Internship Report

Félix Dedek

June 2025

1 Introduction

In my internship I worked under the mentorship of Clément Rey at Centre de Mathématiques Appliqués de Polytechnique (CMAP) whom I would like to thank for this amazing opportunity and his guidance throughout these last 2 months.
The objective was to simulate the densities of various continuous-time stochastic processes using tools from Malliavin calculus. More precisely, we considered the following representation of the density:

$$p_{X_t}(x) = E[(\mathbf{1}_{y \leq x \leq X_t} - \mathbf{1}_{y > x > X_t})H(X_t)]$$

where $H(X_t)$ is a random variable and $y \in \mathbb{R}$. The goal of my internship is to find cases where we can simulate X_t and $H(X_t)$, and hence find $p_{X_t}(x)$ numerically by Monte-Carlo methods. In section 2 we will introduce those mathematical tools : Integration by parts, Malliavin calculus and Signatures. In section 3 we will then find different formulas computed using Malliavin calculus. In the following section we will look at more specific cases and adapt the previous formulas to something that we can numerically simulate. In the fifth and last section we will discuss convergence and implementation.

2 Mathematical Framework

2.1 Integration by parts

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $F, G \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. We say that the integration by parts formula $IP(F;G)$ holds true if there exists an integrable random variable $H(F;G)$ such that

$$E[\phi'(F)G] = E[\phi(F)H(F;G)] \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R})$$

where $\mathcal{C}_c^\infty(\mathbb{R})$ denotes the space of all smooth functions with compact support.

This definition is crucial for finding density formulas as you will see in the theorem below.
Theorem 2.2. Suppose that F satisfies $IP(F;1)$: Then the law of F is absolutely continuous with respect to the Lebesgue measure and the density of the law is given by the following expression:

$$p(x) = E[\mathbf{1}_{F \geq x} H(F;1)]$$

Moreover for all $y \in \mathbb{R}$ we have

$$p(x) = E[(\mathbf{1}_{y \leq x \leq F} - \mathbf{1}_{y > x > F})H(F;1)]$$

Proof. We will prove the second formula as the first one is analogous and simpler. We remind that p is the density of F if and only if for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ it holds $E[\phi(F)] = \int_{\mathbb{R}} \phi(x)p(x)\mathrm{d}x$. So let $y \in \mathbb{R}$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$

and set $\Phi(x) := \int_y^x \phi(t)\mathrm{d}t$. Hence $\Phi'(x) = \phi(x)$ and $\Phi \in \mathcal{C}_c^\infty(\mathbb{R})$. So

$$\begin{aligned}
E[\phi(x)] &= E[\Phi'(F)] && (2.1)\\
&= E[\Phi(F)H(F;1)] && (2.2)\\
&= E\left[\left(\mathbf{1}_{y\le F}\int_y^F \phi(x)\mathrm{d}x - \mathbf{1}_{y>F}\int_F^y \phi(x)\mathrm{d}x\right)H(F;1)\right] && (2.3)\\
&= \int_{\mathbb{R}} \phi(x)E[(\mathbf{1}_{y\le x\le F} - \mathbf{1}_{y>x>F})H(F;1)]\mathrm{d}x && (2.4)
\end{aligned}$$

Which proves the claim. □

So it becomes clear the in order to simulate the density of F, we need to be able to simulate F and the weight $H(F;1)$. In the next subsection we will introduce Malliavin derivatives and skorohod Integrals as well as their dual property, which will play a crucial role to compute p.

2.2 Malliavin Calculus

We briefly introduce Malliavin calculus, a tool that will be essential to compute the weights $H(X_t,1)$ of the density formulas. For more detailed approach, the reader may refer to https://perso.math.u-pem.fr/bally.vlad/RR-4718.pdf. Let E be a separable Hilbert space. We introduce the simple functionals as follows,

$$\begin{aligned}
\mathcal{S} = \mathcal{S}(E) :=& \{F : \Omega \to E, F(\omega) = f(W^1_{t_{k+1}} - W^1_{t_k}, \dots, W^d_{t_{k+1}} - W^d_{t_k}, k = 0, \dots 2^n - 1),\\
& f \in \mathcal{C}_p^\infty(\mathbb{R}^{d\times 2^n}; E), t_k = \frac{kT}{2^n}, n \in \mathbb{N})\}
\end{aligned}$$

Here and throughout this document W refers to a d dimensional standard brownian motion $W_t = (W^1_t, \dots, W^d_t)$. So these simple functional are functionals of increments of a brownian motion over a partition of the time axis $[0,T]$.

For $F \in \mathcal{S}$, we introduce the Malliavin derivatives of F taking values in E^d, as (for component j)

$$\mathbf{D}^j_t F = \partial_{x^{l,j}} f(W_{t_{k+1}} - W_{t_k}, k = 0, \dots 2^n - 1)$$

for $j \in \{1, \dots, d\}$ and $t \in [t_l, t_{l+1})$. Notice that $\mathcal{S}$ is dense in $L^p(\Omega, |.|_E)$, $p \geqslant 1$, and $\mathbf{D}$ is a closable operator on $L^p(\Omega, |.|_E)$. We can define higher order Malliavin derivatives (at some point $\mathbf{t} \in \mathbb{R}^\ell_+$, $\ell \in \mathbb{N}$, instead of point t) by

$$\mathbf{D}_{(t_1,\dots,t_\ell)} F = \mathbf{D}_{t_1} \dots \mathbf{D}_{t_\ell} F, \qquad \mathbf{t} = t_1, \dots t_\ell.$$

A trivial example which will be useful is the functional $F = W_{t_k} \in \mathcal{S}$ having the Malliavin derivative

$$\mathbf{D}_t F = \mathbf{1}_{t\le t_k}$$

Moreover, for $p \geqslant 1$, $m \in \mathbb{N}^*$ we define $\mathbb{D}^{m,p} = \mathbb{D}^{m,p}(E)$ as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{\mathbb{D}^{m,p}(E)} := \Big(\sum_{\ell=0}^m \mathbb{E}[\int_0^T |\mathbf{D}_{(t_1,\dots,t_\ell)} F|^2_{(E^d)^{\otimes \ell}} \mathrm{d}t_1 \dots \mathrm{d}t_\ell|^{\frac{p}{2}}]\Big)^{\frac{1}{p}}.$$

The Malliavin derivatives for functionals in $\mathbb{D}^{m,p}$ are defined as the limit of sequences of Malliavin derivatives of elements of $\mathcal{S}$ with respect to $\|\cdot\|_{\mathbb{D}^{m,p}(E)}$. Notice in particular that $\mathbb{D}^{1,2}(E)$ equipped with the inner product

$$\langle F, G\rangle_{1,2} = \mathbb{E}[\langle F, G\rangle_E] + \mathbb{E}[\int_0^T \langle \mathbf{D}_t F, \mathbf{D}_t G\rangle_{E^d} \mathrm{d}t]$$

is a Hilbert space.

