18.303 Problem Set

Due Wednesday, 10 November 2010.

Problem 1: Stability

(a) For the 1d diffusion equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ on $x \in [-\infty, \infty]$, use Von Neumann analysis to analyze the stability of the following discretization:

$$\frac{u_m^{n+1} - \frac{1}{2}(u_{m+1}^n + u_{m-1}^n)}{\Delta t} = \frac{u_{m+1}^n - 2u_m^{n+1} + u_{m-1}^n}{\Delta x^2}.$$

That is, assume that the solution on the n-th timestep is $\mathbf{u}^n = g^n \mathbf{u}$ for an eigenvalue ("growth factor") g and eigenvector $\mathbf{u}$, and solve for g under the assumption that $u_m = e^{ikx}$. Then find conditions on Δt (if any) such that it is stable (no exponential growth, i.e. $|g| \leq 1$). What is the order of accuracy, in time and space, of this discretization? If you get $|g| = 1$ anywhere, what does this mean about the solutions (especially as Δx and $\Delta t \to 0$)?

(b) Suppose we are discretizing a wave equation $\frac{\partial \mathbf{w}}{\partial t} = \hat{D}\mathbf{w}$ for some $\hat{D}^* = -\hat{D}$. First, we discretize $\hat{D}$ in space, replacing it by some matrix D to obtain an ODE $\frac{d\mathbf{u}}{dt} = D\mathbf{u}$; assume D is real and $D^T = -D$ (i.e, we have chosen the discretization to mimic the properties of $\hat{D}$), and thus D has purely imaginary eigenvalues $\lambda = i\omega$. Now consider the Crank–Nicolson discretization:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = D\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}.$$

(As shown in class, this is 2nd-order accurate in time and space.) Show that this is unconditionally stable (different from the proof in class: in class, we did it for a matrix A that was negative-definite). Do the solutions of this discretized equation grow, decay, or...? How does this compare to the exact equation $\frac{\partial \mathbf{w}}{\partial t} = \hat{D}\mathbf{w}$?

Problem 2: Waves, boundary conditions, and conservation laws

In class (and notes), we showed that we can turn the scalar wave equation $b\nabla \cdot (a\nabla u) = \frac{\partial^2 u}{\partial t^2}$ ($a > 0$ and $b > 0$) into two coupled first-derivative equations: $\frac{\partial u}{\partial t} = b\nabla \cdot \mathbf{v}$, $\frac{\partial \mathbf{v}}{\partial t} = a\nabla u$ by introducing a new (vector) unknown $\mathbf{v}(\mathbf{x}, t)$. By defining $\mathbf{w} = (u, \mathbf{v})^T$, we obtained the form

$$\frac{\partial \mathbf{w}}{\partial t} = \begin{pmatrix} & b\nabla\cdot \\ a\nabla & \end{pmatrix} \mathbf{w} = \hat{D}\mathbf{w},$$

where $\hat{D}$ was anti-Hermitian under the inner product $\langle \mathbf{w}, \mathbf{w}' \rangle = \int_\Omega \left(\frac{1}{b}\bar{u}u' + \frac{1}{a}\bar{\mathbf{v}} \cdot \mathbf{v}'\right)$, for appropriate boundary conditions (e.g. $u|_{d\Omega} = 0$).

(a) Suppose $u|_{d\Omega} = 0$. What boundary condition does this imply for $\mathbf{v}$?

(b) Suppose $\mathbf{v} \cdot \hat{n}|_{d\Omega} = 0$, where $\hat{n}$ is a unit vector perpendicular to the $d\Omega$ boundary. What boundary conditions does this imply for u?

(c) In class, we showed that $\hat{D}^* = -\hat{D}$ implies "conservation of energy:" $\langle \mathbf{w}, \mathbf{w} \rangle$ must be constant in time if $\frac{\partial \mathbf{w}}{\partial t} = \hat{D}\mathbf{w}$. Here, you will give a different conservation law: just as we saw in class for the in the diffusion equation, for any function $\mathbf{n}(\mathbf{x})$ in the left nullspace $\hat{D}^*\mathbf{n} = 0 = -\hat{D}\mathbf{n}$, show that $\langle \mathbf{n}, \mathbf{w} \rangle$ is constant in time if $\frac{\partial \mathbf{w}}{\partial t} = \hat{D}\mathbf{w}$. Describe the left nullspace of the operator $\hat{D}$ above, for the boundary conditions in part (b), and give the resulting conservation law(s). [Tip 1: the nullspace of $\nabla\cdot$ in 3d is the curl of any vector field. Tip 2: for any vector fields $\mathbf{F}$ and $\mathbf{G}$, $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$, which allows you to "integrate by parts" on $\mathbf{F} \cdot (\nabla \times \mathbf{G})$.]