# 18.303 Problem Set 4

Due Wednesday, 6 October 2010.

## Problem 1: Vive la différence

In class, we derived the 2d center-difference approximation $A$ of the operator $\hat{A} = -\nabla^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ in a $L_x \times L_y$ box (Dirichlet boundaries $u = 0$) with $M \times N$ points [$\Delta x = L_x/(M+1)$, $\Delta y = L_y/(N+1)$], writing it in the form of the $MN \times MN$ matrix:

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} 2I & -I & & & \\ -I & 2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & 2I & I \\ & & & -I & 2I \end{pmatrix} + \frac{1}{\Delta y^2}\begin{pmatrix} K & & & & \\ & K & & & \\ & & \ddots & & \\ & & & K & \\ & & & & K \end{pmatrix} = A_x + A_y,$$

where $I$ is the $N \times N$ identity matrix and $K$ is the $N \times N$ matrix $K = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$.

As in class, this assumes that the points $u_{m,n} \approx u(m\Delta x, n\Delta y)$ are converted into size-$MN$ vectors $[u_{1,:}; u_{2,:}; u_{3,:}; \cdots; u_{M,:}]$ by concatenating contiguous "columns" in the $y$ direction. (See also section 3.5 in the Strang book.) This form, however, is not necessarily the most convenient one; as we saw in 1d, it is often nicer to write such matrices in the form $D^T D$ to make it clear that they are positive-definite, etcetera, and to make it easier to implement non-constant coefficients $\nabla \cdot [c(\mathbf{x})\nabla]$.

(a) Suppose $A_x = D_x^T D_x$ and $A_y = D_y^T D_y$ for some (as yet unknown) 1st-derivative matrices $D_x$ and $D_y$. How many columns must $D_x$ and $D_y$ have? Show that $A = D^T D$ for some $D$ written in terms of $D_x$ and $D_y$ (and hence $A$ is real-symmetric, definite, etcetera). [Hint 1: $D$ can be a much bigger matrix than $D_x$ or $D_y$. Hint 2: think of $\nabla^2 u = \nabla \cdot (\nabla u)$; what vector space does $\nabla u$ live in?]

(b) Using the `diff1` function from pset 1, the correct $D_x$ matrix is `Dx=kron(diff1(M),speye(N,N))` and the correct $D_y$ matrix is `Dy=kron(speye(M,M),diff1(N))`. For $M = N = 10$ and $\Delta x = \Delta y = 1$, give a command to form your $D$ matrix in Matlab.[1] Compare $D^T D$ (`D'*D`) to the $A$ matrix produced by the command `A=delsq(numgrid('S',12))` using `A-D'*D` to check that the result is the zero matrix.

## Problem 2: Brute-force Bessel

In this problem, we will solve the Laplacian eigenproblem $-\nabla^2 u = \lambda u$ in a 2d cylinder $r \leq R$ with Dirichlet boundary conditions $u|_{d\Omega} = 0$ by brute force, and compare to the analytical Bessel solutions from class. Recall from class that the analytical solutions are of the form $J_m(k_n^{(m)} r) \times [\alpha \cos(m\theta) + \beta \cos(m\theta)]$ for eigenvalues $\lambda = [k_n^{(m)}]^2$ (positive since we have $-\nabla^2$), where $m = 0, 1, 2, \ldots$ and $k_n^{(m)} R$ is the $n$-th root of $J_m(x)$, and $\alpha$ and $\beta$ are arbitrary constants. Set $R = 1$ and form a corresponding finite-difference matrix $A \approx -\nabla^2$ by the commands:

```
N = 50
G = numgrid('D', N+2);
dx = 2/(N+2);
A = delsq(G) / dx^2;
```

[1]Note: if you have two matrices $B$ and $C$, you can make a new matrix $(B\,C)$ in Matlab by `[B,C]` and a matrix $\begin{pmatrix} B \\ C \end{pmatrix}$ by `[B;C]`.

(a) Using `fzero` as in pset 3 to find roots of $J_m(x)$ to high accuracy, say what the smallest 3 eigenvalues of $-\nabla^2$ should be, counting repeated eigenvalues only once each. Be sure to compare more than one $m$ value!

