18.303 Problem Set 3 Solutions

Problem 1: (5+5+10+10)

(a) $\frac{d}{dt}u_m\Delta x$ is the net rate at which mass diffuses into piece m minuse the rate at which it dissociates:

$$\frac{du_m}{dt}\Delta x = D\frac{u_{m+1}-u_m}{\Delta x} + D\frac{u_{m-1}-u_m}{\Delta x} - Ru_m\Delta x$$

which can be rearranged into

$$\frac{du_m}{dt} = D\frac{u_{m+1}-2u_m+u_{m-1}}{\Delta x^2} - Ru_m.$$

In the $\Delta x \to 0$ limit, the first term is exactly the second derivative, from class, while the second term doesn't change, so we obtain the following PDE for $u(x,t)$:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - Ru$$

(b) This corresponds to $u_0 = u_{M+1} = 0$, or Dirichlet boundary conditions $u(0) = u(L) = 0$.

(c) If there is no flow through the ends, this corresponds to setting

$$D\frac{u_{M+1}-u_M}{\Delta x} = 0 = D\frac{u_0-u_1}{\Delta x},$$

or equivalently $u_{M+1} = u_M$ and $u_0 = u_1$. This makes the u_1 and u_M equations

$$\frac{du_1}{dt} = D\frac{u_2-2u_1+u_1}{\Delta x^2} - Ru_1,$$

$$\frac{du_M}{dt} = D\frac{u_M-2u_M+u_{M-1}}{\Delta x^2} - Ru_M.$$

If we write this in matrix form $d\mathbf{u}/dt = A\mathbf{u}$, we get

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix} - RI$$

where I is the identity matrix, and the first matrix is the same as our second-derivative matrix from class except for the first and last rows. In the continuum limit, setting $\frac{u_{m+1}-u_m}{\Delta x} = 0$ corresponds to setting the first derivative to zero at the ends, i.e. we have the (Neumann) boundary conditions $\frac{\partial u}{\partial x} = 0$ at $x = 0, L$.

(d) The operator in both cases is $\hat{A} = \frac{\partial^2}{\partial x^2} - R$. We will use the usual inner product $\langle u, v\rangle = \int_0^L \bar{u}v$. The R term is just a real number and hence is Hermitian ($\langle u, Rv\rangle = \langle Ru, v\rangle$ by inspection). The $\frac{\partial^2}{\partial x^2}$ term was already shown in class to be Hermitian for Dirichlet boundary conditions. For Neumann boundary conditions, we integrate by parts

$$\langle u, v''\rangle = \int_0^L \bar{u}v'' = \bar{u}\cancel{v'}\Big|_0^L - \int_0^L \overline{u'}v' = -\cancel{\overline{u'}}v\Big|_0^L + \int_0^L \overline{u''}v = \langle u'', v\rangle,$$

in which the boundary terms again vanish so we obtain that $\hat{A} = \hat{A}^*$. To show definiteness, as in class we just integrate by parts once, which with either boundary condition gives

$$\langle u, \hat{A}u \rangle = -\int_0^L \left(|u'|^2 + R|u|^2 \right) dx \leq 0,$$

which is < 0 unless $u = 0$ almost everywhere, due to the R term alone (since $R > 0$ was given). Hence it is negative-definite. If we set $R = 0$, then with Dirichlet boundary conditions it is still negative-definite as proved in class. For $R = 0$ and Neumann boundary conditions, however, $\langle u, \hat{A}u \rangle = 0 \implies u' = 0 \implies u = \text{constant} \cancel{\implies u = 0}$: nonzero constants are allowed by the boundary conditions, so, e.g. $\langle 1, \hat{A}1 \rangle = 0$ is allowed. Hence $\hat{A}$ is negative semidefinite for Neumann boundaries.

