18.303 Midterm, Fall 2010

Each problem has **equal weight**. You have 55 minutes.

Problem 1: Finite differences (20 points)

From class and homework, the d^2/dx^2 operator on $[0, L]$ can be discretized into values $u_m \approx u(m\Delta x)$ at points $x = m\Delta x$ [for $\Delta x = L/(M+1)$] as $A = -D^T D/\Delta x^2$, where

$$D_{\text{Dirichlet}} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}, \qquad D_{\text{Neumann}} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}$$

when the boundary conditions are Dirichlet $u(0) = u(L) = 0$ [D an $(M+1)\times M$ matrix] and Neumann $u'(0) = u'(L) = 0$ [D an $(M-1)\times M$ matrix], respectively.

(a) Write down a new D matrix that implements the boundary conditions: $u(0)+u'(0) = 0$, $u(L) = 0$. For simplicity, apply the left boundary condition at $\Delta x/2$ rather than at 0, using $u(\Delta x/2) \approx (u_0 + u_1)/2$. Be sure to indicate how many rows and columns your D matrix has, and for what m values you have degrees of freedom u_m.

(b) If you use your D matrix from the previous part to solve $u''(x) = f(x)$ approximately via $A = -D^T D/\Delta x^2$, how fast would you expect the errors to vanish as $\Delta x \to 0$? [i.e. errors proportional to Δx^n for what power n?]

Problem 2: Adjoints and stuff (20 points)

Let $\Omega \subseteq \mathbb{R}^2$ be the rectangular 2d region $x \in [0, L_x]$, $y \in [0, L_y]$, with Dirichlet boundaries $u|_{d\Omega} = 0$. Consider the operator

$$\hat{A}u = \nabla^2 u + \frac{\partial}{\partial x}\left[c(x,y)\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial y}\left[c(x,y)\frac{\partial u}{\partial x}\right]$$

for some real-valued function $c(x, y)$. Let $\hat{B} = \frac{\partial}{\partial x} c \frac{\partial}{\partial y}$, i.e. $\hat{B}u = \frac{\partial}{\partial x}\left[c(x,y)\frac{\partial u}{\partial y}\right]$, the second term in $\hat{A}$.

(a) Find $\hat{B}^* =$______________ under the inner product $\langle u, v\rangle = \int_\Omega uv = \int_0^{L_x}\int_0^{L_y} dx\, dy\, u(x, y)\, v(x, y)$ [use real-valued functions for simplicity]. Hence conclude that $\hat{A}^* =$_________.

(b) Under what conditions on $c(x, y)$ will $\hat{A}u = \frac{\partial u}{\partial t}$ have solutions $u(x, y, t)$ that $\to 0$ as $t \to \infty$ for any initial condition $u(x, y, 0)$? Hint: consider $\langle u, \hat{A}u\rangle$ for $u \neq 0$, and note that the eigenvalues of the 2×2 matrix $\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$ are $1 \pm c$. (From class: $\langle u, \nabla^2 u\rangle = -\int_\Omega \nabla u \cdot \nabla u$.)

Problem 3: Thinking Green (20 points)

Consider the operator $\hat{A} = -c(\mathbf{x})\nabla^2$ in some 2d region $\Omega \subseteq \mathbb{R}^2$ with Dirichlet boundaries ($u|_{d\Omega} = 0$), where $c(\mathbf{x}) > 0$. Suppose the eigenfunctions of $\hat{A}$ are $u_n(\mathbf{x})$ with eigenvalues λ_n [that is, $\hat{A}u_n = \lambda_n u_n$] for $n = 1, 2, \ldots$, numbered in order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Let $G(\mathbf{x}, \mathbf{x}')$ be the Green's function of $\hat{A}$.

(a) If $f(\mathbf{x}) = \sum_n \alpha_n u_n(\mathbf{x})$ for some coefficients $\alpha_n =$____________________ (expression in terms of f and u_n), then $\int_\Omega G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') d^2\mathbf{x}' =$____________________ (in terms of α_n and u_n).

(b) The maximum possible value of

$$\frac{\int_\Omega \int_\Omega \frac{1}{c(\mathbf{x})} \overline{u(\mathbf{x})} G(\mathbf{x}, \mathbf{x}') u(\mathbf{x}')\, d^2\mathbf{x}\, d^2\mathbf{x}'}{\int_\Omega \frac{|u(\mathbf{x}'')|^2}{c(\mathbf{x}'')} d^2\mathbf{x}''},$$

for any possible $u(\mathbf{x})$, is ________________________ (in terms of quantities mentioned above). [Hint: min–max.]