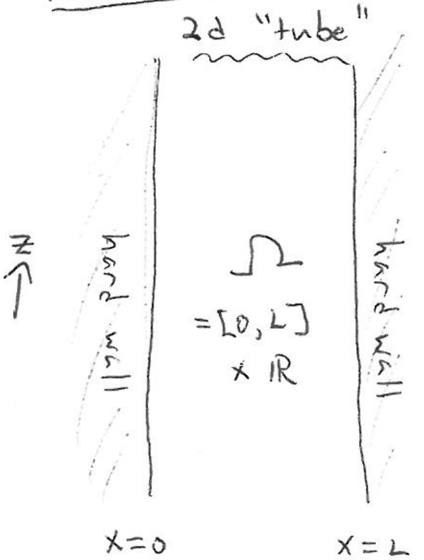
Waveguide Modes and Evanescent Waves



$$\hat{A} = \nabla^2$$

$$u|_{\partial\Omega} = 0$$

$$\Downarrow$$

$$\hat{A} = \hat{A}^* < 0$$

under

$$\langle u, v \rangle = \int_\Omega \bar{u} v$$

waves: $\hat{A} u = \frac{\partial^2 u}{\partial t^2}$

normal modes: $\hat{A} u = \lambda u = -\omega^2 u$

translational symmetry in z

$\Longrightarrow$ separable eigensolutions

$$u(x, z) = e^{ikz} u_k(x) \,:$$

$$\hat{A}\, e^{ikz} u_k = -\omega^2(k)\, e^{ikz} u_k$$

$$\Rightarrow \underbrace{\left(e^{-ikz} \hat{A}\, e^{ikz}\right)}_{\hat{A}_k} u_k = -\omega^2 u_k$$

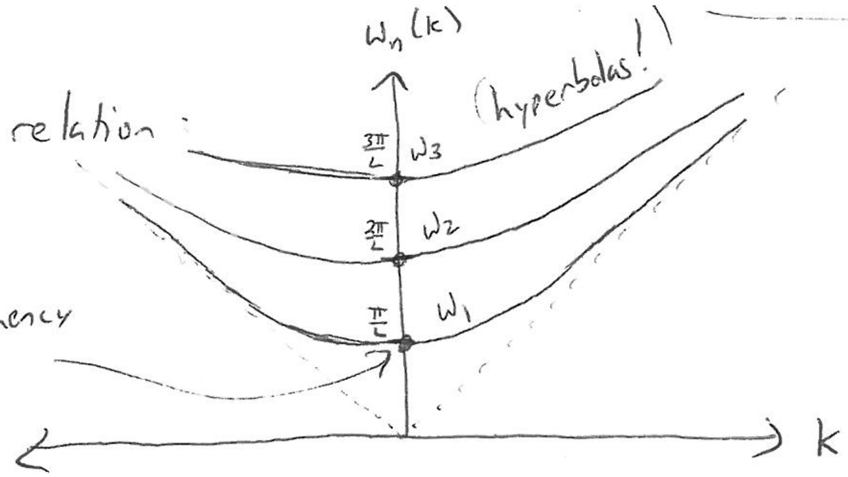
Solution:

$$e^{-ikz}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\left(e^{ikz} u_k(x)\right)$$

$$= \cancel{e^{-ikz}}\ \cancel{e^{ikz}}\left(\frac{\partial^2}{\partial x^2} - k^2\right) u_k \qquad = -\omega^2 u_k$$

$$\Longrightarrow \quad u_k'' = -(\omega^2 - k^2)\, u_k \quad \Longrightarrow \quad \boxed{\begin{aligned} u_{k,n}(x) &= \sin\left(\frac{n\pi x}{L}\right) \\ \omega_n(k) &= \pm\sqrt{\left(\frac{n\pi}{L}\right)^2 + k^2} \end{aligned}}$$

Dispersion relation:



each mode has a low-frequency cutoff

(also called "β" = "propagation constant")

as $k \to 0$ (cutoff):

$v_g = \frac{d\omega}{dk} \to 0$ (standing waves)

$v_p = \frac{\omega}{k} \to \infty$

* Evanescent waves: exponentially decaying / growing solutions

for real $\omega \Rightarrow k$:

$$k_n = \pm\sqrt{\omega^2 - \left(\frac{n\pi}{L}\right)^2} = \begin{cases} \text{real } (\underline{\text{propagating}}) & |\omega| \geq \frac{n\pi}{L} : \text{above cutoff} \\ \text{imaginary } (\underline{\text{evanescent}}) & |\omega| < \frac{n\pi}{L} : \text{below cutoff} \end{cases}$$

— <u>not</u> <u>allowed</u> in ∞ waveguide <u>unless</u> we break translational symmetry

... we never admit solutions that grow exponentially as $z \to \pm\infty$

— <u>needed</u> if we break translation invariance by changing geometry <u>or</u> by introducing a <u>source</u>

