18.303 Problem Set 4

Due Wednesday, 3 October 2012.

Problem 1: Separability and symmetry

Often, separability of the solutions is a consequence of symmetry. In this problem, you will show this for the case of continuous translational symmetry: a PDE that is invariant under translation in z. In particular, suppose that we have the operator

$$\hat{A} = -\frac{1}{w(x,y)}\nabla\cdot c(x,y)\nabla$$

for real coefficients $c, w > 0$ that are invariant in z. We also have a z-invariant domain Ω which is an infinitely long cylinder (not necessarily circular) parallel to the z axis with some z-invariant cross-section; equivalent, Ω could be described as the where $s(x,y) < 0$ for some z-invariant function s whose contour $s(x,y) = 0$ gives the boundary $\partial\Omega$. Under the boundary condition $u(x,y,z)|_{\partial\Omega} = 0$, this operator is self-adjoint under the inner product $\langle u, v\rangle = \int_\Omega w\bar{u}v$. We wish to to find the eigenfunctions $\hat{A}u = \lambda u$ in a *separable* form $u(x,y,z) = v(x,y)Z(z)$. As claimed in class, it turns out that $Z(z)$ always has the same form e^{-ikz} for bounded solutions of *any* z-invariant problem.

(a) Define the *translation* operator $\hat{T}_a$ by $\hat{T}_a u(x,y,z) = u(x,y,z-a)$: $\hat{T}_a$ shifts functions by a in the z direction.

  (i) Show that $\hat{T}_a$ is unitary: $\hat{T}_a^* = \hat{T}_a^{-1} = \hat{T}_{-a}$.

  (ii) Show that $\hat{T}_a$ *commutes* with $\hat{A}$ for all a: $\hat{T}_a\hat{A} = \hat{A}\hat{T}_a$, or equivalently (multiplying both sides by $\hat{T}_{-a}$ on the left), that $\hat{T}_{-a}\hat{A}\hat{T}_a = \hat{A}$. (The domain Ω and the boundary conditions are also obviously invariant under $\hat{T}_a$.) (In fact, one can *define* translation invariance of $\hat{A}$ as commuting with $\hat{T}_a$!)

(b) Suppose that we have an eigenfunction $\hat{A}u = \lambda u$, and suppose for simplicity that λ is nondegenerate (has multiplicity 1, i.e. there is only one linearly independent eigenfunction of this λ). Show that u must also be an eigenfunction of $\hat{T}_a$ for all a. [Hint: consider $\hat{A}(\hat{T}_a u)$ and use the fact that they commute.]

(More generally, one can show that we can find "simultaneous" eigenfunctions of any commuting operators.)

(c) If u is an eigenfunction of $\hat{T}_a$ for all a, that means $\hat{T}_a u = \alpha(a)u$ for some eigenvalues $\alpha(a)$. Show that the functions $\alpha(a)$ and $u(\mathbf{x})$ must have the properties:

  (i) $\alpha(0) = 1$

  (ii) $u(x,y,z) = v(x,y)\alpha(-z)$ for $v(x,y) = u(x,y,0)$.

  (iii) $\alpha(a_1 + a_2) = \alpha(a_1)\alpha(a_2)$.

  (iv) If α is differentiable,[1] then $\alpha(a) = e^{\beta a}$ for some β. [Hint: use the definition of derivative $\alpha'(x) = \lim_{h\to 0}\frac{\alpha(x+h)-\alpha(x)}{h}$, combined with the above properties, to show that $\alpha'(x) = \alpha'(0)\alpha(x)$.] If we do not allow exponentially growing solutions u, it follows that $\beta = -ik$ for some real k.

  Hence we have the separable form $u(x,y,z) = v(x,y)e^{ikz}$ as a consequence of $\hat{T}$ commuting with $\hat{A}$!

[1] Actually, we can impose a much weaker condition on α and still obtain that $\alpha(a) = e^{-ika}$. For example, it is enough to assume that α is *continuous anywhere*. Or, even weaker, we merely have to assume that α is "measurable" (that its integral exists in the Lebesgue sense over any finite interval).

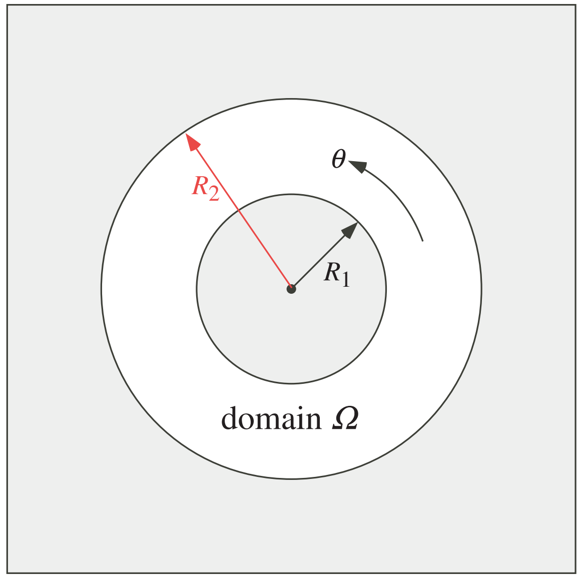

Figure 1: Schematic of the domain Ω for problem 3: an annular region in two dimensions, with radii $r \in [R_1, R_2]$ and angles $\theta \in [0, 2\pi]$.

