18.303 Midterm Exam, Fall 2016

November 9, 2016

Problem 1: Hermitian (33 points)

In homework, you showed that $\nabla\times$ was Hermitian under the inner product $\langle\mathbf{F},\mathbf{G}\rangle = \int \overline{\mathbf{F}}\cdot\mathbf{G}$; that is, $\int \overline{\nabla\times\mathbf{F}}\cdot\mathbf{G} = \int \overline{\mathbf{F}}\cdot\nabla\times\mathbf{G}$, for appropriate boundary conditions, and from that you went on to show that $\nabla\times\nabla\times$ was Hermitian and positive-semidefinite, and hence Maxwell's equations $\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla\times\nabla\times\mathbf{E}$ had oscillating solutions.

The case you analyzed in homework only applied to Maxwell's equations in vacuum, however. In materials, the equations become

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = \varepsilon^{-1}\nabla\times\left(\mu^{-1}\nabla\times\mathbf{E}\right),$$

where $\mu(\mathbf{x})$ and $\varepsilon(\mathbf{x})$ are material properties related to the magnetic and electric polarizability of matter.

Assuming μ and ε are real, positive scalar functions, **choose an inner product and show** that the operator $\hat{A} = \varepsilon^{-1}\nabla\times\mu^{-1}\nabla\times$ is Hermitian and positive semidefinite, and hence that we still have oscillating solutions. Don't worry about the boundary conditions—just assume that we have chosen boundary conditions so that $\nabla\times$ is still Hermitian as above (i.e. so that boundary terms vanish when you integrate by parts, i.e. you can just use the integral identity from above).

Problem 2: Timestepping (34 points)

Suppose we have a PDE $\frac{\partial u}{\partial t} = \hat{A}u$, that we discretize via finite-differences as in class into a "Crank-Nicolson" scheme:

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} = A\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2},$$

where $\mathbf{u}^n$ denotes the (discretized) $u(\mathbf{x})$ at time $t = n\Delta t$, and A is a discretized version of the operator $\hat{A}$ (e.g. finite differences in space if $\hat{A}$ consists of spatial derivatives like ∇^2).

1. From class, the left-hand side is a second-order accurate (errors $\sim \Delta t^2$) center-difference approximation for $\frac{\partial u}{\partial t}$ at time $t = (n+\frac{1}{2})\Delta t$. That requires the right-hand side to also be at time $n+\frac{1}{2}$. If we treat $\mathbf{u}^n$ as an approximation for $\mathbf{u}(n\Delta t)$, show that $\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2} \approx \mathbf{u}([n+\frac{1}{2}]\Delta t) + O(\Delta t^2)$, i.e. it is second-order accurate. (This was claimed in class but not proved. Use the Taylor series.)

2. Suppose our matrix A (independent of n) satisfies $A = A^* \prec 0$ (negative definite). Show that the solutions $\mathbf{u}^n$ of our finite-difference scheme above go to zero as $n\to\infty$. (Hint: write $\mathbf{u}^n = (\text{something})^n\mathbf{u}^0$ as in class and write the eigenvalues of the "something" in terms of the eigenvalues of A.) i.e. show it is unconditionally stable (for any $\Delta t > 0$).

Problem 3: Born (33 points)

Consider the operator $\hat{A}(p) = -\nabla^2 + c(p,\mathbf{x})$ in some domain Ω with Dirichlet boundary conditions $u|_{\partial\Omega} = 0$, where $c(p,\mathbf{x})$ is a real-valued function that depends on some parameter p (p is a real number). Suppose that I tell you that I have a computer program that can quickly and accurately solve

$$\hat{A}(0)u = f$$

for $u(\mathbf{x})$ given any right-hand side $f(\mathbf{x})$. That is, I can apply $\hat{A}(0)^{-1}$.

Explain how, using my $\hat{A}(0)^{-1}$ computer program (without modifying it), you can compute (for a given f)

$$\left.\frac{\partial u}{\partial p}\right|_{p=0}.$$

Hint: write the solution for a *small* Δp approximately using a Born approximation, then take the $\Delta p \to 0$ limit. Assume you can Taylor-expand $c(p, \mathbf{x})$ around $p = 0$.