Considering $F \in E^N$, $F^i \in \mathbb{D}^{1,2}(E)$, the Malliavin covariance matrix of F is defined as

$$\sigma_F = (\int_0^T \langle \mathbf{D}_t F^i, \mathbf{D}_t F^j \rangle_{E^d} \mathrm{d}t)_{(i,j)\in\{1,\dots,N\}^2}$$

We also introduce the Skorohod integral δ satisfying the dual relationship

$$\mathbb{E}[\int_0^T \langle \mathbf{D}_t F, u_t \rangle_{E^d}]\mathrm{d}t = \mathbb{E}[\langle \delta(u), F\rangle_E] \tag{2.5}$$

for $F \in \mathbb{D}^{1,2}(E)$ and $u \in \mathrm{Dom}(\delta) := \{u \in L^2(\Omega \times [0,T]; E^d), |\mathbb{E}[\int_0^T \langle \mathbf{D}_t F, u_t\rangle_{E^d} \mathrm{d}t]| \leqslant C\|F\|_{L^2(\Omega)}, \forall F \in \mathbb{D}^{1,2}(E)\}$. Moreover, we can write for $u = (u^1, \dots, u^d) \in \mathrm{Dom}(\delta)$,

$$\delta(u) = \sum_{j=1}^d \delta^j(u^j)$$

with, for every $j \in \{1, \dots, d\}$ and every $F \in \mathbb{D}^{1,2}(E)$ and $u \in \mathrm{Dom}(\delta)$,

$$\mathbb{E}[\int_0^T \langle \mathbf{D}_t^j F, u_t^j \rangle_E]\mathrm{d}t = \mathbb{E}[\langle \delta_j(u_j), F\rangle_E].$$

We denote $\mathbb{D}^{M,\infty}(E) := \cap_{p=1}^{+\infty} \mathbb{D}^{M,p}(E)$ and for $p \in [1, +\infty) \cup \{+\infty\}$, $\mathbb{D}^{\infty,p}(E) = \cap_{M=1}^{+\infty} \mathbb{D}^{M,p}(E)$.

We also have that for $F \in \mathbb{D}^{1,2}(\mathbb{R})$, $u \in \mathrm{Dom}(\delta)$ such that $Fu \in \mathrm{L}_2(\Omega \times [0,T]; \mathbb{R}^d)$, $T > 0$

$$\delta(Fu) = F\delta(u) - \int_0^T \langle \mathbf{D}_s F, u_s \rangle_{\mathbb{R}^d} \mathrm{d}s, \tag{2.6}$$

2.3 Signatures

For $d \in \mathbb{N}$ we denote by $(\mathbb{R}^d)^{\otimes n}$ the space of n dimensional tensors and set $T((\mathbb{R}^d)) := \{\boldsymbol{\ell} = (\boldsymbol{\ell}_n)_{n\in\mathbb{N}} : \boldsymbol{\ell}_n \in (\mathbb{R}^d)^{\otimes n}\}$. We also write $\ell \in T^n(\mathbb{R}^d)$ when $\ell_i = 0$ for all $i > n$. Let e_i be the i-th standard basis vector of $\mathbb{R}^d$ so that $e_{i_1} \otimes \cdots \otimes e_{i_n}$ with $i_1, \dots, i_n \in \{1, \dots, d\}$ for a basis of $T^n(\mathbb{R}^d)$. We identify e_i with the letter i and $e_{i_1} \otimes \cdots \otimes e_{i_n}$ with the word $i_1 \dots i_n \in \{1, \dots, d\}$. For $\ell \in T((\mathbb{R}^d))$ we write $\ell = \sum_{n\in\mathbb{N}} \sum_{v \in V_n} \ell^v v$ where V_n denotes all n-letter words. The empty word $\varnothing$ is then identified with 1. The signature of a process X at time t $\mathbb{X}_t = (1, \mathbb{X}_t^1, \dots, \mathbb{X}_t^n, \dots)$ is the following object:

$$\mathbb{X}_t^n = \int_{0<t_1<\cdots<t_n} \circ \mathrm{d}X_{t_1} \otimes \cdots \otimes \circ \mathrm{d}X_{t_n} \in T^n(\mathbb{R}^d)$$

were $\circ \mathrm{d}X_t$ denotes the Stratonovich Integral with respect to X_t. We will only use this in the case were X_t is a Brownian motion or an augmented Brownian motion. We also define the scalar product

$$\langle \ell, \mathbb{X}_t \rangle := \sum_{n\in\mathbb{N}} \sum_{v\in V_n} \ell^v \mathbb{X}_t^v$$

.

The following theorem holds (for a proof see future works of Clement Rey).

Theorem 2.3. Let $\widehat{W}_t = (t, W_t^1, \dots, W_t^{d-1})$ and augmented Brownian motion. Then for all $i_1, \dots, i_n = 2, \dots, d$ and $j_1, \dots, j_n = 1, \dots, d$, we have

$$\int_{0\le s_1 \le \cdots \le s_n} \mathbf{D}_{s_1}^{i_1} \cdots \mathbf{D}_{s_n}^{i_n} \langle \boldsymbol{\ell}, \widehat{\mathbb{W}}_t \rangle \circ \mathrm{d}\widehat{W}_{t_1}^{j_1} \otimes \cdots \otimes \circ \mathrm{d}\widehat{W}_{t_n}^{j_n} = \langle \Psi_{j_1\cdots j_n}^{i_1\cdots i_n}(\boldsymbol{\ell}), \widehat{\mathbb{W}}_t \rangle$$

where $\Psi_{j_1\cdots j_n}^{i_1\cdots i_n} \colon T((\mathbb{R}^d)) \longrightarrow T((\mathbb{R}^d))$ is a linear operator defined recursively by

$$\Psi_{j_1\cdots j_n}^{i_1\cdots i_n}(\varnothing) = 0, \qquad \Psi_\varnothing^\varnothing(v) = v$$
$$\Psi_{j_1\cdots j_n}^{i_1\cdots i_n}(v) = \Psi_{j_1\cdots j_n}^{i_1\cdots i_n}(v')k + \mathbf{1}_{k=i_n} \Psi_{j_1\cdots j_{n-1}}^{i_1\cdots i_{n-1}}(v')j_n, \qquad v = v'k.$$

3 Integration by parts formulas using Malliavin calculus

In this section we will find some general IBP forumlas using the Malliavin calculus tools defined above. We will adapt those to more specific cases in later sections.

3.1 General Case

Consider $(X_t)_{t\geq 0}$ a real process and $G, X_t \in \mathbb{D}^{1,2}$ for all $t \geq 0$.
Theorem 3.1. It holds

$$H_1(X_t, G) = \frac{GW_t^i}{\int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} - \frac{\int_0^t \mathbf{D}_s^i G \mathrm{d}s}{\int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} + G\frac{\int_0^t \int_0^t \mathbf{D}_s^i \mathbf{D}_r^i X_t \mathrm{d}r\mathrm{d}s}{(\int_0^t \mathbf{D}_s^i X_t \mathrm{d}s)^2} \tag{3.1}$$

for every $i = 1, \ldots, d$ and also

$$H_2(X_t, G) = (\frac{G\sum_{i=1}^d W_t^i}{\sum_{i=1}^d \int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} - \frac{\sum_{i=j}^d \int_0^t \mathbf{D}_s^j G \mathrm{d}s}{\sum_{i=1}^d \int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} + G\frac{\sum_{i,j=1}^d \int_0^t \int_0^t \mathbf{D}_s^j \mathbf{D}_r^i X_t \mathrm{d}r\mathrm{d}s}{(\sum_{i=1}^d \int_0^t \mathbf{D}_s^i X_t \mathrm{d}s)^2}) \tag{3.2}$$

Note that we write $H_i(X_t, G)$ indstead of $H(X_t, G)$ to differentiate the different formulas more easily.

