18.303 Problem Set 1 Solutions

Problem 1: (5+10+(5+5) points)

(a) $A = A^T$ and A^{-1} (the *unique* matrix such that $A^{-1}A = AA^{-1} = I$) exists. Then $I = I^T = (AA^{-1})^T = (A^{-1})^T A^T = (A^{-1})^T A$ and, since $(A^{-1})^T A = I$, we have $(A^{-1})^T = A^{-1}$. Q.E.D.

(b) In order to show that the eigenvalues are real, we must first allow the eigenvectors and eigenvalues to be complex and then *prove* that they are real. For complex vectors, the inner product must be generalized to $\langle \mathbf{x}, \mathbf{y} \rangle_B = \mathbf{x}^* B \mathbf{y}$. Then, given a (possibly complex) eigenvector $\mathbf{x}$ ($\neq 0$) satisfying $B^{-1}A\mathbf{x} = \lambda \mathbf{x}$, we can take the $\langle \mathbf{x}, \cdots \rangle_B$ inner product with both sides (i.e. multiply both sides by $\mathbf{x}^* B$) to get

$$\begin{aligned} \mathbf{x}^* B B^{-1} A \mathbf{x} &= \lambda \mathbf{x}^* B x \\ \mathbf{x}^* A \mathbf{x} = (A\mathbf{x})^* \mathbf{x} &= \quad \text{(since } A \text{ is real-symmetric)} \\ (BB^{-1}A\mathbf{x})^* \mathbf{x} = (\lambda B \mathbf{x})^* \mathbf{x} &= \\ \bar{\lambda} \mathbf{x}^* B \mathbf{x} &= \quad \text{(since } B \text{ is real-symmetric).} \end{aligned}$$

Then, since B is positive-definite and $\mathbf{x} \neq 0$, we know that $\mathbf{x}^* B \mathbf{x} > 0$ and we can cancel it from both sides to show that $\lambda = \bar{\lambda}$, hence λ is real. (We can then trivially show that $\mathbf{x}$ can be chosen real, though I didn't ask you to do this: given a complex $\mathbf{x}$, simply take the real or imaginary parts of both sides of $B^{-1}A\mathbf{x} = \lambda \mathbf{x}$ to show that both the real and imaginary parts of $\mathbf{x}$ are also eigenvectors.)

To show that they are orthogonal, we consider two eigenvectors satisfying $B^{-1}A\mathbf{x}_{1,2} = \lambda_{1,2}\mathbf{x}_{1,2}$ for distinct eigenvalues $\lambda_1 \neq \lambda_2$, and take the inner product of the $\mathbf{x}_2$ equation with $\mathbf{x}_1$:

$$\begin{aligned} \mathbf{x}_1^* \cancel{BB^{-1}} A \mathbf{x}_2 &= \lambda_2 \mathbf{x}_1^* B \mathbf{x}_2 \\ (BB^{-1}A\mathbf{x}_1)^* \mathbf{x}_2 &= \\ \lambda_1 \mathbf{x}_1^* B \mathbf{x}_2 &= \end{aligned}$$

similar to above. Hence $(\lambda_1 - \lambda_2)\mathbf{x}_1^* B \mathbf{x}_2 = 0$, and since $\lambda_1 - \lambda_2 \neq 0$ we have $\mathbf{x}_1^* B \mathbf{x}_2 = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle_B = 0$. Q.E.D.

(c) A is a real-symmetric 4×4 matrix with eigenvalues $-1, -4, -9, -25$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4$, respectively.

(i) We solved a problem very similar to $\frac{d^2}{dt^2}\mathbf{x} = A\mathbf{x}$ in class for the *infinite*-dimensional case (sine series). Since A is necessarily diagonalizable and we can expand $\mathbf{x}(t)$ in this basis at every t: $\mathbf{x}(t) = \sum_k c_k(t)\mathbf{x}_k$. Plugging this into $\frac{d^2}{dt^2}\mathbf{x} = A\mathbf{x}$, we get $\sum_k \ddot{c}_k \mathbf{x}_k = \sum_k \lambda_k c_k \mathbf{x}_k$. Since A is real-symmetric, the eigenvectors are orthogonal, so we can take the dot product of both sides with $\mathbf{x}_n$ to get[1] an ODE for c_n: $\ddot{c}_n = \lambda_n c_n$. The solutions of this are sines and cosines (or exponentials), and we immediately find $c_n = \alpha_n \cos(t\sqrt{-\lambda_n}) + \beta_n \sin(t\sqrt{-\lambda_n})$ for some coefficients α_n and β_n to be determined. Notice that $-\lambda_n$ is positive so the square roots are *real*, so we have oscillating "normal modes:"

$$\mathbf{x}(t) = \sum_{k=1}^{4} \left[\alpha_k \cos(t\sqrt{-\lambda_k}) + \beta_k \sin(t\sqrt{-\lambda_k}) \right] \mathbf{x}_k.$$

