MATH130165h: Homework 3

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$\boxed{\text{Problem 1}}$

1. First, want to do block-wise Gaussian elimination:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}' \end{bmatrix}$$

we can do: $\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{bmatrix}$

$$\left(\begin{aligned} &\because A^T = A \\ &\therefore A_{11}^T = A_{11} \\ &\quad A_{21}^T = A_{21} \end{aligned} \right) \qquad = \begin{bmatrix} A_{11} & 0 \\ 0 & -A_{21}A_{11}^{-1}A_{12} + A_{22} \end{bmatrix} \begin{bmatrix} I & -A_{11}^{-1}A_{21} \\ 0 & I \end{bmatrix}$$

Thus, we've got the $L = \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$ $\boxtimes$

2. m-step Cholesky factorization:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} L_{11}^T & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix} = \begin{bmatrix} L_{11}L_{11}^T & L_{11}L_{21}^T \\ L_{21}L_{11}^T & L_{21}L_{21}^T + L_{22}L_{22}^T \end{bmatrix}$$

$$\begin{cases} L_{11}L_{11}^T = A_{11} \\ L_{21} = A_{21}\left(L_{11}^{-1}\right)^T \\ L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T \end{cases} \Rightarrow A_{22} - L_{21}L_{21}^T = \underbrace{A_{22} - A_{21}A_{11}^{-1}A_{12}}_{\text{Schur Complement}}$$

$\boxtimes$

Problem 3

A is symmetric, thus it has orthogonormal eigenvectors $[q_1 \cdots q_n]$

Therefore, suppose $x = \sum b_i q_i$ $\quad \rho(x, A) = \frac{x^T A x}{x^T x}$

$$Ax - \rho(x, A)x = r \Rightarrow A \cdot \sum b_i q_i - \rho(x, A) \sum b_i q_i = r$$

$$b_i A q_i = b_i \alpha_i q_i \quad \Downarrow$$

$$\sum \alpha_i b_i q_i - \rho(x, A) \sum b_i q_i = r$$

$\Rightarrow r = \sum [\alpha_i - \rho(x, A)] b_i \cdot q_i$ (r decomposed w.r.t base $\{q_i\}$)

$\Rightarrow \|r\|_2^2 = \sum |\alpha_i - \rho(x, A)|^2 |b_i|^2$

Now, we abstract the problem as follows:

- Since $|\alpha_i - \rho(x, A)|$ represents the gap between eigen and RQ, I name $G_i \triangleq |\alpha_i - \rho(x, A)|$

  Suppose $G_n \geq G_{n-1} \geq \cdots \geq G_2 \geq G_1 \geq 0$

- Since $x = \sum b_i q_i$ and $\|x\|_2^2 = 1 = \|q_i\|_2^2 \Rightarrow \sum |b_i|^2$

- Now we want to show:

$$\frac{\|r\|_2^2}{G_2} \geq G_1, \text{ where } \|r_2\|_2^2 = \sum G_i^2 |b_i|^2. \quad \sum |b_i|^2 = 1$$

$$\frac{\sum G_i |b_i|^2}{G_2 G_1} = \frac{G_1^2 |b_1|^2 + \sum_{i \geq 2} G_i^2 |b_i|^2}{G_2 G_1} \geq \frac{G_1^2 |b_1|^2 + G_2^2 \sum_{i \geq 2} |b_i|^2}{G_2 G_1}$$

$$= \frac{G_1^2 |b_1|^2 + G_2^2 (1 - |b_1|^2)}{G_2 G_1} = \frac{G_1}{G_2} |b_1|^2 + \frac{G_2}{G_1} (1 - |b_1|^2)$$

$$\geq 2\sqrt{|b_1|^2 (1 - |b_1|)^2} = 1 \Rightarrow \frac{\|r\|_2^2}{G_2} \geq G_1$$

Since in our problem,

$$G_2 = \min_{\alpha_j \neq \alpha} |\alpha_j - \rho(X, A)|$$

$$G_1 = |\alpha - \rho(X, A)|$$

We have proven the theorem.

Problem 4

$\forall \lambda$ that is the eigenvalue of matrix A:

$\because$ all super- and sub- diagonal elements of A are non-zero

$\therefore \operatorname{rank}(A-\lambda I) = m-1$

$\Rightarrow \dim[\ker(A-\lambda I)] = 1$

$\because$ For hermitian matrix, the geometric multiplicity and algebraic multiplicity is equal for its eigenvalues,

$\therefore$ all λ's for A has algebraic multiplicity of 1.

$\therefore$ A has distinct eigenvalues.

$\blacksquare$