MATH130165h: Homework 1

Due Apr 7, 2024

Problem 1. [10 pt] For $x \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, verify the following inequalities and give examples of a nonzero vector or matrix for which equality is achieved.

(a) $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{m} \|x\|_\infty$;

(b) $\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m} \|A\|_\infty$;

(c) Show the equivalence between induced matrix norms $\|\cdot\|_p$ and $\|\cdot\|_q$ for $1 \leq p < q \leq \infty$.

Problem 2. [10 pt] For each of the following statement, find an example matrix.

(a) a matrix whose singular values are the same as its eigenvalues;

(b) a matrix whose singular values are the same as the absolute values of its eigenvalues (the matrix has at least one negative eigenvalue);

(c) a matrix whose singular values are not the same as the absolute values of its eigenvalues;

Theorem I. Given a matrix $A \in \mathbb{R}^{m \times n}$. The singular values of A are denoted as $\{\sigma_i\}_{i=1}^{\min(m,n)}$ with their corresponding left and right singular vectors being $\{u_i\}_{i=1}^{\min(m,n)}$ and $\{v_i\}_{i=1}^{\min(m,n)}$. For any r with $0 \leq r \leq \min(m,n)$, the matrix A_r as defined,

$$A_r = \sum_{i=1}^{r} u_i \sigma_i v_i^*,$$

satisfies

$$\|A - A_r\|_{\mathrm{F}} = \inf_{\substack{B \in \mathbb{C}^{m \times n} \\ \mathrm{rank}(B) \leq r}} \|A - B\|_{\mathrm{F}}.$$

Problem 3. [20 pt] Prove the Theorem I.

Problem 4. [15 pt] Let $P \in \mathbb{C}^{m \times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

Problem 5. [30 pt] Implement Modified Gram-Schmidt (MGS) algorithm in C/C++.

Problem 6. [40 pt] Implement Householder QR algorithm and Implicit Q and Q^* application in C/C++, i.e., calculating Qx and Q^*x.

Problem 7. [15 pt] Describe an algorithm for QR factorization based on Givens rotations. And show that the Givens-QR is asymptotically 50% more expensive than Householder QR.

Problem 8. [20 pt] Generate a matrix with exponentially graded singular values. Apply your own MGS and Householder QR to the matrix. Plot the computed r_{jj} verses j in logy style. You could generate your matrices and plot in Matlab or Python.