18.303 Final Exam Solutions, Fall 2010

Problem 1: Derivatives and differences (30 points)

(a) Let's choose the usual inner produce $\langle u, v\rangle = \int_0^L \bar{u}v$. We want to move $\hat{A}$ from one side to the other of the inner product by integrating by parts, and will choose the boundary conditions to make the boundary terms zero. Integrating by parts and choosing the boundary conditions as we go along, we obtain:

$$
\begin{aligned}
\langle u, \hat{A}v\rangle = \int \bar{u}v'''' &= \cancel{\bar{u}v'''|_0^L} - \int \bar{u}'v''' \\
&= -\cancel{\bar{u}'v''|_0^L} + \int \bar{u}''v'' \\
&= \cancel{\bar{u}''v'|_0^L} - \int \bar{u}'''v' \\
&= -\cancel{\bar{u}'''v|_0^L} + \int \bar{u}''''v \\
&= \langle \hat{A}u, v\rangle,
\end{aligned}
$$

where in the first line we cancelled the boundary term by choosing $\boxed{u|_{0,L} = 0}$, and in the second line we cancelled the boundary term by choosing $\boxed{u'|_{0,L} = 0}$. Since v and u have the same boundary conditions, this automatically zeroes the boundary terms in the next two lines as well.

Note that it is *not* correct to simply choose $u|_{0,L} = 0$ and say that we can use the Hermitian property of $\frac{d^2}{dx^2}$ from class twice. The problem is that u'' does not satisfy the same boundary conditions as u, so you can't use the result from class to say that $\langle u, \frac{d^2}{dx^2}v''\rangle = \langle \frac{d^2}{dx^2}u, v''\rangle$.

(b) We will just do center differences four times. As in class, taking a center-difference of u_m yields $u'_{m+1/2}$, i.e. on a staggered grid. This will now happen four times, alternating between half-integer and integer grids to maintain the centering:

$$
\begin{aligned}
u'_{m+1/2} &\approx \frac{u_{m+1} - u_m}{\Delta x}, \\
u''_m &\approx \frac{u'_{m+1/2} - u'_{m-1/2}}{\Delta x} = \frac{u_{m+1} - 2u_m + u_{m-1}}{\Delta x^2}, \\
u'''_{m+1/2} &\approx \frac{u''_{m+1} - u''_m}{\Delta x} = \frac{u_{m+2} - 3u_{m+1} + 3u_m - u_{m-1}}{\Delta x^3}, \\
u''''_m &\approx \frac{u'''_{m+1/2} - u'''_{m-1/2}}{\Delta x} = \boxed{\frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{\Delta x^4}}.
\end{aligned}
$$

Problem 2: No more scalars (30 points)

(a) We want exponentially decaying solutions, so we want $\hat{A}$ to be negative-semidefinite (with zero eigenvalues corresponding to nonzero limiting values). This means we want $\langle \mathbf{u}, \hat{A}\mathbf{u}\rangle \leq 0$ for all $\mathbf{u}$, and hence:

$$
\langle \mathbf{u}, \hat{A}\mathbf{u}\rangle = \langle \mathbf{u}, \nabla \times c\nabla \times \mathbf{u}\rangle = \langle \nabla \times \mathbf{u}, c\nabla \times \mathbf{u}\rangle = \int_\Omega c|\nabla \times \mathbf{u}|^2 \leq 0,
$$

where we have used the self-adjoint property of $\nabla\times$ to move it to the other side of the inner product. For this to be true for all u, we clearly require $c \leq 0$ everywhere (except at isolated points, but we never worry about those).

(b) We know from class that $\langle \mathbf{v}, \mathbf{u}\rangle$ is conserved for any $\mathbf{v} \in N(\hat{A}^*) = N(\hat{A})$ [since we can move $\nabla\times$ from right-to-left twice to obtain $\hat{A}^* = \hat{A}$], so the only question is what is $\mathbf{v}$ and what does this mean?

