MATH130165h: Homework 3

Due Apr 28, 2024

Problem 1. [20 pt] Given an $(m+n) \times (m+n)$ symmetric positive definite matrix A of block form,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is of size $m \times m$ and A_{22} is of size $n \times n$.

1. Find a matrix L such that
$$LAL^\top = \begin{pmatrix} A_{11} & \\ & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}.$$
The bottom-right block is known as the *Schur complement* of A_{11} in A.

2. Show that the Schur complement is the same as m steps of Cholesky factorization algorithm.

Problem 2. [30 pt] Implement an LDLT factorization with pivoting. Verify the correctness of your implementation using a random matrix of size 10×10.

Problem 3. [30 pt] Let A be a symmetric matrix, x be a unit vector, $\rho(x, A) = \frac{x^\top Ax}{x^\top x}$ be the Rayleigh quotient. Let $r = Ax - \rho(x, A)x$ and α be the eigenvalue of A closest to $\rho(x, A)$. Prove,

$$|\alpha - \rho(x, A)| \leq \frac{\|r\|_2^2}{\min_{\alpha_i \neq \alpha} |\alpha_j - \rho(x, A)|},$$

where α_js are eigenvalues of A.

Problem 4. [20 pt] Let $A \in \mathbb{C}^{m\times m}$ be tridiagonal and hermitian, with all its sub- and super diagonal entries nonzero. Prove that the eigenvalues of A are distinct.

Problem 5. [30 pt] Implement a Householder reduction to Hessenberg form for Hermitian matrices. Your implementation should fully explore the Hermitian property. Verify the correctness of your implementation using a random matrix of size 10×10.