

Solution to Homework 3

Date: March 17, 2023

Scribe: Jianliang He

Warning: This note is only used as a reference solution for the homework, and the solution to each question is not unique. The solution may contain factual and/or typographic errors and comments and criticism are kindly welcomed.

Problem 1 Let $X_1, \dots, X_n \stackrel{iid}{\sim} N(0, 1) = F_X$ and $Y_1, \dots, Y_n \stackrel{iid}{\sim} t_{10} = F_Y$, where t_{10} is the t -distribution with 10 degrees of freedom. Use simulation to answer the following: 1) Suppose $n = 1000$, does the Kolmogorov-Smirnov Two-Sample Test rejects the hypothesis at $\alpha = 0.05$? 2) On average, how large does n has to be for the null hypothesis to be rejected? 3) Suppose we change the degree of freedom from 10 to d . How does n in 2) change with d ? Plot a graph to illustrate their relationship.

Solution:

```

1  # nrep: the number of samples for simulation
2  ks2.stats <- function(x,y) {
3    Fn_x.hat <- ecdf(x)
4    Fn_y.hat <- ecdf(y)
5    ks.stats <- max(max(abs(Fn_x.hat(x) - Fn_y.hat(x))),
6                    max(abs(Fn_x.hat(y) - Fn_y.hat(y))))
7    return(ks.stats)
8  }
9  ks2.pvalue<-function(x,y,nrep) {
10   # Calculate the Two-Sample KS-statistics
11   n <- length(x)

```

```
12  ks.stats <- ks2.stats(x,y)
13  # Simulation under the null
14  ks.sim <- c()
15  for (i in 1:nrep) {
16    ks.sim[i] <- ks2.stats(rnorm(n,0,1), rnorm(n,0,1))
17  }
18  return(mean(ks.sim > ks.stats))
19 }
20
21 ### Problem 1
22 # Calculate the p-value of the statistics under the problem setting
23 set.seed(217)
24 data.norm = rnorm(1000); data.t10 = rt(1000, 10); nrep = 5000
25 ks2.pvalue(data.norm, data.t10,nrep)
26
27 ### Problem 2
28 # The understanding of "average" is tricky
29 # Here we take it as "average p-value"
30 nrep <- 100; nsim <- 50; nseq <- seq(4000,12000,500)
31 ks.psim <- c()
32 for (j in 1:length(nseq)){
33   n <- nseq[j] # Sample Size
34   ks.temp <- 0
35   for (i in 1:nsim){
36     data.norm = rnorm(n); data.t10 = rt(n, 10);
37     ks.temp <- ks.temp + ks2.pvalue(data.norm, data.t10,nrep)
38   }
39   ks.psim[j] <- ks.temp/nsim
40 }
41 plot(x = nseq, y = ks.psim, ylab = 'Average_p-value', xlab = 'Sample_Size')
```

```

42 abline(h=0.05,col="red")
43
44 ### Problem 3
45 nrep <- 100; nsim <- 50;
46 nseq <- seq(500,10000,500)
47 tseq <- seq(1,9,1)
48 t.n <- c()
49 for (k in 1:length(tseq)){
50   t <- tseq[k]
51   for (j in 1:length(nseq)){
52     n <- nseq[j] # Sample Size
53     ks.temp <- 0
54     for (i in 1:nsim){
55       data.norm = rnorm(n); data.t10 = rt(n,t);
56       ks.temp <- ks.temp+ks2.pvalue(data.norm, data.t10,nrep)
57     }
58     if (ks.temp/nsim<0.05){
59       t.n[k] <- n; break
60     }
61   }
62 }
63 plot(x = tseq, y =t.n, ylab = 'Sample_Size', xlab = 'Degree_of_freedom')

```

In Problem 1.1, the p -value is 0.254 and we fail to reject the null hypothesis using Kolmogorov-Smirnov Two-Sample Test; In Problem 1.2, under the significance level $\alpha = 0.05$, the least sample size to reject the null is around 10000; The result of Problem 1.3 is shown in the Figure 3.1 (b). \square

Remark In Problem 1.1, I only choose one specific data setting and if you want the overall performance of testing on such two distributions, please repeat such procedure and get an average result. Besides, there exists various understanding on the "average" in Problem 1.2 and in the