Problem 2: (5+5+5+10+5)

(a) Writing out the derivation in 18.02 fashion, this is tedious but straightforward:

$$\begin{aligned}
\nabla \cdot (\mathbf{u} \times \mathbf{v}) &= \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix} \\
&= \frac{\partial(u_y v_z - u_z v_y)}{\partial x} + \frac{\partial(u_z v_x - u_x v_z)}{\partial y} + \frac{\partial(u_x v_y - u_y v_x)}{\partial z} \\
&= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) v_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \right) v_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) v_z \\
&\quad u_x \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right) + u_y \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right) + u_z \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \\
&= (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v}).
\end{aligned}$$

(A much more compact derivation is possible using Einstein notation and the Levi–Civitá tensor, but probably most of you haven't seen this notation.)

(b) Given the above identity, integration by parts is straightforwards:

$$\begin{aligned}
\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle &= \int_\Omega \bar{\mathbf{u}} \cdot (\nabla \times \mathbf{v}) = \int_\Omega \left[\nabla \cdot (\bar{\mathbf{u}} \times \mathbf{v}) + \overline{\nabla \times \mathbf{u}} \cdot \mathbf{v} \right] \\
&= \oiint_{\partial\Omega} (\bar{\mathbf{u}} \times \mathbf{v}) \cdot d\mathbf{S} + \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle,
\end{aligned}$$

applying the divergence theorem in the second line. So, the surface term $\oiint_{\partial\Omega} \mathbf{w} \cdot d\mathbf{S}$ is for $\boxed{\mathbf{w} = \bar{\mathbf{u}} \times \mathbf{v}}$.

(c) We must have $(\bar{\mathbf{u}} \times \mathbf{v}) \cdot d\mathbf{S} = 0$. Let $d\mathbf{S} = \mathbf{n}\, dS$, where $\mathbf{n}$ is the outward unit normal vector at each point on $\partial\Omega$. Then we must have $\bar{\mathbf{u}} \times \mathbf{v} \perp \mathbf{n}$, which is true if, for example, both $\mathbf{u}$ and $\mathbf{v}$ are parallel to $\mathbf{n}$ at the boundary. i.e. if $\boxed{\mathbf{u} \times \mathbf{n}|_{\partial\Omega} = 0}$. It is *not* necessary to require $\mathbf{u}|_{\partial\Omega} = 0$ on the boundary, and that answer will not be accepted as I specifically requested that you not constrain all the components of $\mathbf{u}$ on the boundary.

Another way to see this is to write $(\bar{\mathbf{u}} \times \mathbf{v}) \cdot d\mathbf{S} = \mathbf{n} \cdot (\bar{\mathbf{u}} \times \mathbf{v}) dS = \mathbf{v} \cdot (\mathbf{n} \times \bar{\mathbf{u}}) dS = \bar{\mathbf{u}} \cdot (\mathbf{v} \times \mathbf{n}) dS$ by elementary triple-product identities, and hence we again see that it is sufficient to have $\mathbf{u} \times \mathbf{n} = 0$ on the boundary.

Although I will accept the above answer, it is actually possible to contrive a slightly weaker condition: $\mathbf{u}$ and $\mathbf{v}$ can have components $\perp$ to $\mathbf{n}$ on the boundary, as long as those surface-parallel components are *in the same direction* for both $\mathbf{u}$ and $\mathbf{v}$ (to obtain zero cross product). That is, suppose $\mathbf{p}(\mathbf{x})$ is some surface-parallel ($\mathbf{p} \perp \mathbf{n}$) vector field on $\partial\Omega$.[1] Then it is sufficient for the surface-parallel component of $\mathbf{u}$ to be $\perp$ to $\mathbf{p}$ everywhere on the boundary, or equivalently $\mathbf{u} \cdot \mathbf{p}|_{\partial\Omega} = 0$.

