18.303 Problem Set 4

Due Wednesday, 15 October 2014.

Problem 1:

Consider the operator $\hat{A} = -c(\mathbf{x})\nabla^2$ in some 2d region $\Omega \subseteq \mathbb{R}^2$ with Dirichlet boundaries ($u|_{\partial\Omega} = 0$), where $c(\mathbf{x}) > 0$. Suppose the eigenfunctions of $\hat{A}$ are $u_n(\mathbf{x})$ with eigenvalues λ_n [that is, $\hat{A}u_n = \lambda_n u_n$] for $n = 1, 2, \ldots$, numbered in order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Let $G(\mathbf{x}, \mathbf{x}')$ be the Green's function of $\hat{A}$.

(a) If $f(\mathbf{x}) = \sum_n \alpha_n u_n(\mathbf{x})$ for some coefficients $\alpha_n =$____________________ (expression in terms of f and u_n), then $\int_\Omega G(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')d^2\mathbf{x}' =$____________________ (in terms of α_n and u_n).

(b) The maximum possible value of

$$\frac{\int_\Omega \int_\Omega \frac{1}{c(\mathbf{x})}\overline{u(\mathbf{x})}G(\mathbf{x}, \mathbf{x}')u(\mathbf{x}')\,d^2\mathbf{x}\,d^2\mathbf{x}'}{\int_\Omega \frac{|u(\mathbf{x}'')|^2}{c(\mathbf{x}'')}d^2\mathbf{x}''},$$

for any possible $u(\mathbf{x})$, is ________________________ (in terms of quantities mentioned above). [Hint: min–max. Use the fact, from the handout, that if $\hat{A}$ is self-adjoint then $\hat{A}^{-1}$ is also self-adjoint.]

Problem 2:

In this problem, we will solve the Laplacian eigenproblem $-\nabla^2 u = \lambda u$ in a 2d radius-1 cylinder $r \leq 1$ with Dirichlet boundary conditions $u|_{r=1\Omega} = 0$ by "brute force" in Julia with a 2d finite-difference discretization, and compare to the analytical Bessel solutions. You will find the IJulia notebooks posted on the 18.303 website for Lecture 9 and Lecture 11 extremely useful! (Note: when you open the notebook, you can choose "Run All" from the Cell menu to load all the commands in it.)

(a) Using the notebook for a 100×100 grid, compute the 6 smallest-magnitude eigenvalues and eigenfunctions of A with `i, Ui=eigs(Ai,nev=6,which=''SM'')`. The eigenvalues are given by `i`. The notebook also shows how to compute the exact eigenvalue from the square of the root of the Bessel function. Compared with the high-accuracy λ_1 value, compute the error $\Delta\lambda_1$ in the corresponding finite-difference eigenvalue from the previous part. Also compute $\Delta\lambda_1$ for $N_x = N_y = 200$ and 400. How fast is the convergence rate with Δx? Can you explain your results, in light of the fact that the center-difference approximation we are using has an error that is supposed to be $\sim \Delta x^2$? (Hint: think about how accurately the boundary condition on $\partial\Omega$ is described in this finite-difference approximation.)

(b) Modify the above code to instead discretize $\nabla \cdot c\nabla$, by writing A_0 as $-G^T C_g G$ for some G matrix that implements ∇ and for some C_g matrix that multiplies the gradient by $c(r) = r^2 + 1$. Draw a sketch of the grid points at which the components of ∇ are discretized—these will *not* be the same as the (n_x, n_y) where u is discretized, because of the centered differences. Be careful that you need to evaluate c at the ∇ grid points now! Hint: you can make the matrix $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ in Julia by the syntax `[M1;M2]`.

Hint: Notice in the IJulia notebook from Lecture 11 how a matrix `r` is created from a column-vector of `x` values and a row-vector of `y` values. You will need to modify these `x` and/or `y` values to evaluate r on a new grid(s). Given the r matrix `rc` on this new grid, you can evaluate

$c(r)$ on the grid by `c = rc.^2 + 1`, and then make a diagonal sparse matrix of these values by `spdiagm(reshape(c, prod(size(c))))`.

(c) Using this $A \approx \nabla \cdot c\nabla$, compute the smallest-$|\lambda|$ eigensolution and plot it. Given the eigenfunction converted to a 2d $N_x \times N_y$ array `u`, as in the Lecture 11 notebook, plot $u(r)$ as a function of r, along with a plot of the exact Bessel eigenfunction $J_0(k_0 r)$ from the $c = 1$ case for comparison.

```
plot(r[Nx/2:end,Ny/2], u[Nx/2:end,Ny/2])
k0 = so.newton(x -> besselj(0,x), 2.0)
plot(0:0.01:1, besselj(0, k0 * (0:0.01:1))/50)
```

Here, I scaled $J_0(k_0 r)$ by 1/50, but you should change this scale factor as needed to make the plots of comparable magnitudes. Note also that the r array here is the radius evaluated on the original u grid, as in the Lecture 11 notebook.

Can you qualitatively explain the differences?

Problem 3:

Recall that the displacement $u(x, t)$ of a stretched string [with fixed ends: $u(0, t) = u(L, t) = 0$] satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} + f(x, t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x, t)$ is an external force density (pressure) on the string.

(a) Suppose that $f(x, t) = \Re[g(x)e^{-i\omega t}]$, an oscillating force with a frequency ω. Show that, instead of solving the wave equation with this $f(x, t)$, we can instead use a complex force $\tilde{f}(x, t) = g(x)e^{-i\omega t}$, solve for a complex $\tilde{u}(x, t)$, and then take $u = \Re\tilde{u}$ to obtain the solution for the original $f(x, t)$.

(b) Suppose that $f(x, t) = g(x)e^{-i\omega t}$, and we want to find a *steady-state* solution $u(x, t) = v(x)e^{-i\omega t}$ that is oscillating everywhere at the same frequency as the input force. (This will be the solution after a long time if there is any dissipation in the system to allow the initial transients to die away.) Write an equation $\hat{A}v = g$ that v solves. Is $\hat{A}$ self-adjoint? Positive/negative definite/semidefinite?

(c) Solve for the Green's function $G(x, x')$ of this $\hat{A}$, assuming that $\omega \neq n\pi/L$ for any integer n (i.e. assume ω is not an eigenfrequency [why?]). [Write down the continuity conditions that G must satisfy at $x = x'$, solve for $x \neq x'$, and then use the continuity conditions to eliminate unknowns.]

(d) Form a finite-difference approximation A of your $\hat{A}$. Compute an approximate $G(x, x')$ in Matlab by `A \ dk`, where $\mathbf{d}_k$ is the unit vector of all 0's except for one $1/\Delta x$ at index $k = x'/\Delta x$, and compare (by plotting both) to your analytical solution from the previous part for a couple values of x' and a couple of different frequencies ω (one $< \pi/L$ and one $> \pi/L$) with $L = 1$.

(e) Show the limit $\omega \to 0$ of your G relates in some expected way to the Green's function of $-\frac{d^2}{dx^2}$ from class.