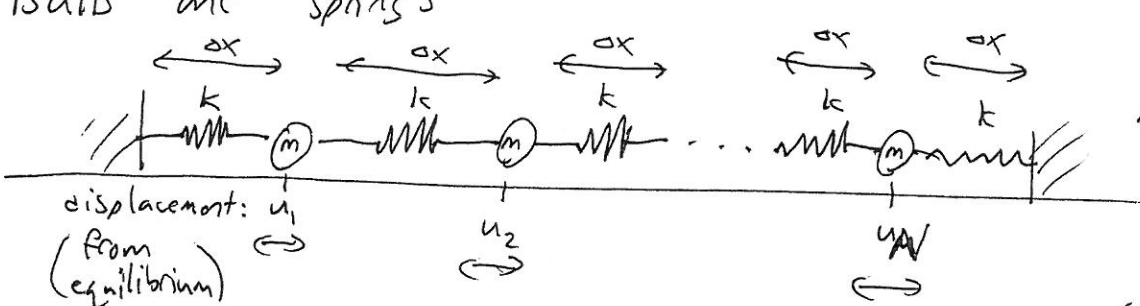
Lecture 5.5 : from discrete to continuum

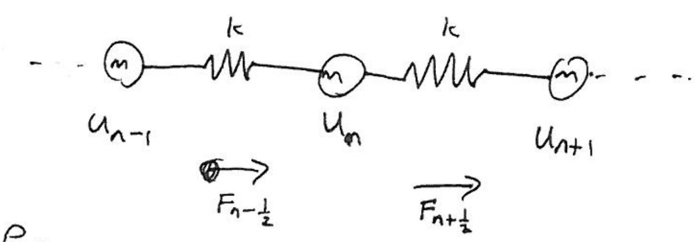
* before, we viewed the discretized eq as an approx for the continuum equations — now we will do the reverse: start with the discrete problem, & derive continuum problem as a limit or approximation

* Balls and springs



N coupled masses (m) sliding without friction, with springs (k)

($u_n = 0$ at equilibrium)



net force on u_n: $\underbrace{k(u_{n+1}-u_n)}_{\text{"}F_{n+\frac{1}{2}}\text{"}} - k\underbrace{\overbrace{(u_n - u_{n-1})}^{\text{change in length}}}_{\text{"}F_{n-\frac{1}{2}}\text{"}}$ (Hooke's Law)

$$= k\underbrace{(u_{n+1} - 2u_n + u_{n-1})}_{\text{(looks like} \approx \frac{d^2}{dx^2} \text{ without the } \Delta x^2 \text{!)}}$$

more systematically:

(i) get $F_{n+1/2}$'s from $k \times$ (differences in u_n's)

(ii) get net force from differences in $F_{n+\frac{1}{2}}$'s

(i) in matrix form:

$$\underbrace{\begin{pmatrix} F_{\frac{1}{2}} \\ F_{3/2} \\ \vdots \\ F_{N+\frac{1}{2}} \end{pmatrix}}_{\substack{\vec{F} \\ (N+1 \text{ components})}} = K \underbrace{\begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}}_{(N+1)\times N} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}}_{\substack{\vec{u} \\ (N \text{ comp.'s})}}$$

$$\Rightarrow \vec{F} = K D \vec{u} \cdot \Delta x$$

same D as for FD approx!

(ii) in matrix form:

$$\underbrace{m\ddot{\vec{u}}}_{N \text{ components}} = \text{net force} = \underbrace{\begin{pmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & -1 & 1 \end{pmatrix}}_{\substack{= -\Delta x D^T \\ \text{from before!}}} \begin{pmatrix} F_{\frac{1}{2}} \\ F_{3/2} \\ \vdots \\ F_{N+\frac{1}{2}} \end{pmatrix}$$

$$\Rightarrow \ddot{\vec{u}} = \underbrace{-\frac{K}{m}\Delta x^2 D^T D}_{A} \vec{u} = A \vec{u}$$

real-symmetric negative definite

$$= \frac{K}{m}\Delta x^2 \underbrace{\frac{\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{pmatrix}}{\Delta x^2}}_{\text{same discrete Laplacian}}$$

- Solution to $\ddot{\vec{u}} = A u$:

A diagonalizable : N eigenvectors $\vec{u}_n$ ← choose orthonormal: $\vec{u}_n^{\,*} \vec{u}_m = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

and eigenvalues $\underbrace{\lambda_n}_{\text{real, } < 0}$

expand $\vec{u}(t)$ in this basis:

$$\vec{u}(t) = \sum_{n=1}^{N} \underbrace{c_n(t)}_{\text{some coefficients}} \vec{u}_n$$

some coefficients : $c_n(t) = \vec{u}_n^{\,*} \vec{u}(t)$ by orthonormality

plug in : $\ddot{\vec{u}} = A u$

$$\sum_n \ddot{c}_n \vec{u}_n = \sum_n c_n \lambda_n \vec{u}_n \Rightarrow \ddot{c}_n = \lambda_n c_n$$

$$\Rightarrow c_n(t) = \alpha_n \cos\left(\underbrace{\sqrt{-\lambda_n}}_{} \, t\right) + \beta_n \sin\left(\underbrace{\sqrt{-\lambda_n}}_{\omega_n} \, t\right)$$

$\omega_n = \sqrt{-\lambda_n}$
= "eigen frequency"
(real since $\lambda_n < 0$)

where α_n, β_n are some coefficients determined by initial conditions

$$\Rightarrow \boxed{\vec{u}(t) = \sum_{n=1}^{N} \left[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\right] \underbrace{\vec{u}_n}_{}}$$

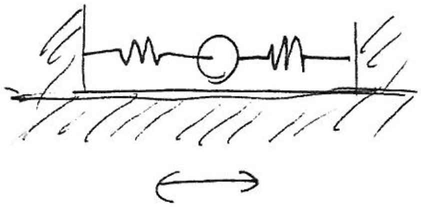
"normal modes"
oscillating with frequencies ω_n

$$\vec{u}(0) = \sum_n \alpha_n \vec{u}_n \Rightarrow \boxed{\alpha_n = \vec{u}_n^{\,*} \vec{u}(0)}$$

$$\dot{\vec{u}}(0) = \sum_n \omega_n \beta_n \vec{u}_n \Rightarrow \boxed{\beta_n = \frac{\vec{u}_n^{\,*} \dot{\vec{u}}(0)}{\omega_n}}$$

examples :

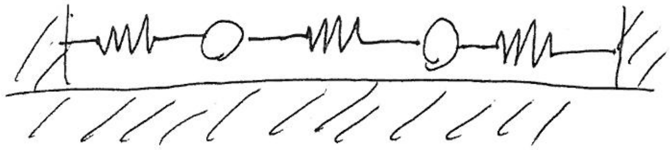
☆ $N=1$:



$A = \frac{k}{m}(-2)$

$\Rightarrow$ one mode, one $\omega_n = \sqrt{+\frac{2k}{m}}$

☆ $N=2$:



$A = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}$: $\longrightarrow \quad \longrightarrow$ (moving together)

$\perp$

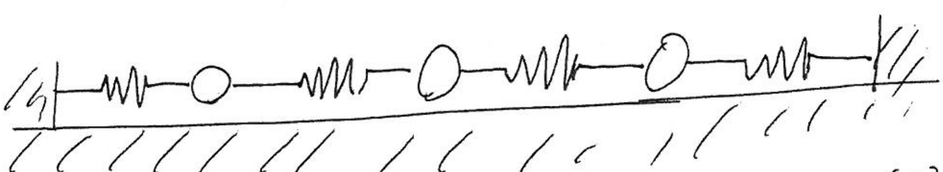
$\vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} / \sqrt{2}$ $\longrightarrow \quad \longleftarrow$ (moving opposite)

$\Rightarrow 2$ modes $\vec{u}_1, \vec{u}_2$

$\omega_1 = \sqrt{\frac{k}{m}}$

$< \omega_2 = \sqrt{\frac{3k}{m}}$

☆ $N=3$:



$A = \frac{k}{m}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

$\vec{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ $\longrightarrow \quad \longrightarrow \quad \longrightarrow$ $\omega_1 \approx 0.765\sqrt{\frac{k}{m}}$

$<$

(note $\perp$)

$\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\longrightarrow \quad \cdot \quad \longleftarrow$ $\omega_2 \approx 1.414\sqrt{\frac{k}{m}}$

