Combinatorics Through Guided Discovery

Kenneth P. Bogart

Preface

This book is an introduction to combinatorial mathematics, also known as combinatorics. The book focuses especially but not exclusively on the part of combinatorics that mathematicians refer to as "counting." The book consist almost entirely of problems. Some of the problems are designed to lead you to think about a concept, others are designed to help you figure out a concept and state a theorem about it, while still others ask you to prove the theorem. Other problems give you a chance to use a theorem you have proved. From time to time there is a discussion that pulls together some of the things you have learned or introduces a new idea for you to work with. Many of the problems are designed to build up your intuition for how combinatorial mathematics works. There are problems that some people will solve quickly, and there are problems that will take days of thought for everyone. Probably the best way to use this book is to work on a problem until you feel you are not making progress and then go on to the next one. Think about the problem you couldn't get as you do other things. The next chance you get, discuss the problem you are stymied on with other members of the class. Often you will all feel you've hit dead ends, but when you begin comparing notes and listening *carefully* to each other, you will see more than one approach to the problem and be able to make some progress. In fact, after comparing notes you may realize that there is more than one way to interpret the problem. In this case your first step should be to think together about what the problem is actually asking you to do. You may have learned in school that for every problem you are given, there is a method that has already been taught to you, and you are supposed to figure out which method applies and apply it. That is not the case here. Based on some simplified examples, you will discover the method for yourself. Later on, you may recognize a pattern that suggests you should try to use this method again.

The point of learning from this book is that you are learning how to discover ideas and methods for yourself, not that you are learning to apply methods that someone else has told you about. The problems in this book are designed to lead you to discover for yourself and prove for yourself the main ideas of combinatorial mathematics. There is considerable evidence that this leads to deeper learning and more understanding.

You will see that some of the problems are marked with bullets. Those are the problems that I feel are essential to having an understanding of what comes later, whether or not it is marked by a bullet. The problems with bullets are the problems in which the main ideas of the book are developed. Your instructor may leave out some of these problems because he or she plans not to cover future problems that rely on them. Many problems, in fact entire sections, are not marked in this way, because they use an important idea rather than developing one. Some other special symbols are described in what follows; a summary appears in ⟨⟨xref without ref or provisional attribute, check spelling⟩⟩.

$\bullet$	essential
$\circ$	motivational material
$+$	summary
$\Rightarrow$	especially interesting
$*$	difficult
$\cdot$	essential for this section or the next

Table 0.0.1: The meaning of the symbols to the left of problem numbers.

Some problems are marked with open circles. This indicates that they are designed to provide motivation for, or an introduction to, the important concepts, motivation with which some students may already be familiar. You will also see that some problems are marked with arrows. These point to problems that I think are particularly interesting. Some of them are also difficult, but not all are. A few problems that summarize ideas that have come before but aren't really essential are marked with a plus, and problems that are essential if you want to cover the section they are in or, perhaps, the next section, are marked with a dot (a small bullet). If a problem is relevant to a much later section in an essential way, I've marked it with a dot and a parenthetical note that explains where it will be essential. Finally, problems that seem unusually hard to me are marked with an asterisk. Some I've marked as hard only because I think they are difficult in light of what has come before, not because they are intrinsically difficult. In particular, some of the problems marked as hard will not seem so hard if you come back to them after you have finished more of the problems.

If you are taking a course, your instructor will choose problems for you to work on based on the prerequisites for and goals of the course. If you are reading the book on your own, I recommend that you try all the problems in a section you want to cover. Try to do the problems with bullets, but by all means don't restrict yourself to them. Often a bulleted problem makes more sense if you have done some of the easier motivational problems that come before it. If, after you've tried it, you want to skip over a problem without a bullet or circle, you should not miss out on much by not doing that problem. Also, if you don't find the problems in a section with no bullets interesting, you can skip them, understanding that you may be skipping an entire branch of combinatorial mathematics! And no matter what, read the textual material that comes before, between, and immediately after problems you are working on!

One of the downsides of how we learn math in high school is that many of us come to believe that if we can't solve a problem in ten or twenty minutes, then we can't solve it at all. There will be problems in this book that take hours of hard thought. Many of these problems were first conceived and solved by professional mathematicians, and *they* spent days or weeks on them. How can you be expected to solve them at all then? You have a context in which to work, and even though some of the problems are so open ended that you go into them without any idea of the answer, the context and the leading examples that preceded them give you a structure to work with. That doesn't mean you'll get them right away, but you will find a real sense of satisfaction when you see what you can figure out with concentrated thought. Besides, you can get hints!

Some of the questions will appear to be trick questions, especially when you get the answer. They are not intended as trick questions at all. Instead they are designed so that they don't tell you the answer in advance. For example the answer to a question that begins "How many..." might be "none." Or there might be just one example (or even no examples) for a problem that asks you to find all examples of something. So when you read a question, unless it directly tells you what the answer is and asks you to show it is true, don't expect the wording of the problem to suggest the answer. The book isn't designed this way to be cruel. Rather, there is evidence that the more open-ended a question is, the more deeply you learn from working on it. If you do go on to do mathematics later in life, the problems that come to you from the real world or from exploring a mathematical topic are going to be open-ended problems because nobody will have done them before. Thus working on open-ended problems now should help to prepare you to do mathematics later on.

You should try to write up answers to all the problems that you work on. If you claim something is true, you should explain why it is true; that is you should prove it. In some cases an idea is introduced before you have the tools to prove it, or the proof of something will add nothing to your understanding. In such problems there is a remark telling you not to bother with a proof. When you write up a problem, remember that the instructor has to be able to "get" your ideas and understand exactly what you are saying. Your instructor is going to choose some of your solutions to read carefully and give you detailed feedback on. When you get this feedback, you should think it over carefully and then write the solution again! You may be asked not to have someone else read your solutions to some of these problems until your instructor has. This is so that the instructor can offer help which is aimed at your needs. On other problems it is a good idea to seek feedback from other students. One of the best ways of learning to write clearly is to have someone point out to you where it is hard to figure out what you mean. The crucial thing is to make it clear to your reader that you really want to know where you may have left something out, made an unclear statement, or failed to support a statement with a proof. It is often very helpful to choose people who have not yet become an expert with the problems, as long as they realize it will help you most for them to tell you about places in your solutions they do not understand, even if they think it is their problem and not yours!

As you work on a problem, think about why you are doing what you are doing. Is it helping you? If your current approach doesn't feel right, try to see why. Is this a problem you can decompose into simpler problems? Can you see a way to make up a simple example, even a silly one, of what the problem is asking you to do? If a problem is asking you to do something for every value of an integer n, then what happens with simple values of n like 0, 1, and 2? Don't worry about making mistakes; it is often finding mistakes that leads mathematicians to their best insights. Above all, don't worry if you can't do a problem. Some problems are given as soon as there is one technique you've learned that might help do that problem. Later on there may be other techniques that you can bring back to that problem to try again. The notes have been designed this way on purpose. If you happen to get a hard problem with the bare minimum of tools, you will have accomplished much. As you go along, you will see your ideas appearing again later in other problems. On the other hand, if you don't get the problem the first time through, it will be nagging at you as you work on other things, and when you see the idea for an old problem in new work, you will know you are learning.

There are quite a few concepts that are developed in this book. Since most of the intellectual content is in the problems, it is natural that definitions of concepts will often be within problems. When you come across an unfamiliar term in a problem, it is likely it was defined earlier. Look it up in the index, and with luck (hopefully no luck will really be needed!) you will be able to find the definition.

Above all, this book is dedicated to the principle that doing mathematics is fun. As long as you know that some of the problems are going to require more than one attempt before you hit on the main idea, you can relax and enjoy your successes, knowing that as you work more and more problems and share more and more ideas, problems that seemed intractable at first become a source of satisfaction later on.

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Contents

Chapter 1

What is Combinatorics?

Combinatorial mathematics arises from studying how we can *combine* objects into arrangements. For example, we might be combining sports teams into a tournament, samples of tires into plans to mount them on cars for testing, students into classes to compare approaches to teaching a subject, or members of a tennis club into pairs to play tennis. There are many questions one can ask about such arrangements of objects. Here we will focus on questions about *how many ways* we may combine the objects into arrangements of the desired type. These are called *counting problems*. Sometimes, though, combinatorial mathematicians ask if an arrangement is possible (if we have ten baseball teams, and each team has to play each other team once, can we schedule all the games if we only have the fields available at enough times for forty games?). Sometimes they ask if all the arrangements we might be able to make have a certain desirable property (Do all ways of testing 5 brands of tires on 5 different cars [with certain additional properties] compare each brand with each other brand on at least one common car?). Problems of these sorts come up throughout physics, biology, computer science, statistics, and many other subjects. However, to demonstrate all these relationships, we would have to take detours into all these subjects. While we will give some important applications, we will usually phrase our discussions around everyday experience and mathematical experience so that the student does not have to learn a new context before learning mathematics in context!

1.1 About These Notes

These notes are based on the philosophy that you learn the most about a subject when you are figuring it out directly for yourself, and learn the least when you are trying to figure out what someone else is saying about it. On the other hand, there is a subject called combinatorial mathematics, and that is what we are going to be studying, so we will have to tell you some basic facts. What we are going to try to do is to give you a chance to discover many of the interesting examples that usually appear as textbook examples and discover the principles that appear as textbook theorems. Your main activity will be solving problems designed to lead you to discover the basic principles of combinatorial mathematics. Some of the problems lead you through a new idea, some give you a chance to describe what you have learned in a sequence of problems, and some are quite challenging. When you find a problem challenging, don't give up on it, but don't let it stop you from going on with other problems. Frequently you will find an idea in a later problem that you can take back to the one you skipped over or only partly finished in order to finish it off. With that in mind, let's get started. In the problems that follow, you will see some problems marked on the left with various symbols. The preface gives a full explanation of these symbols and discusses in greater detail why the book is organized as it is! Table 1.1.1, which is repeated from the preface, summarizes the meaning of the symbols.

$\bullet$	essential
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Table 1.1.1: The meaning of the symbols to the left of problem numbers.

1.2 Supplementary Chapter Problems

1. (interesting) Remember that we can write n as a sum of n ones. How many plus signs do we use? In how many ways may we write n as a sum of a list of k positive numbers? Such a list is called a *composition* of n into k parts. We use $n-1$ plus signs. Write down such a sum and choose $k-1$ of the plus signs. Then each string of ones and plusses between two chosen plus signs, before the first chosen plus sign or after the last chosen one corresponds to a part of a composition of n. Thus the number of compositions of n with k parts is the number of ways to choose the $k-1$ places, which is $\binom{n-1}{k-1}$.

2. In Problem 1 we defined a composition of n into k parts. What is the total number of compositions of n (into any number of parts). The total number of compositions is the number of ways to choose a subset of the plus signs which is 2^{n-1}.

3. (essential for this or the next section) Write down a list of all 16 0-1 sequences of length four starting with 0000 in such a way that each entry differs from the precious one by changing just one digit. This is

called a Grey Code. That is, a *Grey Code* for 0-1 sequences of length n is a list of the sequences so that each entry differs from the previous one in exactly one place. Can you describe how to get a Grey Code for 0-1 sequences of length five from the one you found for sequences of length 4? Can you describe how to prove that there is a Grey code for sequences of length n? (One of many) 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000. To get a code for sequences of length 5, put a zero at the end of each of the sequences we have. Follow that revised sequence by 10001, and write the remainder of the sequence in reverse order with a 1 at the end of each term. (Don't reverse the individual length four sequences, just the sequence of sequences!) We just, in essence, described the inductive step of an inductive proof that Grey Codes exist for sequences of any length.

4. (interesting) Use the idea of a Grey Code from Problem 3 to prove bijectively that the number of even-sized subsets of an n-element set equals the number of odd-sized subsets of an n-element set. Each sequence in the Grey Code is the characteristic function of a set, and the number of elements of the set is the number of ones in the sequence. Since each sequence differs in just one place from the preceding one, the sequences alternate between having an even number of ones and an odd number of ones. Since the first sequence is all zeros and there are 2^n sequences, the last one has an odd number of zeros. Thus the map that takes each sequence except the last to the next one, and takes the last to the first is a bijection between the characteristic functions of sets with an even number of elements and sets with an odd number of elements.

5. (interesting) A list of parentheses is said to be balanced if there are the same number of left parentheses as right, and as we count from left to right we always find at least as many left parentheses as right parentheses. For example, (((()()))()) is balanced and ((()) and (()()))(() are not. How many balanced lists of n left and n right parentheses are there? The number is the Catalan number: we get a bijection between balanced lists of parentheses and Catalan paths by sending each left paren to an upstep and each right paren to a downstep. The condition that there are always as many left parentheses as right ensures we never go below the x axis.

