Lecture 6 : Elliptic operators + friends

* 1d "Sturm-Liouville" operator:

$$\hat{A} = \underbrace{\frac{1}{w(x)}}_{>0}\left[-\frac{\partial}{\partial x} c(x) \frac{\partial}{\partial x} + p(x)\right], \quad u(x) \text{ on } [0, L] = \Omega$$

(c, p real)

Dirichlet boundaries $u\big|_{\partial\Omega} = 0$

let $\langle u, v \rangle = \int_0^L w(x)\, \overline{u(x)}\, v(x)\, dx = \int_\Omega w \bar{u} v$

$$\Rightarrow \langle u, \hat{A} v \rangle = \int_\Omega \cancel{w}\, \bar{u}\, \frac{1}{\cancel{w}}\left[-\frac{\partial}{\partial x}(c v') + p v\right]$$

$$= -\int_\Omega \bar{u}\,(c v')' + \int_\Omega \bar{u}\, p v$$

$$= \int_\Omega \bar{u}'\, c v' + \int_\Omega \overline{p u}\, v = -\int_\Omega \overline{(c u')'}\, v + \int_\Omega \overline{p u}\, v$$

$$\cancelto{0}{- \bar{u} c v'\Big|_{\partial\Omega}} \qquad \cancelto{0}{+ c \bar{u}' v\Big|_{\partial\Omega}}$$

$$= \int_\Omega w\, \overline{\frac{1}{w}\left[-(c u')' + p u\right]}\, v = \langle \hat{A} u, v \rangle$$

$\Longrightarrow \hat{A} = \hat{A}^* \Longrightarrow$ real λ, orthogonal eigenfunctions

(+ "diagonalizable" for "reasonable" w, c, p)

= "Sturm-Liouville theory"

$$\langle u, \hat{A} u \rangle \underset{\text{(same steps)}}{=} \cdots = \int_\Omega \Big(c\,\underbrace{|u'|^2}_{\substack{>0 \\ \text{for } u' \neq 0 \\ \Updownarrow \\ u \neq 0 \\ \text{since } u = \text{constant} \Rightarrow u = 0}} + \; p\,\underbrace{|u|^2}_{\substack{>0 \\ \text{for } u \neq 0}} \Big)$$

> 0 for $u \neq 0$

if $\underline{\underline{c > 0, \;\; p \geq 0}}$

$\Rightarrow \hat{A}$ positive-definite for $c > 0$, $p \geq 0$

= "<u>elliptic operator</u>" (also elliptic if negative-definite)

* Higher dimensions:

- even more useful to do such analysis in $> 1d$, since analytical solutions are even harder, so ability to say general things from $\hat{A}$ is crucial to <u>understanding</u>

a "simple" case: $\hat{A} = -\nabla^2 = -\nabla \cdot \nabla = -\text{div. grad.}$
(still very hard in $> 1d$!)

a generalization: (non-uniform media)

$$\boxed{\hat{A} = \underbrace{\frac{1}{w(\vec{x})}}_{>0} \Big[-\nabla \cdot \underbrace{c(\vec{x})}_{\text{real}} \nabla + \underbrace{p(\vec{x})}_{\text{real}} \Big]}$$

on functions $u(\vec{x})$ on some (finite) domain Ω

+ Dirichlet boundaries (for now): $\boxed{u\big|_{\partial\Omega} = 0}$

(even more general: c could be a self-adjoint matrix) consider: $\langle u, v \rangle = \int_\Omega w \bar{u} v$

$$\Rightarrow \langle u, \hat{A} v \rangle = \underbrace{-\int_\Omega \bar{u} \nabla \cdot (c \nabla v)}_{\text{need to integrate by parts}} + \underbrace{\int_\Omega \bar{u} p v}_{= \int_\Omega \frac{w}{w} \overline{p u}\, v}$$

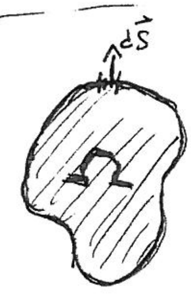
$\Rightarrow p$ term is self-adjoint

★ review of integration by parts:

$$1d: \int_\Omega f g' = \int_\Omega \Big[\underbrace{(fg)'}_{\substack{\text{integral of} \\ \text{derivative} \\ = \text{easy}}} - \underbrace{f' g}_{\substack{\text{from} \\ \text{product} \\ \text{rule}}} \Big] = fg \Big|_{\partial\Omega} - \int f' g$$

$$> 1d: \boxed{\int_\Omega f \nabla \cdot \vec{g} = \int_\Omega \Big[\underbrace{\nabla \cdot (f \vec{g})}_{\substack{\text{integral} \\ \text{of divergence} \\ = \text{easy by} \\ \text{divergence theorem}}} - \underbrace{(\nabla f) \cdot \vec{g}}_{\substack{\text{from "product rule"} \\ \nabla \cdot (f\vec{g}) = \\ (\nabla f) \cdot \vec{g} + f \nabla \cdot g}} \Big]}$$

$$= \underbrace{\oiint_{\partial\Omega}}_{\substack{\text{surface} \\ \text{integral} \\ \text{(1 less dimension)}}} f \vec{g} \cdot \underbrace{d\vec{S}}_{\substack{\text{outward} \\ \text{normal}}} - \int_\Omega (\nabla f) \cdot \vec{g}$$



$$\Longrightarrow\; -\int_\Omega \overbrace{\bar{u}}^{f}\, \nabla\cdot \overbrace{(c\nabla v)}^{\vec{g}} = \cancel{-\oint_{\partial\Omega} \bar{u}\,(c\nabla v)\cdot d\vec{S}} + \int_\Omega \underbrace{\overline{c\nabla u}}_{\text{new: } \vec{g}} \cdot \underbrace{\nabla v}_{f}$$

