18.303 Problem Set 7

Due Wednesday, 27 October 2010.

Problem 1: Reciprocity

In class, we showed that if $\hat{A}$ is self-adjoint ($\hat{A} = \hat{A}^*$), then $\hat{A}^{-1}$ is as well (if it exists). From the fact that $-\nabla^2$ is real-symmetric, we concluded that the Green's function $G_0(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}', \mathbf{x})$, a property called "reciprocity."

(a) In the notes, section 2.1, we considered the operator $-c\nabla^2$ for a real function $c(\mathbf{x}) > 0$ on some Ω with Dirichlet boundaries, and showed that its Green's function is simply $G(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}, \mathbf{x}')/c(\mathbf{x}')$, where G_0 is the Green's function of $-\nabla^2$ on Ω. If we define the integral operator $\hat{G}u = \int_\Omega G(\mathbf{x}, \mathbf{x}')u(\mathbf{x}')d^n\mathbf{x}'$, show explicitly from $G_0(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}', \mathbf{x}) =$ real (*not* as in class by using the fact that $\hat{G}$ is the inverse of a self-adjoint operator) that $\hat{G} = \hat{G}^*$ for an appropriately chosen inner product $\langle u, v\rangle$. [Hint: remember problem 2(c) of pset 2.]

(b) In electrostatics, the potential V of a charge density ρ satisfies $-\nabla^2 V = \rho/\varepsilon_0$. The potential energy of a charge distribution is $\int \rho V$, since V is the potential energy per unit charge; this is the energy to bring the charges together from infinity. If $\rho(\mathbf{x}) = \rho_1(\mathbf{x}) + \rho_2(\mathbf{x})$ and $V = V_1 + V_2$ with $-\nabla^2 V_{1,2} = \rho_{1,2}/\varepsilon_0$, then $\int \rho_1 V_2$ is the energy required to bring the ρ_1 charges in from infinity with the ρ_2 charges already there, and $\int \rho_2 V_1$ is the energy required to bring the ρ_2 charges in from infinity with the ρ_1 charges already there. Using reciprocity, show that $\int \rho_1 V_2 = \int \rho_2 V_1$, which means that the energy doesn't depend on which charges you hold fixed and which ones you bring in from infinity. (ρ_1 and ρ_2 are purely real, of course.)

Problem 2: Born approximations

Consider the operator $\hat{A} = -\nabla^2 + c(\mathbf{x})$ for some real function $c(\mathbf{x}) \geq 0$, on a domain Ω with Dirichlet boundary conditions. Let the $c = 0$ Green's function be $G_0(\mathbf{x}, \mathbf{x}')$, satisfying $-\nabla^2 G_0(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$. Let $u_0(\mathbf{x}) = \int G_0(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')$ be the solution to $-\nabla^2 u_0 = f$, as in the notes.

(a) Suppose we want to solve $\hat{A}u = f$. Show that we can write $u = u_0 + \hat{B}u = \sum_{k=1}^\infty \hat{B}^k u_0$ for some integral operator $\hat{B}$ (similar to what we did in the notes for $-\nabla \cdot c\nabla$). Explicitly write $\hat{B}u$ as some integral of u and G_0.

(b) Similar to section 3 of the notes, suppose that $c = 0$ everywhere except in a small volume V (centered at $\mathbf{x}_1$) where $c(\mathbf{x}) = c_1$. Suppose $f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$ and we want the solution at $\mathbf{x}$, where $\mathbf{x}$ and $\mathbf{x}_0$ are far from $\mathbf{x}_1$ compared to the diameter of V. Suppose $\Omega = \mathbb{R}^3$ so that $G_0(\mathbf{x}, \mathbf{x}') = 1/4\pi|\mathbf{x} - \mathbf{x}'|$. Apply the Born approximation $u \approx u_0 + \hat{B}u_0$, and explicitly write the (approximate) solution $u(\mathbf{x})$ as a sum of G_0 term from the source at $\mathbf{x}_0$ and some term from a "source" at $\mathbf{x}_1$ due the inhomogeneity, keeping only the lowest-order term with respect to the small parameter (the diameter of V compared to $|\mathbf{x}_0 - \mathbf{x}_1|$ and $|\mathbf{x} - \mathbf{x}_1|$).

Problem 3: Born again

In this problem, you will compare a Born approximation to an exact numerical solution for a 1d problem, using the finite-difference approximation from the previous problem sets, considering the 1d operator $\hat{A}u = -\frac{d}{dx}\left(c\frac{d}{dx}u\right)$ on $x \in [0, L]$, for $c(x) > 0$ and Dirichlet boundaries $u(0) = u(L) = 0$. From class, we already know the Green's function $G_0(x, x')$ for $-\frac{d^2}{dx^2}$:

$$G_0(x, x') = \begin{cases} x'(1 - x/L) & x > x' \\ x(1 - x'/L) & x < x' \end{cases}.$$

We will consider $L = 1$ and a function $c(x) = e^{\alpha x}$ that is *nearly* constant if α is small; take $\alpha = 0.01$. Use an $n \times n$ finite-difference approximation with $n = 100$.

(a) Give Matlab commands to form a finite-difference approximation A to $\hat{A}$, similar to problem 3 of pset 2 except now the $c(x)$ function is in *between* the derivatives (between the D matrices). Careful: remember that D is $(n + 1) \times n$, and produces the derivatives not at the grid points but halfway between the grid points. [Note: it looks like there was a typo on pset 2: it computed $D^T D/\Delta x^2$ and said this was a discrete d^2/dx^2, but of course that is actually a discrete $-d^2/dx^2$. I've fixed this online.]

(b) Using section 2.2.2 of the notes, write down the Born approximation for the solution u to $\hat{A}u = \delta(x - x_0)$ [that is, $u = G(x, x_0)$, the Green's function of $\hat{A}$). You should be able to perform all the integrals explicitly (by breaking it up into two cases, $x < x_0$ and $x > x_0$, you should get a few easy integrals). Plot your approximate "scattered" part $u - u_0 = \hat{B}u_0$, defined as in the notes, versus x for $x_0 = L/2 = 1/2$.

(c) In Matlab, use a discrete "delta" function $\delta(x - L/2)$ vector $\mathbf{b}$ where $b_j = \begin{cases} (\Delta x)^{-1} & j = n/2 \\ 0 & \text{otherwise} \end{cases}$ to find the finite-difference solution $\mathbf{u} = A^{-1}\mathbf{b}$ (`A \ b` in Matlab). Also compute the finite-difference $\mathbf{u}_0$ solution $A_0^{-1}\mathbf{b}/e^{\alpha n \Delta x/2}$, where A_0 is the finite-difference approximation to $-\frac{d^2}{dx^2}$. Plot $\mathbf{u} - \mathbf{u}_0$, and compare with your Born approximation from the previous part—plot them on the same plot so that you can compare quantitatively.