18.303 Problem Set 5

Due Friday, 16 October 2015.

Problem 1: Hermitian Green's functions

(a) Show that if $\hat{A}$ is Hermitian for some $\langle u, v\rangle$, then $\hat{A}^{-1}$ is also Hermitian. Hint: consider $\langle u, \hat{A}\hat{A}^{-1}v\rangle$.

(b) Suppose that $u = \hat{A}^{-1}f$ can be expressed in terms of a Green's function $G(\mathbf{x}, \mathbf{x}')$, i.e. $\hat{A}^{-1}u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')d^n\mathbf{x}'$. If $\hat{A}^{-1}$ is Hermitian under the usual inner product $\langle u, v\rangle = \int \bar{u}v$, show that you can relate $G(\mathbf{x}, \mathbf{x}')$ to $G(\mathbf{x}', \mathbf{x})$. (This relationship is sometimes called *reciprocity*.)

(c) Consider a diffusion process where u is concentration and $f(\mathbf{x})$ is a source/sink (concentration per time added/removed), related by the diffusion equation $\frac{\partial u}{\partial t} = \nabla \cdot D\nabla u + f$, where $D(\mathbf{x}) > 0$ is a diffusivity constant of different materials at different points in space. In steady state, u solves $-\nabla \cdot D\nabla u = f$, and we showed in class that $\hat{A} = -\nabla \cdot D\nabla$ is Hermitian and invertible (positive-definite) for Dirichlet boundary conditions. Using your reciprocity relationship from the previous part, explain why there is no possible arrangement of materials [no possible $D(\mathbf{x})$] that allows "one-way diffusion" — diffusion from a source at $\mathbf{x}'$ to a point $\mathbf{x}$ but not vice-versa.

Problem 2: Distributions

This problem concerns distributions as defined in the notes: continuous linear functionals $f\{\phi\}$ from test functions $\phi \in \mathcal{D}$, where $\mathcal{D}$ is the set of infinitely differentiable functions with compact support (i.e. $\phi = 0$ outside some region with finite diameter [differing for different ϕ], i.e. outside some finite interval $[a, b]$ in 1d). Recall that the distributional or "weak" derivative is $f'\{\phi\} = f\{-\phi'\}$.

Consider the function $f(x) = \begin{cases} \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$ and its (weak) derivative, which is connected to something called the Cauchy Principal Value. This defines a regular distribution, even though f blows up as $x \to 0$, because this singularity is "integrable:" $\int \ln x = x\ln x - x$ and $x \ln x$ is finite for $x \to 0$.

(a) Consider the 18.01 derivative of $f(x)$, which gives $f'(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}$. Suppose we just set "$f'(0) = 0$" at the origin to define $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that this $g(x)$ is *not* locally integrable, and hence does not define a distribution.

But the weak derivative $f'\{\phi\}$ *must* exist, so this means that we have to do something different from the 18.01 derivative, and moreover $f'\{\phi\}$ is *not* a regular distribution. What is it?

(b) Write $f\{\phi\} = \lim_{\epsilon\to 0^+} f_\epsilon\{\phi\}$ where $f_\epsilon\{\phi\} = \int_{-\infty}^{-\epsilon} \ln(-x)\phi(x)dx + \int_\epsilon^\infty \ln(x)\phi(x)dx$, since this limit exists and equals $f\{\phi\}$ for all ϕ from your proof in the previous part.[1] Compute the distributional derivative $f'\{\phi\} = \lim_{\epsilon\to 0^+} f'_\epsilon\{\phi\}$, and show that $f'\{\phi\}$ is precisely the *Cauchy Principal Value* (google the definition, e.g. on Wikipedia) of the integral of $g(x)\phi(x)$.

(c) Alternatively, show that $f'\{\phi(x)\} = g\{\phi(x) - \phi(0)\} = \int_{-\infty}^\infty g(x)[\phi(x) - \phi(0)]dx$ (which *is* a well-defined integral for all $\phi \in \mathcal{D}$).

[1] More explicitly, $f\{\phi\} - f_\epsilon\{\phi\} = \int_{-\epsilon}^\epsilon \ln|x|\phi(x)dx \leq (\max\phi)\int_{-\epsilon}^\epsilon \ln|x|dx \to 0$, since you should have done the something like the last integral explicitly in the previous part.