18.303 Final Exam, Fall 2010

There are **five problems** with **equal weight**. You have 3 hours, and may bring **one page** of notes.

Problem 1: Derivatives and differences (30 points)

Consider functions $u(x)$ on $x \in [0, L]$, and the operator $\hat{A} = \frac{d^4}{dx^4}$.

(a) Give *one* example of boundary conditions that make $\hat{A}$ self-adjoint.

(b) If we make a finite difference approximation $u(m\Delta x) \approx u_m$, give a second-order accurate finite-difference approximation of $\hat{A}$. (Hint: use a second-order accurate difference approximation four times.)

Problem 2: No more scalars (30 points)

Let $\Omega \subseteq \mathbb{R}^3$ be some 3d region, and consider 3-component vector-valued functions $\mathbf{u}(\mathbf{x})$ with Dirichlet boundary conditions $\mathbf{u}|_{d\Omega} = 0$. In class, we showed that the curl operator $\nabla\times$ is then self-adjoint for the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \bar{\mathbf{u}} \cdot \mathbf{v}$. Consider the operator $\hat{A}$ for some real-valued function $c(\mathbf{x})$, where:

$$\hat{A}\mathbf{u} = \nabla \times (c \nabla \times \mathbf{u})$$

(a) Under what conditions on $c(\mathbf{x})$ does the equation $\hat{A}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t}$ have only solutions that decay exponentially to some limiting values (possibly nonzero)? (Hint: you should not need to do any messy integrals; the fact that $\nabla\times$ is self-adjoint should simplify things.)

(b) What quantities are conserved over time by solutions of $\hat{A}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t}$? (Hint: the nullspace of $\nabla\times$ is $\nabla\phi$ for any ϕ.)

Problem 3: Guided waves (30 points)

Consider the scalar wave equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$ in two dimensions for $x \in [0, L]$ and $y \in (-\infty, \infty)$, with the Neumann boundary conditions $\frac{\partial u}{\partial x}\big|_{x=0,L} = 0$. That is, Ω is a width-L strip extending infinitely in the y direction, with Neumann boundaries.

(a) If we look for separable eigenfunctions $u(x, y, t) = u_k(x) e^{i(ky - \omega t)}$, what equation and what boundary conditions does u_k satisfy?

(b) Solve your equation from the previous part to obtain the eigenfunctions and the dispersion relation $\omega(k)$.

(c) In this geometry, it possible to propagate a wavepacket (e.g. a Gaussian-envelope pulse) in the y direction without it spreading out (becoming broader in time and/or y)? Why or why not?

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Problem 4: Timestepping and stability (30 points)

Consider the equation $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$, where c is a constant, on an infinite domain $x \in (-\infty, \infty)$. Suppose that we discretize this as $u(m\Delta x, n\Delta t) \approx u_m^n$ by

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = -c\left[\alpha\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + (1-\alpha)\frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right],$$

where α is some real constant with $0 \le \alpha \le 1$.

(a) Show that this discretization is unconditionally stable when $\alpha = 0.5$. (Recall von Neumann analysis. e.g. look for solutions $u_m^n = \lambda^n e^{ikm}$ and show that λ satisfies....)

(b) For what other values of α is it unconditionally stable?

(c) For the remaining values of α, is it conditionally stable (and if so, what are the conditions on Δx and Δt?) or always unstable?

Problem 5: Green's functions (30 points)

Suppose that we have an operator $\hat{A}$ on a domain Ω with Dirichlet boundaries $u_0|_{d\Omega} = 0$, and we know the corresponding Green's function $G_0(\mathbf{x}, \mathbf{x}')$ [i.e. $\hat{A}G_0(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$ and $G_0(\mathbf{x}, \mathbf{x}') = 0$ for $\mathbf{x} \in d\Omega$]. Suppose that we now want to solve the problem with some nonzero Dirichlet boundary condition: $u|_{d\Omega} = b(\mathbf{x})$ for some given function $b(x)$.

(a) Write the solution u of $\hat{A}u = f$ (satisfying the b boundary condition on u) as some integral expression involving G_0, f, and b.

(b) Suppose $\hat{A} = \nabla^2$. In your expression from the previous part, integrate by parts on the term involving b to show that your total solution u is exactly the zero boundary-condition solution $\int G_0(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')$ plus a bunch of extra "source" terms from $d\Omega$ involving b. (Careful with integration by parts: b is not zero on $d\Omega$.)

Useful "integration by parts" formula from class: $\int_\Omega u\nabla \cdot \mathbf{v} = \oiint_{d\Omega} u\mathbf{v} \cdot d\mathbf{A} - \int_\Omega (\nabla u) \cdot \mathbf{v}$.