18.303 Problem Set 4

Due Friday, 7 October 2016.

Problem 1: The Chronicles of Kronecker

In class, we defined the *Kronecker product* $A \otimes B$ of two matrices as the matri

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where a_{ij} is the (row i, column j) entry of A. Derive the following properties of Kronecker products from this definition:

(a) $(A \otimes B)^* = A^* \otimes B^*$. (Hence the Kronecker product of Hermitian matrices is Hermitian.)

(b) $(A \otimes B)(C \otimes D) = AC \otimes BD$, as long as the dimensions of the matrices are matched such that AB and CD are defined. This is the "mixed-product property." You can use the fact, from class (and 18.06), that if you divide two matrices into submatrices and multiply them, you can still use the high-school "row times column" formula on the submatrices (where instead of multiplying numbers and adding them up you multiply the submatrices and add them up).

(c) Suppose that A is $M \times M$, B is $N \times N$, the eigensolutions of A are $Ax_m = \lambda_m x_m$ for $m = 1 \ldots M$, and the eigensolutions of B are $By_n = \mu_n y_n$ for $n = 1 \ldots N$. Show that you can construct *all* the eigenvectors of $I_N \otimes A + B \otimes I_M$ in terms of x, y, λ, μ (where I_K is the $K \times K$ identity matrix, as in class). (Hint: use the mixed-product property from above, and notice that Kronecker products also work for vectors $x \otimes y$: just view vectors as matrices with a single column.)

Explain how your solution is the discrete analogue of a "separable" PDE solution.

Problem 2: Separability and symmetry

Often, separability of the solutions is a consequence of symmetry. In this problem, you will show a related property for the case of discrete translational symmetry: a PDE that is invariant under rotation by $2\pi/N$. In particular, suppose that we have the circular system of N springs and masses, with identical spring constants k, depicted in Figure 1. Suppose that the equation of motion of the n-th mass is

$$m\ddot{\phi}_n = \kappa(\phi_{n+1} - \phi_n) - \kappa(\phi_n - \phi_{n-1}).$$

(a) If we let $\mathbf{x} = (\phi_1, \phi_2, \cdots, \phi_N)$ be the N-component column vector of ϕ_m values, then we can write the equation of motion as $A\mathbf{x} = \ddot{\mathbf{x}}$. What is A?

(b) Your A should be real-symmetric. Is it positive/negative definite/semidefinite?

(c) Define the *rotation* operator R by

$$R \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \\ \phi_1 \end{pmatrix},$$

i.e. R rotates each mass to the position of the next one. It could also be called a *cyclic shift* operator. By inspection, R is the permutation matrix:

$$R = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

Furthermore, R is orthogonal (unitary): $R^{-1} = R^*(= R^T)$. There are a variety of ways to show this, but the simplest is probably to note that

$$R^T = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

performs the permutation

$$R^T \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix} = \begin{pmatrix} \phi_N \\ \phi_1 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \end{pmatrix},$$

which is the rotation in the *opposite direction* compared to R. Hence, the two operations cancel one another out and $R^T R = I$.

Show that R *commutes* with A: $RA = AR$, or equivalently (multiplying both sides by R^{-1} on the left), that $A = R^{-1}AR$. We then say that A is **rotation-invariant** (or shift-invariant).

(d) Suppose that we have an eigenvector $A\mathbf{x} = \lambda\mathbf{x}$, and suppose for simplicity that λ is nondegenerate (has multiplicity 1, i.e. there is only one linearly independent eigenfunction of this λ). Show that $\mathbf{x}$ must also be an eigenfunction of R. (Hint: use the commutation result from the previous part.)

(More generally, one can show that we can find "simultaneous" eigenvectors of any commuting matrices.)

(e) If $\mathbf{x}$ is an eigenvector of R, this means $R\mathbf{x} = \alpha\mathbf{x}$ eigenvalues α. Since R is unitary, recall from 18.06 that $|\alpha| = 1$. (Proof: consider $\mathbf{x}^*\mathbf{x} = \mathbf{x}^*R^*R\mathbf{x} = (R\mathbf{x})^*(R\mathbf{x}) = (\alpha\mathbf{x})^*(\alpha x) = |\alpha|^2\mathbf{x}^*\mathbf{x}$. Since $\mathbf{x}^*\mathbf{x} \neq 0$, this means $|\alpha|^2 = 1$, hence $\alpha = e^{ik}$ for some real k.) Show that:

(i) Choose $x_1 = 1$. Give a formula for all the other components of $\mathbf{x}$.

(ii) Since $R^N = I$ (rotating N times returns to the original vector), show that $\alpha = e^{ik}$ where k must have the form: ____________

(iii) Combining the previous two parts, plug your eigenvector back into $A\mathbf{x} = \lambda\mathbf{x}$ and give a formula for the eigenvalues λ of A.

