MATH130165h: Homework 5

Due May 9, 2024

Problem 1. [10 pt] Let $A \in \mathbb{R}^{n\times n}$ be a *general* square matrix and $x, b \in \mathbb{R}^n$ be vectors. Compute the gradient and Hessian of $f_1(x) = b^\top x$ and $f_2(x) = x^\top Ax$.

Problem 2. [20 pt] Let $A \in \mathbb{R}^{n\times n}$ be a *general* square matrix and $X, B \in \mathbb{R}^{n\times k}$ be two matrices. Compute the gradient of $f_1(X) = \mathrm{tr}(B^\top X)$, $f_2(X) = \mathrm{tr}(X^\top AX)$, and $f_3(X) = \mathrm{tr}(BX^\top AXX^\top BX^\top)$

Problem 3. [20 pt] Show that the following conditions are equivalent for $\mu > 0$.

1. $g(x) = f(x) - \frac{\mu}{2}\|x\|^2$ is convex for all x;
2. $(\nabla f(x) - \nabla f(y))^\top (x - y) \geq \mu \|x - y\|^2$ for all x, y;
3. $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \frac{\alpha(1-\alpha)\mu}{2}\|x - y\|^2$ for all x, y and $\alpha \in [0, 1]$.

Problem 4. [10 pt] Suppose that f is a convex function. Show that the set of global minimizers of f is a convex set.

Problem 5. [30 pt] Consider the following minimization problem,

$$\min_{x\in\mathbb{R}^n} f(x) = \frac{1}{2}x^\top Ax - x^\top b,$$

where $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix, and $b \in \mathbb{R}^n$ is a vector. We apply the steepest descent method with a fixed stepsize to address the minimization problem, i.e.,

$$x_{k+1} = x_k - \alpha \nabla f(x_k),$$

where α is the constant stepsize. Please find a range for α such that the steepest descent method converges. For a fixed stepsize in the range, please prove that x_k converges to $x^* = A^{-1}b$ linearly.