6. (difficult) Suppose we plan to put six distinct computers in a network as shown in Figure 1.2.1. The lines show which computers can communicate directly with which others. Consider two ways of assigning computers to the nodes of the network different if there are two computers that communicate directly in one assignment and that don't communicate directly in the other. In how many different ways can we assign computers to the network?

Figure 1.2.1: A computer network.

We consider two assignments of computers to be equivalent if in both assignments, each computer communicates directly with exactly the same computers. This partitions the set of all 6! computer assignments into blocks of 48 computers each. Thus we have $720/48 = 15$ ways to assign the computers to the network.

7. (interesting) In a circular ice cream dish we are going to put four distinct scoops of ice cream chosen from among twelve flavors. Assuming we place four scoops of the same size as if they were at the corners of a square, and recognizing that moving the dish doesn't change the way in which we have put the ice cream into the dish, in how many ways may we choose the ice cream and put it into the dish? Each ice cream arrangement is equivalent to three others, the ones we get by rotating the dish. This divides the arrangements of four flavors of ice cream into blocks of size 4. Thus we may arrange the ice cream we have chosen in the dish in $4!/4 = 6$ ways. We may choose the ice cream in $\binom{12}{4} = 495$ ways, and so we may choose it and put it into the dish in 2970 ways.

8. (interesting) In as many ways as you can, show that $\binom{n}{k}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{k}$. You can prove this by plugging in the formula for $\binom{n}{k}$ on both sides and cancelling stuff until you get the same thing on both sides. However a much more interesting proof is that the right hand side counts the number of ways to choose a k-element set form an n-element set and then choose an m-element set from what remains. The left hand side counts the number of ways to first chose a k-element subset from the n-element set and then choose an m-element subset from what remains. Thus in both cases you are counting the number of ways to choose an ordered pair consisting of an m-element subset and a disjoint k-element subset from an n-element set. You can also base a proof on the observation that $(x + y + x)^n = \sum_{k=0}^{n} \binom{n}{k}(x + y)^k z^{n-k}$ and $(x+y+z)^n = \sum_{m=0}^{n} \binom{n}{m} x^m (y+z)^{n-m}$ and asking for the coefficient of $x^m y^{n-m-k} z^k$. You do have to use the binomial theorem with an eye to the result you are looking for, however.

9. (interesting) A tennis club has $4n$ members. To specify a doubles match, we choose two teams of two people. In how many ways may we arrange the members into doubles matches so that each player is in one doubles match? In how many ways may we do it if we specify in addition who serves first on each team? We now have many methods for solving this problem. Perhaps the easiest is to list all $(4n)$ people and take them in groups of four for doubles matches, with the first two in a group of four as one team and the second two as another team. We note that interchanging the n blocks of 4 does not change the matches, nor does interchanging the two people on a team nor interchanging the two teams. Thus we have $(4n)!/n!2^{3n}$ ways to arrange the matches. If we are to say who serves first on each team, we might as well say it is the first of the two listed, so now we have $(4n)!/n!2^n$ ways to arrange the matches.

10. A town has n streetlights running along the north side of main street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green? We can think of first choosing the set of even size of poles to be painted green, and the painting the remaining poles red, white, and blue. We may do this in $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{n-2k}$ ways.

11. We have n identical ping-pong balls. In how many ways may we paint them red, white, blue, and green? We can line up the identical ping-pong balls and break them into four groups, those of each color, by inserting dividers. If we want to paint at least one in each color, we can choose three of the spaces between the balls in which to insert dividers, so we can paint them in $\binom{n-1}{k}$. But the problem didn't require us to use each color, so we can put two dividers adjacent to each other. Thus there are $n+1$ places where we can put the first divider (putting it before all the balls means we use no red, and putting it after all of them means we use no green. Now there are $n+2$ places where we can put the second divider, including before or after the first, and $n+3$ places where we can put the third divider. However if we interchange two dividers we still paint the balls before the first divider red, those between then next two white, and so on. Thus $3! = 6$ of these arrangements of balls and dividers correspond to the same paint job, so the number of ways to paint the balls is $\frac{(n+1)(n+2)(n+3)}{6} = \binom{n+3}{3}$. This suggests that another way to think of the problem is to consider $n+3$ slots in a row, and fill n of them with balls and 3 of them with dividers; since the balls are identical and the dividers might as well be identical, the number of ways to do this is the number of ways to choose the slots that get dividers.

12. We have n identical ping-pong balls. In how many ways may we paint them red, white, blue, and green if we use green paint on an even number of them? We first decide how many balls to paint green, then paint the remainder with the other three colors as in Problem 11 This gives us

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-2k+2}{2}$$

ways to paint the balls.

Chapter 2

Distribution Problems

2.1 The idea of a distribution

Many of the problems we solved in Chapter 1 may be thought of as problems of distributing objects (such as pieces of fruit or ping-pong balls) to recipients (such as children). Some of the ways of viewing counting problems as distribution problems are somewhat indirect. For example, in ⟨⟨Unresolved xref, reference "ping-pong"; check spelling or use "provisional" attribute⟩⟩ you probably noticed that the number of ways to pass out k ping-pong balls to n children so that no child gets more than one is the number of ways that we may choose a k-element subset of an n-element set. We think of the children as recipients and objects we are distributing as the identical ping-pong balls, distributed so that each recipient gets at most one ball. Those children who receive an object are in our set. It is helpful to have more than one way to think of solutions to problems. In the case of distribution problems, another popular model for distributions is to think of putting balls in boxes rather than distributing objects to recipients. Passing out identical objects is modeled by putting identical balls into boxes. Passing out distinct objects is modeled by putting distinct balls into boxes.

The twenty-fold way

When we are passing out objects to recipients, we may think of the objects as being either identical or distinct. We may also think of the recipients as being either identical (as in the case of putting fruit into plastic bags in the grocery store) or distinct (as in the case of passing fruit out to children). We may restrict the distributions to those that give at least one object to each recipient, or those that give exactly one object to each recipient, or those that give at most one object to each recipient, or we may have no such restrictions. If the objects are distinct, it may be that the order in which the objects are received is relevant (think about putting books onto the shelves in a bookcase) or that the order in which the objects are received is irrelevant (think about dropping a handful of candy into a child's trick or treat bag). If we ignore the possibility that the order in which objects are received matters, we have created $2 \cdot 2 \cdot 4 = 16$ distribution problems. In the cases where a recipient can receive more than one distinct object, we also have four more problems when the order objects are received matters. Thus we have 20 possible distribution problems.

The Twentyfold Way: A Table of Distribution Problems

k objects and conditions on how they are received	n recipients and mathem Distinct
1. Distinct no conditions	n^k functions
2. Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations
3. Distinct Each gets at least one	? onto functions
4. Distinct Each gets exactly one	$k! = n!$ bijections
5. Distinct, order matters	? ?
6. Distinct, order matters Each gets at least one	? ?
7. Identical no conditions	? ?
8. Identical Each gets at most one	$\binom{n}{k}$ subsets
9. Identical Each gets at least one	? ?
10. Identical Each gets exactly one	1 if $k = n$; 0 otherwise

Table 2.1.1: An incomplete table of the number of ways to distribute k objects to n recipients, with restrictions on how the objects are received

We describe these problems in Table 2.1.1. Since there are twenty possible distribution problems, we call the table the “Twentyfold Way," adapting terminology suggested by Joel Spencer for a more restricted class of distribution problems. In the first column of the table we state whether the objects are distinct (like people) or identical (like ping-pong balls) and then give any conditions on how the objects may be received. The conditions we consider are whether each recipient gets at most one object, whether each recipient gets at least one object, whether each recipient gets exactly one object, and whether the order in which the objects are received matters. In the second column we give the solution to the problem and the name of the mathematical model for this kind of distribution problem when the recipients are distinct, and in the third column we give the same information when the recipients are identical. We use question marks as the answers to problems we have not yet solved and models we have not yet studied. We give explicit answers to problems we solved in Chapter 1 and problems whose answers are immediate. The goal of this chapter is to develop methods that will allow us to fill in the table with formulas or at least quantities we know how to compute, and we will give a completed table at the end of the chapter. We will now justify the answers that are not question marks and replace some question marks with answers as we cover relevant material.

If we pass out k distinct objects (say pieces of fruit) to n distinct recipients (say children), we are saying for each object which recipient it goes to. Thus we are defining a function from the set of objects to the recipients. We saw the following theorem in ⟨⟨Unresolved xref, reference "numberoffunctionsconjecture"; check spelling or use "provisional" attribute⟩⟩.

Theorem 2.1.2. *There are n^k functions from a k-element set to an n-element set.*

We proved it in ⟨⟨Unresolved xref, reference "provenumberoffunctionsconjecture"; check spelling or use "provisional" attribute⟩⟩. If we pass out k distinct objects (say pieces of fruit) to n indistinguishable recipients (say identical paper bags) then we are dividing the objects up into disjoint sets; that is we are forming a partition of the objects into some number, certainly no more than the number k of objects, of parts. Later in this chapter (and again in the next chapter) we shall discuss how to compute the number of partitions of a k-element set into n parts. This explains the entries in row one of our table.

If we pass out k distinct objects to n recipients so that each gets at most one, we still determine a function, but the function must be one-to-one. The number of one-to-one functions from a k-element set to an n element set is the same as the number of one-to-one functions from the set $[k] = \{1, 2, \ldots, k\}$ to an n-element set. In ⟨⟨Unresolved xref, reference "kelementpermutation"; check spelling or use "provisional" attribute⟩⟩ we proved the following theorem.

Theorem 2.1.3. *If $0 \le k \le n$, then the number of k-element permutations of an n-element set is*

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1) = n!/(n-k)!.$$

If $k > n$ there are no one-to-one functions from a k element set to an n element, so we define $n^{\underline{k}}$ to be zero in this case. Notice that this is what the indicated product in the middle term of our formula gives us. If we are supposed to distribute k distinct objects to n identical recipients so that each gets at most one, we cannot do so if $k > n$, so there are 0 ways to do so. On the other hand, if $k \leq n$, then it doesn't matter which recipient gets which object, so there is only one way to do so. This explains the entries in row two of our table.

If we distribute k distinct objects to n distinct recipients so that each recipient gets at least one, then we are counting functions again, but this time functions from a k-element set *onto* an n-element set. At present we do not know how to compute the number of such functions, but we will discuss how to do so later in this chapter and in the next chapter. If we distribute k identical objects to n recipients, we are again simply partitioning the objects, but the condition that each recipient gets at least one means that we are partitioning the objects into exactly n blocks. Again, we will discuss how compute the number of ways of partitioning a set of k objects into n blocks later in this chapter. This explains the entries in row three of our table.

If we pass out k distinct objects to n recipients so that each gets exactly one, then $k = n$ and the function that our distribution gives us is a bijection. The number of bijections from an n-element set to an n-element set is $n!$ by Theorem 2.1.3. If we pass out k distinct objects of n identical recipients so that each gets exactly 1, then in this case it doesn't matter which recipient gets which object, so the number of ways to do so is 1 if $k = n$. If $k \neq n$, then the number of such distributions is zero. This explains the entries in row four of our table.

We now jump to row eight of our table. We saw in ⟨⟨Unresolved xref, reference "ping-pong"; check spelling or use "provisional" attribute⟩⟩ that the number of ways to pass out k identical ping-pong balls to n children is simply the number of k-element subsets of an n-element set. In ⟨⟨Unresolved xref, reference "formulanchoosek"; check spelling or use "provisional" attribute⟩⟩ we proved the following theorem.

Theorem 2.1.4. *If $0 \leq k \leq n$, the number of k-element subsets of an n-element set is given by*

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

We define $\binom{n}{k}$ to be 0 if $k > n$, because then there are no k-element subsets of an n-element set. Notice that this is what the middle term of the formula in the theorem gives us. This explains the entries of row 8 of our table. For now we jump over row 9.

In row 10 of our table, if we are passing out k identical objects to n recipients so that each gets exactly one, it doesn't matter whether the recipients are identical or not; there is only one way to pass out the objects if $k = n$ and otherwise it is impossible to make the distribution, so there are no ways of distributing the objects. This explains the entries of row 10 of our table. Several other rows of our table can be computed using the methods of Chapter 1.

Ordered functions

Activity 1.

