18.303 Midterm, Fall 2013

Each problem has **equal weight**. You have 55 minutes.

Problem 1: Definite (30 points)

In class, we only defined definiteness for Hermitian operators. However, if $\hat{A} \neq \hat{A}^*$, we can more generally define $\hat{A}$ to be "negative-definite" if $\Re\langle u, \hat{A}u\rangle < 0$ for all $u \neq 0$, where $\Re\langle u, \hat{A}u\rangle$ denotes the real part: $\Re\langle u, \hat{A}u\rangle = \frac{\langle u,\hat{A}u\rangle + \overline{\langle u,\hat{A}u\rangle}}{2}$.

(a) For such a negative-definite non-Hermitian operator, show that the *real parts* of its eigenvalues are negative.

(b) Show that this generalization of negative-definiteness is equivalent to saying that the Hermitian operator $\hat{A} + \hat{A}^*$ is negative definite according to the definition from class.

(c) Consider the system of equations $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} - \alpha u$ and $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x} - \beta v$ for some $\alpha(x)$ and $\beta(x)$, for $\Omega = [0, L]$ with Dirichlet boundary conditions $u(0) = u(L) = 0$.

   (i) Write this as equation $\frac{\partial \mathbf{w}}{\partial t} = \hat{A}\mathbf{w}$ in terms of $\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix}$.

   (ii) Under what conditions on α and β will this system of equations have *decaying* solutions (i.e. which $\to 0$ as $t \to \infty$)? You can use the fact, derived in class, that $\hat{D} = \begin{pmatrix} & \partial/\partial x \\ \partial/\partial x & \end{pmatrix} = -\hat{D}^*$ (i.e. is anti-Hermitian) under the inner product $\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \rangle = \int \bar{u}u' + \bar{v}v'$.

Problem 2: Finite differences (30 points)

In class, we discretized $\hat{A}u = \frac{d^2u}{dx^2}$ on $u_m = u(m\Delta x)$ by $u''_m \approx \frac{u_{m+1} - 2u_m + u_{m-1}}{\Delta x^2} + O(\Delta x^2)$ with Dirichlet boundaries $u(0) = u(L) = 0$. Now, we will consider the same operator with **Neumann** boundaries $u'(0) = u'(L) = 0$.

(a) Describe (sketch if needed) a finite-difference grid with spacing Δx between adjacent points, with N unknown values of u for $x \in (0, L)$, such that $u'(0)$ and $u'(L)$ can be evaluated to second-order-accuracy by center differences (with spacing Δx, not $2\Delta x$) on the grid. What is Δx in terms of L and N?

(b) Using this grid, write the discrete version $A\mathbf{u}$ of the operator $\hat{A}u$, applying the Neumann boundary conditions to get the first and last rows.

(c) Show that your matrix is symmetric negative semidefinite, like $\hat{A}$.