

18.303 Problem Set 1

Due Wednesday, 14 September 2011.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

- (a) Suppose A is a real matrix with a nullspace spanned by the vector \mathbf{n} , and a left nullspace spanned by the vector \mathbf{m} . In terms of \mathbf{m} , \mathbf{n} , \mathbf{b} , and/or \mathbf{x} only (*not* A), give an equation that will tell you whether $A^T A \mathbf{x} = \mathbf{b}$ has a solution. Supposing it does have *one* solution \mathbf{x} , give an expression for *all* solutions. [Hint: recall that $C(A^T A) = C(A^T) = N(A)^\perp$.]
- (b) Suppose that A is a real 21×21 matrix with eigenvalues $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^{20}}$ and corresponding eigenvectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{20}$, respectively (that is, $\lambda_k = 2^{-k}$ for $k = 0, \dots, 20$).
 - (i) Is A necessarily diagonalizable? Symmetric?
 - (ii) Suppose A is symmetric. If a sequence of vectors $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n, \dots$ satisfies the recurrence $\mathbf{y}_{n+1} = A \mathbf{y}_n$ with some “random” starting vector \mathbf{y}_0 , what is likely to be true about the solution \mathbf{y}_n at very large n ? (You should be able to give an explicit, approximate formula relating \mathbf{y}_n to \mathbf{y}_0 and one or two of the eigenvectors.)
- (c) In 18.06, you were shown a simple (2–3 line) proof that the eigenvalues λ of A (solutions of $A \mathbf{x} = \lambda \mathbf{x}$) must be real numbers if A is real-symmetric (if you forgot, look it up). Adapt a similar proof to show that the eigenvalues λ of $A \mathbf{x} = \lambda \mathbf{x}$ must have *purely imaginary* λ if A is a real *antisymmetric* matrix: $A^T = -A$.

Problem 2: Les Poissons, les Poissons, how I love les Poissons

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2} u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the “Dirichlet” boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

- (a) How do the eigenfunctions of $\frac{d^2}{dx^2}$ change if we change the boundary conditions to $u(0) = 0$ and $u'(L) = 0$ (note prime = 1st derivative)? Check by explicit integration that different-eigenvalue eigenfunctions u_n and u_m are still orthogonal in the sense that $\int_0^L u_n(x) u_m(x) dx = 0$.
- (b) If we instead consider $\frac{d^2}{dx^2} v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = 2$, $v'(L) = 1$, do these functions form a vector space? Why or why not?
- (c) Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2} u(x) = f(x)$ problem with $u(0) = 0$, $u'(L) = 0$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions q and r . (Transforming a new problem into an old, solved one is always a useful thing to do!)

Problem 3: Nondimensionalization

In 18.303, we will mostly solve PDEs in convenient arbitrary units where we don’t have any pesky dimensionful constants flapping around the equations and annoying us. This represents no loss of generality because, even if someone hands you a PDE in SI units, you can always rescale things to get back into dimensionless form.

- (a) In quantum mechanics, they study the Schrodinger equation:

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right] \psi = i\hbar \frac{\partial \psi}{\partial t}$$

in the function $\psi(\mathbf{x}, t)$. Show how, by a change of variables $\tilde{\mathbf{x}} = \mathbf{x}/X$ and $\tilde{t} = t/T$, for appropriately chosen scale factors X and T ($=?$), you can transform this into a simplified equation:

$$\left[-\tilde{\nabla}^2 + V(X\tilde{\mathbf{x}}) \right] \tilde{\psi} = i \frac{\partial \tilde{\psi}}{\partial \tilde{t}}$$

where $\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) = \psi(X\tilde{\mathbf{x}}, T\tilde{t})$ and $\tilde{\nabla}^2$ is the Laplacian in the $\tilde{\mathbf{x}}$ coordinates. (Hint: first figure out how $\tilde{\nabla}^2 \tilde{\psi}$ and $\partial \tilde{\psi} / \partial \tilde{t}$ relate to $\nabla^2 \psi$ and $\partial \psi / \partial t$, respectively.)¹

- (b) Suppose a future version of yourself—one who has completed 18.303 and understands all the sacred mysteries of PDEs—travels back in time and hands you the solution $u(x, y, t)$ to the 2d scalar wave equation $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$ on a unit-circle domain Ω centered at $(x, y) = 0$, with $u = 0$ on the boundary of the circle, and with the initial condition $u(x, y, 0) = \sin[1 - (x^2 + y^2)]$. However, your engineering professor Alyssa P. Hacker gives you a “different” problem, in “real” units: Prof. Hacker wants you to solve the wave equation $\nabla^2 h = \frac{1}{c^2} \frac{\partial^2 h}{\partial t^2}$ describing the height $h(x, y, t)$, in meters, of a stretched circular drum head, with $c = 150$ m/s, where the drum has a radius $R = 0.2$ m centered at $(x, y) = 0$ with $h = 0$ on the boundary, for the initial condition $h(x, y, 0) = (0.1 \text{ m}) \sin[1 - (x^2 + y^2)/R^2]$. Give Prof. Hacker an equation relating $h(x, y, t)$ to $u(x, y, t)$. (Like in the previous part, you may find it easier to keep track of the changes of variables if you switch notations to $\tilde{x}, \tilde{y}, \tilde{t}$ in the u equations.)

¹Prof. Ed Farhi (MIT Physics) once told me a story of how, after explaining at a party that he was a theoretical physicist, was challenged: “Really? Prove it! What’s the value of \hbar ?” He failed to establish his credentials, because (like most theorists) he always sets \hbar to 1.