18.303 Problem Set 1

Due Wednesday, 12 September 2012.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

(a) Suppose that A is a real square matrix, and consider the linear system of ODEs $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. Suppose that we have a *conserved quantity*: there exists some constant vector $\mathbf{v}$ such that $\frac{d}{dt}(\mathbf{v}^T\mathbf{x}) = 0$ for *all* solutions $\mathbf{x}(t)$ of the ODE (that is, for any initial condition $\mathbf{x}(0)$ and for all t). Explain how $\mathbf{v}$ is related to one or more of the four fundamental subspaces of A.

(b) In 18.06, you were shown a simple (2–3 line) proof that the eigenvalues λ of A (solutions of $A\mathbf{x} = \lambda\mathbf{x}$) must be real numbers if A is real-symmetric (if you've forgot it, look it up), and eigenvectors of distinct λ must be orthogonal. Adapt a similar proof to show that eigenvectors of distinct λ are orthogonal and $|\lambda| = 1$ (not necessarily real!) if A is *unitary*: $A^* = A^{-1}$. (Notation: $A^* = \overline{A^T}$, the complex conjugate of the transpose. The special case of a real unitary matrix is called an *orthogonal* matrix.) (Note: $|\lambda| = 1 = \bar{\lambda}\lambda \Longleftrightarrow 1/\lambda = \bar{\lambda}$.)

(c) Suppose that A is a real 8×8 matrix with eigenvalues $2, -2, 1, -1, 0.5, -0.5, 0.25, -0.25$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_8$, respectively.

   (i) If $\mathbf{x}^{(n)}$ solves the recurrence relation $\mathbf{x}^{(n+1)} = A\mathbf{x}^{(n)}$ with initial condition $\mathbf{x}^{(0)} = \mathbf{b}$ for some random vector $\mathbf{b}$, what is likely to be true about $\mathbf{x}^{(n)}$ for large positive n?

   (ii) If A is symmetric, give an explicit formula for $\mathbf{x}^{(n)}$ in terms of $\mathbf{b}$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_8$ (with no unknown coefficients or unsolved linear systems). (Hint: orthogonal basis.)

Problem 2: Life on a torus

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

(a) Suppose that we we change the boundary conditions to the *periodic* boundary condition $u(0) = u(L)$.

   (i) What are the eigenfunctions of $\frac{d^2}{dx^2}$ now?

   (ii) Will Poisson's equation have unique solutions? Why or why not?

   (iii) Under what conditions (if any) on $f(x)$ would a solution exist? (You can restrict yourself to f with a convergent Fourier series.)

(b) If we instead consider $\frac{d^2}{dx^2}v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = v(L) + 1$, do these functions form a vector space? Why or why not?

(c) Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2}u(x) = f(x)$ problem with $u(0) = u(L)$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions q and r. (Transforming a new problem into an old, solved one is always a useful thing to do!)