Homework Two

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Problem 1

1. First, we analyze the rounding error of dot product:

Suppose $x, y \in \mathbb{R}^n$ $\quad x^T y \in \mathbb{R}$ $\quad x^T y = \sum_{i=1}^{n} x_i y_i$

Let $g_k = fl(\sum_{i=1}^{k} x_i y_i)$

1.1 Show the relationship between g_k and g_{k-1}:

$$g_k = fl\left[\sum_{i=1}^{k} x_i y_i\right] = fl\left[g_{k-1} + fl(x_k y_k)\right]$$

$$= \left[g_{k-1} + x_k y_k (1+\delta_k)\right](\varepsilon_k + 1)$$

1.2 Get the formula of g_n with respect to δ_i and ε_i $(i=1,\cdots,n)$:

$$g_1 = fl(x_1 y_1) = x_1 y_1 (1+\delta_1) \quad , \varepsilon_1 = 0$$

$$g_2 = fl\left[g_1 + fl(x_2 y_2)\right] = (1+\varepsilon_2)\left[x_2 y_2 (1+\delta_2) + x_1 y_1 (1+\delta_1)\right]$$

$$g_3 = fl\left[g_2 + fl(x_3 y_3)\right] = (1+\varepsilon_3)\Big[(1+\varepsilon_2)(1+\delta_1)\, x_1 y_1 + (1+\varepsilon_2)(1+\delta_2)\, x_2 y_2 + (1+\delta_3)\, x_3 y_3\Big]$$

If $g_k = \sum_{i=1}^{k}\left[x_i y_i \cdot (1+\delta_i) \cdot \prod_{j=i}^{n} (1+\varepsilon_j)\right]$

$$g_{k+1} = fl\left[g_k + fl(x_{k+1} y_{k+1})\right] = (1+\varepsilon_{k+1})\left[g_k + x_{k+1} y_{k+1} (1+\delta_{k+1})\right]$$

$$= \sum_{i=1}^{k+1}\left[x_i y_i \cdot (1+\delta_i) \cdot \prod_{j=i}^{n} (1+\varepsilon_j)\right]$$

$$\therefore g_n = \sum_{i=1}^{n}\left[x_i y_i \cdot (1+\delta_i) \cdot \prod_{j=i}^{n} (1+\varepsilon_j)\right] = fl(x^T y)$$

1.3 Bound dot product error :

Notice that $|\delta_i| \le u \quad |\varepsilon_j| \le u \quad \forall i, j$

For i from 1 to n :

$$(1+\delta_i)\cdot\prod_{j=i}^{n}(1+\varepsilon_j) \le (1+u)^n \quad \left(\begin{array}{c}\text{Not } (1+u)^{n+1} \\ \text{because } \varepsilon_1 = 0\end{array}\right)$$

$$\le 1 + nu + O(u^2)$$

$$\Rightarrow \left|fl(x^T y) - x^T y\right| = \left|g_n - x^T y\right| = \left|\sum_{i=1}^{n} x_i y_i \cdot \alpha_i\right|$$

where $\alpha_i = (1+\delta_i)\cdot\prod_{j=i}^{n}(1+\varepsilon_j) - 1 \le nu + O(u^2)$

$$\Rightarrow \left|fl(x^T y) - x^T y\right| \le \sum_{i=1}^{n}|x_i y_i|\left(nu + O(u^2)\right)$$

$$\le |x|^T|y|\left(nu + O(u^2)\right) \quad \blacksquare$$

2. Bound Matrix product error using bound in 1.3 :

In 1.3 we have $\left|fl(x^T y) - x^T y\right| \le |x|^T|y|\left(nu + O(u^2)\right)$

When $C = AB \quad C_{ij} = a_i^T b_j \quad A \in \mathbb{R}^{m\times k} \quad B \in \mathbb{R}^{k\times n}$.

$$\therefore \left|fl(C_{ij}) - C_{ij}\right| \le |a_i|^T|b_j|\left(ku + O(u^2)\right)$$

$$\therefore fl(AB) = C + \delta C \quad |\delta C| \le \underline{ku|A||B| + O(u^2)}$$

($|A|$ and $|B|$ is taking element-wise abstract value of A and B)

Of course, the norm version of the bound is:

$$\|\delta C\|_1 \leq k u \|A\|_1 \|B\|_1 + O(u^2)$$

(Obvious from definition of $\|\cdot\|_1$ and element wise bound) ▨

Problem 2

Let $A \in \mathbb{R}^{m \times n}$

We have the axiom $fl(a \odot b) = (a \odot b)(1+\delta)$, $|\delta| \leq \varepsilon_{machine}$

$\Rightarrow fl(\tilde{P}A) = \tilde{P}A(1+\delta)$

- Check $\tilde{P}A$:

$\tilde{P}A = fl(P) \cdot A$ Let $[fl(P)]_{ij} = P_{ij}(1+\delta_{ij})$

$$\therefore [\tilde{P}A]_{ij} = \sum_{k=1}^{m} [fl(P)]_{ik} a_{kj}$$

$$= \sum_{k=1}^{m} P_{ik}(1+\delta_{ik}) a_{kj} = \sum_{k=1}^{m} P_{ik} a_{kj} + \sum_{k=1}^{m} P_{ik} a_{kj} \delta_{ik}$$

$\because |\delta_{ik}| \leq \varepsilon_{machine}$

$$\therefore [\tilde{P}A]_{ij} = [PA]_{ij} + [PA]_{ij} O(\varepsilon)$$

$$\Rightarrow \tilde{P}A = PA + PA\, O(\varepsilon)$$

$$\Rightarrow fl(\tilde{P}A) = \big(PA + PA\, O(\varepsilon)\big)(1+\delta) \qquad \|\delta\| \leq \varepsilon_{machine}$$

$$= P(A + A\delta + A\, O(\varepsilon)\delta)$$

$$= P(A + A\, O(\varepsilon))$$

$\| A\, O(\varepsilon) \|_2 = O(\varepsilon) \|A\|_2)$

$\therefore fl(\tilde{P}A) = P(A+E)$, where $\|E\| = O(\varepsilon)\|A\|_2$ $\boxtimes$