Properties of the Fourier Series and Transform

Chris Rackauckas

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1 Questions of the Day

1. Is the tranformation of a function into a Fourier series linear?

2 Fourier Series

For simplicity we will assume that our functins are defined on $x \in [0, 1]$ and that $u(0) = u(1) = 0$. This means that $u(x)$ sufficiently nice has a Fourier series:

$$u(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

2.1 What are some properties of this series?

2.1.1 Orthogonal

Recall that the dot product is $(f, g) = \int_0^1 f(x)g(x)dx$. Let's take the dot product between basis elements:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x) = \int_0^1 \frac{[\cos((n-m)\pi x) - \cos(n+m)\pi x]}{2} dx$$

But $n - m \neq 0$ means that the integral is zero since cos is even. If 0, $\int_0^1 \frac{\cos(2\pi x)}{2} = \frac{1}{2}$.

2.1.2 Computing the coefficients

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} b_n \sin{(n\pi x)} \\
\int_0^1 u(x) \sin{(m\pi x)}\, dx &= \int_0^1 \sum_{n=1}^{\infty} b_n \sin{(n\pi x)} \sin{(m\pi x)}\, dx \\
\int_0^1 u(x) \sin{(m\pi x)}\, dx &= b_n \int_0^1 \sin^2{(m\pi x)}\, dx \\
2 \int_0^1 u(x) \sin{(m\pi x)}\, dx &= b_n
\end{aligned}
$$

since $\int_0^1 \sin^2{(m\pi x)}\, dx = 2$.

2.1.3 Spanning

For any function $f(x)$ where $\int |f(x)|^p \, dx < \infty$ for some $p > 1$, the Fourier series converges almost everywhere to $f(x)$. At jump discontinuities, the Fourier series converges to the midpoint.

Example: Fourier Series of 1

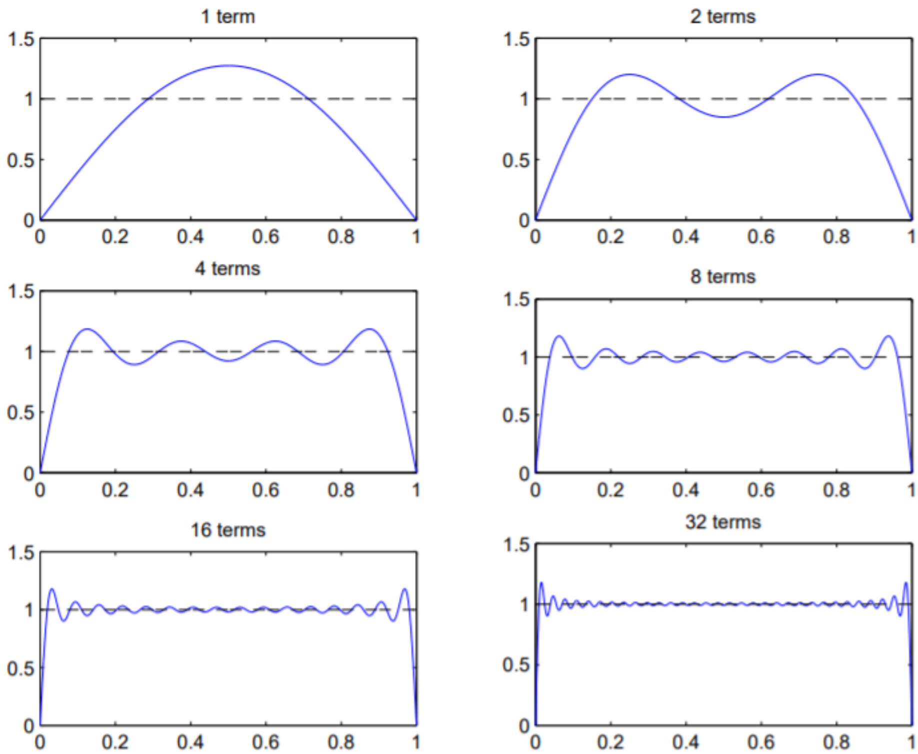

Figure 1: Fourier sine series for $f(x) = 1$, truncated to a finite number of terms (from 1 to 32), showing that the series indeed converges everywhere to $f(x)$, except exactly at the endpoints, as the number of terms is increased.

$$f(x) = 1 = \frac{4}{\pi}\sin(\pi x) + \frac{4}{3\pi}\sin(3\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \ldots$$

The bumps near the discontinuity are known as Gibb's phonomenon.