(a) Suppose we wish to place k distinct books onto the shelves of a bookcase with n shelves. For simplicity, assume for now that all of the books would fit on any of the shelves. Also, let's imagine pushing the books on a shelf as far to the left as we can, so that we are only thinking about how the books sit relative to each other, not about the exact places where we put the books. Since the books are distinct, we can think of a the first book, the second book and so on. How many places are there where we can place the first book? When we place the second book, if we decide to place it on the shelf that already has a book, does it matter if we place it to the left or right of the book that is already there? How many places are there where we can place the second book? Once we have $i-1$ books placed, if we want to place book i on a shelf that already has some books, is sliding it in to the left of all the books already there different from placing it to the right of all the books already or between two books already there? In how many ways may we place the ith book into the bookcase? In how many ways may we place all the books?

Solution. There are n places where we can place the first book. Once we have placed it, there are $n+1$ places where we can place the second book, because on the shelf that has one book, we could put the second book to the left or to the right of the book already there. Once we have $i-1$ books on the shelves the ith book could go on any shelf to the left of all books there, if any, giving us n places, or it could go to the immediate right of any book already there, giving us another $i-1$ places. Thus there are $n+i-1$ places where we could place book i. From this, we can see that the number of ways to place all the books is

$$\prod_{i=1}^{k}(n+i-1).$$

(b) Suppose we wish to place the books in Problem 1 (satisfying the assumptions we made there) so that each shelf gets at least one book. Now in how many ways may we place the books? (Hint: how can you make sure that each shelf gets at least one book before you start the process described in Problem 1?)

Solution. Choose n books from the k books in $\binom{k}{n}$ ways, and assign them to the n places shelves in $n!$ ways, giving us $k!/(k-n)!$ ways to put a book on each shelf. Now leaving these books at the far left of each shelf, place the remaining books in

$$\prod_{i=1}^{k-n}(n+i-1) = \frac{(n+(k-n)-1)!}{(n-1)!} = \frac{(k-1)!}{(n-1)!}$$

ways. Thus we have

$$\frac{k!(k-1)!}{(k-n)!(n-1)!} = k!\binom{k-1}{n-1}$$

ways to place the books. Of course the right hand side of that equation cries out for a combinatorial explanation. Here it is. Imagine lining up the k books in a row. Then there are $k-1$ places in between them. Choose $n-1$ of these places, and slide a piece of paper in there as a divider. Now put the books before the first divider on shelf one, and the books after divider i on shelf $i+1$. This gives an arrangement of the books on the shelves so that every shelf has a book!

The assignment of which books go to which shelves of a bookcase is simply a function from the books to the shelves. But a function doesn't determine which book sits to the left of which others on the shelf, and this information is part of how the books are arranged on the shelves. In other words, the order in which the shelves receive their books matters. Our function must thus assign an ordered list of books to each shelf. We will call such a function an ordered function. More precisely, an *ordered function* from a set S to a set T is a function that assigns an (ordered) list of elements of S to some, but not necessarily all, elements of T in such a way that each element of S appears on one and only one of the lists.[1] (Notice that although it is not the usual definition of a function from S to T, a function can be described as an assignment of subsets of S to some, but not necessarily all, elements of T so that each element of S is in one and only one of these subsets.) Thus the number of ways to place the books into the bookcase is the entry in the middle column of row 5 of our table. If in addition we require each shelf to get at least one book, we are discussing the entry in the middle column of row 6 of our table. An *ordered onto function* is one which assigns a list to each element of T. In Problem 1 you showed that the number of ordered functions from a k-element set to an n-element set is $\prod_{i=1}^{n}(n+i-1)$. This product occurs frequently enough that it has a name; it is called the kth *rising factorial power* of n and is denoted by $n^{\overline{k}}$. It is read as "n to the k rising." (This notation is due to Don Knuth, who also suggested the notation for falling factorial powers.) We can summarize with a theorem that adds two more formulas for the number of ordered functions.

[1] The phrase ordered function is not a standard one, because there is as yet no standard name for the result of an ordered distribution problem.

Theorem 2.1.5. *The number of ordered functions from a k-element set to an n-element set is*

$$n^{\overline{k}} = \prod_{i=1}^{n}(n+i-1) = \frac{(n+i-1)!}{(n-1)!} = (n+k-1)^{\underline{k}}.$$

Broken permutations and Lah numbers

Activity 2. In how many ways may we stack k distinct books into n identical boxes so that there is a stack in every box? (Hint: Imagine taking a stack of k books, and breaking it up into stacks to put into the boxes in the same order they were originally stacked. If you are going to use n boxes, in how many places will you have to break the stack up into smaller stacks, and how many ways can you do this?) (Alternate hint: How many different bookcase arrangements correspond to the same way of stacking k books into n boxes so that each box has at least one book?). The hints may suggest that you can do this problem in more than one way!

Solution. We can make a list of the k distinct books in $k!$ ways. Then we have to choose $n-1$ of the $k-1$ places between the lists as the places where we will break the list. However the order in which we list the boxes is irrelevant, so we have equivalence classes of $n!$ arrangements for each way of putting the books into boxes. Thus we can put the books in boxes in $k!\binom{k-1}{n-1}/n!$ ways.

Alternately, we can take the number of ways to put k books onto n bookshelves so that each shelf gets at least one, and then divide by the number of shelves factorial. That gives us $k!\binom{k-1}{n-1}/n!$ ways to arrange the books.

We can think of stacking books into identical boxes as partitioning the books and then ordering the blocks of the partition. This turns out not to be a useful computational way of visualizing the problem because the number of ways to order the books in the various stacks depends on the sizes of the stacks and not just the number of stacks. However this way of thinking actually led to the first hint in Problem 2. Instead of dividing a set up into nonoverlapping parts, we may think of dividing a *permutation* (thought of as a list) of our k objects up into n ordered blocks. We will say that a set of ordered lists of elements of a set S is a *broken permutation* of S if each element of S is in one and only one of these lists.[2] The number of broken permutations of a k-element set with n blocks is denoted by $L(k,n)$. The number $L(k,n)$ is called a *Lah Number* and, from our solution to Problem 2, is equal to $k!\binom{k-1}{n-1}/n!$.

The Lah numbers are the solution to the question "In how many ways may we distribute k distinct objects to n identical recipients if order matters and each recipient must get at least one?" Thus they give the entry in row 6 and column 6 of our table. The entry in row 5 and column 6 of our table will be the number of broken permutations with less than or equal to n parts. Thus it is a sum of Lah numbers.

[2]The phrase broken permutation is not standard, because there is no standard name for the solution to this kind of distribution problem.

We have seen that ordered functions and broken permutations explain the entries in rows 5 and 6 of our table.

Compositions of integers

Activity 3. In how many ways may we put k identical books onto n shelves if each shelf must get at least one book?

Solution. In ⟨⟨Unresolved xref, reference "bookcaseeveryshelf"; check spelling or use "provisional" attribute⟩⟩ we showed that with k distinct books we could place the books in $k!\binom{k-1}{n-1}$ ways. We can partition these arangements of distinct books into blocks, where each block consists of all arrangements that we get just by permuting the books among themselves. Thus each block has $k!$ arrangements in it, and each arrangement corresponds to an arrangement of identical books. Thus there are $\binom{k-1}{n-1}$ ways to arrange identical books.

Activity 4. A *composition* of the integer k into n parts is a list of n positive integers that add to k. How many compositions are there of an integer k into n parts?

Solution. There is a bijection between compositions of k into n parts and arrangements of k identical books on n shelves so that each shelf gets a book. Namely, the number of books on shelf i is the ith element of the list. Thus the number of compositions of k into n parts is $\binom{k-1}{n-1}$.

Activity 5. Your answer in Problem 4 can be expressed as a binomial coefficient. This means it should be possible to interpret a composition as a subset of some set. Find a bijection between compositions of k into n parts and certain subsets of some set. Explain explicitly how to get the composition from the subset and the subset from the composition.

Solution. If we line up k identical books, there are $k-1$ places in between two books. If we choose $n-1$ of these places and slip dividers into those places, then we have a first clump of books, a second clump of books, and so on. The ith element of our list is the number of books in the ith clump. Clearly using books is irrelevant; we could line up any k identical objects and make the same argument. Our bijection is between compositions and $n-1$-element subsets of the set of $k-1$ spaces between our objects.

Activity 6. Explain the connection between compositions of k into n parts and the problem of distributing k identical objects to n recipients so that each recipient gets at least one.

Solution. Since the recipients are distinct, we can think of them as a first recipient, a second, and so on. Given a composition of k into n parts, let the ith element of the list be the number of objects given to recipient number i.

The sequence of problems you just completed should explain the entry in the middle column of row 9 of our table of distribution problems.

Multisets

In the middle column of row 7 of our table, we are asking for the number of ways to distribute k identical objects (say ping-pong balls) to n distinct recipients (say children).

Activity 7. In how many ways may we distribute k identical books on the shelves of a bookcase with n shelves, assuming that any shelf can hold all the books?

Solution. We saw that we could arrange k distinct books on n shelves in $\prod_{i=1}^{k}(n+i-1)$ ways. We partition these arrangements into blocks by putting two arrangements in the same block if we can get one from the other by permuting the books among themselves. Then the number of blocks is the number of ways to place identical books on the shelves. However, there are $k!$ arrangements per block, so there are

$$\frac{\prod_{i=1}^{k}(n+i-1)}{k!} = (n+k-1)^{\underline{k}} = \binom{n+k-1}{k}$$

ways to arrange identical books.

Activity 8. A *multiset* chosen from a set S may be thought of as a subset with repeated elements allowed. For example the multiset of letters of the word Mississippi is $\{i, i, i, i, m, p, p, s, s, s, s\}$. To determine a multiset we must say how many times (including, perhaps, zero) each member of S appears in the multiset. The number of times an element appears is called its *multiplicity*. The size of a multiset chosen from S is the total number of times any member of S appears. For example, the size of the multiset of letters of Mississippi is 11. What is the number of multisets of size k that can be chosen from an n-element set?

Solution. There is a bijection between arrangements of identical books on n shelves and multisets chosen from an n-element set: the multiplicity of element i is the number of books on shelf i. Thus we have $\binom{n+k-1}{k}$ ways to choose a k-element multiset from an n-element set by Problem 7.

Activity 9.

(a) Your answer in the previous problem should be expressible as a binomial coefficient. Since a binomial coefficient counts subsets, find a bijection between subsets of something and multisets chosen from a set S.

Solution. Note that $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$. We will show a bijection between ways of choosing $n-1$ things our of $n+k-1$ things and multisets. Namely, take $n+k-1$ objects and line them up in a row. Choose $n-1$ of them. Now let the multiplicity of element 1 of our multiset be the number of objects before the first thing we chose. if $1 < i < n$, let the multiplicity of element i of our multiset be the number of things between the $i-1$th thing we chose and the ith thing we choose. Let the multiplicity of the nth element of our multiset be the number of objects after the last one we chose. Another way to say the essentially same thing is to make a list of $n+k-1$ blank spaces. We choose k of them in which we put ones and $n-1$ of them in which we put plus signs. Then the multiplicity of element 1 is the number of ones before the first plus sign, the multiplicity of element n is the number of ones after the last plus sign and if $1 < i < n$, the multiplicity of element i is the number of ones between the $(i-1)$th plus sign and the ith plus sign.

(b) How many solutions are there in nonnegative integers to the equation $x_1 + x_2 + \cdots + x_m = r$, where m and r are constants?

Solution. We can think of x_i as the multiplicity of element i of a multiset chosen from among m things. The total number of elements of the multiset will be r. Thus we have $\binom{m+r-1}{r}$ solutions.

A more precise definition of a *multiset* chosen from a set S is that it is a function m, called a *multiplicity function*, from S to the nonnegative integers. For each x in S, $m(x)$ specifies how many times x appears in the multiset. In our example of the word Mississippi above, our set S can be taken to be the set of alphabet letters and the multiplicity function m is given by $m(i) = 4$, $m(m) = 1$, $m(p) = 2$, $m(s) = 4$, and m of any other member of S is 0. When we list the members of a multiset in braces, it will be clear from context that we are thinking of a multiset. However when we use braces in another way, it may not be clear what we mean. For example, when we write

$$\{x|x \text{ is a letter of Mississippi}\},$$

do we mean the set $\{i, m, p, s\}$ or the multiset $\{i, i, i, i, m, p, p, s, s, s, s\}$? To resolve this, whenever it is not clear from context whether we are talking about a set or multiset we will use the subscript multi on the right brace enclosing the multiset to distinguish a multiset. Thus we write

$$\{x|x \text{ is a letter of Mississippi}\}_{\text{multi}} = \{i, i, i, i, m, p, p, s, s, s, s\}.$$

In this case it is probably clear from the right-hand side of the equation that we are thinking of the left-hand side as a multiset, but we will always try to err in the direction of clarity rather than brevity.

