18.303 Midterm, Fall 2012

Problem 1: Adjoints and operators (20 points)

Consider the operator $\hat{A} = \frac{\partial^2}{\partial x^2}$ on functions $u(x)$ over the interval $[0, L]$, with the boundary conditions $u(0) = 0$ and $u'(L) - \alpha u(L) = 0$ for some real number α (where u' is the derivative).

(a) Show that $\hat{A}^* = \hat{A}$ under $\langle u, v\rangle = \int_0^L \bar{u}(x)v(x)\,dx$.

(b) For what values of α, if any, can you show that $\hat{A}u = \frac{\partial u}{\partial t}$ will have decaying solutions as $t \to \infty$?

Problem 2: Finite differences (20 points)

Suppose we have a 1d diffusion-like equation $\hat{A}u = \frac{\partial^2 u}{\partial x^2} - cu = \frac{\partial u}{\partial t}$, for some $c(x) > 0$, on the domain Ω=$[0, L]$ in a system with a solute flux $u'(0) = 1$ entering Ω from the left, and the solute is removed immediately when it reaches the right, i.e. $u(L) = 0$.

Suppose we are solving for the steady-state solution $u(x)$ with $\partial u/\partial t = 0$. Using the center-difference approximation $u''_m \approx \frac{u_{m+1} - 2u_m + u_{m-1}}{\Delta x^2}$, write down a matrix approximation of this problem $Au = b$, for $u_m \approx u(m\Delta x)$. Let $c_m = c(m\Delta x)$. That is, what is the matrix A and what is the right-hand side vector b? ($b \neq 0$ due to the boundary conditions!)

Problem 3: Green-ish functions (20 points)

Consider $\hat{A} = -\frac{\partial^2}{\partial x^2}$ on $[0, L]$ with Dirichlet boundaries $u(0) = u(L) = 0$.

(a) Find the solution $u(x)$ to $\hat{A}u = \delta'(x - y)$ for any $0 < y < L$. [Hint: first solve for u at $x \neq y$ in terms of some unknown coefficient(s), and then plug your coefficients into $\hat{A}u = \delta'(x - y)$ to find the coefficients.] Recall that the distributional derivative is $f'\{\phi\} = f\{-\phi'\}$, so that $\delta'(x - y)\{\phi(x)\} = \delta(x - y)\{-\phi'\} = -\phi'(y)$, and that the derivative of a discontinuous function gives a delta function at the discontinuity multiplied by the amplitude of the discontinuity.

(b) Since your solution to the previous part depends on both x and y, call it $D(x, y)$. Now, suppose we are solving $\hat{A}u = f'$ for some $f'(x)$ which is the derivative of a function $f'(x)$. Show that you can write $u(x)$ as some integral of D and f.