18.303 Midterm, Fall 2011

Problem 1: (20 points)

In homework, you showed that $\nabla \times \nabla \times$ is self-adjoint and positive semidefinite for vector fields $\mathbf{u}(\mathbf{x})$ in 3d domains Ω where $\mathbf{u}$ is normal to the boundary $\partial\Omega$, with the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \bar{\mathbf{u}} \cdot \mathbf{v}$. The key step in the proof was that $\nabla\times$ is self-adjoint under this inner product, with appropriate boundary conditions: $\int_\Omega \overline{(\nabla \times \mathbf{u})} \cdot \mathbf{v} = \int_\Omega \bar{\mathbf{u}} \cdot (\nabla \times \mathbf{v}) +$ (~~surface terms~~).

Suppose that instead we generalize to an operator $\hat{A}$ given by:

$$\hat{A}\mathbf{u} = C(\mathbf{x}) \nabla \times [B(\mathbf{x}) \nabla \times \mathbf{u}],$$

where C and B are 3×3 *matrices* that may vary with position. Under **what conditions** on C and B, and for **what inner product**, is $\hat{A}$ self-adjoint and positive semidefinite? *(Don't worry about the surface terms from integration by parts; assume that boundary conditions are chosen that make these surface terms vanish.)* Be sure to define a **valid inner product**!

Problem 2: (20 points)

Suppose we have a stretched drum with a position-varying "stretchiness" $c(\mathbf{x}) > 0$, with the displacement $u(\mathbf{x}) = u(x, y)$ satisfying

$$-\nabla \cdot (c \nabla u) = f(\mathbf{x})$$

where f is the force per unit area, where the edges of the drum are fixed: $u|_{\partial\Omega} = 0$.

(a) Suppose $f(\mathbf{x}) = F_0 \delta(\mathbf{x} - \mathbf{x}_0)$: we are pressing the drum at "one point" $\mathbf{x}_0$ with a force F_0. What happens to the solution u in the limit where the point $\mathbf{x}_0$ where we are pressing approaches the boundary $\partial\Omega$? Justify your answer. (Hint: reciprocity.)

(b) How does your answer change if we twist the edges of the drum into a hyperbolic paraboloid, so that $u|_{\partial\Omega} = x^2 - y^2$?

Problem 3: (20 points)

In class, we discretized $\hat{A} = \frac{d^2}{dx^2}$ on $[0, L]$ with Dirichlet boundary conditions $u(0) = u(L) = 0$ by $u_n \approx u(n\Delta x)$ as shown in figure 1(a), with the center-difference approximation $u_n'' \approx \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2}$. Setting $\Delta x = \frac{L}{N+1}$ and setting $u_0 = u_{N+1} = 0$, we obtained the matrix

$$A = \left(\frac{N+1}{L}\right)^2 \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}.$$

Instead, suppose that we discretize it as shown in figure 1(b). Again we will use N points with spacing Δx, but in this case $u_n \approx u([n - 0.5]\Delta x)$. In this way, the first point u_1 and the last point u_N are now $\Delta x/2$ from the boundaries $x = 0$ and $x = L$, which lie "halfway between" two grid points.

(a) Using this discretization and the same center-difference approximation (and the same Dirichlet boundary conditions), what is the new (2nd-order accurate) matrix A? (One approach: recall the connection of Dirichlet and Neumann boundary conditions to symmetric/antisymmetric, i.e. even/odd, boundary conditions.)

(b) Write the new A as $-D^T D$ for some D.

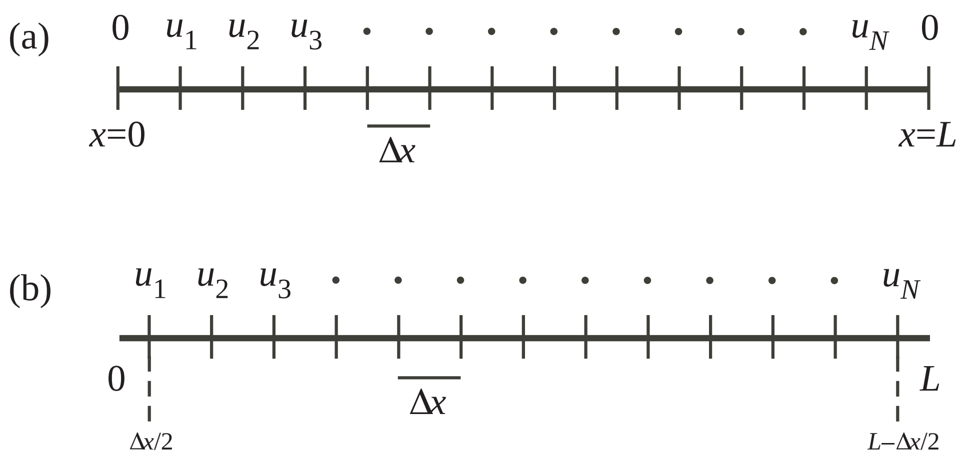

Figure 1: Two possible grids for discretizing $[0, L]$ with Dirichlet boundary conditions via N points with spacing Δx. (a) the grid from class, where the boundaries are "on grid points," located "at" u_0 and u_{N+1}. (b) the grid from problem 3, where the boundaries are "halfway between grid points," located "at" $u_{-0.5}$ and $u_{N+0.5}$.