18.303 Problem Set 6

Due Monday, 26 October 2015.

Problem 1: Scalar Helmholtz equation

Recall that usual scalar wave equation [e.g. for pressure waves $u(x,t)$] is equation $\hat{A}u + f(x,t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x,t)$ is an external force density (pressure) on the string and $\hat{A} = \nabla \cdot c\nabla$ for $c(\mathbf{x}) > 0$ (e.g. c ~ springiness, or "bulk modulus").

(a) Suppose that $f(x,t) = \mathrm{Re}[g(x)e^{-i\omega t}]$, an oscillating force with a frequency ω. Show that, instead of solving the wave equation with this $f(x,t)$, we can instead use a complex force $\tilde{f}(x,t) = g(x)e^{-i\omega t}$, solve for a complex $\tilde{u}(x,t)$, and then take $u = \mathrm{Re}\,\tilde{u}$ to obtain the solution for the original $f(x,t)$.

(b) Suppose that $f(x,t) = g(x)e^{-i\omega t}$, and we want to find a *steady-state* solution $u(x,t) = v(x)e^{-i\omega t}$ that is oscillating everywhere at the same frequency as the input force. (This will be the solution after a long time if there is any dissipation in the system to allow the initial transients to die away.) Write an equation $\hat{H}v = g$ that v solves. Is $\hat{H}$ self-adjoint (e.g. for Dirichlet boundaries $u|_{\partial\Omega} = 0$)? Positive/negative definite/semidefinite?

Problem 2: Green's functions

Consider Green's functions of the self-adjoint indefinite operator $\hat{A} = -\nabla^2 - \omega^2$ ($\kappa > 0$) over all space ($\Omega = \mathbb{R}^3$ in 3d), with solutions that $\to 0$ at infinity. (This is related to the previous problem.) As in class, thanks to the translational and rotational invariance of this problem, we can find $G(\mathbf{x}, \mathbf{x}') = g(|\mathbf{x} - \mathbf{x}'|)$ for some $g(r)$ in spherical coordinates.

(a) Solve for $g(r)$ in 3d, similar to the procedure in class.

(i) Similar to the case of $\hat{A} = -\nabla^2$ in class, first solve for $g(r)$ for $r > 0$, and write $g(r) = \lim_{\epsilon\to 0^+} f_\epsilon(r)$ where $f_\epsilon(r) = 0$ for $r \le \epsilon$. [Hint: although Wikipedia writes the spherical $\nabla^2 g(r)$ as $\frac{1}{r^2}(r^2 g')'$, it may be more convenient to write it equivalently as $\nabla^2 g = \frac{1}{r}(rg)''$, as in class, and to solve for $h(r) = rg(r)$ first. Hint: if you get sines and cosines from this differential equation, it will probably be easier to use complex exponentials, e.g. $e^{i\omega r}$, instead.]

(ii) In the previous part, you should find two solutions, both of which go to zero at infinity. To choose between them, remember that this operator arose from a $e^{-i\omega t}$ time dependence. Plug in this time dependence and impose an "outgoing wave" boundary condition (also called a Sommerfield or radiation boundary condition): require that waves be traveling *outward* far away, not *inward*.

(iii) Then, evaluate $\hat{A}g = \delta(\mathbf{x})$ in the distributional sense: $(\hat{A}g)\{q\} = g\{\hat{A}q\} = q(0)$ for an arbitrary (smooth, localized) test function $q(\mathbf{x})$ to solve for the unknown constants in $g(r)$. [Hint: when evaluating $g\{\hat{A}q\}$, you may need to integrate by parts on the radial-derivative term of $\nabla^2 q$; don't forget the boundary term(s).]

(b) Check that the $\omega \to 0^+$ limit gives the answer from class for the Green's function of $-\nabla^2$.

Problem 3: Born again

Consider the application of the Born approximation in section 3 of the course notes on Green's functions in inhomogeneous media, where we approximately solved $-\nabla \cdot c\nabla u = \delta(\mathbf{x} - \mathbf{x}_0)$ where $c = c_2$ except for a small volume V far away where $c = c_1$. We found that the solution was approximately the "bare" Green's function G_0/c_2 plus a "dipole" correction, which in electrostatics you can interpret as the potential of the induced dipole moment when V is "polarized" by the electric field of the charge at $\mathbf{x}_0$.

Repeat this calculation, except use the operator $\hat{A} = -\nabla^2 - \omega(\mathbf{x})^2$, where $\omega = \omega_2$ everywhere except in a small volume V where it is ω_1. i.e. calculate the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ as in the notes, up to the first Born approximation. [This problem arises e.g. in the Schrodinger operator of quantum mechanics $\hat{A} = -\nabla^2 + V(\mathbf{x})$ for scattering of waves off a small change in the potential V.] Use your Green's function from problem 2.