18.303 Problem Set 10

Due Monday, 29 November 2010.

Problem 1: Energy flow and group velocity

In class (and notes), we showed that we can turn the scalar wave equation $b\nabla \cdot (a\nabla u) = \frac{\partial^2 u}{\partial t^2}$ ($a > 0$ and $b > 0$) into two coupled first-derivative equations: $\frac{\partial u}{\partial t} = b\nabla \cdot \mathbf{v}$, $\frac{\partial \mathbf{v}}{\partial t} = a\nabla u$ by introducing a new (vector) unknown $\mathbf{v}(\mathbf{x}, t)$. By defining $\mathbf{w} = (u, \mathbf{v})^T$, we obtained the form

$$\frac{\partial \mathbf{w}}{\partial t} = \begin{pmatrix} & b\nabla\cdot \\ a\nabla & \end{pmatrix} \mathbf{w} = \hat{D}\mathbf{w},$$

where $\hat{D}$ was anti-Hermitian ($\hat{D}^* = -\hat{D}$) under the inner product $\langle \mathbf{w}, \mathbf{w}' \rangle = \int_\Omega \left(\frac{1}{b}\bar{u}u' + \frac{1}{a}\bar{\mathbf{v}} \cdot \mathbf{v}' \right)$, for appropriate boundary conditions (e.g. $u|_{d\Omega} = 0$). This gave us conservation of "energy": $\frac{\partial}{\partial t}\langle \mathbf{w}, \mathbf{w} \rangle = \langle \mathbf{w}, \hat{D}\mathbf{w} \rangle + \langle \hat{D}\mathbf{w}, \mathbf{w} \rangle = \langle \hat{D}^*\mathbf{w}, \mathbf{w} \rangle + \langle \hat{D}\mathbf{w}, \mathbf{w} \rangle = 0$.

(a) To show that $\hat{D}^* = -\hat{D}$, we integrated by parts and relied on the boundary conditions to ensure that the surface-integral terms from integration by parts (the divergence theorem) vanished. Now, instead, assume that you aren't given any information about u or $\mathbf{v}$ on $d\Omega$, so that we cannot drop the surface integrals when integrating by parts in $\langle \mathbf{w}, \hat{D}\mathbf{w} \rangle$ to move $\hat{D}$ to the other side. Show that we now obtain

$$\frac{\partial}{\partial t}\langle \mathbf{w}, \mathbf{w} \rangle = \oiint_{d\Omega} \mathbf{F} \cdot d\mathbf{A}$$

for some vector field $\mathbf{F}$ in terms of u and $\mathbf{v}$. We can therefore interpret $\mathbf{F}$ as a flow rate ("flux") of energy per unit area per unit time. (The electromagnetic version of this is called the "Poynting vector".)

Reminder: the key identity for integration by parts is that, for any function ϕ and any vector field $\mathbf{f}$, we have $\nabla \cdot (\phi \mathbf{f}) = (\nabla\phi) \cdot \mathbf{f} + \phi\nabla \cdot \mathbf{f}$. e.g. this allows us to write $\int_\Omega \phi\nabla \cdot \mathbf{f} = \int_\Omega [\nabla \cdot (\phi\mathbf{f}) - (\nabla\phi) \cdot \mathbf{f}]$, where the first term turns into a surface integral by the divergence theorem, and vice versa for $\int_\Omega (\nabla\phi) \cdot \mathbf{f}$.

(b) Suppose that we are in 3d have a and b invariant in the z direction (e.g. in a waveguide along z), and the boundary conditions also being independent of z (Ω being infinite in z). Then we can look for separable eigenfunctions $\mathbf{w}(x, y, z, t) = \mathbf{w}_k(x, y)e^{i(kz-\omega t)}$, similar to class, satisfying $\hat{D}\mathbf{w} = -i\omega\mathbf{w} = \frac{\partial \mathbf{w}}{\partial t}$, which leads to the reduced 2d eigenproblem:

$$\hat{D}_k \mathbf{w}_k = -i\omega\mathbf{w}_k.$$

What is $\hat{D}_k$? $\mathbf{w}_k$ is only a function of (x, y), so all we need is a 2d domain Ω_{xy}. Suppose $u|_{d\Omega} = 0$, and so $u_k|_{d\Omega_{xy}} = 0$. Show that $\hat{D}_k^* = -\hat{D}_k$ under the 2d version of the inner product $\langle \mathbf{w}_k, \mathbf{w}_k' \rangle$, same as above but only integrated over Ω_{xy}.

(c) For an eigenfunction $\hat{D}_k\mathbf{w}_k = -i\omega\mathbf{w}_k$, we can write the frequency $\omega(k)$ as the Rayleigh quotient:

$$\omega(k) = \frac{\langle \mathbf{w}_k, i\hat{D}_k\mathbf{w}_k \rangle}{\langle \mathbf{w}_k, \mathbf{w}_k \rangle}.$$

Show that:

$$\frac{\partial \omega}{\partial k} = \frac{\langle \mathbf{w}_k, i\frac{\partial \hat{D}_k}{\partial k}\mathbf{w}_k \rangle}{\langle \mathbf{w}_k, \mathbf{w}_k \rangle}$$

by differentiating via the usual product rule etcetera. You should find that the $\frac{\partial \mathbf{w}_k}{\partial k}$ terms all cancel, due to the fact that $i\hat{D}_k$ is Hermitian $[(i\hat{D}_k)^* = (-i)\hat{D}_k^* = i\hat{D}_k]$.

(d) Show that the numerator $\langle \mathbf{w}_k, i\frac{\partial \hat{D}_k}{\partial k}\mathbf{w}_k \rangle$ from the part (c) is exactly an integral of $\mathbf{F} \cdot \hat{z}$ [the $\mathbf{F}$ from part (a)].

Hence, the group velocity $\partial\omega/\partial k$ is exactly equal to the power flowing in z divided by the energy density $\langle \mathbf{w}_k, \mathbf{w}_k \rangle$ (= energy per unit z), and can be interpreted as an **energy velocity**.