18.303 Problem Set

Due Wednesday, 24 November 2010.

Problem 1: Stability

Suppose that you are discretizing the 1d Schrodinger wave equation (nondimensionalized and with zero potential):

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial x^2}.$$

Suppose that we discretize in space at intervals Δx and in time with some timestep Δt, denoting $u(m\Delta x, n\Delta t) \approx u_m^n$ as in class. We will discretized $\partial^2/\partial x^2$ with the usual center-difference approximation $\frac{u_{m+1}-2u_m+u_{m-1}}{\Delta x^2}$.

(a) Using Von Neumann analysis for infinite domains, analyze the stability of the following four possibilities for the $\partial/\partial t$ discretization (either showing unconditional stability or giving a condition for stability):

(i) Forward difference

(ii) Backward difference

(iii) Center-difference, implicit

(iv) Center-difference, explicit

(b) Modify the animwave.m file from class (see the web site) to implement one of your schemes, for Dirichlet boundary conditions $u(0) = u(L) = 0$, and plot $|u(x,t)|^2$ for several representative times t given the initial condition $u(x,0) = \delta(x - L/2)$ [implement by setting u_m at one position to $1/\Delta x$ and to 0 for other m's].

Problem 2: Waveguide modes

Suppose we have a 3d wave equation $\hat{A}u = \frac{\partial^2 u}{\partial t^2}$, with $\hat{A} = \hat{A}^*$ negative-semidefinite, that is invariant in z. As in class, we can then look for separable waveguide modes of the form $u_k(x,y)e^{i(kz-\omega t)}$, solving the z-independent "reduced" equation:

$$\hat{A}_k u_k = -\omega(k)^2 u_k$$

for the eigenfrequencies $\omega_n(k) \geq 0$ and waveguide modes $u_{k,n}(x,y)$ for all real k, where $\hat{A}_k = e^{-ikz}\hat{A}e^{ikz}$.

(a) Suppose that at time $t = 0$ we have initial u and $\partial u/\partial t$ given by:

$$u(x,y,z,0) = f(x,y,z)$$

$$\dot{u}(x,y,z,0) = g(x,y,z)$$

for some f and g. Write the solution $u(x,y,z,t)$ for all t as a superposition of waveguide modes $u_{k,n}(x,y)e^{i[kz\pm\omega_n(k)t]}$ for all n and k (assuming these form a complete basis for any f and g of interest), giving explicit formulas for the coefficients of this superposition in terms of f and g.

(b) Suppose that $u(x,y,z,t)$ forms some wavepacket whose envelope stays the same shape (or changes only very slowly) and moves at a speed v in the $+z$ direction. What can you conclude about the initial conditions f and g (or your superposition) from the previous part?