Hw8 : MATH130165h

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$\boxed{\text{Problem 1}}$

$f$ is convex so $f(\theta x+(1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$,

$\forall x, y \in dom(f)$, $\theta \in [0,1]$

$\Omega$ is convex so $\theta x + (1-\theta) y \in \Omega$, $\forall x, y \in dom(f)$, $\theta \in [0,1]$

If $x^*$ is a local solution of $\min\limits_{x\in\Omega} f(x)$ :

There exists a neighbourhood $N$ of $x^*$, such that

$f(x^*) \le f(x)$, $\forall x \in N \cap \Omega$

And if $x^*$ is not a global solution :

$\exists \bar{x} \in \Omega$, such that $f(\bar{x}) < f(x^*)$

$\forall \theta \in (0,1)$, Let $x_\theta = \theta x^* + (1-\theta)\bar{x}$

$\because \Omega$ is convex, $x_\theta \in \Omega$

$\because f$ is convex, $f(x_\theta) < \theta f(x^*) + (1-\theta) f(\bar{x}) < f(x^*)$

Since $\theta \in (0,1)$, $x_\theta$ can be arbitrarily close to $x^*$.

Therefore, $\exists x_\theta \in N \cap \Omega$, such that $f(x_\theta) < f(x^*)$

This contradicts with $x^*$ is a local solution.

$\therefore$ Any local solution to $\min\limits_{x\in\Omega} f(x)$ is a global solution

If $x_1^*, x_2^*$ is two local solutions (and thus two global solutions):

Let $x_\theta = \theta x_1^* + (1-\theta) x_2^*$, $\forall \theta \in [0, 1]$

$f(x_\theta) \le \theta f(x_1^*) + (1-\theta) f(x_2^*) = f^*$

Since $f^* = \min_{x \in \Omega} f(x)$, $f(x_\theta) = f^*$

$\Rightarrow x_\theta$ is also a global solution.

$\therefore$ The set of all global solution is convex $\blacksquare$

## Problem 2

The problem can be reformulated as:

$$\min_{x \in \mathbb{R}^n,\, t \in \mathbb{R}} \; t$$

$$\text{s.t.} \quad \begin{aligned} \nu_i(x) &\le t \\ -\nu_i(x) &\le t \end{aligned}, \quad \forall i = 1, 2, \dots, m$$

$\because$ $\nu$ is a smooth vector function

$\therefore$ $\nu_i(x) - t$ and $-\nu_i(x) - t$ is smooth

And $t$ is smooth.

Thus the reformulated problem is a smooth constrained optimization problem. $\blacksquare$