18.303 Problem Set 6

Due Wednesday, 20 October 2010.

Problem 1: Regularization and Green's functions

(a) Let $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$, and $f(x) = 0$ for $x \leq 0$.

(i) Explain why f defines a regular distribution, even though $f(x)$ blows up as $x \to 0^+$.

(ii) Let $g(x) = -\frac{1}{2}\frac{1}{x^{3/2}}$ for $x > 0$, and $g(x) = 0$ for $x \leq 0$: $g(x)$ matches the ordinary derivative $f'(x)$ everywhere $f'(x)$ is defined (i.e. everywhere but $x = 0$). Explain why $g(x)$ does *not* correspond to any regular distribution.

(iii) Viewed as a distibution, f must have a derivative. Give an explicit formula for $f'\{\phi\}$ in terms of an integral of $\phi(x) - \phi(0)$ (*not* ϕ'). Hint: $f\{\phi\} = \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{\phi(x)}{\sqrt{x}} dx$ (why does this limit exist?), and integrate by parts using $\phi'(x) = \frac{d}{dx}[\phi(x) - \phi(0)]$. How is this different from trying to define a distribution directly from $g(x)$?

(b) Consider the $-\nabla^2$ operator in two dimensions, with domain $\Omega = \mathbb{R}^2$ (the whole space). [In Cartesian and cylindrical coordinates, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.] This has translational symmetry and rotational symmetry, so as in class the Green's function $-\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$ is of the form $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x} - \mathbf{x}', \mathbf{0}) = g(|\mathbf{x} - \mathbf{x}'|) = g(r)$, with $-\nabla^2 g(r) = \delta(\mathbf{x})$. Solve for $g(r)$ in three steps, as in class:

(i) For $r > 0$, $-\nabla^2 g(r) = 0$; show that this gives a solution proportional to $\ln(r)$ with some unknown proportionality constant c [determined in (iii) below], not including additive constants (which are in the nullspace of $-\nabla^2$ and hence make the solution nonunique, even if we impose rotational symmetry). [The fact that $\ln(r)$ blows up at $r \to \infty$ makes 2d more tricky than 3d.]

(ii) Explain why your $g(r)$ defines a regular distribution (i.e. why its integral against any (smooth, localized) test function ϕ is defined, even though the logarithm blows up as $r \to 0$ and $r \to \infty$. Suggestion: write $g\{\phi\}$ as $\lim_{\epsilon \to 0} \int_0^{2\pi} d\theta \int_\epsilon^\infty g(r)\phi(r, \theta) r\, dr$, where the limit avoids the need to define the value of $g(r)$ at $r = 0$.

(iii) Using the distributional derivative $(-\nabla^2 g)\{\phi\} = \int g(r)[-\nabla^2 \phi]$ for a smooth localized test function $\phi(\mathbf{x})$, show that $-\nabla^2 g = \delta(\mathbf{x})$ as desired [i.e. show $(-\nabla^2 g)\{\phi\} = \delta\{\phi\} = \phi(\mathbf{0})$] for an appropriate choice of c. Suggestion: write ϕ in cylindrical coordinates $\phi(r, \theta)$, where $\int = \lim_{\epsilon \to 0} \int_0^{2\pi} d\theta \int_\epsilon^\infty r\, dr$, as in the previous part.

Problem 2: A simple integral-equation solver

In this problem, you will implement a simple integral-equation solver in Matlab to solve for the Green's function $-\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$ in two dimensions, in the domain Ω given by the *exterior* of radius-1 circle at the origin, with Dirichlet boundaries (G vanishes at the surface $d\Omega$ of the circle). This problem is diagrammed in figure 1(left). A physical interpretation in electrostatics (for example) would be that $\delta(\mathbf{x} - \mathbf{x}')$ corresponds to a line charge at $\mathbf{x}'$, and $G(\mathbf{x}, \mathbf{x}')$ is the resulting potential ("voltage") at $\mathbf{x}$ [in units where $\varepsilon_0 = 1$], where the circle is a metal cylinder that is grounded (voltage = 0).

If the circle weren't there ("empty space"), the Green's function would be $c \ln |\mathbf{x} - \mathbf{x}'|$ from the previous problem, for some c that you found; for simplicity, in this problem just suppose $c = 1$. With the circle there, the solution is changed—it not only has the $\ln |\mathbf{x} - \mathbf{x}'|$ term coming directly from the $\delta(\mathbf{x} - \mathbf{x}')$, but there are also terms coming from the cylinder—compared to the empty-space solution, there are additional source terms on $d\Omega$ so that the total G satisfies the boundary condition. The (first-kind) integral-equation formulation of this (as discussed in class in 3d) is to put *unknown* sources with density $\sigma(x)$ all around $d\Omega$, so that the total solution:

$$G(\mathbf{x}, \mathbf{x}') = \ln |\mathbf{x} - \mathbf{x}'| + \int_{d\Omega} \sigma(\mathbf{x}'') \ln |\mathbf{x} - \mathbf{x}''| d\mathbf{x}''$$

satisfies the boundary condition $G(\mathbf{x}, \mathbf{x}') = 0$ for $\mathbf{x} \in d\Omega$. We then try to find $\sigma(\mathbf{x}'')$, discretized/approximated in some way, to make this happen. (In the electrostatic interpretation, σ is an induced charge density on the cylinder surface.)

