18.303 Problem Set 1

Due Friday, 18 September 2015.

Note: For computational (Julia-based) homework problems in 18.303, turn in with your solutions a printout of any commands used and their results (please edit out extraneous/irrelevant stuff), and a printout of any graphs requested; alternatively, you can **email your notebook (.ipynb) file** to the grader huetter@math.mit.edu. **Always label** the axes of your graphs (with the `xlabel` and `ylabel` commands), add a title with the `title` command, and add a legend (if there are multiple curves) with the `legend` command. (Labelling graphs is a good habit to acquire.) Because IJulia notebooks let you combine code, plots, headings, and formatted text, it should be straighforward to turn in well-documented solutions.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

(a) In 18.06, you showed that the eigenvectors of a real-symmetric or Hermitian matrix A are orthogonal. You may want to review that proof before doing this problem, which deals with general non-symmetric square $m \times m$ matrices A.

  (i) Show that A and $A^* = \overline{A^T}$ have complex-conjugate eigenvalues. That is, for every eigenvalue λ_n of A, with a corresponding eigenvector x_n ($Ax_n = \lambda_n x_n$), there is a corresponding eigenvalue $\overline{\lambda_n}$ of A^* and a *left eigenvector* y_n satisfying $A^* y_n = \overline{\lambda_n} y_n$.

  (ii) Show that if $\lambda_j \neq \lambda_k$ then $y_j^* x_k = 0$. (This is called a "bi-orthogonality" relation between the left and right eigenvectors.)

  (iii) If $A = A^T$ (but A is *not* necessarily real), relate x_n to y_n and hence give an orthogonality relationship between x_j and x_k for $\lambda_j \neq \lambda_k$.

(b) The solutions $y(t)$ of the ODE $y'' - 2y' - cy = 0$ are of the form $y(t) = C_1 e^{(1+\sqrt{1+c})t} + C_2 e^{(1-\sqrt{1+c})t}$ for some constants C_1 and C_2 determined by the initial conditions. Suppose that A is a real-symmetric 4×4 matrix with eigenvalues $3, 8, 15, 24$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4$, respectively.

  (i) If $\mathbf{x}(t)$ solves the system of ODEs $\frac{d^2}{dt^2}\mathbf{x} - 2\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \mathbf{a}_0$ and $\mathbf{x}'(0) = \mathbf{b}_0$, write down the solution $\mathbf{x}(t)$ as a closed-form expression (no matrix inverses or exponentials) in terms of the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4$ and $\mathbf{a}_0$ and $\mathbf{b}_0$. [Hint: expand $\mathbf{x}(t)$ in the basis of the eigenvectors with unknown coefficients $c_1(t), \ldots, c_4(t)$, then plug into the ODE and solve for each coefficient using the fact that the eigenvectors are _________.]

  (ii) After a long time $t \gg 0$, what do you expect the approximate form of the solution to be?

Problem 2: Les Poisson, les Poisson

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

(a) Suppose that we we change the boundary conditions to the *periodic* boundary condition $u(0) = u(L)$.

  (i) What are the eigenfunctions of $\frac{d^2}{dx^2}$ now?

(ii) Will Poisson's equation have unique solutions? Why or why not?

(iii) Under what conditions (if any) on $f(x)$ would a solution exist? (You can restrict yourself to f with a convergent Fourier series.)

(b) If we instead consider $\frac{d^2}{dx^2}v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = v(L) + 1$, do these functions form a vector space? Why or why not?

(c) Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2}u(x) = f(x)$ problem with $u(0) = u(L)$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions q and r. (Transforming a new problem into an old, solved one is always a useful thing to do!)

Problem 3: Finite-difference approximations

For this question, you may find it helpful to refer to the notes, IJulia notebook, and reading from lecture 3. Consider a "forward" (only uses points $\geq x$) finite-difference approximation of the form:

$$u'(x) \approx \frac{a \cdot u(x + 2\Delta x) + b \cdot u(x + \Delta x) - u(x)}{c \cdot \Delta x}.$$

(a) Substituting the Taylor series for $u(x + \Delta x)$ etcetera (assuming u is a smooth function with a convergent Taylor series, blah blah), show that by an appropriate choice of the constants c and d you can make this approximation *second-order accurate*: that is, the errors are proportional to $(\Delta x)^2$ for small Δx.

(b) Check your answer to the previous part by numerically computing $u'(1)$ for $u(x) = \sin(x)$, as a function of Δx, exactly as in the handout from class (refer to the notebook posted in lecture 3 for the relevant Julia commands, and adapt them as needed). Verify from your log-log plot of the |errors| versus Δx that you obtained the expected second-order accuracy.

(c) Try computing $u'(1)$ for $u(x) = x^2$ and plot the errors vs Δx as above. Explain the results.