (SVD)

U - Matrix

```
-0.01,  0.70,  0.67, -0.19,  0.02

-0.02,  0.70, -0.67,  0.19, -0.03

-0.57, -0.01, -0.16, -0.48,  0.63

-0.57, -0.02, -0.04, -0.29, -0.75

-0.57, -0.02,  0.21,  0.77,  0.12
```

```python
import numpy as np

image = [[255, 255,   2,   2,   2],
         [255, 255,   3,   5,  10],
         [4,    4, 255, 253, 255],
         [3,    2, 255, 255, 255],
         [1,    3, 255, 255, 254]]

U, S, V_T = np.linalg.svd(image)
print(U)

[[-0.01951968  0.70687216  0.67998675 -0.19152823  0.02976065]
 [-0.02857923  0.70651131 -0.67862357  0.19592488 -0.03329408]
 [-0.57631894 -0.01635574 -0.16367682 -0.48361327  0.63790052]
 [-0.57773825 -0.02059233 -0.04545047 -0.29639971 -0.7588649 ]
 [-0.57695512 -0.02195382  0.21961838  0.77665681  0.12334014]]
```