18.303 Final Topic Summary

December 13, 2010

- Eigenfunctions and eigenvalues: given the eigenfunctions/eigenvalues of $\hat{A}$, using them to solve equations such as $\hat{A}u = f$, $\hat{A}u = \partial u/\partial t$, and $\hat{A}u = \partial u^2/\partial t^2$ (or other variations).
  - Important special solvable cases: Fourier series and Bessel functions. Separation of variables in space (and time, but we view separation in time as decomposition into eigenfunctions).
- Inner products $\langle u, v\rangle$, adjoints, definiteness, and properties of $\hat{A}$. Given $\hat{A}$ and an inner product, how to find $\hat{A}^*$ and how to show whether $\hat{A}$ is self-adjoint (Hermitian or real-symmetric), positive/negative definite or semidefinite, and what the consequences of these facts are for eigenvalues and eigenfunctions or for equations in terms of $\hat{A}$ like $\hat{A}u = \partial u/\partial t$ and so on.
  - e.g. why the heat/diffusion equation has exponentially decaying solutions (and why oscillations are damped out especially rapidly).
  - understand how to integrate by parts with ∇.
- Null spaces of operators (left and right) and their consequences on solvability, uniqueness and (in the case of left nullspaces) conservation laws for $\hat{A}u = \partial u/\partial t$ (e.g. conservation of mass for difusion equation).
- Effect of boundary conditions on definiteness, self-adjointness, nullspaces. Turning general Dirichlet and Neumann boundaries into zero Dirichlet and Neumann boundary conditions (i.e. boundary conditions as modifying the right-hand-side).
- The Rayleigh quotient and the min–max (variational) theorem, and its consequences. Guessing the form of the smallest-$|\lambda|$ solutions using the min–max theorem.
- Finite-difference discretizations:
  - Analyzing the order of accuracy of a given discretization (with Taylor expansions).
  - Writing the discretized A to have the same self-adjoint/definite properties as $\hat{A}$ (e.g. by writing A in terms of $D^T D$.
  - Effect of boundary conditions.
  - Finite-difference discretizations in more than 1 dimension.
- Green's functions and inverse operators.
  - Relationship between $\hat{A}^{-1}$ and Green's function $\hat{A}G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$. (And how properties of $\hat{A}$ effect properties of G, e.g. self-adjointness gives reciprocity.)

- Solving G in simple cases (using the delta function, not using ugly limits), especially empty space $\Omega = \mathbb{R}^d$.
- How solutions $\hat{A}^{-1}f$ are made from the Green's function.
- How the Green's function G_0 of empty space relates to Green's functions in inhomogeneous systems or systems with boundaries. Born–Dyson series and Born approximations.

- Delta functions and distributions.
  - Definition, regular vs. singular distributions. Differences from ordinary functions.
  - Distributional derivatives.
  - Solving PDEs with δ on the right-hand side (e.g. finding Green's functions, above).

- Time-stepping and stability.
  - Convergence, consistency, stability (conditional and unconditional). Implicit vs. explicit schemes.
  - Von Neumann analysis. (e.g. forward, backward, and centered differences, Crank-Nicolson, and leap-frog schemes.)

- Wave equations
  - $\hat{A}u = \frac{\partial^2 u}{\partial t^2}$ vs. $\hat{D}\vec{w} = \frac{\partial \vec{w}}{\partial t}$ formulations. (e.g. $\hat{A} = \nabla^2$ vs. $\vec{w} = [u; \vec{v}]$ forms of scalar wave equations).
  - Conservation of energy from $\hat{D} = \hat{D}^*$, correct choice of inner product $\langle \vec{w}, \vec{w}' \rangle$.
  - D'Alembert's solution $f(x - ct) + g(x + ct)$. Planewave solutions $e^{i(\vec{k}\cdot\vec{x} - \omega t)}$.
  - Staggered-grid and leap-frog discretizations, and other discretizations; stability and the CFL condition $c\Delta t \leq \Delta x / \sqrt{\#\text{dimensions}}$.
  - Wave velocity: group velocity $\partial\omega/\partial k$ and phase velocity ω/k. Fourier-transform picture (from class) and energy-transport picture (from homework).
    - Wave dispersion (ω-dependent velocity): numerical, material, and geometric.
  - Waveguides:
    - Separable eigenfunctions $u_k e^{i(kz-\omega t)}$, the reduced eigenproblem $\hat{A}_k$ or $\hat{D}_k$, and the dispersion relation $\omega(k)$.
    - Hard-wall waveguides (e.g. a hollow pipe with Dirichlet or Neumann boundaries), and the corresponding $\omega(k)$ dispersion relation.
    - Snell's law, total internal reflection evanescent waves, and "slow-light" waveguides from low-c regions. The light cone and dispersion relations.
  - The frequency-domain problem: a source $\sim e^{-i\omega t}$ and the resulting steady-state solution $\sim e^{-i\omega t}$.
    - Scalar waves: the Helmholtz equation $(\nabla^2 + \omega^2)u = f$. The corresponding Green's function $\sim e^{i\omega r/c}/r$ in 3d, $\sim e^{i\omega|x|}$ in 1d, etcetera.