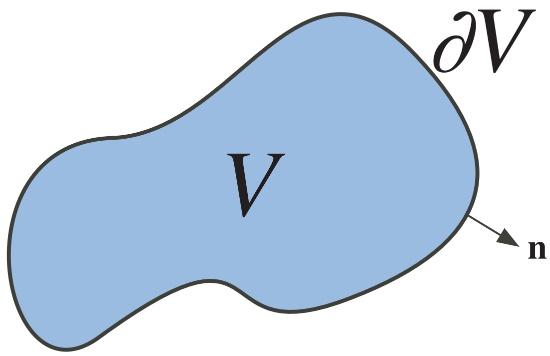

Figure 1: A volume V with a surface ∂V, and an outward unit normal vector $\mathbf{n}$ at each point on ∂V.

18.303 Problem Set 7

Due Monday, 31 October 2013.

Problem 1: Distributions

In class, we only looked explicitly at 1d distributions, but a distribution in d dimensions $\mathbb{R}^d$ can obviously be defined similarly, as (continuous linear) maps $f\{\phi\}$ from smooth (infinitely differentiable) localized (compact supoport) functions $\phi(\mathbf{x})$ to numbers. Analogous to class, define the distributional gradient ∇f by $\nabla f\{\phi\} = f\{-\nabla\phi\}$, i.e. the vector whose components are $f\{-\partial\phi/\partial x_i\}$ for $i = 1, \ldots, d$.

Consider some finite volume V with a surface ∂V, and assume ∂V is differentiable so that at each point it has an outward-pointing unit normal vector $\mathbf{n}$, as shown in figure 1. Define a "surface delta function" $\delta(\partial V)\{\phi\} = \oint_{\partial V} \phi(\mathbf{x}) d^{d-1}\mathbf{x}$ to give the surface integral $\oint_{\partial V}$ of the test function.

Suppose we have a regular distribution $f\{\phi\}$ defined by the function $f(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \mathbf{x} \in V \\ f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases}$, where we may have a discontinuity $f_2 - f_1 \neq 0$ at ∂V.

(a) Show that the distributional gradient of f is

$$\nabla f = \delta(\partial V)\left[f_2(\mathbf{x}) - f_1(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) + \begin{cases} \nabla f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases},$$

where the second term is a regular distribution given by the ordinary 18.02 gradient of f_1 and f_2 (assumed to be differentiable), while the first term is the singular distribution

$$\delta(\partial V)\left[f_1(\mathbf{x}) - f_2(\mathbf{x})\right]\mathbf{n}(\mathbf{x})\{\phi\} = \oint_{\partial V} \left[f_1(\mathbf{x}) - f_2(\mathbf{x})\right]\mathbf{n}(\mathbf{x})\phi(\mathbf{x}) d^{d-1}\mathbf{x}.$$

You can use the integral identity that $\int_V \nabla\psi d^d\mathbf{x} = \oint_{\partial V} \psi\mathbf{n} d^{d-1}\mathbf{x}$ to help you integrate by parts.

(b) Defining $\nabla^2 f\{\phi\} = f\{\nabla^2\phi\}$, derive a similar expression to the above for $\nabla^2 f$. Note that you should have one term from the discontinuity $f_1 - f_2$, and another term from the discontinuity $\nabla f_1 - \nabla f_2$. (Recall how we integrated ∇^2 by parts in class some time ago.)

Problem 2: Green's functions

Consider Green's functions of the self-adjoint indefinite operator $\hat{A} = -\nabla^2 - \omega^2$ ($\kappa > 0$) over all space ($\Omega = \mathbb{R}^3$ in 3d), with solutions that $\to 0$ at infinity. (This is related to the previous problem.) As in class, thanks to the translational and rotational invariance of this problem, we can find $G(\mathbf{x}, \mathbf{x}') = g(|\mathbf{x} - \mathbf{x}'|)$ for some $g(r)$ in spherical coordinates.

(a) Solve for $g(r)$ in 3d, similar to the procedure in class.

(i) Similar to the case of $\hat{A} = -\nabla^2$ in class, first solve for $g(r)$ for $r > 0$, and write $g(r) = \lim_{\epsilon\to 0^+} f_\epsilon(r)$ where $f_\epsilon(r) = 0$ for $r \leq \epsilon$. [Hint: although Wikipedia writes the spherical $\nabla^2 g(r)$ as $\frac{1}{r^2}(r^2 g')'$, it may be more convenient to write it equivalently as $\nabla^2 g = \frac{1}{r}(rg)''$, as in class, and to solve for $h(r) = rg(r)$ first. Hint: if you get sines and cosines from this differential equation, it will probably be easier to use complex exponentials, e.g. $e^{i\omega r}$, instead.]

(ii) In the previous part, you should find two solutions, both of which go to zero at infinity. To choose between them, remember that this operator arose from a $e^{-i\omega t}$ time dependence. Plug in this time dependence and impose an "outgoing wave" boundary condition (also called a Sommerfield or radiation boundary condition): require that waves be traveling *outward* far away, not *inward*.

(iii) Then, evaluate $\hat{A}g = \delta(\mathbf{x})$ in the distributional sense: $(\hat{A}g)\{q\} = g\{\hat{A}q\} = q(0)$ for an arbitrary (smooth, localized) test function $q(\mathbf{x})$ to solve for the unknown constants in $g(r)$. [Hint: when evaluating $g\{\hat{A}q\}$, you may need to integrate by parts on the radial-derivative term of $\nabla^2 q$; don't forget the boundary term(s).]

(b) Check that the $\omega \to 0^+$ limit gives the answer from class for the Green's function of $-\nabla^2$.