18.303 Midterm Exam Solutions, Fall 2016

November 9, 2016

Problem 1: Hermitian (33 points)

We just choose the inner product

$$\langle \mathbf{F}, \mathbf{G} \rangle_\varepsilon = \int \overline{\mathbf{F}} \cdot \varepsilon \mathbf{G}$$

in order to cancel the ε^{-1} factor in $\hat{A}$. (Note that this is a valid inner product since $\varepsilon > 0$, much like the weighted inner product we used for $c\nabla^2$ in class.) Then

$$\begin{aligned}
\langle \mathbf{E}, \hat{A}\mathbf{E}' \rangle_\varepsilon &= \langle \mathbf{E}, \nabla \times \mu^{-1} \nabla \times \mathbf{E}' \rangle = \langle \nabla \times \mathbf{E}, \mu^{-1} \nabla \times \mathbf{E}' \rangle \\
&= \langle \mu^{-1} \nabla \times \mathbf{E}, \nabla \times \mathbf{E}' \rangle = \langle \nabla \times \mu^{-1} \nabla \times \mathbf{E}, \mathbf{E}' \rangle \\
&= \langle \varepsilon^{-1} \nabla \times \mu^{-1} \nabla \times \mathbf{E}, \mathbf{E}' \rangle_\varepsilon = \langle \hat{A}\mathbf{E}, \mathbf{E}' \rangle_\varepsilon,
\end{aligned}$$

where we have integrated by parts twice with $\nabla\times$, using the identity from homework, assuming we have boundary conditions such that the boundary terms vanish as in homework. Hence $\hat{A} = \hat{A}^*$. To check definiteness, we just look at the "middle" step from the end of the first line to see that

$$\langle \mathbf{E}, \hat{A}\mathbf{E} \rangle_\varepsilon = \langle \nabla \times \mathbf{E}, \mu^{-1} \nabla \times \mathbf{E} \rangle = \int_\Omega \mu^{-1} \left| \nabla \times \mathbf{E} \right|^2 \geq 0$$

since $\mu > 0$. Hence $\hat{A} = \hat{A}^* \succeq 0$ and we will obtain oscillating solutions for $\frac{\partial^2 \mathbf{E}}{\partial t^2} = \hat{A}\mathbf{E}$.

A common mistake in this problem was to choose an inner product $\langle \mathbf{F}, \mathbf{G} \rangle = \int \varepsilon\mu \bar{\mathbf{F}} \cdot \mathbf{G}$, and then to claim that $\langle \mathbf{E}, \hat{A}\mathbf{E}' \rangle_\varepsilon = \int \cancel{\varepsilon\mu\varepsilon^{-1}} \bar{\mathbf{E}} \cdot \nabla \times \cancel{\mu^{-1}} \nabla \times \mathbf{E}' = \int \bar{\mathbf{E}} \cdot \nabla \times \nabla \times \mathbf{E}'$, which is not true since $\mu(\mathbf{x})$ is not a constant (you can't interchange it with $\nabla \times$).

Problem 2: Timestepping (34 points)

1. We use the Taylor series around $n + \frac{1}{2}$:

$$\mathbf{u}^{n+\frac{1}{2}\pm\frac{1}{2}} = \mathbf{u}([n + \frac{1}{2}]\Delta t) \pm \Delta t \, \dot{\mathbf{u}}([n + \frac{1}{2}]\Delta t) + O(\Delta t^2),$$

where $+$ gives $\mathbf{u}^{n+1}$ and $-$ gives $\mathbf{u}^n$. Then

$$\begin{aligned}
\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} &= \frac{\mathbf{u}([n + \frac{1}{2}]\Delta t) + \cancel{\Delta t \, \dot{\mathbf{u}}([n + \frac{1}{2}]\Delta t)} + O(\Delta t^2) + \mathbf{u}([n + \frac{1}{2}]\Delta t) - \cancel{\Delta t \, \dot{\mathbf{u}}([n + \frac{1}{2}]\Delta t)} + O(\Delta t^2)}{2} \\
&= \mathbf{u}([n + \frac{1}{2}]\Delta t) + O(\Delta t^2),
\end{aligned}$$

as desired.

