18.303 Midterm Solutions, Fall 2015

November 12, 2015

Problem 1: Adjoint (33 points)

As in class, let's define the inner product as $\langle u, v\rangle = \int_\Omega \bar{u}v/c$, where the division by $c(\mathbf{x})$ will cancel the asymmetrical-looking c factor in $\hat{A}$. Then we have:

$$\begin{aligned}
\langle u, \hat{A}v\rangle &= -\int_\Omega \bar{u}\nabla^2 v = +\int_\Omega \overline{\nabla u}\cdot\nabla v - \oint_{\partial\Omega} \bar{u}\nabla v\cdot\hat{\mathbf{n}}da \\
&= -\int_\Omega \overline{c\nabla^2 u}v/c + \oint_{\partial\Omega} [v\nabla\bar{u} - \bar{u}\nabla v]\cdot\hat{\mathbf{n}}da \\
&= \langle \hat{A}u, v\rangle + \cancel{\oint_{\partial\Omega} [v\nabla\bar{u} - \bar{u}\nabla v]\cdot\hat{\mathbf{n}}da}
\end{aligned}$$

where in the last line we have used the boundary conditions to cancel the integrand:

$$[v\nabla\bar{u} - \bar{u}\nabla v]\cdot\hat{\mathbf{n}} = \alpha\,[v\bar{u} - \bar{u}v] = 0.$$

Problem 2: Finite differences (34 points)

Since we have a boundary condition $u'(L) = \alpha u(L)$ that requires us to know the slope at $x = L$, the most convenient thing is to put $x = L$ *halfway between* two grid points so that we can use a center-difference approximation. In particular, we set up the grid so that $u_m \approx u(m\Delta x)$ as usual, but $\boxed{(M + 0.5)\Delta x = L}$. Then we have the discretized boundary conditions:

$$u_0 = 0$$

$$\frac{u_{M+1} - u_M}{\Delta x} = \alpha\frac{u_M + u_{M+1}}{2},$$

in which the left-hand side is the 2nd-order accurate center-difference approximation for $u'(L)$ and the right-hand side is the 2nd-order accurate approximation for $\alpha u(L)$. (The 2nd-order accuracy of both of these was shown in class.) We have the M unknowns $u_1, \ldots, u_M$, and we can solve the second equation above for

$$\boxed{u_{M+1} = \frac{1 + \alpha\Delta x/2}{1 - \alpha\Delta x/2}u_M = \beta u_M},$$

defining the number β.

If we use the usual center-difference approximation for the second derivative, we have

$$-u''_m = \frac{-u_{m-1} + 2u_m - u_{m+1}}{\Delta x^2},$$

which for the specific cases of $m = 1$ and $m = M$ (the first and last matrix rows) gives (from the boundary conditions):

$$-u''_1 = \frac{2u_m - u_{m+1}}{\Delta x^2},$$

$$-u''_M = \frac{-u_{m-1} + (2 - \beta)u_m}{\Delta x^2}.$$

Hence, we have the matris

$$\boxed{A = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2-\beta \end{pmatrix}}.$$

Since this matrix is obviously real-symmetric, we have self-adjointness under the usual inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^* \mathbf{v}$.

Problem 3: Green (33 points)

We want to solve the *nonlinear* equation

$$\hat{A}u + \alpha u^2 = f$$

for some small $|\alpha(\mathbf{x})| \ll 1$. We begin by writing write $u = u_0 + u_1 + u_2 + \cdots$ as a power series in α, where u_n denotes all the terms proportional to α^n. If we plug this in to our equation, we get $\hat{A}u_0 + \hat{A}u_1 + \hat{A}u_2 + \cdots + \alpha u_0^2 + 2\alpha u_0 u_1 + \alpha u_1^2 + \cdots = f$. If we collect terms by power of α, we get the equations:

$$\hat{A}u_0 = f,$$

$$\hat{A}u_1 + \alpha u_0^2 = 0,$$

$$\hat{A}u_2 + 2\alpha u_0 u_1 = 0,$$

and so on. From the first equation, we get $u_0 = \hat{A}^{-1} f$, i.e.

$$u_0(\mathbf{x}) = \int_\Omega G_0(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')\, d^3\mathbf{x}'.$$

1. We want $u \approx u_0 + u_1$. From the second equation above, we get $u_1 = -\hat{A}^{-1}\alpha u_0^2$, i.e.

   $$u_1(\mathbf{x}) = -\int_\Omega G_0(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}') u_0(\mathbf{x}')^2\, d^3\mathbf{x}'.$$

   Note that if we substitute the u_0 integral, we would have two nested integrals.

2. We want $u \approx u_0 + u_1 + u_2$. From the third equation above, we get $u_2 = -2\hat{A}^{-1}\alpha u_0 u_1$, i.e.

   $$u_2(\mathbf{x}) = -2\int_\Omega G_0(\mathbf{x}, \mathbf{x}')\alpha(\mathbf{x}') u_0(\mathbf{x}') u_1(\mathbf{x}')\, d^3\mathbf{x}'.$$

   Note that if we substituted the u_0 and u_1 integrals, we would have a nested integral for u_0 and another two nested integrals for u_1.

It would be an amusing exercise to devise a Feynman-diagram notation for these integrals. Graphically, the nonlinear term αu^2 appears as a source term that arises when two Green's functions "collide".