

18.303 Problem Set 3

Due Wednesday, 1 October 2014.

Problem 1:

Consider the space of three-component vector fields $\mathbf{u}(\mathbf{x})$ on some finite-volume 3d domain $\Omega \subset \mathbb{R}^3$. One linear operator on these fields is the curl $\nabla \times$, which is important in electromagnetism (which we will study in more detail later in 18.303). Define the inner product of two vector fields \mathbf{u} and \mathbf{v} by the volume integral $\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \bar{\mathbf{u}} \cdot \mathbf{v}$.

- (a) An 18.02 exercise: derive the identity $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$.
- (b) Figure out how to do integration by parts with the curl: show that $\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle + \oint_{\partial\Omega} \mathbf{w} \cdot d\mathbf{S}$, where $d\mathbf{S}$ is the usual outward surface-normal area element, and the \mathbf{w} appearing in the surface integral over the boundary ($\partial\Omega$) is some vector field to be determined. (*Hint*: use the identity you derived in the previous part, combined with the divergence theorem.)
- (c) Give a possible boundary condition on our space of vector fields such that $\nabla \times$ is self-adjoint with this inner product. (Boundary conditions can only involve one vector field at a time! No fair giving an equation that relates \mathbf{u} to \mathbf{v} !) You should *not* have to specify *all* the components of \mathbf{u} on the boundary. It may be convenient to define a vector field $\mathbf{n}(\mathbf{x})$ on $\partial\Omega$ to denote the outward normal vector at each point on the boundary.¹
- (d) Show that $\nabla \times \nabla \times$ is self-adjoint for this inner product under *either* some boundary condition on \mathbf{u} (similar to above) or some boundary condition on *derivatives* of \mathbf{u} . Is it positive or negative definite or semidefinite?
- (e) Two of Maxwell's equations in vacuum are $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$ where c is the speed of light. Take the curl of both sides of the second equation to obtain a PDE in \mathbf{E} alone. Suppose that Ω is the interior of a hollow metal container, where the boundary conditions are that \mathbf{E} is perpendicular to the metal at the surface (i.e. $\mathbf{E} \times \mathbf{n}|_{\partial\Omega} = 0$).² Combining these facts with the previous parts, explain why you would expect to obtain *oscillating* solutions to Maxwell's equations (*standing electromagnetic waves*, essentially light bouncing around inside the container).

(This kind of system exists, for example, in the microwave regime where metals have high conductivity, and such containers are called *microwave resonant cavities*.)

Problem 2:

In class, we solved for the eigenfunctions of ∇^2 in two dimensions, in a cylindrical region $r \in [0, R]$, $\theta \in [0, 2\pi]$ using separation of variables, and obtained Bessel's equation and Bessel-function solutions. Although Bessel's equation has two solutions $J_m(kr)$ and $Y_m(kr)$ (the *Bessel functions*), the second solution (Y_m) blows up as $r \rightarrow 0$ and so for that problem we could only have $J_m(kr)$ solutions (although we still needed to solve a transcendental equation to obtain k).

In this problem, you will solve for the 2d eigenfunctions of ∇^2 in an **annular** region Ω that does *not* contain the *origin*, as depicted schematically in Fig. 1, between radii R_1 and R_2 , so that

¹Technically, we are assuming here that the boundary is a differentiable surface so that the normal vector is well defined. Actually, it is sufficient to assume that it is differentiable except for isolated corners and edges (a "set of measure zero") since those isolated kinks won't contribute anything to the $\oint_{\partial\Omega}$ surface integral.

²In a perfect conductor, any nonzero component of \mathbf{E} parallel to the surface would generate an infinite current parallel to the surface, in which charges instantaneously rearrange to cancel the field. For a real conductor with finite conductivity, matters are more complicated because we must consider what the fields do inside the conductor, but a perfect conductor is a good approximation at microwave and lower frequencies where the penetration of the field into the conductor (the *skin depth*) is much smaller than the wavelength of light.

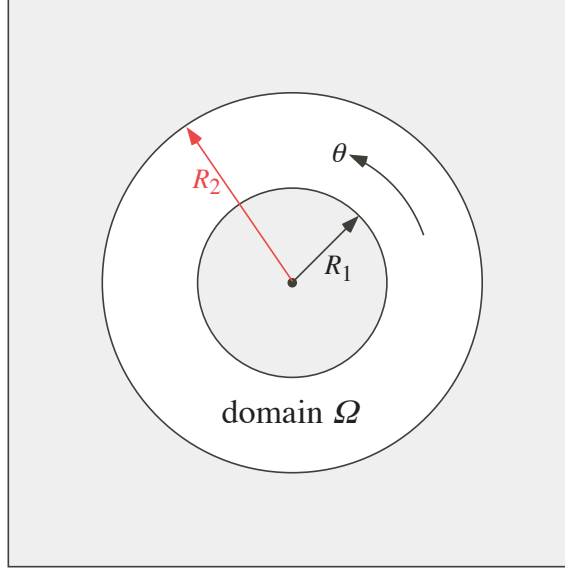


Figure 1: Schematic of the domain Ω for problem 3: an annular region in two dimensions, with radii $r \in [R_1, R_2]$ and angles $\theta \in [0, 2\pi]$.

you will need *both* the J_m and Y_m solutions. Exactly as in class, the separation of variables ansatz $u(r, \theta) = \rho(r)\tau(\theta)$ leads to functions $\tau(\theta)$ spanned by $\sin(m\theta)$ and $\cos(m\theta)$ for integers m , and functions $\rho(r)$ that satisfy Bessel's equation. Thus, the eigenfunctions are of the form:

$$u(r, \theta) = [\alpha J_m(kr) + \beta Y_m(kr)] \times [A \cos(m\theta) + B \sin(m\theta)]$$

for arbitrary constants A and B , for integers $m = 0, 1, 2, \dots$, and for constants α , β , and k to be determined.