Proof. We will only prove the second formula. The first one is analogous to the second by removing sums. Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$. By the chain rule we have for all $i = 1, \ldots, d$

$$\mathbf{D}_s^i f(X_t) = \nabla f(X_t) \mathbf{D}_s^i X_t$$

so by integrating over s on both sides we obtain

$$\nabla f(X_t) = \frac{\int_0^t \mathbf{D}_s^i f(X_t) \mathrm{d}s}{\int_0^t \mathbf{D}_s^i X_t \mathrm{d}s}.$$

So

$$\begin{aligned}
\mathbb{E}[\nabla f(X_t) G] =& \mathbb{E}[\frac{\int_0^t \mathbf{D}_s^i f(X_t) \mathrm{d}s}{\int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} G] \\
=& \mathbb{E}[f(X_t) \delta_i (\frac{G}{\int_0^t \mathbf{D}_s^i X_t \mathrm{d}s})] \\
=& \mathbb{E}[f(X_t)(\frac{GW_t^i}{\sum_{i=1}^d \int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} - \int_0^t \mathbf{D}_s^i (\frac{G}{\int_0^t \mathbf{D}_s^i X_t \mathrm{d}s}) \mathrm{d}s)] \\
=& \mathbb{E}[f(X_t)(\frac{G\sum_{i=1}^d W_t^i}{\sum_{i=1}^d \int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} - \frac{\sum_{i=j}^d \int_0^t \mathbf{D}_s^j G \mathrm{d}s}{\sum_{i=1}^d \int_0^t \mathbf{D}_s^i X_t \mathrm{d}s} + G\frac{\sum_{i,j=1}^d \int_0^t \int_0^t \mathbf{D}_s^j \mathbf{D}_r^i X_t \mathrm{d}r\mathrm{d}s}{(\sum_{i=1}^d \int_0^t \mathbf{D}_s^i X_t \mathrm{d}s)^2})].
\end{aligned}$$

In the second equality we used the duality 2.5 and in the third equality the property 2.6 with $u_s = \mathbf{1}_{s\leq t}$ and the fact that $\delta(u) = W_t^i$. In the last equality we simply applied the chain rule for Malliavin derviatives. $\square$

We can obtain a different formula by applying the same tricks differently:
Theorem 3.2. Let $u \in \mathrm{Dom}(\delta)$ and define $\tilde{\sigma}_t := \int_0^t \sum_{j=1}^d u^j(s) \cdot \mathbf{D}_s^j(X_t)\mathrm{d}s$. Then we have

$$H_3(X_t, G) = G(\tilde{\sigma}_t)^{-1}\delta(u) - \int_0^t \langle \mathbf{D}_s(G(\tilde{\sigma}_t)^{-1}), u_s \rangle_{\mathbb{R}^d} \mathrm{d}s \tag{3.3}$$

Proof. Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$. Using

$$\mathbf{D}_s^i f(X_t) = \nabla f(X_t) \mathbf{D}_s^i X_t$$

we get

$$\begin{aligned}\int_0^t \langle u_s, \mathbf{D}_s f(X_t)\rangle_{\mathbb{R}^d} \mathrm{d}s = & \int_0^t \sum_{j=1}^d u_s^j \mathbf{D}_s^j f(X_t) \mathrm{d}s \\ =& \nabla f(X_t) \int_0^t \sum_{j=1}^d u_s \mathbf{D}_s^j X_t \mathrm{d}s \\ =& \tilde{\sigma}_t \nabla f(X_t)\end{aligned}$$

So using this we get

$$\begin{aligned}\mathbb{E}[\nabla f(X_t) G] =& \mathbb{E}\left[(\tilde{\sigma}_t)^{-1} \int_0^t \langle u_s, \mathbf{D}_s f(X_t)\rangle_{\mathbb{R}^d} \mathrm{d}s \cdot G\right] \\ =& \mathbb{E}\left[\int_0^t \langle G(\tilde{\sigma}_t)^{-1} u_s, \mathbf{D}_s f(X_t)\rangle_{\mathbb{R}^d} \mathrm{d}s\right] \\ =& \mathbb{E}\left[f(X_t)\delta(G(\tilde{\sigma}_t)^{-1} u.)\right] \\ =& \mathbb{E}\left[f(X_t)\left(G(\tilde{\sigma}_t)^{-1}\delta(u) - \int_0^t \langle \mathbf{D}_s(G(\tilde{\sigma}_t)^{-1}), u_s\rangle_{\mathbb{R}^d}\rangle \mathrm{d}s\right)\right]\end{aligned}$$

Where again we used 2.5 and 2.6 in the third and fourth equalities respectively. □

Corollary 3.3. Using $u_s = \mathbf{D}_s X_t$ we have

$$H_4(X_t, G) = G(\sigma_{X_t})^{-1}\delta(\mathbf{D}.X_t) - \int_0^t \langle \mathbf{D}_s(G(\sigma_{X_t})^{-1}), \mathbf{D}_s X_t\rangle_{\mathbb{R}^d} \mathrm{d}s \tag{3.4}$$

3.2 X_t as a solution to a SDE

Consider X_t solution to $X_t = x + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s$ where W is a 1-dimensional Brownian motion ($d = 1$) and the starting point $x \in \mathbb{R}$.

Theorem 3.4 (Variation of Constants Formula)**.** It holds for any $s \in [0, t]$

$$\nabla_x X_t = \mathbf{D}_s X_t \sigma^{-1}(s, X_s) \nabla_x X_s \tag{3.5}$$

Proof. It holds for $0 < u < t$: $X_t = X_u + \int_u^t b(s, X_s)\mathrm{d}s + \int_u^t \sigma(s, X_s)\mathrm{d}W_s$ (this can be seen by computing $X_t = X_u + X_t - X_u$). It holds by chain rule and dominated convergence

$$\nabla_x X_t = \nabla_x X_u + \int_u^t \nabla_x X_s b'(s, X_s)\mathrm{d}s + \int_u^t \nabla_x X_s \sigma'(s, X_s)\mathrm{d}W_s$$

where we denote by σ' and b' the partial derivatives in the second argument. On the other hand we have applying the Malliavin differential operator on the SDE started at 0:

$$\mathbf{D}_u X_t = \int_u^t \mathbf{D}_u X_s b'(s, X_s)\mathrm{d}s + \int_u^t \mathbf{D}_u X_s \sigma'(s, X_s)\mathrm{d}W_s + \sigma(u, X_u)$$

So, by multiplying $\sigma^{-1}(u, X_u)\nabla_x X_u$ on both sides of the previous equation we get

$$\begin{aligned}\sigma^{-1}(u, X_u)\nabla_x X_u \mathbf{D}_u X_t = & \nabla_x X_u + \int_u^t \sigma^{-1}(u, X_u)\nabla_x X_u \mathbf{D}_u X_s b'(s, X_s)\mathrm{d}s \\ & + \int_u^t \sigma^{-1}(u, X_u)\nabla_x X_u \mathbf{D}_u X_s \sigma'(s, X_s)\mathrm{d}W_s\end{aligned}$$

So $\nabla_x X_t$ and $\sigma^{-1}(u, X_u)\nabla_x X_u \mathbf{D}_u X_t$ satisfy the same SDE which proves the claim. □

Theorem 3.5. It holds

$$H_5(X_t, G) = \frac{1}{t}\left((\nabla_x X_t)^{-1} G\delta(\sigma^{-1}(\cdot, X.)\nabla_x X.) - \int_0^t \langle \mathbf{D}_s((\nabla_x X_s)^{-1}G, \sigma^{-1}(s, X_s)\nabla_x X_s\rangle_{\mathbb{R}^d} \mathrm{d}s\right) \tag{3.6}$$

Proof. Using the variation of constants formula we get

$$\begin{aligned}\nabla f(X_t) =&(\nabla_x X_t)^{-1}\nabla_x X_t \nabla f(X_t)\\ =&(\nabla_x X_t)^{-1}\nabla f(X_t)\mathbf{D}_s X_t \sigma^{-1}(s, X_s)\nabla_x X_s\\ =&(\nabla_x X_t)^{-1}\frac{1}{t}\int_0^t \mathbf{D}_s f(X_t)\sigma^{-1}(s, X_s)\nabla_x X_s \mathrm{d}s\end{aligned}$$

where the last line is obtained by integrating on both sides over s and dividing by t toghether with the Malliavin chain rule.