The sequence of problems you just completed should explain the entry in the middle column of row 7 of our table of distribution problems. In the next two sections we will give ways of computing the remaining entries.

2.2 Partitions and Stirling Numbers

We have seen how the number of partitions of a set of k objects into n blocks corresponds to the distribution of k distinct objects to n identical recipients.

While there is a formula that we shall eventually learn for this number, it requires more machinery than we now have available. However there is a good method for computing this number that is similar to Pascal's equation. Now that we have studied recurrences in one variable, we will point out that Pascal's equation is in fact a *recurrence in two variables*; that is it lets us compute $\binom{n}{k}$ in terms of values of $\binom{m}{i}$ in which either $m < n$ or $i < k$ or both. It was the fact that we had such a recurrence and knew $\binom{n}{0}$ and $\binom{n}{n}$ that let us create Pascal's triangle.

Stirling Numbers of the second kind

We use the notation $S(k, n)$ to stand for the number of partitions of a k element set with n blocks. For historical reasons, $S(k, n)$ is called a *Stirling number of the second kind.*

Activity 10.

(a) In a partition of the set $[k]$, the number k is either in a block by itself, or it is not. How does the number of partitions of $[k]$ with n parts in which k is in a block with other elements of $[k]$ compare to the number of partitions of $[k-1]$ into n blocks? Find a two variable recurrence for $S(n, k)$, valid for n and k larger than one.

Solution. The number of partitions of $[k]$ into n parts in which k is in a block with other elements of $[k]$ is equal n times the number of partitions of $[k-1]$ into n blocks, because k could be in any of the n parts, and since it is in a block with other elements of $[k-1]$, removing it leaves a partition of $[k-1]$ into n blocks. The number of partitions of $[k]$ into n blocks in which k is in a block by itself is the number of partitions of $[k]$ into $n-1$ blocks, because you can get any such partition in by deleting the block containing k from a partition of $[k]$ in which k is in a block by itself. Thus $S(k, n)) = S(k-1, n-1) + nS(k-1, n)$.

(b) What is $S(k, 1)$? What is $S(k, k)$? Create a table of values of $S(k, n)$ for k between 1 and 5 and n between 1 and k. This table is sometimes called *Stirling's Triangle (of the second kind)* How would you define $S(k, n)$ for the nonnegative values of k and n that are not both positive? Now for what values of k and n is your two variable recurrence valid?

Solution. $S(k, 1) = 1$ and $S(k, k) = 1$.

$k\backslash n$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	1	3	1	0	0	0
4	0	1	7	6	1	0	0
5	0	1	15	25	10	1	0

As you see in the table, we define $S(0,0)=1$, and $S(0,n)$ or $S(k,0)$ to be 0 otherwise. This makes sense because for $n>0$ there is no partition of an empty set into n parts, and for $k>0$ there is no partition of a k-element set into no parts, but saying there is one partition of the empty set into no parts allows us to use our recurrence to compute $S(1,1)$. This makes our recurrence valid for all nonnegative values of k and n.

(c) Extend Stirling's triangle enough to allow you to answer the following question and answer it. (Don't fill in the rows all the way; the work becomes quite tedious if you do. Only fill in what you need to answer this question.) A caterer is preparing three bag lunches for hikers. The caterer has nine different sandwiches. In how many ways can these nine sandwiches be distributed into three identical lunch bags so that each bag gets at least one?

Solution. We need $S(9,3)$. Thus we need to extend our table for four more rows, but only out to the column labelled 3. These rows are 6,0,1,31,90, 7,0,1,63,301, 7,0,1,127,966, 8,0,1,255,3025, 9,0,1,511,9330. Thus there are 9330 ways to distribute the sandwiches into the lunch bags.

(d) The question in ⟨⟨Unresolved xref, reference "sandwiches"; check spelling or use "provisional" attribute⟩⟩ naturally suggests a more realistic question; in how many ways may the caterer distribute the nine sandwich's into three identical bags so that each bag gets exactly three? Answer this question. (Hint, what if the question asked about six sandwiches and two distinct bags? How does having identical bags change the answer?)

Solution. $\binom{9}{3}\binom{6}{3}\binom{3}{3}/3!$. First we choose three sandwiches for bag 1, then three for bag 2, and put the remainder in bag 3. However, it doesn't matter which bags the sandwiches are in so we have counted each partition 3! times.

Activity 11.

(a) In how many ways can we partition k items into n blocks so that we have k_i blocks of size i for each i? (Notice that $\sum_{i=1}^{k} k_i = n$ and $\sum_{i=1}^{k} ik_i = k$.) The sequence $k_1, k_2, \ldots, k_n$ is called the *type vector* of the partition.

Solution. $\frac{n!}{\prod_{i=1}^{n}(i!)^{k_i}k_i!}$. We can make a list in $n!$ ways, and then break it into first k_1 blocks of size 1, then k_2 blocks of size 2, k_3 blocks of size 3 up to k_n blocks of size n. But then we realize that we get the same partition if we permute the $i!$ elements of a block of size i and we get the same partition if we permute the k_i blocks of size i so we apply the quotient principle.

(b) Describe how to compute $S(n,k)$ in terms of quantities given by the formula you found in Problem 11.

Solution. We can find $S(n,k)$ by summing $\frac{n!}{\prod_{i=1}^{n}(i!)^{k_i}k_i!}$ over all type vectors $(k_1,k_2,\ldots,k_n)$ such that $k_1+k_2+\cdots+k_n=k$.

Activity 12. Find a recurrence similar to the one in Problem 10 for the Lah numbers $L(k,n)$.

Solution. $L(k,n)$ is the number of broken permutations of a k-element set into n parts. Either k is in an ordered block by itself or it is not. If it is, it can go after any of the $k-1$ other elements, or it can go at the beginning of any of the n blocks. If it is not, deleting it gives a broken permutation of a $k-1$-element set into $n-1$ blocks. Thus $L(k,n)=L(k-1,n-1)+(n+k-1)L(k,n)$.

Activity 13. (Relevant in ⟨⟨Unresolved xref, reference "expogenfun"; check spelling or use "provisional" attribute⟩⟩.) The total number of partitions of a k-element set is denoted by $B(k)$ and is called the k-th *Bell number*. Thus $B(1)=1$ and $B(2)=2$.

(a) Show, by explicitly exhibiting the partitions, that $B(3)=5$.

Solution. The five partitions of [3] are the sets $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1\},\{2,3\}\}$, and $\{\{1,2,3\}\}$.

(b) Find a recurrence that expresses $B(k)$ in terms of $B(n)$ for $n<k$ and prove your formula correct in as many ways as you can.

Solution. If we delete the block containing k, we get a partition of a subset of $[k-1]$. Thus $B(k)$ is the sum over all subsets of $[k-1]$ of the number of partitions of that subset. This gives us $B(k)=\sum_{n=0}^{k-1}\binom{k-1}{n}B(n)$.
Alternatively, we can show by the same sort of argument that $S(k,n)=\sum_{i=0}^{k-1}\binom{k-1}{i}S(i,n-1)$ and then use the fact that $B(k)=\sum_{n=0}^{k}S(k,n)$ to get the recurrence for $B(k)$.

(c) Find $B(k)$ for $k=4,5,6$.

Solution.

$$B(4)=\binom{3}{0}B_0+\binom{3}{1}B_1+\binom{3}{2}B_2+\binom{3}{3}B_3=1+3+3\cdot 2+5=15$$

$$B(5)=\sum_{n=0}^{4}\binom{4}{n}B_n=1+4+6\cdot 2+4\cdot 5+15=52$$

$$B(6)=\sum_{n=0}^{5}\binom{5}{n}B_n=1+5+10\cdot 2+10\cdot 5+5\cdot 15+52=203$$

Stirling Numbers and onto functions

Activity 14. Given a function f from a k-element set K to an n-element set, we can define a partition of K by putting x and y in the same block of the partition if and only if $f(x) = f(y)$. How many blocks does the partition have if f is onto? How is the number of functions from a k-element set onto an n-element set related to a Stirling number? Be as precise in your answer as you can.

Solution. If f is onto, the number of blocks of the partition is n. The number of onto functions from a k-element set onto an n-element set is $S(k,n)n!$, because we have a one-to-one function from the blocks to the n-element set.

Activity 15. Each function from a k-element set K to an n-element set N is a function from K onto *some* subset of N. If J is a subset of N of size j, you know how to compute the number of functions that map onto J in terms of Stirling numbers. Suppose you add the number of functions mapping onto J over all possible subsets J of N. What simple value should this sum equal? Write the equation this gives you.

Solution. The sum should equal the number of functions, n^k. Thus we get $\sum_{j=0}^{n} \binom{n}{j} S(k,j)j! = n^k$. By using the fact that $\binom{n}{j} = n^{\underline{j}}/j!$, this may be rewritten as $\sum_{j=0}^{n} n^{\underline{j}} S(k,j) = n^k$.

Activity 16. In how many ways can the sandwiches of ⟨⟨Unresolved xref, reference "sandwiches"; check spelling or use "provisional" attribute⟩⟩ be placed into three distinct bags so that each bag gets at least one?

Solution. $S(9,3) \cdot 3! = 55,980$.

Activity 17. In how many ways can the sandwiches of ⟨⟨Unresolved xref, reference "caterer2"; check spelling or use "provisional" attribute⟩⟩ be placed into distinct bags so that each bag gets exactly three?

Solution. Choose three sandwiches for bag one in $\binom{9}{3}$ ways, three for bag two in $\binom{6}{3}$ ways and put the remainder in bag 3. This gives us $\binom{9}{3}\binom{6}{3} = \frac{9!}{3!3!3!} = 1680$ ways.

The $\frac{9!}{3!3!3!}$ suggests another solution. We can line up the sandwiches in 9! ways. We take the first three for bag one, the second three for bag two and the last three for bag 3. The order of the sandwiches in the bag does not matter though, so each there are 3!3!3! listings corresponding to each way of putting sandwiches in bags, giving us $\frac{9!}{3!3!3!}$ ways to put the sandwiches in bags.

Activity 18.

(a) How many functions are there from a k-element set K to a set $N = \{y_1, y_2, \ldots y_n\}$ so that y_i is the image of j_i elements of K for each i from 1 to n? This number is called a *multinomial coefficient* and denoted by

$$\binom{k}{j_1, j_2, \ldots, j_n}.$$

Solution. $\binom{k}{j_1,j_2,\ldots,j_n} = \frac{k!}{j_1!j_2!\cdots j_n!}$. We get this either as the product of binomial coefficients

$$\binom{k}{j_1}\binom{k-j_1}{j_2}\binom{k-j_1-j_2}{j_3}\cdots\binom{j_n}{j_n},$$

or more elegantly, by lining up the elements of the domain in $k!$ ways, taking the first j_1 elements to y_1, the next j_2 elements to y_2 and so on. However the order of the j_i elements that go to y_i is irrelevant, so $j_1!j_2!\cdots j_n!$ lists all correspond to the same function, giving us $\frac{k!}{j_1!j_2!,\cdots j_n!}$ functions.

(b) Explain how to compute the number of functions from a k-element set K onto an n-element set N by using multinomial coefficients.

Solution. Add the multinomial coefficients $\binom{k}{j_1,j_2,\ldots,j_n}$ in which each j_i is different from zero. To see why, let $N = \{y_1, y_2, \ldots, y_n\}$ and note that we are counting functions that send at least one element of K to each element y_i.

Activity 19. What do multinomial coefficients have to do with expanding the kth power of a multinomial $x_1 + x_2 + \cdots + x_n$? This result is called the *multinomial theorem.*

Solution. When we use the distributive law to multiply out $(x_1 + x_2 + \cdots + x_n)^k$, we will get a sum of a bunch of terms of the form $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ where $i_1 + i_2 + \cdots + i_n = k$. The terms with a given sequence $i_1, i_2, \ldots, i_n$ of exponents will arise from choosing, as we apply the distributive law over and over again, x_1 from i_1 of the factors, x_2 from i_2 of the factors, and so on. Thus the number of terms $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ will be the number of ways to label i_1 of the factors with a 1, i_2 of the factors with a 2, …, and i_n of the factors with an n. The number of ways to do this is a multinomial coefficient, as we now explain. This labeling gives us a function from $[k]$ to $[n]$ as follows. If factor i is labelled j we let $f(i) = j$. Further each function f from $[k]$ to $[n]$ gives us that maps i_j elements of $[k]$ to j will give us such a labelling. Thus the coefficient of $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ will be the multinomial coefficient $\binom{k}{i_1,i_2,\ldots,i_n}$.

