

18.303 Problem Set 4

Due Friday, 9 October 2015.

Problem 1: Adjoints in higher dimensions

- (a) Suppose that we have an *anisotropic* wave equation in d dimensions, $\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho(\mathbf{x})} \nabla \cdot C(\mathbf{x}) \nabla u$, where $\rho(\mathbf{x})$ is a scalar density (real, > 0) and $C(\mathbf{x})$ is a $d \times d$ matrix: a “stiffness” that depends on direction. (e.g. think of a stretched drum in 2 dimensions, made of a material that is easier to stretch in one direction than in the other). Suppose we have Dirichlet boundary conditions $u|_{\partial\Omega} = 0$. If C is a real-symmetric positive-definite matrix at every \mathbf{x} , show that we still expect a superposition of “normal” (orthogonal) oscillating modes (real frequencies) by choosing the correct inner product and analyzing the operators, similar to class.
- (b) Consider 2-component vector fields $\mathbf{u}(\mathbf{x})$ in two dimensions, under the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{u}^* \mathbf{v}$ for some domain Ω (where $*$ applied to a vector is just the conjugate transpose). Give an example of boundary conditions under which the operator

$$\hat{A}\mathbf{u} = \nabla^2 \mathbf{u} + \nabla(\nabla \cdot \mathbf{u}) = \begin{pmatrix} \nabla^2 u_x + \frac{\partial}{\partial x} \nabla \cdot \mathbf{u} \\ \nabla^2 u_y + \frac{\partial}{\partial y} \nabla \cdot \mathbf{u} \end{pmatrix}$$

is self-adjoint under this inner product. (This operator shows up, for example, as part of the Lamé–Navier equations of elastic solids.)

Problem 2: Maxwell

The key two Maxwell equations for the electric field \mathbf{E} and the magnetic field \mathbf{B} are $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{B} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$, where $\epsilon(\mathbf{x})$ is the dielectric permittivity (the square of the index of refraction) and \mathbf{J} is the electric current density. If we look for time-harmonic fields $\mathbf{E} = \mathbf{E}(\mathbf{x})e^{-i\omega t}$ and $\mathbf{B} = \mathbf{B}(\mathbf{x})e^{-i\omega t}$ for some frequency ω [with time-harmonic current $\mathbf{J}(\mathbf{x})e^{-i\omega t}$], then the equations simplify to equations of space (\mathbf{x}) alone: $\nabla \times \mathbf{E} = i\omega \mathbf{B}$ and $\nabla \times \mathbf{B} = -i\omega \epsilon \mathbf{E} + \mathbf{J}$

- (a) Show that, for $\omega \neq 0$, the other two Maxwell equations $\nabla \cdot \epsilon \mathbf{E} = 0$ (Gauss’ law in matter) and $\nabla \cdot \mathbf{B} = 0$ are redundant with the curl equations above.
- (b) If we set $\mathbf{J} = 0$ (no current sources), show that we can find an eigenproblem $\hat{A}\mathbf{E} = \lambda \mathbf{E}$ in terms of \mathbf{E} alone. Give \hat{A} and λ .
- (c) Give possible boundary conditions and an inner product such that $\hat{A} = \hat{A}^*$, assuming $\epsilon(\mathbf{x})$ is real and positive. (Dirichlet boundaries are okay, but you don’t need to make *all* the components of \mathbf{E} zero at the boundaries!)
- (d) Show that you also have $\hat{A} \geq 0$ (positive semidefinite), i.e. $\langle \mathbf{E}, \hat{A}\mathbf{E} \rangle \geq 0$ for all \mathbf{E} . Hence $\lambda \geq 0$ and ω is _____.

Problem 3: Bessels

In class, we solved for the eigenfunctions of ∇^2 in two dimensions, in a cylindrical region $r \in [0, R]$, $\theta \in [0, 2\pi]$ using separation of variables, and obtained Bessel’s equation and Bessel-function solutions. Although Bessel’s equation has two solutions $J_m(kr)$ and $Y_m(kr)$ (the *Bessel functions*), the second solution (Y_m) blows up as $r \rightarrow 0$ and so for that problem we could only have $J_m(kr)$ solutions (although we still needed to solve a transcendental equation to obtain k). See also the “Julia Bessel-function notebook” from lecture 10.