Using this toghether with (2.5) and (2.6), we obtain (3.6) similarly to the proof of (3.4). □

We see a final formula which is computed using (3.3).

Theorem 3.6. It holds

$$H_6(X_t, G) = G\frac{(\nabla_x X_t)^{-1}W_t + \int_0^t \mathbf{D}_s(\nabla_x X_t)^{-1}\mathrm{d}s}{\int_0^t \sigma(s, X_s)(\nabla_x X_s)^{-1}\mathrm{d}s} \tag{3.7}$$

$$- \int_0^t \mathbf{D}_s\left(\frac{G}{\int_0^t \sigma(u, X_u)(\nabla_x X_u)^{-1}\mathrm{d}u}\right)(\nabla_x X_t)^{-1}\mathrm{d}s \tag{3.8}$$

Proof. Consider formula (3.3) and take $u_s = (\nabla_x X_t)^{-1}$. First, using the Variation of constants (3.5) we get

$$\tilde{\sigma}_t = \int_0^t \nabla_x X_t \mathbf{D}_s X_t \mathrm{d}s = \int_0^t \sigma(s, X_s)(\nabla_x X_s)^{-1}\mathrm{d}s.$$

Moreover, $\delta(u.) = \delta(\nabla_x X_t) = (\nabla_x X_t)^{-1}W_t - \int_0^t \mathbf{D}_s((\nabla_x X_t)^{-1}\mathrm{d}s$, so by plugging in both of these results into (3.3) we get the desired result. □

4 Applying IBP to Specific cases

In the last section we found some very abstract mathematical formulas. In this section we will apply them to more specific cases where they can numerically simulated.

4.1 Brownian motion

In this subsection we consider $X_t = \langle \ell_t, W_t\rangle = \sum_{i=1}^d \ell_t^i W_t^i$ where W is a d-dimensional brownian motion and ℓ is deterministic and known. This example is the most trivial case of a signature (when excluding constant processes of course). It is especially useful since the distribution of X_t is known, allowing us verify our results. We will adapt formulas (3.1),(3.2),(3.4) to this case. Moreover, since for the density formulas we need to compute $H(X_t; 1)$ will only consider $G = 1$, which simplifies the formulas a lot since $\mathbf{D}_s 1 = 0$. We state the main result of this section

Theorem 4.1. In the context of this section, formula (3.1) simplifies to

$$H_1(X_t, 1) = \frac{W_t^i}{t\cdot \ell_t^i}, \quad i \in \{1, \dots, d\}\,. \tag{4.1}$$

Formula (3.2) simplifies to

$$H_2(X_t, 1) = \frac{\sum_{i=1}^d W_t^i}{t\sum_{i=1}^d \ell_t^i}. \tag{4.2}$$

Formula (3.4) simplifies to

$$H_3(X_t, 1) = \frac{X_t}{t\|\ell\|_2^2}. \tag{4.3}$$

Remark 4.1. It is clear that all three weights can be easily simulated since it is not difficult to simulate a brownian motion.

Proof. To prove the first two formulas let us do some computations using the Malliavin chain rule when $0 < s < t$:

$$\mathbf{D}_s^i X_t = \sum_{j=1}^{d} \ell_t^j \mathbf{D}_s^i W_t^j = \ell_t^i$$

So that

$$\mathbf{D}_s^i \mathbf{D}_t^j X_t = 0 \quad \forall i, j = 1, \ldots, d$$

Since moreover $G = 1$, the second and third fractions in (3.1) and (3.2) disappear and the result becomes clear.
To prove (4.3), we need to compute σ_{X_t}:

$$\begin{aligned}\sigma_{X_t} &= \int_0^t \sum_{j=1}^{d} \mathbf{D}_s^j X_t \mathbf{D}_s^j X_t \mathrm{d}s \\ &= \int_0^t \sum_{j=1}^{d} (\ell_t^j)^2 \mathrm{d}s \\ &= t\|\ell_t\|_2^2\end{aligned}$$

So $\mathbf{D}_s \sigma_{X_t}^{-1} = 0$ and hence the integral part in formula (3.4) is null. Moreover,

$$\delta(\mathbf{D}_\cdot X_t) = \sum_{i=1}^{d} \delta^i(\mathbf{D}_\cdot^i X_t) = \sum_{i=1}^{d} \delta^i(\ell_t^i \mathbf{1}_{s<t}) = X_t$$

which shows the last element of this theorem. □

In the next section we will discuss how to effectively implement these formulas in python and hence simulate the density of X_t. But before this we will look at more examples.

4.2 Polynomials of a Brownian Motion

In this subsection we generalize the last example by considering general polynomials of Brownian motions: Let

$$X_t = \langle \boldsymbol{\ell}, W_t \rangle = \sum_{n=0}^{N} \sum_{a \in A_n} \boldsymbol{\ell}^a W_t^a,$$

where $A_n = \left\{ a \in \mathbb{N}^N \middle| \quad |a| := \sum_{i=1}^N |a_i| = n \right\}$, $W_t^a = \Pi_{i=1}^N W_t^{a_i}$ and $\boldsymbol{\ell}^a \in \mathbb{R}$ for all a.

Theorem 4.2. The formulas (3.1),(3.2) and (3.4) become

$$H_1(X_t, 1) = \frac{W_t^i}{t\langle \varphi_i(\boldsymbol{\ell}), W_t \rangle} + \frac{\langle \varphi_i(\varphi_i(\boldsymbol{\ell})), W_t \rangle}{(\langle \varphi_i(\boldsymbol{\ell}), W_t \rangle)^2}, \tag{4.4}$$

$$H_2(X_t, 1) = \frac{\sum_{i=1}^d W_t^i}{t\langle \sum_{i=1}^d \varphi_i(\boldsymbol{\ell}), W_t \rangle} + \frac{\langle \sum_{i,j=1}^d \varphi_j(\varphi_i(\boldsymbol{\ell})), W_t \rangle}{(\langle \sum_{i=1}^d \varphi_i(\boldsymbol{\ell}), W_t \rangle)^2}, \tag{4.5}$$

$$H_3(X_t,1) = \frac{\sum_{i=1}^d \left(W_t^i \langle \varphi_i(\boldsymbol{\ell}), W_t\rangle - t\langle \varphi_i(\varphi_i(\boldsymbol{\ell})), W_t\rangle\right)}{t\sum_{i=1}^d \langle \varphi_i(\boldsymbol{\ell}), W_t\rangle^2} + \frac{2\sum_{i,j=1}^d \langle \varphi_i(\varphi_i(\boldsymbol{\ell})), W_t\rangle \langle \varphi_j(\varphi_i(\boldsymbol{\ell})), W_t\rangle \langle \varphi_i(\varphi_i(\boldsymbol{\ell})), W_t\rangle}{\left(\sum_{i=1}^d \langle \varphi_i(\boldsymbol{\ell}), W_t\rangle^2\right)^2} \tag{4.6}$$

where $\varphi_i(\boldsymbol{\ell})^a = (a_j+1)\boldsymbol{\ell}^{a+e_j}$, $e_j \in \mathbb{N}^N, (e_j)_i = \delta_{ij}$. If we write $X_t = p(W_t)$, with $p(x) = \langle \boldsymbol{\ell}, x\rangle$ being the polynomial in question, $\partial_i p(x) = \langle \varphi_i(\boldsymbol{\ell}), x\rangle$. This notation is chosen this way since it corresponds to the notation of signatures which we will see in future sections. It is also the way I implemented the polynomials in python.

