18.303 Problem Set 5

Due Wednesday, 17 October 2012.

Problem 1: Conquering Inversity

Consider the operator $\hat{A} = -c(\mathbf{x})\nabla^2$ in some 2d region $\Omega \subseteq \mathbb{R}^2$ with Dirichlet boundaries ($u|_{\partial\Omega} = 0$), where $c(\mathbf{x}) > 0$. Suppose the eigenfunctions of $\hat{A}$ are $u_n(\mathbf{x})$ with eigenvalues λ_n [that is, $\hat{A}u_n = \lambda_n u_n$] for $n = 1, 2, \ldots$, numbered in order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Let $G(\mathbf{x}, \mathbf{x}')$ be the Green's function of $\hat{A}$.

(a) If $f(\mathbf{x}) = \sum_n \alpha_n u_n(\mathbf{x})$ for some coefficients $\alpha_n =$____________________ (expression in terms of f and u_n), then $\int_\Omega G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') d^2\mathbf{x}' =$____________________ (in terms of α_n and u_n).

(b) The maximum possible value of

$$\frac{\int_\Omega \int_\Omega \frac{1}{c(\mathbf{x})} \overline{u(\mathbf{x})} G(\mathbf{x}, \mathbf{x}') u(\mathbf{x}')\, d^2\mathbf{x}\, d^2\mathbf{x}'}{\int_\Omega \frac{|u(\mathbf{x}'')|^2}{c(\mathbf{x}'')} d^2\mathbf{x}''},$$

for any possible $u(\mathbf{x})$, is ________________________ (in terms of quantities mentioned above). [Hint: min–max. Use the fact, from the handout, that if $\hat{A}$ is self-adjoint then $\hat{A}^{-1}$ is also self-adjoint.]

Problem 2: More Green stuff

Recall that the displacement $u(x, t)$ of a stretched string [with fixed ends: $u(0, t) = u(L, t) = 0$] satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} + f(x, t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x, t)$ is an external force density (pressure) on the string.

(a) Suppose that $f(x, t) = \mathrm{Re}[g(x)e^{-i\omega t}]$, an oscillating force with a frequency ω. Show that, instead of solving the wave equation with this $f(x, t)$, we can instead use a complex force $\tilde{f}(x, t) = g(x)e^{-i\omega t}$, solve for a complex $\tilde{u}(x, t)$, and then take $u = \mathrm{Re}\, \tilde{u}$ to obtain the solution for the original $f(x, t)$.

(b) Suppose that $f(x, t) = g(x)e^{-i\omega t}$, and we want to find a *steady-state* solution $u(x, t) = v(x)e^{-i\omega t}$ that is oscillating everywhere at the same frequency as the input force. (This will be the solution after a long time if there is any dissipation in the system to allow the initial transients to die away.) Write an equation $\hat{A}v = g$ that v solves. Is $\hat{A}$ self-adjoint? Positive/negative definite/semidefinite?

(c) Solve for the Green's function $G(x, x')$ of this $\hat{A}$, assuming that $\omega \neq n\pi/L$ for any integer n (i.e. assume ω is not an eigenfrequency [why?]). [Write down the continuity conditions that G must satisfy at $x = x'$, solve for $x \neq x'$, and then use the continuity conditions to eliminate unknowns.]

(d) Form a finite-difference approximation A of your $\hat{A}$. Compute an approximate $G(x, x')$ in Matlab by `A \ dk`, where $\mathbf{d}_k$ is the unit vector of all 0's except for one $1/\Delta x$ at index $k = x'/\Delta x$, and compare (by plotting both) to your analytical solution from the previous part for a couple values of x' and a couple of different frequencies ω (one $< \pi/L$ and one $> \pi/L$) with $L = 1$.

(e) Show the limit $\omega \to 0$ of your G relates in some expected way to the Green's function of $-\frac{d^2}{dx^2}$ from class.