18.303 Midterm Solutions, Fall 2011

Problem 1: (20 points)

- Just like $c\nabla^2$ from class, we will need a C^{-1} factor in our inner product to cancel the asymmetric-looking C factor in $\hat{A}$. In particular, when we write $\langle \mathbf{u}, \hat{A}\mathbf{v}\rangle_C$, we want the C in $\hat{A}$ to cancel, so there must be a C^{-1} factor multiplying $\mathbf{v}$.

  $$\langle \mathbf{u}, \mathbf{v}\rangle_C = \int_\Omega \bar{\mathbf{u}} \cdot \left(C^{-1}\mathbf{v}\right).$$

  Obviously, in order for this to work **C must be invertible** (at all $\mathbf{x}$). But we require more than that. To have a valid inner product, we must have:

  $$\begin{aligned}\langle \mathbf{v}, \mathbf{u}\rangle_C = \int_\Omega \bar{\mathbf{v}} \cdot \left(C^{-1}\mathbf{u}\right) = \overline{\langle \mathbf{u}, \mathbf{v}\rangle_C} = \int_\Omega \overline{\bar{\mathbf{u}} \cdot \left(C^{-1}\mathbf{v}\right)} \\ = \int_\Omega \overline{\left(\left[C^{-1}\right]^* \mathbf{u}\right)} \cdot \mathbf{v} = \int_\Omega \bar{\mathbf{v}} \cdot \left(\left[C^{-1}\right]^* \mathbf{u}\right)\end{aligned}$$

  and hence we must have $C^{-1} = \left[C^{-1}\right]^* = \left[C^*\right]^{-1}$ and hence $\boxed{C = C^*}$ (we showed in class that the inverse of the adjoint is the adjoint of the inverse) . Furthermore an inner product must satisfy

  $$0 < \langle \mathbf{u}, \mathbf{u}\rangle_C = \int_\Omega \bar{\mathbf{u}} \cdot \left(C^{-1}\mathbf{u}\right)$$

  for $\mathbf{u} \neq 0$, and hence C^{-1} and thus C must be **positive definite** at all $\mathbf{x}$ (excepting isolated points / sets of measure zero). (The condition that C be invertible is therefore redundant.)

- Integrating by parts using the rule for $\nabla\times$, assuming surface terms are zero, we then obtain

  $$\begin{aligned}\langle \mathbf{u}, \hat{A}\mathbf{v}\rangle_C &= \int_\Omega \bar{\mathbf{u}} \cdot (\nabla \times B\nabla \times \mathbf{v}) = \int_\Omega \overline{\nabla \times \mathbf{u}} \cdot (B\nabla \times \mathbf{v}) \\ &= \int_\Omega \overline{B^*\nabla \times \mathbf{u}} \cdot (\nabla \times \mathbf{v}) = \int_\Omega \overline{\nabla \times B^*\nabla \times \mathbf{u}} \cdot \mathbf{v} \\ &= \int_\Omega \overline{C^{-1}C\nabla \times B^*\nabla \times \mathbf{u}} \cdot \mathbf{v} == \int_\Omega \overline{C\nabla \times B^*\nabla \times \mathbf{u}} \cdot C^{-1}\mathbf{v} \\ &= \langle \hat{A}^*\mathbf{u}, \mathbf{v}\rangle_C\end{aligned}$$

  where $\hat{A}^* = C\nabla \times B^*\nabla\times$. For $\hat{A}$ to be self-adjoint, we must therefore have $\boxed{B = B^*}$ (B is self-adjoint everywhere).

- To have $\hat{A}$ positive semidefinite, we must have

  $$0 \leq \langle \mathbf{u}, \hat{A}\mathbf{u}\rangle = \int_\Omega \overline{\nabla \times \mathbf{u}} \cdot (B\nabla \times \mathbf{u})$$

  from which it follows that B must be **positive semidefinite**.

Problem 2: (20 points)

Suppose we have a stretched drum with a position-varying "stretchiness" $c(\mathbf{x}) > 0$, with the displacement $u(\mathbf{x}) = u(x, y)$ satisfying

$$-\nabla \cdot (c\nabla u) = f(\mathbf{x})$$

where f is the force per unit area, where the edges of the drum are fixed: $u|_{\partial\Omega} = 0$.