Here, we will approximate the continuous distribution $\sigma(\mathbf{x}'')$ by N point sources $\sigma_n \delta(\mathbf{x}'' - \mathbf{x}_n)$ for N equally-spaced points $\mathbf{x}_n$ around the circle, with unknown amplitudes σ_n, and we will enforce the Dirichlet boundary condition approximately, only at another set of N points $\mathbf{y}_m$ that are halfway in between the $\mathbf{x}_n$ points. (Caveat: there are much

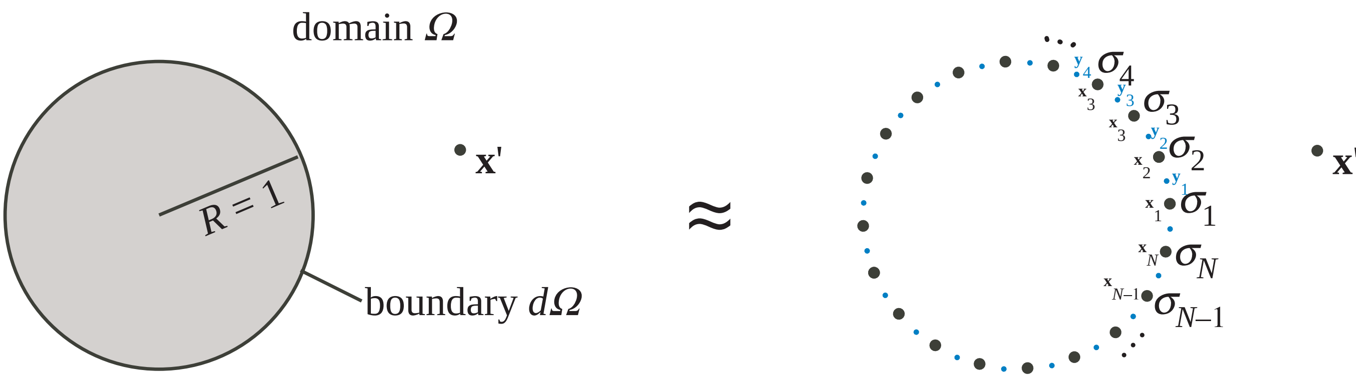

Figure 1: Schematic of 2d Green's function problem. *Left*: for a domain Ω that is the *exterior* of a radius-1 cylinder, we want to solve for the Green's function $-\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x}, \mathbf{x}')$ with Dirichlet (zero) boundary conditions at $d\Omega$. *Right*: approximate integral-equation problem for the Green's function, in which we replace the surface by a set of *unknown* point sources $\sigma_n \delta(\mathbf{x} - \mathbf{x}_n)$ at N points $\mathbf{x}_n$. We will then enforce the Dirichlet boundary conditions at N points $\mathbf{y}_n$ on the cylinder, halfway in between the $\mathbf{x}_n$ points.

better ways to set up this approximation in a "serious" calculation; these are generally called Nyström methods.) This is shown schematically in figure 1(right). That is, we will solve:

$$G(\mathbf{y}_m, \mathbf{x}') = 0 \approx \ln|\mathbf{y}_m - \mathbf{x}'| + \sum_{n=1}^{N} \sigma_n \ln|\mathbf{y}_m - \mathbf{x}_n|$$

for $m = 1, 2, \ldots, N$, giving N equations for the N unknowns σ_n. The key quantities will be the vector $\mathbf{b}$ with components $b_m = \ln|\mathbf{y}_m - \mathbf{x}'|$ and the matrix A with $A_{mn} = \ln|\mathbf{y}_m - \mathbf{x}_n|$. These are created in Matlab by the following commands, for $N = 1001$[1] and $\mathbf{x}' = (2, 0)$:

```
N = 1001;
dtheta = 2*pi / N;
x_theta = [0:N-1]' * dtheta;
y_theta = x_theta + dtheta/2;
x_x = cos(x_theta);
x_y = sin(x_theta);
y_x = cos(y_theta);
y_y = sin(y_theta);
b = log(sqrt((y_x - 2).^2 + y_y.^2));
o = ones(1, N);
A = log(sqrt((y_x * o - o' * x_x').^2 + (y_y * o - o' * x_y').^2));
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(a) Why didn't we just choose $\mathbf{y}_n = \mathbf{x}_n$? (Caveat: as mentioned above, there are much better ways to handle this difficulty than what we are doing here.)

(b) If we want to solve for the vector $\mathbf{s} = (\sigma_1, \sigma_2, \ldots)^T$, what equation should $\mathbf{s}$ solve to enforce the boundary conditions above? (Hint: not quite $A\mathbf{s} = \mathbf{b}$.)

(c) Plot your solution $\mathbf{s}$ versus θ (`x_theta`) [it will be easier to read if you convert θ to degrees $180\theta/\pi$].

(d) Given your solution $\mathbf{s}$, compute $G(\mathbf{x}, \mathbf{x}')$ at $\mathbf{x} = (1, 1)$.

(e) Another possible boundary condition, instead of setting $G = 0$ at the cylinder, would be to require the "total charge" $\int \sigma \approx \Sigma \sigma_n$ to be zero and the cylinder to be at a constant (but arbitrary) potential: this would be what happens if you bring a point (line) charge at $\mathbf{x}'$ close to a *neutral*, isolated conductor. Explain a simple modification to your $\mathbf{s}$ from the previous part that will solve this problem (zero total charge, constant potential at the $\mathbf{y}_m$ points). Using this modification and your previous solution, compute the new potential of the cylinder (for the same $\mathbf{x}'$) at the $\mathbf{y}_m$ points.

[1] We choose N to be odd because, if N is even, A is singular with a nullspace spanned by $(+1, -1, +1, -1, \ldots)^T$. This is just an unfortunate side-effect of the somewhat simplistic way we are going about this integral equation.