1 Theoretical Part

Problem 1.

(a)

Proof. Let $x = [x_1, \cdot, c_m] \in \mathbb{R}^m$, we have:

$$\|x\|_\infty = \sqrt{(\max |x_i|)^2} \leq \|x\|_2 = \sqrt{\sum x_i^2} \leq \sqrt{\sum (\max |x_i|)^2} = \sqrt{m}\|x\|_\infty$$

□

(b)

Proof. Now $A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n, Ax \in \mathbb{R}^m$. According to the result of (a), we have:

$$\|A\|_2 = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|_2}{\|x\|_2} \leq \sup_{x\in\mathbb{R}^n} \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_2} \leq \sup_{x\in\mathbb{R}^n} \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{m}\|A\|_\infty$$

$$\|A\|_2 = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|_2}{\|x\|_2} \geq \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|_2}{\sqrt{n}\|x\|_\infty} \geq \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|_\infty}{\sqrt{n}\|x\|_\infty} = \frac{1}{\sqrt{n}}\|A\|_\infty$$

□

(c)

Proof. We prove (c) in two steps.

Firstly, we show that $\forall p, q \in \mathbb{R}, p, q \geq 1$, vector norm $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent:

To prove this, it is sufficient to show $\forall p \in \mathbb{R}, p \geq 1$, vector norm $\|\cdot\|_p$ and $\|\cdot\|_1$ are equivalent.

Moreover, we have:

$$\begin{aligned}\|x\|_p - \|x'\|_p = \|x' + (x - x')\|_p - \|x'\|_p \leq \|x - x'\|_p \\ \|x'\|_p - \|x\|_p = \|x - (x - x')\|_p - \|x\|_p \leq \|x - x'\|_p\end{aligned}$$

Therefore, suppose $x = \sum \alpha_i e_i$, we have:

$$\begin{aligned}\Big|\|x'\|_p - \|x\|_p\Big| &\leq \|x - x'\|_p \\ &= \Big\|\sum (\alpha_i - \alpha_i')e_i\Big\|_p \\ &\leq \sum_{i=1}^n |\alpha_i - \alpha_i'| \cdot \|e_i\|_p = \sum_{i=1}^n |\alpha_i - \alpha_i'| = \|x - x'\|_1\end{aligned}$$

Thus, $\forall \delta > 0, \exists \epsilon > 0$ $s.t$ $\|x - x'\|_1 < \delta \implies \left| \|x\|_p - \|x'\|_p \right| < \epsilon = \delta$. That is, any norm $\|\cdot\|_p$ is a continuous function under the topology induced by $\|\cdot\|_1$. So we can always get $C_1 = \max_{\|u\|_1=1} \|u\|_p$, $C_2 = \min_{\|u\|_1=1} \|u\|_p$ such that $C_1 \le \|u\|_p \le C_2$ because the set $\{u : \|u\|_1 = 1\}$ is compact. Multiply each side of the equality by $\|x\|_1$, we get:

$$C_1\|x\|_1 \le \|x\|_p \le C_2\|x\|_1$$

So far we know $\exists \alpha, \beta$ such that $\beta\|x\|_q \le \|x\|_p \le \alpha\|x\|_q$ for $1 \le p < q \le \infty$

Secondly, we can show that $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent for $1 \le p < q \le \infty$ using the same strategy as (b):

$$\|A\|_p = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|_p}{\|x\|_p} \le \sup_{x\in\mathbb{R}^n} \frac{\alpha\|Ax\|_q}{\beta\|x\|_q} = \frac{\alpha}{\beta}\|A\|_q$$

$$\|A\|_p = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|_p}{\|x\|_p} \ge \sup_{x\in\mathbb{R}^n} \frac{\beta\|Ax\|_q}{\alpha\|x\|_q} = \frac{\beta}{\alpha}\|A\|_q$$

$\square$

Example of (a):

