18.303 Problem Set 6

Due Friday, 21 October 2015.

Problem 1: Hermitian Green's functions

Recall that if $\hat{A}$ is Hermitian for some $\langle u, v\rangle$ and is invertible, then $\hat{A}^{-1}$ is also Hermitian; the proof is essentially identical to the one for matrices in pset 1.

(a) Suppose that $u = \hat{A}^{-1} f$ can be expressed in terms of a Green's function $G(\mathbf{x}, \mathbf{x}')$, i.e. $\hat{A}^{-1} f(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') d^n\mathbf{x}'$. If $\hat{A}^{-1}$ is Hermitian under the usual inner product $\langle u, v\rangle = \int \bar{u} v$, show that you can relate $G(\mathbf{x}, \mathbf{x}')$ to $G(\mathbf{x}', \mathbf{x})$. (This relationship is sometimes called *reciprocity*.)

(b) Consider a diffusion process where u is concentration and $f(\mathbf{x})$ is a source/sink (concentration per time added/removed), related by the diffusion equation $\frac{\partial u}{\partial t} = \nabla \cdot D \nabla u + f$, where $D(\mathbf{x}) > 0$ is a diffusivity constant of different materials at different points in space. In steady state, u solves $-\nabla \cdot D \nabla u = f$, and we showed in class that $\hat{A} = -\nabla \cdot D \nabla$ is Hermitian and invertible (positive-definite) for Dirichlet boundary conditions. Using your reciprocity relationship from the previous part, explain why there is no possible arrangement of materials [no possible $D(\mathbf{x})$] that allows "one-way diffusion" — diffusion from a source at $\mathbf{x}'$ to a point $\mathbf{x}$ but not vice-versa.

Problem 2: More Green stuff

Recall that the displacement $u(x, t)$ of a stretched string [with fixed ends: $u(0, t) = u(L, t) = 0$] satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} + f(x, t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x, t)$ is an external force density (pressure) on the string.

(a) Suppose that $f(x, t) = \operatorname{Re}[g(x) e^{-i\omega t}]$, an oscillating force with a frequency ω. Show that, instead of solving the wave equation with this $f(x, t)$, we can instead use a complex force $\tilde{f}(x, t) = g(x) e^{-i\omega t}$, solve for a complex $\tilde{u}(x, t)$, and then take $u = \operatorname{Re} \tilde{u}$ to obtain the solution for the original $f(x, t)$.

(b) Suppose that $f(x, t) = g(x) e^{-i\omega t}$, and we want to find a *steady-state* solution $u(x, t) = v(x) e^{-i\omega t}$ that is oscillating everywhere at the same frequency as the input force. (This will be the solution after a long time if there is any dissipation in the system to allow the initial transients to die away.) Write an equation $\hat{A} v = g$ that v solves. Is $\hat{A}$ self-adjoint? Positive/negative definite/semidefinite?

(c) Solve for the Green's function $G(x, x')$ of this $\hat{A}$, assuming that $\omega \neq n\pi/L$ for any integer n (i.e. assume ω is not an eigenfrequency [why?]). [Write down the continuity conditions that G must satisfy at $x = x'$, solve for $x \neq x'$, and then use the continuity conditions to eliminate unknowns.]