(f) In the continuum limit, what is the form of the eigenfunctions $\phi(\theta)$ in terms of the angle θ?

This is a simplified case of a much deeper theory about how symmetry relates to the solutions of PDEs. More generally, especially for the case of degenerate (multiplicity > 1) eigenvalues and combinations of multiple symmetry operators (rotations, reflections, translations, ...), one uses the tool of *group theory* to describe the interactions of the symmetry operators and the tool of *group representation theory* to describe the consequences for the eigenfunctions and other solutions.

Problem 3: *Fiat lux.*

Consider the space of three-component vector fields $\mathbf{u}(\mathbf{x})$ on some finite-volume 3d domain $\Omega \subset \mathbb{R}^3$. One linear operator on these fields is the curl $\nabla\times$, which is important in electromagnetism (which we will study in more detail later in 18.303). Define the inner product of two vector fields $\mathbf{u}$ and $\mathbf{v}$ by the volume integral $\langle \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \bar{\mathbf{u}} \cdot \mathbf{v}$. In this problem, you will need the 18.02 identity: $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot (\nabla \times \mathbf{v})$, which is easily derived.

(a) Figure out how to do integration by parts with the curl: show that $\langle \mathbf{u}, \nabla \times \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \mathbf{v} \rangle + \oiint_{\partial\Omega} \mathbf{w} \cdot d\mathbf{S}$, where $d\mathbf{S}$ is the usual outward surface-normal area element, and the $\mathbf{w}$ appearing in the surface integral over the boundary ($\partial\Omega$) is some vector field to be determined. (*Hint:* use the given 18.02 identity, combined with the divergence theorem.)

(b) Show that $\nabla\times$ is Hermitian with this inner product if we impose the boundary condition that $\mathbf{u}$ is *perpendicular* to the boundary. It is convenient to define a vector field $\mathbf{n}(\mathbf{x})$ on $\partial\Omega$ to denote the outward normal vector at each point on the boundary, in which case our boundary condition is that $\mathbf{u} \times \mathbf{n} = 0$ on $\partial\Omega$.[1]

(c) Two of Maxwell's equations in vacuum are $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ where c is the speed of light. Take the curl of both sides of the second equation to obtain a PDE in $\mathbf{E}$ alone. Suppose that Ω is the interior of a hollow metal container, where the boundary conditions are that $\mathbf{E}$ is perpendicular to the metal at the surface (i.e. $\mathbf{E} \times \mathbf{n}|_{\partial\Omega} = 0$).[2] Combining these facts with the previous parts, explain why you would expect to obtain *oscillating* solutions to Maxwell's equations (*standing electromagnetic waves*, essentially light bouncing around inside the container). Hint: write down a Hermitian positive semidefinite eigenproblem with eigenvalues $\omega^2 \geq 0$.

(This kind of system exists, for example, in the microwave regime where metals have high conductivity, and such containers are called *microwave resonant cavities*.)

[1] Technically, this is easiest if we assume that the boundary is a differentiable surface so that the normal vector is uniquely defined everywhere. Actually, it is sufficient to assume that it is differentiable except for isolated corners and edges (a "set of measure zero") since those isolated kinks won't contribute anything to the $\oiint_{\partial\Omega}$ surface integral.

[2] In a perfect conductor, any nonzero component of $\mathbf{E}$ parallel to the surface would generate an infinite current parallel to the surface, in which charges instantaneously rearrang to cancel the field. For a real conductor with finite conductivity, matters are more complicated because we must consider what the fields do inside the conductor, but a perfect conductor is a good approximation at microwave and lower frequencies where the penetration of the field into the conductor (the *skin depth*) is much smaller than the wavelength of light.

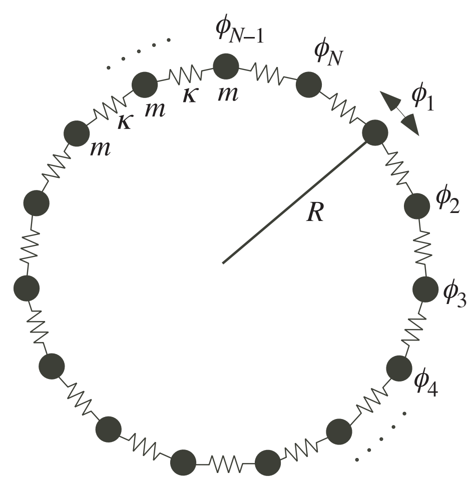

Figure 1: Circular systems of N identical masses m and springs κ. ϕ_n is the angular displacement of the n-th mass ($\phi_m = 0$ for all springs when they are at rest). Imagine that the springs can move in the ϕ direction, but cannot move in the radial direction (for example, if they are sliding without friction on the surface of a cylinder of radius R).