

18.303 Problem Set 6

Due Wednesday, 31 October 2012.

Problem 1: Distributions

This problem concerns distributions as defined in the notes: continuous linear functionals $f\{\phi\}$ from test functions $\phi \in \mathcal{D}$, where \mathcal{D} is the set of infinitely differentiable functions with compact support (i.e. $\phi = 0$ outside some region with finite diameter [differing for different ϕ], i.e. outside some finite interval $[a, b]$ in 1d).

- (a) For a classical function $f(x)$ to form a regular distribution $f\{\phi\} = \int_{-\infty}^{\infty} f(x)\phi(x)dx$, we must have a finite integral $|f\{\phi\}| < \infty$ for all $\phi \in \mathcal{D}$. In this notes, it was claimed to be sufficient that f be “locally integrable:” $\int_a^b |f(x)|dx < \infty$ for all intervals $[a, b]$. Show this. (That is, prove that $\int_a^b |f(x)|dx < \infty$ for all $[a, b]$ implies $|f\{\phi\}| < \infty$ for all ϕ . Hint: just bound $|f\{\phi\}|$ above. You may use the fact that $|\phi(x)|$ has a maximum value since ϕ is continuous and compactly supported, by the “extreme value theorem” of analysis.)

The converse is also true: if $f(x)$ is *not* locally integrable, then $f\{\phi\}$ is not defined for all ϕ , but you need not prove this.

- (b) In this part, you will consider the function $f(x) = \begin{cases} \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$ and its (weak) derivative, which is connected to something called the Cauchy Principal Value.

- (i) Show that $f(x)$ defines a regular distribution, by showing that $f(x)$ is locally integrable for all intervals $[a, b]$.

- (ii) Consider the 18.01 derivative of $f(x)$, which gives $f'(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}$. Suppose we just set

“ $f'(0) = 0$ ” at the origin to define $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that this $g(x)$ is *not* locally integrable, and hence does not define a distribution.

But the weak derivative $f'\{\phi\}$ *must* exist, so this means that we have to do something different from the 18.01 derivative, and moreover $f'\{\phi\}$ is *not* a regular distribution. What is it?

- (iii) Write $f\{\phi\} = \lim_{\epsilon \rightarrow 0^+} f_\epsilon\{\phi\}$ where $f_\epsilon\{\phi\} = \int_{-\infty}^{-\epsilon} \ln(-x)\phi(x)dx + \int_{\epsilon}^{\infty} \ln(x)\phi(x)dx$, since this limit exists and equals $f\{\phi\}$ for all ϕ from your proof in the previous part.¹ Compute the distributional derivative $f'\{\phi\} = \lim_{\epsilon \rightarrow 0^+} f'_\epsilon\{\phi\}$, and show that $f'\{\phi\}$ is precisely the *Cauchy Principal Value* (google the definition, e.g. on Wikipedia) of the integral of $g(x)\phi(x)$.

- (iv) Alternatively, show that $f'\{\phi(x)\} = g\{\phi(x) - \phi(0)\} = \int_{-\infty}^{\infty} g(x)[\phi(x) - \phi(0)]dx$ (which is a well-defined integral for all $\phi \in \mathcal{D}$).

- (c) In class, we only looked explicitly at 1d distributions, but a distribution in d dimensions \mathbb{R}^d can obviously be defined similarly, as maps $f\{\phi\}$ from smooth localized functions $\phi(\mathbf{x})$ to numbers. Analogous to class, define the distributional gradient ∇f by $\nabla f\{\phi\} = f\{-\nabla\phi\}$.

Consider some finite volume V with a surface ∂V , and assume ∂V is differentiable so that at each point it has an outward-pointing unit normal vector \mathbf{n} , as shown in figure 1. Define a “surface delta function” $\delta(\partial V)\{\phi\} = \oint_{\partial V} \phi(\mathbf{x})d^{d-1}\mathbf{x}$ to give the surface integral $\oint_{\partial V}$ of the test function.

Suppose we have a regular distribution $f\{\phi\}$ defined by the function $f(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \mathbf{x} \in V \\ f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases}$, where we may have a discontinuity $f_2 - f_1 \neq 0$ at ∂V .

