18.303 Midterm Solutions, Fall 2013

Problem 1: (7+8+(5+10) = 30 points)

(a) Consider an eigensolution $\hat{A}u = \lambda u$. Then $\Re\langle u, \hat{A}u\rangle = \Re\langle u, \lambda u\rangle = \Re\left(\lambda\langle u, u\rangle\right) = \langle u, u\rangle\Re\lambda < 0$, hence $\Re\lambda < 0$ (since $\langle u, u\rangle$ is real and positive).

(b) From the properties of inner products, $\Re\langle u, \hat{A}u\rangle = \frac{\langle u,\hat{A}u\rangle + \overline{\langle u,\hat{A}u\rangle}}{2} = \frac{\langle u,\hat{A}u\rangle + \langle \hat{A}u,u\rangle}{2} = \frac{\langle u,\hat{A}u\rangle + \langle u,\hat{A}^*u\rangle}{2} = \frac{\langle u,(\hat{A}+\hat{A}^*)u\rangle}{2} < 0$ for all $u \neq 0$, and hence $\hat{A} + \hat{A}^*$ is negative definite according to the definition from class, and conversely if $\hat{A} + \hat{A}^*$ is negative definite then $\Re\langle u, \hat{A}u\rangle < 0$.

(c) Consider the system of equations $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} - \alpha u$ and $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x} - \beta v$ for some $\alpha(x)$ and $\beta(x)$, for $\Omega = [0, L]$ with Dirichlet boundary conditions $u(0) = u(L) = 0$.

(i) Similar to class, in terms of $\mathbf{w} = \begin{pmatrix} u \\ v \end{pmatrix}$, we have

$$\frac{\partial \mathbf{w}}{\partial t} = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -\alpha & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\beta \end{pmatrix}\mathbf{w} = \hat{A}\mathbf{w},$$

so

$$\hat{A} = \begin{pmatrix} -\alpha & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & -\beta \end{pmatrix} = \hat{D} - \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$$

where $\hat{D} = \begin{pmatrix} & \partial/\partial x \\ \partial/\partial x & \end{pmatrix} = -\hat{D}^*$ is the scalar-wave operator from class, using $\langle\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix}\rangle = \int \bar{u}u' + \bar{v}v'$.

(ii) There are a couple of ways to approach this.

We can look at eigensolutions. For an eigensolution $\hat{A}\mathbf{w} = \lambda\mathbf{w}$, this has the solution $e^{\lambda t}\mathbf{w}(t = 0)$, which is decaying if $\Re\lambda < 0$. If we assume (as usual) that we have a basis of eigenfunctions, then (from above) it is sufficient for $\hat{A} + \hat{A}^*$ to be negative-definite in order for the solutions to be exponentially decaying. Then

$$\hat{A} + \hat{A}^* = \cancel{\hat{D} + \hat{D}^*} - \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} - \begin{pmatrix} \bar{\alpha} & \\ & \bar{\beta} \end{pmatrix} = -\Re\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix},$$

(where by inspection we have seen the adjoint of α and β is just $\bar{\alpha}$ and $\bar{\beta}$). For this to be negative definite, we need $-\Re\int\left(\alpha|u|^2 + \beta|v|^2\right) < 0$ for all $u, v \neq 0$, which implies

$$\Re\alpha > 0, \quad \Re\beta > 0$$

almost everywhere.

Another way to derive the same sufficient condition is to consider $\frac{\partial}{\partial t}\langle\mathbf{w}, \mathbf{w}\rangle$ for any solution $\mathbf{w}(x, t)$ of our PDE $\frac{\partial \mathbf{w}}{\partial t} = \hat{A}\mathbf{w}$. If $\frac{\partial}{\partial t}\langle\mathbf{w}, \mathbf{w}\rangle < 0$ for all $\mathbf{w} \neq 0$, then the solution is decaying, and $\frac{\partial}{\partial t}\langle\mathbf{w}, \mathbf{w}\rangle = \cdots = \langle\mathbf{w}, (\hat{A} + \hat{A}^*)\mathbf{w}\rangle$, similar to the derivation in class of conservation of energy for wave equations, hence we arrive again at $\hat{A} + \hat{A}^* < 0$.

Problem 2: (10+10+10 = 30 points)

(a) Let u_m denote $u(m\Delta x)$, as usual. We want to be able to write $u'(0) = u'_0$ as a center difference with spacing Δx, i.e. $u'_0 = \frac{u_{0.5} - u_{-0.5}}{\Delta x}$. Hence, we need to store our unknowns at grid points $\boxed{u_{m+0.5}}$ for $m = 0, 1, 2, \ldots, N - 1$ (for N unknowns). Similarly, we should put the other boundary at N so that we can write $u'_N = \frac{u_{N+0.5} - u_{N-0.5}}{\Delta x}$. Hence $N\Delta x = L$ or $\boxed{\Delta x = L/N}$ (which is different from the $\frac{L}{N+1}$ we used with Dirichlet boundary conditions!).

(b) First, we apply the boundary conditions $u'_0 = 0$ and $u'_N = 0$ from above to obtain $u_{-0.5} = u_{+0.5}$ and $u_{N+0.5} = u_{N-0.5}$. Then we can write the second derivative, similar to class, by

$$u''_{m+0.5} = \frac{u_{m+1.5} - 2u_{m+0.5} + u_{m-0.5}}{\Delta x^2},$$

where for the first row ($m = 0$) and the last row ($m = N - 1$) of the matrix we will have

$$u''_{0.5} = \frac{u_{1.5} - 2u_{0.5} + u_{-0.5}}{\Delta x^2} = \frac{u_{1.5} - u_{0.5}}{\Delta x^2},$$

$$u''_{N-0.5} = \frac{u_{N+0.5} - 2u_{N-0.5} + u_{N-1.5}}{\Delta x^2} = \frac{-u_{N-0.5} + u_{N-1.5}}{\Delta x^2}$$

via the boundary conditions. Hence, A looks like

$$\boxed{A = \frac{1}{\Delta x^2} \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}}.$$

(c) This A is obviously real-symmetric. To show that it is definite, the easiest thing to do, similar to class, is to factorize A as the product of two first-derivative operations. Let us construct the matrix D that computes $u'_1, u'_2, \ldots, u'_{N-1}$ from $u_{0.5}, u_{1.5}, \ldots, u_{N-0.5}$. (Since $u'_0 = 0$ and $u'_N = 0$, we need not compute them.)

$$\begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{N-1} \end{pmatrix} = D\mathbf{u} = \frac{1}{\Delta x} \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} u_{0.5} \\ u_{1.5} \\ \vdots \\ u_{N-1.5} \\ u_{N-0.5} \end{pmatrix}.$$

Notice that D is $(N-1) \times N$. Similarly, to get $u''_{0.5}, u''_{1.5}, \ldots, u''_{N-0.5}$ from $u'_1, u'_2, \ldots, u'_{N-1}$, we do:

$$\begin{pmatrix} u''_{0.5} \\ u''_{1.5} \\ \vdots \\ u''_{N-1.5} \\ u''_{N-0.5} \end{pmatrix} = \frac{1}{\Delta x} \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{N-1} \end{pmatrix} = -D^T \begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{N-1} \end{pmatrix},$$

where we have used the Neumann boundary conditions for the first and last rows. Hence $A = -D^T D$, which is at least negative semidefinite. It is *not* negative definite, because it is easy to see that $N(A) = N(D)$ contains the constant vector $(1, 1, \ldots, 1, 1)^T$.