18.303 Problem Set 4

Due Wednesday, 2 October 2013.

Problem 1: Separability and symmetry

Often, separability of the solutions is a consequence of symmetry. In this problem, you will show a related property for the case of discrete translational symmetry: a PDE that is invariant under rotation by $2\pi/N$. In particular, suppose that we have the circular system of N springs and masses, with identical spring constants k, depicted in Figure 1. Suppose that the equation of motion of the n-th mass is

$$m\ddot{\phi}_n = \kappa(\phi_{n+1} - \phi_n) - \kappa(\phi_n - \phi_{n-1}).$$

(a) If we let $\mathbf{x} = (\phi_1, \phi_2, \cdots, \phi_N)$ be the N-component column vector of ϕ_m values, then we can write the equation of motion as $A\mathbf{x} = \ddot{\mathbf{x}}$. What is A?

(b) Your A should be real-symmetric. Is it positive/negative definite/semidefinite?

(c) Define the *rotation* operator R by

$$R \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \\ \phi_1 \end{pmatrix},$$

i.e. R rotates each mass to the position of the next one. It could also be called a *cyclic shift* operator.

(i) Write down R

(ii) Show that R is orthogonal (unitary): $R^{-1} = R^*(= R^T)$.

(iii) Show that R *commutes* with A: $RA = AR$, or equivalently (multiplying both sides by R^{-1} on the left), that $A = R^{-1}AR$. We then say that A is rotation-invariant (or shift-invariant).

(d) Suppose that we have an eigenvector $A\mathbf{x} = \lambda\mathbf{x}$, and suppose for simplicity that λ is nondegenerate (has multiplicity 1, i.e. there is only one linearly independent eigenfunction of this λ). Show that $\mathbf{x}$ must also be an eigenfunction of R. (Hint: use the commutation result from the previous part.)

(More generally, one can show that we can find "simultaneous" eigenvectors of any commuting matrices.)

(e) If $\mathbf{x}$ is an eigenvector of R, this means $R\mathbf{x} = \alpha\mathbf{x}$ eigenvalues α. Show that:

(i) Since R is unitary, show that we must have $|\alpha| = 1$. (Hint: somewhat similar to the proof that Hermitian matrices have real eigenvalues.) Hence we can write $\alpha = e^{ik}$ for some real k.

(ii) Choose $x_1 = 1$. Give a formula for all the other components of $\mathbf{x}$.

(iii) Since $R^N = I$ (rotating N times returns to the original vector), show that $\alpha = e^{ik}$ where k must have the form: ____________

(iv) Combining the previous two parts, plug your eigenvector back into $A\mathbf{x} = \lambda\mathbf{x}$ and give a formula for the eigenvalues λ of A.

(f) In the continuum limit, what is the form of the eigenfunctions $\phi(\theta)$ in terms of the angle θ?

This is a simplified case of a much deeper theory about how symmetry relates to the solutions of PDEs. More generally, especially for the case of degenerate (multiplicity > 1) eigenvalues and combinations of multiple symmetry operators (rotations, reflections, translations, ...), one uses the tool of *group theory* to describe the interactions of the symmetry operators and the tool of *group representation theory* to describe the consequences for the eigenfunctions and other solutions.

Problem 2: Bessel, Bessel, toil and mess...el

In class, we solved for the eigenfunctions of ∇^2 in two dimensions, in a cylindrical region $r \in [0, R]$, $\theta \in [0, 2\pi]$ using separation of variables, and obtained Bessel's equation and Bessel-function solutions. Although Bessel's equation has two solutions $J_m(kr)$ and $Y_m(kr)$ (the *Bessel functions*), the second solution (Y_m) blows up as $r \to 0$ and so for that problem we could only have $J_m(kr)$ solutions (although we still needed to solve a transcendental equation to obtain k).

In this problem, you will solve for the 2d eigenfunctions of ∇^2 in an **annular** region Ω that does *not contain the origin*, as depicted schematically in Fig. 2, between radii R_1 and R_2, so that you will need *both* the J_m and Y_m solutions. Exactly as in class, the separation of variables ansatz $u(r, \theta) = \rho(r)\tau(\theta)$ leads to functions $\tau(\theta)$ spanned by $\sin(m\theta)$ and $\cos(m\theta)$ for integers m, and functions $\rho(r)$ that satisfy Bessel's equation. Thus, the eigenfunctions are of the form:

$$u(r, \theta) = [\alpha J_m(kr) + \beta Y_m(kr)] \times [A \cos(m\theta) + B \sin(m\theta)]$$

for arbitrary constants A and B, for integers $m = 0, 1, 2, \ldots$, and for constants α, β, and k to be determined.