Proof. Simply apply basic Malliavin calculus to the formulas above. □

Remark 4.2. Since in my internship I had very limited time, I did not pay to much attention to the conditions under which one can apply the above formulas. One important one being that $1/\int_0^t \mathbf{D}_s X_t \mathrm{d}s \in \mathrm{dom}(\delta)$. However, when considering certain polynomials, they clearly do not fullfill this condition. Consider for example $X_t = W_t^2/2$ $(d=1)$, $1/\int_0^t \mathbf{D}_s X_t \mathrm{d}s = \frac{1}{tW_t}$ which is not in the domain of δ. To see this take for example $F = X_t$, then $E[\int_0^t \mathbf{D_s} F \frac{1}{tW_t}\mathrm{d}s] = E[\frac{1}{W_t}]$ which is not well defined. If we ignore this, the previous formulas give $H(X_t,1) = 1/t + 1/W_t^2$, which is clearly not integrable either. In particular, since $X_t \geq 0$, $p(x) = E[1/t + 1/W_t^2] = \infty$ for all $x < 0$. So the formula no longer works. We will however consider examples of polynomials which do work in the implementation section.

4.3 Geometric Brownian motion

Another example whose density can be explicitely computed is the geometric brownian motion. Like above, we will use this example to verify our results. We consider $X_t = x \cdot e^{bt+\sigma W_t}$ with some constants $\sigma > 0$ and $b \in \mathbb{R}$

By applying any of the formulas of Section 3 to this case, we obtain the same following result:

$$H(X_t,1) = X_t^{-1}\left(\frac{W_t}{t\sigma} + 1\right) \tag{4.7}$$

Proof. To prove this we will use formula (3.1) as it is gives the simplest and shortest proof. First remark that

$$\mathbf{D}_s X_t = \sigma X_t \quad \forall s < t.$$

Hence we get

$$\begin{aligned} H(X_t,1) &= \frac{W_t}{\int_0^t \sigma X_t \mathrm{d}s} + \frac{\int_0^t\int_0^t \sigma^2 X_t \mathrm{d}r\mathrm{d}s}{\left(\int_0^t \sigma X_t \mathrm{d}s\right)^2} \\ &= \frac{W_t}{t\sigma X_t} - \frac{\sigma^2 t^2 X_t}{\sigma^2 t^2 X_t^2} \\ &= X_t^{-1}\left(\frac{W_t}{t\sigma} + 1\right) \end{aligned}$$

□

4.4 X_t as a signature

We now consider a more general and abstract case, as we assume that $X_t = \langle \boldsymbol{\ell}, \widehat{\mathbb{W}}_t\rangle$, where $\boldsymbol{\ell} \in T((\mathbb{R}^d))$ is known and $\widehat{\mathbb{W}}$ is the signature of an augmented $d-1$-dimensional Brownian motion $\widehat{W}_t = (t, W_t^1, \ldots, W_t^{d-1}) \in \mathbb{R}^d$.

Theorem 4.3. As long as *everything is well defined*:

$$H_1(X_t,1) = \frac{W_t^i}{\langle \Psi_1^i(\boldsymbol{\ell}), \widehat{\mathbb{W}}_t\rangle} + \frac{\langle \Psi_1^i(\Psi_1^i(\boldsymbol{\ell})), \widehat{\mathbb{W}}_t\rangle}{(\langle \Psi_1^i(\boldsymbol{\ell}), \widehat{\mathbb{W}}_t\rangle)^2}, \qquad i = 2,\ldots,d \tag{4.8}$$

$$H_2(X_t,1) = \frac{\sum_{i=1}^d W_t^i}{\sum_{i=1}^d \langle \Psi_1^i(\boldsymbol{\ell}), \widehat{\mathbb{W}}_t\rangle} + \frac{\sum_{i,j=1}^d \langle \Psi_1^i(\Psi_1^j(\boldsymbol{\ell})), \widehat{\mathbb{W}}_t\rangle}{\sum_{i=1}^d (\langle \Psi_1^i(\boldsymbol{\ell}), \widehat{\mathbb{W}}_t\rangle)^2} \tag{4.9}$$

4.5 Stochastic Volatility model

We finally consider the case were X_t is a one dimensional Process such that $\frac{\mathrm{d}X_t}{X_t} = \sigma_t \mathrm{d}t$ where σ is a solution of $\mathrm{d}\sigma_t = a(\sigma_t)\mathrm{d}t + b(\sigma_t)\mathrm{d}W_t$ for some functions a and b. We can assume that $X_t = x\exp(-\int_0^t \frac{\sigma_s^2}{2}\mathrm{d}t + \int_0^t \sigma_s \mathrm{d}W_s)$ Using the notation of the SDE Framework section we have

$$\begin{cases} b(t, X_t) = 0 \\ \sigma(t, X_t) = \sigma_t X_t \end{cases}$$

So using that $\nabla_x X_t = \frac{X_t}{x}$, (3.8) becomes

$$H(X_t, G) = X_t^{-1}\left(G\int_0^t \sigma_s \mathrm{d}s\left(W_t - \int_0^t \sigma_s \mathrm{d}s - G\int_0^t\int_s^t \mathbf{D}_s\sigma_u \mathrm{d}W_u \mathrm{d}s\right.\right. \tag{4.10}$$

$$\left.\left.+\int_0^t\int_s^t \sigma_u \mathbf{D}_s\sigma_u \mathrm{d}u\mathrm{d}s\right) - \int_0^t \sigma_s \mathrm{d}s \int_0^t \mathbf{D}_s G\mathrm{d}s - G\int_0^t\int_s^t \mathbf{D}_s\sigma_u \mathrm{d}u\mathrm{d}s\right) \tag{4.11}$$

