18.303 Problem Set 1

Due Friday, 16 September 2017.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

(a) Show that if A is real-symmetric and invertible, then A^{-1} is real-symmetric too.

(b) In 18.06, you were shown a simple (2–3 line) proof that the eigenvalues λ of A (solutions of $A\mathbf{x} = \lambda\mathbf{x}$) must be real numbers if A is real-symmetric (if you've forgot it, look it up), eigenvectors of distinct λ must be orthogonal. Now consider the eigenproblem $B^{-1}A\mathbf{x} = \lambda\mathbf{x}$ where A and B are both real-symmetric and B is positive-definite. In this class, we denote "dot products" by $\langle \mathbf{x}, \mathbf{y} \rangle$, where the familiar dot product of column vectors is $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^*\mathbf{y}$ (recall that $*$ denotes conjugate-transpose) . Consider a modified "dot product" $\langle \mathbf{x}, \mathbf{y} \rangle_B = \mathbf{x}^* B\mathbf{y}$ of vectors $\mathbf{x}$ and $\mathbf{y}$. Using this new dot product, adapt your 18.06 proof to show that the eigenvalues of $B^{-1}A$ are also real, and its eigenvectors of distinct λ are orthogonal under this new dot product.

(c) Suppose that a vector y is in the left nullspace of A, i.e. $A^*y = 0$. Show that, for any solution $x(t)$ to $\frac{dx}{dt} = Ax$, the quantity y^*x is constant in time ("conserved").

Problem 2: 18.06 + Julia warmup

Note: For computational (Julia-based) homework problems in 18.303, turn in with your solutions a printout of any commands used and their results (please edit out extraneous/irrelevant stuff); alternatively, you can **email your notebook (.ipynb) file** to the grader seturton@mit.edu. Because IJulia notebooks let you combine code, plots, headings, and formatted text, it should be straighforward to turn in well-documented solutions. (Alternatively, you can use Matlab or Python or whatever, but then you are on your own when it comes to writing your code.)

Using Julia, construct a random 5×5 real-symmetric matrix A with the command `A = full(Symmetric(randn(5,5)))`.

(a) The eigenvalues of A are printed by `eigvals(A)`. The `Q,R = qr(A)` function computes the QR factorization $A = QR$ Show that `Q,R = qr(A); eigvals(R*Q)` gives the same eigenvalues. i.e. the matrices A and RQ are *similar*. (You may instead want to use `sort(eigvals(A))` to sort the eigenvalues, since eigenvalues are returned in random order.) Prove that A and RQ are similar. (Hint: the QR factorization constructs an orthonormal basis of the columns of A, so that Q is unitary, i.e. $Q^* = Q^{-1}$.)

(b) Make a copy `B = A`, and repeatedly execute the command `Q,R = qr(B); B = R*Q`. This constructs the QR factorization $B = QR$ and then sets B to the new matrix RQ. To execute this process 1000 times, you can do `for i = 1:1000; Q,R = qr(B); B = R*Q; end`. After many iterations, compare the matrix B to the eigenvalues of A (computed above). What do you notice? (You should observe a remarkable fact ... this fact is the basis for the "QR algorithm" to compute eigenvalues of matrices.)

Problem 3: Quasiperiodic boundary conditions

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a small variation on this:

Suppose that we we change the boundary conditions to the *quasiperiodic* boundary condition $u(0) = e^{i\phi}u(L)$ for some real number ϕ.

(a) What are the eigenfunctions and eigenvalues of $\frac{d^2}{dx^2}$ now?

(b) For what values of ϕ will Poisson's equation *not* have unique solutions, and why?

(c) Under what conditions (if any) on $f(x)$ and ϕ would a solution exist? (You can restrict yourself to f with a convergent Fourier series.)