18.303 Problem Set 4 Solutions

Problem 1: (5+10+10 points)

In class, we defined the *Kronecker product* $A \otimes B$ of two matrices as the matri

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where a_{ij} is the (row i, column j) entry of A. Derive the following properties of Kronecker products from this definition:

(a) We have

$$(A \otimes B)^* = \begin{pmatrix} \overline{a_{11}}B^* & \overline{a_{21}}B^* & \cdots \\ \overline{a_{12}}B^* & \overline{a_{22}}B^* & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

by swapping rows and columns of $A \otimes B$ and conjugating. By inspection, this is the same as $A^* \otimes B^*$, since the entries of A^* are

$$A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots \\ \overline{a_{12}} & \overline{a_{22}} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

(b) We have

$$(A \otimes B)(C \otimes D) = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_{11}D & c_{12}D & \cdots \\ c_{21}D & c_{22}D & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

As shown in class, when we multiply the two "block" matrices like this, we can use the ordinary "row times column" matrix-multiplication formula where we multiply blocks and add them up, i.e. the product is

$$\begin{pmatrix} \sum_{k=1}^n a_{1k}Bc_{k1}D & \sum_{k=1}^n a_{1k}Bc_{k2}D & \cdots \\ \sum_{k=1}^n a_{2k}Bc_{k2}D & \sum_{k=1}^n a_{2k}Bc_{k2}D & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where n is the number of columns of A (and rows of C). That is, the (i,j)-th block is

$$\sum_{k=1}^n a_{ik}Bc_{kj}D = \left(\sum_{k=1}^n a_{ik}c_{kj}\right) BD = (AC)_{ij}BD$$

where we have noticed that $\sum_{k=1}^n a_{ik}c_{kj}$ is simply the formula for the i,j element of AC. But this means

$$(A \otimes B)(C \otimes D) = \begin{pmatrix} (AC)_{11}BD & (AC)_{12}BD & \cdots \\ (AC)_{21}BD & (AC)_{22}BD & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = (AC) \otimes (BD).$$

Q.E.D.

(c) Consider the vector $y_n \otimes x_m$. Applying linearity and the mixed-product formula from the previous part, we have

$$\begin{aligned}(I_N \otimes A + B \otimes I_M)(y_n \otimes x_m) &= (I_N y_n) \otimes (A x_m) + (B y_n) \otimes (I_M x_m) \\ &= y_n \otimes (\lambda_m x_m) + (\mu_n y_n) \otimes x_m \\ &= (\lambda_m + \mu_n) y_n \otimes x_m,\end{aligned}$$

hence this is a "separable" eigenvector of $I_N \otimes A + B \otimes I_M$ with eigenvalue $\lambda_m + \mu_n$. There are MN of these $y_n \otimes x_m$ eigenvectors, and $I_N \otimes A + B \otimes I_M$ is $MN \times MN$, so that is all of the eigenvectors and eigenvalues.

As discussed in class, an MN-row column vector $y_n \otimes x_m$ can be thought of as a "two-dimensional $M \times N$ array" that has been written in column-major order, and the matrix $I_N \otimes A + B \otimes I_M$ can be thought of as a "two-dimensional" operator that acts with A in the the M direction and B in the N direction. If we reverse this "one-dimensionalization" process, $y_n \otimes x_m$ corresponds to the "two-dimensional array" $x_m y_m^T$, which varies like x_m in the M direction and like y_m in the N direction. This is exactly the analogue of a 2d separable PDE solution $X(x)Y(y)$ that is a product of one-dimensional functions $X(x)$ and $Y(y)$ along each direction.

Problem 2: (5+10+5+5+(5+5+5)+5 points)

Often, separability of the solutions is a consequence of symmetry. In this problem, you will show a related property for the case of discrete translational symmetry: a PDE that is invariant under rotation by $2\pi/N$. In particular, suppose that we have the circular system of N springs and masses, with identical spring constants k, depicted in Figure 1. Suppose that the equation of motion of the n-th mass is

$$m\ddot{\phi}_n = \kappa(\phi_{n+1} - \phi_n) - \kappa(\phi_n - \phi_{n-1}).$$

(a) Since $\ddot{\phi}_n = \frac{\kappa}{m}(\phi_{n+1} - 2\phi_n + \phi_{n-1})$, we can write

$$A = \frac{\kappa}{m}\begin{pmatrix} -2 & 1 & & & & & 1 \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \end{pmatrix}.$$

Note the first and last rows! This is a consequence of the periodicity of the system, since we can identify $\phi_0 = \phi_N$ and $\phi_{N+1} = \phi_1$.

