18.303 Problem Set 3

Due Wednesday, 26 September 2012.

Problem 1: Deriving diffusion

In this problem, you will be deriving the fact that a diffusion equation results from any microscopic picture in which the flow of solute is proportional to the gradient or difference in solute concentration ("Fick's law").

(a) Consider the case of salt diffusing in water within a narrow straw of length L, which we will approximate as a one-dimensional system. Imagine dividing the straw into M small pieces of length Δx, and call the mass of salt in the m-th piece $u_m \Delta x$ for $m = 1, \ldots, M$, where u_m is a concentration (mass per Δx). Suppose that the rate at which mass flows from one piece to the next is proportional to the difference in concentrations (u_m), and is inversely proportional to the distance to the mass must travel (Δx), so that the *mass per unit time* flowing from m to $m+1$ is given by:

$$D \frac{u_{m+1} - u_m}{\Delta x}$$

for some constant *diffusion coefficient* D. Derive an equation for the net rate of change du_m/dt of the concentration in piece m. Take the limit $\Delta x \to 0$ and show that you obtain a diffusion equation for $u(x)$, where $u(m\Delta x) = u_m$. Don't worry about the boundary conditions; just look at the interior $1 < m < M$.

(b) Suppose that any salt that reaches the ends of the straw is immediately removed (there is a little salt-eating demon sitting at each end of the straw). What boundary condition on u_0 and u_{M+1} does that imply in the discrete model? What boundary condition on $u(x)$ in the continuum limit?

(c) Suppose that we seal the ends of the straw, so that no salt can enter or leave through the ends. Write down a boundary condition on u_0 and u_{M+1} that reflects this situation in the discrete model, write the corresponding matrix A in the discrete system ($d\mathbf{u}/dt = A\mathbf{u}$), and write the corresponding boundary condition on $u(x)$ in the continuum limit.

Problem 2: Adjoints in higher dimensions

(a) Suppose that we have an *anisotropic* wave equation in d dimensions, $\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho(\mathbf{x})} \nabla \cdot C(\mathbf{x}) \nabla u$, where $\rho(\mathbf{x})$ is a scalar density (real, > 0) and $C(\mathbf{x})$ is a $d \times d$ matrix: a "stiffness" that depends on direction. (e.g. think of a stretched drum in 2 dimensions, made of a material that is easier to stretch in one direction than in the other). Suppose we have Dirichlet boundary conditions $u|_{\partial\Omega} = 0$. If C is a real-symmetric positive-definite matrix at every $\mathbf{x}$, show that we still expect a superposition of "normal" (orthogonal) oscillating modes (real frequencies) by choosing the correct inner product and analyzing the operators, similar to class.

(b) Consider 2-component vector fields $\mathbf{u}(\mathbf{x})$ in two dimensions, under the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \mathbf{u}^* \mathbf{v}$ for some domain Ω (where $*$ applied to a vector is just the conjugate transpose). Give an example of boundary conditions under which the operator

$$\hat{A}\mathbf{u} = \nabla^2 \mathbf{u} + \nabla(\nabla \cdot \mathbf{u}) = \begin{pmatrix} \nabla^2 u_x + \frac{\partial}{\partial x} \nabla \cdot \mathbf{u} \\ \nabla^2 u_y + \frac{\partial}{\partial y} \nabla \cdot \mathbf{u} \end{pmatrix}$$

is self-adjoint under this inner product. (This operator shows up, for example, as part of the Lamé–Navier equations of elastic solids.)

Problem 3: Symmetric matrices

In class, when looking at the balls-and-springs system for varying masses and spring constants, we showed that the $M \times M$ matrix

$$A = -M^{-1} D^* K D \Delta x^2$$

is self-adjoint and negative-definite under the inner product $\langle \mathbf{x}, \mathbf{y} \rangle_M = \mathbf{x}^* M \mathbf{y}$, where M and K were positive diagonal matrices (in fact, any positive-definite matrices would do), and $D\Delta x$ was our $(M+1) \times M$ difference matrix.

(a) Construct a 5×5 example of such an A in Matlab, using the `diff1(5)` command from the last pset to construct $D\Delta x$, and pick some M and K with distinct diagonal elements. [e.g. the command `diag([1 2 3 4 5])` makes a 5×5 diagonal matrix with the numbers 1 to 5 on its diagonal.) Using the [V,S] = eig(A) command, as in the last pset, to get the matrix V of eigenvectors and the diagonal matrix S of eigenvalues, verify that the eigenvalues are real and negative and that the eigenvectors are orthogonal under the M inner product above. e.g. `V(:,1)' * M * V(:,2)` computes the M inner product of the first two eigenvectors, and `V'*M*V` computes a matrix ($V^* M V$) of all the inner products of all the eigenvectors.

(b) Show that *any* diagonalizable matrix A with real eigenvalues is self-adjoint under $\langle \mathbf{x}, \mathbf{y} \rangle_W = \mathbf{x}^* W \mathbf{y}$ for $W = (VV^*)^{-1}$, in terms of the diagonalization $A = V \Lambda V^{-1}$ of A. Compute this in Matlab for your V from (a); do you get $W = M$? Explain why you can get many different W matrices from the same $W = (VV^*)^{-1}$ formula.