(b) Compute the first 5 eigenvalues and eigenvectors of $A$ with the command `[V,S]=eigs(A,5,'SA')`, where `diag(S)` are the eigenvalues and the columns of $V$ are the eigenvectors.

(i) Compare (*quantitatively*) to your predicted eigenvalues in the previous part.

(ii) Plot each eigenfunction with the commands: `U=G; U(G>0)=V(G(G>0),k); surf(U)` where $k = 1, 2, 3, 4, 5$ for the 5 eigenvectors, respectively and compare *qualitatively* with the analytical solution above (it is enough to check that the pattern of peaks and valleys looks right). [`surf` produces pretty 3d plots, but sometimes 2d plots are easier to understand. e.g. you can do a 2d contour plot with `contour(U); colorbar` if you prefer.]

(c) If you double the resolution to $N = 100$, the errors in these 5 eigenvalues should decrease. Do they decrease by roughly 1/2, as if the error is $\sim \Delta x$, or by roughly 1/4, as if the error is $\sim \Delta x^2$? (If the trend isn't clear, try doubling again to $N = 200$.) How can you relate this to the fact that center-difference approximations are supposed to have $\sim \Delta x^2$ errors? (Hint: remember the previous pset, and use `spy(G)` to view the grid.)

## Problem 3: Physical schmisical

This question involves mathematical equations similar to those of problem 2, but asks some physical questions if we interpret $u(x, y)$ as the vertical displacement of a stretched surface on a circular drum of radius $R$. This displacement satisfies the 2d wave equation $\nabla^2 u = \frac{1}{c^2}\partial^2 u/\partial t^2$ (this is derived from $F = ma$ where $\partial^2 u/\partial t^2$ is acceleration) where $c$ is a constant (the "wave speed" as we will see later in the semester) that depends on the tension and density; here, say $c = 200$ m/s. If the edges of the drum are held flat, then the boundary condition is the Dirichlet $u|_{r=R} = 0$.

(a) You bang your drum a few times with your drumstick, and your friend Elise the electrical engineer analyzes the sound frequency components $f$ after each hit with her spectrum analyzer [i.e. breaking the sound into components with time dependence $\sim \sin(2\pi f t + \text{phase})$], finding that the lowest observed frequency component is roughly 100 Hz. What is the radius $R$ of the drum?

(b) Setting $R = 1$ again, suppose that your drum surface is at rest, but that you have twisted the edges so that the drum edges are at a height $x^2y^2 = R\cos^2(\theta)\sin^2(\theta)$ (for simplicity, assume they are still at a radius $R$).

(i) What equation and what boundary conditions does the height $u(x, y)$ of the drum surface now satisfy?

(ii) Approximately solve this equation using your finite-difference approximation for $-\nabla^2$ above. Plot the solution $u(x, y)$ similarly to how you plotted the eigenfunctions above, and check that it satisfies the boundary conditions by also plotting $u - x^2y^2$. Hint: in Matlab, you can solve $Av = b$ efficiently by the command `v=A\b`. Hint: to define the function $g(x, y) = x^2y^2$ as a 2d matrix `g2`, do: `x2=ones(N+2,1)*linspace(-1,1,N+2); y2=x2'; g2=x2.^2 .* y2.^2;` and to convert this to a column vector `g` in the grid coordinates you would do `g=g2(G>0);` ... you can do similarly for other functions of $x$ and $y$.

(c) Hold the edges of the drum flat again, so that $u = 0$ on the boundaries. Change to $N = 20$ and recompute your $A$ matrix to make it a bit smaller. Compute the $A^{-1}$ matrix `Ainv=inv(full(A));` and plot a couple columns $k$ via the commands `U=G; U(G>0)=Ainv(G(G>0),k); surf(U)` for the columns `k=G(10,10)` [a point in the middle of the cylinder] and `k=G(10,15)` [a point halfway to one side]. What do these plots correspond to, physically, and why? (Think about what equations the columns of $A^{-1}$ each solve.)