18.303 Problem Set 5

Due Wednesday, 12 October 2011.

Problem 1: Kronecker products

Suppose that we flatten an $N_x \times N_y$ grid u_{n_x,n_y} into $N_x N_y \times 1$ column vectors $\mathbf{u}$ using column-major order as in class. Let I_x and I_y be $N_x \times N_x$ and $N_y \times N_y$ identity matrices respectively, and let A_x and A_y be negative-definite and self-adjoint (under the usual dot product $\mathbf{u}^*\mathbf{v}$) $N_x \times N_x$ and $N_y \times N_y$ matrices, respectively.

(a) Show that $(A \otimes C)(\mathbf{b} \otimes \mathbf{d}) = (A\mathbf{b}) \otimes (C\mathbf{d})$ for any matrices A and C and any vectors $\mathbf{b}$ and $\mathbf{d}$ (with sizes equal to the number of columns in A and C, respectively). [The generalization of this is the identity $(A \otimes C)(B \otimes D) = (AB) \otimes (CD)$, but you need not prove the general case.]

(b) If A and C are self-adjoint under the usual dot product, show that $A \otimes C$ is self-adjoint. If A and C are furthermore positive-definite, show that $A \otimes C$ is positive-definite.

(c) Suppose that we have eigensolutions $A_x\mathbf{x} = \lambda_x\mathbf{x}$ and $A_y\mathbf{y} = \lambda_y\mathbf{y}$ of A_x and A_y. Construct an eigensolution of $A = \alpha I_y \otimes A_x + \beta A_y \otimes I_x + \gamma A_y \otimes A_x$ (where α, β, γ are scalars). (This is a matrix analogue of *separable solutions*.) Given all of the eigensolutions of A_x and A_y, can you get all of the eigensolutions of A in this way? Under what conditions on α, β, γ is A positive-definite?

Problem 2: Gridded cylinders

In this problem, we will solve pset 4's Laplacian eigenproblem $c\nabla^2 u = \lambda u$ in a 2d cylinder $r \le R$ with Dirichlet boundary conditions $u|_{d\Omega} = 0$ by "brute force" with a 2d finite-difference discretization, and compare to the analytical Bessel solutions from pset 4. Here, $R = 1$, $c(r) = 5$ for $r < 0.5$ and $= 1$ for $r \ge 0.5$. Recall that the first two $m = 0$ eigenvalues were $\lambda_1 \approx -10.841794631$ and $\lambda_2 \approx -67.14978273775$. You can form a matrix $A \approx c\nabla^2$ with the commands (similar to class):

```
Lx = 2; Ly = 2; Nx = 100; Ny = 100; dx = Lx/(Nx+1); dy = Ly/(Ny+1); N = Nx*Ny;
Dx = diff1(Nx)/dx; Dy = diff1(Ny)/dy;
[y,x] = meshgrid([1:Ny]*dy - Ly/2, [1:Nx]*dx - Lx/2);
r = sqrt(x.^2 + y.^2); theta = atan2(y,x);
c = 5 * (r < 0.5) + 1 * (r >= 0.5);
C = spdiags(reshape(c,N,1), 0, N,N);
A0 = C * (kron(speye(Ny,Ny),-Dx'*Dx) + kron(-Dy'*Dy,speye(Nx,Nx)));
i = find(r < 1);
A = A0(i,i);
```

(a) Compute the 10 smallest-magnitude eigenvalues and eigenfunctions of A with `[V,S]=eigs(A,10,'sm')`. The eigenvalues are given by `diag(S)`. Download bluered.m from the course web site. You can plot the eigenfunctions with:

```
u = zeros(N,1); u(i) = V(:,k); u = reshape(u, Nx, Ny); pcolor(x,y,u); shading interp;
bluered;
```

where k ranges from 1 to 10. (You can use `surf` instead of `pcolor` to make a 3d plot.) Figure out which ones correspond to λ_1 and λ_2 from pset 4. You don't need to turn in printouts of all 10 plots, just the ones for λ_1 and λ_2 and perhaps one or two others. Also, for the λ_1 and λ_2 solutions, plot $u(x, 0)$ vs. $x \in [0, 1]$ for comparison with the $u(r)$ plots in pset 4, by the command: `plot(x(end/2:end,end/2), u(end/2:end,end/2))`

(b) Compared with the high-accuracy λ_1 value from pset 4 (above), compute the error $\Delta\lambda_1$ in the corresponding finite-difference eigenvalue from the previous part. Also compute $\Delta\lambda_1$ for $N_x = N_y = 200$ and 400. [Just use `eigs(A,1,'sm')` to get the smallest-magnitude eigenvalue.] How fast is the convergence rate with Δx? Can you explain your results, in light of the fact that the center-difference approximation we are using has an error that is supposed to be $\sim \Delta x^2$? (Hint: think about how accurately the boundary condition on $\partial\Omega$ is described in this finite-difference approximation.)

(c) Modify the above code to instead discretize $\nabla \cdot c\nabla$, by writing A_0 as $-G^T C_g G$ for some G matrix that implements ∇ and for some C_g matrix that multiplies the gradient by c (different from the C matrix above, which multiplies a scalar field by c). Draw a sketch of the grid points at which the components of ∇ are discretized—these will *not* be the same as the (n_x, n_y) where u is discretized, because of the centered differences. Be careful that you need to evaluate c at the ∇ grid points now! Hint: you can make the matrix $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ in Matlab by the syntax `[M1;M2]`.

(d) Using this $A \approx \nabla \cdot c\nabla$, compute the smallest-$|\lambda|$ eigensolution and plot $u(r) = u(x, 0)$ as in part (a). What is the continuity condition at $r = 0.5$? (Compare to pset 4, where the condition was that u and u' were continuous.)

Problem 3: Min–max theorem

(a) Consider the operator $\hat{A} = -\nabla^2$ on the space of functions $u(r, \theta)$ where Ω is the unit-radius circle and $u|_{\partial\Omega} = 0$. We found the eigenvalues of this operator analytically in class, and the smallest eigenvalue is $\lambda_1 \approx 5.783$ (the square of the first root of J_0). Here, you will estimate λ_1 from the Rayleigh quotient $R(u) = \langle u, \hat{A}u \rangle / \langle u, u \rangle = \int |\nabla u|^2 / \int |u|^2$, by plugging in some trial function $u(r)$. In cylindrical coordinates, for $u(r)$, $R(u) = \int_0^1 r|u'|^2 dr / \int_0^1 r|u|^2 dr$. In particular, consider the function:

$$u_a(r) = (1 - r)^a.$$

For what a ($a > 0.5$) is $R(u_a)$ minimized? How does the minimum of $R(u_a)$ compare with λ_1? (Just plot R vs. a in Matlab and minimize it graphically.) Useful integral: $\int x(1-x)^p dx = -(1-x)^{p+1}(px + x + 1)/(p^2 + 3p + 2)$.]

(b) Consider $-\nabla^2 u = \lambda u$ for functions $u(\mathbf{x})$ in the 2d triangular domain Ω given by $x \geq 0$, $y \geq 0$, $|x| + |y| \leq 1$ (a square cut in half diagonally) with **Neumann** boundary conditions $\mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0$. Sketch contour plots of the eigenfunctions for the smallest 4 eigenvalues, making reasonable guesses based on the fact that these minimize $R(u) = \int |\nabla u|^2 / \int |u|^2$ (constrained by the fact that they must be orthogonal). (In your plots, label peaks with a "+" and dips with a "−".)