18.303 Problem Set 9

Due Wednesday, 17 November 2010.

Problem 1: Discretizing a 2d wave equation

Consider the scalar wave equation with constant coefficients, which can be written as in the notes as two coupled equations $\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{v}$, $\frac{\partial \mathbf{v}}{\partial t} = \nabla u$. Now, let us consider this in two dimensions (for an infinite spatial domain $\Omega = \mathbb{R}^2$, don't worry about boundary conditions).

(a) Suppose that we discretize $u(x, y, t)$ as $u^n_{m_x,m_y} \approx u(m_x\Delta x, m_y\Delta y, n\Delta t)$ for integers m_x, m_y, n. Explain how (i.e. where/when) we should discretize $v_x(x, y, t)$ and $v_y(x, y, t)$ so that our equations $\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{v}$ and $\frac{\partial \mathbf{v}}{\partial t} = \nabla u$ can all be implemented as explicit center differences, similar to the staggered-grid leap-frog scheme in class for the 1d case. Give the discretized equations. (Hint: v_x and v_y don't have to be discretized on the same spatial grid.)

(b) Combine your discretized equations back into a discretization of $\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$, as in class, by writing:

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u^{n+1}_{m_x,m_y} - 2u^n_{m_x,m_y} + u^{n-1}_{m_x,m_y}}{\Delta t^2} = \frac{\frac{u^{n+1}_{m_x,m_y} - u^n_{m_x,m_y}}{\Delta t} - \frac{u^n_{m_x,m_y} - u^{n-1}_{m_x,m_y}}{\Delta t}}{\Delta t} = \cdots,$$

where the right-hand side is some spatial derivatives (finite differences) of u. (Do *not* re-discretize $\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$ from the beginning; make sure you plug in your discretizations from the previous part.)

(c) Using the previous part (which helpfully is only in terms of u), perform a Von Neumann analysis to relate Δt to Δx and Δy: plug in $u^n_{m_x,m_y} = e^{i(k_x m_x \Delta x + k_y m_y \Delta y - \omega n \Delta t)}$ and solve for $\omega(k_x, k_y)$, and find out under what conditions (on Δt) ω is real for all possible values of k_x and k_y.

Problem 2: Bouncing waves

In class, I gave an animwave.m file (available on the web site) that animated the 1d wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with periodic boundary conditions. For example, to animate a Gaussian-shaped pulse propagating to the right, you can use the command:

```
animwave(@(x) exp(-(x-0.5).^2/0.1^2), 100, 1, 5, 0.9);
```

(a) Modify the animwave.m file implement Dirichlet boundary conditions $u(0) = u(L) = 0$ instead of periodic boundary conditions. Hint: define u_m for $m = 1 \ldots M$, and $v_{m+0.5}$ for $m = 0 \ldots M$ (that is, v will have one more point than u), with $u_0 = u_{M+1} = 0$. You shouldn't need to explicitly implement any boundary condition for v. Run it for the gaussian pulse as above, and show the graph at a few timesteps to illustrate what happens when your pulse hits a boundary.

(b) In an infinite 1d problem ($\Omega = \mathbb{R}$) for the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with constant c, there is a solution $u(x, t) = f(x - ct)$ for a right-going wave. Suppose instead that the domain is $x < 0$ (i.e. semi-infinite), with Dirichlet boundary conditions $u(0) = 0$. Construct a solution $u(x, t)$ using $f(x)$ and $f(-x)$ that satisfies both the wave equation and the boundary condition, and relate it to what you observed in the previous part.