If we assume that we know a solution of σ of the form $\sigma_t = \langle \boldsymbol{\ell}_t, \widehat{\mathbb{W}}_t\rangle$, $\widehat{\mathbb{W}}_t$ is the signature of $\widehat{W}_t = (t, W_t^1, \ldots, W_t^d)$ and $(W_t)_{t\geq 0}$ is d-dimensional brownian motion, we can approximate this formula using the following computation rules:

$$\int_0^t\int_s^t \mathbf{D}_s\sigma_u \mathrm{d}W_u \mathrm{d}s = \int_0^t\int_0^t \mathbf{D}_s\langle \boldsymbol{\ell}_u, \widehat{\mathbb{W}}_u\rangle \mathrm{d}s\mathrm{d}W_u \tag{4.12}$$

$$= \int_0^t \langle \Psi_1^2(\boldsymbol{\ell}_u), \widehat{\mathbb{W}}_u\rangle \mathrm{d}W_u \tag{4.13}$$

and

$$\int_0^t\int_s^t \sigma_u \mathbf{D}_s\sigma_u \mathrm{d}W_u \mathrm{d}s = \int_0^t \sigma_u \int_0^t \mathbf{D}_s\langle \boldsymbol{\ell}_u, \widehat{\mathbb{W}}_u\rangle \mathrm{d}s\mathrm{d}W_u \tag{4.14}$$

$$= \int_0^t \sigma_u\langle \Psi_1^2(\boldsymbol{\ell}_u), \widehat{\mathbb{W}}_u\rangle \mathrm{d}W_u \tag{4.15}$$

and

$$\int_0^t\int_s^t \mathbf{D}_s\sigma_u \mathrm{d}u\mathrm{d}s = \int_0^t\int_0^t \mathbf{D}_s\langle \boldsymbol{\ell}_u, \widehat{\mathbb{W}}_u\rangle \mathrm{d}s\mathrm{d}u \tag{4.16}$$

$$= \int_0^t \langle \Psi_1^2(\boldsymbol{\ell}_u), \widehat{\mathbb{W}}_u\rangle \mathrm{d}u \tag{4.17}$$

So, with $G = 1$, (4.11) becomes

$$H(X_t, 1) = \left(X_t\int_0^t \sigma_s \mathrm{d}s\right)^{-1}\left(W_t + \int_0^t \langle \Psi_1^2(\boldsymbol{\ell}), \widehat{\mathbb{W}}_u\rangle \mathrm{d}W_u\right. \tag{4.18}$$

$$\left.-\int_0^t \sigma_u\langle \Psi_1^2(\boldsymbol{\ell}), \widehat{\mathbb{W}}_u\rangle \mathrm{d}u + \frac{\int_0^t \langle \Psi_1^2(\boldsymbol{\ell}), \widehat{\mathbb{W}}_u\rangle \mathrm{d}u}{\int_0^t \sigma_s \mathrm{d}s}\right) + X_t^{-1} \tag{4.19}$$

5 Implementation

In Theorem 2.2 we presented an infinite amount of formulas given a weight which we will simply call H. It is of course a viable option to simply take one randomly since they are all valid. Let us however first discuss how to optimally choose the formula and how to implement it. First let us reformulate the second formula to make things a bit more clear. For fixed $x, y \in \mathbb{R}$ it holds

$$\begin{aligned} p(x) &= E[(\mathbf{1}_{y\leq x\leq F} - \mathbf{1}_{y>x>F})H] \\ &= \mathbf{1}_{y\leq x}E[\mathbf{1}_{x\leq F}H] - \mathbf{1}_{y>x}E[\mathbf{1}_{y>x}H] \end{aligned}$$

So for each x, one could try to choose the optimal y. We can in fact find a every effective method to do so. Let us remind that we are estimating the expectation by Monte-Carlo simulation, i.e. we are generating $n \in \mathbb{N}$ paths of W_i, computing the weigths H_i an the processes F_i associated with each path and applying the LLN on the random variables $Y_{1i} = \mathbf{1}_{x \leq F_i}$ or $Y_{2i} = -\mathbf{1}_{x > F_i} H_i$. So to compare the convergence of both of these formulas we should compare the Variance of the Y's: since the Y_i's are i.i.d generated, with the CLT we can construct the confidence interval $\pm \frac{q\sigma}{\sqrt{n}}$ where q is the 95% quantile of $\mathcal{N}(0,1)$ and σ the standard deviation of Y, hence wanting to minimize the variance. Since $\text{Var}(Y) = E[Y^2] - E[Y]^2 = E[Y^2] - p(x)$ we simply need to minimize the second moment. In the large majority of cases it is however very difficult to compute this second moment by hand. Instead, let us show an effective recipe to choose p by estimating this second moment without needing too much extra computation:

For $x \in \mathbb{R}$

1. Compute $(\mathbf{1}_{x \leq L_i})_i$
2. Compute $E[H] = \frac{1}{n} \sum_{i=1}^{n} H_i$
3. Compute $E[\mathbf{1}_{x \leq L} H^2] = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{x \leq L_i} H_i^2$
4. Compute $E[\mathbf{1}_{x \leq L} H] = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{x \leq L_i} H_i$
5. If $E[\mathbf{1}_{x \leq L} H^2] \leq \frac{E[H^2]}{2}$: $p(x) := E[\mathbf{1}_{x \leq L} H]$, with empirical variance $E[\mathbf{1}_{x \leq L} H^2] - p(x)^2$.
   Else $p(x) = -E[\mathbf{1}_{x \geq L} H] = E[\mathbf{1}_{x \leq L} H] - E[H]$ with empirical variance $E[\mathbf{1}_{x \geq L} H^2] - p(x)^2 = E[H^2] - E[\mathbf{1}_{x \geq L} H] - p(x)^2$

This can be easily parallelized for a discrete intervale $(x_k)_{1 \leq k \leq K}$ in Python. The reason we choose the threshold $E[\mathbf{1}_{x \leq L} H^2] \leq \frac{E[H^2]}{2}$ is because this is equivalent to $E[\mathbf{1}_{x \leq L} H^2] \leq E[\mathbf{1}_{x > L} H^2]$. Also note that all expectations in the above recipe are empirical. In fact we will see below that in some cases, simply choosing a fixed y can give the same results. This due to the fact that $E[\mathbf{1}_{x \leq L} H^2]$ is monotically increasing in x while $E[\mathbf{1}_{x \geq L} H^2]$ is monotically decreasing. Hence there should exist an $x_0 \in \mathbb{R}$ s.t. $E[\mathbf{1}_{x \leq L} H^2] \leq E[\mathbf{1}_{x > L} H^2]$ for all $x > x_0$ and $E[\mathbf{1}_{x \leq L} H^2] \geq E[\mathbf{1}_{x > L} H^2]$ for all $x < x_0$, hence yielding optimal results when choosing $y = x_0$. However, as stated earlier, finding that optimal x_0 analytically is very difficult since it would be necessary to compute $E[\mathbf{1}_{x \leq L} H^2]$ by hand.

Let us briefly discuss the implementation of Brownian motions as it is used in all cases. We use Euler Scheme: to simulate a 1-dimensional path of a Brownian motion over $[0,t]$ we generate a discrete Interval $(x_k)_{0 \leq k \leq K}$ with $x_k = \frac{k}{t}$ as well as $(Z_k)_{1 \leq k \leq K} \overset{iid}{\sim} \mathcal{N}(0,t)$. We also set $dt := \frac{1}{K}$ and $W_0 = 0$ and set recursively $W_{x_{k+1}} = W_{x_k} + \sqrt{dt} \cdot Z_k$.

In the following subsections we will consider some examples of implementation and their results.

5.1 Implementation Brownian Motion

This case is very interesting since we can compute the density of X_t explicitly and hence observe the accuracy of our convergence. I implemented everything as described above and in the previous section and compared both the different formulas for the weights and the different density formula. Here we generate $bs = 10000$ Brownian motions of dimension $d = 15$ with a path of $K = 2000$ points. We generate $\ell = (\ell^i)_{i=1\ldots d}$ normally. We have the plots of H_1, H_2, and H_3 respectively in colors with their associated confidence interval of 95% with the actual density in black.

