

## 18.303 Problem Set 5

Due Monday, 17 October 2016.

### Problem 1: Bessels

In class, we solved for the eigenfunctions of  $\nabla^2$  in two dimensions, in a cylindrical region  $r \in [0, R]$ ,  $\theta \in [0, 2\pi]$  using separation of variables, and obtained Bessel's equation and Bessel-function solutions. Although Bessel's equation has two solutions  $J_m(kr)$  and  $Y_m(kr)$  (the *Bessel functions*), the second solution ( $Y_m$ ) blows up as  $r \rightarrow 0$  and so for that problem we could only have  $J_m(kr)$  solutions (although we still needed to solve a transcendental equation to obtain  $k$ ). See also the "IJulia Bessel-function notebook" from lecture 11.

In this problem, you will solve for the 2d eigenfunctions of  $\nabla^2$  in an **annular** region  $\Omega$  that does *not* contain the origin, as depicted schematically in Fig. 1, between radii  $R_1$  and  $R_2$ , so that you will need *both* the  $J_m$  and  $Y_m$  solutions. Exactly as in class, the separation of variables ansatz  $u(r, \theta) = \rho(r)\tau(\theta)$  leads to functions  $\tau(\theta)$  spanned by  $\sin(m\theta)$  and  $\cos(m\theta)$  for integers  $m$ , and functions  $\rho(r)$  that satisfy Bessel's equation. Thus, the eigenfunctions are of the form:

$$u(r, \theta) = [\alpha J_m(kr) + \beta Y_m(kr)] \times [A \cos(m\theta) + B \sin(m\theta)]$$

for arbitrary constants  $A$  and  $B$ , for integers  $m = 0, 1, 2, \dots$ , and for constants  $\alpha$ ,  $\beta$ , and  $k$  to be determined.

For fun, we will also **change the boundary conditions** somewhat. We will impose "Neumann" boundary condition  $\frac{\partial u}{\partial r} = 0$  at  $R_1$  and  $R_2$ . That is, for a function  $u(r, \theta)$  in cylindrical coordinates,  $\frac{\partial u}{\partial r}|_{r=R_1, R_2} = 0$ . The following *exact* identities for the derivatives of the Bessel functions will be helpful:

$$J'_m(x) = \frac{J_{m-1}(x) - J_{m+1}(x)}{2}, \quad Y'_m(x) = \frac{Y_{m-1}(x) - Y_{m+1}(x)}{2}$$

- (a) Using the boundary conditions, write down two equations for  $\alpha$ ,  $\beta$  and  $k$ , of the form  $E \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$  for some  $2 \times 2$  matrix  $E$ . This only has a solution when  $\det E = 0$ , and from this fact obtain a single equation for  $k$  of the form  $f_m(k) = 0$  for some function  $f_m$  that depends on  $m$ . This is a transcendental equation; you can't solve it by hand for  $k$ . In terms of  $k$  (which is still unknown), write down a possible expression for  $\alpha$  and  $\beta$ , i.e. a basis for  $N(E)$ .
- (b) Assuming  $R_1 = 1$ ,  $R_2 = 2$ , plot your function  $f_m(k)$  versus  $k \in [0, 20]$  for  $m = 0, 1, 2$ . Note that Julia provides the Bessel functions built-in:  $J_m(x)$  is `besselj(m, x)` and  $Y_m(x)$  is `bessely(m, x)`. You can plot a function with the `plot` command. See the IJulia notebook posted on the course web page for lecture 11 for some examples of plotting and finding roots in Julia.
- (c) For  $m = 0$ , find the first three (smallest  $k > 0$ ) solutions  $k_1$ ,  $k_2$ , and  $k_3$  to  $f_0(k) = 0$ . Get a rough estimate first from your graph (zooming if necessary), and then get an accurate answer by calling the `scipy.optimize.newton` function as illustrated in the lecture-11 IJulia notebook. (Note that there is *also* a  $k = 0$  eigenfunction for  $m = 0$ , corresponding to the *constant* function: the nullspace of  $\hat{A}$  with Neumann boundary conditions, as in class.)
- (d) Because  $\nabla^2$  is self-adjoint under  $\langle u, v \rangle = \int_{\Omega} \bar{u} v$  (we showed in class, in general  $\Omega$ , that this is still true with these boundary conditions), we know that the eigenfunctions must be orthogonal. From class, this implies that the radial parts must also be orthogonal when integrated via  $\int r dr$ . Check that your Bessel solutions for  $k_1$  and  $k_2$  are indeed orthogonal, by numerically integrating their product via the `quadgk` function as in the IJulia notebook from class.

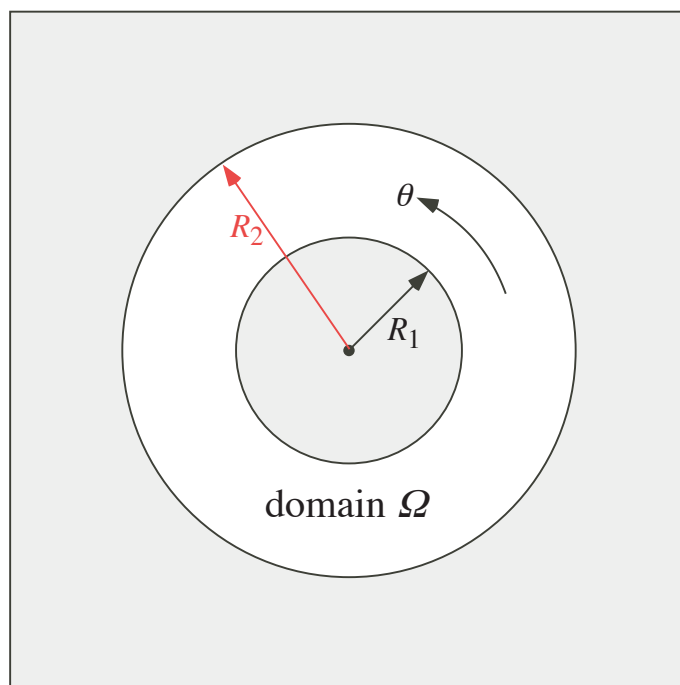


Figure 1: Schematic of the domain  $\Omega$  for problem 3: an annular region in two dimensions, with radii  $r \in [R_1, R_2]$  and angles  $\theta \in [0, 2\pi]$ .