Advection and Upwinding

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$$u_t + a u_x = 0$$

1 Stability Analysis

Let's start with FTCS

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

Von Neumann

$$u_i^n = e^{ikx_i}$$

so then

$$G...$$

and then

$$|G|^2 = 1 + \frac{a^2 \Delta t^2}{\Delta x^2} \sin^2(k\Delta x)$$

Notice that, for all values, $|G| > 1$. Therefore FTCS is UNCONDITIONALLY UNSTABLE.

2 Upwinding

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

for $a > 0$, while

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u_{i+1}^n - u_i^n}{\Delta x} = 0$$

for $a < 0$. Second order upwind scheme

$$u_x^- = \frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x}$$

How is this derived? Go back to our polynomial interpolation

$$g(x) = \frac{u_3 - 2u_2 - u_1}{2\Delta x^2}x^2 + \frac{-u_3 + 4u_2 - 3u_1}{2\Delta x}x + u_1$$

and then

$$g'(x) = \frac{u_3 - 2u_2 - u_1}{\Delta x^2}x + \frac{-u_3 + 4u_2 - 3u_1}{2\Delta x}$$

so then

$$g'(0) = \frac{-u_3 + 4u_2 - 3u_1}{2\Delta x}$$

2.1 Stability of first order

Van Neurmann...

$$|G| = 1 - 2\mu\left(1 - \mu\right)\left(1 - \cos\left(k\Delta x\right)\right)$$

where

$$\mu = \frac{a\Delta t}{\Delta x}$$

and thus

$$a\frac{\Delta t}{\Delta x} \leq 1$$