18.303 Midterm Solutions, Fall 2010

Problem 1: Finite differences (20 points)

(a) At the right boundary it is the same as the Dirichlet matrix before. At the left boundary, evaluating at $\Delta x/2$, we have

$$u'_{1/2} \approx \frac{u_1 - u_0}{\Delta x} \approx -\frac{u_0 + u_1}{2} \approx -u_{1/2},$$

Solving for u_0, we have:

$$\frac{u_1 - u_0}{\Delta x} = -\frac{u_0 + u_1}{2} \implies u_0\left(\frac{1}{2} - \frac{1}{\Delta x}\right) = -u_1\left(\frac{1}{2} + \frac{1}{\Delta x}\right) \implies u_0 = u_1 \frac{\frac{1}{\Delta x} + \frac{1}{2}}{\frac{1}{\Delta x} - \frac{1}{2}} = u_1 \frac{2 + \Delta x}{2 - \Delta x},$$

so that the first row of D, which computes $u'_{1/2}$, now computes $u'_{1/2} = \frac{u_1 - u_0}{\Delta x} = \frac{u_1}{\Delta x}\left(1 - \frac{2+\Delta x}{2-\Delta x}\right) = \frac{u_1}{\Delta x}\frac{-2\Delta x}{2-\Delta x}$, and hence D could be written as the following $(M+1) \times M$ matrix that computes the derivatives from M unknowns $(u_1, u_{2,} \ldots, u_M)^T$:

$$D = \begin{pmatrix} \frac{-2\Delta x}{2-\Delta x} & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -1 \end{pmatrix}.$$

(b) Since we applied the left boundary condition at the wrong place $\Delta x/2$, this will introduce an error $\sim \Delta x$.

[Like in the Neumann pset, however, we can correct this by changing Δx to "stretch" the grid so that the left boundary is at 0: $\Delta x = L/(M + 0.5)$, in which case we would get errors $\sim \Delta x^2$ since the difference formulas are otherwise 2nd-order.]

Problem 2: Adjoints and stuff (20 points)

(a) **Solution 1:** Writing out the integrals:

$$\begin{aligned}
\left\langle u, \hat{B}v \right\rangle &= \int_0^{L_x}\int_0^{L_y} dxdy\, u\frac{\partial}{\partial x}\left[c(x,y)\frac{\partial v}{\partial y}\right] \\
&= -\int_0^{L_x}\int_0^{L_y} dxdy\, \frac{\partial u}{\partial x}\left[c(x,y)\frac{\partial v}{\partial y}\right] + \int_0^{L_y} dy\, \cancel{u(x,y)\left[c(x,y)\frac{\partial v}{\partial y}\right]\Big|_{x=0}^{L_x}} \\
&= +\int_0^{L_x}\int_0^{L_y} dxdy\, \frac{\partial}{\partial y}\left[c(x,y)\frac{\partial u}{\partial x}\right] v - \int_0^{L_x} dx\, \cancel{\left[c(x,y)\frac{\partial u}{\partial x}\right] v(x,y)\Big|_{y=0}^{L_y}} \\
&= \left\langle \frac{\partial}{\partial y}\left[c\frac{\partial u}{\partial x}\right], v \right\rangle,
\end{aligned}$$

where we have integrated by parts twice (flipping the sign twice to get back to +) and the boundary terms vanish as usual from Dirichlet. Note that integrating by parts **switches the order** of the derivatives! Hence:

$$\hat{B}^* = \left(\frac{\partial}{\partial x}\left[c(x,y)\frac{\partial}{\partial y}\right]\right)^* = \frac{\partial}{\partial y}\left[c(x,y)\frac{\partial}{\partial x}\right],$$

and thus

$$\hat{A} = \nabla^2 + \hat{B} + \hat{B}^*.$$

It follows immediately that $\hat{A}^* = \nabla^2 + \hat{B}^* + \hat{B} = \hat{A}$ (since we already know ∇^2 is self-adjoint with this inner product and boundary condition).

