18.303 Problem Set 5

Due Friday, 15 October 2010.

Problem 1: Min–max theorem

(a) Consider the operator $-\frac{d^2}{dx^2}$ on the space of functions $u(x)$ for $x \in [0,1]$ and $u(0) = u(1) = 0$, with the usual inner product $\langle u, v\rangle = \int_0^1 \bar{u}v$. Integrating by parts, the Rayleigh quotient can be written $R(u) = \langle u', u'\rangle / \langle u, u\rangle$ as in class. Consider the function:

$$u_a(x) = \begin{cases} x/a & x \leq a \\ \frac{1-x}{1-a} & x > a \end{cases},$$

where a is some number in $(0, 1)$. For what a is $R(u_a)$ minimized? How does the minimum of $R(u_a)$ compare with the smallest eigenvalue of $-d^2/dx^2$?

(b) Consider $-\nabla^2 u = \lambda u$ for functions $u(\mathbf{x})$ in the 2d triangular domain Ω given by $x \geq 0$, $y \geq 0$, $|x| + |y| \leq 1$ (a square cut in half diagonally) with Dirichlet boundary conditions $u|_{d\Omega} = 0$. Sketch contour plots of the eigenfunctions for the smallest 3 eigenvalues, making reasonable guesses based on the fact that these minimize $R(u) = \int |\nabla u|^2 / \int |u|^2$ (constrained by the fact that they must be orthogonal). (In your plots, label peaks with a "+" and dips with a "-".)

Problem 2: Green's functions

In this problem, you will solve for the 1d Green's function of the operator $\hat{A} = -\frac{d^2}{dx^2} + \kappa^2$ for some real κ, on the space of functions $u(x)$ for $x \in [0, L]$ and Dirichlet boundaries $u(0) = u(L) = 0$. Note that this operator is real-symmetric positive-definite. *Helpful information:* the ODE $-y'' + \kappa^2 y = \alpha$, where α is a constant, is solved by functions $y(x)$ of the form $y(x) = c_1 e^{-\kappa x} + c_2 e^{+\kappa x} + \frac{\alpha}{\kappa^2}$ for arbitrary constants c_1 and c_2.

(a) First, as in class, solve the "finite-delta" problem $\hat{A}g(x) = \frac{s(x-x')}{\Delta x}$ where $s(x) = \begin{cases} 1 & x \in [0, \Delta x] \\ 0 & \text{otherwise} \end{cases}$, and $x' \in [0, L - \Delta x]$ for $g(x)$ by breaking it up into three regions ($x < x'$, $x \in [x', x' + \Delta x]$, $x > x'$) and enforcing continuity of g and g'. [*Trick:* you will get four equations in four unknowns, but by dividing two of the equations by the other two you can eliminate two of the unknowns immediately.] Plot your solution in Matlab for $x' = 0.25$, 0.5, and 0.75 with $\Delta x = 0.1$ and 0.01, with the parameters $L = 1$, $\kappa = 5$.

(b) Take the limit $\Delta x \to 0$ to obtain the Green's function $G(x, x')$. Alternatively, you may compute $G(x, x')$ directly by solving $\hat{A}G(x, x') = \delta(x - x')$ as in class and the notes (breaking G into two regions where $\hat{A}G = 0$, and then matching the two regions by requiring G to be continuous and its slope to have a discontinuity $= 1$ at x'). Plot it for $L = 1$, $\kappa = 5$.

(c) Verify that $G(x, x') = G(x', x)$.

(d) Verify that $\hat{A}G(x, x') = \delta(x - x')$, using the rules for differentiating discontinuous functions in the distribution sense.

(e) **Optional:** Using $u(x) = \int G(x, x')f(x')dx'$, solve $\hat{A}u = f$ for $f(x) = e^x$.

Problem 3: Distributions

Define $s_\varepsilon(x)$ by:

$$s_{\Delta x}(x) = \begin{cases} 0 & |x| > \Delta x \\ \frac{1}{2\Delta x} & |x| \leq \Delta x \end{cases}.$$

As a distribution, $s_{\Delta x}\{\phi\} \to \delta\{\phi\} = \phi(0)$ as $\Delta x \to 0$. What is the distributional derivative $s'_{\Delta x}\{\phi\}$? Show that $s'_{\Delta x}\{\phi\} \to \delta'\{\phi\} = \delta\{-\phi'\} = -\phi'(0)$ as $\Delta x \to 0$.