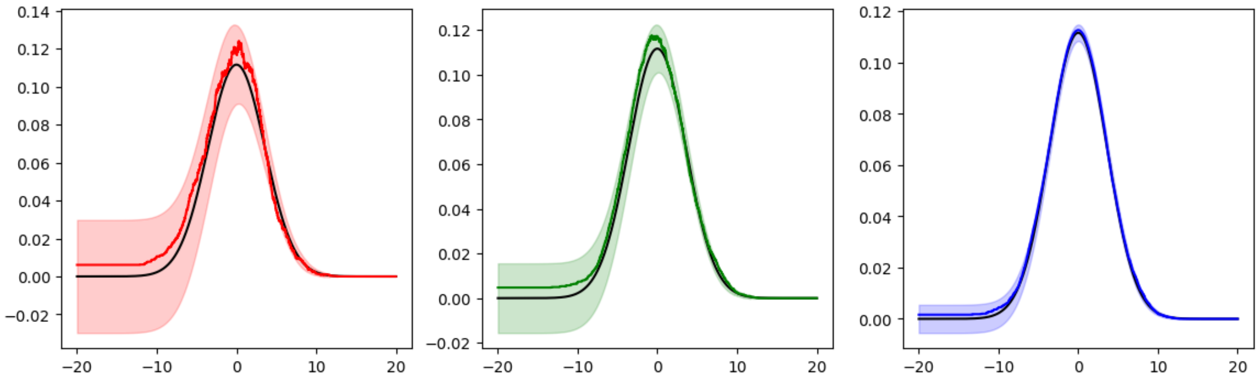

Figure 1: Plots using $p(x) = E[\mathbf{1}_{x \leq X_t} H(X_t, 1)]$

	average 1-norm	average 2-norm	∞-norm
H_1	0.0046	$8.2 \cdot 10^{-5}$	0.0134
H_2	0.0030	$5.5 \cdot 10^{-5}$	0.0099
H_3	0.0010	$1.6 \cdot 10^{-5}$	0.0028

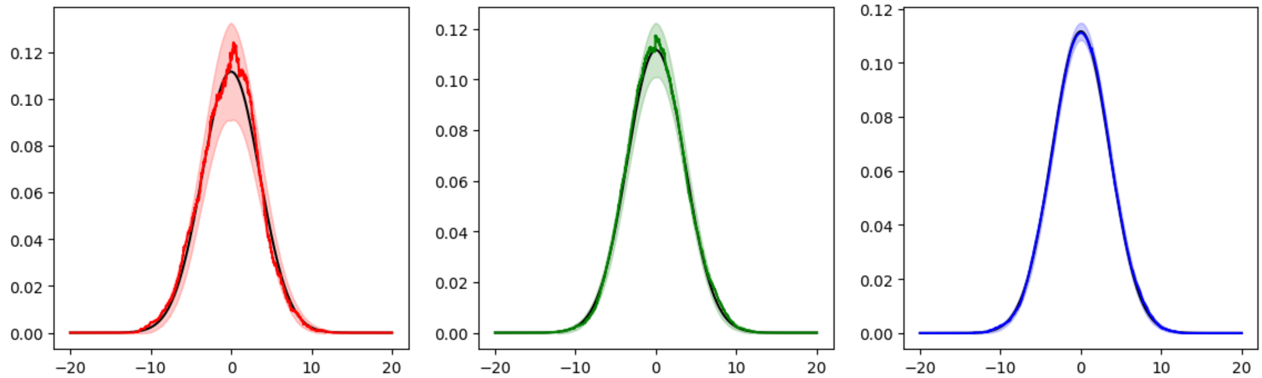

Figure 2: Plots using the more advanced algorithm

	average 1-norm	average 2-norm	∞-norm
H_1	0.0024	$5.7 \cdot 10^{-5}$	0.014
H_2	0.0015	$3.3 \cdot 10^{-5}$	0.0078
H_3	0.0004	$8.9 \cdot 10^{-6}$	0.0020

We can clearly see that H_3 is by far the best formula in terms of weight, and that the more advanced recipe for the density has a much better convergence in general. Note that in this case, when taking the intuitive value $y = 0$ for the density formula we get the same results as for the algorithm. This shows that the optimal value x_0 discussed previously is likely very close to, if not 0.

5.2 Implementation of Polynomials

As discussed earlier, a lot of issues arise because of domain definitions and necessary conditions that are not being respected. Let us however focus on cases where the density formula seems to have worked. We define three polynomials of different degree but similar structure:

$$X_t^1 = 2W_t^3 + 2W_t^2 + W_t + 1$$
$$X_t^2 = W_t^5 - 2W_t^4 + 2W_t^3 + 2W_t^2 + W_t + 1$$
$$X_t^3 = W_t^{13} - 2W_t^{10} + 2W_t^3 + 2W_t^2 + W_t + 1$$

Here the index i of X_t^i denotes which different processes and W_t^i denotes the i-th power of a one dimensional Brownian motion W_t at time $t > 0$. By simply blindly applying formula (4.4) and implementing it in python using a custom class of polynomials, we get the following graphs for $t = 1$:

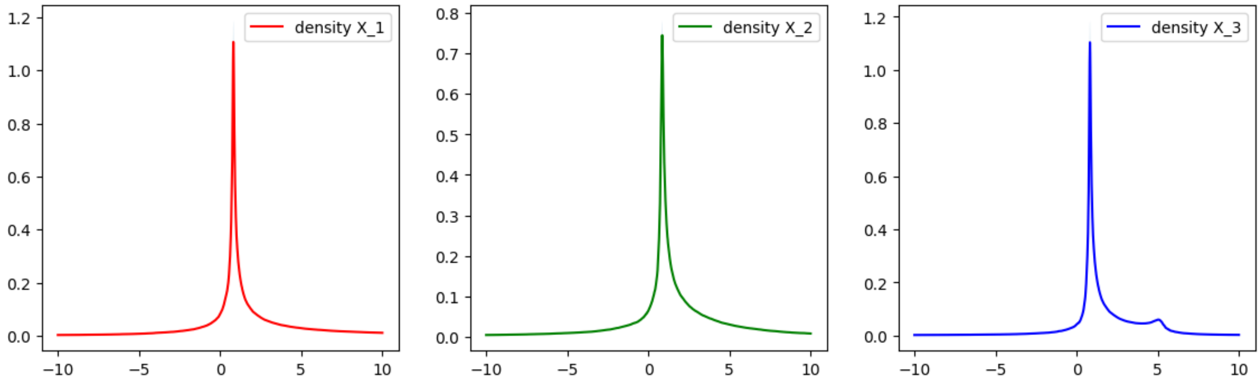

Figure 3: Enter Caption

Note that a confidence interval of level 5% is drawn but almost invisible: only a slight line at the very peek of each curve is visible. That is because on the rest of the graph, the emperical variance is very small:

	average 2-norm	∞-norm
$\mathrm{Var}(Y_t^1)$	5.01	0.0025
$\mathrm{Var}(Y_t^2)$	2.6	0.0014
$\mathrm{Var}(Y_t^3)$	5.9	0.0028

So the confidence interval is to small to be seen by the naked eye, which indicates good convergence. Also note that we used $bs = 10000$ Brownian motions on a path of $K = 5000$ points, which is quick to compute. In all 3 cases the integral value of these curves is of around 0.85, which is likely due to the fact that we integrated over a small interval $(-20, 20)$ or to numerical inaccuracies.