For fun, we will also **change the boundary conditions** somewhat. We will impose "Neumann" boundary condition $\frac{\partial u}{\partial r} = 0$ at R_1 and R_2. That is, for a function $u(r, \theta)$ in cylindrical coordinates, $u(R_1, \theta) = 0$ and $\frac{\partial u}{\partial r}|_{r=R_2} = 0$. The following *exact* identities for the derivatives of the Bessel functions will be helpful:

$$J_m'(x) = \frac{J_{m-1}(x) - J_{m+1}(x)}{2}, \quad Y_m'(x) = \frac{Y_{m-1}(x) - Y_{m+1}(x)}{2}$$

(a) Using the boundary conditions, write down two equations for α, β and k, of the form $E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ for some 2×2 matrix E. This only has a solution when $\det E = 0$, and from this fact obtain a single equation for k of the form $f_m(k) = 0$ for some function f_m that depends on m. This is a transcendental equation; you can't solve it by hand for k. In terms of k (which is still unknown), write down a possible expression for α and β, i.e. a basis for $N(E)$.

(b) Assuming $R_1 = 1$, $R_2 = 2$, plot your function $f_m(k)$ versus $k \in [0, 20]$ for $m = 0, 1, 2$. Note that Julia provides the Bessel functions built-in: $J_m(x)$ is `besselj(m,x)` and $Y_m(x)$ is `bessely(m,x)`. You can plot a function with the `plot` command. See the IJulia notebook posted on the course web page for lecture 8 for some examples of plotting and finding roots in Julia.

(c) For $m = 0$, find the first three (smallest $k > 0$) solutions k_1, k_2, and k_3 to $f_0(k) = 0$. Get a rough estimate first from your graph (zooming if necessary), and then get an accurate answer by calling the scipy.optimize.newton function as in pset 1, and also as illustrated in the lecture-8 IJulia notebook. (Note that there is *also* a $k = 0$ eigenfunction for $m = 0$, corresponding to the *constant* function: the nullspace of $\hat{A}$ with Neumann boundary conditions, as in class.)

(d) Because ∇^2 is self-adjoint under $\langle u, v \rangle = \int_\Omega \bar{u}v$ (we showed in class, in general Ω, that this is still true with these boundary conditions), we know that the eigenfunctions must be orthogonal. From class, this implies that the radial parts must also be orthogonal when integrated via

$\int r\,dr$. Check that your Bessel solutions for k_1 and k_2 are indeed orthogonal, by numerically integrating their product via the `quadgk` function in Julia as in pset 2 and as in the lecture-8 IJulia notebook.

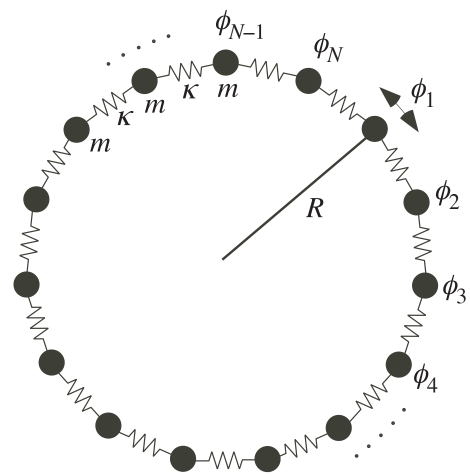

Figure 1: Circular systems of N identical masses m and springs κ. ϕ_n is the angular displacement of the n-th mass ($\phi_m = 0$ for all springs when they are at rest). Imagine that the springs can move in the ϕ direction, but cannot move in the radial direction (for example, if they are sliding without friction on the surface of a cylinder of radius R).

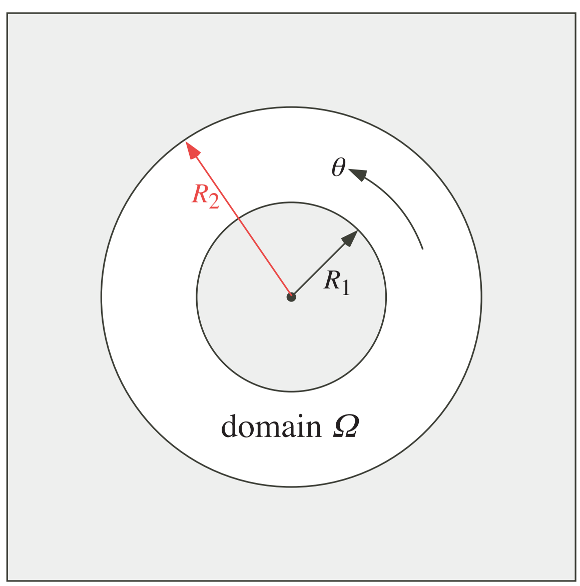

Figure 2: Schematic of the domain Ω for problem 3: an annular region in two dimensions, with radii $r \in [R_1, R_2]$ and angles $\theta \in [0, 2\pi]$.