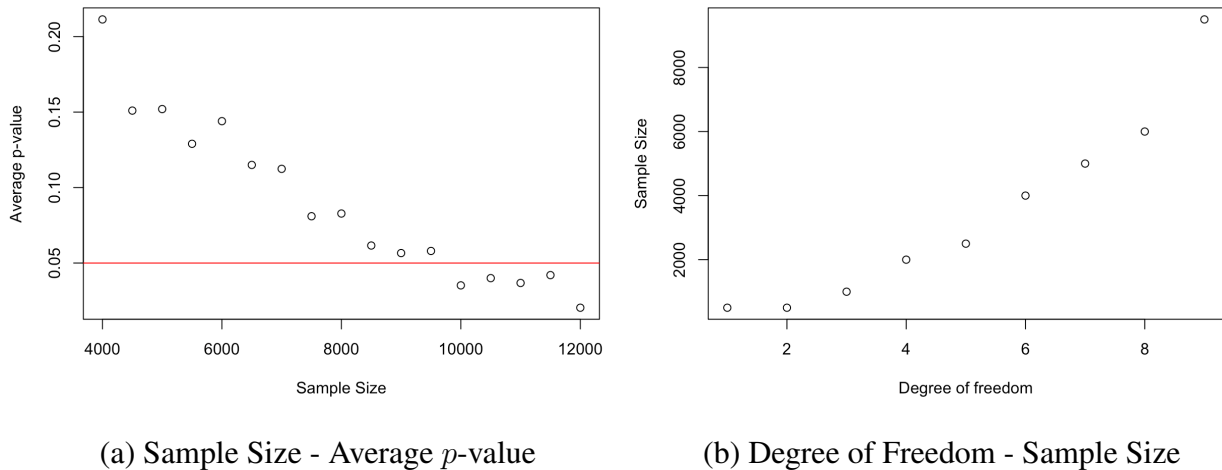


Figure 3.1: Reference to Problem 1.

solution above "average p -value" is taken for the simplicity in the simulation. Any reasonable understanding including average K-S statistics etc. is also allowed and the solution is not unique.

Problem 2 1) Generate samples ($n = 30$) according to the following model: $X_1, \dots, X_n \stackrel{iid}{\sim} N(0, 1) = F_X$ and $Y_1, \dots, Y_n \stackrel{iid}{\sim} N(0, 2^2) = F_Y$, Does the Mann-Whitney U test reject the null hypothesis $H_0 : F_X = F_Y$ at $\alpha = 0.1$? 2) Generate samples ($n = 30$) according to the following model: $X_1, \dots, X_n \stackrel{iid}{\sim} N(0, 1) = F_X$ and $Y_1, \dots, Y_n \stackrel{iid}{\sim} N(0.5, 1) = F_Y$, Does the Mann-Whitney U test reject the null hypothesis $H_0 : F_X = F_Y$ at $\alpha = 0.1$?

Solution:

```

1 mwu.stats <- function(x, y) {
2   data <- merge(x, y)
3   return(sum(data[, 1] > data[, 2]))
4 }
5
6 mwu.pvalue <- function(x, y, nrep) {
7   stats <- mwu.stats(x, y)
8   n <- length(x)
9   record <- c()

```

```

10   for (i in 1:nrep){
11     record[i] <- mwu.stats(rnorm(n), rnorm(n))
12   }
13   # Calculate the two-sided p-value
14   return(2*min(mean(record < stats), mean(record > stats)))
15 }
16
17 # Calculate the p-value of the statistics under the problem setting
18 # Problem 1
19 x <- rnorm(30); y <- rnorm(30, 0, 2); nrep <- 5000
20 mwu.pvalue(x, y, nrep)
21
22 # Problem 2
23 x <- rnorm(30); y <- rnorm(30, 0.5, 1); nrep <- 5000
24 mwu.pvalue(x, y, nrep)

```

In Problem 1, the p -value of Mann-Whitney U Test is $0.2732 > 0.1$ and we fail to reject the null;

In Problem 2, the p -value of Mann-Whitney U Test is $0.0048 < 0.1$ and thus we reject the null,

showing that the distribution of two sample is significantly different;. \square

Problem 3 Suppose you have the following sample 1, 2, 2.5, 3.3, 10, 15, 15.5, 17, 20. Let the population median be M . Does the Wilcoxon signed rank test reject $H_0 : M = 14$ at $\alpha = 0.1$?

Solution: (Solution is not unique and the following code is based on the number of run)

```

1 wilcox.stats <- function(x, med){
2   s <- sign(x-med)
3   s.abs <- abs(x-med)
4   return(sum(rank(s.abs)[s>0]) - sum(rank(s.abs)[s<0]))
5 }
6

```

```
7 wilcox.pvalue <- function(x, med, nrep){
8   stats <- wilcox.stats(x, med)
9   record <- c()
10  for (i in 1:nrep){
11    record[i] <- wilcox.stats(rnorm(n), 0)
12  }
13  # Calculate the two-sided p-value
14  return(2*min(mean(record < stats), mean(record > stats)))
15 }
16
17 # Calculate the p-value of the statistics under the problem setting
18 x <- c(1, 2, 2.5, 3.3, 10, 15, 15.5, 17, 20); med <- 14; nrep <- 50000
19 wilcox.pvalue(x, med, nrep)
```

The corresponding p -value is $0.164 > 0.1$ and we fail to reject the null.

□