Stirling Numbers and bases for polynomials

Activity 20. Find a way to express n^k in terms of $k^{\underline{j}}$ for appropriate values j. You may use Stirling numbers if they help you. Notice that $x^{\underline{j}}$ makes sense for a numerical variable x (that could range over the rational numbers, the real numbers, or even the complex numbers instead of only the nonnegative integers, as we are implicitly assuming n does), just as x^j does. Find a way to express the power x^k in terms of the polynomials $x^{\underline{j}}$ for appropriate values of j and explain why your formula is correct.

Solution. In Problem 15, we saw that $\sum_{j=0}^{n}\binom{n}{j}S(k,j)j! = n^k$. Using the relationship between binomial coefficients and falling factorials, we may rewrite this as $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = n^k$. This expresses n^k in terms of $k^{\underline{j}}$. To be precise, we define $x^{\underline{j}}$ to be $x(x-1)\cdots(x-j+1)$. At first glance it looks like we could express $x^{\underline{j}}$ in terms of powers of x by simply substituting x for n in the equation $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = n^k$. However this gives us $\sum_{j=0}^{x} x^{\underline{j}}S(k,j) = x^k$, and we have never defined what we mean by a sum whose upper limit is a variable x. Thus we need to examine the equation $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = n^k$ to see if we can replace the n that is the upper limit of the sum with something else. Notice that $S(k,j) = 0$ when $j > k$. This means that if $k \le j$, then $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = \sum_{j=0}^{k} n^{\underline{j}}S(k,j)$. Notice also that $n^{\underline{j}} = n(n-1)\cdots(n-j+1)$ is zero when $j > n$ because one of its factors is zero then. This implies that if $k > j$, then $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = \sum_{j=0}^{k} n^{\underline{j}}S(k,j)$. Therefore, regardless of the relative size of k and n, we have that $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = \sum_{j=0}^{k} n^{\underline{j}}S(k,j)$. Therefore

$$\sum_{j=0}^{k} n^{\underline{j}}S(k,j) = n^k. \tag{2.1}$$

It makes sense to write the polynomial $\sum_{j=0}^{k} x^{\underline{j}}S(k,j)$; this is simply a polynomial of degree k in the variable x. The expression $\sum_{j=0}^{k} x^{\underline{j}}S(k,j) - x^k$ is also a polynomial in x, but it might not be of degree k since we are subtracting a degree k term from a degree k polynomial. In fact for every positive integer value n of x, this polynomial is zero. That is, $\sum_{j=0}^{k} n^{\underline{j}}S(k,j) - n^k = 0$, which is just a restatement of Equation (2.1). But it is a fact of algebra that the number of solutions of a nontrivial polynomial equation is no more than the degree of the polynomial. Since the polynomial equation $\sum_{j=0}^{k} x^{\underline{j}}S(k,j) - x^k$ has infinitely many different solutions, it must be a trivial equation; that is $\sum_{j=0}^{k} x^{\underline{j}}S(k,j) - x^k$ must be zero for every real (and even every complex) number x. Thus $\sum_{j=0}^{k} x^{\underline{j}}S(k,j) = x^k$, and we have expressed x^k in terms of $x^{\underline{j}}$ for $j \le k$.

You showed in Problem 20 how to get each power of x in terms of the falling factorial powers $x^{\underline{j}}$. Therefore every polynomial in x is expressible in terms of a sum of numerical multiples of falling factorial powers. Using the language of linear algebra, we say that the ordinary powers of x and the falling factorial powers of x each form a basis for the "space" of polynomials, and that the numbers $S(k,n)$ are "change of basis coefficients." If you are not familiar with linear algebra, a *basis* for the *space of polynomials*[3] is a set of polynomials such that each polynomial, whether in that set or not, can be expressed in one and only one way as a sum of numerical multiples of polynomials in the set.

Activity 21. Show that every power of $x+1$ is expressible as a sum of numerical multiples of powers of x. Now show that every power of x (and

[3]The space of polynomials is just another name for the set of all polynomials.

thus every polynomial in x) is a sum of numerical multiples (some of which could be negative) of powers of $x+1$. This means that the powers of $x+1$ are a basis for the space of polynomials as well. Describe the change of basis coefficients that we use to express the binomial powers $(x+1)^n$ in terms of the ordinary x^j explicitly. Find the change of basis coefficients we use to express the ordinary powers x^n in terms of the binomial powers $(x+1)^k$.

Solution. We know that

$$(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \tag{2.2}$$

from the binomial theorem. (In the way we stated the binomial theorem, instead of $\binom{n}{i}x^i$ we would have gotten $\binom{n}{i}x^{n-i}$. There are two ways to fix this. One is to observe that the coefficient of x^i in that expansion is $\binom{n}{n-i}$, which equals $\binom{n}{i}$. The other is to observe that when we expand $(1+x)^n$ according to the binomial theorem we get exactly what we wrote on the right hand side in Equation (2.2).) Therefore every power of $x+1$ is expressible in terms of powers of x.

How do we express powers of x in terms of powers of $x+1$? Some experimentation would help us guess how to do so; however there is a really nice trick that also isn't hard to see. Namely, we can write

$$\begin{aligned} x^n = (x+1-1)^n = [(x+1)-1]^n = & \sum_{i=0}^{n} \binom{n}{i} (x+1)^{n-i}(-1)^i \\ = & \sum_{i=0}^{n} \binom{n}{i} (x+1)^{i}(-1)^{n-i} \end{aligned}$$

This means that every power of x is expressible in terms of powers of $x+1$ and the change of basis coefficients to express powers of x in terms of powers of $x+1$ are $(-1)^{n-i}\binom{n}{i}$ while the change of basis coefficients used to express powers of $x+1$ in terms of powers of x are $\binom{n}{i}$.

Activity 22.

(a) By multiplication, we can see that every falling factorial polynomial can be expressed as a sum of numerical multiples of powers of x. In symbols, this means that there are numbers $s(k,n)$ (notice that this s is lower case, not upper case) such that we may write $x^{\underline{k}} = \sum_{n=0}^{k} s(k,n)x^n$. These numbers $s(k,n)$ are called Stirling Numbers of the first kind. By thinking algebraically about what the formula

$$x^{\underline{k}} = x^{\underline{k-1}}(x-k+1) \tag{2.3}$$

means, we can find a recurrence for Stirling numbers of the first kind that gives us another triangular array of numbers called Stirling's triangle of the first kind. Explain why Equation (2.3) is true and use it to derive a recurrence for $s(k,n)$ in terms of $s(k-1,n-1)$ and $s(k-1,n)$.

Solution. Equation (2.3) is effectively the inductive step of an inductive definition of $x^{\underline{k}}$. With this equation we can write

$$\begin{aligned}
\sum_{n=0}^{k} s(k,n)x^n &= x^{\underline{k}} = x^{\underline{k-1}}(x-k+1) \\
&= \left(\sum_{n=0}^{k-1} s(k-1,n)x^n\right)(x-k+1) \\
&= \sum_{n=0}^{k-1} s(k-1,n)x^{n+1} - \sum_{n=0}^{k-1}(k-1)s(k-1,n)x^n \\
&= \sum_{n=1}^{k} s(k-1,n-1)x^n - \sum_{n=0}^{k-1}(k-1)s(k-1,n)x^n.
\end{aligned}$$

Equating the coefficients of x^n in the first and last line of this equation, we get $s(k,n) = s(k-1,n-1) - (k-1)s(k-1,n)$, for n between 1 and $k-1$..

(b) Write down the rows of Stirling's triangle of the first kind for $k = 0$ to 6.

Solution.

$k\backslash n$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	-1	1	0	0	0	0
3	0	2	-3	1	0	0	0
4	0	-6	11	-6	1	0	0
5	0	24	-50	35	-10	1	0
6	0	-120	274	-225	85	-15	1

By definition, the Stirling numbers of the first kind are also change of basis coefficients. The Stirling numbers of the first and second kind are change of basis coefficients from the falling factorial powers of x to the ordinary factorial powers, and vice versa.

Activity 23. Explain why every rising factorial polynomial $x^{\overline{k}}$ can be expressed in terms of the falling factorial polynomials $x^{\underline{n}}$. Let $b(k,n)$ stand for the change of basis coefficients that allow us to express $x^{\overline{k}}$ in terms of the falling factorial polynomials $x^{\underline{n}}$; that is, define $b(k,n)$ by the equations

$$x^{\overline{k}} = \sum_{n=0}^{k} b(k,n)x^{\underline{n}}.$$

(a) Find a recurrence for $b(k,n)$.

Solution.

$$
\begin{aligned}
\sum_{n=0}^{k} b(k,n)x^{\underline{n}} = x^{\overline{k}} &= x^{\overline{k-1}}(x+k-1)\\
&= \left(\sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n}}\right)(x+k-1)\\
&= \sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n}}(x+k-1)\\
&= \sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n}}(x-n+n+k-1)\\
&= \sum_{n=0}^{k-1} b(k-1,n)x^{\underline{n+1}} + (n+k-1)b(k-1,n)x^{\underline{n}}\\
&= \sum_{n=1}^{k} b(k-1,n-1)x^{\underline{n}} + \sum_{n=0}^{k-1}(n+k-1)b(k-1,n)x^{\underline{n}}
\end{aligned}
$$

Thus if n is not 0 or k, we equate the coefficient of $x^{\underline{n}}$ in the first line and last line to get

$$b(k,n) = b(k-1,n-1) + (n+k-1)b(k-1,n).$$

The trick of subtracting n and adding n in the middle of the computation was the result of wanting to mimic the way in which we increased the power on x in the solution to Problem 22.

(b) Find a formula for $b(k,n)$ and prove the correctness of what you say in as many ways as you can.

Solution. We will answer the next part of the problem here! The recurrence for $b(k,n)$ is exactly the same as the recurrence for $L(k,n)$. Further, $b(0,0) = 1 = L(0,0)$, $b(0,n) = 0 = L(0,n)$ for $n > 0$, and $b(k,k) = L(k,k) = 1$. Thus $b(k,n)$ and $L(k,n)$ are identical. This and the formula from Problem 2 gives one proof that $b(k,n) = k!\binom{k-1}{n-1}/n!$.

A second proof that the change of basis coefficients are Lah numbers goes as follows. $n^{\overline{k}}$ counts the number of ordered functions from a k-element set to an n-element set. One way to determine such an ordered function is to take a broken permutation of the k-element set into n or fewer parts, and then take a one-to-one function from the parts to the n-element set. More informally we assign the parts of the broken permutation to distinct elements of the n-element set. If the broken permutation has i parts, the number of ways to do this assignment is the number of i-element permutations of an n-element set, $n^{\underline{i}}$. Thus $n^{\overline{k}} = \sum_{i=0}^{n} L(k,i)n^{\underline{i}}$. However we can change the upper limit of the sum to k because $L(k,i)$ is zero when $i > k$ and $n^{\underline{i}}$ is zero when $i > n$. Now we change n to x because we have a polynomial equality which is valid for infinitely many of the values of the variable. This gives us $x^{\overline{k}} = \sum_{i=0}^{k} L(k,i)x^{\underline{i}}$. Thus $b(k,i) = L(k,i)$.

(c) Is $b(k,n)$ the same as any of the other families of numbers (binomial coefficients, Bell numbers, Stirling numbers, Lah numbers, etc.) we have studied?

Solution. As we said in our solution to the previous part, $b(k,n)$ is the Lah number $L(k,n)$.

(d) Say as much as you can (but say it precisely) about the change of basis coefficients for expressing $x^{\underline{k}}$ in terms of $x^{\overline{n}}$.

Solution. There are several ways of finding this relationship, but the most concise way is to observe that $(-x)^{\underline{k}} = (-1)^k x^{\overline{k}}$ and $(-x)^{\overline{k}} = (-1)^k x^{\underline{k}}$. This lets us write

$$\begin{aligned}
(-x)^{\overline{k}} &= \sum_{n=0}^{k} b(k,n)(-x)^{\underline{n}} \\
(-1)^k x^{\underline{k}} &= \sum_{n=0}^{k} (-1)^n b(k,n)x^{\overline{n}} \\
x^{\underline{k}} &= \sum_{n=0}^{k} (-1)^{n-k} b(k,n)x^{\overline{n}}.
\end{aligned}$$

Therefore the change of basis coefficients are $(-1)^{n-k}b(k,n)$.