When $x = [1, 0, 0, \cdots 0]^T$, $\|x\|_\infty = \|x\|_2 = 1$

When $x = [\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}, \cdots \frac{1}{\sqrt{m}}]$, $\|x\|_2 = \sqrt{m}\|x\|_\infty = 1$

Example of (b):

When $A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$, $\|A\|_\infty = 1$, $\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 = \sqrt{m}$

$\therefore \|A\|_2 = \sqrt{m}\|A\|_\infty$

When $A = \begin{bmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \vdots & & \\ \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix}_{m\times n}$, $\|A\|_\infty = \frac{n}{\sqrt{n}} = \sqrt{n}$

$\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 = 1$

$\therefore \|A\|_2 = \frac{1}{\sqrt{n}}\|A\|_\infty$

Problem 2.

(a) $A = I$ A's singular values and eigenvalues are all 1.

(b) $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ A's singular values are 2, 3 and eigenvalues are -2, 3.

(c) $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ A's singular values are $\sqrt{2}$, $2+\sqrt{2}$ and eigenvalues are both 1.

Problem 3

Lemma 1: If $A, B \in \mathbb{R}^{m \times n}$ and $\operatorname{rank}(B) \leq k$, then $\forall i$, $\sigma_{k+i} \leq \sigma_i(A-B)$

proof: When $i=1$, $\dim(\operatorname{Ker}(B)) = n-k \Rightarrow$ There exists a unit-length vector $w \in \operatorname{Ker}(B) \cap \operatorname{span}\langle v_1, \cdots, v_{k+1} \rangle \Rightarrow$

$\|Aw\| = \|(A-B)w\| \leq \sigma_1(A-B)\|w\|$

And $\|Aw\|^2 = \left\| \sum_{i=1}^{k+1} \sigma_i u_i (v_i^* w) \right\|^2$ $(w \in \operatorname{span}\langle v_1, \cdots v_{k+1} \rangle)$

$$= \sum \sigma_i^2 (v_i^* w)^2 \geq \sigma_{k+1}^2 \sum (v_i^* w)^2$$

$$= \sigma_{k+1}^2 \|w\|^2$$

Therefore, $\sigma_{k+1}(A)\|w\| \leq \sigma_1(A-B)\|w\| \Rightarrow \sigma_{k+1}(A) \leq \sigma_1(A-B)$

For i in general, $\sigma_i(A-B) = \sigma_i(A-B) + \sigma_1(B-B_k)$

$= \sigma_1(A-B-(A-B)_{i-1}) + \sigma_1(B-B_k)$

$\geq \sigma_1(A-B-(A-B)_{i-1} + B - B_k)$

$= \sigma_1(A-(A-B)_{i-1} - B_k) \geq \sigma_{i+k}(A)$ ∎

According to the Lemma and Resurt in the class:

$$\|A-A_r\|_F^2 = \sum_{i=r+1}^{\mathrm{rank}(A)} \sigma_i(A)^2 \leq \sum_{i=1}^{\mathrm{rank}(A)-r} \sigma_i(A-B)^2 \leq \|A-B\|_F^2$$

∎

Problem 4

$$\|P\|_2 = \sup_{x\in\mathbb{C}^m} \frac{\|Px\|_2}{\|x\|_2} \geq \frac{\|P(Px)\|_2}{\|Px\|_2} = \frac{\|Px\|_2}{\|Px\|_2} = 1$$

take $x = Px \in \mathbb{C}^m$.

When P is an orthogonal projector, $P^* = P$

$$\|Pv\|^2 = v^*P^*Pv = v^*P^2v = v^*Pv$$

$$= |\langle v, Pv\rangle| \leq \|v\|\cdot\|Pv\|$$

$$\Rightarrow \forall v, \frac{\|Pv\|}{\|v\|} \leq 1.$$

Thus, when $P^*=P$, $\|P\| = 1$

∎

Problem 7.

