18.303 Midterm Topic Summary

November 1, 2010

- Eigenfunctions and eigenvalues: given the eigenfunctions/eigenvalues of $\hat{A}$, using them to solve equations such as $\hat{A}u = f$, $\hat{A}u = \partial u/\partial t$, and $\hat{A}u = \partial u^2/\partial t^2$ (or other variations).
  - Important special solvable cases: Fourier series and Bessel functions. Separation of variables in space (and time, but we view separation in time as decomposition into eigenfunctions).
- Inner products $\langle u, v\rangle$, adjoints, definiteness, and properties of $\hat{A}$. Given $\hat{A}$ and an inner product, how to find $\hat{A}^*$ and how to show whether $\hat{A}$ is self-adjoint (Hermitian or real-symmetric), positive/negative definite or semidefinite, and what the consequences of these facts are for eigenvalues and eigenfunctions or for equations in terms of $\hat{A}$ like $\hat{A}u = \partial u/\partial t$ and so on.
  - e.g. why the heat/diffusion equation has exponentially decaying solutions (and why oscillations are damped out especially rapidly).
  - understand how to integrate by parts with ∇.
- Null spaces of operators (left and right) and their consequences on solvability, uniqueness and (in the case of left nullspaces) conservation laws for $\hat{A}u = \partial u/\partial t$ (e.g. conservation of mass for difusion equation).
- Effect of boundary conditions on definiteness, self-adjointness, nullspaces. Turning general Dirichlet and Neumann boundaries into zero Dirichlet and Neumann boundary conditions (i.e. boundary conditions as modifying the right-hand-side).
- The Rayleigh quotient and the min–max (variational) theorem, and its consequences. Guessing the form of the smallest-$|\lambda|$ solutions using the min–max theorem.
- Finite-difference discretizations:
  - Analyzing the order of accuracy of a given discretization (with Taylor expansions).

- Writing the discretized A to have the same self-adjoint/definite properties as $\hat{A}$ (e.g. by writing A in terms of $D^T D$.
- Effect of boundary conditions.
- Finite-difference discretizations in more than 1 dimension.

- Green's functions and inverse operators.
  - Relationship between $\hat{A}^{-1}$ and Green's function $\hat{A}G(\vec{x}, \vec{x}') = \delta(\vec{x} - \vec{x}')$. (And how properties of $\hat{A}$ effect properties of G, e.g. self-adjointness gives reciprocity.)
  - Solving G in simple cases (using the delta function, not using ugly limits), especially empty space $\Omega = \mathbb{R}^d$.
  - How solutions $\hat{A}^{-1}f$ are made from the Green's function.
  - How the Green's function G_0 of empty space relates to Green's functions in inhomogeneous systems or systems with boundaries. Born–Dyson series and Born approximations.

- Delta functions and distributions.
  - Definition, regular vs. singular distributions. Differences from ordinary functions.
  - Distributional derivatives.
  - Solving PDEs with δ on the right-hand side (e.g. finding Green's functions, above).