(a) The solution $u(\mathbf{x})$ is just $F_0 G(\mathbf{x}, \mathbf{x}_0)$ in terms of the Green's function G. By the boundary conditions, $G(\mathbf{x}, \mathbf{x}_0) \to 0$ as $\mathbf{x} \to \partial\Omega$. However, reciprocity means that $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$ [since this $\hat{A}$ uses an unweighted inner product, and is real], which means that G also goes to zero at all $\mathbf{x}$ as $\mathbf{x}_0 \to \partial\Omega$. Hence $\boxed{u(\mathbf{x}) \to 0}$ as $\mathbf{x}_0 \to \partial\Omega$.

Physically, this is pretty obvious: when you press on the drum at the edge, it does nothing because the edge is fixed.

(b) To handle general Dirichlet boundary conditions, we write $u = v + (x^2 - y^2)$. In this way, v satisfies

$$\hat{A}v = -\nabla \cdot (c\nabla v) = f - \hat{A}(x^2 - y^2) = f + g$$

where $v|_{\partial\Omega} = 0$ and we let $g = \nabla \left(c\nabla \left[x^2 - y^2\right]\right)$. We now solve for v using G as above, to find

$$u(\mathbf{x}) = (x^2 - y^2) + F_0 G(\mathbf{x}, \mathbf{x}_0) + \int_{\mathbf{x}' \in \Omega} G(\mathbf{x}, \mathbf{x}')g(\mathbf{x}').$$

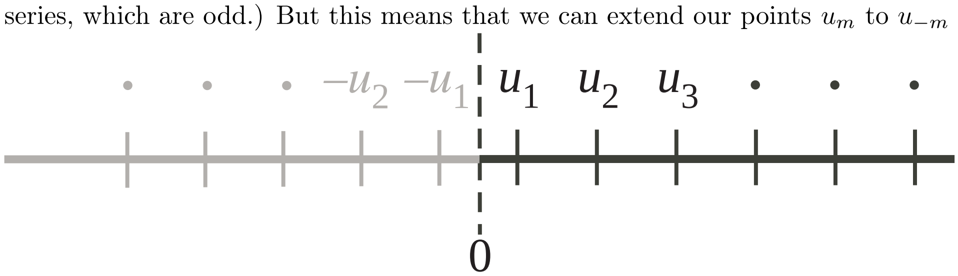
Therefore, as $\mathbf{x}_0 \to \partial\Omega$, the F_0 term goes to zero as before so u goes to the (Laplace-equation) solution with **no external force**.

Note that if $c = 1$, then $\hat{A}(x^2 - y^2) = 0$ and $u \to x^2 - y^2$.

Problem 3: (20 points)

(a) There are two main differences. First, we now have $\boxed{\Delta x = L/N}$ since there are only N intervals Δx for N points ($N - 1$ intervals plus two halves at the ends).

Second, the **first and last rows** of A will be different because of the different handling of the boundary conditions (the u_n'' in the interior is the same). The easiest way to analyze this problem is to recall that Dirichlet boundaries are equivalent to antisymmetric/odd functions around the boundary. e.g. the Dirichlet boundary at $x = 0$ is equivalent to requiring $u(-x) = -u(x)$. (Alternatively, recall that the solutions can be expanded in sine series, which are odd.) But this means that we can extend our points u_m to $u_{-m} = -u_m$ as shown:



So, when we compute u_1'', we find

$$u_1'' \approx \frac{u_{-1} - 2u_1 + u_2}{\Delta x^2} = \frac{-u_1 - 2u_1 + u_2}{\Delta x^2} = \frac{-3u_1 + u_2}{\Delta x^2}.$$

Similarly for the last row

$$u_N'' \approx \frac{u_{N-1} - 2u_N + u_{N+1}}{\Delta x^2} = \frac{u_{N-1} - 2u_N - u_N}{\Delta x^2} = \frac{u_{N-1} - 3u_N}{\Delta x^2}$$

and hence we obtain

$$\boxed{A = \left(\frac{N}{L}\right)^2 \begin{pmatrix} -3 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -3 \end{pmatrix}}.$$

