18.303 Problem Set 4 Solutions

Problem 1: (5+10+(3+3+4)+10+(5+5+5+5)+5)

(a) Since $\ddot{\phi}_n = \frac{\kappa}{m}(\phi_{n+1} - 2\phi_n + \phi_{n-1})$, we can write

$$A = \frac{\kappa}{m}\begin{pmatrix} -2 & 1 & & & & & 1 \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \end{pmatrix}.$$

Note the first and last rows! This is a consequence of the periodicity of the system, since we can identify $\phi_0 = \phi_N$ and $\phi_{N+1} = \phi_1$.

(b) To check definiteness, the easiest way is to factorize A. Similar to class, we write $\ddot{\phi}_n$ in two steps: first we compute $\psi_{n+0.5} = \phi_{n+1} - \phi_n$, then we compute $\ddot{\phi}_n = \frac{\kappa}{m}(\psi_{n+0.5} - \psi_{n-0.5})$. Unlike the 1d case in class, however, there are only N values $\psi_{n+0.5}$, equal to the number of springs! Hence, we obtain an $N \times N$ matrix D given by:

$$\begin{pmatrix} \psi_{1.5} \\ \psi_{2.5} \\ \vdots \\ \psi_{N-0.5} \\ \psi_{N+0.5} \end{pmatrix} = D\mathbf{x} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix},$$

where we must be careful to get the periodicity right for the last row $\psi_{N+0.5} = \phi_1 - \phi_N$. Similarly, noting that $\ddot{\phi}_1 = \frac{\kappa}{m}(\psi_{1.5} - \psi_{N+0.5})$, we have:

$$\ddot{\mathbf{x}} = \frac{\kappa}{m}\begin{pmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}\begin{pmatrix} \psi_{1.5} \\ \psi_{2.5} \\ \vdots \\ \psi_{N-0.5} \\ \psi_{N+0.5} \end{pmatrix} = -\frac{\kappa}{m}D^T D\mathbf{x},$$

where we have identified that the matrix to take the differences of the $\psi_{n+0.5}$ is precisely $-D^T$. Hence, $A = -\frac{\kappa}{m}D^T D$, which by inspection is at least **negative semidefinite** (from class).

It is **not** negative-definite, however. This can be checked in a variety of ways, most easily by noticing that

$$D\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} = 0,$$

and hence D is not full-rank (and similarly for A).

(c) We defined the *rotation* operator R by

$$R \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \\ \phi_1 \end{pmatrix}.$$

(i) By inspection, R is the permutation matrix:

$$R = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}.$$

(ii) There are a variety of ways to show this, but the simplest is probably to note that

$$R^T = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

performs the permutation

$$R^T \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix} = \begin{pmatrix} \phi_N \\ \phi_1 \\ \vdots \\ \phi_{N-2} \\ \phi_{N-1} \end{pmatrix},$$

which is the rotation in the *opposite direction* compared to R. Hence, the two operations cancel one another out and $R^T R = I$.

(iii) Multiplying RA acts R on each of the *columns* of A, i.e. it permutes each column, giving:

$$RA = \frac{\kappa}{m} \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \\ -2 & 1 & & & & & 1 \end{pmatrix}.$$

Multiplying $AR = (R^T A^T)^T = (R^T A)^T$ is equivalent to permuting each *row* of A by R^T

(i.e. in the opposite direction), hence

$$R^T A = \frac{\kappa}{m} \begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \\ -2 & 1 & & & & & 1 \end{pmatrix},$$

which $= RA$. Q.E.D.

(d) Consider the vector $\mathbf{y} = R\mathbf{x}$. Using $RA = AR$, we obtain: $A\mathbf{y} = AR\mathbf{x} = RA\mathbf{x} = \lambda R\mathbf{x} = \lambda\mathbf{y}$. Therefore, $\mathbf{y}$ is an eigenvector of A with eigenvalue λ. But we were told that λ has multiplicity 1: this means that $\mathbf{y}$ must be linearly dependent on $\mathbf{x}$, i.e. $\mathbf{y} = \alpha\mathbf{x}$ for some scalar α. Hence $\mathbf{y} = R\mathbf{x} = \alpha\mathbf{x}$, and $\mathbf{x}$ is an eigenvector of R with eigenvalue α. Q.E.D.

(e) Let $R\mathbf{x} = \alpha\mathbf{x}$.

(i) We need to use the fact that $R^*R = I$. Therefore, consider $\mathbf{x}^*\mathbf{x} = \mathbf{x}^*R^*R\mathbf{x} = (R\mathbf{x})^*(R\mathbf{x}) = (\alpha\mathbf{x})^*(\alpha x) = |\alpha|^2\mathbf{x}^*\mathbf{x}$. Since $\mathbf{x}^*\mathbf{x} \neq 0$ (eigenvectors are nonzero), this means $|\alpha|^2 = 1$, hence $\alpha = e^{ik}$ for some real k.

