18.303 Midterm Solutions, Fall 2014

Problem 1:

Define the inner product $\langle u,v\rangle_c = \int_\Omega c\bar{u}v = \int_\Omega \overline{cu}v = \langle cu, v\rangle$ where $\langle u,v\rangle = \int_\Omega \bar{u}v$. Then $\langle u,\hat{A}v\rangle_c = \langle cu,(cv)''\rangle = \langle (cu)'', cv\rangle = \langle c\hat{A}u, v\rangle = \langle \hat{A}u, v\rangle_c$, where we have used the self-adjointness of d^2/dx^2 under $\langle\cdot,\cdot\rangle$ from class. Therefore, $\hat{A} = \hat{A}^*$ under the $\langle u,v\rangle_c$ inner product (which is a proper inner product for real $c > 0$).

Problem 2:

We need $-\nabla^2 g = \delta(\mathbf{x})$, and we determine this by evaluating both sides with an arbitrary test function ψ, using the distributional derivative $(-\nabla^2 g)\{\psi\} = g\{-\nabla^2\psi\}$ as in class. In cylindrical coordinates:

$$\begin{aligned}
g\{-\nabla^2\psi\} &= -\lim_{\epsilon\to 0^+}\int_\epsilon^\infty r\,dr\int_0^{2\pi} d\phi\, c\ln r\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right]\\
&= -c\int_0^{2\pi} d\phi\lim_{\epsilon\to 0^+}\int_\epsilon^\infty dr\,\ln r\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) - \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\frac{c\ln r}{r}\,dr\,\cancel{\left.\frac{\partial\psi}{\partial\phi}\right|_0^{2\pi}}\\
&= -c\int_0^{2\pi} d\phi\lim_{\epsilon\to 0^+}\left[\cancel{\left. r\ln r\frac{\partial\psi}{\partial r}\right|_\epsilon^\infty} - \int_\epsilon^\infty dr\,\frac{\partial(\ln \cancel{r})}{\partial r}\left(\cancel{r}\frac{\partial\psi}{\partial r}\right)\right]\\
&= c\int_0^{2\pi} d\phi\lim_{\epsilon\to 0^+}\left.\psi\right|_{r=\epsilon}^{r=\infty} = -2\pi c\psi(\mathbf{0}).
\end{aligned}$$

To get $\delta\psi = \psi(\mathbf{0})$, therefore, we need $\boxed{c = -1/2\pi}$.

Problem 3:

It is convenient to write $\hat{D}_\sigma = \hat{D} - \sigma I$, where I is the 2×2 identity matrix. Then it follows from $\hat{D}^* = -\hat{D}$ and $(\sigma I)^* = \sigma I$ (since σ is a real scalar and I is obviously self-adjoint) under the usual inner product $\langle \mathbf{w},\mathbf{w}'\rangle = \int_\Omega \mathbf{w}^*\mathbf{w}'$ that we have $\hat{D}_\sigma^* = -\hat{D} - \sigma I$ and $\hat{D}_\sigma + \hat{D}_\sigma^* = -2\sigma I$.

(a) For a solution $\mathbf{w}$ of $\hat{D}_\sigma\mathbf{w} = \partial\mathbf{w}/\partial t$, we have

$$\begin{aligned}
\partial\langle\mathbf{w},\mathbf{w}\rangle/\partial t &= \langle\partial\mathbf{w}/\partial t,\mathbf{w}\rangle + \langle\mathbf{w},\partial\mathbf{w}/\partial t\rangle\\
&= \langle\hat{D}_\sigma\mathbf{w},\mathbf{w}\rangle + \langle\mathbf{w},\hat{D}_\sigma\mathbf{w}\rangle\\
&= \langle\mathbf{w},\hat{D}_\sigma^*\mathbf{w}\rangle + \langle\mathbf{w},\hat{D}_\sigma\mathbf{w}\rangle = \langle\mathbf{w},\left(\hat{D}_\sigma^*+\hat{D}_\sigma\right)\mathbf{w}\rangle\\
&= -2\langle\mathbf{w},\sigma\mathbf{w}\rangle = -2\int\sigma(x)\|\mathbf{w}(x)\|^2 < 0
\end{aligned}$$

and hence $\|\mathbf{w}\|^2 = \langle\mathbf{w},\mathbf{w}\rangle$ is **decreasing** in time.