2.1.4 Example: Fourier Series of $f(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|$

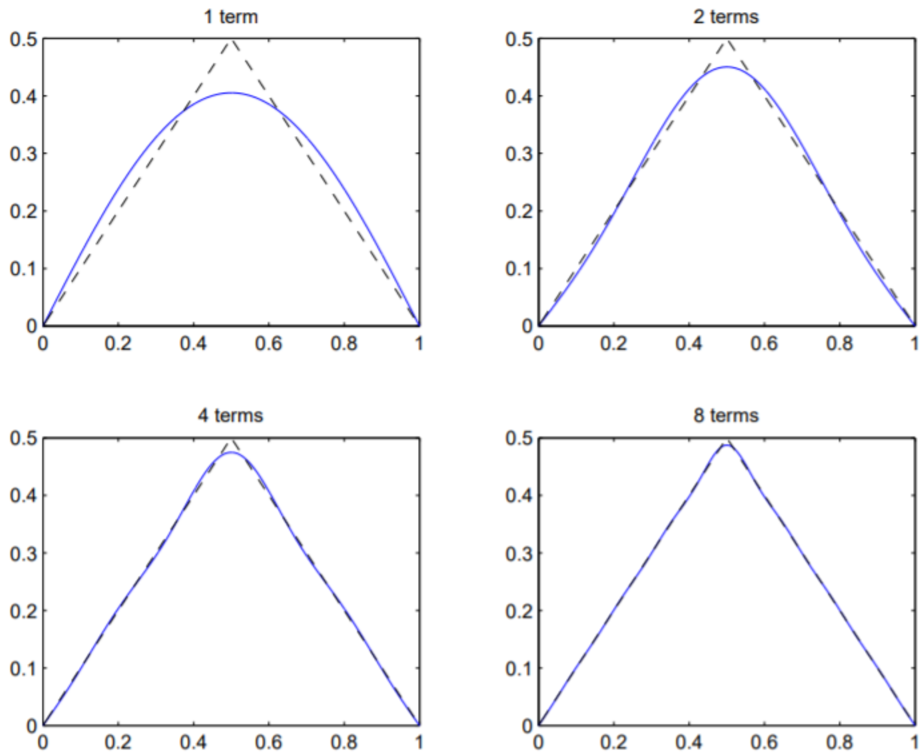

Figure 2: Fourier sine series (blue lines) for the triangle function $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$ (dashed black lines), truncated to a finite number of terms (from 1 to 32), showing that the series indeed converges everywhere to $f(x)$.

$$f(x) = \frac{4}{\pi^2} \sin(\pi x) - \frac{4}{(3\pi)^2} \sin(3\pi x) + \frac{4}{(5\pi)^2} \sin(5\pi x) + \dots$$

3 Fourier Transform

Let $\mathcal{F}(u)$ be the Fourier Transform as the function $\mathcal{F} : u \to [b_1, b_2, \dots]$.

3.1 What are some properties of the Fourier Transform?

3.1.1 Linearity

$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ and $g(x) = \sum_{n=1}^{\infty} \tilde{b}_n \sin(n\pi x)$. But

$$\int_0^1 (\alpha f(x) + \beta g(x)) \sin(n\pi x)\, dx = \alpha \int_0^1 f(x) \sin(n\pi x)\, dx + \beta \int_0^1 g(x) \sin(n\pi x)\, dx$$

which means $\alpha f(x) + \beta g(x) = \sum_{n=1}^{\infty} \left(\alpha b_n + \beta \tilde{b}_n\right) \sin(n\pi x)$. This means that

$$\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g)$$

3.1.2 Inverse

Given an infinite vector $[b_1, b_2, \ldots]$, the inverse Fourier Transform $\mathcal{F}^{-1}$ is defined as $\mathcal{F}^{-1}([b_1, b_2, \ldots]) = \sum_n b_n \sin(n\pi x)$.

