Problem Set 1

February 8, 2019

1 Linear Algebra Warm Up

1. Let $(x, y) = x^T y$ be the real dot product in $\mathbb{R}^n$. A matrix A is real-symmetric if and only if $(Ax, y) = (x, Ay)$ for all $x, y \in \mathbb{R}^n$. Show that if a matrix A is real-symmetric and invertible, then A^{-1} is real-symmetric too.

2. Consider the eigenvalue problem $B^{-1}Ax = \lambda x$ where A and B are both real-symmetric matrices and B is positive-definitie, i.e. $(Bx, x) > 0$ for any $x \neq 0$. Define the modified dot product $(x, y)_B = x^T By$.

   (a) Show that $(\cdot, \cdot)_B$ is an inner product.

   (b) Show that $C = B^{-1}A$ is symmetric with respect to this dot product.

2 Quasi-Periodic Boundary Conditions

In lecture we investigated the 1D Poisson equation $Au = -\frac{d^2}{dx^2}u(x) = f(x)$ with Dirichlet boundary conditions $u(0) = u(L) = 0$. In this case, we solved for the eigenfunctions of A to get the Fourier sine series. Now consider a variation of the problem. Suppose the boundary conditions are instead $u(0) = e^{i\phi}u(L)$ and $u'(0) = e^{i\phi}u'(L)$ for some $\phi \in \mathbb{R}$.

1. What are the eigenfunctions and eigenvalues of A when considering this set of functions?

2. For what values of ϕ will the Poisson equation with these boundary conditions not have unique solutions? Why?

3. Under what conditions (if any) on $f(x)$ and ϕ would a solution exist? (Assume that f has a convergent Fourier series.)

3 Spectral PDE Solvers

In this problem you will code a solver for the 1D Poisson equation with Dirichlet boundary conditions using the method Fourier sine series. This kind of PDE solver is known as a spectral method. Consider the 1D Poisson equation on the domain $[0, 1]$ Dirichlet boundary conditions $u(0) = u(1) = 0$.

1. Recall that when expanding a function $f(x) = \sum_{n=1}^{\infty} b_n \psi_n(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$, the coefficients can be computed as $b_m = 2 \int_0^1 f(x) \sin(m\pi x)\, dx$. Write a function which uses computational quadrature (QuadGK.jl) to transform a function $f(x)$ into an array $b \in \mathbb{R}^n$ of coefficients for the sine series. Test this on the function on the input $h(x) = x\,(x-1)(x-2)$. Plot $f(x)$ and the sine expansion approximation with $n = 1, 2$, and 5 in the same graph.

2. The Discrete Fast Fourier Transform (DFT) is an optimized way of computing Fourier series coefficients from a function. For an array $y \in \mathbb{R}^n$ of evenly-spaced samples of $f(x)$, it holds that $b = \frac{1}{N} DFT(y)$. Use FFTW.jl and its dft function to write a more optimized function that computes the sine series of a function $f(x)$.

3. Since the sine functions are the eigenfunctions of the Poisson equation, it holds that $A\psi_n = \lambda_n \psi_n$. Use this relation to write a function that computes the sine series coefficients of the PDE's solution given the sine series of the input f. Test your function on h.