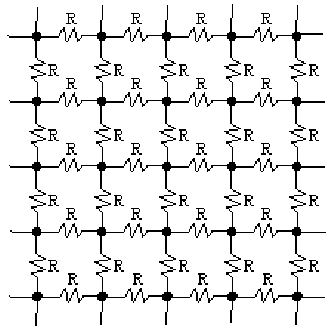

Figure 1: A 2d grid of resistors R.

18.303 Problem Set 3

Due Friday, 2 October 2015.

Problem 1:

Consider a 2d grid of identical resistors R as shown in figure 1. Recall from 8.02 that the current I through a resistor is given by $\Delta V/R$, where ΔV is the voltage difference across the resistor.

Let $V_{i,j}$ denote the voltage at the (i,j)-th node (= dot in the figure). Let $I_{i+0.5,j}$ denote the current from node (i,j) to node $(i+1,j)$, and let $I_{i,j+0.5}$ denote the current from node (i,j) to node $(i,j+1)$. Suppose that we also inject an "external" current $C_{i,j}$ into each node (i,j), e.g. by connecting a wire from "above" the grid to the node ($C_{i,j} < 0$ if we extract current rather than injecting it, and $C_{i,j} = 0$ for nodes that we don't touch).

(a) Write difference equations relating $V_{i,j}$ to $I_{i+0.5,j}$ and $I_{i,j+0.5}$, and difference equations relating $I_{i+0.5,j}$ and $I_{i,j+0.5}$ to $C_{i,j}$ (note that $C_{i,j}$ must equal the net current leaving node i,j, by Kirchhoff's current law). Combine these to a difference equation relating $V_{i,j}$ to $C_{i,j}$.

(b) Supposed that the nodes (i,j) are separated by Δx in space from $(i+1,j)$ or $(i,j+1)$. Take the limit $\Delta x \to 0$ of an infinitesimally fine grid, with $R = \rho\Delta x \to 0$ for some constant "resistivity" ρ (resistance per unit length). Show that if you appropriately scale $C_{i,j}$ as well, you obtain a Poisson equation $\nabla^2 V =????$ for the voltage $V(x,y)$, where you think of $V_{i,j}$ as $\approx V(i\Delta x, j\Delta y)$.

(c) Suppose that you instead have a variable resistivity R (different for each edge in the grid), which you think of as discretizing a resistivity function $\rho(x,y)$. [i.e. $R_{i+0.5,j} = \rho([i+0.5]\Delta x, j\Delta x)\Delta x$ etcetera.] In the continuum limit, what happens to your Poisson equation from the previous part?

(d) Implement the difference equations from the first part in Julia as a matrix equation $Av = c$ for a vector v of the voltages V and a vector c of the currents C, for an $N \times N$ grid with Dirichlet boundaries ($V = 0$ at the edges of the grid.) (Hint: use the Kronecker-product code from class.) Put in a current $C_{i,j} = 1$ and $C_{i+1,j} = -1$ (i.e. inject a current at one node and remove the current from the adjacent node) for an (i,j) near the center of the grid. Solve $Av = c$, and find the voltage difference $V_{i+1,j} - V_{i,j}$: this (via $V = IR$) is the "equivalent resistance" across that pair of nodes in your resistor grid. Plot this equivalent resistance vs N (try doubling N a few times) on a semilog scale—does it appear to be asymptoting to something for $N \to \infty$?