18.303 Problem Set 1 Solutions

Problem 1: (5+(5+5)+(5+5))

(a) Since $\mathbf{v}$ is a constant, $\frac{d}{dt}(\mathbf{v}^T\mathbf{x}) = v^T\frac{d\mathbf{x}}{dt} = \mathbf{v}^T A\mathbf{x} = 0$. Since this is given to be true for all times, including all $t = 0$, and for all initial conditions $\mathbf{x}(0)$, it means that $\mathbf{v}^T A\mathbf{x} = 0$ for all $\mathbf{x}$, and hence $\mathbf{v}^T A = 0$, or (taking the transpose) $A^T\mathbf{v} = 0$. Hence $\mathbf{v}$ is in the **left nullspace** $N(A^T)$. Equivalently, $\mathbf{v}$ is orthogonal to the column space $C(A)$.

(b) Given an eigensolution $A\mathbf{x} = \lambda\mathbf{x}$ ($\mathbf{x} \neq 0$) and $A^* = A^{-1}$, consider $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*\lambda\mathbf{x} = \lambda|\mathbf{x}|^2 = (\mathbf{x}^*A)\mathbf{x} = (A^*\mathbf{x})^*\mathbf{x} = (A^{-1}\mathbf{x})^*\mathbf{x} = (\lambda^{-1}\mathbf{x})^*\mathbf{x} = \bar{\lambda}^{-1}\mathbf{x}^*\mathbf{x} = \bar{\lambda}^{-1}|\mathbf{x}|^2$. $|\mathbf{x}| \neq 0$, so $\lambda = \bar{\lambda}^{-1}$ and hence $\lambda\bar{\lambda} = |\lambda|^2 = 1$, thus $|\lambda| = 1$ as desired.

Suppose we have two eigensolutions $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$ with distinct eigenvalues $\lambda_1 \neq \lambda_2$. Then $\mathbf{x}_2^*A\mathbf{x}_1 = \lambda_1\mathbf{x}_2^*\mathbf{x}_1 = (A^*\mathbf{x}_2)^*\mathbf{x}_1 = \bar{\lambda}_2^{-1}\mathbf{x}_2^*\mathbf{x}_1$, similar to above. Since $|\lambda_2| = 1$, we can use the hint to conclude $\bar{\lambda}_2^{-1} = \lambda_2$,and hence $(\lambda_1 - \lambda_2)\mathbf{x}_2^*\mathbf{x}_1 = 0$. Since $\lambda_1 - \lambda_2 \neq 0$, this means $\mathbf{x}_2^*\mathbf{x}_1 = 0$: the eigenvectors are orthogonal.

(i) First, a familiar property of such recurrences from 18.06: $\mathbf{x}^{(1)} = A\mathbf{b}$, $\mathbf{x}^{(2)} = A\mathbf{x}^{(1)} = A^2\mathbf{b}$,$\mathbf{x}^{(3)} = A^3\mathbf{b}$, and so on, so that $\mathbf{x}^{(n)} = A^n\mathbf{b}$. (This could be proved more formally by induction.) Since we have an 8×8 matrix with 8 eigenvectors, it is diagonalizable, so we can expand $\mathbf{b}$ in the basis of eigenvectors: $\mathbf{b} = \sum_{i=1}^{8} c_i\mathbf{x}_i$ for some coefficients c_i. Hence

$$\mathbf{x}^{(n)} = A^n\mathbf{b} = \sum_{i=1}^{8} c_i\lambda_i^n\mathbf{x}_i,$$

since A^n multiplies each eigenvector by the corresponding eigenvalue to the n-th power. For large positive n, this is dominated by the two eigenvectors with largest $|\lambda|$: $\mathbf{x}_1$ and $\mathbf{x}_2$, so

$$\mathbf{x}^{(n)} \approx c_1 2^n\mathbf{x}_1 + c_2(-2)^n\mathbf{x}_2 = 2^n\left[c_1\mathbf{x}_1 + (-1)^n c_2\mathbf{x}_2\right],$$

with all other terms being exponentially smaller.

(ii) If A is (real) symmetric, then the eigenvectors are orthogonal (since all the eigenvalues are distinct). Hence we can solve for c_i just by taking the dot product:

$$\mathbf{x}_i^*\mathbf{b} = c_i\mathbf{x}_i^*\mathbf{x}_i,$$

since all other terms in the sum are zero, and thus $c_i = \mathbf{x}_i^*\mathbf{b}/\mathbf{x}_i^*\mathbf{x}_i = \mathbf{x}_i^*\mathbf{b}/|\mathbf{x}_i|^2$. So, from above:

$$\mathbf{x}^{(n)} \approx 2^n\left[\frac{\mathbf{x}_1\mathbf{x}_1^*\mathbf{b}}{|\mathbf{x}_1|^2} + (-1)^n\frac{\mathbf{x}_2\mathbf{x}_2^*\mathbf{b}}{|\mathbf{x}_2|^2}\right].$$

Note that while it is common to assume that eigenvectors are normalized to length 1, this is *not* automatic and you were *not* given such a normalization.

