Properties of the Fourier Series and Transform

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1 Questions of the Day

1. Is the tranformation of a function into a Fourier series linear?

2 Fourier Series

For simplicity we will assume that our functins are defined on $x \in [0, 1]$ and that $u(0) = u(1) = 0$. This means that $u(x)$ sufficiently nice has a Fourier series:

$$u(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

2.1 What are some properties of this series?

2.1.1 Orthogonal

Recall that the dot product is $(f, g) = \int_0^1 f(x)g(x)dx$. Let's take the dot product between basis elements:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x) = \int_0^1 \frac{[\cos((n-m)\pi x) - \cos(n+m)\pi x]}{2} dx$$

But $n - m \neq 0$ means that the integral is zero since cos is even. If 0, $\int_0^1 \frac{\cos(2\pi x)}{2} = \frac{1}{2}$.

2.1.2 Computing the coefficients

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} b_n \sin(n\pi x) \\
\int_0^1 u(x)\sin(m\pi x)\,dx &= \int_0^1 \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sin(m\pi x)\,dx \\
\int_0^1 u(x)\sin(m\pi x)\,dx &= b_n \int_0^1 \sin^2(m\pi x)\,dx \\
2\int_0^1 u(x)\sin(m\pi x)\,dx &= b_n
\end{aligned}
$$

since $\int_0^1 \sin^2(m\pi x)\,dx = 2$.

2.1.3 Spanning

For any function $f(x)$ where $\int |f(x)|^p\,dx < \infty$ for some $p > 1$, the Fourier series converges almost everywhere to $f(x)$. At jump discontinuities, the Fourier series converges to the midpoint.

Example: Fourier Series of 1

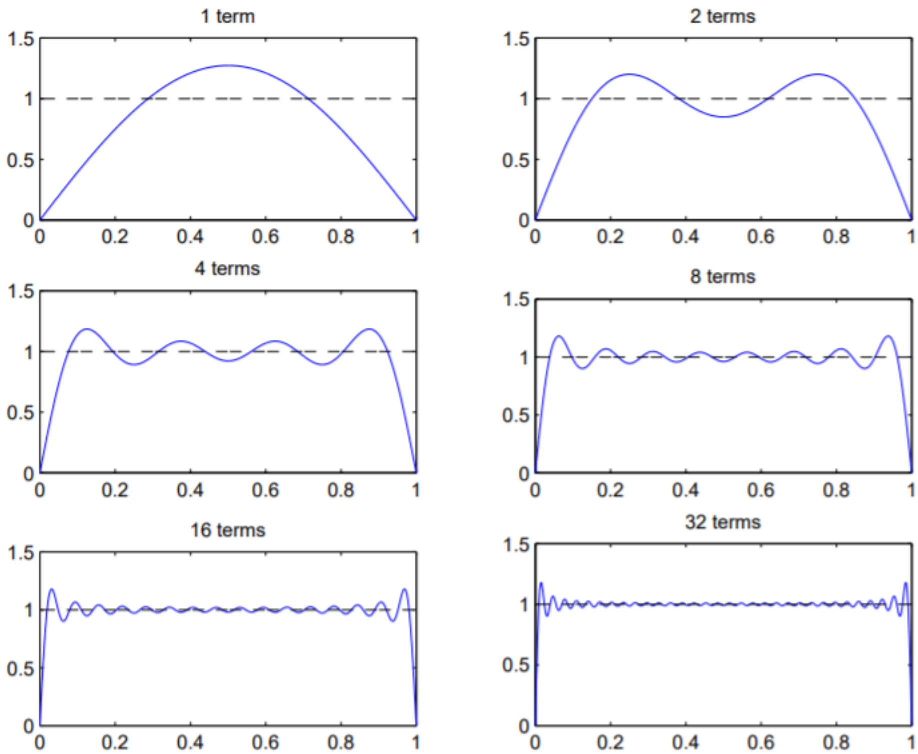

Figure 1: Fourier sine series for $f(x) = 1$, truncated to a finite number of terms (from 1 to 32), showing that the series indeed converges everywhere to $f(x)$, except exactly at the endpoints, as the number of terms is increased.

$$f(x) = 1 = \frac{4}{\pi}\sin(\pi x) + \frac{4}{3\pi}\sin(3\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \ldots$$

The bumps near the discontinuity are known as Gibb's phonomenon.