It is also possible to do this by Taylor-expanding around $\mathbf{u}(n\Delta t)$, and comparing the result to the Taylor series of $\mathbf{u}([n + \frac{1}{2}]\Delta t)$:

$$\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} = \frac{\mathbf{u}(n\Delta t) + \Delta t \, \dot{\mathbf{u}}(n\Delta t) + O(\Delta t^2) + \mathbf{u}(n\Delta t)}{2} = u(n\Delta t) + \frac{\Delta t}{2} \dot{\mathbf{u}}(n\Delta t) + O(\Delta t^2) = \mathbf{u}([n + \frac{1}{2}]\Delta t) + \mathbf{u}([n + \frac{1}{2}]\Delta t).$$

2. Solving for $\mathbf{u}^{n+1}$, we have

$$\mathbf{u}^{n+1} = \left(I - \frac{A\Delta t}{2}\right)^{-1}\left(I + \frac{A\Delta t}{2}\right)\mathbf{u}^n = B\mathbf{u}^n,$$

where we have defined the matrix B, and hence

$$\mathbf{u}^n = B^n \mathbf{u}^0$$

as in class. If λ is an eigenvalue of A for some eigenvector, then the *same* vector is an eigenvector of B with eigenvalue $\mu = (1 + \lambda\Delta t/2)/(1 - \lambda\Delta t/2)$. If $A = A^* \prec 0$, then $\lambda < 0$, and it follows that $|\mu| < 1$ for any $\Delta t > 0$ (the denominator of μ is bigger than the numerator, since addition gives a bigger number than subtraction). Hence $B^n \to 0$ as $n \to \infty$, and the scheme is unconditionally stable.

By the way, a common mistake here is to write $\mathbf{u}^{n+1} = \frac{I+A\Delta t/2}{I-A\Delta t/2}\mathbf{u}^n$, which is "not even wrong:" if B and C are matrices, the expression $\frac{B}{C}$ is meaningless because it is not clear whether you mean $B^{-1}C$ or CB^{-1} (unless they happen to commute, which they don't in this case). Another common mistake is to check that $\mu < 1$, which is not sufficient: you need $|\mu| < 1$ for the solutions to decay.

Problem 3: Born (33 points)

We write

$$\hat{A}(\Delta p) = -\nabla^2 + c(\Delta p, \mathbf{x}) = \hat{A}(0) + \left.\frac{\partial c}{\partial p}\right|_{p=0} \Delta p + O(\Delta p^2)$$

by Taylor-expanding c around $p = 0$. Then, by moving the $\partial c/\partial p$ term to the right-hand-side, we see that $\hat{A}(\Delta p)u = f$ solves

$$u = \hat{A}(0)^{-1}\left[f - \left.\frac{\partial c}{\partial p}\right|_{p=0} u\Delta p + O(\Delta p^2)\right].$$

Now, plugging in the right-hand-side for u, as in the derivation of the Born–Dyson series in class, we obtain

$$u = \hat{A}(0)^{-1}\left[f - \left.\frac{\partial c}{\partial p}\right|_{p=0} \hat{A}(0)^{-1} f\Delta p + O(\Delta p^2)\right],$$

where we have lumped all terms of order Δp^2 or higher together, and the second term is the first Born approximation. Now, to get the derivative, we do

$$\begin{aligned}
\left.\frac{\partial u}{\partial p}\right|_{p=0} &= \lim_{\Delta p \to 0} \frac{u|_{p=\Delta p} - u|_{p=0}}{\Delta p} \\
&= \lim_{\Delta p \to 0} \frac{\hat{A}(0)^{-1}\left[\cancel{f} - \left.\frac{\partial c}{\partial p}\right|_{p=0}\hat{A}(0)^{-1} f\Delta p + O(\Delta p^2)\right] - \cancel{\hat{A}(0)^{-1} f}}{\Delta p} \\
&= \boxed{-\hat{A}(0)^{-1}\left.\frac{\partial c}{\partial p}\right|_{p=0}\hat{A}(0)^{-1} f}.
\end{aligned}$$

In fact, one can easily generalize this approach to show that, for any invertible operator that depends in a differentiable way on a parameter p, the derivative of the inverse of the operator is:

$$\frac{\partial}{\partial p}\hat{A}^{-1} = -\hat{A}^{-1}\frac{\partial \hat{A}}{\partial p}\hat{A}^{-1},$$

which is a geralization of the chain rule $\frac{\partial}{\partial p}a(p)^{-1} = -\frac{\partial a/\partial p}{a^2}$ from first-year calculus.