$$\cancel{-\oint_{\partial\Omega} \bar{u}\,(c\nabla v)\cdot d\vec{S}} \xrightarrow{} 0 \text{ if } u\big|_{\partial\Omega} = 0$$

$$\left(\underline{\text{or}} \text{ if } (\nabla v)\cdot d\vec{S}\big|_{\partial\Omega} = 0\right)$$

$$= \cancel{\oint_{\partial\Omega} v\,\overline{c\nabla u}\cdot d\vec{S}} \xrightarrow{} 0 \;-\; \int_\Omega v\, \overline{\nabla\cdot(c\nabla u)} \cdot w \cdot \frac{1}{w}$$

$$\Longrightarrow\; \langle u, \hat{A} v\rangle = \langle \hat{A} u, v\rangle$$

$$\Longrightarrow\; \boxed{\hat{A} = \hat{A}^*} \;\Longrightarrow\; \boxed{\begin{array}{c}\text{real } \lambda, \text{ orthogonal eigenvectors} \\ (\text{+ usually "diagonalizable"})\end{array}}$$

also: $$\langle u, \hat{A} u\rangle = \underbrace{\cdots}_{\substack{\text{same steps,}\\ \text{stopping halfway}}} = \int_\Omega \Big[c\underbrace{|\nabla u|^2}_{\substack{>0 \text{ for}\\ u \neq \text{constant}\\ =0 \text{ by Dirichlet}}} + p\underbrace{|u|^2}_{\substack{>0\\ \text{for } u\neq 0}} \Big]$$

$$> 0 \text{ for } u \neq 0$$

$$\text{if } c > 0,\; p \geq 0$$

$$\Longrightarrow\; \boxed{\begin{array}{c}\hat{A} \quad (\text{"elliptic"}) \\ \text{positive-definite} \\ \text{for } c>0,\, p\geq 0\end{array}} \;\Longrightarrow\; \boxed{\lambda > 0,\; N(\hat{A}) = \{0\}}$$

examples :

heat / diffusion : $\underbrace{\frac{1}{w}}\nabla\cdot(\underbrace{c}\nabla \underbrace{u}) = \frac{\partial u}{\partial t} \quad (+ \underbrace{f(\vec{x},t)})$

$w \sim$ heat capacity > 0 | c: thermal conductivity > 0 | u: temperature | f: sources / sinks

$=$ $\boxed{\text{"parabolic" equation}}$: $\hat{A} u = \frac{\partial u}{\partial t}$ $\quad \hat{A}$ negative-definite

$\Longrightarrow$ exponentially decaying solutions

(smoothing: faster spatial oscillations decay faster)

$$\left(u(\vec{x},t) = \sum_{n=1}^{\infty} \langle u_n, u\rangle\big|_{t=0}\, u_n(\vec{x})\, e^{\lambda_n t} \qquad \lambda_n < 0 \right.$$

$$\left. \text{normalization} \quad \langle u_n, u_m\rangle = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases} \right)$$

Poisson : $\underbrace{\frac{1}{w}}_{>0}\nabla\cdot(\underbrace{c}_{>0 \text{ or } <0}\nabla u) = f$ $\quad$ (eg. steady-state heat eq. $\frac{\partial u}{\partial t} = 0$)

$=$ $\boxed{\text{"elliptic equation"}}$: $\hat{A} u = f$ $\quad \hat{A}$ definite (pos. or neg.)

$N(\hat{A}) = \{0\} \Longrightarrow$ unique solution

(if any : existence if f in span of eigenfunctions)

$$\left(u(\vec{x}) = \sum_{n=1}^{\infty} \frac{\langle u_n, f\rangle\, u_n(\vec{x})}{\lambda_n} \right)$$

note "smoothing" property :

- u is smoother than f since large λ = fast spatial oscillations are suppressed

scalar wave equation: (e.g. pressure waves)

$$\underbrace{\frac{1}{w}}_{\sim \frac{1}{\text{density}} > 0} \nabla \cdot (\underbrace{c}_{\substack{\sim \text{springiness} \\ > 0}} \nabla u) = \frac{\partial^2 u}{\partial t^2} \quad (+ \underbrace{f(\vec{x},t)}_{\substack{\text{external} \\ \text{force}}})$$

$= \boxed{\text{"hyperbolic equation"}}$: $\hat{A} u = \frac{\partial^2 u}{\partial t^2}$ $\quad \hat{A}$ negative definite (or maybe semidefinite)

$\Longrightarrow$ oscillating solutions:

$$\hat{A} u_n = \underbrace{\lambda_n}_{<0} u_n = -\omega_n^2 u_n$$

$$(\omega_n = \sqrt{-\lambda_n})$$

choose $\langle u_n, u_m \rangle = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

$$\Longrightarrow u(\vec{x},t) = \sum_{n=1}^{\infty} \left[\langle u_n, u|_{t=0} \rangle \cos(\omega_n t) + \frac{\langle u_n, \dot{u}|_{t=0} \rangle}{\omega_n} \sin(\omega_n t) \right] u_n(\vec{x})$$

"normal modes"

Laplace's equation:

$$\underbrace{\frac{1}{w}}_{>0} \nabla \cdot (\underbrace{c}_{>0} \nabla u) = 0$$

: e.g. heat equation for $\frac{\partial u}{\partial t} = 0$ or wave equation for $\frac{\partial^2 u}{\partial t^2} = 0$ and no external force or sources: $f = 0$

$\Longrightarrow u = 0$ (boring!) for $u|_{\partial\Omega} = 0$

... more interesting if $u|_{\partial\Omega} \neq 0$ ...