4 Solving the Poisson Equation

4.1 Representation of Δ in the Fourier Basis

$\Delta = \frac{d^2}{dx^2}$. The claim is that

$$\Delta = \begin{bmatrix} -(\pi)^2 & 0 & 0 \\ 0 & -(2\pi)^2 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} = D$$

is diagonal in the Fourier basis.

$$\frac{d^2}{dx^2} \sin(n\pi x) = -(n\pi)^2$$

and so

$$\begin{aligned} \frac{d^2}{dx^2} \sum_{n=1}^{\infty} b_n \sin(n\pi x) &= \sum_{n=1}^{\infty} b_n \frac{d^2}{dx^2} \sin(n\pi x) \\ &= -\sum_{n=1}^{\infty} (n\pi)^2 b_n \sin(n\pi x) \end{aligned}$$

and thus for $B = [b_1, b_2, \ldots]^T$, $\Delta B = \left[-\pi^2 b_1, -(2\pi)^2 b_2, \ldots\right]$ which we see is multiplication by that diagonal matrix.

4.2 Solution to the Poisson Equation

$$\Delta u = f$$

Now diagonalize Δ. Notice that it is diagonal in the Fourier basis, and so we write the diagonalization of $\Delta = \mathcal{F}^{-1} D \mathcal{F}$ and get

$$\begin{aligned} \mathcal{F}^{-1} D \mathcal{F} u &= f \\ u &= \mathcal{F}^{-1} D^{-1} \mathcal{F} f. \end{aligned}$$

4.3 Example Solution

Let $f(x) = \frac{1}{2} - \left| x - \frac{1}{2} \right|$. Then

$$f(x) = \frac{4}{\pi^2} \sin(\pi x) - \frac{4}{(3\pi)^2} \sin(3\pi x) + \frac{4}{(5\pi)^2} \sin(5\pi x) + \dots$$

So then

$$\Delta u = f$$

implies that

$$u(x) = -\frac{4}{\pi^4} \sin(\pi x) + \frac{1}{9\pi^4} \sin(3\pi x) + \frac{4}{15^2\pi^4} \sin(5\pi x) + \dots$$

4.4 Solving the Heat Equation

Let's start using some notation.

$$u_t = \frac{du}{dt}.$$

$$u_t = \Delta u + f(t,x)$$

and let $u(0,x) = u_0(x)$ be given. We want to solve for $u(t,x)$. Let's diagonalize Δ:

$$u_t = \mathcal{F}^{-1} D \mathcal{F} u + f(t,x)$$

apply $\mathcal{F}$ to both sides:

$$(\mathcal{F}u)_t = \mathcal{F} u_t = D\mathcal{F}u + \mathcal{F} f(t,x)$$

Define $v = \mathcal{F}u$, then

$$v_t = Dv + \mathcal{F} f(t,x)$$

with the initial condition $v_0(x) = \mathcal{F} u_0(x)$. Notice that this is just a system of ordinary differential equations. To see this more clearly, let's write out what this means. Now, at any given point in time u has a Fourier series, so let's write this as

$$u(t,x) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x).$$

Then $u(0,x) = \sum_{n=1}^{\infty} b_n(0)\sin(n\pi x)$, so $b_n(0)$ is given. Let $\sum_{n=1}^{\infty} c_n(t)\sin(n\pi x)$. So now write out the equation:

$$\left(\sum_{n=1}^{\infty} b_n(t)\sin(n\pi x)\right)_t = \Delta\sum_{n=1}^{\infty} b_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} c_n(t)\sin(n\pi x)$$
$$\sum_{n=1}^{\infty} b_n'(t)\sin(n\pi x) = -\sum_{n=1}^{\infty}(n\pi)^2 b_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} c_n(t)\sin(n\pi x)$$

thus by matching terms we see that

$$b_n'(t) = -(n\pi)^2 b_n(t) + c_n(t)$$

is an infinite system of decoupled ODEs, where $b_n(0)$ is given. The solution to the Heat Equation is just given by the solution of these ODEs.

