18.303: Self-adjointness (*reciprocity*) and definiteness (positivity) in Green's functions

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0 Review

Suppose we have some vector space V of functions $u(\mathbf{x})$ on a domain Ω, an inner product $\langle u, v\rangle$, and a linear operator $\hat{A}$. [More specifically, V forms a Sobolev space, in that we require $\langle u, \hat{A}u\rangle$ to be finite.] $\hat{A}$ is self-adjoint if $\langle u, \hat{A}v\rangle = \langle \hat{A}u, v\rangle$ for all $u, v \in V$, in which case its eigenvalues λ_n are real and its eigenfunctions $u_n(\mathbf{x})$ can be chosen orthonormal. $\hat{A}$ is positive definite (or semidefinite) if $\langle u, \hat{A}u\rangle > 0$ (or ≥ 0) for all $u \neq 0$, in which case its eigenvalues are > 0 (or ≥ 0); suppose that we order them as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$.

Suppose that $\hat{A}$ is positive definite, so that $N(\hat{A}) = \{0\}$ and $\hat{A}u = f$ has a unique solution for all f in some suitable space of functions $C(\hat{A})$. Then, for scalar-valued functions u and f, we can typically write

$$\boxed{u(\mathbf{x}) = \hat{A}^{-1} f = \int_{\mathbf{x}' \in \Omega} G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')}, \tag{1}$$

in terms of a **Green's function** $G(\mathbf{x}, \mathbf{x}')$, where $\int_{\mathbf{x}' \in \Omega}$ denotes integration over $\mathbf{x}'$. In this note, we don't address how to find G, but instead ask what properties it must have from the self-adjointness and definiteness of $\hat{A}$. [This generalizes in a straightforward way to vector-valued $\mathbf{u}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$, in which case $G(\mathbf{x}, \mathbf{x}')$ is matrix-valued.]

1 Self-adjointness of $\hat{A}^{-1}$ and reciprocity of G

We can show that $(\hat{A}^{-1})^* = (\hat{A}^*)^{-1}$, from which it follows that if $\hat{A} = \hat{A}^*$ ($\hat{A}$ is self-adjoint) then $\hat{A}^{-1}$ **is also self-adjoint**. In particular, consider $\hat{A}^{-1}\hat{A} = 1$: $\langle u, v\rangle = \langle u, \hat{A}^{-1}\hat{A}v\rangle = \langle (\hat{A}^{-1})^* u, \hat{A}v\rangle = \langle \hat{A}^*(\hat{A}^{-1})^* u, v\rangle$, hence $\hat{A}^*(\hat{A}^{-1})^* = 1$ and $(\hat{A}^{-1})^* = (\hat{A}^*)^{-1}$. And of course, we already knew that the eigenvalues of $\hat{A}^{-1}$ are λ_n^{-1} and the eigenfunctions are $u_n(\mathbf{x})$.

What are the consequences of self-adjointness for G? Suppose the u are scalar functions, and that the inner product is of the form $\langle u, v\rangle = \int_\Omega w\bar{u}v$ for some weight $w(\mathbf{x}) > 0$. From the fact that $\langle u, \hat{A}^{-1}v\rangle = \langle \hat{A}^{-1}u, v\rangle$, substituting equation (1), we must therefore have:

$$\begin{aligned}
\langle u, \hat{A}^{-1}v\rangle &= \iint_{\mathbf{x}, \mathbf{x}' \in \Omega} w(\mathbf{x}) \overline{u(\mathbf{x})} G(\mathbf{x}, \mathbf{x}') v(\mathbf{x}') \\
&= \langle \hat{A}^{-1}u, v\rangle \\
&= \iint_{\mathbf{x}, \mathbf{x}' \in \Omega} w(\mathbf{x}) \overline{G(\mathbf{x}, \mathbf{x}') u(\mathbf{x}')} v(\mathbf{x}) = \iint_{\mathbf{x}, \mathbf{x}' \in \Omega} w(\mathbf{x}') \overline{u(\mathbf{x}) G(\mathbf{x}', \mathbf{x})} v(\mathbf{x}'),
\end{aligned}$$

where in the last step we have interchanged/relabeled $\mathbf{x} \leftrightarrow \mathbf{x}'$. Since this must be true for all u and v, it follows that

$$\boxed{w(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') = w(\mathbf{x}') \overline{G(\mathbf{x}', \mathbf{x})}}$$

for all $\mathbf{x}, \mathbf{x}'$. This property of G (or its analogues in other systems) is sometimes called **reciprocity**. In the common case where $w = 1$ and $\hat{A}$ and G are real (so that the complex conjugation can be omitted), it says that *the effect at* $\mathbf{x}$ *from a source at* $\mathbf{x}'$ *is the same as the effect at* $\mathbf{x}'$ *from a source at* $\mathbf{x}$.

There are many interesting consequences of reciprocity. For example, its analogue in linear electrical circuits says that the current at one place created by a voltage at another is the same as if the locations of the current and voltage are swapped. Or, for antennas, the analogous theorem says that a given antenna works equally well as a transmitter or a receiver.

1.1 Example: $\hat{A} = -\frac{d^2}{dx^2}$ on $\Omega = [0,L]$

For this simple example (where $\hat{A}$ is self-adjoint under $\langle u,v\rangle = \int \bar{u}v$), with Dirichlet boundaries, we previously obtained a Green's function,

$$G(x,x') = \begin{cases} \left(1-\frac{x'}{L}\right)x & x < x' \\ (1-\frac{x}{L})x' & x \geq x' \end{cases},$$

which obviously obeys the $G(x,x') = G(x',x)$ reciprocity relation.