2.1.4 Example: Fourier Series of $f(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|$

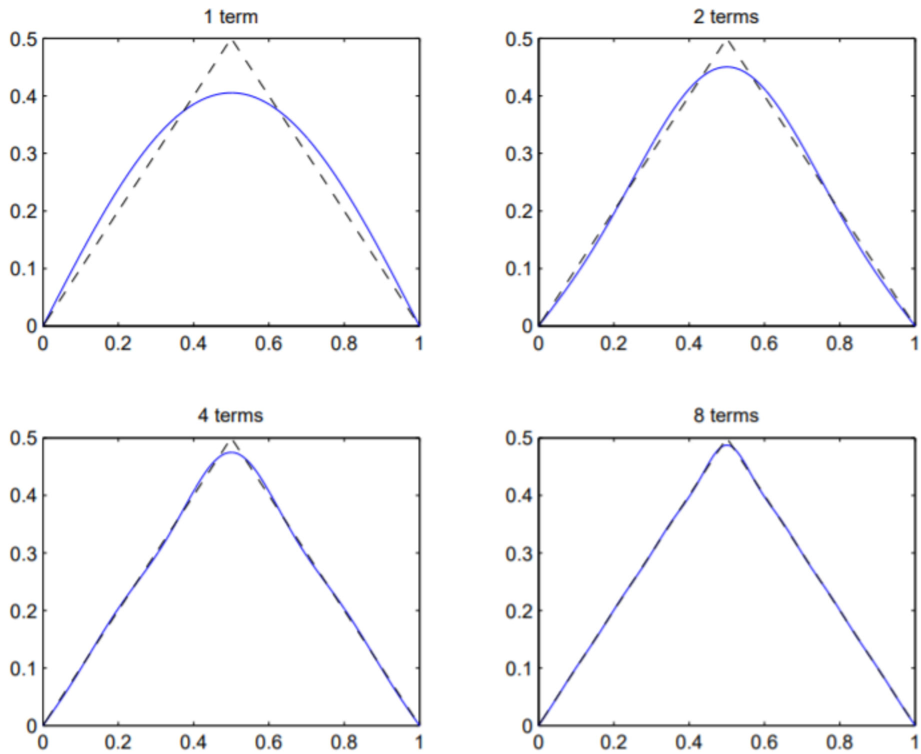

Figure 2: Fourier sine series (blue lines) for the triangle function $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$ (dashed black lines), truncated to a finite number of terms (from 1 to 32), showing that the series indeed converges everywhere to $f(x)$.

$$f(x) = \frac{4}{\pi^2}\sin(\pi x) - \frac{4}{(3\pi)^2}\sin(3\pi x) + \frac{4}{(5\pi)^2}\sin(5\pi x) + \dots$$

3 Fourier Transform

Let $\mathcal{F}(u)$ be the Fourier Transform as the function $\mathcal{F} : u \to [b_1, b_2, \dots]$.

3.1 What are some properties of the Fourier Transform?

3.1.1 Linearity

$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$ and $g(x) = \sum_{n=1}^{\infty} \tilde{b}_n \sin(n\pi x)$. But

$$\int_0^1 (\alpha f(x) + \beta g(x)) \sin(n\pi x)\, dx = \alpha \int_0^1 f(x) \sin(n\pi x)\, dx + \beta \int_0^1 g(x) \sin(n\pi x)\, dx$$

which means $\alpha f(x) + \beta g(x) = \sum_{n=1}^{\infty} \left(\alpha b_n + \beta \tilde{b}_n\right) \sin(n\pi x)$. This means that

$$\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g)$$

3.1.2 Inverse

Given an infinite vector $[b_1, b_2, \ldots]$, the inverse Fourier Transform $\mathcal{F}^{-1}$ is defined as $\mathcal{F}^{-1}([b_1, b_2, \ldots]) = \sum_n b_n \sin(n\pi x)$.

4 Solving the Poisson Equation

4.1 Representation of Δ in the Fourier Basis

$\Delta = \frac{d^2}{dx^2}$. The claim is that

$$\Delta = \begin{bmatrix} -(\pi)^2 & 0 & 0 \\ 0 & -(2\pi)^2 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} = D$$

is diagonal in the Fourier basis.

$$\frac{d^2}{dx^2} \sin(n\pi x) = -(n\pi)^2$$

and so

$$\begin{aligned} \frac{d^2}{dx^2} \sum_{n=1}^{\infty} b_n \sin(n\pi x) &= \sum_{n=1}^{\infty} b_n \frac{d^2}{dx^2} \sin(n\pi x) \\ &= -\sum_{n=1}^{\infty} (n\pi)^2 b_n \sin(n\pi x) \end{aligned}$$

and thus for $B = [b_1, b_2, \ldots]^T$, $\Delta B = \left[-\pi^2 b_1, -(2\pi)^2 b_2, \ldots\right]$ which we see is multiplication by that diagonal matrix.