To start with, for simplicity take $c < 0$ (i.e. $c \neq 0$). From the hint, $\nabla\phi$ for any ϕ is the nullspace of $\nabla\times$, so this is clearly in the nullspace of $\hat{A}$. The converse is also true, since $\hat{A}$ is of the form $(\nabla\times)^*c(\nabla\times)$. In particular (re-iterating the proof from class for $A = B^*B$), if $\mathbf{v} \in N(\hat{A})$ then $\hat{A}\mathbf{v} = 0 \implies 0 = \langle \mathbf{v}, \hat{A}\mathbf{v}\rangle =$

$\int c|\nabla \times \mathbf{v}|^2 \implies \nabla \times \mathbf{v} = 0$, and so $\mathbf{v} = \nabla\phi$ for some ϕ. Therefore $\langle \nabla\phi, \mathbf{u}\rangle$ is conserved for any test function ϕ. We can simplify this further: integrating by parts, we obtain $\langle \phi, \nabla \cdot \mathbf{u}\rangle$ is conserved for any ϕ, which means that $\nabla \cdot \mathbf{u}$ is conserved. (This is also easy to show directly: $\frac{\partial \nabla\cdot\mathbf{u}}{\partial t} = \nabla \cdot (\hat{A}\mathbf{u}) = 0$ since the divergence of a curl is zero.

Now, suppose we include the case where $c = 0$ in some region R (not just a set of zero measure). In that case, there are additional functions in the nullspace: any $\mathbf{v}$ such that $\nabla \times \mathbf{v}$ is nonzero only in R. These give an additional conservation law that is not quite so simple to write down, but I didn't actually expect you to notice this case.

Problem 3: Guided waves (30 points)

Consider the scalar wave equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$ in two dimensions for $x \in [0, L]$ and $y \in (-\infty, \infty)$, with the Neumann boundary conditions $\frac{\partial u}{\partial x}\big|_{x=0,L} = 0$. That is, Ω is a width-L strip extending infinitely in the y direction, with Neumann boundaries.

(a) If we look for separable eigenfunctions $u(x, y, t) = u_k(x)e^{i(ky-\omega t)}$, what equation and what boundary conditions does u_k satisfy?

(b) Starting with the boundary conditions, since $\frac{\partial u}{\partial x} = \frac{\partial u_k}{\partial x} e^{i(ky-\omega t)} = 0$ implies $\frac{\partial u_k}{\partial x} = 0$, the boundary conditions are simply Neumann on u_k : $\frac{\partial u_k}{\partial x}\big|_{x=0,L} = 0$.

The equation works out exactly as in class:

$$\begin{aligned}\nabla^2[u_k e^{i(ky-\omega t)}] &= \frac{\partial^2}{\partial t^2} u_k e^{i(ky-\omega t)} = -\omega^2 u_k e^{i(ky-\omega t)} \\ &= (u_k'' - k^2 u_k)e^{i(ky-\omega t)},\end{aligned}$$

yielding: $\boxed{u_k'' = -(\omega^2 - k^2)u_k}$.

(c) Since this equation is of the same form as one we have seen many times in class ($u'' = -u \cdot$ number), we know immediately that the solutions are sines/cosines/exponentials. To satisfy the Neumann boundary conditions, much like in one of the psets, the solutions must be cosines:

$$u_{k,n}(x) = \cos\left(\frac{n\pi x}{L}\right)$$

for $n = 0, 1, 2, 3, \ldots$ (of course, u_k could be multiplied by any constant too). Note that $n = 0$ is a valid solution! The corresponding eigenfunctions are obtained by plugging this into the eigenequation, obtaining $\omega^2 - k^2 = (n\pi/L)^2$, i.e.

$$\omega_n(k) = \pm\sqrt{k^2 + (n\pi/L)^2}$$

for $n = 0, 1, 2, 3, \ldots$.

(d) To obtain a wavepacket solution that propagates without spreading, we must have a group velocity $\frac{\partial \omega_k}{\partial k}$ that is constant (independent of k). This is *not* true of $\omega_n(k)$ from the previous part for $n > 0$, similar to class. However, for $n = 0$ we obtain $\omega = \pm|k|$, a straight line, and in this case the group velocity is a constant (± 1) and so wavepackets will indeed propagate without spreading *if we make them out of the $n = 0$ modes.*

Problem 4: Timestepping and stability (30 points)

(a) Since the coefficients are constant and the domain is infinite, we can look for von-Neumann solutions $u_m^n = \lambda^n e^{ikm}$ and check whether $|\lambda| \le 1$ for all k. (Note that the original PDE is a wave equation, since $(\frac{\partial}{\partial x})^* = \frac{\partial}{\partial x}$, so there are no legitimate exponentially growing solutions.) Plugging this in, and dividing both sides by $\lambda^n e^{ikm}$, we obtain:

$$\begin{aligned}\frac{\lambda - 1}{\Delta t} &= -c\left[\alpha\lambda \frac{e^{ik} - e^{-ik}}{2\Delta x} + (1-\alpha)\frac{e^{ik} - e^{-ik}}{2\Delta x}\right] \\ &= -c\left[\alpha\lambda + (1-\alpha)\right]\frac{i\sin(k)}{\Delta x},\end{aligned}$$

using the fact that $e^{ik} - e^{-ik} = 2i\sin(k)$. Solving for λ, we find:

$$\lambda(k) = \frac{1 - ic(1-\alpha)\frac{\Delta t}{\Delta x}\sin k}{1 + ic\alpha\frac{\Delta t}{\Delta x}\sin k}.$$