[Actually, there are other possible conditions on $\mathbf{u}$ if we allow non-local boundary conditions, where $\mathbf{u}$ at one point on the boundary is related to $\mathbf{u}$ at another point. For example, if Ω is a cube and $\mathbf{u}$ is *periodic* (i.e. $\mathbf{u}$ on one face equals $\mathbf{u}$ on the opposite face), then the $\oiint_{\partial\Omega}$ vanishes because each face of the cube cancels the opposite face, without requiring any component of $\mathbf{u}$ to be zero.]

(d) We just "integrate by parts" twice:

$$\langle \mathbf{u}, \nabla\times\nabla\times\mathbf{v}\rangle = \int_\Omega \bar{\mathbf{u}}\cdot(\nabla\times\nabla\times\mathbf{v}) = \cancel{\oiint_{\partial\Omega}[\bar{\mathbf{u}}\times(\nabla\times\mathbf{v})]\cdot d\mathbf{S}} + \int_\Omega (\nabla\times\bar{\mathbf{u}})\cdot(\nabla\times\mathbf{v})$$
$$= \cancel{\oiint_{\partial\Omega}[(\nabla\times\bar{\mathbf{u}})\times\mathbf{v}]\cdot d\mathbf{S}} + \int_\Omega (\nabla\times\nabla\times\bar{\mathbf{u}})\cdot\mathbf{v} = \langle\nabla\times\nabla\times\mathbf{u}, \mathbf{v}\rangle,$$

where the surface terms cancel if *either* $\boxed{\mathbf{u}\times\mathbf{n}|_{\partial\Omega} = 0}$ or $\boxed{(\nabla\times\mathbf{u})\times\mathbf{n}|_{\partial\Omega} = 0}$, that is if either $\mathbf{u}$ or its curl are normal to the surface. (As in the previous part, one can actually weaken this slightly, but this is sufficient for our purposes.)

To check definiteness, carry integration by parts "halfway" through:

$$\langle \mathbf{u}, \nabla\times\nabla\times\mathbf{u}\rangle = \int_\Omega |\nabla\times\mathbf{u}|^2 \geq 0,$$

so it is **positive semidefinite**. It is *not* positive-definite since $\nabla\times\mathbf{u} = 0$ for $\mathbf{u} = \nabla\phi \neq 0$ for *any* non-constant scalar field ϕ , and we can easily choose such a ϕ such that $\nabla\phi$ satisfies the boundary conditions (e.g., choose ϕ so that it is constant in a neighborhood of $\partial\Omega$).

(e) Taking the curl of both sides of $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, we obtain $\nabla\times\nabla\times\mathbf{E} = -\nabla\times\frac{\partial\mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B} = -\frac{\partial^2\mathbf{E}}{\partial t^2}$. That is, we have

$$\frac{\partial^2\mathbf{E}}{\partial t^2} = \hat{A}\mathbf{E}$$

where $\hat{A} = -\nabla\times\nabla\times$. From the previous parts, for $\mathbf{E}\perp\mathbf{n}$ at the surface, this $\hat{A}$ is self-adjoint and negative semidefinite, and hence we have a **hyperbolic equation**.

As in class, we therefore expect orthogonal eigenfunctions and real $\lambda \leq 0$, and hence oscillating "normal mode" solutions with eigenfrequencies $\omega = \sqrt{-\lambda}$.

[Technically, we also obtain $\lambda = 0$ solutions which are non-oscillatory—from above, these are nullspace solutions $\mathbf{E} = \nabla\phi$ for some ϕ, which physically correspond to the time-independent fields of fix charge distributions, where $-\phi$ is the potential and $\nabla\cdot\mathbf{E} = \nabla^2\phi$ is the charge density. Everything else, all of the other eigenfunctions, are oscillating solutions.]

[1] By the "hairy ball theorem" of topology, $\mathbf{p}$ must vanish somewhere on $\partial\Omega$ if Ω is simply connected (no holes), at which point $\mathbf{u}$ must be parallel to $\mathbf{n}$.