2 Positive-definiteness of $\hat{A}^{-1}$ and positivity of G

Not only is $\hat{A}^{-1}$ self-adjoint, but since its eigenvalues are the inverses λ_n^{-1} of the eigenvalues of $\hat{A}$, then if $\hat{A}$ is positive-definite ($\lambda_n > 0$) then $\hat{A}^{-1}$ **is also positive-definite** ($\lambda_n^{-1} > 0$). From another perspective, if $\hat{A}u = f$, then positive-definiteness of $\hat{A}$ means that $0 < \langle u, \hat{A}u\rangle = \langle u, f\rangle = \langle \hat{A}^{-1}f, f\rangle = \langle f, \hat{A}^{-1}f\rangle$ for $u \neq 0 \Leftrightarrow f \neq 0$, hence $\hat{A}^{-1}$ is positive-definite. (And if $\hat{A}$ is a PDE operator with an ascending sequence of unbounded eigenvalues, then the eigenvalues of $\hat{A}^{-1}$ are a descending sequence $\lambda_1^{-1} > \lambda_2^{-1} > \cdots > 0$ that approaches 0 asymptotically from above.[1])

If $\hat{A}$ is a real operator (real u give real $\hat{A}u$), then $\hat{A}^{-1}$ should also be a real operator (real f give real $u = \hat{A}^{-1}f$). Furthermore, under fairly general conditions for real positive-definite (elliptic) PDE operators $\hat{A}$, especially for second-derivative ("order 2") operators, then one can often show $\boxed{G(\mathbf{x},\mathbf{x}') > 0}$ (except of course for $\mathbf{x}$ or $\mathbf{x}'$ at the boundaries, where G vanishes for Dirichlet conditions).[2] The analogous fact for matrices A is that if A is real-symmetric positive-definite *and* it has *off-diagonal entries* ≤ 0 — like our $-\nabla^2$ second-derivative matrices (recall the $-1, 2, -1$ sequences in the rows) and related finite-difference matrices — it is called a **Stieltjes matrix**, and such matrices can be shown to have inverses with nonnegative entries.[3]

2.1 Example: $\hat{A} = -\nabla^2$ with $u|_{\partial\Omega} = 0$

Physically, the positive-definite problem $-\nabla^2 u = f$ can be thought of as the displacement u in response to an applied pressure f, where the Dirichlet boundary conditions correspond to a material pinned at the edges. The Green's function $G(\mathbf{x},\mathbf{x}')$ is the limit of the displacement u in response to a force concentrated at a single point $\mathbf{x}'$. The Green's function $G(\mathbf{x},\mathbf{x}')$ for some example points $\mathbf{x}'$ is shown for a 1d domain $\Omega = [0,1]$ in figure 1(left) (a "stretched string"), and for a 2d domain $\Omega = [-1,1] \times [-1,1]$ in figure 1(right) (a "square drum"). As expected, $G > 0$ everywhere except at the edges where it is zero: the whole string/membrane moves in the positive/upwards direction in response to a positive/upwards force.

[1] Such $\hat{A}^{-1}$ integral operators are typically what are called "compact" operators. Functional analysis books often prove diagonalizability (a "spectral theorem") for compact operators *first* and only later consider diagonalizability of PDE-like operators by viewing them as the inverses of compact operators.

[2] See, for example, "Characterization of positive reproducing kernels. Application to Green's functions," by N. Aronszajn and K. T. Smith [*Am. J. Mathematics*, vol. 79, pp. 611–622 (1957), `http://www.jstor.org/stable/2372564`]. However, as usual there are pathological counter-examples.

[3] There are many books with "nonnegative matrices" in their titles that cover this fact, usually as a special case of a more general class of something called "M matrices," but I haven't yet found an elementary presentation at an 18.06 level. Note that the *diagonal* entries of a positive-definite matrix P are always positive, thanks to the fact that $P_{ii} = \mathbf{e}_i^T P \mathbf{e}_i > 0$ where $\mathbf{e}_i$ is the unit vector in the i-th coordinate.

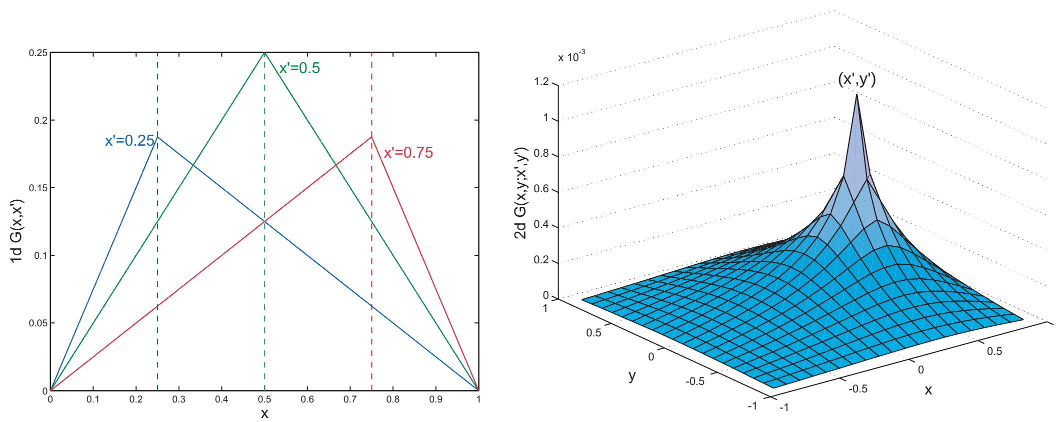

Figure 1: Examples illustrating the positivity of the Green's function $G(\mathbf{x}, \mathbf{x}')$ for a positive-definite operator ($-\nabla^2$ with Dirichlet boundaries). *Left:* a "stretched string" 1d domain $[0, 1]$. *Right:* a "stretched square drum" 2d domain $[-1, 1] \times [-1, 1]$.