$<$

$\vec{u}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ +1 \end{pmatrix}$ $\longrightarrow \quad \longleftarrow \quad \longrightarrow$ $\omega_3 \approx 1.848\sqrt{\frac{k}{m}}$

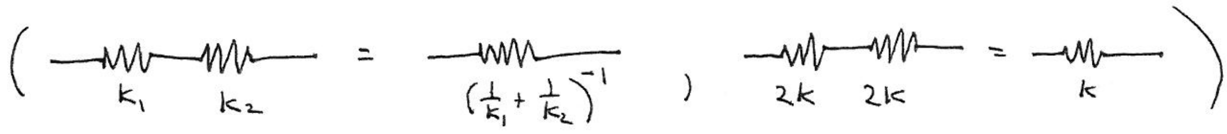
* The Continuum Limit $N \to \infty$

- make Δx smaller + smaller

$\Longrightarrow$ make mass m smaller : let $m = \rho \Delta x$

$\uparrow$ density (per length)

- shortening springs increases k !



$\Longrightarrow$ let $k = c/\Delta x$

$$\Longrightarrow \ddot{\vec{u}} = -\frac{k}{m} \Delta x^2 D^T D \vec{u} = -\frac{c}{\rho} D^T D \vec{u}$$

$$\xrightarrow[\Delta x \to 0]{} \boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = +\frac{c}{\rho} \frac{\partial^2 u(x,t)}{\partial x^2}} \qquad \left(-D^T D \to \frac{\partial^2}{\partial x^2}\right)$$

scalar wave equation !

$$\ddot{u} = \hat{A} u \ , \qquad \text{fixed ends: } u(0,t) = u(L,t) = 0$$

$$\hat{A} = +\frac{c}{\rho} \frac{\partial^2}{\partial x^2}$$

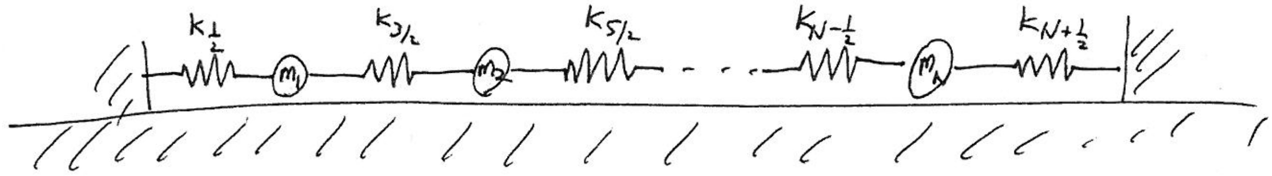
negative ~~positive~~-definite + self-adjoint for usual $\langle u, v \rangle = \int \bar{u} v$

$\Longrightarrow$ real, $\lambda_n < 0$

$\Longrightarrow$ oscillating sols with $\omega_n = \sqrt{-\lambda_n}$

* Inhomogeneous materials:

– suppose each m, k is <u>different</u>:



$$\Rightarrow \quad \text{(i)} \quad \vec{F} = K D \vec{u} \, \Delta x$$

$$K = \begin{pmatrix} k_{1/2} & & & \\ & k_{3/2} & & \\ & & \ddots & \\ & & & k_{N+\frac{1}{2}} \end{pmatrix}$$

$(N+1) \times (N+1)$ diagonal matrix of k's

$$\text{(ii)} \quad \ddot{\vec{u}} = -\underbrace{M^{-1}} \Delta x^2 D^T K D \vec{u} = A \vec{u}$$

$$M^{-1} = \begin{pmatrix} 1/m_1 & & & \\ & 1/m_2 & & \\ & & \ddots & \\ & & & 1/m_N \end{pmatrix} = N \times N \text{ diagonal matrix of } 1/m\text{'s}$$

$$A = -\Delta x^2 M^{-1} D^T K D = A^*$$

under $\langle \vec{u}, \vec{v} \rangle = \vec{u}^* M \vec{v}$ & negative-def for $m, k > 0$

$$\xrightarrow[N \to \infty]{} \quad \hat{A} = \frac{1}{\rho(x)} \frac{\partial}{\partial x} c(x) \frac{\partial}{\partial x}$$

$$= \hat{A}^* \text{ under } \langle u, v \rangle = \int \rho \bar{u} v$$

& negative-definite for $\rho, c > 0$