(b) To check definiteness, the easiest way is to factorize A. Similar to class, we write $\ddot{\phi}_n$ in two steps: first we compute $\psi_{n+0.5} = \phi_{n+1} - \phi_n$, then we compute $\ddot{\phi}_n = \frac{\kappa}{m}(\psi_{n+0.5} - \psi_{n-0.5})$. Unlike the 1d case in class, however, there are only N values $\psi_{n+0.5}$, equal to the number of springs! Hence, we obtain an $N \times N$ matrix D given by:

$$\begin{pmatrix} \psi_{1.5} \\ \psi_{2.5} \\ \vdots \\ \psi_{N-0.5} \\ \psi_{N+0.5} \end{pmatrix} = D\mathbf{x} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix},$$

where we must be careful to get the periodicity right for the last row $\psi_{N+0.5} = \phi_1 - \phi_N$. Similarly, noting that $\ddot{\phi}_1 = \frac{\kappa}{m}(\psi_{1.5} - \psi_{N+0.5})$, we have:

$$\ddot{\mathbf{x}} = \frac{\kappa}{m} \begin{pmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} \psi_{1.5} \\ \psi_{2.5} \\ \vdots \\ \psi_{N-0.5} \\ \psi_{N+0.5} \end{pmatrix} = -\frac{\kappa}{m} D^T D\mathbf{x},$$

where we have identified that the matrix to take the differences of the $\psi_{n+0.5}$ is precisely $-D^T$. Hence, $A = -\frac{\kappa}{m} D^T D$, which by inspection is at least **negative semidefinite** (from class).

It is **not** negative-definite, however. This can be checked in a variety of ways, most easily by noticing that

$$D \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} = 0,$$

and hence D is not full-rank (and similarly for A).

(c) Multiplying RA acts R on each of the *columns* of A, i.e. it permutes each column, giving:

$$RA = \frac{\kappa}{m} \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \\ -2 & 1 & & & & & 1 \end{pmatrix}.$$

Multiplying $AR = (R^T A^T)^T = (R^T A)^T$ is equivalent to permuting each *row* of A by R^T (i.e. in the opposite direction), hence

$$R^T A = \frac{\kappa}{m} \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \\ -2 & 1 & & & & & 1 \end{pmatrix},$$

which $= RA$. Q.E.D.

(d) Consider the vector $\mathbf{y} = R\mathbf{x}$. Using $RA = AR$, we obtain: $A\mathbf{y} = AR\mathbf{x} = RA\mathbf{x} = \lambda R\mathbf{x} = \lambda\mathbf{y}$. Therefore, $\mathbf{y}$ is an eigenvector of A with eigenvalue λ. But we were told that λ has multiplicity 1: this means that $\mathbf{y}$ must be linearly dependent on $\mathbf{x}$, i.e. $\mathbf{y} = \alpha\mathbf{x}$ for some scalar α. Hence $\mathbf{y} = R\mathbf{x} = \alpha\mathbf{x}$, and $\mathbf{x}$ is an eigenvector of R with eigenvalue α. Q.E.D

(e) (i) We just write out $R\mathbf{x} = e^{ik}\mathbf{x}$:

$$R \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_N \\ 1 \end{pmatrix} = e^{ik} \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix}$$

and hence $x_2 = e^{ik}$, $x_3 = e^{ik}x_2 = e^{2ik}$, and so on, or

$$\mathbf{x} = \begin{pmatrix} 1 \\ e^{ik} \\ \vdots \\ e^{i(N-2)k} \\ e^{i(N-1)k} \end{pmatrix},$$

or more simply:

$$\boxed{x_n = e^{i(n-1)k}}.$$

(ii) On an eigenvector, $R^N\mathbf{x} = e^{iNk}\mathbf{x} = \mathbf{x}$, and hence $e^{iNk} = 1$. This means that Nk is an integer multiple of 2π, i.e. $Nk = 2\pi m$ for $m = 0, 1, 2, \ldots$, giving eigenvalues

$$\boxed{\alpha_m = e^{i\frac{2\pi m}{N}}}.$$

A little more carefully, we notice that $\alpha_N = \alpha_0$, so we have N distinct eigenvalues $\boxed{m = 0, 1, \ldots, N-1}$.

(iii) Now that we know the eigenvectors x_n, we can plug it back into $A\mathbf{x} = \lambda\mathbf{x}$. Each row of this equation has the form

$$\frac{\kappa}{m}\left(x_{n+1} - 2x_n + x_{n-1}\right) = \lambda x_n$$

and plugging in the form of $x_n = e^{ik(n-1)} = e^{ikn}e^{-ik}$ and dividing both sides by x_n gives:

$$\frac{\kappa}{m}\left(e^{ik} - 2 + e^{ik}\right) = \lambda = \frac{\kappa}{m}\left[2\cos(k) - 2\right].$$

Hence, plugging in the equation for k from above, we have:

$$\boxed{\lambda_m = \frac{2\kappa}{m}[\cos(2\pi m/N) - 1] = -\frac{4\kappa}{m}\sin^2\left(\frac{\pi m}{N}\right)}$$

for $\boxed{m = 0, 1, \ldots, N-1}$, where we have used the half-angle identity $1 - \cos(k) = 2\sin^2(k/2)$ to simplify the final expression. Note that the eigenvalues are real and ≤ 0 as expected, with exactly one zero eigenvalue $\lambda_0 = 0$.

