18.303 Problem Set 5 Solutions

Problem 1: (10+10+10+10)

See also the IJulia notebook posted with the solutions.

(a) Setting the slopes to be zero at R_1 and R_2 simply gives

$$\alpha J_m'(kr) + \beta Y_m'(kr) = 0$$

at the two radii, or $E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ where

$$\boxed{E = \begin{pmatrix} J_m'(kR_1) & Y_m'(kR_1) \\ J_m'(kR_2) & Y_m'(kR_2) \end{pmatrix}}.$$

Hence, writing $f_m(k) = \det E$, we get

$$\boxed{f_m(k) = J_m'(kR_1)Y_m'(kR_2) - J_m'(kR_2)Y_m'(kR_1)}.$$

Given a k for which $f_m(k) = 0$, then we can solve for the nullspace of E by arbitrarily choosing a scaling such that $\boxed{\alpha = 1}$ and solving for β from the first *or* second rows of $E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$:

$$\boxed{\beta = -\frac{J_m'(kR_1)}{Y_m'(kR_1)} = -\frac{J_m'(kR_2)}{Y_m'(kR_2)}}.$$

(b) Note that $f_m(k)$ for $m > 0$ has a divergence as $k \to 0$, so we used the `ylim` command to rescale the vertical axis (otherwise it would be hard to read the plot!); see the solution IJulia notebook

(c) We'll use the Scilab `newton` function, similar to class, to find the roots, with initial guesses provided by our plot in the notebook. We find $k_1 \approx 3.196578$, $k_2 \approx 6.31234951$, and $k_3 \approx 9.4444649$. See the solutions notebook.

(d) See the IJulia notebook. Using our k_1 and k_2 from part (c) and our α and β from part (a), we find that $\int_{R_1}^{R_2} r u_{0,1}(r) u_{0,2}(r) dr \approx 10^{-15}$, which is zero up to roundoff errors.