18.303 Problem Set 6

Due Wednesday, 20 October 2011.

Problem 1: Another minimization theorem

Suppose $\hat{A}$ is a positive-definite (or semidefinite) self-adjoint operator with respect to some inner product $\langle\cdot,\cdot\rangle$ on a domain Ω with some boundary conditions, and we are solving $\hat{A}u = f$. Show that the solution u minimizes the functional $F\{v\} = \langle v, \hat{A}v\rangle - \langle f, v\rangle - \langle v, f\rangle$ over all $v(\mathbf{x})$ in our function space. Hint: show that $F\{u+s\} > F\{u\}$ (or $\geq F\{u\}$ if $\hat{A}$ is semidefinite) for any function $s(\mathbf{x}) \neq 0$. At the minimum, $F\{u\} =$?

Problem 2: Reciprocity

Recall that, in the steady state, the equilibrium heat/temperature distribution $u(\mathbf{x})$ in a diffusion process satisfies $c\nabla\cdot\kappa\nabla u = -f$ where $c, \kappa > 0$ are related to the heat capacity and thermal conductivities, respectively. and $f(\mathbf{x})$ describes sources/sinks of heat. Your friend Sal U. Short claims to have invented a new device, a *one-way thermal conductor*: simply by arranging different materials in a secret geometry $c(\mathbf{x})$ and $\kappa(\mathbf{x})$, Sal can make a rod where a placing a heat source at end 1 conducts heat to (raises the temperature of) end 2, but a heat source at end 2 does not conduct heat to (raise the temperature of) end one. Such a "one-way conductor" (or equivalently a "one-way insulator") would have lots of useful applications (for example, it could be used to violate the second law of thermodynamics and transfer heat from a cold body to a hot one!). Sal just needs a little investment to commercialize this marvelous invention. Is there anything we have learned in 18.303 which would cast doubt on the validity of Sal's claim?

Problem 3: Green's functions

Recall that the displacement $u(x,t)$ of a stretched string [with fixed ends: $u(0,t) = u(L,t) = 0$] satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} + f(x,t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x,t)$ is an external force density (pressure) on the string.

(a) Suppose that $f(x,t) = \mathrm{Re}[g(x)e^{-i\omega t}]$, an oscillating force with a frequency ω. Show that, instead of solving the wave equation with this $f(x,t)$, we can instead use a complex force $\tilde{f}(x,t) = g(x)e^{-i\omega t}$, solve for a complex $\tilde{u}(x,t)$, and then take $u = \mathrm{Re}\,\tilde{u}$ to obtain the solution for the original $f(x,t)$.

(b) Suppose that $f(x,t) = g(x)e^{-i\omega t}$, and we want to find a *steady-state* solution $u(x,t) = v(x)e^{-i\omega t}$ that is oscillating everywhere at the same frequency as the input force. (This will be the solution after a long time if there is any dissipation in the system to allow the initial transients to die away.) Write an equation $\hat{A}v = g$ that v solves. Is $\hat{A}$ self-adjoint? Positive/negative definite/semidefinite?

(c) Solve for the Green's function $G(x,x')$ of this $\hat{A}$, assuming that $\omega \neq n\pi/L$ for any integer n (i.e. assume ω is not an eigenfrequency [why?]). [Write down the continuity conditions that G must satisfy at $x = x'$, solve for $x \neq x'$, and then use the continuity conditions to eliminate unknowns.]

(d) Form a finite-difference approximation A of your $\hat{A}$. Compute an approximate $G(x,x')$ in Matlab by `A \ dk`, where $\mathbf{d}_k$ is the unit vector of all 0's except for one $1/\Delta x$ at index $k = x'/\Delta x$, and compare (by plotting both) to your analytical solution from the previous part for a couple values of x' and a couple of different frequencies ω (one $< \pi/L$ and one $> \pi/L$) with $L = 1$.

(e) Show the limit $\omega \to 0$ of your G relates in some expected way to the Green's function of $-\frac{d^2}{dx^2}$ from class.