18.303 Problem Set 4

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May 6, 2019

Due Monday, May 20th

1 Hermitian Green's functions

Recall that if $\hat{A}$ is Hermitian for some $\langle u, v\rangle$ and is invertible, then $\hat{A}^{-1}$ is also Hermitian; the proof is essentially identical to the one for matrices in pset 1.

(a) Suppose that $u = \hat{A}^{-1}f$ can be expressed in terms of a Green's function $G(\mathbf{x}, \mathbf{x}')$, i.e. $\hat{A}^{-1}f(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')d^n\mathbf{x}'$. If $\hat{A}^{-1}$ is Hermitian under the usual inner product $\langle u, v\rangle = \int \bar{u}v$, show that you can relate $G(\mathbf{x}, \mathbf{x}')$ to $G(\mathbf{x}', \mathbf{x})$. (This relationship is sometimes called *reciprocity.*)

(b) Consider a diffusion process where u is concentration and $f(\mathbf{x})$ is a source/sink (concentration per time added/removed), related by the diffusion equation $\frac{\partial u}{\partial t} = \nabla\cdot D\nabla u + f$, where $D(\mathbf{x}) > 0$ is a diffusivity constant of different materials at different points in space. In steady state, u solves $-\nabla\cdot D\nabla u = f$, and we showed in class that $\hat{A} = -\nabla\cdot D\nabla$ is Hermitian and invertible (positive-definite) for Dirichlet boundary conditions. Using your reciprocity relationship from the previous part, explain why there is no possible arrangement of materials [no possible $D(\mathbf{x})$] that allows "one-way diffusion" — diffusion from a source at $\mathbf{x}'$ to a point $\mathbf{x}$ but not vice-versa.

2 Probabilistic PDE Solvers

Use the Forward-Kolmogorov relationship to solve

$$u_t = -\mu u_x + \frac{1}{2}\sigma^2 u_{xx}$$

on $t \in [0, 1]$ with $u_0 = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)$, using individual trajectories of the stochastic differential equation:

$$dX_t = \mu dt + \sigma dW_t$$

computed by the Euler-Maruyama method with $\Delta t = 0.01$ where $\mu = \frac{1}{2}$ and $\sigma = \frac{5}{4}$. Show the solution of $u(1, x)$ via a histogram or kernal density estimate (KDE). You may want to check out KernelDensity.jl and StatPlots.jl.

Compare this solution against a forward-time centered-space with upwinding on the advection term discretization of the PDE, where $\Delta x = 0.1$ and $\Delta t = 0.00001$ on $x \in [-20, 20]$. Show that the two solutions coincide with one another by plotting the two together.