Problem 2: ((5+5+10)+5+5)

(a) Suppose that we we change the boundary conditions to the *periodic* boundary condition $u(0) = u(L)$.

(i) As in class, the eigenfunctions are sines, cosines, and exponentials, and it only remains to apply the boundary conditions. $\sin(kx)$ is periodic if $k = \frac{2\pi n}{L}$ for $n = 1, 2, \ldots$ (excluding $n = 0$ because we do not allow zero eigenfunctions and excluding $n < 0$ because they are not linearly independent), and $\cos(kx)$ is periodic if $n = 0, 1, 2, \ldots$ (excluding $n < 0$ since they are the same functions). The eigenvalues are $-k^2 = -(2\pi n/L)^2$.

e^{kx} is periodic only for imaginary $k = i\frac{2\pi n}{L}$, but in this case we obtain $e^{i\frac{2\pi n}{L}x} = \cos(2\pi nx/L) + i\sin(2\pi nx/L)$, which is *not linearly independent* of the sin and cos eigenfunctions above. Recall from 18.06 that the eigenvectors for a given eigenvalue form a vector space (the null space of $A - \lambda I$), and when asked for eigenvectors we only want a *basis* of this vector space. Alternatively, it is acceptable to start with exponentials and call our eigenfunctions $e^{i\frac{2\pi n}{L}x}$ for all integers n, in which case we wouldn't give sin and cos eigenfunctions separately.

Similarly, $\sin(\phi + 2\pi nx/L)$ is periodic for any ϕ, but this is not linearly independent since $\sin(\phi + 2\pi nx/L) = \sin\phi\cos(2\pi nx/L) + \cos\phi\sin(2\pi nx/L)$.

(ii) No, any solution will not be unique, because we now have a nonzero nullspace spanned by the constant function $u(x) = 1$ (which is periodic): $\frac{d^2}{dx^2}1 = 0$. Equivalently, we have a 0 eigenvalue corresponding to $\cos(2\pi nx/L)$ for $n = 0$ above.

(iii) As suggested, let us restrict ourselves to $f(x)$ with a convergent Fourier series. That is, as in class, we are expanding $f(x)$ in terms of the eigenfunctions:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x}.$$

(You could also write out the Fourier series in terms of sines and cosines, but the complex-exponential form is more compact so I will use it here.) Here, the coefficients c_n, by the usual orthogonality properties of the Fourier series, are $c_n = \frac{1}{L}\int_0^L e^{-\frac{2\pi n}{L}x} f(x)dx$.

In order to solve $\frac{d^2u}{dx^2} = f$, as in class we would divide each term by its eigenvalue $-(2\pi n/L)^2$, but we can only do this for $n \neq 0$. Hence, we can only solve the equation if the $n = 0$ term is absent, i.e. $c_0 = 0$. Appling the explicit formula for c_0, the equation is solvable (for f with a Fourier series) if and only if:

$$\boxed{\int_0^L f(x)dx = 0}.$$

There are other ways to come to the same conclusion. For example, we could expand $u(x)$ in a Fourier series (i.e. in the eigenfunction basis), apply d^2/dx^2, and ask *what is the column space* of d^2/dx^2? Again, we would find that upon taking the second derivative the $n = 0$ (constant) term vanishes, and so the column space consist of Fourier series missing a constant term.

The same reasoning works if you write out the Fourier series in terms of sin and cos sums separately, in which case you find that f must be missing the $n = 0$ cosine term, giving the same result.

(b) No. For example, the function 0 (which must be in any vector space)does not satisy those boundary conditions. (Also adding functions doesn't work, scaling them by constants, etcetera.)

(c) We merely pick any twice-differentiable function $q(x)$ with $q(L)-q(0) = -1$, in which case $u(L)-u(0) = [v(L)-v(0)]+[q(L)-q(0)] = 1-1 = 0$ and u is periodic. Then, plugging $v = u-q$ into $\frac{d^2}{dx^2}v(x) = f(x)$, we obtain

$$\frac{d^2}{dx^2}u(x) = f(x) + \frac{d^2q}{dx^2},$$

which is the (periodic-u) Poisson equation for u with a (possibly) modified right-hand side.

For example, the simplest such q is probably $q(x) = x/L$, in which case $d^2q/dx^2 = 0$ and u solves the Poisson equation with an *unmodified* right-hand side.