18.303 Problem Set 1

Due Wednesday, 11 September 2013.

Problem 1: 18.06 warmup

Here are a few questions that you should be able to answer based only on 18.06:

(a) Show that if A is real-symmetric and invertible, then A^{-1} is real-symmetric too.

(b) In 18.06, you were shown a simple (2–3 line) proof that the eigenvalues λ of A (solutions of $A\mathbf{x} = \lambda\mathbf{x}$) must be real numbers if A is real-symmetric (if you've forgot it, look it up), eigenvectors of distinct λ must be orthogonal. Now consider the eigenproblem $B^{-1}A\mathbf{x} = \lambda\mathbf{x}$ where A and B are both real-symmetric and B is positive-definite. In this class, we denote "dot products" by $\langle \mathbf{x}, \mathbf{y} \rangle$, where the familiar dot product of real column vectors is $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$. Consider a modified "dot product" $\langle \mathbf{x}, \mathbf{y} \rangle_B = \mathbf{x}^T B\mathbf{y}$ of real vectors $\mathbf{x}$ and $\mathbf{y}$. Using this new dot product, adapt your 18.06 proof to show that the eigenvalues of $B^{-1}A$ are also real, and its eigenvectors of distinct λ are orthogonal under this new dot product.

(c) Suppose that A is a real-symmetric 4×4 matrix with eigenvalues $-1, -4, -9, -25$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4$, respectively.

  (i) If $\mathbf{x}(t)$ solves the system of ODEs $\frac{d^2}{dt^2}\mathbf{x} = A\mathbf{x}$ with initial conditions $\mathbf{x}(0) = \mathbf{a}_0$ and $\mathbf{x}'(0) = \mathbf{b}_0$, write down the solution $\mathbf{x}(t)$ as a closed-form expression (no matrix inverses or exponentials) in terms of the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_4$ and $\mathbf{a}_0$ and $\mathbf{b}_0$. [Hint: expand $\mathbf{x}(t)$ in the basis of the eigenvectors with unknown coefficients $c_1(t), \ldots, c_4(t)$, then plug into the ODE and solve for each coefficient. The ODE $\ddot{y} = -\omega^2 y$ has a general solution $y(t) = u\cos(\omega t) + v\sin(\omega t)$ for some constants u and v.]

  (ii) Compare and contrast the solutions of $\frac{d^2}{dt^2}\mathbf{x} = A\mathbf{x}$ and $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$ for this A.

Problem 2: Les Poisson, les Poisson

In class, we considered the 1d Poisson equation $\frac{d^2}{dx^2}u(x) = f(x)$ for the vector space of functions $u(x)$ on $x \in [0, L]$ with the "Dirichlet" boundary conditions $u(0) = u(L) = 0$, and solved it in terms of the eigenfunctions of $\frac{d^2}{dx^2}$ (giving a Fourier sine series). Here, we will consider a couple of small variations on this:

(a) Suppose that we we change the boundary conditions $u(0) = 0$, $u'(L) = u(L)/L$.

  (i) What are the eigenfunctions of $\frac{d^2}{dx^2}$ now? Explain why there are an infinite number of solutions. Give the first three eigenvalues approximately (to at least two decimal places multiplied by some power of L). (Newton's method may be helpful here, e.g. via `using PyCall; @pyimport scipy.optimize as s; s.newton(x -> cos(x) - x, 1)` in Julia.)

  (ii) Do you expect Poisson's equation have unique solutions? Why or why not?

(b) If we instead consider $\frac{d^2}{dx^2}v(x) = g(x)$ for functions $v(x)$ with the boundary conditions $v(0) = 0$, $v'(L) = v(L)/L + 1$, do these functions form a vector space? Why or why not?

(c) Explain how we can transform the $v(x)$ problem of the previous part back into the original $\frac{d^2}{dx^2}u(x) = f(x)$ problem with $u(0) = u(L)$, by writing $u(x) = v(x) + q(x)$ and $f(x) = g(x) + r(x)$ for some functions q and r. (Transforming a new problem into an old, solved one is always a useful thing to do!)