

18.303 Problem Set 2

Due Wednesday, 19 September 2012.

Note: For Matlab homework problems in 18.303, turn in with your solutions a printout of any commands used and their results (please edit out extraneous/irrelevant stuff; e.g. don't turn in pages and pages of numbers if you accidentally output the contents of a large matrix), and a printout of any graphs requested. **Always label** the axes of your graphs (with the `xlabel` and `ylabel` commands), add a title with the `title` command, and add a legend (if there are multiple curves) with the `legend` command. (Labelling graphs is a good habit to acquire.)

Note: You can run Matlab on Athena with `add matlab && matlab` at the Athena prompt. You can also get a free Matlab for your own computer. See <http://web.mit.edu/matlab/www/>

Problem 1: Finite-difference approximations

For this question you may find it helpful to refer to the notes and readings from lecture 3. Consider a finite-difference approximation for the second derivative of the form:

$$u''(x) \approx \frac{a \cdot u(x + 2\Delta x) + b \cdot u(x + \Delta x) + c \cdot u(x) + b \cdot u(x - \Delta x) + a \cdot u(x - 2\Delta x)}{d \cdot (\Delta x)^2}.$$

- (a) By substituting the Taylor expansion for $u(x + \delta)$, solve for coefficients a, b, c, d that make this expression *fourth-order accurate*: errors proportional to $(\Delta x)^4$ for small Δx .
- (b) Check your answer to the previous part by numerically computing $u''(1)$ for $u(x) = \sin(x)$, as a function of Δx , exactly as in the handout from class (refer to the handout for the relevant Matlab commands and adapt them as needed). Verify from your log-log plot of the |errors| versus Δx that you obtained the expected rate of convergence.

Problem 2: Inner products, adjoints, definiteness

Here, we consider inner products $\langle u, v \rangle$ on some vector space of complex-valued functions and the corresponding adjoint \hat{A}^* of linear operators \hat{A} , where the adjoint is defined, as in class, by whatever satisfies $\langle u, \hat{A}v \rangle$ for all u and v . Usually, \hat{A}^* is obtained from \hat{A} by some kind of integration by parts. In particular, suppose V consists of functions¹ $u(x)$ on $x \in [0, L]$ with *periodic* boundary conditions $u(0) = u(L)$ as in pset 1, and define the inner product $\langle u, v \rangle = \int_0^L \overline{u(x)} v(x) dx$ as in class.

- (a) Show that $\hat{A} = -\frac{d}{dx} \left(c \frac{d}{dx} \right)$ is self-adjoint ($\hat{A}^* = \hat{A}$) and positive semi-definite ($\langle u, \hat{A}u \rangle \geq 0$) for any function $c(x) > 0$,² but is *not* positive definite ($\langle u, \hat{A}u \rangle = 0$ for some $u \neq 0$ in V).
- (b) Suppose that we define the *integral* operator \hat{B} as follows: if $v = \hat{B}u$, $v(x) = \int_0^L G(x, x') u(x') dx'$ for some function $G(x, x')$. This is a perfectly good linear operator (which we will return to later in 18.303). (You can assume that G is sufficiently well-behaved for all such integrals to exist.) What condition must G satisfy in order for \hat{B} to be self-adjoint?

Problem 2: Finite-difference matrices in Matlab

For this problem, the Matlab function `diff1.m` linked on the web site will be helpful. If you download this file and put it in your directory (the one you run Matlab from), the command

¹As mentioned in class, we technically need to restrict ourselves to a Sobolev space of u where $\langle u, \hat{A}u \rangle$ is defined and finite, i.e. we are implicitly assuming that u is sufficiently differentiable and not too divergent, etcetera, but for the most part we will pass over such technicalities in 18.303 (which are important for rigorous proofs, but only serve to exclude counter-examples that have no physical relevance).

²Again, assuming c is sufficiently well-behaved that the various integrals exist.

```
D = diff1(M);
```

will return the $(M+1) \times M$ first derivative matrix D from class for Dirichlet boundary conditions $u(0) = u(L) = 0$, omitting the Δx factor, so that the discrete 1d Laplacian $\approx -\frac{d^2}{dx^2}$ is approximated by $A = D^T D / \Delta x$ (`D'*D/dx^2` in Matlab) for $\Delta x = L/(M+1)$.

- (a) If, instead of Dirichlet boundaries, we have periodic boundaries $u(0) = u(L)$, how is A modified? Assume you discretize $[0, L]$ into the points $u_m \approx u(m\Delta x)$ for $m = 0, 1, \dots, M$, with $u_0 = u_M$ (so that we have M degrees of freedom u_1, \dots, u_M) and $\Delta x = L/M$. Write down what the new A matrix looks like (hint: only the first and last rows are changed). (Careful: the A in class was for d^2/dx^2 , whereas now we have $-d^2/dx^2$.)

- (b) Write Matlab code to create such an A matrix for $M = 100$ and $L = 1$, of the form:

```
L=1; M=100;
dx = L/M; D = diff1(M); A = D'*D / dx^2;
??????
```

where `??????` is code to change the Dirichlet A matrix into the periodic one. Hint: you can change the i, j entry of A (row i , column j) in Matlab by `A(i,j)=...new value...`; Note: your A should still be symmetric!

- (c) You can get a list of the eigenvalues in Matlab by `eig(A)`, and sort them in ascending order by `sort(eig(A))`. Check that they are ≥ 0 (semidefinite A), that there is a zero eigenvalue (or at least one that is very close to zero...there will be some roundoff error to make it slightly nonzero, and it may even be slightly negative, e.g. -10^{-12}), that the nonzero eigenvalues are doubly repeated (as predicted in pset 1), and that the first few nonzero eigenvalues are approximately $(2\pi n/L)^2$ for $n = 1, 2, \dots$ (as predicted in pset 1).
- (d) Get the eigenvectors *and* eigenvalues by `[V,S]=eig(A)`; (the columns of V are the eigenvectors, and S is a diagonal matrix of eigenvalues). Again, make sure these are sorted in order of λ by using the commands: `[lambda,i]=sort(diag(S))`; and `V = V(:,i)`; Now, plot the first few eigenvectors by the command `x=linspace(0,L,M+1); x=x(2:end); plot(x,V(:,1),'r-', x,V(:,2),'b-', x,V(:,3),'k:', x,V(:,4),'c.-');`; `legend('1','2','3','4');`; `xlabel('x');`; `ylabel('eigenfunctions')` ...do they match your predictions from pset 1?
- (e) It is actually possible to solve the discrete Laplacian eigenproblem *analytically*. In particular, at each point m we have the discrete eigenequation $\frac{-u_{m+1} + 2u_m - u_{m-1}}{\Delta x^2} = \lambda u_m$, with periodic boundary conditions ($u_0 = u_M$, $u_{M+1} = u_1$).

- (i) Show that an eigenvector of the form

$$u_m = e^{ikm\Delta x}$$

is an *exact* solution to this discrete eigenproblem for certain values of k (which k satisfy the discrete boundary condition?).

- (ii) Use this to find an *exact* formula for the eigenvalues λ , and explicitly check that it *exactly* matches (up to roundoff errors, 12-13 digits) the Matlab eigenvalues from part (c) above.
- (iii) Show that as $\Delta x \rightarrow 0$, your eigenvalues λ approach the eigenvalues $(2\pi n/L)^2$ of the continuous problem (hint: Taylor expand), with an error proportional to Δx^2 for small Δx .