For fun, we will also **change the boundary conditions** somewhat. We will impose “Neumann” boundary condition $\frac{\partial u}{\partial r} = 0$ at R_1 and R_2 . That is, for a function $u(r, \theta)$ in cylindrical coordinates, $\frac{\partial u}{\partial r}|_{r=R_1} = 0$ and $\frac{\partial u}{\partial r}|_{r=R_2} = 0$. The following *exact* identities for the derivatives of the Bessel functions will be helpful:

$$J'_m(x) = \frac{J_{m-1}(x) - J_{m+1}(x)}{2}, \quad Y'_m(x) = \frac{Y_{m-1}(x) - Y_{m+1}(x)}{2}$$

- (a) Using the boundary conditions, write down two equations for α , β and k , of the form $E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ for some 2×2 matrix E . This only has a solution when $\det E = 0$, and from this fact obtain a single equation for k of the form $f_m(k) = 0$ for some function f_m that depends on m . This is a transcendental equation; you can't solve it by hand for k . In terms of k (which is still unknown), write down a possible expression for α and β , i.e. a basis for $N(E)$.
- (b) Assuming $R_1 = 1$, $R_2 = 2$, plot your function $f_m(k)$ versus $k \in [0, 20]$ for $m = 0, 1, 2$. Note that Julia provides the Bessel functions built-in: $J_m(x)$ is `besselj(m,x)` and $Y_m(x)$ is `bessely(m,x)`. You can plot a function with the `plot` command. See the IJulia notebook posted on the course web page for lecture 8 for some examples of plotting and finding roots in Julia.
- (c) For $m = 0$, find the first three (smallest $k > 0$) solutions k_1 , k_2 , and k_3 to $f_0(k) = 0$. Get a rough estimate first from your graph (zooming if necessary), and then get an accurate answer

by calling the `scipy.optimize.newton` function as in pset 1, and also as illustrated in the lecture-8 IJulia notebook. (Note that there is *also* a $k = 0$ eigenfunction for $m = 0$, corresponding to the *constant* function: the nullspace of \hat{A} with Neumann boundary conditions, as in class.)

- (d) Because ∇^2 is self-adjoint under $\langle u, v \rangle = \int_{\Omega} \bar{u}v$ (we showed in class, in general Ω , that this is still true with these boundary conditions), we know that the eigenfunctions must be orthogonal. From class, this implies that the radial parts must also be orthogonal when integrated via $\int r dr$. Check that your Bessel solutions for k_1 and k_2 are indeed orthogonal, by numerically integrating their product via the `quadgk` function in Julia as in pset 2 and as in the lecture-8 IJulia notebook.
- (e) Let's change the problem. Suppose that the domain is now $0 \leq r \leq R_2$, and the operator is $\hat{A} = c(r)\nabla^2$ with $c(r) = 2$ for $r < R_1$ and $c(r) = 1$ for $r \geq R_1$. Suppose we impose Dirichlet boundary conditions $u(R_2) = 0$.
 - (i) What is the form of the eigenfunctions? (Define them in terms of $J_m(kr)$ and $Y_m(kr)$ with unknown coefficients in the $r < R_1$ and $r \geq R_1$ regions; don't try to solve for the coefficients.)
 - (ii) If we solve for eigenfunctions $\hat{A}u = \lambda u$, and u is everywhere finite, then what continuity conditions must u satisfy at $r = R_1$ in order for $\hat{A}u$ to be well defined and finite? If you combine these continuity conditions with the boundary condition at R_2 , you should find that the number of equations that u must satisfy matches the number of unknown coefficients in the previous part.
 - (iii) As before, write down a condition $f_m(k) = 0$ that must be satisfied in order for the above equations to have a solution. The roots of this function then give the eigenvalues. (You would have to solve it numerically as above, but you need not do this here; just write down f_m , which you can leave in the form of a determinant.)

Problem 3:

The Bessel functions $u(x) = J_m(kx)$, from class, solve the eigenproblem:

$$\hat{A}u = u'' + \frac{u'}{r} - \frac{m^2}{r^2}u = -k^2u = \lambda u$$

on $[0, R]$ where $u(R) = 0$ and $u(0) = 0$ for $m > 0$.

- (a) Show that this operator \hat{A} is of the form of a Sturm-Liouville operator (from the class notes), and is therefore self-adjoint for an appropriate inner product (which?). (Hint: rewrite the first two terms of \hat{A} as a single term.)
- (b) Show that \hat{A} is negative definite, hence $\lambda < 0$ and we are entitled to write $\lambda = -k^2$ for a real k .
- (c) Write down a center-difference discretization of this operator \hat{A} for $u_n = u(n\Delta x)$ with $n = 1, \dots, R/\Delta x = R/(N+1)$. Be careful of *where* you evaluate the $1/r$ factors (both to maintain second-order accuracy, and to avoid dividing by zero.)
- (d) In Julia, form the matrix approximation A of \hat{A} for $m = 1$ (with $N = 100$ and $R = 1$) using code similar to problem 2 from pset 2. Compare its smallest-magnitude eigenfunction to $J_1(k_{1,1}r/R)$ (where $k_{1,1}$ is the first root of J_1), evaluated with the help of the Julia code posted in Lecture 9. They should be the same up to some overall scale factor, within the discretization accuracy.