18.303 Problem Set 6 Solutions

Problem 1: (10+10 points)

(a) $\hat{A}^{-1}$ is Hermitian, i.e.,$\langle u, \hat{A}^{-1}v\rangle = \langle \hat{A}^{-1}u, v\rangle$. Substituting $\hat{A}^{-1}u(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{x}')u(\mathbf{x}')d^n\mathbf{x}'$, we must therefore have:

$$\begin{aligned}\langle u, \hat{A}^{-1}v\rangle &= \iint_{\mathbf{x},\mathbf{x}'\in\Omega} \overline{u(\mathbf{x})}G(\mathbf{x},\mathbf{x}')v(\mathbf{x}') \\ &= \langle \hat{A}^{-1}u, v\rangle \\ &= \iint_{\mathbf{x},\mathbf{x}'\in\Omega} \overline{G(\mathbf{x},\mathbf{x}')u(\mathbf{x}')}v(\mathbf{x}) = \iint_{\mathbf{x},\mathbf{x}'\in\Omega} \overline{u(\mathbf{x})G(\mathbf{x}',\mathbf{x})}v(\mathbf{x}'),\end{aligned}$$

where in the last step we have interchanged/relabeled $\mathbf{x} \leftrightarrow \mathbf{x}'$. Since this must be true for all u and v, it follows that

$$\boxed{G(\mathbf{x},\mathbf{x}') = \overline{G(\mathbf{x}',\mathbf{x})}}.$$

(b) A (time-independent) source term at $\mathbf{x}'$ corresponds to $f(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}')$ or some multiple thereof, in which case the steady-state solution ($\partial u/\partial t = 0$) is $u(\mathbf{x}) = G(\mathbf{x},\mathbf{x}')$, where G is the Green's function of $-\nabla\cdot D\nabla$. If there is diffusion from $\mathbf{x}'$ to $\mathbf{x}$, that means that $u(\mathbf{x}) = G(\mathbf{x},\mathbf{x}') \neq 0$ (i.e. a source at $\mathbf{x}'$ eventually produces a nonzero concentration at $\mathbf{x}$). By reciprocity, however, this means that $G(\mathbf{x}',\mathbf{x}) = G(\mathbf{x},\mathbf{x}') \neq 0$ (G is real since the operator is real), and this is precisely the concentration at $\mathbf{x}$ from a source at $\mathbf{x}'$. That is, reciprocity means that diffusion from $\mathbf{x}'$ to $\mathbf{x}$ is exactly the same (produces the same concentration) as diffusion from $\mathbf{x}$ to $\mathbf{x}'$, so one-way diffusion (or even *slightly* asymmetrical diffusion) is not possible.

Problem 2: (5+5+10 points)

Recall that the displacement $u(x,t)$ of a stretched string [with fixed ends: $u(0,t) = u(L,t) = 0$] satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} + f(x,t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x,t)$ is an external force density (pressure) on the string.

(a) Suppose that $\tilde{u}$ solves $\frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}(x,t) = \frac{\partial^2 \tilde{u}}{\partial t^2}$ and satisfies $\tilde{u}(0,t) = \tilde{u}(L,t) = 0$. Now, consider $u = \operatorname{Re}\tilde{u} = \frac{\tilde{u}+\bar{\tilde{u}}}{2}$. Clearly, $u(0,t) = u(L,t) = 0$, so u satisfies the same boundary conditions. It also satisfies the PDE:

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{1}{2}\left[\frac{\partial^2 \tilde{u}}{\partial t^2} + \overline{\frac{\partial^2 \tilde{u}}{\partial t^2}}\right] \\ &= \frac{1}{2}\left[\frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f} + \overline{\frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{f}}\right] \\ &= \frac{\partial^2 u}{\partial x^2} + f,\end{aligned}$$

since $f = \frac{\tilde{f}+\bar{\tilde{f}}}{2} = \operatorname{Re}\tilde{f}$. The key factors that allowed us to do this are (i) *linearity*, and (ii) the *real-ness* of the PDE (the PDE itself contains no i factors or other complex coefficients).

(b) Plugging $u(x,t) = v(x)e^{-i\omega t}$ and $f(x,t) = g(x)e^{-i\omega t}$ into the PDE, we obtain

$$\frac{\partial^2 v}{\partial x^2}\cancel{e^{-i\omega t}} + g\cancel{e^{-i\omega t}} = -\omega^2 v\cancel{e^{-i\omega t}},$$

and hence

$$\left(-\frac{\partial^2}{\partial x^2} - \omega^2\right)v = g$$

and $\boxed{\hat{A} = -\frac{\partial^2}{\partial x^2} - \omega^2}$. The boundary conditions are $v(0) = v(L) = 0$, from the boundary conditions on u.

