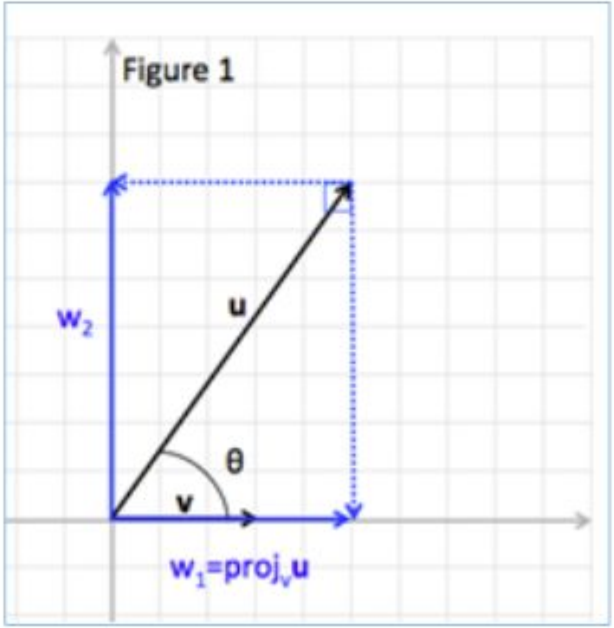
Decomposing a Vector Into Two Orthogonal Vectors



Support u and v are vectors

Vector **u** decomposed into orthogonal components w_1 and w_2.

Want to decompose **u** as: $u = w_1 + w_2$

w_1 is parallel to vector **v** and w_1 is perpendicular/orthogonal to w_2

The vector component w_1 is also called the projection of vector **u** onto vector **v**:

- $w_1 = \text{proj}_v \mathbf{u}$.
- $w_2 = u - w_1$

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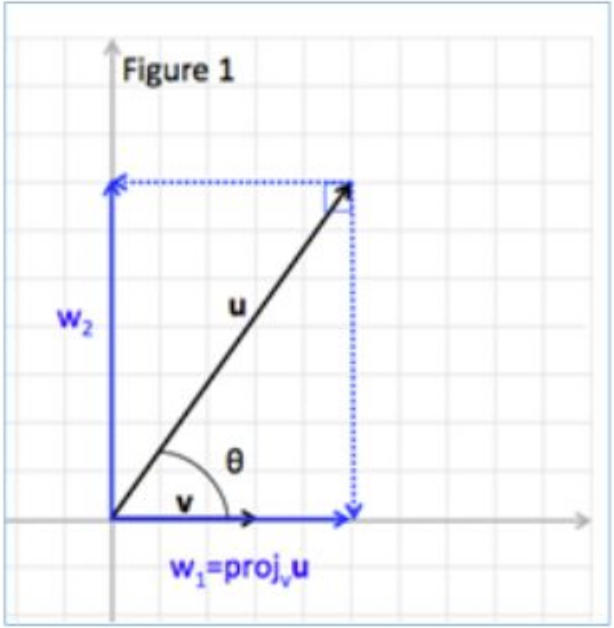


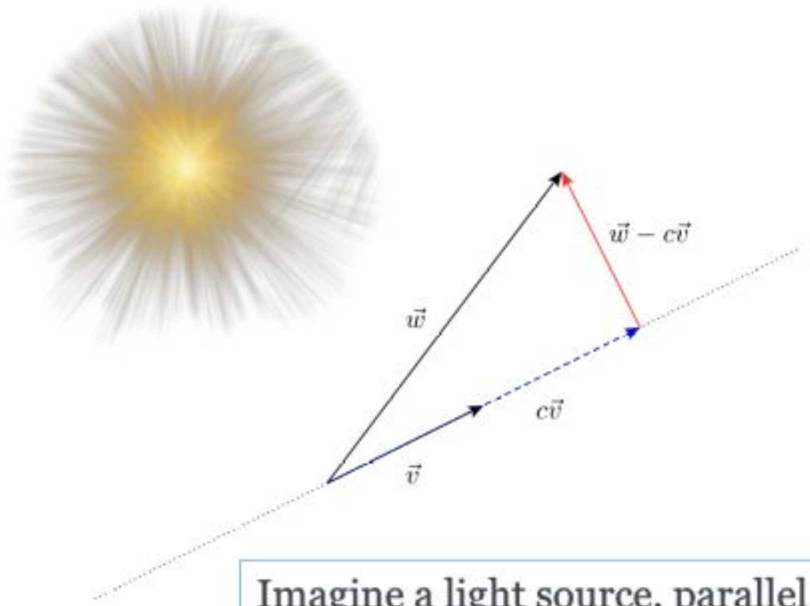
The proj$_v$**u** can be calculated as follows:

$$proj_v\ u = \left[\frac{u \cdot v}{v^2}\right] v$$

New Terminology

Vector Projection

Vector Projection

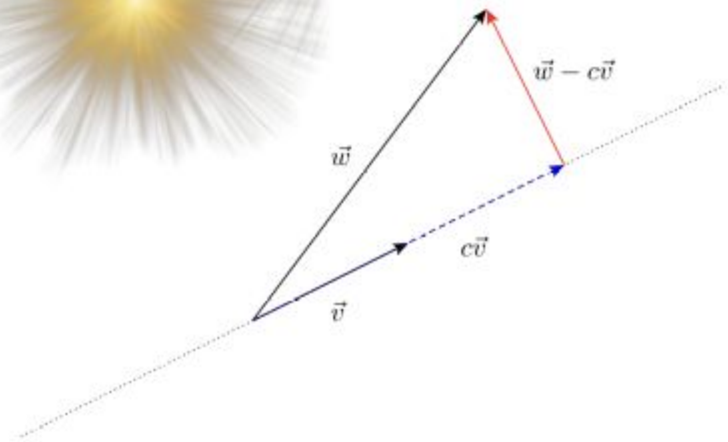


Why is it called projection?

Imagine a light source, parallel to $\vec{v}$, above $\vec{w}$. The light would cast rays perpendicular to $\vec{v}$.

$\text{proj}_{\vec{v}}\,\vec{w}$ is the shadow cast by $\vec{w}$ on the line defined by $\vec{v}$.

Vector Projection



The vector connecting $\vec{w}$ and $c\vec{v}$ is $\vec{w} - c\vec{v}$.

We want to find c such that $\vec{w} - c\vec{v}$ is perpendicular to $\vec{v}$.

Two perpendicular vectors have vector dot product of zero, so:

$$(\vec{w} - c\vec{v}) \cdot \vec{v} = 0$$

By distribution over addition of dot products:

$$\begin{aligned} (\vec{w} - c\vec{v}) \cdot \vec{v} = 0 &\implies \\ \vec{w} \cdot \vec{v} - c\vec{v} \cdot \vec{v} = 0 &\implies \\ \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} &= c \end{aligned}$$

Because $\|\vec{v}\| = \sqrt{(\vec{v} \cdot \vec{v})}$:

$$c = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}$$

So:

$$\operatorname{proj}_{\vec{v}} \vec{w} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

We can also write the projection in terms of the unit vector defined by $\vec{v}$:

$$\begin{aligned} \hat{u} \triangleq \frac{\vec{v}}{\|\vec{v}\|} &\implies \\ \operatorname{proj}_{\vec{v}} \vec{w} &= \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|} \vec{u} \end{aligned}$$

Decomposing a Vector Into Two Orthogonal Vectors

Example 1: Let $u = \langle -2, 2\rangle$ and $v = \langle 3, 5\rangle$. Write vector u as the sum of two orthogonal vectors one of which is a projection of u onto v.

Step 1: Find the proj_v u.

$$proj_v u = \left[\frac{u \cdot v}{\| v \|^2}\right] v = w_1$$

$$proj_v u = \left[\frac{u \cdot v}{\| v \|^2}\right] v$$

$$proj_v u = \left[\frac{(-2 \cdot 3) + (2 \cdot 5)}{\sqrt{3^2 + 5^2}^{\,2}}\right] \langle 3, 5\rangle$$