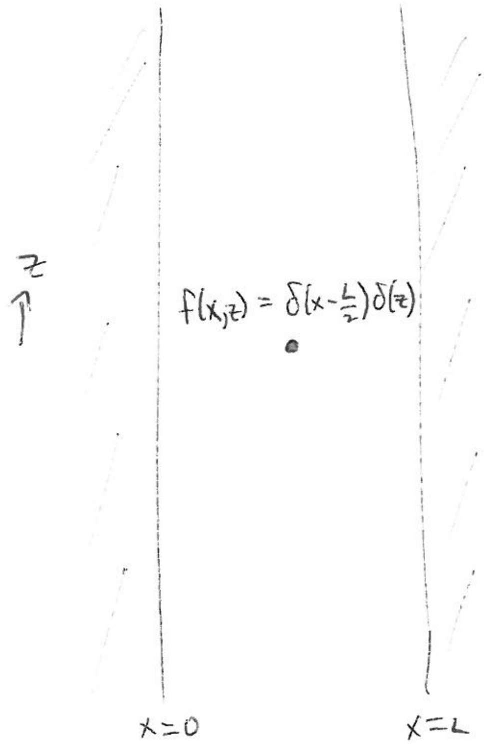
* Scalar <u>Helmholtz equation</u>:

$$\hat{A}u = \frac{\partial^2 u}{\partial t^2} + \underbrace{f(x,t)}_{\text{source term}} \xrightarrow[\text{in } t]{\text{Fourier transform}} \hat{A}\hat{u} = -\omega^2\hat{u} + \hat{f}$$

(equivalently: time-harmonic response $\hat{u}e^{-i\omega t}$ to a time-harmonic source $\hat{f}e^{-i\omega t}$)

$$\Rightarrow \boxed{(\hat{A} + \omega^2)\, u(\vec{x}) = f(\vec{x})} \quad [\text{dropping the hats}]$$

<u>example</u> : δ- function source f in $(\hat{A}+\omega^2)u = f$



look for solutions : (sum of propagating + evanescent)

$$\boxed{u(x,z) = \begin{cases} \sum_n c_n^+ \sin\left(\frac{n\pi x}{L}\right) e^{+ik_n z} & z>0 \\ \sum_n c_n^- \sin\left(\frac{n\pi x}{L}\right) e^{-ik_n z} & z<0 \end{cases} \quad \text{where } k_n = \sqrt{\omega^2 - \left(\frac{n\pi}{L}\right)^2}}$$

$\Longrightarrow$ by construction,

$$\boxed{(\hat{A}+\omega^2)u = 0 \quad \text{for} \quad z \neq 0}$$

$$\left[= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2\right) \sum_n c_n^\pm \sin\left(\frac{n\pi x}{L}\right) e^{\pm ik_n z} \right.$$

$$\left. = \sum_n c_n^\pm \sin\left(\frac{n\pi x}{L}\right) e^{\pm ik_n z} \underbrace{\left(-\left(\frac{n\pi}{L}\right)^2 - k_n^2 + \omega^2\right)}_{0} \right]$$

* u is continuous at $z=0$, if $\boxed{c_n^+ = c_n^-} = c_n$

but $\frac{\partial u}{\partial z}$ is discontinuous

$$\Longrightarrow \frac{\partial^2 u}{\partial z^2} = \delta(z) \sum_n c_n \sin\left(\frac{n\pi x}{L}\right) \cdot 2ik_n \quad \left(\text{only source of a } \delta(z)!\right)$$

$$= \delta\left(x-\frac{L}{2}\right)\delta(z) \quad (= f(x))$$

$$\Longrightarrow c_n = \frac{\cancel{2}}{L} \frac{1}{\cancel{2}ik_n} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \delta\left(x-\frac{L}{2}\right) dx = \frac{1}{ik_n L} \sin\left(\frac{n\pi \cancel{L}}{2\cancel{L}}\right)$$

$$= \frac{1}{ik_n L} \begin{cases} 0 & n \text{ even (antisymmetric } \sin\left(\frac{n\pi x}{L}\right)) \\ (-1)^{\frac{n-1}{2}} & n \text{ odd (symmetric } \sin) \end{cases}$$

$$\Rightarrow \boxed{u(x,z) = \sum_{\substack{n \\ \text{odd}}} \frac{(-1)^{\frac{n-1}{2}}}{iL\underbrace{\sqrt{w^2 - \left(\frac{n\pi}{L}\right)^2}}_{= k_n}} \sin\left(\frac{n\pi x}{L}\right) e^{i k_n |z|}}$$

* sum of waves <u>propagating</u> <u>away</u> from source and <u>evanescent</u> waves <u>decaying</u> <u>away</u> from source

* solution <u>blows</u> <u>up</u> as $\omega \to$ cutoff $\frac{n\pi}{L}$ of any mode!
(real effect!)

— <u>but</u> remember that this is generally just a single Fourier component $\hat{u}$ of solution:

suppose $f(x,t) = \delta(z)\,\delta(x - \frac{L}{2})\, f(t)$

for some "pulse" $f(t)$
(usually square-integrable)

$\xleftrightarrow[\text{Fourier}]{} \hat{f}(w)$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(w) e^{-iwt}\, dw$$

where $\hat{u} = \sum_{\text{odd } n} \frac{(-1)^{\frac{n+1}{2}}}{i k_n L} \sin\left(\frac{n\pi x}{L}\right) e^{i k_n |z|} \hat{f}(w)$

and this <u>integral</u> is <u>finite</u> because $\frac{1}{k_n}$ is an <u>integrable</u> <u>singularity</u> ($\int \frac{1}{\sqrt{w}}\, dw$ is finite)