¹More explicitly, $f\{\phi\} - f_\epsilon\{\phi\} = \int_{-\epsilon}^{\epsilon} \ln|x|\phi(x)dx \leq (\max \phi) \int_{-\epsilon}^{\epsilon} \ln|x|dx \rightarrow 0$, since you should have done the something like the last integral explicitly in the previous part.

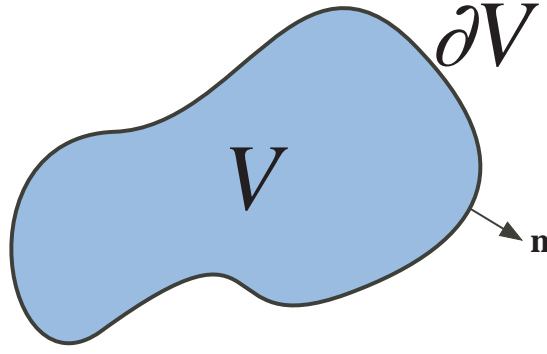


Figure 1: A volume V with a surface ∂V , and an outward unit normal vector \mathbf{n} at each point on ∂V .

- (i) Show that the distributional gradient of f is

$$\nabla f = \delta(\partial V) [f_1(\mathbf{x}) - f_2(\mathbf{x})] \mathbf{n}(\mathbf{x}) + \begin{cases} \nabla f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases},$$

where the second term is a regular distribution given by the ordinary 18.02 gradient of f_1 and f_2 (assumed to be differentiable), while the first term is the singular distribution

$$\delta(\partial V) [f_1(\mathbf{x}) - f_2(\mathbf{x})] \mathbf{n}(\mathbf{x}) \{ \phi \} = \oint_{\partial V} [f_1(\mathbf{x}) - f_2(\mathbf{x})] \mathbf{n}(\mathbf{x}) \phi(\mathbf{x}) d^{d-1} \mathbf{x}.$$

You can use the integral identity that $\int_V \nabla \psi d^d \mathbf{x} = \oint_{\partial V} \psi \mathbf{n} d^{d-1} \mathbf{x}$ to help you integrate by parts.

- (ii) Defining $\nabla^2 f \{ \phi \} = f \{ \nabla^2 \phi \}$, derive a similar expression to the above for $\nabla^2 f$. Note that you should have one term from the discontinuity $f_1 - f_2$, and another term from the discontinuity $\nabla f_1 - \nabla f_2$. (Recall how we integrated ∇^2 by parts in class some time ago.)

Problem 2: Green's functions

Consider Green's functions of the self-adjoint indefinite operator $\hat{A} = -\nabla^2 - \omega^2$ ($\kappa > 0$) over all space ($\Omega = \mathbb{R}^3$ in 3d), with solutions that $\rightarrow 0$ at infinity. (This is the multidimensional version of problem 2 from pset 5.) As in class, thanks to the translational and rotational invariance of this problem, we can find $G(\mathbf{x}, \mathbf{x}') = g(|\mathbf{x} - \mathbf{x}'|)$ for some $g(r)$ in spherical coordinates.

- (a) Solve for $g(r)$ in 3d, similar to the procedure in class.

- (i) Similar to the case of $\hat{A} = -\nabla^2$ in class, first solve for $g(r)$ for $r > 0$, and write $g(r) = \lim_{\epsilon \rightarrow 0^+} f_\epsilon(r)$ where $f_\epsilon(r) = 0$ for $r \leq \epsilon$. [Hint: although Wikipedia writes the spherical $\nabla^2 g(r)$ as $\frac{1}{r^2} (r^2 g')'$, it may be more convenient to write it equivalently as $\nabla^2 g = \frac{1}{r} (r g)''$, as in class, and to solve for $h(r) = r g(r)$ first. Hint: if you get sines and cosines from this differential equation, it will probably be easier to use complex exponentials, e.g. $e^{i\omega r}$, instead.]
- (ii) Then, evaluate $\hat{A} g = \delta(\mathbf{x})$ in the distributional sense: $(\hat{A} g) \{ q \} = g \{ \hat{A} q \} = q(0)$ for an arbitrary (smooth, localized) test function $q(\mathbf{x})$ to solve for the unknown constants in $g(r)$. [Hint: when evaluating $g \{ \hat{A} q \}$, you may need to integrate by parts on the radial-derivative term of $\nabla^2 q$; don't forget the boundary term(s).]