(ii) We just write out $R\mathbf{x} = e^{ik}\mathbf{x}$:

$$R \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_N \\ 1 \end{pmatrix} = e^{ik} \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix}$$

and hence $x_2 = e^{ik}$, $x_3 = e^{ik}x_2 = e^{2ik}$, and so on, or

$$\mathbf{x} = \begin{pmatrix} 1 \\ e^{ik} \\ \vdots \\ e^{i(N-2)k} \\ e^{i(N-1)k} \end{pmatrix},$$

or more simply:

$$\boxed{x_n = e^{i(n-1)k}}.$$

(iii) On an eigenvector, $R^N\mathbf{x} = e^{iNk}\mathbf{x} = \mathbf{x}$, and hence $e^{iNk} = 1$. This means that Nk is an integer multiple of 2π, i.e. $Nk = 2\pi m$ for $m = 0, 1, 2, \ldots$, giving eigenvalues

$$\boxed{\alpha_m = e^{i\frac{2\pi m}{N}}}.$$

A little more carefully, we notice that $\alpha_N = \alpha_0$, so we have N distinct eigenvalues $\boxed{m = 0, 1, \ldots, N-1}$.

(iv) Now that we know the eigenvectors x_n, we can plug it back into $A\mathbf{x} = \lambda\mathbf{x}$. Each row of this equation has the form

$$\frac{\kappa}{m}\left(x_{n+1} - 2x_n + x_{n-1}\right) = \lambda x_n$$

and plugging in the form of $x_n = e^{ik(n-1)} = e^{ikn}e^{-ik}$ and dividing both sides by x_n gives:

$$\frac{\kappa}{m}\left(e^{ik} - 2 + e^{ik}\right) = \lambda = \frac{\kappa}{m}\left[2\cos(k) - 2\right].$$

Hence, plugging in the equation for k from above, we have:

$$\boxed{\lambda_m = \frac{2\kappa}{m}[\cos(2\pi m/N) - 1] = -\frac{4\kappa}{m}\sin^2\left(\frac{2\pi m}{N}\right)}$$

for $\boxed{m = 0, 1, \ldots, N-1}$, where we have used the half-angle identity $1 - \cos(k) = 2\sin^2(k/2)$ to simplify the final expression. Note that the eigenvalues are real and ≤ 0 as expected, with exactly one zero eigenvalue $\lambda_0 = 0$.

(f) The angular difference between each mass is $\Delta\theta = \frac{2\pi}{N}$, and hence $x_n = e^{i\Delta\theta m(n-1)} = e^{im\theta}$ where we define the angle $\theta = (n-1)\Delta\theta$. Hence the eigenfunctions in the continuum limit are simply

$$\phi(\theta) = e^{im\theta}$$

for integers m (or any constant multiple thereof, of course).

Problem 2: (15+10+5+5)

See also the IJulia notebook posted with the solutions.

(a) Setting the slopes to be zero at R_1 and R_2 simply gives

$$\alpha J'_m(kr) + \beta Y'_m(kr) = 0$$

at the two radii, or $E\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0$ where

$$\boxed{E = \begin{pmatrix} J'_m(kR_1) & Y'_m(kR_1) \\ J'_m(kR_2) & Y'_m(kR_2)\end{pmatrix}}.$$

Hence, writing $f_m(k) = \det E$, we get

$$\boxed{f_m(k) = J'_m(kR_1)Y'_m(kR_2) - J'_m(kR_2)Y'_m(kR_1)}.$$

Given a k for which $f_m(k) = 0$, then we can solve for the nullspace of E by arbitrarily choosing a scaling such that $\boxed{\alpha = 1}$ and solving for β from the first *or* second rows of $E\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0$:

$$\boxed{\beta = -\frac{J'_m(kR_1)}{Y'_m(kR_1)} = -\frac{J'_m(kR_2)}{Y'_m(kR_2)}}.$$

(b) The plot is shown in Figure 1. Note that $f_m(k)$ for $m > 0$ has a divergence as $k \to 0$, so we used the `ylim` command to rescale the vertical axis (otherwise it would be hard to read the plot!); see the solution IJulia notebook.

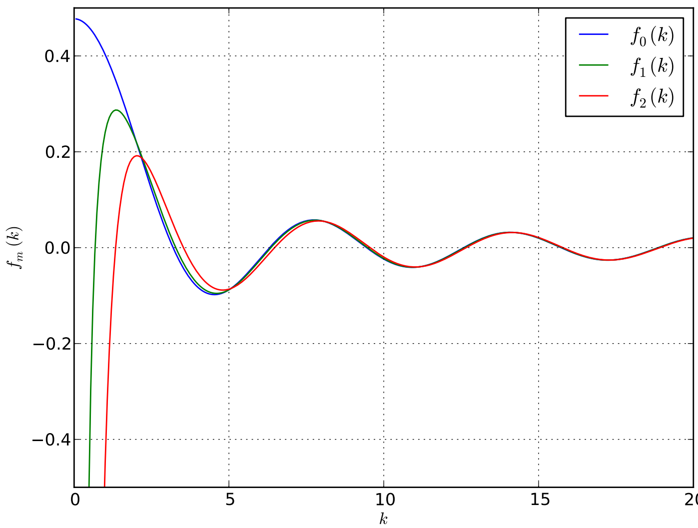

Figure 1: Plot of $f_m(k)$ for $m = 0, 1, 2$.

(c) We'll use the Scilab `newton` function, similar to class, to find the roots, with initial guesses provided by our plot in Figure 1. We find $k_1 \approx 3.196578$, $k_2 \approx 6.31234951$, and $k_3 \approx 9.4444649$. See the solutions notebook.

(d) See the IJulia notebook. Using our k_1 and k_2 from part (c) and our α and β from part (a), we find that $\int_{R_1}^{R_2} r u_{0,1}(r) u_{0,2}(r) dr \approx 10^{-15}$, which is zero up to roundoff errors.