2.3 Partitions of Integers

We have now completed all our distribution problems except for those in which both the objects and the recipients are identical. For example, we might be putting identical apples into identical paper bags. In this case all that matters is how many bags get one apple (how many recipients get one object), how many get two, how many get three, and so on. Thus for each bag we have a number, and the multiset of numbers of apples in the various bags is what determines our distribution of apples into identical bags. A multiset of positive integers that add to n is called a *partition* of n. Thus the partitions of 3 are 1+1+1, 1+2 (which is the same as 2+1) and 3. The number of partitions of k

is denoted by $P(k)$; in computing the partitions of 3 we showed that $P(3) = 3$. It is traditional to use Greek letters like λ (the Greek letter λ is pronounced LAMB duh) to stand for partitons; we might write $\lambda = 1, 1, 1$, $\gamma = 2, 1$ and $\tau = 3$ to stand for the three partitions we just described. We also write $\lambda = 1^3$ as a shorthand for $\lambda = 1, 1, 1$, and we write $\lambda \dashv 3$ as a shorthand for "λ is a partition of three."

Activity 24. Find all partitions of 4 and find all partitions of 5, thereby computing $P(4)$ and $P(5)$.

Solution. $4 = 1 + 1 + 1 + 1$, $4 = 2 + 1 + 1$, $4 = 2 + 1$, $4 = 3 + 1$, $4 = 4$, so that $P(4) = 5$. $5 = 1 + 1 + 1 + 1 + 1$, $5 = 2 + 1 + 1 + 1$, $5 = 2 + 2 + 1$, $5 = 3 + 1 + 1$, $5 = 3 + 2$, $5 = 4 + 1$, $5 = 5$, so that $P(5) = 7$.

The number of partitions of k into n parts

A *partition of the integer k into n parts* is a multiset of n positive integers that add to k. We use $P(k, n)$ to denote the number of partitions of k into n parts. Thus $P(k, n)$ is the number of ways to distribute k identical objects to n identical recipients so that each gets at least one.

Activity 25. Find $P(6, 3)$ by finding all partitions of 6 into 3 parts. What does this say about the number of ways to put six identical apples into three identical bags so that each bag has at least one apple?

Solution. $6 = 4 + 1 + 1$, $6 = 3 + 2 + 1$, $6 = 2 + 2 + 2$, so $P(6, 3) = 3$. This says there are three ways to put six identical apples into three identical bags so that each bag gets at least one apple.

Representations of partitions

Activity 26.

(a) How many solutions are there in the positive integers to the equation $x_1 + x_2 + x_3 = 7$ with $x_1 \geq x_2 \geq x_3$?

Solution. This problem is asking for $P(7, 3)$ and suggests an organized way to go about finding it: list the partitions starting with the largest part and work down. $7 = 5 + 1 + 1$, $7 = 4 + 2 + 1$, $7 = 3 + 3 + 1$, $7 = 3 + 2 + 2$, and if we have three numbers that add to seven, one must be larger than two, so there are four such solutions.

(b) Explain the relationship between partitions of k into n parts and lists $x_1, x_2, \ldots, x_n$ of positive integers with $x_1 \geq x_2 \geq \ldots \geq x_n$. Such a representation of a partition is called a *decreasing list* representation of the partition.

Solution. There is a bijection between partitions of k into n parts and lists, in nonincreasing order, of n positive integers that add to k, because each multiset of numbers that adds to k can be listed in nonincreasing order in exactly one way.

Activity 27.

(a) Describe the relationship between partitions of k and lists or vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + 2x_2 + \ldots kx_k = k$. Such a representation of a partition is called a *type vector* representation of a partition, and it is typical to leave the trailing zeros out of such a representation; for example $(2, 1)$ stands for the same partition as $(2, 1, 0, 0)$. What is the decreasing list representation for this partition, and what number does it partition?

Solution. The type vector of a partition of k is a way of representing the multiplicity function of the multiset of integers that adds to k. Thus there is a bijection between type vectors and partitions.

(b) How does the number of partitions of k relate to the number of partitions of $k + 1$ whose smallest part is one?

Solution. They are equal, because if we take two different partitions of $k - 1$ and increase the multiplicity of 1 in each (by one), they are still different; also if we take two different partitions of k that have parts of size one, and decrease the multiplicity of 1 in each (by one), they are still different.

When we write a partition as $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$, it is customary to write the list of λ_is as a decreasing list. When we have a type vector $(t_1, t_2, \ldots, t_m)$ for a partition, we write either $\lambda = 1^{t_1}2^{t_2} \cdots m^{t_m}$ or $\lambda = m^{t_m}(m-1)^{t_{m-1}} \cdots 2^{t_2}1^{t_1}$. Henceforth we will use the second of these. When we write $\lambda = \lambda_1^{i_1}\lambda_2^{i_2} \cdots \lambda_n^{i_n}$, we will assume that $\lambda_i > \lambda_i + 1$.

Ferrers and Young Diagrams and the conjugate of a partition

The decreasing list representation of partitions leads us to a handy way to visualize partitions. Given a decreasing list $(\lambda_1, \lambda_2, \ldots \lambda_n)$, we draw a figure made up of rows of dots that has λ_1 equally spaced dots in the first row, λ_2 equally spaced dots in the second row, starting out right below the beginning of the first row and so on. Equivalently, instead of dots, we may use identical squares, drawn so that a square touches each one to its immediate right or immediately below it along an edge. See Figure 2.3.1 for examples. The figure we draw with dots is called the Ferrers diagram of the partition; sometimes the figure with squares is also called a Ferrers diagram; sometimes it is called a Young diagram. At this stage it is irrelevant which name we choose and which kind of figure we draw; in more advanced work the squares are handy because we can put things like numbers or variables into them. From now on we will use squares and call the diagrams Young diagrams.

Figure 2.3.1: The Ferrers and Young diagrams of the partition (5,3,3,2)

Activity 28. Draw the Young diagram of the partition (4,4,3,1,1). Describe the geometric relationship between the Young diagram of (5,3,3,2) and the Young diagram of (4,4,3,1,1).

Solution. We get the Young diagram of $(5, 3, 3, 2)$ by flipping the Young diagram of $(4, 4, 3, 1, 1)$ around a line that includes the diagonal of the upper left box; if we think of the top left corner of the diagram as being at the origin, we flip around the line $y = -x$.

Activity 29. The partition $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is called the *conjugate* of the partition $(\gamma_1, \gamma_2, \ldots, \gamma_m)$ if we obtain the Young diagram of one from the Young diagram of the other by flipping one around the line with slope -1 that extends the diagonal of the top left square. See Figure 2.3.2 for an example.

Figure 2.3.2: The Ferrers diagram the partition (5,3,3,2) and its conjugate.

What is the conjugate of (4,4,3,1,1)? How is the largest part of a partition related to the number of parts of its conjugate? What does this tell you about the number of partitions of a positive integer k with largest part m?

Solution. $(5, 3, 3, 2)$. The largest part of a partition equals the number of parts of its conjugate. The number of partitions of k with largest part m equals the number of partitions of k with m parts.

Activity 30.

(a) A partition is called *self-conjugate* if it is equal to its conjugate. Find a relationship between the number of self-conjugate partitions of k and the number of partitions of k into distinct odd parts.

Solution. The number of self-conjugate partitons of k equals the number of partitions of k with distinct odd parts. Here is a geometric description of a bijection from self conjugate partitions of k to partitions into distinct odd parts. Take the top row and left column of squares of the Young diagram, and make them into one row in a new diagram. (Only include the square that is in both the row and column once.) Now take the remaining squares in the next row and column and make a new row of the Young diagram of the second partition with them. Continue this process with succeeding rows and columns, not using any squares you have already used. Because the first partition is self conjugate, the diagram has the same number of rows as columns and row i and column i have the same length. Because row i and column i share one square, and we only use that square once when we create a new row, each row we create has odd length. Thus we get a partition with the same number of squares, so it is a partition of k and each part is odd. The parts are distinct because when we take off the squares of a row and column, we reduce the number of squares in each row and column that remains. Given a partition of k into distinct odd parts, we use the fact that each row has a unique middle element, and each is shorter than the one above (by at least two squares) to reverse the process. Thus we have a bijection.

(b) Explain the relationship between the number of partitions of k into even parts and the number of partitions of k into parts of even multiplicity, i.e. parts which are each used an even number of times as in (3,3,3,3,2,2,1,1).

Solution. The number of partitions of k into even parts equals the number of partitions of parts of even multiplicity, because if we take the Young diagram of a partition of k into even parts and conjugate it, the resulting diagram has columns of even length. Thus the difference in heights of two successive columns is an even number, but this difference is the multiplicity of one of the parts of the conjugate. Further the height of the last column of a partition is the multiplicity of the first part. Since the multiplicity of any part of a partition is either the difference in height of two successive columns of the Young diagram or the height of the last column, then each part of the conjugate has even multiplicity. This bijection can be reversed, because if all the differences in height of the columns are even and the height of the last column is even, then when we conjugate this partition, the last row will be an even length, and all differences in length of the rows will be even, so all the parts of the resulting partition will be even.

Activity 31.

(a) Show that the number of partitions of k into 4 parts equals the number of partitions of $3k$ (or $3k + 4$ or $3k - 4$) into 4 parts.

Solution. Think about putting the Young diagram of the partition into the upper left corner of a rectangle that is k units wide and four units high. Subdivide the rectangle into $4k$ squares of unit area. The Young diagram covers k of these squares. The uncovered squares are in rows of length $r_1 \leq r_2 \leq r_3 \leq r_4$. Thus if we list these lengths in the opposite order, we have a decreasing list representation of a partition of $3k$. Even r_1 will have to be positive, because the first part of the original partition will be at most $k-3$. To get partitions of $3k+4$, use a rectangle of width $k + 1$, and to get partitions of $3k - 4$, use a rectangle of width $k - 1$. Since the first row of the Young diagram has at most $k - 3$ squares, we will still have four nonzero parts in the partition that results.

(b) The idea of conjugation of a partition could be defined without the geometric interpretation of a Young diagram, but it would seem far less natural without the geometric interpretation. Another idea that seems much more natural in a geometric context is this. Suppose we have a partition of k into n parts with largest part m. Then the Young diagram of the partition can fit into a rectangle that is m or more units wide (horizontally) and n or more units deep. Suppose we place the Young diagram of our partition in the top left-hand corner of an m' unit wide and n' unit deep rectangle with $m' \geq m$ and $n' \geq n$, as in Figure 2.3.3.

Figure 2.3.3: To complement the partition (5,3,3,2) in a 6 by 5 rectangle: enclose it in the rectangle, rotate, and cut out the original Young diagram.

Why can we interpret the part of the rectangle not occupied by our Young diagram, rotated in the plane, as the Young diagram of another partition? This is called the *complement* of our partition in the rectangle. What integer is being partitioned by the complement? What conditions on m' and n' guarantee that the complement has the same number of parts as the original one? What conditions on m' and n' guarantee that the complement has the same largest part as the original one? Is it possible for the complement to have both the same number of parts and the same largest part as the original one? If we complement a partition in an m' by n' box and then complement that partition in an m' by n' box again, do we get the same partition that we started with?

Solution. If we fill the rectangle with unit squares, those not in the Young diagram of the original partition λ will fall into rows. The lengths of the rows are nonnegative, and are nondecreasing as we move down. Therefore, after we rotate through 180 degrees, these same rows will be listed in the opposite order, lined up along the left sides, and will have nonincreasing length. Thus they will be the Young diagram of a partition. The integer being partitioned will be $m'n' - k$. If $m' > m$ and $n' = n$. the two partitions will have the same number of parts, because we will have a nonzero number of empty squares at the end of each row of the Young diagram of λ. If $m' = m$ and $n' - n$ is the multiplicity of the largest part of λ, they will have the same number of parts. Otherwise, their numbers of parts will differ. If $n' > n$ and $m = m'$, then the two partitions will have the same largest part. If $n' = n$ and $m' - m$ is the smallest part of λ, then they will have the same largest part. Otherwise, their largest parts will differ. Thus for the two partitions to have the same number of parts, either $m' = m$ or $n' = n$. If $m' = m$ and they have the same largest part, then $n' > n$. But this is consistent with $n' - n$ being the multiplicity of the largest part of λ. Thus they can have the same number of parts and the same largest part if $m' = m$ and $n' - n$ is the multiplicity of the largest part of λ, or similarly if $n = n'$ and $m' - m$ is the smallest part of λ.

Activity 32.

(a) Suppose we take a partition of k into n parts with largest part m, complement it in the smallest rectangle it will fit into, complement the result in the smallest rectangle it will fit into, and continue the process until we get the partition 1 of one into one part. What can you say about the partition with which we started?

