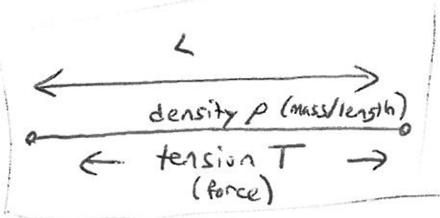
# 18.303 and Music

* stretched string (piano, guitar, violin, ...) :




$\Rightarrow$ scalar wave equation (neglecting friction, dispersion, ...)

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}, \quad u(x,t) = \text{displacement}$$

$$u|_{\partial\Omega} = 0$$

$\Rightarrow$ eigenfunctions of $\hat{A} = \frac{T}{\rho}\frac{\partial^2}{\partial x^2}$ $(= \hat{A}^* < 0)$ are $u_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, $\lambda_n = -\frac{T}{\rho}\left(\frac{n\pi}{L}\right)^2$, $n = 1, 2 \ldots$

$\Rightarrow$ solutions

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\right]}$$

where $\omega_n = \sqrt{\frac{T}{\rho}}\,\frac{n\pi}{L}$, $\alpha_n$ from $u(x,0)$, $\beta_n$ from $\dot{u}(x,0)$

= sum of "normal modes"

$n = 1$ : fundamental frequency $\omega_1 = \sqrt{\frac{T}{\rho}}\,\frac{\pi}{L}$ "the note" "the pitch"

$n > 1$ : harmonics

higher tension and/or lighter string and/or shorter string $\Rightarrow$ higher pitch

e.g. piano $\approx$ 7 octaves $= 2^7$ factor of $\omega$ (highest $\omega$/lowest)

$\Rightarrow$ at same $T, \rho$ the lowest string would be 128x longer than highest

* **Timbre**: the "same" note sounds different on different instruments ... or even on same instrument played differently!

$= $ different amplitudes $\left(\dfrac{\alpha_n^2+\beta_n^2}{\alpha_0^2+\beta_0^2}\right)$ of harmonics $(n>1)$

(e.g. due to different initial conditions)

[... plus other effects (not captured in simplified scalar wave equation):

- decay rates (depending on $n$)
- "harmonics" $\omega_n$ are not exactly integer multiples (due to dispersion and other effects)
- ...]

* **Western scale(s)**: pitch relationships (intervals) = frequency ratios

- octave = factor of 2 in $\omega$ = "same note" because harmonics exactly align
- transposition = multiply all $\omega$'s by same factor $\Rightarrow$ same intervals
- subdivide octave into 12 intervals = semitones (half-steps)

incompatible goals:

modern piano, guitar, ... ① intervals invariant under transposition $\Rightarrow$ equal ratios $\Rightarrow$ semitone $= 2^{1/12}$ ratio "equal temperament"

violin, voice, ... tuned by ear ② small rational ratios $3/2$, $4/3$, $5/4$, etc. $\Rightarrow$ "nice" sounding intervals $\Rightarrow$ various tunings, "wolf" intervals, "Pythagorean comma")

$\Rightarrow$ "fifth" $= 2^{7/12} = 1.498\ldots \neq \frac{3}{2}$ "just" fifth