There is also a harder way to solve this problem and get the same answer. You could re-derive the second derivative operation using the steps in part (b), writing $A = D_2 D_1$, The key point is that, in order to get $u'(0) = u'_{0.5}$, we could use the Taylor expansion:

$$u(\Delta x/2) = \cancel{u(0)} + \frac{\Delta x}{2}u'(0) + \frac{\Delta x^2}{8}u''(0) + \frac{\Delta x^3}{48}u'''(0) + \cdots.$$

We can then write

$$u'_{0.5} = u'(0) \approx \frac{u(\Delta x/2)}{\Delta x/2} = \frac{2u_1}{\Delta x}.$$

Normally, this would be only first-order accurate because of the $u''(0)$ term, but in this case (odd functions, sine series) $u''(0) = 0$ and this is second-order accurate. We therefore obtain the same D_1 matrix as below, and A follows.

(b) As usual, our D matrix should compute u' at points halfway between the grid points by a center-difference approximation:

$$u'_{n+0.5} \approx \frac{u_{n+1} - u_n}{\Delta x}.$$

Again, this will differ at the endpoints by the odd-symmetry/Dirichlet condition:

$$u'_{0.5} \approx \frac{u_1 - u_{-1}}{\Delta x} = \frac{2u_1}{\Delta x},$$

$$u'_{N+0.5} \approx \frac{u_{N+1} - u_N}{\Delta x} = \frac{-2u_N}{\Delta x},$$

giving the $(N+1) \times N$ matrix

$$D_1 = \frac{N}{L} \begin{pmatrix} 2 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -2 \end{pmatrix}$$

However, this does *not* give $A = -D_1^T D_1$! It is easy to check explicitly that $-D_1^T D_1$ gives a $-5 = -2^2 - 1^2$ instead of a -3 in the $a_{1,1}$ and $a_{N,N}$ entries. Alternatively, we can think about how we would compute

$$u''_n = \frac{u'_{n+0.5} - u'_{n-0.5}}{\Delta x}.$$

In this case, there is no funny weighting of the endpoints: $u''_1 = \frac{u'_{1.5} - u'_{0.5}}{\Delta x}$, so this operation corresponds to the $N \times (N+1)$ matrix

$$D_2 = \frac{N}{L} \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix} \neq -D_1^T.$$

We can check that indeed $A = D_2 D_1$, but this is not the desired form. To turn it into the desired form, we first write D_2 in terms of $-D_1^T$:

$$D_2 = -D_1^T \begin{pmatrix} 0.5 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0.5 \end{pmatrix}$$

where we have multiplied by diagonal $(N+1) \times (N+1)$ matrix which scales the first and last columns by 0.5 to turn the 2's into 1's. We then have

$$A = -D_1^T \begin{pmatrix} 0.5 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0.5 \end{pmatrix} D_1 = -D^T D$$

if we write

$$D = \begin{pmatrix} \sqrt{0.5} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \sqrt{0.5} \end{pmatrix} D_1 = \boxed{\frac{N}{L} \begin{pmatrix} \sqrt{2} & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \\ & & & -\sqrt{2} \end{pmatrix}},$$

where the trick was to take the square root of our diagonal matrix and associate half with D_1 and half with D_1^T.

Another way to think about this problem is that this is like your homework problem **(pset 1, problem 4)** where the usual definition of "matrix transpose" was not really the right concept anymore. The problem is that the $u'_{n+0.5}$ includes points **exactly at** the $x = 0$ and $x = L$ boundaries, and furthermore since u is an odd function then u' must be an **even** function at the boundaries. So, just like in homework, the right inner product of the $u'_{n+0.5}$ has a weight 0.5 for the endpoints, and this changes the appropriate definition of "D_1^T" to include the extra 0.5 scalings given here.

Note also that it is perfectly reasonable to leave A in the form $-D_1^T W D_1$ where W is our positive diagonal scaling matrix, since this formulation is still manifestly real-symmetric and positive-definite.