(f) The angular difference between each mass is $\Delta\theta = \frac{2\pi}{N}$, and hence $x_n = e^{i\Delta\theta m(n-1)} = e^{im\theta}$ where we define the angle $\theta = (n-1)\Delta\theta$. Hence the eigenfunctions in the continuum limit are simply

$$\phi(\theta) = e^{im\theta}$$

for integers m (or any constant multiple thereof, of course).

Problem 3: (5+5+10 points)

(a) Given the above identity, integration by parts is straightforwards:

$$\begin{aligned}\langle \mathbf{u}, \nabla \times \mathbf{v}\rangle &= \int_\Omega \bar{\mathbf{u}} \cdot (\nabla \times \mathbf{v}) = \int_\Omega \left[\nabla \cdot (\bar{\mathbf{u}} \times \mathbf{v}) + \overline{\nabla \times \mathbf{u}} \cdot \mathbf{v}\right] \\ &= \oiint_{\partial\Omega} (\bar{\mathbf{u}} \times \mathbf{v}) \cdot d\mathbf{S} + \langle \nabla \times \mathbf{u}, \mathbf{v}\rangle,\end{aligned}$$

applying the divergence theorem in the second line. So, the surface term $\oiint_{\partial\Omega} \mathbf{w} \cdot d\mathbf{S}$ is for $\boxed{\mathbf{w} = \bar{\mathbf{u}} \times \mathbf{v}}$.

(b) If $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$, then $\mathbf{u}$ is parallel to $\mathbf{n}$ and hence $\bar{\mathbf{u}} \times \mathbf{v}$ is perpendicular to $\mathbf{n}$ and $d\mathbf{S}$. Hence the boundary term the integration by parts above vanishes, and $\langle \mathbf{u}, \nabla \times \mathbf{v}\rangle = \langle \nabla \times \mathbf{u}, \mathbf{v}\rangle$, so $\nabla\times$ is Hermitian.

(c) Taking the curl of both sides of Faraday's Law, we have

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Under the same inner product as above, we can just "integrate by parts" twice:

$$\begin{aligned}\langle \mathbf{u}, \nabla \times \nabla \times \mathbf{v}\rangle &= \int_\Omega \bar{\mathbf{u}} \cdot (\nabla \times \nabla \times \mathbf{v}) = \cancel{\oiint_{\partial\Omega} [\bar{\mathbf{u}} \times (\nabla \times \mathbf{v})] \cdot d\mathbf{S}} + \int_\Omega (\nabla \times \bar{\mathbf{u}}) \cdot (\nabla \times \mathbf{v}) \\ &= \cancel{\oiint_{\partial\Omega} [(\nabla \times \bar{\mathbf{u}}) \times \mathbf{v}] \cdot d\mathbf{S}} + \int_\Omega (\nabla \times \nabla \times \bar{\mathbf{u}}) \cdot \mathbf{v} = \langle \nabla \times \nabla \times \mathbf{u}, \mathbf{v}\rangle,\end{aligned}$$

where the boundary terms cancel as before under the boundary condition $\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0$. Hence $\nabla \times \nabla\times$ will have real eigenvalues λ. Furthermore, we can easily show that $\nabla \times \nabla\times$ is positive semidefinite, since from above

$$\langle \mathbf{u}, \nabla \times \nabla \times \mathbf{u}\rangle = \int_\Omega |\nabla \times \mathbf{u}|^2 \geq 0,$$

and hence $\lambda \geq 0$ for some real "eigenfrequencies" ω. Equivalently, we have

$$\hat{A}\mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

where $\hat{A} = -c^2 \nabla \times \nabla \times$ is Hermitian and negative semidefinite. From class, this is a **hyperbolic** equation with oscillating solutions (whose frequencies ω come from the eigenvalues $-\omega^2$ of $\hat{A}$).tals have high conductivity, and such containers are called *microwave resonant cavities*.)

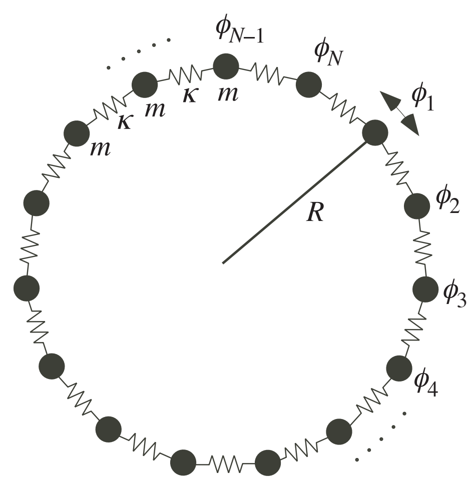

Figure 1: Circular systems of N identical masses m and springs κ. ϕ_n is the angular displacement of the n-th mass ($\phi_m = 0$ for all springs when they are at rest). Imagine that the springs can move in the ϕ direction, but cannot move in the radial direction (for example, if they are sliding without friction on the surface of a cylinder of radius R).