$$proj_v u = \left[\frac{-6 + 10}{\sqrt{34}^{\,2}}\right] \langle 3, 5\rangle$$

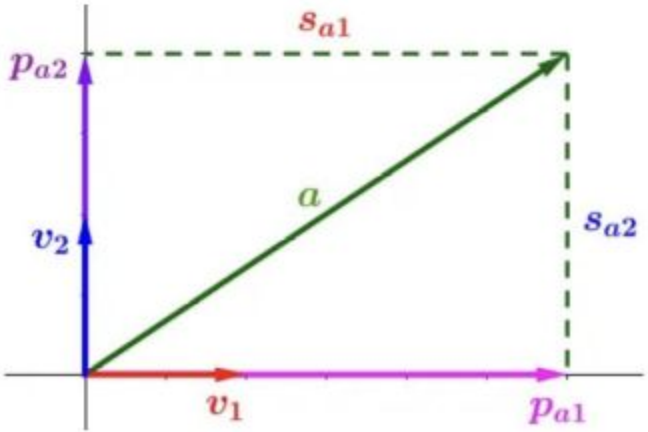
$$proj_v u = \left[\frac{4}{34}\right] \langle 3, 5\rangle = \left[\frac{2}{17}\right] \langle 3, 5\rangle$$

$$proj_v u = \langle \frac{6}{17}, \frac{10}{17}\rangle$$

Decomposing a Vector Into Two Orthogonal Vectors

Step 2: Find the orthogonal component. $\mathbf{w_2 = u - w_1}$	$w_2 = u - w_1$ $w_2 = \langle -2, 2\rangle - \langle \frac{6}{17}, \frac{10}{17}\rangle$ $w_2 = \langle \left(-2 - \frac{6}{17}\right), \left(2 - \frac{10}{17}\right)\rangle$ $w_2 = \langle -\frac{40}{17}, \frac{24}{17}\rangle$
Step 3: Write the vector as the sum of two orthogonal vectors. $\mathbf{u = w_1 + w_2}$	$u = w_1 + w_2$ $u = \langle \frac{6}{17}, \frac{10}{17}\rangle + \langle -\frac{40}{17}, \frac{24}{17}\rangle$

Supposing that, vector ($\boldsymbol{a}$) is decomposed, we get 3 pieces of information:



1. The **directions** of projection — the **unit** vectors (v_1 and v_2) **representing the directions** onto which we project (decompose). In the above they're the x and y axes, but can be any other orthogonal axes.
2. The **lengths** of projection (the **line segments** s_{a1} and s_{a2}) — which tell us how much of the vector is **contained** in each direction of projection (more of vector $\boldsymbol{a}$ is leaning on the direction v_1 than it is on v_2, hence s_{a1}>s_{a2}).
3. The **vectors** of projection (p_{a1} and p_{a2})—which are used to **reconstruct** the original vector $\boldsymbol{a}$ by adding them together (as a vector sum), and for which it's easy to verify that $p_{a1}=s_{a1}*v_1$ and $p_{a2}=s_{a2}*v_2$—**So they're redundant, as they can be deduced from the former 2 pieces.**

Image credits: https://towardsdatascience.com/svd-8c2f72e264f

A unit vector is a vector with magnitude 1

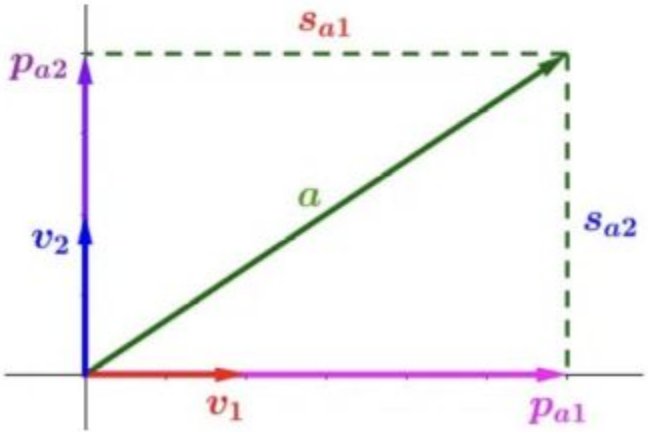
To find a unit vector , u, in the same direction of a vector, v, we divide the vector by its magnitude

$$\boxed{\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{||\vec{v}||}\vec{v}}$$

For a vector $\vec{v}$ = <a,b> its magnitude is given by

$$\boxed{||\vec{v}|| = \sqrt{a^2 + b^2}}$$

Vector Projection



Any vector can be expressed in terms of:
1. Projection directions unit vectors (v_1, v_2, …).
2. The lengths of projections onto them (s_{a1}, s_{a2}, …).