

## 18.303 Problem Set 3

Due Wednesday, 28 September 2011.

### Problem 1: Discrete diffusion

In this problem, you will examine thermal conduction in a system of a finite number  $N$  of pieces, and then take the  $N \rightarrow \infty$  limit to recover the heat equation. In particular:

- You have a metal bar of length  $L$  and cross-sectional area  $a$  (hence a volume  $La$ ), with a varying temperature  $T$  along the rod. We conceptually subdivide the rod into  $N$  (touching) pieces of length  $\Delta x = L/N$ .
- If  $\Delta x$  is small, we can approximate each piece as having a uniform temperature  $T_n$  within the piece ( $n = 1, 2, \dots, N$ ), giving a vector  $\mathbf{T}$  of  $N$  temperatures.
- Suppose that the rate  $q$  (in units of W) at which heat flows across the boundary from piece  $n$  to piece  $n + 1$  is given by  $q = \frac{\kappa a}{\Delta x}(T_n - T_{n+1})$ , where  $\kappa$  is the metal's *thermal conductivity* (in units of W/m·K). That is, piece  $n$  *loses* energy at a rate  $q$ , and piece  $n + 1$  *gains* energy at the same rate, and the heat flows faster across bigger areas, over shorter distances, or for larger temperature differences. Note that  $q > 0$  if  $T_n > T_{n+1}$  and  $q < 0$  if  $T_n < T_{n+1}$ : heat flows from the hotter piece to the cooler piece.
- If an amount of heat  $\Delta Q$  (in J) flows into a piece, its temperature changes by  $\Delta T = \Delta Q / (c\rho a\Delta x)$ , where  $c$  is the specific heat capacity (in J/kg·K) and  $\rho$  is the density (kg/m<sup>3</sup>) of the metal.
- The rod is insulated: no heat flows out the sides or through the ends.

Given these assumptions, you should be able to answer the following:

- (a) “Newton’s law of cooling” says that the temperature of an object changes at a rate (K/s) proportional to the temperature difference with its surroundings. Derive the equivalent here: show that our assumptions above imply that  $\frac{dT_n}{dt} = \alpha(T_{n+1} - T_n) + \alpha(T_{n-1} - T_n)$  for some constant  $\alpha$ , for  $1 < n < N$ . Also give the (slightly different) equations for  $n = 1$  and  $n = N$ .
- (b) Write your equation from the previous part in matrix form:  $\frac{d\mathbf{T}}{dt} = A\mathbf{T}$  for some matrix  $A$ .
- (c) Let  $T(x, t)$  be the temperature along the rod, and suppose  $T_n(t) = T([n - 0.5]\Delta x, t)$  (the temperature at the *center* of the  $n$ -th piece). Take the limit  $N \rightarrow \infty$  (with  $L$  fixed, so that  $\Delta x = L/N \rightarrow 0$ ), and derive a partial differential equation  $\frac{\partial T}{\partial t} = \hat{A}T$ . What is  $\hat{A}$ ? (Don’t worry about the  $x = 0, L$  ends until the next part.)
- (d) What are the boundary conditions on  $T(x, t)$  at  $x = 0$  and  $L$ ? Check that if you go backwards, and form a center-difference approximation of  $\hat{A}$  with these boundary conditions, that you recover the matrix  $A$  from above.
- (e) Instead of temperature  $T$ , we can instead use the energy density  $u = c\rho T$  (units of J/m<sup>3</sup>). What is  $\hat{B}$  in the PDE  $\frac{\partial u}{\partial t} = \hat{B}u$  describing  $u(x, t)$ ?
- (f) Show that  $\hat{B}$  with these boundary conditions on  $u$  is self-adjoint under some appropriate inner product  $\langle u, v \rangle$ , and determine whether it is positive or negative definite or semidefinite. Hence the eigenvalues are \_\_\_\_\_.
- (g) Suppose  $u(x, 0) = f(x)$  for some function  $f(x)$ . Give a simple expression for  $u(x, \infty)$ . [Your final answer should contain no infinite series and no unknown functions other than  $f(x)$ ! But you can write  $u(x, t)$  down as an infinite series first and then take the limit to see what falls out.]

## Problem 2: Give the curl a whirl

Consider the space of three-component vector fields  $\mathbf{u}(\mathbf{x})$  on some finite-volume 3d domain  $\Omega \subset \mathbb{R}^3$ . One linear operator on these fields is the curl  $\nabla \times$ , which is important in electromagnetism (which we will study in more detail later in 18.303). Define the inner product of two vector fields  $\mathbf{u}$  and  $\mathbf{v}$  by the volume integral  $\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \bar{\mathbf{u}} \cdot \mathbf{v}$ .

- (a) An 18.02 exercise: derive the identity  $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$ .
- (b) Figure out how to do integration by parts with the curl: show that  $\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle + \oint_{\partial\Omega} \mathbf{w} \cdot d\mathbf{S}$ , where  $d\mathbf{S}$  is the usual outward surface-normal area element, and the  $\mathbf{w}$  appearing in the surface integral over the boundary ( $\partial\Omega$ ) is some vector field to be determined. (*Hint*: use the identity you derived in the previous part, combined with the divergence theorem.)
- (c) Give a possible boundary condition on our space of vector fields such that  $\nabla \times$  is self-adjoint with this inner product. (Boundary conditions can only involve one vector field at a time! No fair giving an equation that relates  $\mathbf{u}$  to  $\mathbf{v}$ !) You should *not* have to specify *all* the components of  $\mathbf{u}$  on the boundary. It may be convenient to define a vector field  $\mathbf{n}(\mathbf{x})$  on  $\partial\Omega$  to denote the outward normal vector at each point on the boundary.<sup>1</sup>
- (d) Show that  $\nabla \times \nabla \times$  is self-adjoint for this inner product under *either* some boundary condition on  $\mathbf{u}$  (similar to above) or some boundary condition on *derivatives* of  $\mathbf{u}$ . Is it positive or negative definite or semidefinite?
- (e) Two of Maxwell's equations in vacuum are  $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$  and  $\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$  where  $c$  is the speed of light. Take the curl of both sides of the second equation to obtain a PDE in  $\mathbf{E}$  alone. Suppose that  $\Omega$  is the interior of a hollow metal container, where the boundary conditions are that  $\mathbf{E}$  is perpendicular to the metal at the surface (i.e.  $\mathbf{E} \times \mathbf{n}|_{\partial\Omega} = 0$ ).<sup>2</sup> Combining these facts with the previous parts, explain why you would expect to obtain *oscillating* solutions to Maxwell's equations (*standing electromagnetic waves*, essentially light bouncing around inside the container).

(This kind of system exists, for example, in the microwave regime where metals have high conductivity, and such containers are called *microwave resonant cavities*.)

---

<sup>1</sup>Technically, we are assuming here that the boundary is a differentiable surface so that the normal vector is well defined. Actually, it is sufficient to assume that it is differentiable except for isolated corners and edges (a "set of measure zero") since those isolated kinks won't contribute anything to the  $\oint_{\partial\Omega}$  surface integral.

<sup>2</sup>In a perfect conductor, any nonzero component of  $\mathbf{E}$  parallel to the surface would generate an infinite current parallel to the surface, in which charges instantaneously rearrange to cancel the field. For a real conductor with finite conductivity, matters are more complicated because we must consider what the fields do inside the conductor, but a perfect conductor is a good approximation at microwave and lower frequencies where the penetration of the field into the conductor (the *skin depth*) is much smaller than the wavelength of light.