Finally let us look at the case we described earlier: $X_t = \frac{W_t^2}{2}$. We remind that the issue with the formula $p(x) = E[\mathbf{1}_{X_t \geq x} H]$ was that since $X_t \geq 0$, $p(x) = E[1/t + 1/W_t^2] = \infty$ for all $x < 0$. If we however use the improved algorithm we get the following graph:

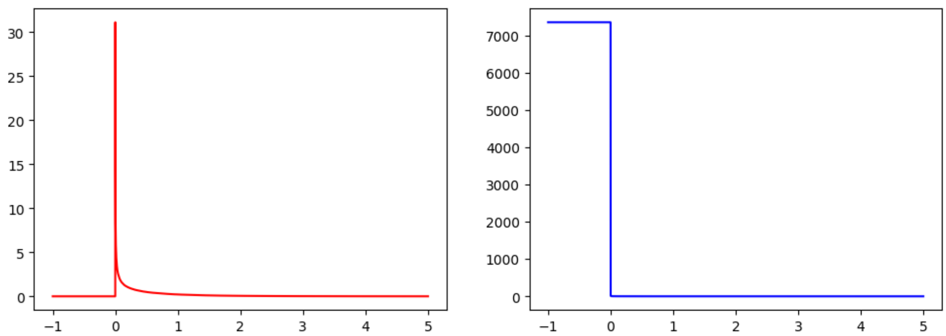

Figure 4: Advanced density formula (right-side), simple density formula (left-side)

Since X_t follows a (rescaled) $\mathcal{X}_1^2$ distribution which corresponds to an exponential distribution, the red plot is most likely correct since it also has an integral of 0.98. So we can see that even in cases where we explicitely know that the IBP formulas are not well defined, they can still work and yield good results.

5.3 Implementation of Geometric Brownian motion

We implement a geometric brownian motion X_t with a drift $b = 3$ and $\sigma = 1$ over the time interval $[0, 1]$. we generate $bs = 10000$ Brownian motions over a $K = 200$ point interval. Here is a plot of a couple of paths of X_t next to a plot of its density:

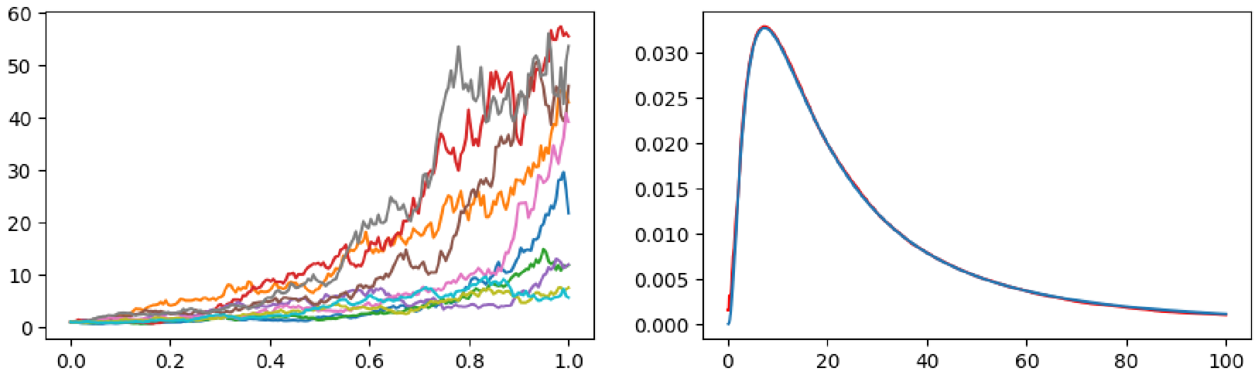

Figure 5: Geometric Brownian motion

On the left graph, the blue line corresponds to the actual density computed by hand and in red the simulate density. They are almost indistinguishable, so I did not bother to add a norm difference or a confidence interval.

5.4 Implementing signatures

To implement signatures in python I used the library Signatory. Due to some domain of definition errors, most advanced cases did not work. Let us see and example of $X_t = \langle \boldsymbol{\ell}, \widehat{\mathbb{W}}_t \rangle$ with $d = 2$, i.e. $\widehat{W}_t = (t, W_t^1)$ and $\boldsymbol{\ell} \in T^2(\mathbb{R}^d)$ having each entry i.i.d generated in $\mathcal{N}(0, 1)$. We generate $bs = 10000$ Brownian motions on a path of $K = 200$ points. We obtain the following density:

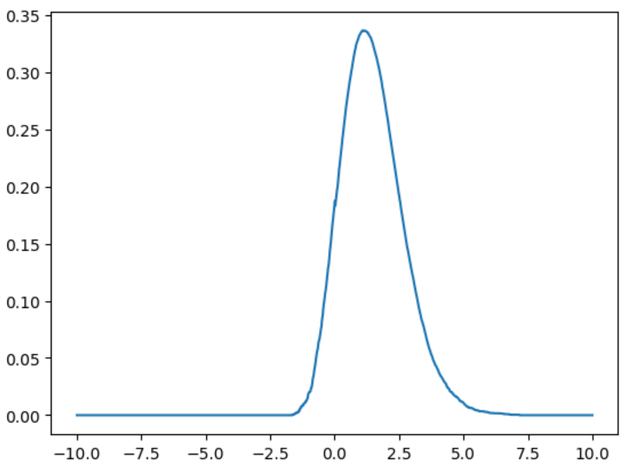

Figure 6: Density of a signature

The integral of this curve gives 0.97 which is a good sign and the curve looks rather smooth, however I have not verified the accuracy of this result.

5.5 Implementing Stochastic Volatility model

Unfortunately, due to the appearance of a lot of integrals in the weights, in most complex cases a lot of abhorrently large values appear leading to infinite values. We will only present a very simple case here: Let $d = 1$ and $\boldsymbol{\ell} \in T^1(\mathbb{R}^d)$. If we generate $bs = 10000$ Brownian motions on $K = 5000$ points, we get the following plots for sigma and the density of X_t:

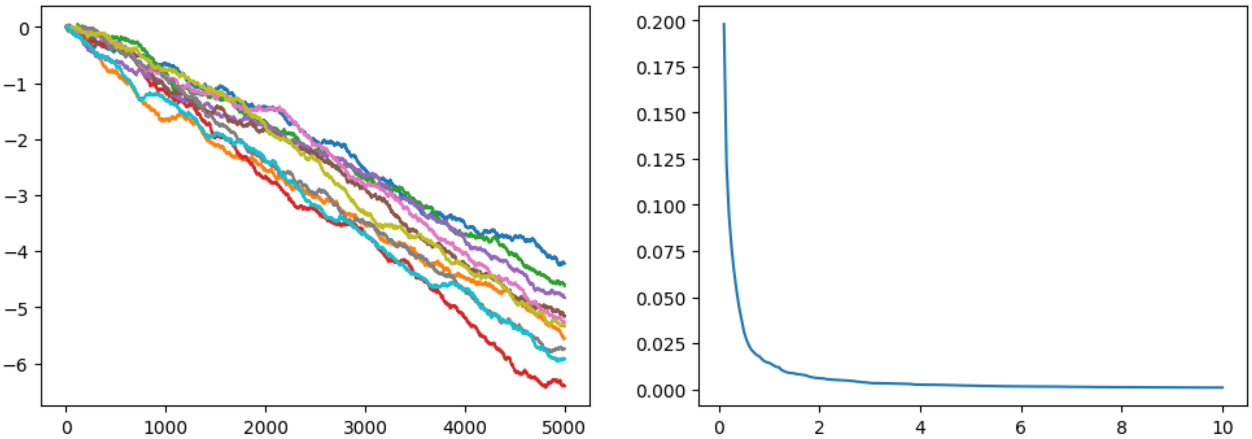

Figure 7: Stochastic Volatility density