

18.303 Problem Set 7 Solutions

Problem 1: (10+10pts)

- (a) Let V^c denote the complement of V (the exterior region outside V , i.e. $V^c = \mathbb{R}^d \setminus V$), and note that $\partial V^c = \partial V$ (but with the outward-normal vector reversed in sign). We write:

$$\begin{aligned}\nabla f\{\phi\} &= f\{-\nabla\phi\} = -\int_V f_1 \nabla\phi - \int_{V^c} f_2 \nabla\phi \\ &= -\int_V [\nabla(f_1\phi) - \phi\nabla f_1] - \int_{V^c} [\nabla(f_2\phi) - \phi\nabla f_2] \\ &= -\oint_{\partial V} f_1 \phi \mathbf{n} + \int_V \phi \nabla f_1 + \oint_{\partial V} f_2 \phi \mathbf{n} + \int_{V^c} \phi \nabla f_2 \\ &= \left[\delta(\partial V) [f_2(\mathbf{x}) - f_1(\mathbf{x})] \mathbf{n}(\mathbf{x}) + \begin{cases} \nabla f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases} \right] \{\phi\}\end{aligned}$$

as desired. **Note: there was a typo in the pset**, which had $f_1 - f_2$ instead of $f_2 - f_1$ in the formula you were supposed to derive.

- (b) In this case, we will need to integrate by parts twice, but we can just quote the results from the “notes on elliptic operators” from class (where we integrated by parts twice with $-\nabla^2$ already), albeit keeping the boundary terms from ∂V that were zero in the notes:

$$\begin{aligned}\nabla^2 f\{\phi\} &= f\{\nabla^2\phi\} = \int_V f_1 \nabla^2\phi + \int_{V^c} f_2 \nabla^2\phi \\ &= \oint_{\partial V} [(f_1 - f_2)\nabla\phi - \phi(\nabla f_1 - \nabla f_2)] \cdot \mathbf{n} + \int_V \phi \nabla^2 f_1 + \int_{V^c} \phi \nabla^2 f_2.\end{aligned}$$

But the first term is $\delta(\partial V) [f_1(\mathbf{x}) - f_2(\mathbf{x})] \{\mathbf{n} \cdot \nabla\phi\} = (\mathbf{n} \cdot \nabla)\delta(\partial V) [f_2(\mathbf{x}) - f_1(\mathbf{x})] \{\phi\}$ by the definition (note the sign change) of the distributional derivative $\mathbf{n} \cdot \nabla$ (note that this is a scalar derivative in the \mathbf{n} direction, not a gradient vector). The second term is a surface delta function weighted by $(\mathbf{n} \cdot \nabla f_1 - \mathbf{n} \cdot \nabla f_2)$, the discontinuity in the normal derivative. And the last terms are just a regular distribution. So, we have

$$\nabla^2 f = (\mathbf{n} \cdot \nabla)\delta(\partial V) [f_2 - f_1] + \delta(\partial V) [\mathbf{n} \cdot \nabla f_1 - \mathbf{n} \cdot \nabla f_2] + \begin{cases} \nabla^2 f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla^2 f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases}.$$

As noted in class, $\mathbf{n} \cdot \nabla$ of a delta function is a “dipole” oriented in the \mathbf{n} direction, so the first term is a “dipole layer”.

Problem 2: ((5+5+5)+5 points)

Consider Green’s functions of the self-adjoint indefinite operator $\hat{A} = -\nabla^2 - \omega^2$ ($\kappa > 0$) over all space ($\Omega = \mathbb{R}^3$ in 3d), with solutions that $\rightarrow 0$ at infinity. (This is related to the previous problem.) As in class, thanks to the translational and rotational invariance of this problem, we can find $G(\mathbf{x}, \mathbf{x}') = g(|\mathbf{x} - \mathbf{x}'|)$ for some $g(r)$ in spherical coordinates.

- (a) Solve for $g(r)$ in 3d, similar to the procedure in class.

- (i) For $r > 0$, $-\nabla^2 g - \omega^2 g = 0$ and hence $-\omega^2 g = \nabla^2 g = \frac{1}{r}(rg)'' \implies h'' = -\omega^2 h$ where $h(r) = rg(r)$. The solution to this is $h(r) = ce^{i\omega r} + de^{-i\omega r}$ for some constants c and d , or $g(r) = \frac{ce^{i\omega r} + de^{-i\omega r}}{r}$ for $r > 0$.
- (ii) This operator arose by assuming a time dependence $e^{-i\omega t}$ multiplying the solution, in which case we are looking at wave solutions $\frac{ce^{i\omega(r-t)} + de^{-i\omega(r+t)}}{r}$, where the c term describes waves moving *out* towards $r \rightarrow \infty$, while the d term describes waves moving *in* from infinity. In wave problems, we typically impose a boundary condition of *outgoing waves at infinity*, in which case we must set $d = 0$. (However, the choice would have been reversed if we picked the opposite sign convention, $e^{+i\omega t}$, for the time dependence.)
- (iii) From above, we have $g(r) = ce^{i\omega r}/r$. As in class, the $1/r$ singularity is no problem in 3d (it is cancelled by the Jacobian factor $r^2 dr$), so g is a regular distribution. Given an arbitrary test function $q(\mathbf{x})$, we now

evaluate

$$\begin{aligned}
(\hat{A}g)\{q\} &= g\{\hat{A}q\} = \int g\hat{A}q \\
&= \int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \left[-\omega^2 gq - \frac{g}{r} \frac{\partial^2}{\partial r^2}(rq) + \underbrace{(\theta, \phi \text{ derivatives of } q)}_{\text{integrate to 0, from class}} \right] \\
&= \iint \sin\theta d\theta d\phi \left[\int_0^\infty c \left(-\omega^2 r q e^{i\omega r} - \underbrace{e^{i\omega r} \frac{\partial^2}{\partial r^2}[rq]}_{\text{int. by parts}} \right) dr \right] \\
&= \iint \sin\theta d\theta d\phi \left[-c e^{i\omega r} \frac{\partial}{\partial r}[rq] \Big|_0^\infty + c \int_0^\infty \left(-\omega^2 r q e^{i\omega r} + \underbrace{i\omega e^{i\omega r} \frac{\partial}{\partial r}[rq]}_{\text{int. by parts}} \right) dr \right] \\
&= \iint \sin\theta d\theta d\phi \left[c q(0) - \cancel{i c \omega e^{i\omega r} [rq] \Big|_0^\infty} + c \int_0^\infty \left(\cancel{-\omega^2 r q e^{i\omega r}} + \cancel{\omega^2 e^{i\omega r} [rq]} \right) dr \right] \\
&= 4\pi c q(0),
\end{aligned}$$

and hence $\boxed{c = 1/4\pi}$. Thus

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{i\omega|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}.$$

(b) The $\omega \rightarrow 0$ limit gives $1/4\pi|\mathbf{x}-\mathbf{x}'|$ as in class, by inspection.