Since ω^2 is real, this is in the general Sturm-Liouville form that we showed in class is **self-adjoint**.

Subtracting a constant from an operator just shifts all of the eigenvalues by that constant, keeping the eigenfunctions the same. Thus $\hat{A}$ is still positive-definite if ω^2 is $<$ the smallest eigenvalue of $-\partial^2/\partial x^2$, and positive semidefinite if $\omega^2 =$ the smallest eigenvalue. In this case, we know analytically that the eigenvalues of $-\partial^2/\partial x^2$ with these boundary conditions are $(n\pi/L)^2$ for $n = 1, 2, \ldots$. So $\hat{A}$ is positive-definite if $\omega^2 < (\pi/L)^2$, it is positive-semidefinite if $\omega^2 = (\pi/L)^2$, and it is indefinite otherwise.

(c) We know that $\hat{A}G(x, x') = 0$ for $x \neq x'$. Also, just as in class and as in the notes, the fact that $\hat{A}G(x, x') = \delta(x - x')$ meas that G must be continuous (otherwise there would be a δ' factor) and $\partial G/\partial x$ must have a jump discontinuity:

$$\left.\frac{\partial G}{\partial x}\right|_{x=x'+} - \left.\frac{\partial G}{\partial x}\right|_{x=x'-} = -1$$

for $-\partial^2 G/\partial x^2$ to give $\delta(x - x')$. We could also show this more explicitly by integrating:

$$\begin{aligned} \int_{x'-0^+}^{x'+0^+} \hat{A}G\, dx &= \int_{x'-0^+}^{x'+0^+} \delta(x - x')\, dx = 1 \\ &= -\left.\frac{\partial G}{\partial x}\right|_{x'-0^+}^{x'+0^+} - \cancel{\int_{x'-0^+}^{x'+0^+} \omega^2 G\, dx}, \end{aligned}$$

which gives the same result.

Now, similar to class, we will solve it separately for $x < x'$ and for $x > x'$, and then impose the continuity requirements at $x = x'$ to find the unknown coefficients.

For $x < x'$, $\hat{A}G = 0$ means that $\frac{\partial^2 G}{\partial x^2} = -\omega^2 G$, hence $G(x, x')$ is some sine or cosine of ωx. But since $G(0, x') = 0$, we must therefore have $G(x, x') = \alpha \sin(\omega x)$ for some coefficient α.

Similarly, for $x > x'$, we also have a sine or cosine of ωx. To get $G(L, x') = 0$, the simplest way to do this is to use a sine with a phase shift: $G(x, x') = \beta \sin(\omega[L - x])$ for some coefficient β.

Continuity now gives two equations in the two unknowns α and β:

$$\begin{aligned} \alpha \sin(\omega x') &= \beta \sin(\omega[L - x']) \\ \alpha\omega \cos(\omega x') &= -\beta\omega \cos(\omega[L - x']) + 1, \end{aligned}$$

which has the solution $\alpha = \frac{1}{\omega} \frac{\sin(\omega[L-x'])}{\cos(\omega x')\sin(\omega[L-x'])+\sin(\omega x')\cos(\omega[L-x'])}$, $\beta = \frac{1}{\omega} \frac{\sin(\omega x')}{\cos(\omega x')\sin(\omega[L-x'])+\sin(\omega x')\cos(\omega[L-x'])}$. This simplifies a bit from the identity $\sin A \cos B + \cos B \sin A = \sin(A + B)$, and hence

$$\boxed{G(x, x') = \frac{1}{\omega \sin(\omega L)} \begin{cases} \sin(\omega x)\ \sin(\omega[L - x']) & x < x' \\ \sin(\omega x')\ \sin(\omega[L - x]) & x \geq x' \end{cases}},$$

which obviously obeys reciprocity.

Note that if ω is an eigenfrequency, i.e. $\omega = n\pi/L$ for some n, then this G blows up. The reason is that $\hat{A}$ in that case is singular (the n-th eigenvalue was shifted to zero), and defining $\hat{A}^{-1}$ is more problematical. (Physically, this corresponds to driving the oscillating string at a resonance frequency, which generally leads to a diverging solution unless there is dissipation in the system.)