Solution. Let us call the process of enclosing λ in the smallest rectangle possible and then forming the complement in that rectangle *encomplementation* (This is short for *en*closure and *complementation* and is not a standard term—there is no standard term for this operation.) and call the result of it the *encomplement* of λ. The result of two encomplementations on the Young diagram of a partition is to remove all rows of maximum length and all columns of maximum length from the Young diagram. Thus the description of the result of an even number $2j$ of encomplementations is straightforward; we remove all the rows of the j largest distinct lengths and all columns of the j largest distinct lengths. So if an even number of encomplementations brings us to a partition with one block of size one, we should be able to describe the original partition fairly easily. To deal with the result of an odd number of encomplementations, we ask what happens if we encomplement just once. If the complement of λ in the smallest rectangle in which if fits has one square, then $\lambda = \lambda_1^{n_1}\lambda_1 - 1$. Thus we are asking for the partitions which, after an even number of encomplementations, give us either the partition with one block or a partition of the form $\lambda_1^{n_1}(\lambda_1 - 1)$. First we ask what kind of partition results in the second one after two encomplementations. If we get $\lambda_1^{n_1}(\lambda_1 - 1)$ from two encomplementations, the partition we started with had the form

$$\lambda_0^{n_0}(\lambda_1 + \lambda_2)^{n_1}(\lambda_1 + \lambda_2 - 1)\lambda_2^{n_2}.$$

If we get $\lambda_1^{n_1}(\lambda_1 - 1)$ from four encomplementations, then we started with a partition of the form

$$\lambda_{-1}^{n_{-1}}(\lambda_0 + \lambda_3)^{n_0}(\lambda_1 + \lambda_2 + \lambda_3)^{n_1}(\lambda_1 + \lambda_2 + \lambda_3 - 1)(\lambda_2 + \lambda_3)^{n_3}\lambda_3^{n_3}.$$

From this pattern we see that a partition that results in $\lambda_1^{n_1}(\lambda_1 - 1)$ after $2j$ encomplementations has the form

$$\lambda_{1-j}^{n_{1-j}}\lambda_{2-j}^{n_{2-j}}\cdots\lambda_0^{n_0}\lambda_1'^{\,n_1}(\lambda_1' - 1)\lambda_2^{n_2}\cdots\lambda_{j+1}^{n_{j+1}}, \tag{2.4}$$

where $\lambda_i > \lambda_{i+1}$ and $\lambda_0 > \lambda_1' > \lambda_2 + 1$.

On the other hand, a partition λ that results in 1 after two encomplementations has the form $\lambda_0^{n_0}(\lambda_1+1)\lambda_1^{n_1}$, and so a partition that results in 1 after j encomplementations is of the form

$$\lambda_{1-j}^{n_{1-j}}\lambda_{2-j}^{n_{2-j}}\cdots\lambda_0^{n_0}(\lambda_1+1)\lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_j^{n_j}, \tag{2.5}$$

where $\lambda_i > \lambda_{i+1}$ and $\lambda_0 > \lambda_1 + 1$. Thus a partition results in a single part of size 1 after some number of encomplementations if and only if it has the form of Equation (2.4) or Equation (2.5).

(b) Show that $P(k,n)$ is at least $\frac{1}{n!}\binom{k-1}{n-1}$.

Solution. The number of compositions of k into n parts is $\binom{k-1}{n-1}$. We can divide the compositions into blocks, where two compositions are in the same block if and only if one is a rearrangement of the other. Then the blocks correspond bijectively to partitions of k into n parts. However we cannot compute the number of blocks by dividing by the number of compositions per block since the number of compositions per block ranges from 1 to $n!$. But then if we divide the number of compositions by $n!$ we will get a number less than the number of blocks because $n!$ times the number of blocks would be, by the sum principle, greater than the number of partitions.

With the binomial coefficients, with Stirling numbers of the second kind, and with the Lah numbers, we were able to find a recurrence by asking what happens to our subset, partition, or broken permutation of a set S of numbers if we remove the largest element of S. Thus it is natural to look for a recurrence to count the number of partitions of k into n parts by doing something similar. Unfortunately, since we are counting distributions in which all the objects are identical, there is no way for us to identify a largest element. However if we think geometrically, we can ask what we could remove from a Young diagram to get a Young diagram. Two natural ways to get a partition of a smaller integer from a partition of n would be to remove the top row of the Young diagram of the partition and to remove the left column of the Young diagram of the partition. These two operations correspond to removing the largest part from the partition and to subtracting 1 from each part of the partition respectively. Even though they are symmetric with respect to conjugation, they aren't symmetric with respect to the number of parts. Thus one might be much more useful than the other for finding a recurrence for the number of partitions of k into n parts.

Activity 33. In this problem we will study the two operations and see which one seems more useful for getting a recurrence for $P(k,n)$.

(a) How many parts does the remaining partition have when we remove the largest part (more precisely, we reduce its multiplicity by one) from a partition of k into n parts? What can you say about the number of parts of the remaining partition if we remove one from each part?

Solution. Reducing the multiplicity of the largest part by one reduces the number of parts by one. Removing 1 from each part reduces the number of parts by the multiplicity of the smallest part, so it strictly reduces the number of parts, perhaps even to one.

(b) If the largest part of a partition is j and we remove it, what integer is being partitioned by the remaining parts of the partition? If we remove one from each part of a partition of k into n parts, what integer is being partitioned by the remaining parts?

Solution. If we remove the largest part, the integer being partitioned is k minus the largest part. Thus it is a number less than k and at least $n - 1$. If we remove one from each part of the partition, the integer being partitioned is $k - n$.

(c) The last two questions are designed to get you thinking about how we can get a bijection between the set of partitions of k into n parts and some other set of partitions that are partitions of a smaller number. These questions describe two different strategies for getting that set of partitions of a smaller number or of smaller numbers. Each strategy leads to a bijection between partitions of k into n parts and a set of partitions of a smaller number or numbers. For each strategy, use the answers to the last two questions to find and describe this set of partitions into a smaller number and a bijection between partitions of k into n parts and partitions of the smaller integer or integers into appropriate numbers of parts.

Solution. Removing the largest part of a partition of k into n parts gives us a bijection between partitions of k into n parts and and partitions of numbers k' between $n - 1$ and $k - 1$ into $n - 1$ parts of size at most $k - k'$. (Removing the largest part gives us such a partition, and adjoining a part of size $k - k'$ to such a partition gives us a partition of k with n parts.)

Removing one from each part of a partition of k into n parts gives us a bijection between partitions of k into n parts and and partitions $k - n$ into n or fewer parts. (Removing one from each part of a partition of k into n parts gives us such a partition, and, given such a partition, we get a partition of k into n parts by adding one to each part and then creating enough parts of size 1 to have n parts.)

(d) Find a recurrence (which need not have just two terms on the right hand side) that describes how to compute $P(k, n)$ in terms of the number of partitions of smaller integers into a smaller number of parts. (Hint: One of the two sets of partitions of smaller numbers from the previous part is more amenable to finding a recurrence than the other.)

Solution. The second bijection is to the set of partitions of $k - 1$ into n or fewer parts, and this makes the second bijection sound easier to work with. We get $P(k, n) = \sum_{i=1}^{n} P(k - n, i)$. The proof is the bijection we already described; in particular a partition of $k - n$ into i parts corresponds to the partition of k we get by adding one to each of the i parts and then creating $n - i$ parts of size one.

(e) What is $P(k, 1)$ for a positive integer k?

Solution. $P(k, 1) = 1$.

(f) What is $P(k, k)$ for a positive integer k?

Solution. $P(k, k) = 1$.

(g) Use your recurrence to compute a table with the values of $P(k, n)$ for values of k between 1 and 7.

Solution.

$k\backslash n$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1	1	1	0	0	0	0
4	1	2	1	1	0	0	0
5	1	2	2	1	1	0	0
6	1	3	3	2	1	1	0
7	1	3	4	3	2	1	1

(h) What would you want to fill into row 0 and column 0 of your table in order to make it consistent with your recurrence. What does this say $P(0, 0)$ should be? We usually define a sum with no terms in it to be zero. Is that consistent with the way the recurrence says we should define $P(0, 0)$?

Solution. We would want to have $P(0, 0) = 1$ and $P(k, 0) = P(0, n) = 0$ for positive integer k or n. Since the sum of the empty multiset of positive integers is zero, this gives us one partition of the number zero, namely the empty multiset of positive integers.

It is remarkable that there is no known formula for $P(k, n)$, nor is there one for $P(k)$. This section was are devoted to developing methods for computing values of $P(n, k)$ and finding properties of $P(n, k)$ that we can prove even without knowing a formula. Some future sections will attempt to develop other methods.

We have seen that the number of partitions of k into n parts is equal to the number of ways to distribute k identical objects to n recipients so that each receives at least one. If we relax the condition that each recipient receives at least one, then we see that the number of distributions of k identical objects to n recipients is $\sum_{i=1}^{n} P(k, i)$ because if some recipients receive nothing, it does not matter which recipients these are. This completes rows 7 and 8 of our table of distribution problems. The completed table is shown in Figure 2.3.4. There are quite a few theorems that you have proved which are summarized by Table 2.3.4. It would be worthwhile to try to write them all down!

The Twentyfold Way: A Table of Distribution Problems	
k objects and conditions on how they are received	n recipients and mathematical Distinct
1. Distinct no conditions	n^k functions
2. Distinct Each gets at most one	$n^{\underline{k}}$ k-element permutations
3. Distinct Each gets at least one	$S(k,n)n!$ onto functions
4. Distinct Each gets exactly one	$k! = n!$ permutations
5. Distinct, order matters	$(k+n-1)^{\underline{k}}$ ordered functions
6. Distinct, order matters Each gets at least one	$(k)^{\underline{n}}(k-1)^{\underline{k-n}}$ ordered onto functions
7. Identical no conditions	$\binom{n+k-1}{k}$ multisets
8. Identical Each gets at most one	$\binom{n}{k}$ subsets
9. Identical Each gets at least one	$\binom{k-1}{n-1}$ compositions (n parts)
10. Identical Each gets exactly one	1 if $k = n$; 0 otherwise

Table 2.3.4: The number of ways to distribute k objects to n recipients, with restrictions on how the objects are received

Partitions into distinct parts

Often $Q(k, n)$ is used to denote the number of partitions of k into distinct parts, that is, parts that are different from each other.

Activity 34. Show that

$$Q(k, n) \leq \frac{1}{n!}\binom{k-1}{n-1}.$$

Solution. The number of compositions of k into n parts is $\binom{k-1}{n-1}$. Thus the number of compositions of k into n distinct parts is less than $\binom{k-1}{n-1}$. Divide the compositions of k into n distinct parts into blocks with two compositions in the same block if one is a rearrangement of the other. Because the parts are distinct, each block has $n!$ members. Further, there is a bijection between the blocks of this partition and the partitions of k into n distinct parts. Since the number of compositions of k into n distinct parts is less than $\binom{k-1}{n-1}$, the number of partitions of k into n distinct parts is less than $\frac{1}{n!}\binom{k-1}{n-1}$.

Activity 35. Show that the number of partitions of 7 into 3 parts equals the number of partitions of 10 into three distinct parts.

Solution. Given a partition λ of 7 in decreasing list form $\lambda_1, \lambda_2, \lambda_3$, if we add 0 to λ_3, 1 to λ_2 and 2 to λ_1 the resulting partition of 10 has distinct parts. If we take a partition λ' of 10 with distinct parts, then $\lambda'_1 \geq \lambda'_2 + 1$, $\lambda'_1 \geq \lambda'_2 + 2$, and $\lambda'_2 \geq \lambda'_3 + 1$. Therefore if we subtract 2 from λ'_1 to get λ_1, subtract 1 from λ'_2 to get λ_2 and let $\lambda_3 = \lambda'_3$, then $\lambda_1, \lambda_2, \lambda_3$ is the decreasing list representation of a partition of $10 - 3 = 7$. Thus there is a bijection between partitions of 7 into three parts and partitions of 10 into three distinct parts.

Activity 36. There is a relationship between $P(k, n)$ and $Q(m, n)$ for some other number m. Find the number m that gives you the nicest possible relationship.

Solution. The number of partitions of k into n parts is equal to the number of partitions of $k + \binom{n}{2}$ into n distinct parts. The bijection from partitions of k with n parts to partitions of $k + \binom{n}{2}$ with n distinct parts that proves this is the one that takes a partition $\lambda_n\lambda_{n-1}\cdots\lambda_1$ of k with $\lambda_i > \lambda_{i+1}$ and adds $i - 1$ to λ_i to get λ'_i. Then λ' is a partition into distinct parts, and the number it partitions is $k + 1 + 2 + \cdots + n - 1 = k + \binom{n}{2}$. The proof that it is a bijection is the fact that subtracting $n - i$ from the ith part of a partition of k into distinct parts yields a partition of k, because part $i + j$ is at least j smaller than part i.

Activity 37. Find a recurrence that expresses $Q(k, n)$ as a sum of $Q(k-n, m)$ for appropriate values of m.