This is a simplified case of a much deeper theory about how symmetry relates to the solutions of PDEs. More generally, especially for the case of degenerate (multiplicity > 1) eigenvalues and combinations of multiple symmetry operators (rotations, reflections, translations, ...), one uses the tool of *group theory* to describe the interactions of the symmetry operators and the tool of *group representation theory* to describe the consequences for the eigenfunctions and other solutions.

Problem 2: Bessel, Bessel, toil and mess...el

In class, we solved for the eigenfunctions of ∇^2 in two dimensions, in a cylindrical region $r \in [0, R]$, $\theta \in [0, 2\pi]$ using separation of variables, and obtained Bessel's equation and Bessel-function solutions. Although Bessel's equation has two solutions $J_m(kr)$ and $Y_m(kr)$ (the *Bessel functions*), the second solution (Y_m) blows up as $r \to 0$ and so for that problem we could only have $J_m(kr)$ solutions (although we still needed to solve a transcendental equation to obtain k).

In this problem, you will solve for the 2d eigenfunctions of ∇^2 in an **annular** region Ω that does *not contain the origin*, as depicted schematically in Fig. 1, between radii R_1 and R_2, so that you will need *both* the J_m and Y_m solutions. Exactly as in class, the separation of variables ansatz $u(r, \theta) = \rho(r)\tau(\theta)$ leads to functions $\tau(\theta)$ spanned by $\sin(m\theta)$ and $\cos(m\theta)$ for integers m, and functions $\rho(r)$ that satisfy Bessel's equation. Thus, the eigenfunctions are of the form:

$$u(r, \theta) = [\alpha J_m(kr) + \beta Y_m(kr)] \times [A \cos(m\theta) + B \sin(m\theta)]$$

for arbitrary constants A and B, for integers $m = 0, 1, 2, \ldots$, and for constants α, β, and k to be determined.

For fun, we will also **change the boundary conditions** somewhat. We will impose a Dirichlet ($u = 0$) condition at the inner radius R_1, and a "Neumann" boundary condition $\frac{\partial u}{\partial r} = 0$ at the outer radius R_2. That is, for a function $u(r, \theta)$ in cylindrical coordinates, $u(R_1, \theta) = 0$ and $\frac{\partial u}{\partial r}|_{r=R_2} = 0$. The following *exact* identities for the derivatives of the Bessel functions will be helpful:

$$J'_m(x) = \frac{J_{m-1}(x) - J_{m+1}(x)}{2}, \quad Y'_m(x) = \frac{Y_{m-1}(x) - Y_{m+1}(x)}{2}$$

(a) Using the boundary conditions, write down two equations for α, β and k, of the form $E\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$ for some 2×2 matrix E. This only has a solution when $\det E = 0$, and from this fact obtain a single equation for k of the form $f_m(k) = 0$ for some function f_m that depends on m. This is a transcendental equation; you can't solve it by hand for k. In terms of k (which is still unknown), write down a possible expression for α and β, i.e. a basis for $N(E)$.

(b) Assuming $R_1 = 1$, $R_2 = 2$, plot your function $f_m(k)$ versus $k \in [0, 20]$ for $m = 0, 1, 2$. Note that Matlab provides the Bessel functions built-in: $J_m(x)$ is `besselj(m,x)` and $Y_m(x)$ is `bessely(m,x)`. You can plot a function with the `fplot` command. For example, to plot $J_m(k) \cdot Y_m(3k)$ as a blue line, you would do `fplot(@(k) besselj(m, k) .* bessely(m, 3*k), [0,20], 'b')`, assuming you have assigned `m` to a value. [The syntax `@(k)` _ _ _ _ _ defines a function of k. Note: `.*` rather than `*` for the multiply; this is to do an element-wise multiplication rather than a matrix multiplication when k is a vector.] It might be helpful to use `hold on` between plots so that you can plot f_0, f_1, and f_2 on the same plot (labelled with the `legend` command, of course).

(c) For $m = 0$, find the first three (smallest $k > 0$) solutions k_1, k_2, and k_3 to $f_0(k) = 0$. Get a rough estimate first from your graph (zooming if necessary), and then get an accurate answer by calling the `fzero` function (Matlab's nonlinear root-finding function). For example, to find a root of $\cos(x) - x$ near $x = 1$, you would call `fzero(@(x) cos(x) - x, 1)` in Matlab. Plot the corresponding functions $\alpha J_0(kr) + \beta Y_0(kr)$.

(d) Because ∇^2 is self-adjoint under $\langle u, v\rangle = \int_\Omega \bar{u}v$ (and it is easy to show that this is still true with these boundary conditions), we know that the eigenfunctions must be orthogonal. From class, this implies that the radial parts must also be orthogonal when integrated via $\int r\,dr$. Check that your Bessel solutions for k_1 and k_2 are indeed orthogonal, by numerically integrating them via the `quadl` function in Matlab. In particular, if you have assigned your α, β, k solutions to variables `a1`,`b1`,`k1` and `a2`,`b2`,`k2` in Matlab, then the integral for $r \in [1, 2]$ is given to at least six digits of accuracy by `quadl(@(r) (a1*besselj(0,k1*r) + b1*bessely(0,k1*r)) .* (a2*besselj(0,k2*r) + b2*bessely(0,k2*r)) .* r, 1,2, 1e-6)` in Matlab.