18.303 Midterm Exam, Fall 2015

November 5, 2015

Problem 1: Adjoint (33 points)

Consider the operator $\hat{A} = -c\nabla^2$ on a d-dimensional domain Ω, with boundary $\partial\Omega$, for a real function $c(\mathbf{x}) > 0$. Let $\hat{\mathbf{n}}(\mathbf{x})$ denote the outward normal vector on $\partial\Omega$. Suppose we have the "Robin" boundary condition

$$\nabla u \cdot \hat{n}|_{\partial\Omega} = \alpha u|_{\partial\Omega}$$

where $\alpha(\mathbf{x})$ is some real-valued function. **Show** that $\hat{A}$ is Hermitian under an appropriate choice of inner product $\langle u, v\rangle$.

Hint: recall from class how to do integration by parts in d dimensions: $\int_\Omega \mathbf{f}\cdot\nabla g = \oint_{\partial\Omega} \mathbf{f} g \cdot \hat{\mathbf{n}} da - \int_\Omega g \nabla \cdot \mathbf{f}$ for any differentiable scalar function $g(\mathbf{x})$ and vector field $\mathbf{f}(\mathbf{x})$.

Problem 2: Finite differences (34 points)

Consider the 1d constant-coefficient version of the operator from the previous problem: $\hat{A} = -\frac{\partial^2}{\partial x^2}$ on the domain $\Omega = [0, L]$, with boundary conditions

$$\begin{aligned} u(0) &= 0, \\ u'(L) &= \alpha u(L) \end{aligned}$$

for some real number α. (From the previous problem, $\hat{A}$ is Hermitian.)

Give a **finite-difference** discretization of $\hat{A}u \approx A\mathbf{u}$ for an $M \times M$ matrix A and a vector $\mathbf{u}$ of components $u_m \approx u(m\Delta x)$. Write down the matrix A. Some things to be careful about:

- Make sure your discretization is second-order accurate, i.e. errors $O(\Delta x^2)$.
- Be explicit about *where* u_M is, i.e. how does Δx relate to L and M? (Hint: $M\Delta x \neq L$). Draw a picture of the grid and label $0, 1, 2, \ldots, M-1, M, M+1$ along with $x = 0$ and $x = L$.
- The $m = 0$ boundary condition is easy. Be careful on the other side: how does u_{M+1} relate to u_M?
- Be sure that your matrix A is self-adjoint (under some inner product).

Problem 3: Green (33 points)

In class, we saw that the Green's function for $\hat{A} = -\nabla^2$ in 3d for $\Omega = \mathbb{R}^3$ is $G_0(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$, such that $\hat{A}u_0 = f$ is solved by $u_0(\mathbf{x}) = \hat{A}^{-1}f = \int G_0(\mathbf{x}, \mathbf{x}')\, f(\mathbf{x}')\, d^3\mathbf{x}'$.

Now, suppose that we want to solve the *nonlinear* equation

$$\hat{A}u + \alpha u^2 = f$$

for some small $|\alpha(\mathbf{x})| \ll 1$.

1. Suppose that we want to solve this *approximately* to *first order* in α, dropping terms of $O(\alpha^2)$ etc. **Write down this approximate solution** in terms of **integrals** of G_0, u_0, f, and α. (Hint: Born approximation.)
2. Improve your approximation by adding the *second-order* term in α, i.e. dropping terms of $O(\alpha^3)$ etc. (Hint: write $u = u_0 + u_1 + u_2 + \ldots$, where u_n is all the terms proportional to α^n.)