In this problem, you will solve for the 2d eigenfunctions of ∇^2 in an **annular** region Ω that does *not contain the origin*, as depicted schematically in Fig. 1, between radii R_1 and R_2 , so that

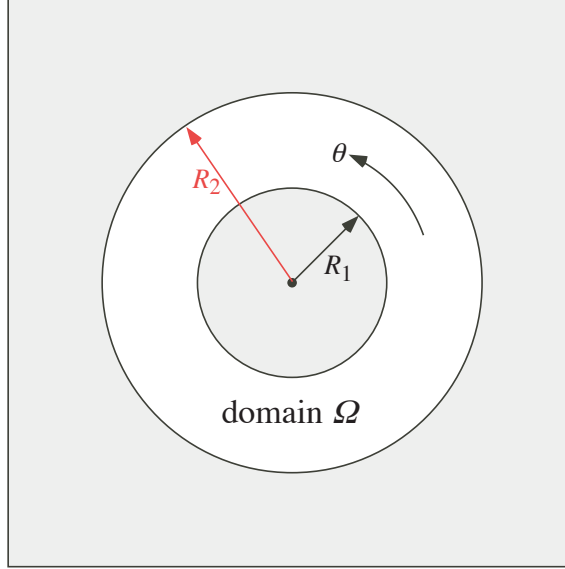


Figure 1: Schematic of the domain Ω for problem 3: an annular region in two dimensions, with radii $r \in [R_1, R_2]$ and angles $\theta \in [0, 2\pi]$.

you will need *both* the J_m and Y_m solutions. Exactly as in class, the separation of variables ansatz $u(r, \theta) = \rho(r)\tau(\theta)$ leads to functions $\tau(\theta)$ spanned by $\sin(m\theta)$ and $\cos(m\theta)$ for integers m , and functions $\rho(r)$ that satisfy Bessel's equation. Thus, the eigenfunctions are of the form:

$$u(r, \theta) = [\alpha J_m(kr) + \beta Y_m(kr)] \times [A \cos(m\theta) + B \sin(m\theta)]$$

for arbitrary constants A and B , for integers $m = 0, 1, 2, \dots$, and for constants α , β , and k to be determined.

For fun, we will also **change the boundary conditions** somewhat. We will impose “Neumann” boundary condition $\frac{\partial u}{\partial r} = 0$ at R_1 and R_2 . That is, for a function $u(r, \theta)$ in cylindrical coordinates, $\frac{\partial u}{\partial r}|_{r=R_1, R_2} = 0$. The following *exact* identities for the derivatives of the Bessel functions will be helpful:

$$J'_m(x) = \frac{J_{m-1}(x) - J_{m+1}(x)}{2}, \quad Y'_m(x) = \frac{Y_{m-1}(x) - Y_{m+1}(x)}{2}$$

- (a) Using the boundary conditions, write down two equations for α , β and k , of the form $E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ for some 2×2 matrix E . This only has a solution when $\det E = 0$, and from this fact obtain a single equation for k of the form $f_m(k) = 0$ for some function f_m that depends on m . This is a transcendental equation; you can't solve it by hand for k . In terms of k (which is still unknown), write down a possible expression for α and β , i.e. a basis for $N(E)$.
- (b) Assuming $R_1 = 1$, $R_2 = 2$, plot your function $f_m(k)$ versus $k \in [0, 20]$ for $m = 0, 1, 2$. Note that Julia provides the Bessel functions built-in: $J_m(x)$ is `besselj(m,x)` and $Y_m(x)$ is `bessely(m,x)`. You can plot a function with the `plot` command. See the IJulia notebook posted on the course web page for lecture 8 for some examples of plotting and finding roots in Julia.
- (c) For $m = 0$, find the first three (smallest $k > 0$) solutions k_1 , k_2 , and k_3 to $f_0(k) = 0$. Get a rough estimate first from your graph (zooming if necessary), and then get an accurate answer

by calling the `scipy.optimize.newton` function as in pset 1, and also as illustrated in the lecture-8 IJulia notebook. (Note that there is *also* a $k = 0$ eigenfunction for $m = 0$, corresponding to the *constant* function: the nullspace of \hat{A} with Neumann boundary conditions, as in class.)

- (d) Because ∇^2 is self-adjoint under $\langle u, v \rangle = \int_{\Omega} \bar{u}v$ (we showed in class, in general Ω , that this is still true with these boundary conditions), we know that the eigenfunctions must be orthogonal. From class, this implies that the radial parts must also be orthogonal when integrated via $\int r dr$. Check that your Bessel solutions for k_1 and k_2 are indeed orthogonal, by numerically integrating their product via the `quadgk` function as in the IJulia notebook from class.