- (b) Check that the $\omega \rightarrow 0^+$ limit gives the answer from class.

- (c) Remember that this operator, in the last pset, came from a wave equation with sinusoidal time dependence. If our "real" problem is a scalar wave equation (e.g. sound waves in air) $-\nabla^2 u + \frac{\partial^2 u}{\partial t^2} = f(\mathbf{x}, t)$, with $f(\mathbf{x}, t) = \delta(\mathbf{x}) \cos(\omega t) = \text{Re}[\delta(\mathbf{x}) e^{-i\omega t}]$ (i.e. an oscillating point source), write u in terms of your G . Does it make physical sense?

Problem 3: Equivalence principles and integral equations

In class, we gave several examples of how inhomogeneities and interfaces in PDEs could be replaced by “sources”, leading to integral equations. This idea is often couched in terms of an “equivalence principle,” in which interfaces are replaced by “equivalent sources”. In this problem, you will derive a couple of versions of this idea for the case of Poisson’s equation $-\nabla^2 u = f$. The geometry will be based on figure 1, in which we have some volume V , and we will give different equations inside and outside of V . You will need to use your answer from problem 1(c) above in order to take the Laplacian ∇^2 of a function u that is defined piecewise inside and outside of V .

- (a) Suppose that , where the domain Ω is the *exterior* of V with Dirichlet boundaries. That is, $-\nabla^2 u = f$ outside of V , and $u = 0$ on ∂V . Let us now extend this solution to all space, via

$$\tilde{u}(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \mathbf{x} \notin V \\ 0 & \mathbf{x} \in V \end{cases}.$$

- (i) If we now apply $-\nabla^2$ *everywhere* to this \tilde{u} , find the right-hand side \tilde{f} that makes

$$-\nabla^2 \tilde{u} = \tilde{f}$$

true (in the distribution sense, of course). Hint: \tilde{f} should equal f plus surface terms.

- (ii) Given the Green’s function $G(\mathbf{x}, \mathbf{x}') = 1/4\pi|\mathbf{x} - \mathbf{x}'|$ of $-\nabla^2$ in infinite space, explain how $u(\mathbf{x})$ can now be written in terms of unknowns only at ∂V (integrated with G). Then, by writing the boundary condition $u|_{\partial V} = 0$, we obtain an “integral equation” for these unknowns, very similar to something I wrote down in class.

- (b) Suppose that our domain is all space, and we have the equation $-\nabla \cdot [c(\mathbf{x})\nabla u] = f$ where $c(\mathbf{x}) = \begin{cases} c_1 & \mathbf{x} \in V \\ c_2 & \mathbf{x} \notin V \end{cases}$ for some constants c_1 and c_2 . If f is a regular distribution, then we must have both u and $\mathbf{n} \cdot \nabla u$ continuous (as in class) at ∂V , so as not to get a delta function on the left-hand-side.

- (i) Construct a solution $u_1(\mathbf{x}) = \begin{cases} u_1(\mathbf{x}) & \mathbf{x} \in V \\ 0 & \mathbf{x} \notin V \end{cases}$ that satisfies $-c_1 \nabla^2 u_1 = f_1$ (in all space!) for some distribution f_1 . Give f_1 .
- (ii) In the same way, you can construct a u_2 that is nonzero outside V and satisfies $-c_2 \nabla^2 u_2 = f_2$ for some f_2 .
- (iii) Given the Green’s function $G_{1,2}(\mathbf{x}, \mathbf{x}') = 1/4\pi c_{1,2}|\mathbf{x} - \mathbf{x}'|$ of $-c_{1,2} \nabla^2$ in infinite space with constant coefficients c_1 or c_2 , explain how $u(\mathbf{x})$ can now be written in terms of unknowns only at ∂V (integrated with G). Then, by writing the conditions that u and $\mathbf{n} \cdot \nabla u$ are continuous, obtain an “integral equation” for these unknowns.