18.303 Problem Set 7

Due Wednesday, 30 October 2013.

Problem 1: Born again

In the class notes on "Green's functions in inhomogeneous media," section 3, we applied the Born approximation to a small "object" far from both the source ($\mathbf{x}_0$) and the observer ($\mathbf{x}$). A key step in the derivation was to approximate the integrand

$$g(\mathbf{x}') = \nabla' G_0(\mathbf{x}, \mathbf{x}') \cdot \nabla' u_0(\mathbf{x}')$$

of $\hat{B}u_0$ as a constant in V and to pull it out of the integral. In this problem, you will carry out that process more carefully, as a Taylor expansion of $g(\mathbf{x}')$ around $\mathbf{x}_1$:

$$g(\mathbf{x}') \approx g(\mathbf{x}_1) + \nabla' g\big|_{\mathbf{x}_1} \cdot (\mathbf{x}' - \mathbf{x}_1) + (\mathbf{x}' - \mathbf{x}_1) \cdot H \cdot (\mathbf{x}' - \mathbf{x}_1) + \ldots$$

where H is the 3×3 second-derivative ("Hessian") matrix with entries $H_{ij} = \frac{\partial^2 g}{\partial x_i' \partial x_j'}$. What we did in class was to simply keep only the *first* term in this expansion.

(a) Show that the *second* term in the Taylor expansion, $\nabla' g\big|_{\mathbf{x}_1} \cdot (\mathbf{x}' - \mathbf{x}_1)$, contributes *zero* to $\hat{B}u_0$ if the point $\mathbf{x}_1$ is chosen correctly. Give a formula for the correct $\mathbf{x}_1$. (Do you recognize this formula?)

(b) Work out the *third* term (the H term) in the Taylor expansion. This is called the "quadrupole" term. How fast does this term's contribution to $u(\mathbf{x})$ decay with $|\mathbf{x} - \mathbf{x}_1|$ and $|\mathbf{x}_0 - \mathbf{x}_1|$? (In comparison, the "dipole" term we examined in class decayed as $\sim 1/|\mathbf{x} - \mathbf{x}_1|^2$ and $\sim 1/|\mathbf{x}_0 - \mathbf{x}_1|^2$.)

Problem 2: Crank-Nicolson vs. Leapfrog

In class, we considered negative-definite discretizations $A = A^* < 0$ of $\frac{\partial u}{\partial t} = \hat{A}u$ and showed that the Crank-Nicolson discretization

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = A\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}$$

is unconditionally stable, where the right-hand side was a way of approximating $\mathbf{u}^{n+0.5}$. Here, we will instead consider using $\mathbf{u}^{n+0.5}$ directly. That is, we will imagine computing $\mathbf{u}$ at timesteps $\mathbf{u}^0, \mathbf{u}^{0.5}, \mathbf{u}^1, \mathbf{u}^{1.5}, \ldots$ where we update $\mathbf{u}$ by the center-difference equations:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = A\mathbf{u}^{n+0.5},$$

$$\frac{\mathbf{u}^{n+0.5} - \mathbf{u}^{n-0.5}}{\Delta t} = A\mathbf{u}^n.$$

This is called "leapfrog" scheme, for reasons that should become apparent (if you know the children's game of leapfrog).

(a) Suppose we are given the initial condition $\mathbf{u}^0$. Outline an algorithm to update $\mathbf{u}$ successively in order to compute $\mathbf{u}^n$ and $\mathbf{u}^{n+0.5}$ for as many $n > 0$ as desired, using the leapfrog scheme above. (You may need to construct a different way to compute $\mathbf{u}^{0.5}$. Preserve second-order accuracy in time!)

(b) Consider $\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}$ and show that you can get an equation in $\mathbf{u}^n$ alone (no half-integer timesteps). Relate this to the original PDE via $\frac{\partial^2 u}{\partial t^2} = \cdots$.

(c) Suppose $\hat{A} = \frac{\partial^2}{\partial x^2}$ and A is the usual center-difference approximation $u_m'' = \frac{u_{m+1} - 2u_m + u_{m-1}}{\Delta x^2}$. Perform a Von Neumann stability analysis by pluggin in $u_m^n = e^{ikm}G^n$ and $u_m^{n+0.5} = e^{ikm}G^{n+0.5}a$ for an unknown growth rate G, an unknown relative amplitude a of u_m^n and $u_m^{n+0.5}$, and arbitrary real k.

  (i) Show that $a = \pm 1$.

  (ii) If we can restrict ourselves to solutions with $a = +1$ (i.e. integer and half-integer timesteps have the same sign), find conditions on Δt and Δx (if any) such that $|G| < 1$ for all real k (stability, since the original PDE is a diffusion equation with decaying solutions).