4.2 Solution to the Poisson Equation

$$\Delta u = f$$

Now diagonalize Δ. Notice that it is diagonal in the Fourier basis, and so we write the diagonalization of $\Delta = \mathcal{F}^{-1} D \mathcal{F}$ and get

$$\begin{aligned} \mathcal{F}^{-1} D \mathcal{F} u &= f \\ u &= \mathcal{F}^{-1} D^{-1} \mathcal{F} f. \end{aligned}$$

4.3 Example Solution

Let $f(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|$. Then

$$f(x) = \frac{4}{\pi^2} \sin(\pi x) - \frac{4}{(3\pi)^2} \sin(3\pi x) + \frac{4}{(5\pi)^2} \sin(5\pi x) + \dots$$

So then

$$\Delta u = f$$

implies that

$$u(x) = -\frac{4}{\pi^4} \sin(\pi x) + \frac{1}{9\pi^4} \sin(3\pi x) + \frac{4}{15^2 \pi^4} \sin(5\pi x) + \dots$$

4.4 Solving the Heat Equation

Let's start using some notation.

$$u_t = \frac{du}{dt}.$$

$$u_t = \Delta u + f(t, x)$$

and let $u(0, x) = u_0(x)$ be given. We want to solve for $u(t, x)$. Let's diagonalize Δ:

$$u_t = \mathcal{F}^{-1} D \mathcal{F} u + f(t, x)$$

apply $\mathcal{F}$ to both sides:

$$(\mathcal{F} u)_t = \mathcal{F} u_t = D \mathcal{F} u + \mathcal{F} f(t, x)$$

Define $v = \mathcal{F} u$, then

$$v_t = D v + \mathcal{F} f(t, x)$$

with the initial condition $v_0(x) = \mathcal{F} u_0(x)$. Notice that this is just a system of ordinary differential equations. To see this more clearly, let's write out what this means. Now, at any given point in time u has a Fourier series, so let's write this as

$$u(t, x) = \sum_{n=1}^{\infty} b_n(t) \sin(n\pi x).$$

Then $u(0,x)=\sum_{n=1}^{\infty}b_n(0)\sin(n\pi x)$, so $b_n(0)$ is given. Let $\sum_{n=1}^{\infty}c_n(t)\sin(n\pi x)$. So now write out the equation:

$$\left(\sum_{n=1}^{\infty}b_n(t)\sin(n\pi x)\right)_t=\Delta\sum_{n=1}^{\infty}b_n(t)\sin(n\pi x)+\sum_{n=1}^{\infty}c_n(t)\sin(n\pi x)$$
$$\sum_{n=1}^{\infty}b_n'(t)\sin(n\pi x)=-\sum_{n=1}^{\infty}(n\pi)^2\,b_n(t)\sin(n\pi x)+\sum_{n=1}^{\infty}c_n(t)\sin(n\pi x)$$

thus by matching terms we see that

$$b_n'(t)=-(n\pi)^2\,b_n(t)+c_n(t)$$

is an infinite system of decoupled ODEs, where $b_n(0)$ is given. The solution to the Heat Equation is just given by the solution of these ODEs.

We can solve this ODE using the integrating factor method. Rewrite this as:

$$b_n'(t)+(n\pi)^2\,b_n(t)=c_n(t)$$

Now multiply both sides of the equation by $e^{(n\pi)^2t}$:

$$b_n'(t)e^{(n\pi)^2t}+(n\pi)^2\,b_n(t)e^{(n\pi)^2t}=c_n(t)e^{(n\pi)^2t}$$

Now check that:

$$\left(b_n(t)e^{(n\pi)^2t}\right)'=b_n'(t)e^{(n\pi)^2t}+(n\pi)^2\,b_n(t)e^{(n\pi)^2t}$$

so then

$$\left(b_n(t)e^{(n\pi)^2t}\right)'=c_n(t)e^{(n\pi)^2t}$$

which means that

$$b_n(t)e^{(n\pi)^2t}=\int_0^t c_n(t)e^{(n\pi)^2t}dt+C$$

and thus

$$b_n(t)=e^{-(n\pi)^2t}\int_0^t c_n(t)e^{(n\pi)^2t}dt+e^{-(n\pi)^2t}b_n(0)$$

is an equation for how the Fourier coefficients evolve over time. Why is $C=b_n(0)$? It has to be: plug in $t=0$.