18.303 Problem Set 1 Solutions

Problem 1: 18.06 warmup (5+10+5 points)

Here are a few questions that you should be able to answer based only on 18.06:

(a) Recall that $(AB)^T = B^T A^T$. Then $I = I^T = (AA^{-1})^T = (A^{-1})^T A^T$, hence $(A^T)^{-1} = (A^{-1})^T$.

(b) Suppose $B^{-1}A\mathbf{x} = \lambda\mathbf{x}$ where $A = A^*$ and $B = B^* \succ 0$ (positive-definite), then

$$\begin{aligned}\langle \mathbf{x}, B^{-1}A\mathbf{x}\rangle_B &= \langle \mathbf{x}, \lambda\mathbf{x}\rangle_B = \lambda(\mathbf{x}^* B\mathbf{x}) \\ &= \mathbf{x}^* B(B^{-1}A\mathbf{x}) = \mathbf{x}^* AB^{-1}B\mathbf{x} = (B^{-1}A\mathbf{x})^* B\mathbf{x} = \bar{\lambda}(\mathbf{x}^* B\mathbf{x}),\end{aligned}$$

where in the second-to-last step we used the facts that $A^* = A$ and $(B^{-1})^* = B^{-1}$ [from part (a)]. Since $B \succ 0$ and $\mathbf{x} \neq 0$ (eigenvectors are not zero), $\mathbf{x}^* B\mathbf{x} > 0$, and we can divide by it to get $\lambda = \bar{\lambda}$, hence λ is real.

If we have a second eigenvector $B^{-1}A\mathbf{y} = \mu\mathbf{y}$ with $\lambda \neq \mu$, then we just take the dot product of both sides with $\mathbf{x}$ to obtain:

$$\begin{aligned}\langle \mathbf{x}, B^{-1}A\mathbf{y}\rangle_B &= \langle \mathbf{x}, \mu\mathbf{y}\rangle_B = \mu(\mathbf{x}^* B\mathbf{y}) \\ &= \mathbf{x}^* B(B^{-1}A\mathbf{y}) = \mathbf{x}^* AB^{-1}B\mathbf{y} = (B^{-1}A\mathbf{x})^* B\mathbf{y} = \lambda(\mathbf{x}^* B\mathbf{y}),\end{aligned}$$

and hence $(\mu - \lambda)(\mathbf{x}^* B\mathbf{y}) = 0$, which implies $\mathbf{x}^* B\mathbf{y} = \langle \mathbf{x}, \mathbf{y}\rangle_B = 0$, i.e. they are orthogonal under this inner product.

(c) (You have to assume that A does not depend on t for this problem.) Since y is a constant vector (independent of t),

$$\frac{d}{dt}(y^* x) = y^* \frac{dx}{dt} = y^* Ax = (A^* y)^* x = 0,$$

hence $y^* x$ is conserved.

Problem 2: 18.06 + Julia warmup (10+10 points)

See the solutions notebook.

Problem 3: Quasiperiodic boundary conditions (10+5+10 points)

We consider $\frac{d^2}{dx^2}u(x) = f(x)$ with *quasiperiodic* boundary condition $u(0) = e^{i\phi}u(L)$ for some real number ϕ. Erratum: I really should have specified $u'(0) = e^{i\phi}u'(L)$ as well.

(a) As in class, the eigenfunctions of second derivatives are sines, cosines, and exponentials. If you go through the three different possibilities, you end up trying $u(x) = e^{ikx}$, which have eigenvalue $-k^2$ since $u'' = -k^2 u$. The only question is whether we can choose k to satisfy the boundary conditions. (You could also do $e^{\alpha x}$, in which case you would find an imaginary α.) From $u(0) = 1 = e^{i\phi}u(L) = e^{ikL + i\phi}$, we immediately find that $k = -\phi/L$ works. But this is only one eigenvalue... where are the others? The trick[1] is that if we add any integer multiple of $2\pi/L$ to k, it doesn't change the value of e^{ikL}. So, the allowed solutions are

$$u_n(x) = e^{i(\frac{2\pi n}{L} - \frac{\phi}{L})x}$$

[1] We could also say that $e^{ikL} = e^{-i\phi}$, so $k = \log(e^{-i\phi})/iL$, and then remember that the complex logarithm is multi-valued (you can add any multiplie of $2\pi i$ to the result).

for any integer n (including $n \leq 0$), and the eigenvalues are $\lambda_n = -\frac{(2\pi n-\phi)^2}{L^2}$.

If you also got $\sin(n\pi x/L)$, this is my fault for not specifying the quasiperiodicity of u' as well as of u.

(b) We don't have unique solutions when there is a nontrivial nullspace, which corresponds to having one or more zero eigenvalues. From above, that only occurs for $\phi = 2\pi n$, in which case $\lambda_n = 0$.

(c) As in class, the key to doing problems like this is to expand the right-hand-side in the basis of the eigenfunctions, i.e., try to write

$$f(x) = \sum_n c_n e^{i(\frac{2\pi n}{L} - \frac{\phi}{L})x}$$

for some coefficients c_n. But if we move the ϕ factor to the left-hand side, this is *exactly* the same as an ordinary Fourier series for

$$f(x)e^{i\phi x/L} = \sum_n c_n e^{i\frac{2\pi n}{L}x},$$

which converges (almost everywhere), from class, whenever $\int |f(x)e^{i\phi x/L}|^{1+\epsilon} = \int |f(x)|^{1+\epsilon} < \infty$ for some $\epsilon > 0$; thanks to the absolute values, this is equivalent to the condition for $f(x)$ to have a convergent Fourier series. So, we can do this expansion.

In terms of this expansion, formally writing $u(x) = \hat{A}^{-1}f(x)$, we can just divide each term in the series by the corresponding eigenvalue, as in class, to obtain a solution:

$$u(x) = \sum_n \frac{c_n}{\lambda_n} e^{i(\frac{2\pi n}{L} - \frac{\phi}{L})x}.$$

The only case in which this would *not* work is if we divided by zero, i.e. some $\lambda_n = 0$. In that case, there is no solution *unless* we also have $c_n = 0$: if that term is missing from the $f(x)$ series, we never divide by zero.

So, in summary, if $f(x)$ has a convergent Fourier series, a solution exists if $\phi \neq 2\pi n$ for any n, *or* if $\phi = 2\pi n$ *and* we have

$$c_n = 0 = \frac{1}{L}\int_0^L f(x)e^{-i(\frac{2\pi n}{L} - \frac{\phi}{L})x}dx,$$

where for the last expression I used the explicit formula for the Fourier series coefficients of $f(x)e^{i\phi x/L}$ (or, later on, we will equivalently think of this as arising from the orthogonality of eigenfunctions of a Hermitian operator).