Solution 2:

By now, you should be used to the fact that integrating by parts moves a first derivative from one side to the other of an inner product with a sign flip, plus a boundary term which is zero for Dirichlet boundaries, hence the above solution could be written in short form as:

$$\begin{aligned}\left\langle u, \hat{B}v \right\rangle &= \left\langle u, \frac{\partial}{\partial x}\left[c\frac{\partial v}{\partial y}\right]\right\rangle \\ &= -\left\langle \frac{\partial u}{\partial x}, c\frac{\partial v}{\partial y}\right\rangle \\ &= -\left\langle c\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}\right\rangle \\ &= +\left\langle \frac{\partial}{\partial y}\left[c\frac{\partial u}{\partial x}\right], v\right\rangle = \left\langle \hat{B}^* u, v\right\rangle .\end{aligned}$$

Solution 3 (not expected): A more beautiful and general way to do this is to write

$$\hat{A}u = \nabla \cdot \left[\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \nabla u\right] = \nabla \cdot (C\nabla u)$$

with the real-symmetric matrix $C = C^T$, in which case we can integrate by parts using the divergence theorem for any shape Ω (and in any number of dimensions if we make C a bigger matrix):

$$\begin{aligned}\langle u, \nabla \cdot (C\nabla v)\rangle &= \langle -\nabla u, C\nabla v\rangle + \cancel{\oint\!\!\!\oint_{d\Omega} uC\nabla v \cdot d\mathbf{A}} = -\int_\Omega \nabla u \cdot C\nabla v \\ &= -\int_\Omega (C^T\nabla u)\cdot \nabla v = -\langle C\nabla u, \nabla v\rangle \\ &= +\langle \nabla \cdot (C\nabla u), v\rangle - \cancel{\oint\!\!\!\oint_{d\Omega} (C\nabla u) v \cdot d\mathbf{A}},\end{aligned}$$

quoting the integration-by-parts for ∇ as in class (where the boundary terms vanish by Dirichlet), hence $\hat{A} = \hat{A}^*$.

(b) It will have (exponentially) decaying solutions if $\hat{A}$ is negative-definite, i.e. $\langle u, \hat{A}u\rangle < 0$ for all $u \neq 0$ (hence negative λ's). To check this, we integrate by parts *once*, to obtain:

$$\begin{aligned}\langle u, \hat{A}u\rangle &= -\langle \nabla u, \nabla u\rangle - \langle \frac{\partial u}{\partial x}, c\frac{\partial u}{\partial y}\rangle - \langle \frac{\partial u}{\partial y}, c\frac{\partial u}{\partial x}\rangle \\ &= -\int_\Omega \nabla u \cdot \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \nabla u,\end{aligned}$$

which is < 0 if $\nabla u \neq 0$ (true because of the boundary conditions: u cannot be a nonzero constant) and if $\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$ is positive-definite, i.e. its eigenvalues $1 \pm c$ must be > 0, i.e. $|c| < 1$..

Problem 3: Thinking Green (20 points)

(a) $\hat{A}$ is self-adjoint under the inner product $\langle u, v\rangle = \int_\Omega \bar{u}v/c$, from class (and the 1d version in homework), and hence the eigenfunctions are orthogonal, hence $\alpha_n = \langle u_n, f\rangle / \langle u_n, u_n\rangle$. The expression $u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')d^2\mathbf{x}'$ is just the solution to $\hat{A}u = f$, i.e. it is $u = \hat{A}^{-1}f = \sum_n \frac{\alpha_n}{\lambda_n} u_n(\mathbf{x})$.

(b) This is just $\langle u, \hat{A}^{-1}u\rangle / \langle u, u\rangle$, which is a Rayleigh quotient for the self-adjoint operator $\hat{A}^{-1}$, and hence (thanks to the min–max theorem) it is bounded above by the **largest** eigenvalue of $\hat{A}^{-1}$, which is $1/\lambda_1$ (the inverse of the **smallest** eigenvalue of $\hat{A}$). (Note that by positive-definiteness $0 < \lambda_1 < \lambda_2 < \cdots$.)