18.303 Problem Set 1

Due Friday, 12 September 2014.

Note: For computational (Julia-based) homework problems in 18.303, turn in with your solutions a printout of any commands used and their results (please edit out extraneous/irrelevant stuff), and a printout of any graphs requested; alternatively, you can **email your notebook (.ipynb) file** to the grader cjfan@math.mit.edu. **Always label** the axes of your graphs (with the `xlabel` and `ylabel` commands), add a title with the `title` command, and add a legend (if there are multiple curves) with the `legend` command. (Labelling graphs is a good habit to acquire.) Because IJulia notebooks let you combine code, plots, headings, and formatted text, it should be straighforward to turn in well-documented solutions.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

(a) Suppose that B is a Hermitian positive-definite matrix. Show that there is a unique matrix $\sqrt{B}$ which is Hermitian positive-definite and has the property $(\sqrt{B})^2 = B$. (Hint: use the diagonalization of B.)

(b) Suppose that A and B are Hermitian matrices and that B is positive-definite.

  (i) Show that $B^{-1}A$ is *similar* (in the 18.06 sense) to a Hermitian matrix. (Hint: use your answer from above.)

  (ii) What does this tell you about the eigenvalues λ of $B^{-1}A$, i.e. the solutions of $B^{-1}A\mathbf{x} = \lambda\mathbf{x}$?

  (iii) Are the eigenvectors $\mathbf{x}$ orthogonal?

  (iv) In Julia, make a random 5×5 real-symmetric matrix via `A=rand(5,5); A = A+A'` and a random 5×5 positive-definite matrix via `B = rand(5,5); B = B'*B` ... then check that the eigenvalues of $B^{-1}A$ match your expectations from above via `lambda,X = eigvals(B\A)` (this will give an array `lambda` of the eigenvalues and a matrix `X` whose columns are the eigenvectors).

  (v) Using your Julia result, what happens if you compute $C = X^T BX$ via `C=X'*B*X`? You should notice that the matrix C is very special in some way. Show that the elements C_{ij} of C are a kind of "dot product" of the eigenvectors i and j, but with a factor of B in the middle of the dot product.

(c) The solutions $y(t)$ of the ODE $y'' - 2y' - cy = 0$ are of the form $y(t) = C_1 e^{(1+\sqrt{1+c})t} + C_2 e^{(1-\sqrt{1+c})t}$ for some constants C_1 and C_2 determined by the initial conditions. Suppose that A is a real-symmetric 4×4 matrix with eigenvalues $3, 8, 15, 24$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4$, respectively.

  (i) If $\mathbf{x}(t)$ solves the system of ODEs $\frac{d^2}{dt^2}\mathbf{x} - 2\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \mathbf{a}_0$ and $\mathbf{x}'(0) = \mathbf{b}_0$, write down the solution $\mathbf{x}(t)$ as a closed-form expression (no matrix inverses or exponentials) in terms of the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4$ and $\mathbf{a}_0$ and $\mathbf{b}_0$. [Hint: expand $\mathbf{x}(t)$ in the basis of the eigenvectors with unknown coefficients $c_1(t), \ldots, c_4(t)$, then plug into the ODE and solve for each coefficient using the fact that the eigenvectors are _________.]

  (ii) After a long time $t \gg 0$, what do you expect the approximate form of the solution to be?

Problem 2: Les Poisson, les Poisson

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

(a) Suppose that we we change the boundary conditions to the *periodic* boundary condition $u(0) = u(L)$.

(i) What are the eigenfunctions of $\frac{d^2}{dx^2}$ now?

(ii) Will Poisson's equation have unique solutions? Why or why not?

(iii) Under what conditions (if any) on $f(x)$ would a solution exist? (You can restrict yourself to f with a convergent Fourier series.)

(b) If we instead consider $\frac{d^2}{dx^2}v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = v(L) + 1$, do these functions form a vector space? Why or why not?

(c) Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2}u(x) = f(x)$ problem with $u(0) = u(L)$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions q and r. (Transforming a new problem into an old, solved one is always a useful thing to do!)

Problem 3: Finite-difference approximations

For this question, you may find it helpful to refer to the notes and reading from lecture 3. Consider a finite-difference approximation of the form:

$$u'(x) \approx \frac{-u(x + 2\Delta x) + c \cdot u(x + \Delta x) - c \cdot u(x - \Delta x) + u(x - 2\Delta x)}{d \cdot \Delta x}.$$

(a) Substituting the Taylor series for $u(x+\Delta x)$ etcetera (assuming u is a smooth function with a convergent Taylor series, blah blah), show that by an appropriate choice of the constants c and d you can make this approximation *fourth-order accurate*: that is, the errors are proportional to $(\Delta x)^4$ for small Δx.

(b) Check your answer to the previous part by numerically computing $u'(1)$ for $u(x) = \sin(x)$, as a function of Δx, exactly as in the handout from class (refer to the notebook posted in lecture 3 for the relevant Julia commands, and adapt them as needed). Verify from your log-log plot of the |errors| versus Δx that you obtained the expected fourth-order accuracy.