If $\alpha = 0.5$, then this is of the form $\lambda = \frac{1-i\#}{1+i\#}$ for $\# = ic\frac{1}{2}\frac{\Delta t}{\Delta x}\sin k$, giving $|\lambda| = 1$ and so the scheme is unconditionally stable.

(b) In general,

$$|\lambda|^2 = \frac{1 + c^2(1-\alpha)^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k}{1 + c^2\alpha^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k},$$

and this is ≤ 1, i.e. the numerator is $\leq$ the denominator, if $1 - \alpha \leq \alpha$, and so it is unconditionally stable for $1 \geq \alpha \geq 1/2$.

(c) For $\alpha < 1/2$, the numerator of $|\lambda|^2$ above is $>$ the denominator, regardless of k or Δt (> 0), and so it is always unstable.

Problem 5: Green's functions (30 points)

(a) We use the trick from class and homework: write $u = u_0 + b$, where $u_0|_{d\Omega} = 0$, to turn it back to an ordinary Dirichlet problem for u_0: $\hat{A}u_0 = f - \hat{A}b$. This satisfies the ordinary G_0 Green's function integral, with the new right-hand side $f - \hat{A}b$:

$$u_0(\mathbf{x}) = \int_\Omega G_0(\mathbf{x}, \mathbf{x}') \left(f - \hat{A}b\right)\Big|_{\mathbf{x}'} d^n\mathbf{x}',$$

and thus

$$u(\mathbf{x}) = u_0 + b = b(\mathbf{x}) + \int_\Omega G_0(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')d^n\mathbf{x}' - \int_\Omega G_0(\mathbf{x}, \mathbf{x}')\hat{A}'b(\mathbf{x}')d^n\mathbf{x}',$$

where by $\hat{A}'$ here I mean the operator $\hat{A}$ acting on the $\mathbf{x}'$ coordinate.

(b) Here, letting $\hat{A}' = \nabla'^2$, the b term from above integrates by parts to:

$$\begin{aligned} -\int_\Omega G_0(\mathbf{x}, \mathbf{x}')\nabla'^2 b(\mathbf{x}') &= \cancel{-\oiint_{d\Omega} G_0(\mathbf{x}, \mathbf{x}')\nabla' b(\mathbf{x}') \cdot d\mathbf{A}'} + \int_\Omega \nabla' G_0(\mathbf{x}, \mathbf{x}') \cdot \nabla' b(\mathbf{x}') \\ &= \oiint_{d\Omega} \nabla' G_0(\mathbf{x}, \mathbf{x}')b(\mathbf{x}') \cdot d\mathbf{A}' - \int_\Omega \nabla'^2 G_0(\mathbf{x}, \mathbf{x}')\, b(\mathbf{x}'), \end{aligned}$$

where in the first line we have used the fact that $G_0(\mathbf{x}, \mathbf{x}') = G_0(\mathbf{x}', \mathbf{x}) = 0$ for $\mathbf{x}' \in d\Omega$ by the given boundary conditions (using reciprocity to swap $\mathbf{x}$ and $\mathbf{x}'$). Also, from reciprocity and the definition of G_0, we have $\nabla'^2 G_0(\mathbf{x}, \mathbf{x}') = \nabla'^2 G_0(\mathbf{x}', \mathbf{x}) = \delta(\mathbf{x}' - \mathbf{x})$. Inserting this into the last $\int_\Omega$ integral, the delta function integrates to give $-b(\mathbf{x})$ to cancel the $+b(\mathbf{x})$ term in $u(\mathbf{x})$ above, leaving:

$$u(\mathbf{x}) = \int_\Omega G_0(\mathbf{x}, \mathbf{x}')f(\mathbf{x}')d^n\mathbf{x}' + \oiint_{d\Omega} \nabla' G_0(\mathbf{x}, \mathbf{x}')b(\mathbf{x}') \cdot d\mathbf{A}',$$

which looks eactly like the zero-boundary-condition solution plus an additional set of "source" terms from the boundary. (In fact, from class, the ∇G_0 source terms correspond to "dipole" sources on the boundaries.)