[1] Actually, we don't need to use orthogonality here; this would work even for a nonsymmetric A. We can write $\mathbf{x}(t) = X\mathbf{c}$ where X is the matrix whose columns are eigenvectors and $\mathbf{c}$ is the vector of the coefficients c_k, and then the ODE gives $X\ddot{\mathbf{c}} = X\Lambda\mathbf{c}$ where Λ is the diagonal matrix of eigenvalues (from $AX = X\Lambda$). Then, since A is diagonalizable (even if it is nonsymmetric, it still has 4 distinct eigenvalues), X is invertible; multiplying both sides by X^{-1} on the left gives $\ddot{\mathbf{c}} = \Lambda\mathbf{c}$, and the n-th row of this gives $\ddot{c}_n = \lambda_n c_n$ as desired.

To get the coefficients, we use the initial conditions as in class: $\mathbf{x}(0) = \mathbf{a}_0 = \sum_k \alpha_k \mathbf{x}_k$ and $\dot{\mathbf{x}}(0) = \mathbf{b}_0 = \sum_k \beta_k \sqrt{-\lambda_k}\mathbf{x}_k$, and taking the inner product of both sides with $\mathbf{x}_n$ (using orthogonality again, gives):

$$\alpha_n = \frac{\mathbf{x}_n^* \mathbf{a}_0}{\|\mathbf{x}_n\|^2},$$

$$\beta_n = \frac{\mathbf{x}_n^* \mathbf{b}_0}{\|\mathbf{x}_n\|^2 \sqrt{-\lambda_n}},$$

which completes our closed-form solution. (Note that you were *not* given that $\|\mathbf{x}_n\| = 1$, although of course we could *choose* that normalization.)

(ii) From above, $\frac{d^2}{dt^2}\mathbf{x} = A\mathbf{x}$ has *oscillating* solutions, and none of the eigenvectors ever dominates in general (unless it dominated in the initial conditions). In contrast, $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ has solutions $e^{At}\mathbf{x}(0) = \sum_k e^{\lambda_k t}\mathbf{x}_k \frac{\mathbf{x}_k^*\mathbf{x}(0)}{\|\mathbf{x}_k\|^2}$, which are *exponentially decaying* and are eventually dominated by the $\mathbf{x}_1$ solution (since it has the largest λ) regardless of the initial conditions (unless they are orthogonal to $\mathbf{x}_1$).

Problem 2: ((10+5)+5+10 points)

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

(a) Suppose that we we change the boundary conditions $u(0) = 0$, $u'(L) = u(L)/L$.

(i) As in class, solving $u'' = \lambda u$ gives sine, cosine, or exponential solutions (at least for $\lambda \neq 0$). $u(0) = 0$ means that the solutions must be $\sin(kx)$ for some k, with eigenvalue $\lambda = -k^2$. $u'(L) = u(L)/L$ means that $k\cos(kL) = \sin(kL)/L$, or

$$kL = \tan(kL),$$

which is a transcendental equation for k. The left- and right-hand sides of this equation are plotted in Fig. 1. Because the tangent function has an infinite number of vertical asymptotes at odd multiples of $\pi/2$, there are an infinite number of intersections and hence an infinite number of solutions. Note, however, that there is no solution in the $[0, \pi/2]$ interval, because kL and $\tan(kL)$ start out tangent and then the latter curves up, so they never intersect in that interval except at $kL = 0$. We don't count $kL = 0$ as a solution, because that would give $\sin(kx) = 0$, which is not a valid eigenfunction.

To solve the equation, you can do so graphically, but it is better to use Newton's method on $z - \tan(z)$. Given an initial guess z for kL, one simply iterates $z \leftarrow z - \frac{z - \tan(z)}{1 - \sec^2(z)}$. Note, however, that Newton's method may not converge unless you give a good enough initial guess, but $n\pi/2 - 0.1$ seems good enough (for $n = 3, 5, 7, \ldots$). This can be automated in Julia or other programs, or you can do it yourself on a calculator. Given a root z, the eigenvalues are then $-k^2 = -z^2/L^2$. The result for the first few eigenvalues, to many significant digits, is:

$$\begin{aligned}
\lambda_1 &= -20.190728556426645/L^2, \\
\lambda_2 &= -59.6795159441095/L^2, \\
\lambda_3 &= -118.89986916362645/L^2.
\end{aligned}$$

