18.303 Midterm Solutions, Fall 2012

Problem 1: Adjoints and operators (20 points)

(a) Integrating by parts, and using the fact that $u'(L) = \alpha u(L)$, have

$$\begin{aligned}
\langle u, \hat{A}v\rangle &= \int_0^L \bar{u}v'' = \bar{u}v'\big|_0^L - \int_0^L \overline{u'}v' = \bar{u}v' - \overline{u'}v\Big|_0^L + \int_0^L \overline{u''}v \\
&= \overline{u(L)}v'(L) - \overline{u'(L)}v(L) - \cancel{\overline{u(0)}}v'(0) + \overline{u'(0)}\cancel{v(0)} + \langle \hat{A}u, v\rangle \\
&= \alpha\left[\cancel{\overline{u(L)}v(L) - \overline{u(L)}v(L)}\right] + \langle \hat{A}u, v\rangle \\
&= \langle \hat{A}u, v\rangle,
\end{aligned}$$

hence $\hat{A} = \hat{A}^*$.

(b) For it to have decaying solutions, as in class we want $\hat{A}$ to be negative definite (or negative semidefinite, if we allow the solutions to decay to a nonzero asymptote). Hence, we need $\langle u, \hat{A}u\rangle < 0$ (or ≤ 0) for $u \neq 0$. As in class, we look at the middle step above where we have integrated by parts once:

$$\begin{aligned}
\langle u, \hat{A}u\rangle &= \bar{u}u'\big|_0^L - \int_0^L \overline{u'}u' \\
&= \alpha\,|u(L)|^2 - \int_0^L |u'(x)|^2\,dx
\end{aligned}$$

This is ≤ 0 if $\alpha \le 0$. It would $= 0$ if $u' = 0$ and $\alpha u(L) = 0$. However, $u' = 0$ implies $u(x) =$ constant, but since $u(0) = 0$ the only constant it can be is 0. Hence, $\langle u, \hat{A}u\rangle < 0$ for $u \neq 0$ for any $\boxed{\alpha \le 0}$.

Problem 2: Finite differences (20 points)

Let $m = M + 1$ correspond to the $x = L$ boundary, in which case we set $u_{M+1} = 0$ similar to class. The discrete form of the left boundary condition will be something like

$$u'(0) = 1 \approx \frac{u_1 - u_0}{\Delta x} \implies u_0 = u_1 - \Delta x.$$

There is some subtlety about where to place the left boundary. If we want this boundary condition to be imposed with second-order accuracy, we need $\frac{u_1 - u_0}{\Delta x}$ to be a center-difference approximation, in which case we should make $x = 0$ correspond to $m = 0.5$. This means that $(M + 1 - 0.5)\Delta x = L$, or $\boxed{\Delta x = \frac{L}{M + 0.5}}$. [However, I didn't specify that second-order accuracy was required, so it is acceptable if you make, e.g., $m = 0$ correspond to $x = 0$ similar to class, in which case we have a first-order forward difference approximation for $u'(0)$ and you will get $\Delta x = L/(M+1)$. I do expect you to specify Δx.]

Then, applying $u''_m \approx \frac{u_{m+1} - 2u_m + u_{m-1}}{\Delta x^2}$ for $m = 1, 2, \ldots, M$ will give us the same matrix as usual except that we need to be careful with the first and last rows, where:

$$\begin{aligned}
u''_1 &= \frac{u_2 - 2u_1 + u_0}{\Delta x^2} = \frac{u_2 - u_1}{\Delta x^2} - \frac{1}{\Delta x}, \\
u''_M &= \frac{\cancel{u_{M+1}} - 2u_M + u_{M-1}}{\Delta x^2}.
\end{aligned}$$

In particular, the $1/\Delta x$ term in the u''_1 equation needs to be moved to the right-hand side of $\hat{A}u - cu = \cancel{\frac{\partial u}{\partial t}}$. The $-cu$ term just gives us a diagonal term $-c_m u_m$. Writing this all out in matrix form, we obtain:

$$Au = b$$

where

$$\boxed{A = \frac{1}{\Delta x^2}\begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} - \begin{pmatrix} c_1 & & & & \\ & c_2 & & & \\ & & \ddots & & \\ & & & c_{M-1} & \\ & & & & c_M \end{pmatrix}}$$

and

$$\boxed{b = \frac{1}{\Delta x}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}}.$$

Notice that, much like in class, the nonzero boundary condition adds a source term to the right-hand-side, and the source term looks much like a (discretized) delta function since it has height $1/\Delta x$ and "width" Δx (one "pixel")

Problem 3: Green-ish functions (20 points)

(a) As in class, for $x \neq y$ we have $u'' = 0$, or u is a straight line. Since it has to go through zero at the endpoints, we have

$$u(x) = \begin{cases} \alpha x & x < y \\ \beta(x - L) & x > y \end{cases}$$

for some constants α and β to be determined.

The first derivative, in the distribution sense, gives

$$u'(x) = [\beta(y - L) - \alpha y]\,\delta(x - y) + \begin{cases} \alpha & x < y \\ \beta & x > y \end{cases},$$

where we have picked up a δ function from the discontinuity at y. The second derivative, in the distribution sense, then gives

$$u''(x) = [\beta(y - L) - \alpha y]\,\delta'(x - y) + (\beta - \alpha)\delta(x - y),$$

where the derivative of δ gave us δ' and the discontinuity $\beta - \alpha$ gave us another δ. Comparing to $-u''(x) = \delta'(x - y)$, we immediately obtain the equations

$$\beta = \alpha,$$

$$\alpha(y - L) - \alpha y = -1 \implies \alpha = \frac{1}{L}.$$

Note that since $y \in [0, L]$, the denominator in α is never zero and so α is always finite. Hence, we obtain the regular distribution

$$\boxed{u(x) = \frac{1}{L}\begin{cases} x & x < y \\ x - L & x > y \end{cases}},$$

which is discontinuous but has continuous slope.

(b) Similar to the Green's functions in class, we can do this by superposition:

$$\boxed{u(x) = -\int_0^L D(x, y) f(y) dy},$$

since

$$\hat{A}u = -\int_0^L [AD(x, y)]\, f(y) dy = \int_0^L [-\delta'(x - y)] f(y) dy = -\delta'(x - y)\{f(y)\} = f'(y).$$