Solution. Suppose λ is a partition of k into n distinct parts. Either 1 is one of those parts or not. Thus if we subtract 1 from each part, we either get a partition of $k-n$ into $n-1$ parts or a partition of $k-n$ into n parts. If λ and λ' are different partitions of k into n distinct parts, they go to different partitions. Each partition of $k-n$ into $n-1$ parts or n parts can be gotten in this way from a corresponding partition of k into n parts. Thus we have a bijective correspondence and $Q(k,n) = Q(k-n, n-1) + Q(k-n, n)$.

Activity 38. Show that the number of partitions of k into distinct parts equals the number of partitions of k into odd parts.

Solution. We start by giving a function from the set of partitions of k to the set of partitions of k with (only) odd parts. Clearly such a function cannot be one to one. Then we show that when restricted to the partitions with distinct parts it is one-to-one and onto by constructing an inverse. Given a partition $\lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_n^{i_n}$, write $\lambda_i = \gamma_i 2^{k_i}$, where γ_i is odd. (Thus 2^{k_i} is the highest power of 2 that is a factor of λ_i, so it is 1 if λ_i is odd.). It is possible that $\gamma_i = \gamma_j$, for example if $\lambda_i = 36$ and $\lambda_j = 18$, then $\gamma_i = \gamma_j = 9$. We construct a new partition π whose parts are the numbers γ_j as follows: Given an odd number p, let the multiplicity $m(p)$ of p in π be $\sum_{j:\gamma_j=m} 2^{k_j}$. Thus $\sum_{p:m(p)\neq 0} m(p)p = k$. Therefore, π is a partition of k whose parts are all odd.

Now consider a partition π of k whose parts are all odd. Let $\pi = \pi_1^{r_1}\pi_2^{r_2}\cdots\pi_t^{r_t}$, with $\pi_i > \pi_{i+1}$. (In terms of the multiplicity function m, $m(\pi_i) = r_i$, and $\sum_{i=1}^{t} r_i\pi_i = k$.) We are going to write the binary expansion of each r_i as $r_i = \sum_{j=0}^{\lfloor \log_2 r_i \rfloor} 2^{j a_{ij}}$, where a_{ij} is 1 if 2^j appears in the binary expansion of r_i, and 0 otherwise. All of the numbers $\pi_i 2^{j a_{ij}}$ are distinct, because a power of two times one odd number cannot equal a power of two times another odd number. The numbers $\pi_i 2^{j a_{ij}}$ add to k, so they are the parts of a partition π' of k into distinct parts. When we apply the function constructed in the first part of the solution to π', we get π, so the correspondence between π and π' is a bijection.

Activity 39. Euler showed that if $k \neq \frac{3j^2+j}{2}$, then the number of partitions of k into an even number of distinct parts is the same as the number of partitions of k into an odd number of distinct parts. Prove this, and in the exceptional case find out how the two numbers relate to each other.

Solution. This solution is taken largely from the book Introduction to Combinatorics by Ioan Tomescu (published in London by Collet's in 1975). Tomescu calls a collection of rows in a Young diagram a "trapezoid" if each row contains one less cell than the row above and the number of cells in the rows above and below the trapezoid differ by two or more from the number of cells in rows of the trapezoid. Thus in (8,6,5,4,2,1) we have 3 trapezoids, the first row, the next three rows, and the last two. Since we are dealing with partitions with distinct parts, we don't have to worry about how two equal rows affect the definition of a trapezoid. We will describe a way to transform a partition with an even number of distinct parts into a partition with an odd number of distinct parts and vice versa.

First we describe a transformation on Young diagrams. Here is the first part of the description. Suppose the smallest part m of λ is less than or equal to the number j of rows in the top trapezoid. Suppose further that if we have only one trapezoid, then $j > m$. Then we construct a partition with one less part by adding 1 to each of the m largest parts and discarding the part m. We still have a diagram for a partition of the same integer, but now the parity of the number of parts has changed, and we *may* have increased the number of trapezoids by 1. The smallest part will now be larger than the number (now m) of rows in the top trapezoid. (Notice that the construction would not work if we had only one trapezoid and $j = m$ because we would first remove one row of the trapezoid and thus have no row to which to attach one of our squares.)

Here is the second part of the description of the transformation. Suppose now that m is larger than the number j of rows of the top trapezoid in the Young diagram. Suppose also that the Young diagram has at least two trapezoids or it has one trapezoid and $j \geq m - 2$. Take one square from each of the j rows of the top trapezoid (which is the whole diagram if there is only one trapezoid) and also add a row of j squares at the bottom of the diagram. (Since $m > j$, this gives us a Young diagram of a partition of the same integer into distinct parts.) The parity of the number of rows has changed, and now the number of rows of the top trapezoid is at least as large as the smallest part of the partition. (Note, two previously distinct trapezoids may have joined together to form one on top.) (Notice that if we have one trapezoid and $j = m + 1$, then the construction yields a partition with two equal parts, which is why we made the special assumption above.) Now let T be the transformation described by the two constructions above. Its domain is all Young diagrams except those with one trapezoid and $m \leq j \leq m + 1$. T^2 is the identity, and so T is a bijection. When restricted to partitions with an odd number of parts, T gives partitions with an even number of parts, so on its domain it gives a bijection between partitions with an even number of parts and partitions with an odd number of parts.

If $m = j$ and the diagram has just one trapezoid, then the diagram has $\frac{3j^2-j}{2}$ squares, and if $m = j + 1$ and the diagram has just one trapezoid, then the diagram has $\frac{3j^2+j}{2}$ squares. Thus if $k \neq \frac{3j^2 \pm j}{2}$, the number of partitions of k into distinct even parts equals the number of partitions of k into distinct odd parts.

If $k = \frac{3j^2 \pm j}{2}$ and j is even, then there is one diagram of a partition of k that is not in the domain of the bijection and has an even number of rows, so in this case there will be one more partition with an even number of parts than with an odd number. If $k = \frac{3j^2 \pm j}{2}$ and j is odd, there is one diagram with an odd number of rows not in the domain and so in this case there is one more partition with an odd number of parts than with an even number. This completes the exceptional cases of the problem.

Supplementary Problems

1. Answer each of the following questions with n^k, k^n, $n!$, $k!$, $\binom{n}{k}$, $\binom{k}{n}$, $n^{\underline{k}}$,

$k^{\underline{n}}$, $n^{\overline{k}}$, $k^{\overline{n}}$, $\binom{n+k-1}{k}$, $\binom{n+k-1}{n}$, $\binom{n-1}{k-1}$, $\binom{k-1}{n-1}$, or "none of the above".

(a) In how many ways may we pass out k identical pieces of candy to n children? $\binom{n+k-1}{k}$

(b) In how many ways may we pass out k distinct pieces of candy to n children? n^k

(c) In how many ways may we pass out k identical pieces of candy to n children so that each gets at most one? (Assume $k \leq n$.) $\binom{n}{k}$.

(d) In how many ways may we pass out k distinct pieces of candy to n children so that each gets at most one? (Assume $k \leq n$.) $n^{\underline{k}}$

(e) In how many ways may we pass out k distinct pieces of candy to n children so that each gets at least one? (Assume $k \geq n$.) None of the above.

(f) In how many ways may we pass out k identical pieces of candy to n children so that each gets at least one? (Assume $k \geq n$.) $\binom{k-1}{n-1}$

2. The neighborhood betterment committee has been given r trees to distribute to s families living along one side of a street.

(a) In how many ways can they distribute all of them if the trees are distinct, there are more families than trees, and each family can get at most one? $s^{\underline{r}}$

(b) In how many ways can they distribute all of them if the trees are distinct, any family can get any number, and a family may plant its trees where it chooses? s^r

(c) In how many ways can they distribute all the trees if the trees are identical, there are no more trees than families, and any family receives at most one? $\binom{s}{r}$

(d) In how many ways can they distribute them if the trees are distinct, there are more trees than families, and each family receives at most one (so there could be some leftover trees)? $\sum_{k=0}^{s}\binom{s}{k}r^{\underline{k}}$ or$\sum_{k=0}^{s} s^{\underline{k}}\binom{r}{k}$

(e) In how many ways can they distribute all the trees if they are identical and anyone may receive any number of trees? $\binom{r+s-1}{r}$

(f) In how many ways can all the trees be distributed and planted if the trees are distinct, any family can get any number, and a family must plant its trees in an evenly spaced row along the road? $s^{\overline{r}} = (r+s-1)^{\underline{r}}$

(g) Answer the question in Part 2.f assuming that every family must get a tree. $r!\binom{r-1}{s-1}$

(h) Answer the question in Part 2.e assuming that each family must get at least one tree. $\binom{r-1}{s-1}$

3. In how many ways can n identical chemistry books, r identical mathematics books, s identical physics books, and t identical astronomy books be arranged on three bookshelves? (Assume there is no limit on the number of books per shelf.) $\frac{(n+r+s+t+2)!}{n!r!s!t!2!}$

4. (interesting) One formula for the Lah numbers is
$$L(k,n) = \binom{k}{n}(k-1)^{\underline{k-n}}$$
Find a proof that explains this product. First choose the n elements which will be the first member of the part they lie in. (This, in effect, labels the n parts.) Then assign the remaining $k-n$ elements to their parts by making an ordered function of $n-k$ objects to n recipients in $(n+(k-n)-1)^{k-n} = (k-1)^{k-n}$ ways.

5. What is the number of partitions of n into two parts? $n/2$ if n is even and $(n-1)/2$ if n is odd, equivalently, $\lfloor n/2 \rfloor$

6. Show that the number of partitions of k into n parts of size at most m equals the number of partitions of $mn-k$ into no more than n parts of size at most $m-1$. If we take the complement of the Young diagram of a partition of k into n parts of size at most m in an rectangle with n rows and m columns, the number we partition will be $mn-k$, and we will have no more than n parts, each of size at most $m-1$. And if we take the complement of a partition of this second kind in the same rectangle, we will get a partition of the first kind.

7. Show that the number of partitions of k into parts of size at most m is equal to the number of partitions of of $k+m$ into m parts. Given the first kind of partition, take the conjugate (giving a partition of k into at most m parts), add one to each part, and then add enough parts of size 1 to get a total of m parts. It is straightforward that this process can be reversed.

8. You can say something pretty specific about self-conjugate partitions of k into distinct parts. Figure out what it is and prove it. With that, you should be able to find a relationship between these partitions and partitions whose parts are consecutive integers, starting with 1. What is that relationship? In a self-conjugate partition, the number of parts is the size of the largest part. If these parts are distinct, this means that each number between 1 and the largest part appears once as a part. That is, the parts are a list of consecutive integers, starting with 1.

9. What is $s(k,1)$? Since s$(k,1)$ is the coefficient of x^1 in
$$x^{\underline{k}} = x(x-1)(x-2)\cdot(x-(k-1)),$$
it is $(-1)^{k-1}(k-1)!$.

10. Show that the Stirling numbers of the second kind satisfy the recurrence

$$S(k,n) = \sum_{i=1}^{k} S(k-i, n-1)\binom{n-1}{i-1}.$$

A partition of $[k]$ into n blocks has a block containing k. If this block has size i, when you remove it, you get a partition of a set of size $k-i$ into $n-1$ blocks. The number of possible sets of size i containing k is $\binom{k-1}{i-1}$, and i can be any number between 1 and k. Each partition of k into n blocks may be constructed exactly once by first choosing the block containing k and then partitioning the remaining elements into $n-1$ blocks. This proves the formula.

11. (interesting) Let $c(k,n)$ be the number of ways for k children to hold hands to form n circles, where one child clasping his or her hands together and holding them out to form a circle is considered a circle. Find a recurrence for $c(k,n)$. Is the family of numbers $c(k,n)$ related to any of the other families of numbers we have studied? If so, how? The kth child is either in a circle by him/her self, and there are $c(k-1,n-1)$ ways for this to happen, or is in a circle with some other children. In the second case child i can be to the immediate right of any of the other $k-1$ children, so there are $(k-1)c(k-1,n)$ ways for this to happen. Thus $c(k,n) = c(k-1,n-1) + (k-1)c(k-1,n)$. This recurrence is almost the same as the recurrence for $s(k,n)$, except it has a plus sign where the recurrence for the Stirling numbers of the first kind has a minus sign. Further $c(k,1) = (k-1)!$ and $c(k,k) = 1$, which agrees, except for sign, with the Stirling numbers of the first kind. If we experiment with applying the recurrence, we see that whenever we use it to compute $c(k,n)$, we get that $c(k,n) = |s(k,n)|$. It is now straightforward to prove by induction that $c(k,n) = |s(k,n)|$.

Bibliography

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