18.303 Problem Set 1

Due Wednesday, 15 September 2010.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

(a) Suppose A has a nullspace spanned by the vector $\mathbf{n}$, and a left nullspace spanned by the vector $\mathbf{m}$. In terms of $\mathbf{m}$, $\mathbf{n}$, $\mathbf{b}$, and/or $\mathbf{x}$ only (*not* A), give an equation that will tell you whether $A\mathbf{x} = \mathbf{b}$ has a solution. Supposing it does have a solution $\mathbf{x}$, give an expression for *all* solutions.

(b) Suppose that A is a real 20×20 matrix with eigenvalues $-1, -2, \ldots, -20$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{20}$, respectively.

  (i) Is A necessarily diagonalizable? Symmetric?

  (ii) Suppose A is symmetric. If $\mathbf{x}(t)$ solves $\frac{d}{dt}\mathbf{x} = Ax$ with $\mathbf{x}(0) = \mathbf{b}$, for some random $\mathbf{b}$, what is likely to be true about the solution at large t?

(c) In 18.06, you were shown a simple (2–3 line) proof that the eigenvalues λ of A (solutions of $A\mathbf{x} = \lambda\mathbf{x}$) must be real numbers if A is real-symmetric (if you forgot, look it up). Adapt the same proof to show that the solutions λ of $A\mathbf{x} = \lambda B\mathbf{x}$ must be real if A and B are real-symmetric and B is positive-definite.[1] Why does the proof fail if B is not positive-definite?

(d) Linearity means that if $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{c}$, then the solution $\mathbf{z}$ of $A\mathbf{z} = 2\mathbf{b} + 4\mathbf{c}$ is $\mathbf{z} =$?

Problem 2: Les Poissons, les Poissons, how I love les Poissons

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

(a) Does the space of all possible right-hand sides $f(x)$ (for any u) form a vector space? Is it the same as the vector space of the $u(x)$ functions? Why or why not?

(b) How do the eigenfunctions of $\frac{d^2}{dx^2}$ change if we change the boundary conditions to $u'(0) = u'(L) = 0$?

(c) If we instead consider $\frac{d^2}{dx^2}v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = 0$, $v(L) = 1$, do these functions form a vector space? Why or why not?

(d) Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2}u(x) = f(x)$ problem with $u(0) = u(L) = 0$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions q and r. Suggest two different possible choices of q and r. (Transforming a new problem into an old, solved one is always a useful thing to do!)

[1] The problem $A\mathbf{x} = \lambda B\mathbf{x}$ is known as a "generalized eigenproblem" and comes up a lot in numerical methods for PDE eigenproblems.

Problem 3: Nullicious

(a) How does the null space of $\frac{d^2}{dx^2}$ differ if we consider the functions $u(x)$ on $x \in [0, L]$ with $u(0) = u(L) = 0$ (as in class) versus functions with different boundary conditions $u'(0) = u'(L) = 0$? What does this imply for the solutions of $\frac{d^2}{dx^2} u(x) = f(x)$, if any, in the latter case?

(b) The linear operator $\frac{d^4}{dx^4}$ is important for something called the *beam equation* (for which Wikipedia has some pretty pictures). If we consider the vector space of functions $u(x)$ on $x \in [0, L]$ with the boundary conditions $u(0) = u(L) = 0$, then what is the null space of $\frac{d^4}{dx^4}$? Give an example of *additional* boundary conditions that you could add that shrink this null space to $\{0\}$ (but make sure it stays a vector space!).