If $\sigma(x) \geq \sigma_0 > 0$ for some σ_0, then we can go further and say that $E(t) = \|\mathbf{w}\|^2$ is decaying at least exponentially fast in time, since in that case $dE/dt \leq -2\sigma_0 E$ and from this one can show that $E(t) \leq E(0)e^{-2\sigma_0 t}$.

(i) Given an eigensolution $\hat{D}_\sigma\mathbf{w}_n = \lambda_n\mathbf{w}_n$, we can consider

$$\begin{aligned}
\langle\mathbf{w}_n,(\hat{D}_\sigma+\hat{D}_\sigma^*)\mathbf{w}_n\rangle &= -2\langle\mathbf{w}_n,\sigma\mathbf{w}_n\rangle\\
&= \langle\mathbf{w}_n,\hat{D}_\sigma\mathbf{w}_n\rangle + \langle\hat{D}_\sigma\mathbf{w}_n,\mathbf{w}_n\rangle\\
&= \langle\mathbf{w}_n,\lambda_n\mathbf{w}_n\rangle + \langle\lambda_n\mathbf{w}_n,\mathbf{w}_n\rangle\\
&= \left(\lambda_n+\overline{\lambda_n}\right)\langle\mathbf{w}_n,\mathbf{w}_n\rangle = \langle\mathbf{w}_n,\mathbf{w}_n\rangle 2\,\mathrm{Re}\,\lambda_n.
\end{aligned}$$

Note that we moved $\hat{D}_\sigma^*$ to act on the left via its adjoint. It is *not in general true* that $\hat{D}_\sigma^*\mathbf{w}_n = \overline{\lambda_n}\mathbf{w}_n$. Then we have:

$$\mathrm{Re}\,\lambda_n = -\frac{\langle\mathbf{w}_n,\sigma\mathbf{w}_n\rangle}{\langle\mathbf{w}_n,\mathbf{w}_n\rangle} < 0$$

since $\sigma > 0$. Hence the eigensolutions are **decaying exponentially** in time (while they oscillate via the imaginary part of λ_n), from their time dependence $e^{\lambda_n t}$.

Problem 4:

We will have $\partial u/\partial t = \partial v/\partial x - \sigma u$ and $\partial v/\partial t = \partial u/\partial x - \sigma v$, so the only new terms are the σ terms. In the discretized $\partial u/\partial t$ equation, the left-hand side is evaluated at point m and time $n{+}0.5$, so we have to get $u_m^{n+0.5} = \frac{u_m^n + u_m^{n+1}}{2} + O(\Delta t^2)$ by averaging (similarly to how we handled the Crank-Nicolson discretization in class). Similarly for the $\partial v/\partial t$ equation. Hence, we obtain:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{v_{m+0.5}^{n+0.5} - v_{m-0.5}^{n+0.5}}{\Delta x} - \sigma \frac{u_m^{n+1} + u_m^n}{2},$$

$$\frac{v_{m+0.5}^{n+0.5} - v_{m+0.5}^{n-0.5}}{\Delta t} = \frac{u_{m+1}^n - u_m^n}{\Delta x} - \sigma \frac{v_{m+0.5}^{n+0.5} + v_{m+0.5}^{n-0.5}}{2}.$$

Solving for u_m^{n+1} and $v_{m+0.5}^{n+0.5}$, we obtain the "leap-frog" equations:

$$u_m^{n+1} = \left(1 + \frac{\sigma \Delta t}{2}\right)^{-1} \left[\left(1 - \frac{\sigma \Delta t}{2}\right) u_m^n + \frac{\Delta t}{\Delta x}\left(v_{m+0.5}^{n+0.5} - v_{m-0.5}^{n+0.5}\right)\right],$$

$$v_{m+0.5}^{n+0.5} = \left(1 + \frac{\sigma \Delta t}{2}\right)^{-1} \left[\left(1 - \frac{\sigma \Delta t}{2}\right) v_{m+0.5}^{n-0.5} + \frac{\Delta t}{\Delta x}\left(u_{m+1}^n - u_m^n\right)\right].$$

Note that $\sigma > 0$, so we are never dividing by zero in $1 + \sigma \Delta t/2$, regardless of Δt, which is comforting.