Algorithm of QR using givens rotation:

for $j = 1$ to n $\quad$ $(A_{m\times n})$

$\quad$ for $i = m$ to $j+1$

$$A_{i-1:i,\,j:n} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}^T A_{i-1:i,\,j:n}$$

$\quad$ end

end

FLOP count:

To calculate c and s, for example we want to sparsify $\begin{bmatrix} a \\ b \end{bmatrix}$:

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}^T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

we do: $\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \Rightarrow \begin{cases} ca - sb = r = \sqrt{a^2+b^2} \\ sa + cb = 0 \end{cases}$

Let $s = \frac{1}{\sqrt{1+r^2}}$ $\quad c = \frac{-a}{b} s$ (But we only calculate $\sqrt{a^2+b^2}$)

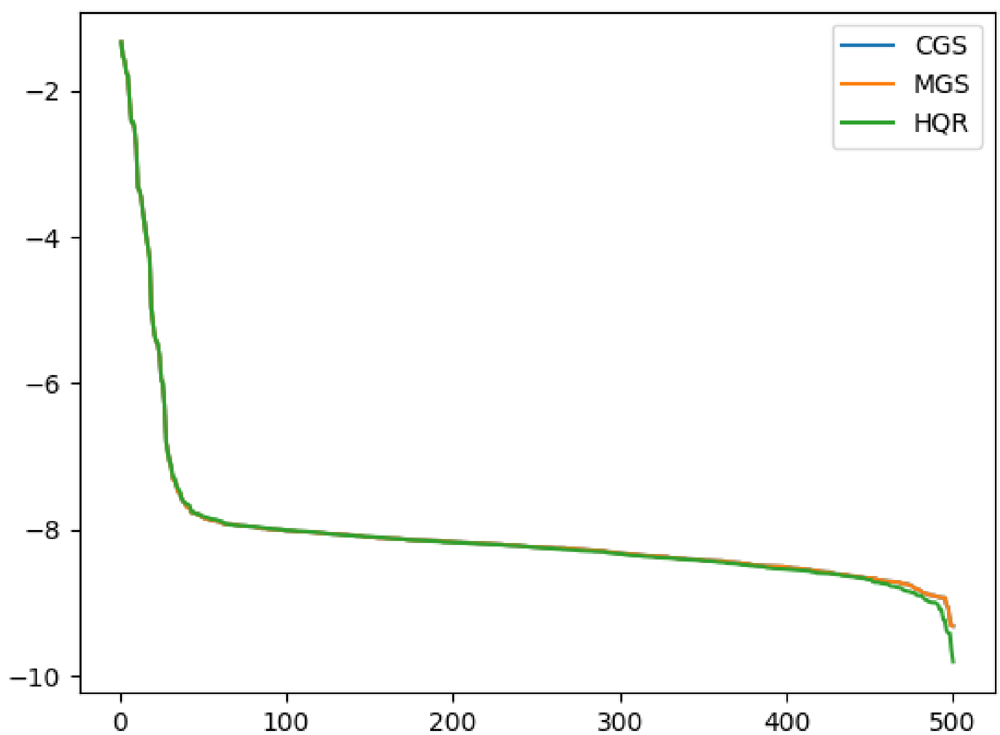
$$\Rightarrow \text{Total FLOPs} = \sum_{j=1}^{n} \sum_{k=j+1}^{m} 3n = \sum_{j=1}^{n} 3n(m-j)$$

$$\sim O(3mn^2)$$

From the class we know: Housholder QR has total FLOPs of $O(2mn^2)$

Problem 8

In this case, I generate A (a matrix with exponentially degraded singular values) with $m=n=500$, and the degrade_rate is 0.6. A is randomly generated (see code for details).



Comparing three QR methods (classical QR, MGS and Householder QR), they all perform numerical instability quickly when SV goes small (see the plateau). But in this case, CGS is not significantly worse than other methods. More sophisticated example is needed to see the difference.