18.303 Problem Set 6

Due Wednesday, 30 October 2013.

Problem 1: Distributions

Let $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$, and $f(x) = 0$ for $x \le 0$.

(a) Explain why f defines a regular distribution, even though $f(x)$ blows up as $x \to 0^+$.

(b) Let $g(x) = -\frac{1}{2}\frac{1}{x^{3/2}}$ for $x > 0$, and $g(x) = 0$ for $x \le 0$: $g(x)$ matches the ordinary derivative $f'(x)$ everywhere $f'(x)$ is defined (i.e. everywhere but $x = 0$). Explain why $g(x)$ does *not* correspond to any regular distribution.

(c) Viewed as a distibution, f must have a derivative. Give an explicit formula for $f'\{\phi\}$ in terms of an integral of $\phi(x) - \phi(0)$ (*not* ϕ'). Hint: $f\{\phi\} = \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\phi(x)}{\sqrt{x}}dx$ (why does this limit exist?), and integrate by parts using $\phi'(x) = \frac{d}{dx}[\phi(x) - \phi(0)]$. How is this different from trying to define a distribution directly from $g(x)$?

(d) Give a similar formula for $f''\{\phi\}$ in terms of $\phi(x) - \cdots$ (no ϕ' or ϕ''), and compare to the 18.01 $f''(x)$ (which exists for $x \neq 0$ only).

Problem 2: Green's function of a 1d Helmholtz equation

Recall that the displacement $u(x, t)$ of a stretched string [with fixed ends: $u(0, t) = u(L, t) = 0$] satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} + f(x, t) = \frac{\partial^2 u}{\partial t^2}$, where $f(x, t)$ is an external force density (pressure) on the string.

(a) Suppose that $f(x, t) = \mathrm{Re}[g(x)e^{-i\omega t}]$, an oscillating force with a frequency ω. Show that, instead of solving the wave equation with this $f(x, t)$, we can instead use a complex force $\tilde{f}(x, t) = g(x)e^{-i\omega t}$, solve for a complex $\tilde{u}(x, t)$, and then take $u = \mathrm{Re}\,\tilde{u}$ to obtain the solution for the original $f(x, t)$.

(b) Suppose that $f(x, t) = g(x)e^{-i\omega t}$, and we want to find a *steady-state* solution $u(x, t) = v(x)e^{-i\omega t}$ that is oscillating everywhere at the same frequency as the input force. (This will be the solution after a long time if there is any dissipation in the system to allow the initial transients to die away.) Write an equation $\hat{A}v = g$ that v solves. Is $\hat{A}$ self-adjoint? Positive/negative definite/semidefinite?

(Your resulting equation is called a *Helmholtz* equation.)

(c) Solve for the Green's function $G(x, x')$ of this $\hat{A}$, assuming that $\omega \neq n\pi/L$ for any integer n (i.e. assume ω is not an eigenfrequency [why?]). [Write down the continuity conditions that G must satisfy at $x = x'$, solve for $x \neq x'$, and then use the continuity conditions to eliminate unknowns.]

(d) Form a finite-difference approximation A of your $\hat{A}$ (code from previous psets and lectures will be helpful here). Compute an approximate $G(x, x')$ in Julia by `A \ dk`, where $\mathbf{d}_k$ is the unit vector of all 0's except for one $1/\Delta x$ at index $k = x'/\Delta x$ [in Julia: `dk=zeros(N); dk[k] = 1/dx`], and compare (by plotting both) to your analytical solution from the previous part for a couple values of x' and a couple of different frequencies ω (one $< \pi/L$ and one $> \pi/L$) with $L = 1$.

(e) Show the limit $\omega \to 0$ of your G relates in some expected way to the Green's function of $-\frac{d^2}{dx^2}$ from class.