Problem Set 2

Due Friday, March 1st

February 20, 2019

1 Inner Products, Adjoints, and Definiteness

1. Show the following identities of the adjoint operator.

   (a) $A = (A^*)^*$

   (b) $(A^*)^{-1} = \left(A^{-1}\right)^*$

2. Consider the inner product $(x, y)_B = x^* B y$ where B is a real-symmetric positive-definite matrix. Show that if A is real-symmetric, then $C = B^{-1}A$ is self-adjoint w.r.t. $(\cdot, \cdot)_B$.

3. Recall that a linear operator A is self-adjoint with respect to an inner product $(\cdot, \cdot)$ if $(u, Av) = (A^*u, v)$ for all u and v in the corresponding vector space. Let $\tilde{A} = -\frac{d^2}{dx^2}$ be the Poisson operator on the space of quasi-periodic function from Problem Set 1, i.e. the functions $u(0) = e^{i\phi}u(L)$ and $u'(0) = e^{i\phi}u(L)$. Let the L^2 inner product be defined as $(u, v) = \int_0^L \overline{u(x)}v(x)dx$. Show that $\tilde{A}$ is self-adjoint. For what values (if any) of ϕ is it positive-definite? Is this consistant with your answer to Problem 2 of Problem Set 1?

4. Let $B = -\frac{d^2}{dx^2} + q(x)$ where $q(x)$ is a real-valued function that satisfies $q(x) \geq q_0$ for some constant q_0. Show that this operator is self-adjoint for the quasi-periodic functions. Additionally, show that all eigenvalues of B are $\geq q_0$ (Hint: consider whether $\tilde{A} - q_0$ is definite).

2 Finite Difference Approximations

Let's use the finite difference method to analyze and solve the 1D Poisson equation. Let

$$\tilde{A}u = -c\frac{d^2}{dx^2}$$

where $c(x) > 0$ is a real-valued positive function. Consider Dirichlet boundary conditions $u(0) = u(L) = 0$ on $[0, L]$. Approximate $u(jh) \approx u_j$ at N evenly spaced points where $j = 1, 2, \ldots, N$, $u_0 = u_{N+1} = 0$, and $h = \frac{L}{N+1}$.

1. Write down the finite difference approximation using second order centered differences to approximate $\tilde{A}u$ by AU, where U is the column vector $[u_j]$ and $A = CD^TD$ where D is the 1st derivative matrix from δ_+ (D^T is δ_-!)

2. Write a function that computes the action of A on a vector U without using the matrix A itself. This is known as a matrix-free implementation of the linear operator.

3. Explain why you expect the matrix A to have real, positive eigenvalues, even though $A \neq A^T$.

4. Use the diagm command in the LinearAlgebra standard library to build the matrix A with $c(x) = e^{3x}$ and $N = 100$. Now use the command eigen to analyze the eigenvalues and eigenvectors of A (notice that $\lambda < 0$ since A is negative-definite)

   (a) Plot the eigenvectors corresponding to the four smallest-magnitude (smallest absolute value) eigenvalues.

   (b) Verify that the eigenvectors for the two smallest eigenvalues are orthogonal w.r.t. the correct inner product $(v_1, v_2)_X$, where X is replaced by the appropriate matrix. (Hint: look at 1.2)

5. Use the \ operator to solve the Poisson equation $\tilde{A}u = f(x)$ with $c(x) = e^{3x}$ and $f(x) = 1 - e^{-2x}$

6. For $c(x) = 1$ we saw that the eigenfunctions are $\sin(n\pi x/L)$. How does this compare to the eigenvalues you plotted in the previous part? Try changing $c(x)$ to some other real positive valued function and see how different you can make the eigenfunctions from $\sin(n\pi x/L)$. Is there some feature that always remains similar, no matter how much you change c?