MANA130083.01 Nonparametric *Spring 2023*

Solution to Homework 4

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***Warning**: This note is only used as a reference solution for the homework, and the solution to each question is not unique. The solution may contain factual and/or typographic errors and comments and criticism are kindly welcomed.*

Remark: In Problem 1 and 3, to compare ”which test is more powerful” you’d better calculate the power of test using simulation (which is omitted in solution and you can try it on your own). Tips: You can estimate the power by repeating the tests under specific data setting and take the average reject rate as an estimation.

Problem 1 *Let $X_1, \dots, X_n \overset{iid}{\sim} N(0, \sigma_X^2)$ and $Y_1, \dots, Y_n \overset{iid}{\sim} N(0, \sigma_Y^2)$. Use simulation to answer the following: 1) Suppose $n = 100$, $\sigma_X^2 = 1$ and $\sigma_Y^2 = 1.2$, use the Mood test and the Sukhatme test to test $H_0 : \sigma_X^2 = \sigma_Y^2$ at level $\alpha = 0.05$. 2) Can you think of a scenario where the Mood test is more powerful than the Sukhatme test? 3) Can you think of a scenario where the Sukhatme test is more powerful than the Mood test?*

Solution:

```
mood.stats <- function(X,Y){
  X <- sort(X); Y <- sort(Y)
  XY <- sort(c(X,Y))
  rank.X <- match(X, XY)
  return(sum((rank.X-length(XY)+1)/2)^2)
}
```

```r

moods.pvalue <- function(X,Y,nrep){
  # Calculate the Mood test statistics
  n <- length(X)
  stats <- mood.stats(X, Y)
  # Simulate under the null
  mood.sim <- c()
  for (i in 1:nrep) {
    mood.sim[i] <- mood.stats(rnorm(n,0,1), rnorm(n,0,1))
  }
  return(2*min(mean(mood.sim < stats), mean(mood.sim> stats)))
}

Sukhatme.stats <- function(X,Y){
  data.pos <- merge(X[which(X>0)],Y[which(Y>0)])
  data.neg <- merge(X[which(X<0)],Y[which(Y<0)])
  return(sum(data.pos[,1]<data.pos[,2])+sum(data.neg[,1]>data.neg[,2]))
}

Sukhatme.pvalue <- function(X,Y,nrep){
  # Calculate the Sukhatme test statistics
  n <- length(X)
  stats <- Sukhatme.stats(X, Y)
  # Simulate under the null
  Sukhatme.sim <- c()
  for (i in 1:nrep) {
    Sukhatme.sim[i] <- Sukhatme.stats(rnorm(n,0,1), rnorm(n,0,1))
  }
  return(return(2*min(mean(Sukhatme.sim < stats), mean(Sukhatme.sim> stats))))
}
```

```
# Calculate the p-value of the statistics under the problem setting
set.seed(217)
X = rnorm(100); Y = rnorm(100,0,1.2); nrep = 500
moods.pvalue(X,Y,nrep)
Sukhatme.pvalue(X,Y,nrep)

## Problem 1.2
# Case 1: False rejection when median is not 0
X = rnorm(100,1,1); Y = rnorm(100,1,1); nrep = 500
# Case 2: n is small
set.seed(217)
X = rnorm(10,0,1); Y = rnorm(10,0,5); nrep = 500
# Case 3: n is extremely large, Sukhatme brings larger computation cost
# and become less powerful than the Mood test

## Problem 1.3
# Case 1: Skewed distribution
set.seed(217)
X = rnorm(50,1,0.5); Y = rgamma(50,0.5,0.5); nrep = 500
```

Both of the test failed to reject the null under the setting of Problem 1.1 with respectively p-value of 0.208 and 0.596. Scenario of Problem 1.2 and 1.3 are listed in codes. $\square$

Problem 2 *Suppose X and Y follows the same continuous distribution and are independent. If the median of the distribution is 0, show that $\mathbb{P}(\{Y < X < 0\} \cup \{0 < X < Y\}) = \frac{1}{4}$.*

Proof: Using the probability of conditional probability, it follows

$$\mathbb{P}(Y < X < 0) = \mathbb{P}(Y < X | X < 0, Y < 0)\mathbb{P}(Y < 0, X < 0) = \frac{1}{4},$$

where the last equality is based on the independency and property of median. Similarly, we have $\mathbb{P}(Y > X > 0) = \frac{1}{4}$ and the result follows.

□

Problem 3 *Let $X_1, \ldots, X_n \overset{iid}{\sim} N(0,1) = F_X$ and $Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu,1) = F_Y$. Use simulation to answer the following: 1) Suppose $n = 100$ and $\mu = 0.1$, use the Wilcoxon Rank-Sum test and the Terry-Hoeffding test to test $H_0 : F_X = F_Y$ at level $\alpha = 0.05$. 2) Can you think of a scenario where the Wilcoxcon Rank-Sum test is more powerful than the Terry-Hoeffding test? 3) Can you think of a scenario where the Terry-Hoeffding test is more powerful than the the Wilcoxcon Rank-Sum test?*

Solution: (Solution is not unique and the following code is based on the number of run)

```
TH.stats <- function(X,Y){
  X <- sort(X); Y <- sort(Y)
  XY <- sort(c(X,Y))
  rank.X <- match(X, XY)
  weight <- normorder.expect(length(XY))
  return(sum(weight[rank.X]))
}

TH.pvalue <- function(X,Y,nrep){
  # Calculate the Terry-Hoeffding Rank Sum test statistics
  n <- length(X)
  stats <- TH.stats(X, Y)
  # Simulate under the null
  TH.sim <- c()
  for (i in 1:nrep) {
    TH.sim[i] <- TH.stats(rnorm(n,0,1), rnorm(n,0,1))
  }
  return(2*min(mean(TH.sim < stats), mean(TH.sim> stats)))
```

```
}

# Calculate the p-value of the statistics under the problem setting
set.seed(217)
X = rnorm(100); Y = rnorm(100,0.1,1); nrep = 500
wilcoxsum.pvalue(X,Y,nrep)
TH.pvalue(X,Y,nrep)

# Problem 3.2
# Case 1: Samplings from some non-normal distributions
X = runif(15,0.5,1); Y = runif(15,-1,2); nrep = 500


# Problem 3.3
# Case 1: Terry-Hoeffding Rank Sum test is theoretically optimal
# under the normal assumption
X = rnorm(50,0,2); Y = rnorm(50,0.1,2); nrep = 500
}
```

Both of the test failed to reject the null under the setting of Problem 1.1 with respectively p-value of 0.808 and 0.772.Scenario of Problem 3.2 and 3.3 are listed in codes and you can read [Hodges Jr and Lehmann, 1960] for more technical details. □

References

[Hodges Jr and Lehmann, 1960] Hodges Jr, J. and Lehmann, E. L. (1960). Comparison of the normal scores and wilcoxon tests. In *Proceedings 4th Berkeley Symposium*, volume 1, pages 307–317.