We can solve this ODE using the integrating factor method. Rewrite this as:

$$b_n'(t) + (n\pi)^2 b_n(t) = c_n(t)$$

Now multiply both sides of the equation by $e^{(n\pi)^2 t}$:

$$b_n'(t)e^{(n\pi)^2 t} + (n\pi)^2 b_n(t)e^{(n\pi)^2 t} = c_n(t)e^{(n\pi)^2 t}$$

Now check that:

$$\left(b_n(t)e^{(n\pi)^2 t}\right)' = b_n'(t)e^{(n\pi)^2 t} + (n\pi)^2 b_n(t)e^{(n\pi)^2 t}$$

so then

$$\left(b_n(t)e^{(n\pi)^2 t}\right)' = c_n(t)e^{(n\pi)^2 t}$$

which means that

$$b_n(t)e^{(n\pi)^2 t} = \int_0^t c_n(t)e^{(n\pi)^2 t}dt + C$$

and thus

$$b_n(t) = e^{-(n\pi)^2 t}\int_0^t c_n(t)e^{(n\pi)^2 t}dt + e^{-(n\pi)^2 t}b_n(0)$$

is an equation for how the Fourier coefficients evolve over time. Why is $C = b_n(0)$? It has to be: plug in $t = 0$.

Now look at what the solution is like. Without a focing term, $c_n(t) = 0$ for all n. When that's the case,

$$b_n(t) = e^{-(n\pi)^2 t}b_n(0)$$

the solutions decay over time, and the higher n gives faster decay.

4.5 Solving the Wave Equation

Now let's solve the Wave Equation

$$u_{tt} = \Delta u$$

We can do so by ansatz. Since we know the eigenvectors of the second derivative operators, know how to diagonalize both derivative operators in this equation. If the t portion of the equation does not depend on the x portion, then differentiation is simplified, i.e.

$$u(t, x) = u_1(t)u_2(x)$$

then

$$u_{tt} = u_{1tt}u_2$$

and

$$u_{xx} = u_1 u_{2xx}$$

Thus let's make this assumption (called separation of variables). Let's make both parts the eigenfunctions of the second derivative. Thus we have:

$$u(t, x) = \sum_n \sum_m b_n \cos(n\pi t) \sin(m\pi x)$$

where b_n are the coefficients of the expansion of the initial condition (notice cos was used since $\cos(0) = 1$ making the initial condition work). Plugging this in, we get

$$-\sum_n \sum_m (n\pi)^2 b_n \cos(n\pi t) \sin(m\pi x) = -\sum_n \sum_m (m\pi)^2 b_n \cos(n\pi t) \sin(m\pi x)$$

and thus matching coefficients we notice that $n = m$. Thus we have that

$$u(t, x) = \sum_n b_n \cos(n\pi t) \sin(n\pi x)$$

are solutions to the Wave Equation. Notice that these solutions are oscilatory in time, whereas the Heat Equation decayed over time!

4.6 Relating back to the Poisson Equation

These equations look similar to the Poisson equation: how are they similar? Let's look at the Heat Equation again, and now let's assume $f(t, x) = f(x)$

$$u_t = \Delta u + f(x)$$

Now let's assume that the Heat Equation has progress long enough into time that it stopped changing. If that's the case, then $u_t = 0$ and then $-\Delta u = f(t, x)$ is the Poisson Equation. This is called a steady state: a state which does not change over time. As the forced heat equation evolves, its long time behavior as $t \to \infty$ is the solution to the Poisson equation!