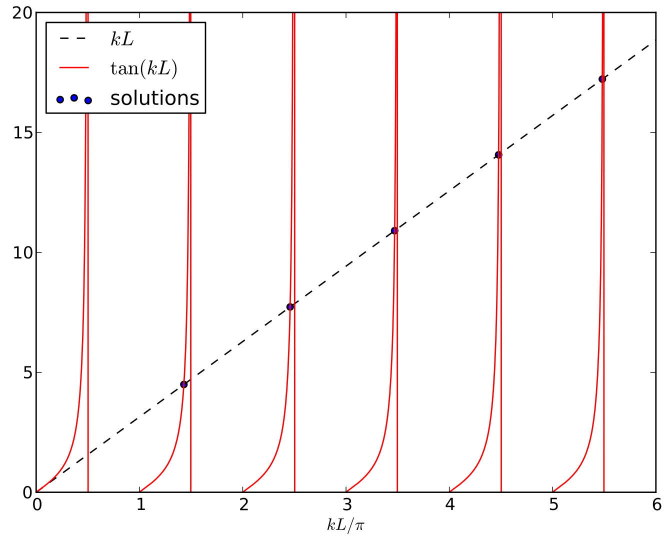

Figure 1: Plot of the left- and right-hand sides of the transcendental equation $kL = \tan(kL)$, along with the solutions (dots). Note that the vertical red lines are not part of the $\tan(kL)$ curve, they are only the vertical asymptotes where the function changes sign. Note also that there is no solution in the $[0, \pi/2]$ interval, because kL and $\tan(kL)$ start out tangent and then the latter curves up, so they never intersect in that interval except at $kL = 0$.

An IJulia notebook illustrating these calculations is posted on the web site with the solutions.

However, this is missing a solution. As in class, if we solve for the null space $u'' = 0$, the solutions are straight lines. To satisfy the left boundary condition we have $u(x) = \alpha x$ (the line must go through the origin), in which case the right boundary condition $u'(L) = \alpha = u(L)/L = \alpha L/L$ is automatically satisfied for all α! So, another eigenfunction is $u(x) = x$ (all other α are simply multiples of this and hence are not linearly independent), with eigenvalue

$$\lambda_0 = 0.$$

(ii) No, we do not expect the solutions to be unique, because there is a nontrivial nullspace spanned by $u_0(x) = x$, or equivalently there is an eigenvalue of zero. Hence, to any solution $u(x)$ of $\hat{A}u = f$ we can add any multiple of x and get another solution satisfying the same boundary conditions.

If you forgot the zero eigenvalue in the last part, you should have noticed it here when you checked for the presence of a nontrivial nullspace $u'' = 0$.

(b) No, the boundary condition $v'(L) = v(L)/L + 1$ cannot be satisfied by a vector space. For example, $v(x) = 0$ does not satisfy it, and any vector space must contain 0.

(c) We have $v'(L) = v(L)/L+1$, we write $u(x) = v(x)+q(x)$, and we want $u'(L) = u(L)/L$. Hence $u'(L) = v'(L) + q'(L) = v(L)/L + 1 + q'(L) = [v(L) + q(L)]/L$, yielding $\boxed{q'(L) = q(L)/L - 1}$. We must also have $u(0) = 0$, and since $v(0) = 0$ this implies $\boxed{q(0) = 0}$. Now we can try a few simple guesses to find such a $q(x)$. The simplest would be polynomials (with no constant term so that $q(0) = 0$). Plugging them in and trying quickly shows that $q(x) = \alpha x$ or $q(x) = \alpha x^2 + \beta x$ cannot work. The next try is $q(x) = \alpha x^3 + \beta x^2 + \gamma x$, which gives $q'(L) = 3\alpha L^2 + 2\beta L + \gamma = q(L)/L - 1 = \alpha L^2 + \beta L + \gamma - 1$, and hence $\beta L = -2\alpha L^2 - 1$ works (regardless of γ). For example, we can set $\gamma = 0$ and $\alpha = 1$ to get

$$q(x) = x^3 - \frac{2L^2 + 1}{L} x^2.$$

Of course, there are infinitely many other possible solutions. Regardless of what $q(x)$ you use, you should plug it into $v'' = g = (u - q)'' \implies u'' = g + q'' = f$, hence

$$f(x) = q''(x) = 6x - 2\frac{2L^2 + 1}{L}$$

where we have plugged in our choice of $q(x)$.